CHAPTER 5
LINE AND SRFACE INTEGRALS

5.1 LINE INTEGRAL

In this section we define an integral that is similar to a single integral except that instead of integrating over an interval $[a, b]$, we integrate over a curve . Such integrals are called ***line integrals***, although "curve integrals" would be better terminology. They were invented in the early 19th century to solve problems involving fluid flow, forces, electricity, and magnetism. Primary here are some of the more basic curves that we will need to know how to do as well as limits on the parameter if they are required.

curves	Parametric Equations		Interval
	Counter-clockwise	clockwise	
Ellipse - $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$x = a\ cost$, $y = b\ sint$	$x = a\ cost$, $y = -b\ sint$	$0 \le t \le 2\pi$
Circle - $x^2+y^2=r^2$	$x = r\ cost$, $y = r\ sint$	$x = r\ cost$, $y = -r\ sint$	$0 \le t \le 2\pi$
$y = f(x)$	$x = t\ , y = f(t)$		For all x the function defined
$x = g(y)$	$y = t\ , x = g(t)$		For all y the function defined
Line segment from $(x_0y_0z_0)$ to $(x_1y_1z_1)$	$x = (1-t)x_0 + tx_1$ $y = (1-t)y_0 + ty_1$ $z = (1-t)z_0 + tz_1$		$0 \le t \le 1$

5.1.1 Line integral of a scalar filed

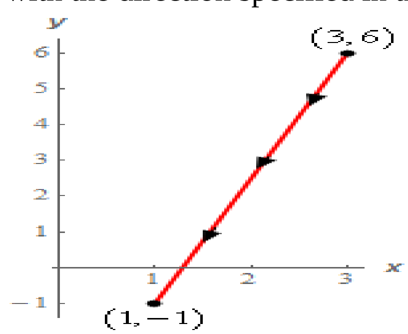
Definition: For some scalar field $f : U \subseteq \mathbb{R}^{\boldsymbol{n}} \to \mathbb{R}$, the line integral along a smooth curve $C \subset U$ is defined as

$$\int_C f ds = \int_a^b f(r(t)||\overrightarrow{r'}(t)||\, dt.$$

where $\boldsymbol{r}$: $[a, b] \to C$ is an arbitrary parameterization of the curve C such that $\boldsymbol{r}(a)$ and $\boldsymbol{r}(b)$ give the endpoints of C and $a < b$.The function f is called the integrand, the curve C is the domain of integration, and the symbol ds may be intuitively interpreted as an elementary arc length.

Example 5.1: Evaluate $\int_C (3x^2 - 2y)\, ds$, where C is the line segment from $(3,6)$ to $(1, -1)$.

Solution: Here is sketch of C with the direction specified in the problem statement shown.



So, we'll need to parameterize this line and we know how to parameterize the equation of a line between two points. Here is the vector form of the parameterization of the line.

$$\vec{r}(t) = (1-t)(3,6) + t(1,-1) = (3-2t, 6-7t), \quad 0 \le t \le 1$$

We could also break this up into parameter form as follows.

$x = 3 - 2t \; and \; y = 6 - 7t,$ for $0 \le t \le 1.$

We'll need the magnitude of the derivative of the parameterization so let's get that.

$$\overrightarrow{r'}(t) = (-2,7) \Rightarrow ||\overrightarrow{r'}(t)|| = \sqrt{(-2)^2 + (-7)^2} = \sqrt{53}$$

We will also need the integrand evaluated at the parameterization.

$$3x^2 - 2y = 3(3-2t\,)^2 - 2(6-7t) = 3(3-2t\,)^2 - 12 + 14t$$

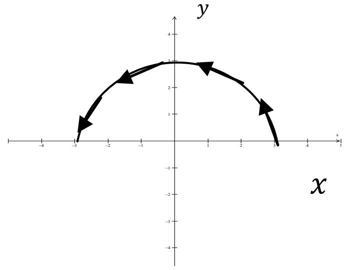
Then the line integral is

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))||\overrightarrow{r'}(t)||dt$$

$$= \int_0^1 [3(3-2t\,)^2 - 12 + 14t]\,\sqrt{53}\,dt = 8\sqrt{53}.$$

Example 5.2: Evaluate $\int_C (2 + x^2 y)\,ds$, where C is the upper half of the unit circle $x^2 + y^2 = 1$ with counterclockwise rotation.

Solution: First, Here is a quick sketch of *C* with the direction specified in the problem statement shown.



we must write C in parametric form. The upper half of the unit circle is

$x(t) = cost, y(t) = sint, 0 \le t \le \pi$, then

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))||\overrightarrow{r'}(t)||dt$$

$$= \int_0^\pi (2 + cos^2t\; sint)\sqrt{cos^2t + sin^2t}dt$$

$$= \int_0^\pi (2 + cos^2t\; sint)dt$$

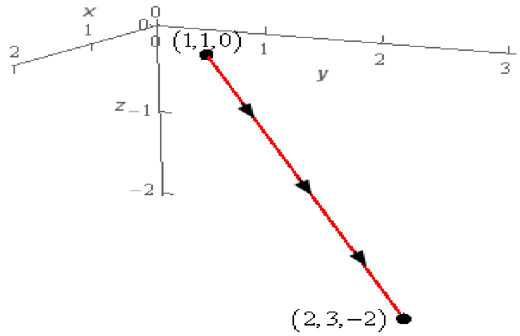
$$= \int_0^\pi 2dt + \int_0^\pi cos^2t\; sintdt$$

$$= \left[2t - \frac{cos^3t}{3}\right]_0^\pi$$

$$= 2\pi + \frac{1}{3} + \frac{1}{3} = 2\left(\pi + \frac{1}{3}\right)$$

Example 5.3: Evaluate $\int_C (xy - 4z)\, ds$ where C is the line segment from to $(1,1,0)$ to $(2,3,-2)$.

Solution: Here is a quick sketch of the curve *C*.



We know how to get the parameterization of a line segment so

$$\vec{r}(t) = (1-t)(1,1,0) + t(2,3,-2) = \big((1+t), 1+2t, -2t\big) \quad 0 \le t \le 1$$

$$= x(t)i + y(t)j + z(t)k$$

We'll need the magnitude of the derivative of the parameterization so let's get that.

$\vec{r'}(t) = (1, 2, -2)$ then $|\vec{r'}(t)| = \sqrt{(1)^2 + (2)^2 + (-2)^2} = \sqrt{9} = 3$

And here is the integrand evaluated at the parameterization.

$xy - 4z = (1+t)(1+2t) - 4(-2t) = 2t^2 + 11t + 1$

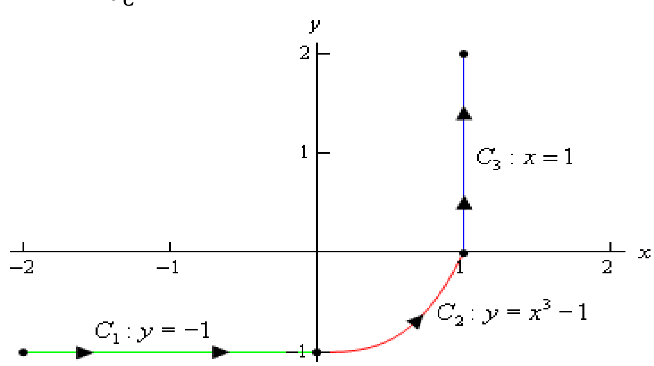
The line integral is then,

$$\int_C (xy - 4z)\, ds = \int_0^1 (2t^2 + 11t + 1)(3)dt = 3\left[\frac{2}{3}t^3 + \frac{11}{2}t^2 + t\right]_0^1 = \frac{43}{2}$$

Remark: It is still possible to compute a line integral when the curve C is not a smooth curve, as long as it is piecewise smooth, that is made of smooth pieces $C_1, C_2, C_3, \dots, C_n$. In this case

$$\int_C f\, ds = \int_{C_1} f\, ds + \int_{C_2} f\, ds + \int_{C_3} f\, ds + \cdots + \int_{C_n} f\, ds = \sum_{i=1}^{n} \int_{C_i} f\, ds$$

Example 5.4: Evaluate $\int_C 4x^3\, ds$ where *C* is the curve shown below.



Solution.

So, first we need to parameterize each of the curves.

$c_1: x = t\ ,\ \ y = -1,\ \ -2 \le t \le 0$
$c_2: x = t\ ,\ \ y = t^3 - 1,\ \ 0 \le t \le 1$
$c_3: x = 1\ ,\ \ y = t,\ \ 0 \le t \le 2$

Now let's do the line integral over each of these curves.

- $\int_{C_1} 4x^3\, ds = \int_{-2}^{0} 4t^3 \sqrt{(1)^2 + (0)^2}\, dt = \int_{-2}^{0} 4t^3 dt = t^4|_{-2}^{0} = -16$
- $\int_{C_2} 4x^3\, ds = \int_0^1 4t^3 \sqrt{(1)^2 + (3t^2)^2}\, dt$

$$= \int_0^1 4t^3 \sqrt{1 + 9t^4}\, dt$$

$$= \left(\frac{1}{9}\right)\left(\frac{2}{3}\right)\left[(1 + 9t^4)^{\frac{3}{2}}\right]_0^1 = 2.268$$

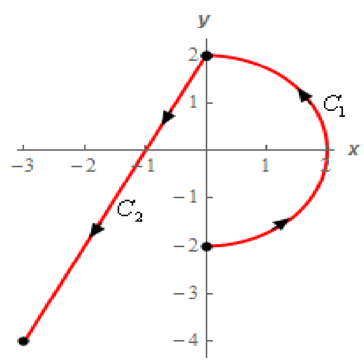
- $\int_{C_3} 4x^3\, ds = \int_0^2 4(1)^3 \sqrt{(0)^2 + (1)^2}\, dt = \int_0^2 4\, dt = 8$

Finally, the line integral that we were asked to compute is,

$$\int_C 4x^3\, ds = \int_{C_1} 4x^3\, ds + \int_{C_2} 4x^3\, ds + \int_{C_3} 4x^3\, ds$$
$$= -16 + 2.268 + 8$$
$$= -5.732$$

Example 5.5: Evaluate $\int_C (1 + x^3)\, dx$ where C is the right half of the circle of radius 2 with counter clockwise rotation followed by the line segment from $(0, 2)$ to $(-3, -4)$.

Solution. The direction is sketch bellow.



Now let's parameterize each of these curves.

$c_1: x = 2\,cost\ ,\ \ y = 2\,sint$, then $\vec{r}(t) = 2\,costi + 2\,sintj,\ -\frac{\pi}{2} \le t \le \frac{\pi}{2}$

$c_2:\ \vec{r}(t) = (1 - t)(0,2) + t(-3,-4) = (-3t, 2 - 6t),\ \ 0 \le t \le 1$

Now we need to evaluate the line integral. Be careful with this type line integral. Note that the differential, in this case, is dx and not ds as they were in the previous section.

All we need to do is recall that $dy = y'dt \ and\ dx = x'dt$ when we convert the line integral into a "standard" integral.

Here is the line integral.

$$\int_{C_1}(1+x^3)\,dx=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}[1+(2\cos t)^3]\,(-2sint)dt$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}[-2\,sint-16cos^3t\sin t]\,dt=0$$

$$\int_{C_2}(1+x^3)\,dx=\int_0^1[1+(-3t)^3]\,(-3)dt$$

$$=\int_0^1-3+81t^3\,dt=\frac{69}{4}$$

Now we need to do is add up the line integrals over these curves to get the full line integral.

$$\int_C(1+x^3)\,dx=\int_{C_1}(1+x^3)\,dx+\int_{C_2}(1+x^3)\,dx$$

$$=0+\frac{69}{4}=\frac{69}{4}$$

Exercise:1

1. Evaluate $\int_C(2yx^2-4x)\,ds$ where C is the lower half of the circle centered at the origin of radius 3 with clockwise rotation.
2. Evaluate $\int_C x\,ds$ for each of the following curves.
   a. $C_1: y=x^2, -1\le x\le 1$
   b. C_2 : the line segment from $(-1,1)$ to $(1,1)$
   c. C_3 :the line segment from $(1,1)$ to $(-1,1)$

5.1.2 Line integral of a vector field

Definition: For a vector field $\vec{F}:U\subseteq\mathbb{R}^{\boldsymbol{n}}\to\mathbb{R}^{\boldsymbol{n}}$,, the line integral along a smooth curve $C\subset U$, in the direction of $\boldsymbol{r}$, is defined as$\int_C\vec{F}(r).\,dr=\int_a^b\vec{F}(r).\left(\overrightarrow{r'}(t)\right)dt$

where ‘ · ’ is the dot product and $\boldsymbol{r}$: $[a,b]\to C$ is a parameterization of the curve C such that $\boldsymbol{r}(a)$ and $\boldsymbol{r}(b)$ give the endpoints of C.

A line integral of a scalar field is thus a line integral of a vector field where the vectors are always tangential to the line.

Line integrals of vector fields are independent of the parameterization **r** in absolute value, but they do depend on its orientation. Specifically, a reversal in the orientation of the parameterization changes the sign of the line integral.

The line integral is also defined by $\int_C\vec{F}(r).\,dr=\int_C\vec{F}(x,y,z).\vec{T}(x,y,z)ds$, where $\vec{r}(t)=xi+yj+zk$ is the parameterization of the oriented smooth curve C,

$\vec{T}(x,y,z)$ is the unit tangent vector to C at (x,y,z) given by $\frac{\frac{d\vec{r}}{dt}}{\left\|\frac{d\vec{r}}{dt}\right\|}$. Recalling that $ds = \left\|\frac{d\vec{r}}{dt}\right\| dt$

, we have that:-

$$\int_C \vec{F}(r).dr = \int_C \vec{F}(x,y,z).\vec{T}(x,y,z)ds = \int_a^b \left[\vec{F}\big(x(t),y(t),z(t)\big).\frac{\frac{d\vec{r}}{dt}}{\left\|\frac{d\vec{r}}{dt}\right\|}\right]\left\|\frac{d\vec{r}}{dt}\right\| dt$$

$$= \int_a^b \left[\vec{F}\big(x(t),y(t),z(t)\big).\frac{d\vec{r}}{dt}\right] dt,$$

where $[a,b]$ is an interval on which C is parameterized by $\vec{r}(t) = xi + yj + zk,\ [a,b]$

Thus $\int_C \vec{F}(r).dr = \int_a^b \left[\vec{F}\big(x(t),y(t),z(t)\big).\frac{d\vec{r}}{dt}\right] dt$

Remark: If $\vec{F}$ represents the force acting on a particle moving along an arc AB then the work done during the small displacement $\delta\vec{r}$ is $\vec{F}.\delta\vec{r}$. Therefore, the total work done by $\vec{F}(r)$ during the displacement from A to B is given by $W_T = \int_{AB} \vec{F}(r).d\vec{r}$.

Example: 5.6 Evaluate the line integral $\int_C \vec{F}(r).dr$, where $F = (x+y)i + (y-x)j$ along each of the paths in the xy-plane.

a. The parabola followed by $x = y^2$ from $(1,1)$ to $(4,2)$

b. The curve $x = 2u^2 + u + 1$ and $y = u^2 + 1$ from $(1,1)$ to $(4,2)$

c. the line $y = 1$ from $(1,1)$ to $(4,1)$, followed by the line $x = 4$ from $(4,1)$ to $(4,2)$.

Solution:

Since each of the path lies entirely in the *xy*-plane, we have $dr = dx\, i + dy\, j$. Therefore,

$$\int_C \vec{F}(r).dr = \int_C (x+y)dx + (y-x)\,dy \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (*)$$

We now evaluate the line integral along each path.

a. Along the parabola $y^2 = x$ we have $2ydy = dx$. Substituting for x in Eq. $(*)$ and using just the limits on y, we obtain $\int_{(1,1)}^{(4,2)} [(x+y)dx + (y-x)dy]$

$$= \int_1^2 [(y^2+y)2y + (y-y^2)]\,dy = \frac{34}{3}$$

b. The second path is given in terms of parameter *u*. We could eliminate *u* between two equations to obtain a relationship between *x* and *y* directly. Along the curve $x = 2u^2 + u + 1, y = 1 + u^2$, we have $dx = (4u+1)\,du$ and $dy = 2u\,du$. Substituting for x and y in Eq. $(*)$ and writing the correct limits on *u*, we obtain

$$\int_{(1,1)}^{(4,2)} [(x+y)dx + (y-x)dy] = \int_0^1 [(3u^2+u+2)(4u+1) - (u^2+u)2u]du = \frac{32}{3}$$

c. For the third path the line integral must be evaluated along the two line segments separately and the results added together. First, along the line $y = 1$, we have $dy = 0$. Substituting this into Eq. $(*)$ and using just the limits on *x* for this segment, we obtain

$$\int_{(1,1)}^{(4,1)} [(x+y)dx + (y-x)dy] = \int_1^4 (x+1)dx = \frac{21}{2}$$

Along the line $x = 4$, we have $dx = 0$. Substituting this into Eq.$(*)$ and using just the limits on y, we obtain $\int_{(4,1)}^{(4,2)} [(x+y)dx + (y-x)dy] = \int_1^2 (y-4)dy = \frac{-5}{2}$

Therefore, the value of the line integral along the whole path is $\frac{21}{2} - \frac{5}{2} = \frac{16}{2} = 8$

Example: 5.7

An object of mass m moves along the curve given by the position Vector $\vec{r}(t) = (at^2, sinbt, cosbt),\ 0 \le t \le 1, a$ and b are constants. Find the total force acting on the object and the work done by this force.

Solution:

You will recall that Newton. s second law of motion says that

$$F = ma(t) = mr''(t)$$

And the work done W is given by $W = \int_0^1 \vec{F}(r).\vec{r}'(t) = \int_0^1 mr''(t).\vec{r}'(t)$

Now $\vec{r}(t) = (at^2, sinbt, cosbt)$

$\vec{r}'(t) = (2at, bcosbt, -bsinbt)$ and

$\vec{r}''(t) = (2a, -b^2 sinbt, -b^2 cosbt)$

Then $F = ma(t) = mr''(t) = m[2ai + (-b^2 sinbt)j, (-b^2 cosbt)k]$

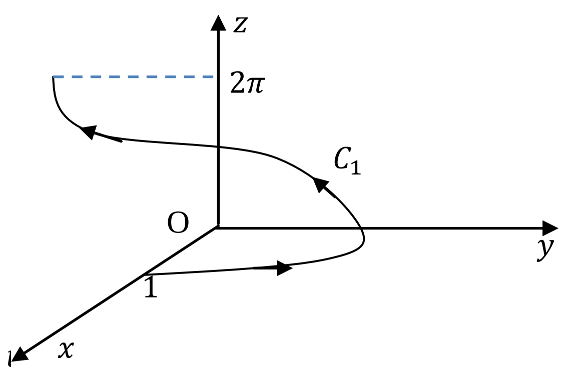
And also $\vec{F}(r)(t).\vec{r}'(t) = m4a^2t$

It follows that $W = \int_0^1 m4a^2t\, dt = 2a^2m$

Example: 5.8 A particle moves up ward along the circular helix C_1 parameterized by $\vec{r_1}(t) = \cos ti + \sin tj + tk$ for $0 \le t \le 2\pi$ under a force give $F(x, y, z) = -zyi + zx + j + xyk$.

Find the work done on the particle by the force.

Solution: Consider the figure



In $C_1 : \vec{r_1}(t) = \cos ti + \sin tj + tk$, for $0 \le t \le 2\pi$

$$\Rightarrow \begin{cases} x(t) = \cos t \\ y(t) = \sin t \Rightarrow \vec{r_1}'(t) = -\sin ti + \cos \\ z(t) = t \end{cases}$$

Therefore, $W_T = \int_{C_1} \vec{F}.d\vec{r} = \int_0^{2\pi} [t \sin ti + t\cos tj + \sin t \cos tj].(-\sin ti + \cos tj + k]dt$

$$= \int_0^{2\pi} \left(t\sin^2 t + t\cos^2 t + \sin t\cos t \right) dt = \int_0^{2\pi} (t + \sin t\cos t) dt$$

$$= \frac{t^2}{2}\Big|_0^{2\pi} + \frac{1}{2}\int_0^{2\pi} \sin(2t) dt = \frac{4\pi^2}{2} - \frac{1}{4}\cos 2t\Big|_0^{2\pi} = 2\pi^2$$

Exercise:2

1. Calculate the line integral of the scalar function $F(x, y) = xy^3$ over the right half of the semi circle $x^2 + y^2 = 4$ along the counterclockwise direction from $(0, -2)$ to $(0, +2)$
2. An object acted on by a force $F\ (x;\ y) = (x^3, y)$ moves along the parabola $y =\ 3x^2$ from $(0, 0)\ to\ (1, 3)$. Calculate the work done by F.

5.2 Fundamental theorem of line integral

Let C be a smooth curve with initial point (x_0, y_0, z_0) and terminal point (x_1, y_1, z_1). Let f be a function of three variables differentiable at every point on C. If $\nabla f\,(x, y, z)$ is continuous on C then $\int_C \nabla f\,.\,d\vec{r} = f(x_1, y_1, z_1) - f(x_0, y_0, z_0)$.

Proof: Let $\vec{r}(t) = x(t)i + y(t)j + z(t)k, a \le t \le b$ be a parameterization of C. Then using the Chain Rule and the Fundamental Theorem of Calculus,

$$\int_C \nabla f.d\vec{r} = \int_a^b \left[\nabla f(x(t), y(t), z(t)).\frac{d\vec{r}}{dt} \right] dt$$

$$= \int_a^b \left(\frac{\partial}{\partial x} f(x(t), y(t), z(t))i + \frac{\partial}{\partial y} f(x(t), y(t), z(t))j + \frac{\partial}{\partial z} f(x(t), y(t), z(t))k \right) . \left(\frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k \right) dt$$

$$= \int_a^b \frac{d}{dt}\left(f(x(t), y(t), z(t)) \right) dt = \left[f(x(t), y(t), z(t)) \right] \Big|_{t=a}^{t=b}$$

$$= f(x(b), y(b)), z(b) - f(x(a), y(a), z(a)) = f(x_1, y_1, z_1) - f(x_0, y_0, z_0)$$

Note: suppose that $\vec{F}$ is a continuous vector field in some domain D.

1. $\vec{F}$ is a **conservative** vector field if there exist a function f such that $\vec{F} = \nabla f$.The function f is called a **potential function** for the vector field.
2. A path C is called **closed** if its initial and final points are the same point. For example a circle is a closed path.
3. A path C is **simple** if it doesn't cross itself. A circle is a simple curve.
4. A region D is **open** if it doesn't contain any of its boundary points.

Example: 5.9 Determine if the following vector field is conservative or not. If it is conservative find the potential function.

A. $\vec{F}(x,y) = (6 - 2xy + y^3)i + (x^2 - 8y + 3xy^2)j$

B. $\vec{F}(x,y,z) = (2z^4 - 2y - y^3)i + (z - 2x - 3xy^2)j + (6 + y + 8xz^3)k$

Solution:

There really isn't all that much to do with this problem. All we need to do is identify P and Q where $\vec{\mathrm{F}}$ is a plane vector field and P,Q and R where $\vec{\mathrm{F}}$ is a space vector field. Then run through the test.

A. $P = 6 - 2xy + y^3$ and $\mathrm{Q} = x^2 - 8y + 3xy^2$

So $P_y = -2x + 3y^2$ and $Q_x = 2x + 3y^2$

we can clearly see that $P_y \neq Q_x$ and so the vector field is **not conservative.**

B. $\vec{F}(x,y,z) = (2z^4 - 2y - y^3)i + (z - 2x - 3xy^2)j + (6 + y + 8xz^3)k$

Here $P = 2z^4 - 2y - y^3$, $\mathrm{Q} = z - 2x - 3xy^2$ and $R = 6 + y + 8xz^3$

So $P_y = -2 - 3y^2$, $P_z = 8z^3$, $Q_x = -2 - 3y^2$, $Q_z = 1$ $R_x = 8z^3$ and $R_y = 1$

we can clearly see that $P_y = Q_x$

$P_z = R_x$

$Q_z = R_y$, hence the vector field is **conservative.**

To find the potential function for this vector field we know that we need to first either integrate P with respect to x, integrate Q with respect to y or R with respect z. So, let's go with the following integration for this problem.

$$f(x,y,z) = \int \mathrm{Q}\,dy = \int (z - 2x - 3xy^2)\,dy$$
$$= zy - 2xy - xy^3 + h(x,z)$$

Next, we can differentiate the function from the previous step with respect to x and set equal to P or differentiate the function with respect to z and set equal to R. So let's differentiate with respect to z.

$$f_z = y + h_z(x,z) = R = 6 + y + 8xz^3$$
$$\Longrightarrow h_z(x,z) = 6 + 8xz^3$$
$$\Longrightarrow h(x,z) = \int (6 + 8xz^3)\,dz = 6z + 2xz^4 + k(x)$$

Then $f(x,y,z) = zy - 2xy - xy^3 + 6z + 2xz^4 + k(x)$

Again let's differentiate with respect to x.

$$f_x = -2y - y^3 + 2z^4 + k'(x) = P = 2z^4 - 2y - y^3$$
$$\Longrightarrow k'(x) = 0$$
$$\Longrightarrow k(x) = C$$

Therefore the potential function for the vector field is

$$f(x,y,z) = zy - 2xy - xy^3 + 6z + 2xz^4 + C.$$

Example: 5.10. Evaluate $\int_C \nabla f.\mathrm{d}\vec{\mathrm{r}}$ where $f(x,y,z) = \cos(\pi x) + \sin(\pi y) - xyz$ and C is any path that starts at $(1, \frac{1}{2}, 2)$ and ends at $(2,1,-1)$.

Solution: Let C, be any path that starts at $(1, \frac{1}{2}, 2)$ and ends at $(2,1,-1)$ on $a \leq t \leq b$.

Then $\vec{r}(a) = (1, \frac{1}{2}, 2)$ and $\vec{r}(b) = (2,1,-1)$

The integral is $\int_C \nabla f.\mathrm{d}\vec{\mathrm{r}} = f(2,1,-1) - f\left(1, \frac{1}{2}, 2\right)$

$$= \cos 2\pi + \sin \pi + 2 - \left(\cos \pi + \sin \frac{\pi}{2} - \left(\frac{1}{2}\right) 2(1)\right)$$

$$= 4$$

Example: 5.11

1. Calculate the line integral of the vector field $\vec{F} = 2xi + 3yj + 4zk$ along the following two paths joining the origin to the point $P(1,1,1)$.
   a. Along a straight line joining the origin to P,
   b. along a path parameterized by $x = t,\ y = t^2,\ z = t^3$
2. From the result of Problem 1, can you conclude that the force is conservative? If so, determine a potential function for this vector field.

Solution:

1.
   a. For the straight line path $x = y = z$

   $$\int_C \vec{F}(r).dr = \int_C f_x dx + f_y dy + f_z dz = \int_0^1 2x dx + \int_0^1 3y dy + \int_0^1 4z dz = \frac{9}{2}$$

   b. For the second path $dx = dt,\ dy = 2t dt,\ dz = 3t^2 dt$, so that

   $$\int_C \vec{F}(r).dr = \int_0^1 (2t + 6t^3 + 12t^5) dt = \frac{9}{2}$$

2. Just from the fact that line integrals along two different paths give the same result, one cannot conclude that the force is conservative. However, in this particular case, the vector field happens to be conservative. Let the potential function be $f,\ \vec{F} = \nabla f$. Equating components of the force, we get

   $$f_x = \frac{\partial f}{\partial x} = 2x \Longrightarrow f = x^2 + C_1(y,z)$$

   $$f_y = \frac{\partial f}{\partial y} = 3y \Longrightarrow f = \frac{3}{2}y^2 + C_2(x,z)$$

   $$f_z = \frac{\partial f}{\partial z} = 4z \Longrightarrow f = 2z^2 + C_3(x,y)$$

Clearly, the function is given by $f = x^2 + \frac{3}{2}y^2 + 2z^2 + C$, where C is an arbitrary constant, which can be taken to be zero. The line integral can therefore be written as

$$\int_C \nabla f.dr = \int_O^P df = f\Big|_{(0,0,0)}^{(1,1,1)} = 1 + \frac{3}{2} + 2 = \frac{9}{2}$$

Path independence of line integral

Theorem: the line integral $\int_C \mathrm{F}(\mathrm{r}).dr = \int_C (F_1dx + F_2dy + F_3dz)$, $dr(dx, dy, dz)$ with continuous F_1, F_2, F_3 in a domain D in space is path independence in D iff $F = (F_1, F_2, F_3)$ is the gradient of some function f in D. i.e. $F = gradf$ thus $F_1 = \frac{\partial f}{\partial x}$, $F_2 = \frac{\partial f}{\partial y}$, $F_3 = \frac{\partial f}{\partial z}$

Corollary:

The line integrals: $\int_A^B (F_1dx + F_2dy + F_3dz) = f(B) - f(A)$, $F = gradf$

Note: $\int_C \vec{\mathrm{F}}.\mathrm{d}\vec{\mathrm{r}}$ is **independent of path** if $\int_{C_1} \vec{\mathrm{F}}.\mathrm{d}\vec{\mathrm{r}} = \int_{C_2} \vec{\mathrm{F}}.\mathrm{d}\vec{\mathrm{r}}$ for any two paths C_1 and C_2 in D with the same initial and final points.

Example: 5.12 Show that the integral $\int_C F.dr = \int_C (2xdx + 2ydy + 4zdz),$ is path independent in any domain in space and find its value in the integral from A: (0,0,0) to B: (2,2,2)

Solution: $F = [2x, 2y, 4z] = grad\ f$, where $f = x^2 + y^2 + 2z^2$, because $\frac{\partial f}{\partial x} = 2x = F_1$, $\frac{\partial f}{\partial y} = 2y = F_2$, $\frac{\partial f}{\partial z} = 4z = F_3$. Hence, the integral is independent of path according to theorem above, then $\int_A^B F.dr = f(B) - F(A) = 4 + 4 + 8 = 16.$

Facts:

1. $\int_C \nabla\mathrm{f}.\mathrm{d}\vec{\mathrm{r}}$ is independent of path.

   This is easy enough to prove since all we need to do is look at the fundamental theorem of line integral above. The theorem tells us that in order to evaluate this integral all we need are the initial and final points of the curve. This in turn tells us that the line integral must be independent of path.

2. If $\vec{\mathrm{F}}$ is a conservative vector field then $\int_C \vec{\mathrm{F}}.\mathrm{d}\vec{\mathrm{r}}$ is independent of path.

   This fact is also easy enough to prove. If $\vec{\mathrm{F}}$ is conservative then it has a potential function, *f*, and so the line integral becomes $\int_C \vec{\mathrm{F}}.\mathrm{d}\vec{\mathrm{r}} = \int_C \nabla\mathrm{f}.\mathrm{d}\vec{\mathrm{r}}$. Then using the first fact we know that this line integral must be independent of path.

3. The integral $\int_C \vec{\mathrm{F}}.\mathrm{d}\vec{\mathrm{r}} = 0$ for every closed path *C* if and only if $\int_C \vec{\mathrm{F}}.\mathrm{d}\vec{\mathrm{r}}$ is independent of path.

Example: 5.13 Given that $\int_C \vec{\mathrm{F}}.\mathrm{d}\vec{\mathrm{r}}$ is path independent, compute $\int_C \vec{\mathrm{F}}.\mathrm{d}\vec{\mathrm{r}}$ where C is an ellipse given by $\frac{(x-5)^2}{4} + \frac{y^2}{9} = 1$ with the counter clockwise rotation.

Solution: There are two important things in the problem statement.

First, and somewhat more importantly that the integral is independent of path.

Second the curve, C, is the full ellipse which is a closed curve.

Then the value of a line integral of this type around a closed path will be zero

$$\int_C \vec{F}.d\vec{r} = 0$$

5.3 Green`s Theorem

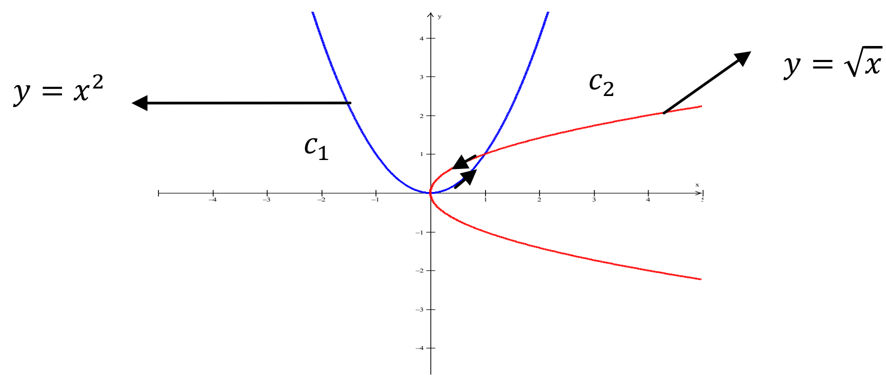
Theorem: Let R be a simple region in the xy – plane with a piece – wise smooth boundary C oriented counter clock wise. Let M and N be function of two variables having continuous partial derivatives on R. Then $\int_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$

Before working some examples there are some alternate notations that we need to acknowledge. When working with a line integral in which the path satisfies the condition of Green's Theorem we will often denote the line integral as,$\oint_C Mdx + Ndy$. This notations do assume that C satisfies the conditions of Green's Theorem so be careful in using them.

Example:5.14 Verify green`s theorem in the plane $\oint_C (3x^2 - 8y^2)\, dx + (4y - 6xy)dy$,where C is the boundary of the region defined by $y = \sqrt{x},\ \ y = x^2$.

Solution: $y = \sqrt{x}\ \ i.e\ y^2 = x$ and $y = x^2$ are two parabolas which intersect at (0, 0) and (1, 1)

We have $\oint_C Mdx + Ndy = \oint_{C_1} Mdx + Ndy + \oint_{C_2} Mdx + Ndy$



Along C_1: $x^2 = y, dy = 2xdx$ and limits of x are 0 and 1.

Line integral along C_1 becomes

$$\oint_{C_1} (3x^2 - 8x^4)\, dx + (4x^2 - 6x^3)2xdx = \int_0^1 (3x^2 - 8x^3 - 20x^4)\, dx = -1$$

Along C_2: $y^2 = x, dx = 2ydy$ and limits of y are 0 and 1.

Line integral along C_2 becomes

$$\oint_{C_2} (3y^4 - 8y^2)\, 2ydy + (4y - 6y^3)dy = \int_1^0 (6y^5 - 22y^3 + 4y)\, dy = \frac{5}{2}$$

Line integral along $C = -1 + \frac{5}{2} = \frac{3}{2}$ (i)

Again $\int_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$

$$= \iint_R \left(\frac{\partial}{\partial x}(4y - 6xy) - \frac{\partial}{\partial y}(3x^2 - 8y^2) \right) dA$$

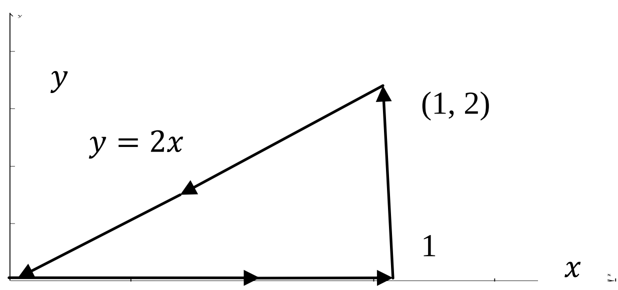
$$= \iint_R (-6y + 16y) dA = \int_0^1 \left(\int_{x^2}^{\sqrt{x}} 10y \; dy \right) dx = \frac{3}{2} \qquad \text{(ii)}$$

Hence from (i) and (ii) the green`s theorem is verified.

Example:5.15 Use Green's Theorem to evaluate $\oint_C xydx + x^2y^3dy$, where C is the triangle with vertices $(0, 0), (1, 0), (1, 2)$ with positive orientation.

Solution:

Let's first sketch C and D for this case to make sure that the conditions of Green's Theorem are met for C and will need the sketch of D to evaluate the double integral.



So, the curve does satisfy the conditions of Green's Theorem and we can see that the following inequalities will define the region enclosed. $0 \leq x \leq 1$ and $0 \leq y \leq 2x$

We can identify M and N from the line integral. Here they are. $M = xy \; and \; N = x^2y^3$

So, using Green's Theorem the line integral becomes,

$$\oint_C xydx + x^2y^3dy = \iint_D (2xy^3 - x)dA$$

$$= \int_0^1 \int_0^{2x} (2xy^3 - x)dydx$$

$$= \int_0^1 \left[\frac{1}{2}xy^4 - xy \right]_0^{2x} dx$$

$$= \int_0^1 (8x^5 - 2x^2)dx$$

$$= \left[\frac{4}{3}x^6 - \frac{2}{3}x^3 \right]_0^1 = \frac{2}{3}$$

Example: 5.16 Evaluate $\oint_C y^3dx - x^3dy$ where C are the two circles of radius 2 and radius 1 centered at the origin with positive orientation.

Solution: In this case the region D will now be the region between these two circles and that will only change the limits in the double integral so we'll not put in some of the details here. Here is the work for this integral.

$$\oint_C y^3dx - x^3dy = -3\iint_D (y^2 + x^2)dA$$
$$= -3\int_0^{2\pi}\int_1^2 r^3 dr d\theta$$
$$= -3\int_0^{2\pi}\left[\frac{r^4}{4}\right]_1^2 d\theta$$
$$= -3\int_0^{2\pi}\frac{15}{4}d\theta = \frac{-45\pi}{2}$$

Exercise: 5.3

1. Use Green's Theorem to evaluate $\int_C \vec{\mathrm{F}}.\mathrm{d}\vec{\mathrm{r}}$ counterclockwise around the boundary curve C of the region R,where $F = \frac{1}{2}xy^4i + \frac{1}{2}yx^4j, R$ is the rectangle with vertices $(0,0),(3,0),(3,2),(0,2)$.
2. Prove that the area of a plane region R is given either by $\int_C xdy$, $\int_C -ydx$ or $\frac{1}{2}\int_C -ydx + xdy$, where C is the boundary of R oriented counter clock - wise.
3. Use Green's Theorem find the area of a disk of radius a.

5.4 Surface integral

Review of Surfaces

Adding one more independent variable to a vector function describing a curve $x = x(t)$, $y = y(t),\ z = z(t)$; we arrive to equations that describe a surface. Thus, a surface in space is a vector function of two variables:

$$r(u,v) = \big(x(u,v), y(u,v), z(u,v)\big).$$

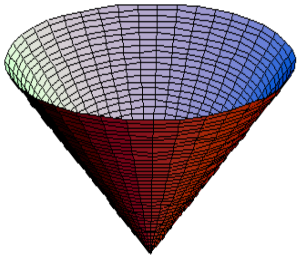
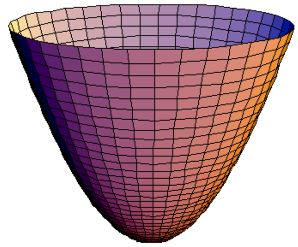
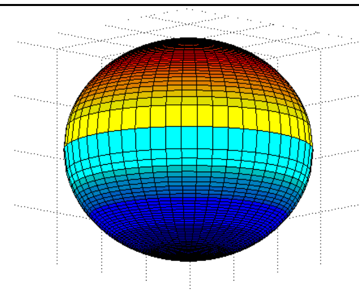
These equations are called parametric equations of the surface and the surface given via parametric equations is called a parametric surface.

✓ The following are examples of parametric surfaces.

A. The cone $z = \sqrt{x^2 + y^2}$ has representation using cylindrical coordinates as $x = r\cos\theta\ , y = r\sin\theta,\ z = r$.

B. The parapoloid $z = x^2 + y^2$ has representation using cylindrical coordinates as $x = r\cos\theta\ , y = r\sin\theta,\ z = r^2$.

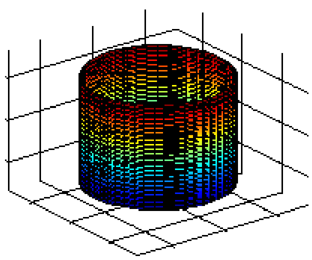
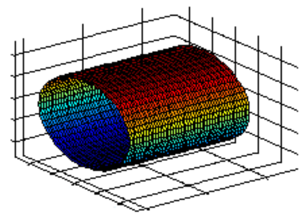
C. The sphere $x^2 + y^2 + z^2 = 9$ has representation using cylindrical coordinates as $x = 3\cos\theta\sin\emptyset\ ,\ \ y = 3\sin\theta\sin\emptyset,\ z = 3\cos\emptyset$.

Cone paraboloid sphere

D. The cylinder $x^2 + y^2 = 4$ has representation using cylindrical coordinates as $x = 2cos\theta$, $y = 2sin\theta$, $z = z$. The parameters here are θ and z.

E. The cylinder $y^2 + z^2 = 4$ has representation using cylindrical coordinates as $x = x$, $y = 3cos\theta$, $z = 2sin\theta$. The parameters here are θ and x.

cylinder $x^2 + y^2 = 4$ cylinder $y^2 + z^2 = 4$

A. Surface integral over non oriented surface

Similarly as for line integrals, we can integrate a scalar or a vector function over a surface. The surface integrals of scalar functions are two dimensional analogue of the line integrals of scalar functions.

Line integral of a scalar function ↔ Length

Surface integral of a scalar function ↔Area

Recall that we computed the surface area of the surface $z = f(x, y)$ over region Σ to be

$$\boxed{\begin{array}{l}\text{Surface area } \iint_\Sigma ds = \iint_\Sigma |r_x \times r_y|\, dxdy = \iint_\Sigma \sqrt{f_x + f_y + 1}\, dxdy, \\ \text{where } r = (x, y, z) = \big(x, y, f(x, y)\big)\end{array}}$$

Definition: Let Σ be the graph of function $z = f(x, y)$ having continuous partial derivatives on a region R in the xy – plane, which is either horizontally or vertically simple.

Let $g(x, y, z)$ be a continuous function on Σ. The surface integral of $g(x, y, z)$ over Σ, is defined by

$$\boxed{\iint_\Sigma g(x,y,z)ds = \iint_R g(x,y,f(x,y))\sqrt{f_x^2 + f_y^2 + 1}\ dA}$$

Example-5.17 Evaluate $\iint_\Sigma (1+z)ds,$ if Σ is the hemisphere $Z = \sqrt{1 - x^2 - y^2}$.

Solution: Σ is the graph of $f(x,y)=\sqrt{1-x^2-y^2}$ on $R: x^2+y^2\le 1$

Here, $f_x=\dfrac{-x}{\sqrt{1-x^2-y^2}}$, $f_y=\dfrac{-y}{\sqrt{1-x^2-y^2}}$ on $x^2+y^2<1$

So, $\iint_\Sigma (1+z)ds=\lim\limits_{b\to 1^-}\iint_{R_b}\left(1+\sqrt{1-x^2-y^2}\right)\sqrt{1+f_x^2+f_y^2}\,dA$

$$=\lim_{b\to 1^-}\iint_{R_b}\left(1+\sqrt{1-x^2-y^2}\right)\left(\frac{1}{\sqrt{1-x^2-y^2}}\right)dA=\lim_{b\to 1^-}\iint_{R_b}\left(\frac{1}{\sqrt{1-x^2-y^2}}+1\right)dA,$$ where $R_b: x^2+y^2\le b,\quad 0<b<1$

$$=\lim_{b\to 1^-}\int_0^{2\pi}\int_0^{b}\left(\frac{1}{\sqrt{1-r^2}}+1\right)r\,dr\,d\theta=\int_0^{2\pi}\lim_{b\to 1^-}\left(-\sqrt{1-r^2}+\frac{1}{2}r^2\right)\Big|_0^b d\theta=\frac{3}{2}\int_0^{2\pi}d\theta=3\pi\cdot$$

Exercise- 5.4: Evaluate:

a. $\iint_\Sigma z^2 ds$, where Σ is the portion of the cone $z=\sqrt{x^2+y^2}$ for which $1\le x^2+y^2\le 4$.

b. $\iint_\Sigma (x+y+z)ds$, where Σ is the portion of the plane $x+y+z=1$ in the 1st octant for which $0\le z\le 1$.

B. Surface integral over oriented surface

Definition: Given a continuous vector field $\vec{F}$ on an oriented surface Σ, the surface integral of the normal component of $\vec{F}$ over Σ, denoted by $\iint_\Sigma \vec{F}.\vec{n}ds$, where $\vec{n}$ is the unit normal vector t Σ, is called the **flux** of $\vec{F}$ across Σ. When the surface is closed, the flux integral is usually denoted by $\oiint_\Sigma \vec{F}.\vec{n}\,ds$.

To evaluate flux integral " $\iint_\Sigma \vec{F}.\vec{n}ds$ ", recall that the normal vector to the surface $\Sigma: Z=f(x,y)$ is given by $f_x i+f_y j-k$ or $-f_x i-f_y j+k$. Then unit normal vector is $\vec{n}=\dfrac{f_x i+f_y j-k}{\sqrt{1+f_x^2+f_y^2}}$, when $\vec{n}$ is directed down ward or $\vec{n}=\dfrac{-f_x i-f_y j+k}{\sqrt{1+f_x^2+f_y^2}}$, when $\vec{n}$ is directed upward. So, if $\vec{F}=Mi+Nj+Pk$, then

$$\iint_\Sigma \vec{F}.\vec{n}ds=\iint_R (Mi+Nj+Pk)\bullet\frac{f_x i+f_y j-k}{\sqrt{1+f_x^2+f_y^2}}\sqrt{1+f_x^2+f_y^2}\,dA=\iint_R \left(Mf_x+Nf_y-P\right)dA\qquad,$$

when $\vec{n}$ is directed down ward, or $\iint_\Sigma \vec{F}.\vec{n}ds=\iint_R (Mi+Nj+Pk)\bullet\dfrac{-f_x i-f_y j+k}{\sqrt{1+f_x^2+f_y^2}}\sqrt{1+f_x^2+f_y^2}\,dA$

$=\iint_R\left(-Mf_x-Nf_y+P\right)dA$, where $\vec{n}$ is directed upward.

Example-5.18: Evaluate $\iint_{\Sigma}\vec{F}.\vec{n}ds, if\ \vec{F}=yi-xj+8k$ and Σ is the portion of the paraboloid $z=9-x^2-y^2$ above the xy – plane and $\vec{n}$ is directed upward.

Solution: $z=9-x^2-y^2=f(x,y)\Rightarrow f_x=-2x,\ f_y=-2y$ &

$\vec{F}=yi-xj+8k=Mi+Nj+Pk\Rightarrow M=y, N=-x\ \&\ P=8$

$\therefore \iint_{\Sigma}\vec{F}.\vec{n}ds=\iint_R((-y)(-2x)-(-x)(-2y)+8)dA=\iint_R 8dA=8\iint_R dA=8\,(\text{area of } R: x^2+y^2\le 9)=72\pi$

Remark: 1. If $\delta(x,y,z)$ is the density of a fluid at a point $P=(x,y,z)$, then the **rate of mass flow** of the fluid with velocity $\vec{V}(x,y,z)$ through the whole surface Σ is given by $\iint_{\Sigma}\delta(x,y,z)\vec{V}(x,y,z).\vec{n}ds$

2. Flux integrals can be defined for a surface $\sum$ composed of several-oriented surface $\sum_1,\sum_2,\dots,\sum_n$, then $\iint_{\sum}\vec{F}.\vec{n}ds=\iint_{\sum_1}\vec{F}.\vec{n}ds+\iint_{\sum_2}\vec{F}.\vec{n}ds+\cdots+\iint_{\sum_n}\vec{F}.\vec{n}ds$

Exercise-5.4:

1. If the velocity of water is $\vec{F}=yi+2j+xzk\ \ m/\sec$, show that the flux of water through the parabolic cylinder $y=x^2, 0\le x\le 3, 0\le Z\le 2$ is $69\ \ m^3/\sec$.
2. Suppose $\sum$ is part of the paraboloid $z=1-x^2-y^2$ that lies above the xy-plane and is oriented by the normal directed upward. The velocity of a fluid with constant density δ is given by
$\vec{V}=xi+yj+2zk$. Determine the rate of mass flow through the surface $\sum$ in the direction of $\vec{n}$.
3. Evaluate $\iint_{\Sigma}\vec{F}.\vec{n}ds$, where Σ is the unit sphere $x^2+y^2+z^2=1$, oriented with normal that is directed out ward, and $\vec{F}=zk$.

5.5 DIVERGENCE'S THEOREM AND STOKES'S THEOREM

5.5.1 STOKES'S THEOREM

Let Σ be an oriented surface with normal $\vec{n}$ and finite surface area. Assume that Σ is bounded by a closed piece – wise smooth curve C whose orientation is induced by Σ.

Let $\vec{F}=Mi+Nj+Pk$ be a continuous vector field on Σ such that its component functions have continuous partial derivatives at each point on the boundary C.

Then $\int_C\vec{F}.d\vec{r}=\int_C Mdx+Ndy+Pdz=\iint_{\Sigma}\nabla X\vec{F}.\vec{n}ds$

Note: $\nabla X\vec{F}=\left(\frac{\partial P}{\partial y}-\frac{\partial N}{\partial z}\right)i+\left(\frac{\partial M}{\partial z}-\frac{\partial P}{\partial x}\right)j+\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)k$, where $\vec{n}=\frac{-f_xi-f_yj+k}{\sqrt{1+f_x^2+f_y^2}}$, $ds=\sqrt{1+f_x^2+f_y^2}dA$

so that $\int_C \vec{F}.d\vec{r} = \int_C Mdx + Ndy + Pdz = \iint_\Sigma \nabla X \vec{F}.\vec{n}ds$

$$= \iint_R \left[-\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) f_x - \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) f_y + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\right] dA,$$

where all partial derivatives are to be evaluate at $(x, y, f(x, y))$.

Example-5.19:

Show that Green's Theorem in the plane is a special case of Stokes's theorem.

Proof: $\vec{F}(x, y) = M(x, y)i + N(x, y)j$ in the xy – plane,

Then $\nabla X \vec{F}(x, y) = \left(\frac{\partial N(x, y)}{\partial x} - \frac{\partial M(x, y)}{\partial y}\right)k$

$\Rightarrow \int_C \vec{F}.d\vec{r} = \int_C Mdx + Ndy = \iint_\Sigma \nabla X \vec{F}.\vec{n}ds = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$, which is Green's theorem in the plane.

Example 5.20: Verify Stoke`s theorem for the function $\vec{F} = zi + xj + yk$ where C is the unit circle in xy- plane bounding the hemisphere $z = \sqrt{1 - x^2 - y^2}$

Solution: Stoke`s theorem $\int_C \vec{F}.dr = \iint_\Sigma curl\,\vec{F}.\vec{n}ds$ where C is the unit circle $x^2 + y^2 = 1, z = 0$ and $\vec{r} = xi + yj + zk$, then $d\vec{r} = dxi + dyj + dzk$

$$\vec{F}.dr = (zi + xj + yk).(dxi + dyj + dzk) = zdx + xdy + ydz$$

So $\int_C \vec{F}.dr = \int_C zdx + \int_C xdy + \int_C ydz$

The parametric equation of the curve is $x = \cos t\,, y = \sin t\,, 0 \le t \le 2\pi,\ z = 0, dz = 0$

$$\int_C \vec{F}.dr = \int_C xdy = \int_0^{2\pi} \cos t \cos t\, dt = \int_0^{2\pi} \cos^2 t\, dt = \pi \qquad \text{(i)}$$

Again curl$\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = i + j + k$

Let $\vec{n}$ be the outward unit normal to the surface $x^2 + y^2 + z^2 = 1$ at point (x, y, z),

Then $F(x, y, z) = x^2 + y^2 + z^2 - 1$

$$\vec{n} = \frac{gradF}{\|gradF\|} = \frac{2xi + 2yj + 2zk}{\sqrt{(2x)^2 + (2y)^2 + (2z)^2}} = xi + yj + zk$$

Therefore $curl\vec{F}.\vec{n} = (i + j + k).(xi + yj + zk) = x + y + z$

Using spherical polar coordinates

$x = r\, sin\theta\, cos\emptyset = sin\theta\, cos\emptyset \qquad .:r = 1$

$y = r\, sin\theta\, sin\emptyset = sin\theta\, sin\emptyset$

$z = r\, cos\theta = cos\theta$ and $ds = sin\theta\, d\theta\, d\emptyset$

In first octant $\theta = 0\ to\ \frac{\pi}{2}$ and $\emptyset = 0\ to\ 2\pi$

$curl\ F.\vec{n} = sin\theta\, cos\emptyset + sin\theta\, sin\emptyset + cos\theta$

$$\iint_{\Sigma} curl\,\vec{F}.\vec{n}ds = \int_{0}^{\frac{\pi}{2}}\int_{0}^{2\pi}(sin\theta\ cos\emptyset + sin\theta\ sin\emptyset + cos\theta)sin\theta\ d\emptyset\ d\theta$$

$$= \int_{0}^{\frac{\pi}{2}}(sin\theta\ sin\emptyset - sin\theta\ cos\emptyset + cos\theta)|_{0}^{2\pi}sin\theta\ \ d\theta$$

$$= 2\pi\int_{0}^{\frac{\pi}{2}}cos\theta sin\theta d\theta$$

$$= \pi\int_{0}^{\frac{\pi}{2}}sin2\theta d\theta$$

$$= \pi\left[-\frac{cos2\theta}{2}\right]_{0}^{\frac{\pi}{2}}$$

$$= \pi \qquad \text{(ii)}$$

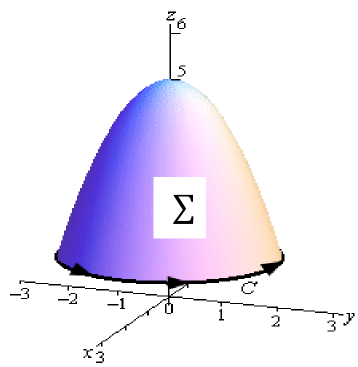
From (i) and (ii) we get $\iint_{\Sigma} curl\,\vec{F}.\vec{n}ds = \int_{C}\vec{F}.dr$

Which verifies Stoke`s theorem.

Example 5.21: Use Stokes' Theorem to evaluate $\iint_{\Sigma} curl\vec{F}\,.ds$ where $\vec{F} = z^2i - 3xyj + x^3y^3k$ and $\sum$ is the part of $z = 5 - x^2 - y^2$ above the plane $z = 1$. Assume that $\sum$ is oriented upwards.

Solution:

Let's start this off with a sketch of the surface.



In this case the boundary curve C will be where the surface intersects the plane $z = 1$ and so will be the curve $1 = 5 - x^2 - y^2 \Longrightarrow x^2 + y^2 = 4$ at $z = 1$.

So, the boundary curve will be the circle of radius 2 that is in the plane z =1. The parameterization of this curve is,

$$\vec{r}(t) = 2costi + 2sintj + k, 0 \leq t \leq 2\pi$$

The first two components give the circle and the third component makes sure that it is in the plane $z = 1$.

Using Stokes' Theorem we can write the surface integral as the following line integral.

$$\iint_{\Sigma} curl\,\vec{F}.\vec{n}ds = \int_{C} \vec{F}.dr = \int_{0}^{2\pi} \vec{F}\left(\vec{r}(t)\right).\vec{r}'(t)dt$$

Let's first get the vector field evaluated on the curve. Remember that this is simply plugging the components of the parameterization into the vector field.

$$\vec{F}\left(\vec{r}(t)\right) = (1)^2 i - 3\left(2(cost)(2sint)\right)j + (2cost)^3(2sint)^3$$
$$= i - 12costsintj + 64(cos^3t)(sin^3t)$$

Next, we need the derivative of the parameterization and the dot product of this and the vector field.

$$\vec{r}'(t) = -2sinti + 2costj$$
$$\vec{F}\left(\vec{r}(t)\right).\vec{r}'(t) = -2sint - 24sintcos^2t$$

We can now do the integral.

$$\iint_{\Sigma} curl\,\vec{F}.\vec{n}ds = \int_{0}^{2\pi} (-2sint - 24sintcos^2t)dt$$
$$= [2cost + 8cos^3t]_0^{2\pi} = 0$$

Remarks:

1. If two oriented surfaces $\sum_1$ and $\sum_2$ are bounded by the same curve C and induce the same orientation on C, and if $\vec{n}_1$, $\vec{n}_2$ are normal of $\sum_1$, $\sum_2$respectively, then by Stokes's theorem: $\int_C \vec{F}.dr = \iint_{\Sigma_1} curl\vec{F}.\vec{n}_1 ds$ and $\int_C \vec{F}.dr = \iint_{\Sigma_2} curl\vec{F}.\vec{n}_2 ds$, which implies that $\iint_{\Sigma_1} curl\,\vec{F}.\vec{n}_1 ds = \iint_{\Sigma_2} curl\,\vec{F}.\vec{n}_2 ds \Longrightarrow \iint_{\Sigma_1} \nabla X\,\vec{F}.\vec{n}_1 ds = \iint_{\Sigma_2} \nabla X\,\vec{F}.\vec{n}_2 ds$
2. If two oriented surfaces $\sum_1$ and $\sum_2$ are bounded by the same curve C but have opposite orientations on C, then $\int_C \vec{F}.dr = \iint_{\Sigma_1} \nabla X\vec{F}.\vec{n}_1 ds = -\iint_{\Sigma_2} \nabla X\vec{F}.\vec{n}_2 ds$

Exercise-5.5.

1. Use Stokes's Theorem to evaluate $\int_C (x+y)dx + (2x-z)dy + (y+z)dz$, where C is the boundary of ΔABC with vertices A = (2, 0, 0), B = (0, 3, 0) and C = (0, 0, 6).
2. Let $\sum$ be the semi – ellipsoid $z = 2\sqrt{1-x^2-y^2}$ oriented so that its normal is directed upward. If $\vec{F} = x^2 i + y^2 j + z^2 \tan(xy)\, k$, evaluate $\iint_{\Sigma} \nabla X\,\vec{F}.\vec{n}ds$.

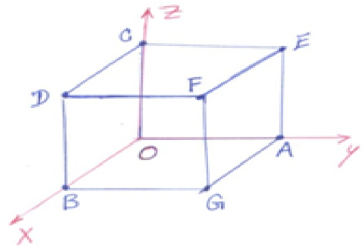
5.5.2 GAUSS DIVERGENCE THEOREM

Definition: A solid region D is said to be simple iff D is the region between graphs of two continuous functions $Z = F_1(x,y)$ and $Z = F_2(x,y)$ on a simple region R in the xy-plane and if D has the corresponding properties with respect the xz – plane and yz –plane. Regions bounded by spheres, hemi – spheres, ellipsoids, cubes, tetrahedrons, etc... are simple solid regions.

Theorem: Let D be a simple solid region whose boundary surface $\sum$ is oriented by the normal $\vec{n}$ directed out ward from D, and let $\vec{F}$ be a vector field whose component functions have continuous partial derivatives on D. Then $\iint_{\Sigma} \vec{F}.\vec{n}ds = \iiint_D \nabla.\vec{F}dv$

Example-5.22 Verify the divergence theorem for the function $\vec{F} = 4xzi - y^2 j + yzk$ and $\sum$ is the surface of the cube $x = 0, x = 1, y = 0, y = 1, z = 0$ and $z = 1$.

Solution: Sketch the solid region **D**.



i. Using surface integral, we have: Face OAGB, out ward normal vector is $\vec{n} = -k, z = 0$

$$\Rightarrow \iint_{\Sigma} \vec{F}.\vec{n}ds = \iint_{R} (4x.0i - y^2 j + y.ok).(-k)dA = \iint_{R} odA = 0$$

ii. Face CEFD, out ward normal vector is $\vec{n} = k, z = 1$

$$\Rightarrow \iint_{\Sigma} \vec{F}.\vec{n}ds = \iint_{R} (4x.1i - y^2 j + y.1k).(k)dA = \iint_{R} ydA = \int_{x=0}^{x=1} \left(\int_{y=0}^{y=1} ydy \right) dx = \frac{1}{2} \int_{x=0}^{x=1} dx = \frac{1}{2}$$

iii. Face AGFE, out ward normal vector is $\vec{n} = j, y = 1$

$$\Rightarrow \iint_{\Sigma} \vec{F}.\vec{n}ds = \iint_{R} (4xzi - 1.zk).(j)dA = \iint_{R} -1dA = -1$$

iv. Face OBDC, outward normal vector is $\vec{n} = -j, y = 0$

$$\Rightarrow \iint_{\Sigma} \vec{F}.\vec{n}ds = \iint_{R} (4xzi - 0^2 j + 0.zk).(-j)dA = \iint_{R} odA = 0$$

v. Face DBGF, out ward normal is $\vec{n} = i, x = 1$

$$\Rightarrow \iint_{\Sigma} \vec{F}.\vec{n}ds = \iint_{R} (4.1zi - y^2 j + yzk).(i)dA = \iint_{R} 4zdA = 4\iint zdA = 4\int_{x=0}^{x=1} \left(\int_{z=0}^{z=1} zdz \right) dx = 2$$

vi. Face OAEC, outward normal is $\vec{n} = -i, x = 0 \Rightarrow \iint_{\Sigma} \vec{F}.\vec{n}ds = \iint_{R} (4.0zi - y^2 j + yzk).(-i)dA = 0$

Thus, $\iint_{\Sigma} \vec{F}.\vec{n}ds = 0 + \frac{1}{2} + (-1) + 0 + 2 + 0 = \frac{3}{2}$

Using the divergence theorem we have that

$$\iint_{\Sigma} \vec{F}.\vec{n}ds = \iiint_{D} \nabla.\vec{F}dv = \iiint_{D} (4z - 2y + y)dv = \iiint_{D} (4z - y)dv$$

$= \int_{x=0}^{x=1} \int_{y=0}^{y=1} \int_{z=0}^{z=1} (4z - y)dzdx = \frac{3}{2}$, which is very short.

Example-5.23:

If Σ is the sphere $x^2 + y^2 + z^2 = 4$ and $\vec{F} = 3xi + 4yj + 5zk$, use the Divergence theorem to evaluate to $\iint_{\Sigma} \vec{F}.\vec{n}ds$.

Solution: Here, $\nabla.\vec{F} = 3 + 4 + 5 = 12$.

$\therefore \iint_{\Sigma} \vec{F}.\vec{n}ds = \iiint_{D} 12dv = 12\iiint_{D} 1dv$, where D is the solid $x^2 + y^2 + z^2 \le 4$

$$= \int_{\theta=0}^{\theta=2\pi} \left[\int_{\phi=0}^{\phi=\pi} \left(\int_{\rho=0}^{\rho=2} (\rho^2 \sin\phi) d\rho \right) d\phi \right] d\theta = 128\pi$$, using spherical coordinates.

Exercise-5.6

1. Verify the Divergence theorem for $\vec{F} = (x^2 - yz)i + (y^2 - zx)j + (z^2 - xy)k$ taken over the rectangular parallelepiped $0 \le x \le a, 0 \le y \le b \ \& \ 0 \le z \le c$
2. Use the Divergence theorem to evaluate $\iint_{\Sigma} \vec{F}.\vec{n}ds$, if $\vec{F} = 4xi - 2y^2 j + z^2 k$ and Σ is the surface bounding the region $x^2 + y^2 = 4, z = 0$ and $z = 3$.