CHAPTER SIX

COMPLEX ANALYTIC FUNCTIONS

6.1 COMPLEX NUMBERS

Definition 6.1: The complex number **z** can be defined as pair of real numbers such that

$\boldsymbol{z = x + iy}$ where $\boldsymbol{x}$ and $\boldsymbol{y}$ be any real numbers and i is called imaginary unit, which is defined as $\boldsymbol{i = \sqrt{-1}\ or\ i^2 = -1}$

REMARK: 1. Given a complex number $z = \boldsymbol{x + iy}$, $\boldsymbol{x}$ is called the real part of $\boldsymbol{z}$ and $\boldsymbol{y}$ is called the imaginary part of $\boldsymbol{z}$ and we write

$$\boldsymbol{x = Re\ (z) \quad and\ \ y = Im(z)}$$

Thus $\boldsymbol{z = Re\ (z) + i\ Im(z)}$ this is so called the standard form of a complex number.

Definition 6.2:

Let $\boldsymbol{z = x + iy}$ be a complex number**,** then the conjugate of $\boldsymbol{z}$ **,** denoted by $\boldsymbol{\bar{z}}$ **,** is defined as $\boldsymbol{\bar{z} = x - iy}$

FUNDAMENTAL OPERATIONS WITH COMPLEX NUMBERS

If $\boldsymbol{z_1 = x_1 + iy_1}$ **and** $\boldsymbol{z_2 = x_2 + iy_2}$ **,** then

1. $\boldsymbol{z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)}$
2. $\boldsymbol{z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2)}$
3. $\boldsymbol{z_1\, z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)}$
4. $\frac{z_1}{z_2} = \frac{x_1+iy_1}{x_2+iy_2} = \frac{x_1+iy_1}{x_2+iy_2} \cdot \frac{x_2-iy_2}{x_2-iy_2} = \frac{x_1x_2+y_1y_2}{{x_2}^2+{y_2}^2} + i\,\frac{x_2y_1-x_1y_2}{{x_2}^2+{y_2}^2}$ for $\boldsymbol{z_2 \neq 0}$

Definition 6.3 :(modulus or norms of complex numbers)

The modulus or norms of a complex number $\boldsymbol{z = x + iy}$ denoted by $\boldsymbol{|z|}$**and** is defined by

$$\boldsymbol{|z| = \sqrt{x^2 + y^2}}$$

Remark: If $z = x + iy$, then

- ❖ $\boldsymbol{\frac{z+\bar{z}}{2} = Re(z)}$
- ❖ $\boldsymbol{\frac{z-\bar{z}}{2i} = Im(z)}$
- ❖ $\boldsymbol{z\bar{z} = x^2 + y^2 = |z|^2}$

Examples 6.1: find the modulus for the following complex numbers:

a) $\frac{2+i}{1-i}$
b) 3-2i

Solution

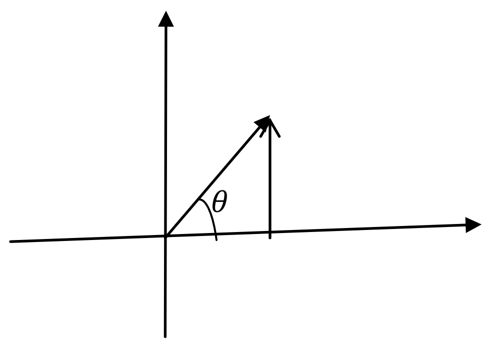
a) $z = \frac{2+i}{1-i} = \frac{2+i}{1-i}\,\frac{1+i}{1+i} = \frac{2(1+i)+i(1+i)}{1(1+i)-i(1+i)} = \frac{2+2i+i+\boldsymbol{i^2}}{1+i-i-\boldsymbol{i^2}} = \frac{2+3i-1}{1+1} = \frac{1+3i}{2} = \frac{1}{2} + i\,\frac{3}{2}$

Then $|\boldsymbol{z}| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{9}{4}} = \sqrt{\frac{10}{4}} = \frac{1}{2}\sqrt{\mathbf{10}}$

b) exercise

POLAR FORM OF COMPLEX NUMBERS

Any complex number $\boldsymbol{z = x + iy}$ can be represented by a polar coordinate $r\ and\ \theta$



But $\boldsymbol{x = r\cos\theta}$ **and** $\boldsymbol{y = r\sin\theta}$

Then $\boldsymbol{z = x + iy = r\cos\theta + ir\sin\theta}$ **which is called the polar form of the complex numbers, where** $\boldsymbol{r = \sqrt{x^2 + y^2}}$ **and** $\boldsymbol{\theta = tan^{-1}\left(\frac{y}{x}\right)}$

- $\boldsymbol{\theta}$ **is** called argument of $\boldsymbol{z}$ and denoted by $\mathbf{arg}\,\boldsymbol{z}$.
- The principal argument of $\boldsymbol{z}$, denoted by $\mathbf{Arg}\,\boldsymbol{z}$ is the value of $\boldsymbol{\theta}$ such that $\boldsymbol{-\pi} < \theta \le \pi$.
- In general $\mathbf{arg}\,\boldsymbol{z} = \mathbf{Arg}\,\boldsymbol{z} \pm \boldsymbol{2n\pi}, \boldsymbol{n} \in \mathbb{Z}$

Example 6.2: find the polar representation for the following complex numbers.

a. $1 + i$ b. $1 + i\sqrt{3}$

Solution:

a. $1 + i$

Then $\boldsymbol{r = \sqrt{x^2 + y^2} = \sqrt{1^2 + 1^2} = \sqrt{2}}$ and $\boldsymbol{\theta = tan^{-1}\left(\frac{y}{x}\right) = tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}}$

Therefore $\mathbf{Arg}\,\boldsymbol{z} = \boldsymbol{\theta} = \boldsymbol{\frac{\pi}{4}}$ and $\mathbf{arg}\,\boldsymbol{z} = \mathbf{Arg}\,\boldsymbol{z} \pm \boldsymbol{2n\pi} = \boldsymbol{\frac{\pi}{4}} \pm \boldsymbol{2n\pi}$

Hence $1 + i = \sqrt{\mathbf{2}}\left(\boldsymbol{cos}\left(\boldsymbol{\frac{\pi}{4}} \pm \boldsymbol{2n\pi}\right) + \boldsymbol{isin}\left(\boldsymbol{\frac{\pi}{4}} \pm \boldsymbol{2n\pi}\right)\right)$

b. $1+i\sqrt{3}$

Then $\boldsymbol{r}=\sqrt{\boldsymbol{x^2+y^2}}=\sqrt{\mathbf{1^2+(\sqrt{3})^2}}=\sqrt{\mathbf{4}}=\mathbf{2}$ and $\boldsymbol{\theta}=\boldsymbol{tan^{-1}}\left(\frac{y}{x}\right)=\boldsymbol{tan^{-1}}\left(\frac{\sqrt{3}}{1}\right)=\frac{\pi}{3}$

Therefore **Arg** $\boldsymbol{z}=\boldsymbol{\theta}=\frac{\pi}{3}$

In general **arg** $\boldsymbol{z}=$ **Arg** $\boldsymbol{z}\pm\mathbf{2}\boldsymbol{n\pi}=\frac{\pi}{3}\ \pm\mathbf{2}\boldsymbol{n\pi}$

Hence $1+i=\mathbf{2}\left(\boldsymbol{cos}\left(\frac{\pi}{3}\ \pm\mathbf{2}\boldsymbol{n\pi}\right)+\boldsymbol{isin}\left(\frac{\pi}{3}\ \pm\mathbf{2}\boldsymbol{n\pi}\right)\right)$

Euler's formula

Form: $e^{i\theta}=cos\theta+i\,sin\theta$

Remark: Let z be any complex number.

Then $z=x+iy=rcos\theta+irsin\theta=r(cos\theta+isin\theta)=re^{i\theta}$

De Moivre's theorem

Theorem:

$$(rcos\theta+irsin\theta)^n=r^n[cos(n\theta)+\ isin(n\theta)]\text{, where } n\in z$$

Proof: left as an exercise (hint use principle of mathematical induction)

Example 6.3: simplify the following

a. $\left(\mathbf{1}+\boldsymbol{i}\sqrt{\mathbf{3}}\right)^{\mathbf{5}}$ **b.** $\left(\mathbf{2}\sqrt{\mathbf{2}}-\boldsymbol{i}\mathbf{2}\sqrt{\mathbf{2}}\right)^{\mathbf{9}}$

Solution:

a. $z=1+i\sqrt{3}$, $\boldsymbol{r}=\sqrt{\mathbf{1^2+(\sqrt{3})^2}}=\sqrt{\mathbf{4}}=\mathbf{2}$ **and** $\boldsymbol{\theta}=\boldsymbol{tan^{-1}}\left(\frac{y}{x}\right)=\boldsymbol{tan^{-1}}\left(\frac{\sqrt{3}}{1}\right)=\frac{\pi}{3}$

Thus $1+i\sqrt{3}\ =2e^{i\frac{\pi}{3}}$

Hence $\left(\mathbf{1}+\boldsymbol{i}\sqrt{\mathbf{3}}\right)^{\mathbf{5}}=\left(2e^{i\frac{\pi}{3}}\right)^{\mathbf{5}}=\mathbf{2^5}\boldsymbol{e}^{\boldsymbol{i}\left(\frac{5\pi}{3}\right)}=\mathbf{32}\left(\boldsymbol{cos}\left(\frac{5\pi}{3}\right)+\boldsymbol{isin}\left(\frac{5\pi}{3}\right)\right)$

$$=32\left(\frac{1}{2}+i\left(-\frac{\sqrt{3}}{2}\right)\right)=16-i16\sqrt{3}$$

b. Left as an exercise

Roots of complex numbers

Let $\boldsymbol{n}\in\boldsymbol{z}^{+}$, then any none zero complex has $\boldsymbol{n}$ distinct $\boldsymbol{n^{th}}$ roots. Let $\boldsymbol{z_0}$ be a none zero complex numbers, we wish to solve $\boldsymbol{z^n}=\boldsymbol{z_0}$ or $\boldsymbol{z}=(\boldsymbol{z_0})^{\frac{1}{n}}$, $\boldsymbol{n}\in\boldsymbol{z}^{+}$.

Let $\boldsymbol{z}=\boldsymbol{r}(\boldsymbol{cos\theta}+\boldsymbol{isin\theta})$ **and** $\boldsymbol{z_0}=\boldsymbol{r_0}(\boldsymbol{cos\theta_0}+\boldsymbol{isin\theta_0})$

Thus $\boldsymbol{z^n}=\boldsymbol{z_0}$

$\Rightarrow(\boldsymbol{rcos\theta}+\boldsymbol{irsin\theta})^{\boldsymbol{n}}=r_0(\boldsymbol{cos\theta}_0+\boldsymbol{isin\theta}_0)$

$\Rightarrow\boldsymbol{r^n}[\boldsymbol{cos}(\boldsymbol{n\theta})+\ \boldsymbol{isin}(\boldsymbol{n\theta})]=r_0(\boldsymbol{cos\theta}_0+\boldsymbol{isin\theta}_0)$

Then $r^n=r_0$ and $\boldsymbol{n\theta}=\boldsymbol{\theta}_0+2k\pi$

$$r=(r_0)^{\frac{1}{n}}\text{ and }\theta=\frac{\theta_0+2k\pi}{n}\text{ for }k=0,1,2,3,\dots,n-1$$

Thus $\boldsymbol{z}=(\boldsymbol{r_0})^{\frac{1}{n}}\left(\boldsymbol{cos}\left(\frac{\theta_0+2k\pi}{n}\right)+\boldsymbol{isin}\left(\frac{\theta_0+2k\pi}{n}\right)\right)$ **for** $k=0,1,2,3,\dots,n-1$

Example 6.4: find the following roots

a. $\left(\mathbf{1}+\boldsymbol{i}\sqrt{\mathbf{3}}\right)^{\frac{1}{4}}$ **b.** $(\boldsymbol{i})^{\frac{1}{5}}$ **c.** $(\mathbf{1})^{\frac{1}{5}}$

Solution:

a. **Her** $\boldsymbol{z_0}=\mathbf{1}+\boldsymbol{i}\sqrt{\mathbf{3}}$ **and** $\boldsymbol{n}=\mathbf{4}$

Then $r_0=\sqrt{\mathbf{1}^2+\left(\sqrt{\mathbf{3}}\right)^2}=\sqrt{\mathbf{4}}=\mathbf{2}$ **and** $\boldsymbol{\theta}_0=\boldsymbol{tan}^{-1}\left(\frac{\sqrt{3}}{1}\right)=\frac{\pi}{3}$

Hence $\boldsymbol{z}=(\boldsymbol{r_0})^{\frac{1}{n}}\left(\boldsymbol{cos}\left(\frac{\theta_0+2k\pi}{n}\right)+\boldsymbol{isin}\left(\frac{\theta_0+2k\pi}{n}\right)\right)$ **for** $k=0,1,2,3,\dots,n-1$

Now $\boldsymbol{z}=\mathbf{2}^{\frac{1}{4}}\left(\boldsymbol{cos}\left(\frac{\frac{\pi}{3}+2k\pi}{4}\right)+\boldsymbol{isin}\left(\frac{\frac{\pi}{3}+2k\pi}{4}\right)\right)$ **for** $k=0,1,2,3$

If $k=0$, $\boldsymbol{z}=\mathbf{2}^{\frac{1}{4}}\left(\boldsymbol{cos}\left(\frac{\frac{\pi}{3}}{4}\right)+\boldsymbol{isin}\left(\frac{\frac{\pi}{3}}{4}\right)\right)=\mathbf{2}^{\frac{1}{4}}\left(\boldsymbol{cos}\left(\frac{\pi}{12}\right)+\boldsymbol{isin}\left(\frac{\pi}{12}\right)\right)$

If $k=1$, $z=\mathbf{2}^{\frac{1}{4}}\left(\boldsymbol{cos}\left(\frac{\frac{\pi}{3}+2\pi}{4}\right)+\boldsymbol{isin}\left(\frac{\frac{\pi}{3}+2\pi}{4}\right)\right)=\mathbf{2}^{\frac{1}{4}}\left(\boldsymbol{cos}\left(\frac{7\pi}{12}\right)+\boldsymbol{isin}\left(\frac{7\pi}{12}\right)\right)$

If $k=2$, $\boldsymbol{z}=\mathbf{2}^{\frac{1}{4}}\left(\boldsymbol{cos}\left(\frac{\frac{\pi}{3}+4\pi}{4}\right)+\boldsymbol{isin}\left(\frac{\frac{\pi}{3}+4\pi}{4}\right)\right)=\mathbf{2}^{\frac{1}{4}}\left(\boldsymbol{cos}\left(\frac{13\pi}{12}\right)+\boldsymbol{isin}\left(\frac{13\pi}{12}\right)\right)$

If $k=3$, $\boldsymbol{z}=\mathbf{2}^{\frac{1}{4}}\left(\boldsymbol{cos}\left(\frac{\frac{\pi}{3}+6\pi}{4}\right)+\boldsymbol{isin}\left(\frac{\frac{\pi}{3}+6\pi}{4}\right)\right)=\mathbf{2}^{\frac{1}{4}}\left(\boldsymbol{cos}\left(\frac{19\pi}{12}\right)+\boldsymbol{isin}\left(\frac{19\pi}{12}\right)\right)$

b and c are left's as an exercise

Example 6.5: solve $z^4+4=0$

Solution: $z^4+4=0$

$$z^4=-4\Rightarrow \boldsymbol{z}=(-\mathbf{4})^{\frac{1}{4}}$$

Her $z_0=-4$ and $n=4$, $r_0=\sqrt{(-4)^2}=\sqrt{16}=4$ and $\theta_0=tan^{-1}\left(\frac{0}{-4}\right)=\pi$

Hence z is given by $\boldsymbol{z}=\mathbf{4}^{\frac{1}{4}}\left(\boldsymbol{cos}\left(\frac{\pi+2k\pi}{4}\right)+\boldsymbol{isin}\left(\frac{\pi+2k\pi}{4}\right)\right)$ **for** $k=0,1,2,3$

If $k=0$, $\boldsymbol{z}=\mathbf{4}^{\frac{1}{4}}\left(\boldsymbol{cos}\left(\frac{\pi}{4}\right)+\boldsymbol{isin}\left(\frac{\pi}{4}\right)\right)=\sqrt{\mathbf{2}}\left(\frac{\sqrt{2}}{2}+\boldsymbol{i}\frac{\sqrt{2}}{2}\right)=\mathbf{1}+\boldsymbol{i}$

If $k=1$, $\boldsymbol{z}=\mathbf{4}^{\frac{1}{4}}\left(\boldsymbol{cos}\left(\frac{\pi+2\pi}{4}\right)+\boldsymbol{isin}\left(\frac{\pi+2\pi}{4}\right)\right)=\sqrt{\mathbf{2}}\left(\boldsymbol{cos}\left(\frac{3\pi}{4}\right)+\boldsymbol{isin}\left(\frac{3\pi}{4}\right)\right)$

$$=\sqrt{\mathbf{2}}\left(\frac{-\sqrt{2}}{2}+\boldsymbol{i}\frac{\sqrt{2}}{2}\right)=-\mathbf{1}+\boldsymbol{i}$$

If $k=2$, $\boldsymbol{z}=\mathbf{4}^{\frac{1}{4}}\left(\boldsymbol{cos}\left(\frac{\pi+4\pi}{4}\right)+\boldsymbol{isin}\left(\frac{\pi+4\pi}{4}\right)\right)=\sqrt{\mathbf{2}}\left(\boldsymbol{cos}\left(\frac{5\pi}{4}\right)+\boldsymbol{isin}\left(\frac{5\pi}{4}\right)\right)$

$$=\sqrt{\mathbf{2}}\left(\frac{\sqrt{2}}{2}-\boldsymbol{i}\frac{\sqrt{2}}{2}\right)=-\mathbf{1}-\boldsymbol{i}$$

If $k=3$, $\boldsymbol{z}=\mathbf{4}^{\frac{1}{4}}\left(\boldsymbol{cos}\left(\frac{\pi+6\pi}{4}\right)+\boldsymbol{isin}\left(\frac{\pi+6\pi}{4}\right)\right)=\sqrt{\mathbf{2}}\left(\boldsymbol{cos}\left(\frac{7\pi}{4}\right)+\boldsymbol{isin}\left(\frac{7\pi}{4}\right)\right)$

$$=\sqrt{\mathbf{2}}\left(\frac{-\sqrt{2}}{2}-\boldsymbol{i}\frac{\sqrt{2}}{2}\right)=\mathbf{1}-\boldsymbol{i}$$

<u>**6.2 functions of complex variables**</u>

A function of the complex variable $\boldsymbol{z}$ is a rule that assigns to each value $\boldsymbol{z}$ in a set D one and only one complex value $\boldsymbol{w}$. We write $\boldsymbol{w = f(z)}$ and call w the image of z under f. The set D is called the domain of definition of f and the set of all image $\boldsymbol{R = \{w = f(z): z \in D\}}$ is called the range of f.

$\boldsymbol{z}$ can be expressed **by** $\boldsymbol{z = u(x,y) + iv(x,y)}$, where $\boldsymbol{u(x,y)}$ is the real parts and $\boldsymbol{v(x,y)}$ is the imaginary part of the function of complex variables. This gives us the representation

$$f(z) = f(x,y) = \boldsymbol{u(x,y) + iv(x,y)}$$

Example 6.6: find the domain of the following functions of complex variable and write as $\boldsymbol{f(x,y) = u(x,y) + iv(x,y)}$

a. $\boldsymbol{f(z) = \frac{1}{z^2+1}}$ **b.** $\boldsymbol{f(z) = \frac{z}{\bar{z}+z}}$ **c.** $\boldsymbol{f(z) = z + \frac{1}{z}}$

Solution:

a. $\boldsymbol{f(z) = \frac{1}{z^2+1}}$

Domain of $f(z) = \{z/z \neq \pm i\}$

Let $z = x + iy$, $\boldsymbol{f(z) = (x,y) = \frac{1}{(x+iy)^2+1} = \frac{1}{x^2+i(2xy)-y^2+1} = \frac{1}{(x^2+-y^2+1)+i(2xy)}}$

$$\boldsymbol{= \frac{1}{(x^2-y^2+1)+i(2xy)} = \frac{1}{(x^2-y^2+1)+i(2xy)}\left(\frac{(x^2-y^2+1)-i(2xy)}{(x^2-y^2+1)-i(2xy)}\right)}$$

$$\boldsymbol{= \frac{(x^2-y^2+1)-i(2xy)}{(x^2-y^2+1)^2+(2xy)^2}}$$

$$\boldsymbol{= \frac{(x^2-y^2+1)}{(x^2-y^2+1)^2+(2xy)^2} - \frac{i(2xy)}{(x^2-y^2+1)^2+(2xy)^2}}$$

Then $\boldsymbol{u(x,y) = \frac{(x^2-y^2+1)}{(x^2-y^2+1)^2+(2xy)^2}}$ And $\boldsymbol{v(x,y) = -\frac{(2xy)}{(x^2-y^2+1)^2+(2xy)^2}}$

Hence $\boldsymbol{f(z) = f(x,y) = u(x,y) + iv(x,y)}$

$$\boldsymbol{= \frac{(x^2 - y^2 + 1)}{(x^2 - y^2 + 1)^2 + (2xy)^2} - \frac{i(2xy)}{(x^2 - y^2 + 1)^2 + (2xy)^2}}$$

b and c are left as an exercise

Remark: if $a_0, a_1, a_2, a_3, \ldots\ldots, a_n$ are complex constant to the function

$f(z) = a_0 + a_1 z + a_2 z^2 + \ \ldots\ a_n z^n$ is said to be a polynomial in z

6.2.1: Limit and continuity(Reading assignment for student's)

Definition of Limit:

Let $\boldsymbol{w = f(z)}$ be a complex function of the complex variable $\boldsymbol{z}$ that is defined for all values of$\boldsymbol{z}$ in some neighborhood of $\boldsymbol{z_0}$, except perhaps at the point$\boldsymbol{z_0}$. We say that $\boldsymbol{f}$ has the limit $\boldsymbol{w_0}$ provided that the value $\boldsymbol{f(z)}$ gets close to the value$\boldsymbol{w_0}$ as $\boldsymbol{z}$ gets close to $\boldsymbol{z_0}$.

Then we write $\boldsymbol{\lim_{z\to z_0} f(z) = w_0}$ **if and only if**

For every number $\boldsymbol{\varepsilon} > 0$ ***there is a number*** $\boldsymbol{\delta} > 0$ ***such that***

$$\boldsymbol{0 < |z - z_0| < \delta \ \Rightarrow |f(z) - w_0| < \varepsilon}$$

Example 6.7: show that $\lim_{z\to 4}(4z-4)=12$

Proof: let $\boldsymbol{\varepsilon}>0$ be given. we must find $\delta>\mathbf{0}$ such that

$$\mathbf{0}<|\boldsymbol{z}-\mathbf{4}|<\delta \Rightarrow |\boldsymbol{f}(\boldsymbol{z})-\mathbf{12}|<\varepsilon$$

Consider

$$|\boldsymbol{f}(\boldsymbol{z})-\mathbf{12}|<\varepsilon \Rightarrow |(\mathbf{4z}-\mathbf{4})-\mathbf{12}|<\varepsilon$$
$$\Rightarrow |\mathbf{4z}-\mathbf{16}|<\varepsilon$$
$$\Rightarrow |\mathbf{4}(\boldsymbol{z}-\mathbf{4})|<\varepsilon$$
$$\Rightarrow \mathbf{4}|\boldsymbol{z}-\mathbf{4}|<\varepsilon$$
$$\Rightarrow |\boldsymbol{z}-\mathbf{4}|<\frac{\boldsymbol{\varepsilon}}{\mathbf{4}}$$

Now choose $\boldsymbol{\delta}=\frac{\boldsymbol{\varepsilon}}{\mathbf{4}}$

Thus $\mathbf{0}<|\boldsymbol{z}-\mathbf{4}|<\delta \Rightarrow |\boldsymbol{z}-\mathbf{4}|<\frac{\boldsymbol{\varepsilon}}{\mathbf{4}} \Rightarrow \mathbf{4}|\boldsymbol{z}-\mathbf{4}|=|(\mathbf{4z}-\mathbf{4})-\mathbf{12}|<\varepsilon$

Exercise: using $\boldsymbol{\varepsilon}-\boldsymbol{\delta}$ definition of limit prove that

a. $\lim_{z\to 1}\frac{z^2-1}{z-1}=2 \;\; for\; z\neq 1$

b. $\lim_{z\to 3}(z^2+z)=12$

Theorem: if limit of $f(z)$ as z approaches z_0 exists, then the limit is unique.

Proof: left as an exercise

Example: evaluate $\lim_{z\to 0}\frac{z}{\bar{z}}$

Solution: let $z=x+iy \Rightarrow \bar{z}=x-iy$

Then $\lim_{z\to 0}\frac{z}{\bar{z}}=\lim_{(x,y)\to(0,0)}\frac{x+iy}{x-iy}$

i. **Along the $x\; axis$(that means $y=o, z=x\; and\; \bar{z}=x$), we have**

$\lim_{z\to 0}\frac{z}{\bar{z}}=\lim_{(x,0)\to(0,0)}\frac{x+i(0)}{x-i(0)}=\lim_{z\to 0}\frac{x}{x}=\lim_{(x,0)\to(0,0)}1=1$

ii. **Along the y _ axis , we have**

$$\lim_{z\to 0}\frac{z}{\bar{z}}=\lim_{(0,y)\to(0,0)}\frac{0+iy}{0-iy}=\lim_{z\to 0}\frac{iy}{-iy}=\lim_{(x,0)\to(0,0)}-1=-1$$

iii. **Let's approach** to the $z_0=0$ through the $y=mx$ where m is the slope. we have

$\lim_{(x,mx)\to(0,0)}\frac{x+i(mx)}{x-i(mx)}=\lim_{(x,mx)\to(0,0)}\frac{x(1+im)}{x(1-im)}=\lim_{(x,mx)\to(0,0)}\frac{1+im}{1-im}=\frac{1+im}{1-im}$

We conclude from $(i)-(iii)$ the limit is different

Hence $\lim_{z\to 0}\frac{z}{\bar{z}}$ does not exist.

Basic limit theorems

If $f(z)$ and $g(z)$ be two complex valued functions $\lim_{z\to z_0}f(z)=L_1$ and $\lim_{z\to z_0}g(z)=L_2$, $L_1, L_2\in\mathbb{C}$, then

1) $\lim_{z\to z_0}(f(z)\pm g(z))=L_1\pm L_2$
2) $\lim_{z\to z_0}\alpha f(z)=\alpha L_2$ where α be any real numbers
3) $\lim_{z\to z_0}(f(z).g(z))=L_1.L_2$
4) $\lim_{z\to z_0}\left(\frac{f(z)}{g(z)}\right)=\frac{L_1}{L_2}$, provided $L_2\neq 0$

Continuity:

Definition: let f be a complex valued function defined on a region $s \subseteq \mathbb{C}$, then f is said to be continuous at $z_0 \in \boldsymbol{s}$ if $\lim_{z \to z_0} f(z) = f(z_0)$

Example: show that

$$f(z) = \begin{cases} \frac{xy}{x^2+y^2} - i\frac{y^2}{x^2+y^2}, & if \;\; z \neq 0 \\ 0 & , if \;\; z = 0 \end{cases} \quad \text{is not continuous at } z = 0$$

Solution: first let's find $\lim_{z \to z_0} f(z)$

Let's approach to $z = 0$ through the line $y = mx \;\; where\; m\; is\; the\; slope$, we have

$$\lim_{z \to 0} f(z) = \lim_{(x,mx) \to (0,0)} \frac{x(mx)}{x^2 + (mx)^2} - i\frac{(mx)^2}{x^2 + (mx)^2}$$

$$= \lim_{(x,mx) \to (0,0)} \frac{mx^2}{x^2+m^2x^2} - i\frac{m^2x^2}{x^2+m^2x^2}$$

$$= \lim_{(x,mx) \to (0,0)} \frac{x^2}{x^2}\left(\frac{m}{1+m^2} - i\frac{m^2}{1+m^2}\right)$$

$$= \lim_{x \to 0} \frac{m}{1+m^2} - i\frac{m^2}{1+m^2} = \frac{m}{1+m^2} - i\frac{m^2}{1+m^2}$$, m is any real number.

Hence the limit does not exist

Therefore f is not continuous at $z = 0$

Example: determine the region in which $f(z)$ is continuous

a. $f(z) = \frac{Rel(z)}{(z-1)(z^2+9)}$ b. $f(z) = \frac{1}{1+(z-\pi)^2}$ c. $f(z) = \frac{z^2+2z+3}{z+1}$

Solution:

a. $f(z) is\; not\; defined\; at\; (z-1)(z^2+9) = 0 \;\Rightarrow z = 1 \; and \; z = \pm 3i$

Hence f is continuous in the region

$$D = \{z/z \neq 1, \pm 3i\}$$

b. $1 + (z-\pi)^2 = 0 \Rightarrow (z-\pi)^2 = -1$

$$\Rightarrow z - \pi = \pm i \;\Rightarrow z = \pi \pm i$$

Hence f is continuous in the region

$$D = \{z/z \neq \pi \pm i\}$$

c. $z + 1 = 0 \Rightarrow z = -1$

Hence f is continuous in the region

$$D = \{z/z \neq -1\}$$

Theorem:

Let $f(z)$ and $g(z)$ be continuous complex function, then $f \pm g$, $f.g$, and $\frac{f}{g}$, provided that $g \neq 0$ are continuous.

Proof: left as an exercise

6.2.2: derivatives of a complex functions

Definition: let z_0 be a fixed complex numbers and $\triangle z$ be a complex variable the derivative of the function $f(z)$ at $z = z_0$ is defined as

$$\boldsymbol{f'(z_0) = \lim_{\triangle z\to 0} \frac{f(z_0+\triangle z)-f(z_0)}{\triangle z}}$$

If this limit exists

In general

$$\boldsymbol{f'(z) = \lim_{\triangle z\to 0} \frac{f(z+\triangle z)-f(z)}{\triangle z}}$$

Example: find the derivative of $f(z) = z^2 - 2z$ at the point $z = z_0$

Solution: be definition

$$\boldsymbol{f'(z_0) = \lim_{\triangle z\to 0} \frac{f(z_0+\triangle z)-f(z_0)}{\triangle z} = \lim_{\triangle z\to 0} \frac{(z_0+\triangle z)^2-2(z_0+\triangle z)-(z_0{}^2-2z_0)}{\triangle z}}$$

$$\boldsymbol{= \lim_{\triangle z\to 0} \frac{z_0{}^2+2z_0\triangle z+(\triangle z)^2-2z_0-2\triangle z-(z_0{}^2-2z_0)}{\triangle z}}$$

$$= \lim_{\triangle z\to 0} \frac{2z_0\triangle z+(\triangle z)^2-2\triangle z}{\triangle z}$$

$$= \lim_{\triangle z\to 0} 2z_0 + \triangle z - 2 = 2z_0 - 2 = 2(z_0 - 1)$$

Exercise:

1. Let $w = f(z) = |z|^2$ is w differentiable at $z = 0$?
2. Show that $\lim_{\boldsymbol{\triangle z\to 0}} \frac{\overline{\boldsymbol{\triangle z}}}{\boldsymbol{\triangle z}}$ does not exist.
3. Differentiate the following using the definition
   a. $f(z) = z^3$ b. $f(Z) = \frac{1}{z^2}$ at $z = z_0$

Properties of derivatives of a complex function

1. $\frac{d}{dz}(c) = 0$ for c be a constant complex number
2. $\frac{d}{dz}(f(z) + g(z)) = \frac{d}{dz}\big(f(z)\big) + \frac{d}{dz}\big(g(z)\big)$
3. $\frac{d}{dz}(f(z)g(z) = g(z)\frac{d}{dz}\big(f(z)\big) + f(z)\frac{d}{dz}\big(g(z)\big)$
4. $\frac{d}{dz}\left(\frac{f(z)}{g(Z)}\right) = \frac{g(z)\frac{d}{dz}(f(z))-f(z)\frac{d}{dz}(g(z))}{(g(z))^2}$ provided that $g(z) \neq 0$
5. $\frac{d}{dz}(z^n) = nz^{n-1}$
6. $\frac{d}{dz}f\big(g(z)\big) = f'(g(z))g'(z)$

Example: evaluate

a. $\frac{d}{dz}(1 + z^4)$ b. $\frac{d}{dz}cos(sinz)$ c. $\frac{d}{dz}(sinz + cosz$

Solution:

a. $\frac{d}{dz}(1 + z^4) = 4z^3$

b. $\frac{d}{dz}cos(sinz) = -sin(sinz)\,cos$

c. Left as an exercise

6.3: Cauchy− Riemann Equations

Objective: to drive basic criteria to identify whether a given function is analytic or not.

Suppose that a function $f(z)$ has a derivative at z and let

$$f(z) = u(x,y) + iv(x,y)$$

The derivative is independent of the path is which $\triangle z$ tends to zero as

$$f'(z) = \lim_{\triangle z \to 0} \frac{f(z+\triangle z)-f(z)}{\triangle z}$$

Case 1: let's approach through $\triangle y = 0$, $\triangle z \to 0$ ($\triangle z = \triangle x + i \triangle y$)

Thus

$$f'(z) = \lim_{\triangle z \to 0} \frac{f(z+\triangle z)-f(z)}{\triangle z}$$

$$= \lim_{\triangle x \to 0} \frac{f(z+\triangle x)-f(z)}{\triangle x}$$

$$= \lim_{\triangle x \to 0} \frac{u(x+\triangle x, y) + iv(x+\triangle x, y) - (u(x,y) + iv(x,y))}{\triangle x}$$

$$= \lim_{\triangle x \to 0} \frac{u(x+\triangle x, y) - (u(x,y)}{\triangle x} + i \lim_{\triangle x \to 0} \frac{v(x+\triangle x, y) - v(x,y))}{\triangle x}$$

$$= u_x + iv_x$$

Then $f'(z) = u_x + iv_x$ **(1)**

<u>**Case 2**</u>: let's approach through $\triangle x = 0$, $\triangle z \to 0$ ($\triangle z = \triangle x + i \triangle y$)

Thus

$$f'(z) = \lim_{\triangle z \to 0} \frac{f(z+\triangle z)-f(z)}{\triangle z}$$

$$= \lim_{i\triangle y \to 0} \frac{f(z+\triangle y)-f(z)}{i\triangle y}$$

$$= \lim_{i\triangle y \to 0} \frac{u(x,y+\triangle y)+iv(x,y+\triangle y)-(u(x,y)+iv(x,y))}{i\triangle y}$$

$$= \lim_{i\triangle y \to 0} \frac{u(x,\ y+\triangle y)-(u(x,y)}{i\triangle y} + i \lim_{i\triangle y \to 0} \frac{v(x,\ y+\triangle y)-v(x,y))}{i\triangle y}$$

$$= (-i) \lim_{i\triangle y \to 0} \frac{u(x,\ y+\triangle y)-(u(x,y)}{\triangle y} + \lim_{i\triangle y \to 0} \frac{v(x,\ y+\triangle y)-v(x,y))}{\triangle y}$$

$$= -iu_y + v_y$$

Hence $f'(z) = -iu_y + v_y$ **(2)**

Equating equation (1) and (2)

Since the limit is unique, then we have

$$u_x + iv_x = -iu_y + v_y$$

$$u_x = v_y \text{ and } v_x = -u_y \qquad \textbf{(3)}$$

Thus equation (3) is said to be **Cauchy−Riemann equations**

Remark: the Cauchy− Riemann conditions are the necessary conditions for the existence of the derivative of the $f(z)$.

Examples: verify that Cauchy Riemann equation for the following.

a. $f(z) = e^z$
b. $f(z) = z^2$
c. $g(z) = cosxcoshy - isinxsinhy$

Solution:

a. $(z) = e^z = e^{x+iy} = e^x e^{iy} = e^x(cosy + isiny)$

Her $u(x,y) = e^x cosy$ and $v(x,y) = e^x siny$

Then $\frac{\partial u}{\partial x} = e^x cosy,\quad \frac{\partial u}{\partial y} = -e^x siny\ , \frac{\partial v}{\partial x} = e^x siny$ and $\frac{\partial v}{\partial y} = e^x cosy$

Hence $\frac{\partial u}{\partial x} = e^x cosy = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = e^x siny = -\frac{\partial u}{\partial y}$

Therefore the Cauchy Riemann equations are satisfied.

b. $f(z) = z^2 = (x+iy)^2 = x^2 + 2ixy - y^2 = (x^2 - y^2) + i(2xy)$

$u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$

$\mathrm{u_x} = 2\mathrm{x} = \mathrm{v_y}$ and $\mathrm{v_x} = 2\mathrm{y} = -\mathrm{u_y}$

Hence the Cauchy Riemann condition is satisfied.

C. $g(z) = cosxcoshy - isinxsinhy$

$u(x,y) = cosxcoshy$ and $v(x,y) = -sinxsinhy$

Then $\frac{\partial u}{\partial x} = -sinxcohsy,\quad \frac{\partial u}{\partial y} = cosxsinhy\ , \frac{\partial v}{\partial x} = -cosxsinhy$

and $\frac{\partial v}{\partial y} = -sinxcoshy$

$\mathrm{u_x} = -sinxcohsy = \mathrm{v_y}\ , \mathrm{v_x} = -cosxsinhy\ = -\mathrm{u_y}$

Hence the C.R.C is satisfied.

Remark: If the Cauchy Riemann conditions are not satisfied for a complex function at any point, then we can conclude that the function is not differentiable.

Theorem:

Let $\boldsymbol{f(z) = u(x,y) + iv(x,y)}$ be a complex function defined in a region D such that u and v their first order partial derivatives $\left(u_x, u_y, v_x, v_y\right)$ exists and are continuous in D.

If the first order partial derivatives satisfy the Cauchy Riemann condition at a point $z = (x,y)$ in D , then f is differentiable at $z = x + iy$

Example:

1. Prove/disprove the following the function are differentiable
   a. $f(z) = x^3 - 3xy^2 + i(3yx^2 - y^3)$
   b. $f(z) = 2 + iz$
   c. $f(z) = sinxcoshy + i(cosxsinhy)$
   d. $f(z) = z(Imz)$
2. Find the value of of $a, b\ and\ c$ such that

$f(z) = x + ay - i(bx + cy)$ is differentiable at every point.

Solution:

1.

a. $f(z) = x^3 - 3xy^2 + i(3yx^2 - y^3)$

$u(x, y) = x^3 - 3xy^2$ and $v(x, y) = 3yx^2 - y^3$

Then $u_x = 3x^2 - 3y^2$, $u_y = -6xy$, $v_x = 6xy$ and $v_y = 3x^2 - 3y^2$

Now $u_x = 3x^2 - 3y^2 = v_y$

$u_y = -6xy = -v_x$

Hence the Cauchy Riemann conditions are satisfied

Therefore f is differentiable.

b. $f(z) = 2 + iz = 2 + i(x + iy) = 2 - y + ix$ (since $z = x + iy$)

her $u(x, y) = 2 - y$ and $v(x, y) = x$

then $u_x = 0$, $u_y = -1$, $v_x = 1$ and $v_y = 0$

now $u_x = 0 = v_y$ and $u_y = -1 = -v_x$

Hence the Cauchy Riemann conditions are satisfied

Therefore f is differentiable

c. Left as an exercise

d. $f(z) = z(Imz) = (x + iy)y = xy + iy^2$

$u(x, y) = xy$ *and* $v(x, y) = y^2$

$u_x = y$, $u_y = x$, $v_x = 0$ and $v_y = 2y$

But $u_x \neq v_y$ and $u_y \neq -v_x$

Hence the Cauchy Riemann conditions are not satisfied

Therefore f is not differentiable

2. Left as an exercise

Cauchy Riemann condition with polar coordinates

Theorem: let $f(z) = u(r, \theta) + iv(r, \theta)$ be a differentiable function at $z = re^{i\theta}$. Then the Cauchy Riemann conditions are

$$\boldsymbol{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}\ ,\ \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}}$$

Example: show that $f(z) = z^2$ satisfies the CRC

Solution: let $z = re^{i\theta}$

$$f(z) = z^2 = \left(re^{i\theta}\right)^2 = r^2e^{i(2\theta)} = r^2\left(\cos(2\theta) + isin(2\theta)\right)$$

$$= r^2\cos(2\theta) + ir^2\sin(2\theta)$$

$u(r, \theta) = r^2\cos(2\theta)$ and $v(r, \theta) = r^2\sin(2\theta)$

$\frac{\partial u}{\partial r} = 2rcos(2\,\theta)$, $\boldsymbol{\frac{\partial u}{\partial \theta}} = -\boldsymbol{2}r^2\sin(2\theta)$

$\frac{\partial v}{\partial r} = 2rsin(2\,\theta)$, $\boldsymbol{\frac{\partial v}{\partial \theta}} = \boldsymbol{2}r^2\cos(2\theta)$

$$\frac{\partial u}{\partial r} = 2rcos(2\,\theta) = \frac{\mathbf{1}}{\boldsymbol{r}}(\mathbf{2}r^2\cos(2\theta)) = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

Therefore $\boldsymbol{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}}$

$$\boldsymbol{\frac{\partial u}{\partial \theta}} = -\mathbf{2}r^2\sin(2\theta) = -r(2rsin(2\,\theta)) = -r\frac{\partial v}{\partial r}$$

Therefore $\boldsymbol{\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}}$

Hence this is satisfied the CRC

Analytic functions

Definition: A function f is said to be analytic at a point $z_0 \in \mathbb{C}$ if it is differentiable at every z in some $\varepsilon -$ neighborhoods of the point z_0.

The function f is said to be analytic in a region if it is analytic at every point in the region. The function f is said to be entire if it is analytic in $\mathbb{C}$.

A point z_0 where a function $f(Z)$ is no analytic is said to be a singular point or briefly singularity of f.

Example:

1. $f(z) = \frac{1}{z}$, her singularity is $z = 0$
2. $f(z) = \frac{1}{z-1}$, singularity is $z = 1$
3. $f(z) = \frac{1}{(z-4)(z-3+i)}$, then the singularity are $z = 4\ and\ z = 3 - i$

Example:

1. Identify which of the following functions are analytic.
   a. $f(z) = z^3 + 2$
   b. $f(z) = e^x(\cos(y) + isin(y))$
   c. $f(z) = e^{-x}(\cos(y) - isin(y))$

Solution:

a. $f(z) = z^3 + z$

Let $z = re^{i\theta}$

$$f(z) = \left(re^{i\theta}\right)^3 + 2 = r^3e^{3i\theta} + 2 = r^3(\cos(3\theta) + isin(3\theta)) + 2$$
$$= (r^3\cos(3\theta) + 2) + ir^3\sin(3\theta)$$

Her $u(r,\theta) = r^3\cos(3\theta) + 2$ and $v(r,\theta) = r^3\sin(3\theta)$

$$\frac{\partial u}{\partial r} = 3r^2cos(3\,\theta)\,,\ \boldsymbol{\frac{\partial u}{\partial \theta}} = -\mathbf{3}r^3\sin(3\theta)$$

$$\frac{\partial v}{\partial r} = 3r^2sin(3\,\theta)\quad,\ \boldsymbol{\frac{\partial v}{\partial \theta}} = \mathbf{3}r^3\cos(3\theta)$$

$$\frac{\partial u}{\partial r} = 3r^2cos(3\,\theta) = \frac{\mathbf{1}}{\boldsymbol{r}}(\mathbf{3}r^2\cos(3\theta)) = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

Therefore $\boldsymbol{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}}$

$$\boldsymbol{\frac{\partial u}{\partial \theta}} = -\mathbf{3}r^3\sin(2\theta) = -r(3r^2sin(3\,\theta)) = -r\frac{\partial v}{\partial r}$$

Therefore $\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$

Hence $f(z)$ is CRC satisfied

$f(z)$ is analytic function.

b and c left as an exercises

Harmonic functions

Definition:

Let $f(x,y)$ be a function of two variables, then the equation

$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ is said to be **Laplace equation**.

Definition: any function that has continuous partial derivatives of the second order and that satisfy the Laplace equation is called the **harmonic functions**.

Theorem: the real and imaginary parts of an analytic function

$f(z) = u(x,y) + iv(x,y)$ is harmonic function.

Example: is $u(x,y) = x^2 + y$ harmonic?

Solution:

$$\frac{\partial u}{\partial x} = 2x \ , \frac{\partial^2 u}{\partial x^2} = 2$$
$$\frac{\partial u}{\partial y} = 1 \ , \frac{\partial^2 u}{\partial y^2} = 0$$
$$\therefore \ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 + 0 = 2 \neq 0$$

Therefore $u(x,y)$ is not harmonic function.

Exercise: prove that $u(x,y) = e^{-x}\big(x\sin(y) - ycos(y)\big)$ is harmonic.

Remark: Laplace equation provides us the necessary condition for a function to be the real or imaginary part of an analytic function.

Harmonic conjugates

If we are given a function $u(x,y)$ that is harmonic in the domain D and if we can find another harmonic function $v(x,y)$ where their first partial derivatives satisfy the Cauchy – Riemann equations throughout D, then we say that $v(x,y)$ *is the harmonic conjugate of* $u(x,y)$. It then follows that the function

$f(z) = u(x,y) + iv(x,y)$ is analytic in D

Theorem: let $f(z) = u(x,y) + iv(x,y)$ be an analytic function, then $\boldsymbol{v}$ is harmonic conjugate of $\boldsymbol{u}$ if and only if $\boldsymbol{u}$ is harmonic conjugate of $-\boldsymbol{v}$

Proof left as an exercise

Example: find a harmonic conjugate of $u(x,y) = e^x cosy$

Solution:

$u(x,y) = e^x cosy$

$$\frac{\partial u}{\partial x} = e^x cosy \ , \frac{\boldsymbol{\partial^2 u}}{\boldsymbol{\partial x^2}} = e^x cosy$$

$$\frac{\partial u}{\partial y} = -e^x siny \ , \frac{\boldsymbol{\partial^2 u}}{\boldsymbol{\partial y^2}} = -e^x cosy$$

$$\therefore \ \frac{\boldsymbol{\partial^2 u}}{\boldsymbol{\partial x^2}} + \frac{\boldsymbol{\partial^2 u}}{\boldsymbol{\partial y^2}} = e^x cosy - e^x cosy = \mathbf{0}$$

Hence $u(x, y)$ is harmonic

Let $\boldsymbol{v}(\boldsymbol{x}, \boldsymbol{y}$ be a harmonic conjugate of $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y})$

$f(z) = u(x, y) + iv(x, y)$ is analytic

Then $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ , \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^x cosy \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\frac{\partial v}{\partial X} = -\frac{\partial u}{\partial y} = e^x siny \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

First integrating equation (1) with respect to y , we have

$$v(x, y) = \int e^x cosydy$$

$$v(x, y) \ = e^x siny \ + g(x) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Differentiating equation (3) with respect to x, we get

$$\frac{\partial v}{\partial X} = e^x siny + g^{'}(x) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Now equating (2) and (3) we have

$$e^x siny = e^x siny + g^{'}(x)$$

$$g^{'}(x) = 0$$

Then $g(x) = c$, where c is any constant

Thus, $\boldsymbol{v}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{e^x siny} \ + \boldsymbol{c}$

Exercise: find a harmonic conjugate for $u(x, y) = y^3 - 3x^2y$

6.4 Elementary Functions

A. **Exponential functions**:

Form: $f(z) = e^z \ = e^x(cosy + isiny)$ where $z = x + iy$

Properties:

- $\frac{d}{dz}(e^z) = e^z$
- $e^{z_1}e^{z_2} \ = e^{z_1+z_2}$
- $|e^z| = e^x$ where $z = x + iy$

B. **Trigonometric functions:**

Recall that Euler's formula

Thus $e^{i\theta} = cos\theta + isin\theta$ ……………………………………………….(1)

$e^{-i\theta} = cos\theta - isin\theta$ ……………………………………………….(2)

From (1) and (2) we have

$cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ and $sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

Thus

- $cosz = \frac{e^{iz}+e^{-iz}}{2}$
- $sinz = \frac{e^{iz}-e^{-iz}}{2i}$
- $tanz = \frac{sinz}{cosz}$

Property:

- $\frac{d}{dz}(cosz) = -sinz$
- $\frac{d}{dz}(sinz) = cosz$
- $\sin(-z) = -sinz$
- $\cos(-z) = cosz$
- $\sin(z_1 + z_2) + \sin(z_1 - z_2) = 2\sin(z_1 z_2)$
- $sin^2(z) + cos^2(z) = 1$
- $\sin(z_1 + z_2) = \sin(z_1)\cos(z_2) + \sin(z_2)\cos(z_1)$
- $\cos(z_1 + z_2) = \cos(z_1)\cos(z_2) - \sin(z_1)\sin(z_2)$

C. **Hyperbolic functions**

Definition: the hyperbolic sine and hyperbolic cosine of complex variables are defined as

$\cosh(z) = \frac{e^z + e^{-z}}{2}$ $\sinh(z) = \frac{e^z - e^{-z}}{2}$

Since e^z *and* e^{-z} are analytic functions, then $\cosh(z)$ and $\sinh(z)$ are analytic everywhere.

Properties:

- $\frac{d}{dz}(\cosh(z)) = \sinh(z)$
- $\frac{d}{dz}(\sinh(z)) = \cosh(z)$
- $\sinh(z_1 + z_2) = \sin h(z_1)\cos h(z_2) + \sin h(z_2)\cos h(z_1)$
- $cosh^2(z) - sinh^2(z) = 1$
- $\cosh(-z) = \cosh(z)$
- $\sinh(-z) = -\sinh(z)$
- $\sinh(iz) = i\sin(z)$
- $\cosh(iz) = \cos(z)$
- $\sin(iz) = i\sinh(z)$
- $\cos(iz) = \cosh(z)$

D. **Logarithmic functions**

Consider $e^w = z\,, z \neq 0$ and $z, w \in \mathbb{C}$

Let $w = u + iv\ \ , z = x + iy = re^{i\theta}$

$where\ r = |z|\ and\ \theta = argz = Argz + 2n\pi, n = 0, \pm 1, \pm 2, \dots..$

Now $e^w = e^{u+iv} = z = re^{i\theta}$

$e^{u+iv} = re^{i\theta}$

$e^u e^{iv} = re^{i\theta}$

$e^u = r\ \ and\ e^{iv} = e^{i\theta}$

$u = lnr$ and $v = \theta + 2n\pi$

Therefore $logz = w = u + iv = lnr + i(\theta + 2n\pi)$

$$\boldsymbol{logz = ln|z| + i(argz)}$$

DEFINTION: the logarithmic function of a non zero complex number z is given by $\boldsymbol{logz = ln|z| + i(argz)}$

Where $\boldsymbol{argz} = \theta + 2n\pi\ and\ |\boldsymbol{z}| = \boldsymbol{r}$

REMARK: 1. If θ is a principal value only, then $\boldsymbol{logz = ln|z| + i(Argz)}$

EXAMPLE: solve the following

a. $log(1)$

b. $log(-1)$

c. $log(-1+\boldsymbol{i})$

Solution:

a. $\boldsymbol{log(1)}$

Be definition we have

$log(1) = ln|1| + i(\arg(1)) = 0 + iarg(1+i0) = i(0^{\circ} + 2n\pi) = 2n\pi i$

b. $log(-1) = ln|-1| + i(\arg(-1)) = 0 + iarg(-1+i0)$

$= i(\pi + 2n\pi)$ where $n = 0, \pm 1, \pm 2, \dots$..

c. $log(-1+\boldsymbol{i}) = ln|-1+i| + i(\arg(-1+i)) = ln\sqrt{2} + iarg(-1+i1)$

$= ln\sqrt{2} + i(\frac{3\pi}{4} + 2n\pi)$

$since\ arg(-1+i1) = \frac{3\pi}{4} + 2n\pi\ , n = 0, \pm 1, \dots$

E. **COMPLEX EXPONENTS**

Note: $\boldsymbol{e^{logz} = e^{ln|z| + i(argz)}}$

EXAMPLE: Simplify the following

a. $(i)^i$

b. $1^{\sqrt{2}}$

c. $Re\left((1+i)^i\right)$

Solution:

a. $(i)^i = \boldsymbol{e^{log\left(e^{i}\right)^{i}}} = \boldsymbol{e^{i(log\,i)}} = \boldsymbol{e^{i(ln|i| + i(\arg(i)))}} = \boldsymbol{e^{i\left(0 + i\left(\frac{\pi}{2} + 2n\pi\right)\right)}}$

$= \boldsymbol{e^{-\left(\frac{\pi}{2} + 2n\pi\right)}}$ **where** $n = 0, \pm 1, \pm 2, \dots$..

b. $1^{\sqrt{2}} = \boldsymbol{e^{log 1^{\sqrt{2}}}} = \boldsymbol{e^{\sqrt{2}(log 1)}} = \boldsymbol{e^{\sqrt{2}(ln|1| + i(\arg(1)))}} = \boldsymbol{e^{\sqrt{2}\left(0 + i\left(0^{\circ} + 2n\pi\right)\right)}}$

$= \boldsymbol{e^{\sqrt{2}\left(i\left(0^{\circ} + 2n\pi\right)\right)}} = \boldsymbol{e^{i(2\sqrt{2}\,n\pi)}}$

c. $Re\left((1+i)^i\right)$

But $(1+i)^i = \boldsymbol{e^{log}}(1+i)^i = \boldsymbol{e^{i(log(1+i))}} = \boldsymbol{e^{i(ln|1+i| + i(\arg(1+i)))}}$

$= \boldsymbol{e^{i\left(ln\sqrt{2} + i\left(\frac{\pi}{4} + 2n\pi\right)\right)}} = \boldsymbol{e^{iln\sqrt{2} - \left(\frac{\pi}{4} + 2n\pi\right)}} = \boldsymbol{e^{-\left(\frac{\pi}{4} + 2n\pi\right)} e^{iln\sqrt{2}}}$

$= \boldsymbol{e^{-\left(\frac{\pi}{4} + 2n\pi\right)}\left(cos\left(ln\sqrt{2}\right) + isin\left(ln\sqrt{2}\right)\right)}$

$= \boldsymbol{e^{-\left(\frac{\pi}{4} + 2n\pi\right)} cos\left(ln\sqrt{2}\right) + ie^{-\left(\frac{\pi}{4} + 2n\pi\right)} sin\left(ln\sqrt{2}\right)}$

Therefore $Re\left((1+i)^i\right) = \boldsymbol{e^{-\left(\frac{\pi}{4} + 2n\pi\right)} cos\left(ln\sqrt{2}\right)}$

6.5 complex integral

Definition: the definite integral of a complex function $\int_{\alpha}^{\beta} f(z)dz$

Between α *and* β is defined in terms of the values of $f(z)$ at the points along the curve from α *to* β. The integral is called line integral or curve linear integral.

Let $f(Z) = u(x,y) + iv(x,y)$

And $z = x + iy$. Then $\frac{dz}{dx} = \frac{d}{dx}(x + iy) = 1 + i\frac{dy}{dx}$

$$dz = dx + idy$$

Thus $\int_{\alpha}^{\beta} f(z)dz$

$$\int_{\alpha}^{\beta}(u(x,y) + iv(x,y)).(dx + idy)$$

$$\int_{\alpha}^{\beta}(u(x,y)dx - v(x,y)dy) + i(v(x,y)dx + u(x,y)dy)$$

$$\int_{\alpha}^{\beta} u(x,y)dx - v(x,y)dy + i\int_{\alpha}^{\beta} v(x,y)dx + u(x,y)dy$$

$$\int_{c} u(x,y)dx - v(x,y)dy + i\int_{c} v(x,y)dx + u(x,y)dy$$

Provided this integral exists and C represents the curve from α *to* β.

Definition: let $f(z): [a,b] \to \mathbb{C}$ be a complex function and let $f(x) = u(x) + iv(x)$ for $a \le x \le b$, then

$$\int_{a}^{b} f(x)dx = \int_{a}^{b} u(x)dx + i\int_{a}^{b} v(x)dx$$

Example: let $f(x) = 1 - ix^2$ on $1 \le x \le 2$. Then find $\int_{1}^{2} f(x)dx$

Solution: $\int_{1}^{2} f(x)dx = \int_{1}^{2}(1 - ix^2)dx = \int_{1}^{2} dx - i\int_{1}^{2} x^2dx = 1 - i\frac{7}{3}$

Exercise: let $f(x) = \cos(2x) + isin(2x)$ on $0 \le x \le \frac{\pi}{4}$. Then find $\int_{0}^{\frac{\pi}{4}} f(x)dx$

Definition: let $f(z)$ be a complex function and let $C: [a,b] \to \mathbb{C}$ be as a smooth curve is the plane. Assume that $f(z)$ is continuous at all points on C. Then the integral of over C is defined to be

$$\int_{C} f(z)dz = \int_{a}^{b} f\big(z(t)\big).r'(t)dt\ , \quad \text{where} \quad z = z(t) \ \text{and}\ a \le x \le b$$

Example: evaluate $\int_{c} \bar{z}dz$ where $C: r(t) = e^{it}$ on $0 \le t \le 2\pi$

Solution:

$$\int_c \bar{z}dz = \int_0^{2\pi} \overline{(e^{\imath t})} \left(e^{it}\right)' dt = \int_0^{2\pi} e^{-it} \; ie^{it}dt = i\int_0^{2\pi} e^{-it+it}\, dt = 2\pi i$$

Exercise:

1. evaluate $\int_c z^2dz$, where $C: r(t) = t + it$ for $0 \le t \le 1$
2. evaluate $\int_c z^2dz$ on the curve $C: y = x^2$ for $0 \le x \le 2$
3. integrate $\int_c f(z)dz$, where $f(z) = Rel(z)$ along the line segment from $z = 0 \; to \; z = 1 + i$
4. compute the value of the integral $\int_0^{2+i} z^2dz$

Definition:

1. A domain D is said to be simply connected in a complex plane if every simple closed path in D end closes only points of D.
2. A domain D which is not simply connected is called connected .

CACHY'S INTEGRAL THEOREM

THEOREM: (THE CACHY'S INTEGRAL THEOREM)

Let C be a simple closed curve. If $f(z)$is analytic within the region bounded by C as well as on curve C. then we have Cauchy's theorem that

$$\int_c f(z)dz = \oint_c f(z)dz = 0$$

Where the second integral emphasizes the fact that C is a simple closed curve.

Proof: left as an exercise

Example:

1. Evaluate $\int_c z^2\, dz$, where C is a unit circle in counter clockwise

   Solution: since $f(Z) = z^2$ is analytic

   By the above theorem

   $$\int_c z^2\, dz = 0$$

2. $\int_c (z^2 + z + 2)\, dz$, $C: |Z - 1| = 2$

   Left as an exercise

CAUCHY INTEGRAL FORMULA

THEOREM: let $f(z)$ is analytic in a simply connected domain D, then for any point $\boldsymbol{z_0}$ **in D** and any simple closed path C in D that enclosed $\boldsymbol{z_0}$

$$\oint_c \frac{f(z)}{z-z_0} dz = (2\pi i) f(z_0)$$

Where C is transverse in the positive (counter clockwise) sense.

Proof: left as an exercise.

Example: evaluate

a. $\oint_c \frac{e^z}{z-2} dz$ where $C: |z-2| = 1$

b. $\oint_c \frac{z^2-4}{z^2+4} dz$, where C is

I. $|z-i| = 2$

II. $|z-1| = 2$

III. $|z+3i| = 2$

Solution:

a. $|z-2| = 1$ here center = (2,0) and radius is 1

Then $f(z) = e^z$ **is analytic** and $z_0 = 2$ is inside the curve

Therefore $\oint_c \frac{e^z}{z-2} = (2\pi i) f(z_0) = 2\pi i e^2$

b. $\oint_c \frac{z^2-4}{z^2+4} dz = \oint_c \frac{z^2-4}{(z-2i)(z+2i)} dz$

i. $\oint_c \frac{z^2-4}{(z-2i)(z+2i)} dz = \oint_c \frac{\frac{z^2-4}{z+2i}}{z-2i} dz$, since $|z-i| = 2$

Then $f(z) = \frac{z^2-4}{z+2i}$ and $z_0 = 2i$ is inside the curve

Hence $\oint_c \frac{z^2-4}{(z-2i)(z+2i)} dz = \oint_c \frac{\frac{z^2-4}{z+2i}}{z-2i} dz = (2\pi i) f(z_0) = 2\pi i f(2i)$

But $f(2i) = \frac{(2i)^2-4}{2i+2i} = \frac{-8}{4i} = 2i$

Then $\oint_c \frac{z^2-4}{(z-2i)(z+2i)} dz = \oint_c \frac{\frac{z^2-4}{z+2i}}{z-2i} dz = 2\pi i f(2i) = (2\pi i)(2i) = -4\pi$

ii. Since $|z-1| = 2$

When $z_0 = 2i$, then $|2i-1| = \sqrt{5}$

$\therefore \sqrt{5} > 2$

$\therefore z_0 = 2i$ is out of the domain

When $z_0 = -2i$, then $|-2i - 1| = \sqrt{5}$

$\therefore \sqrt{5} > 2$

$\therefore z_0 = -2i$ is out of the domain

The points $2i \; and - 2i$ are outside the curve

Thus $\oint_c \frac{z^2-4}{z^2+4} dz = 0$ by Cauchy integral theorem

iii. Left as an exercise