CHAPTER 4

Vector Differential Calculus

The vector differential calculus extends the basic concepts of (ordinary) differential calculus to vector functions, by introducing derivative of a vector function and the new concepts of gradient, divergence and curl.

4.1 Vector Calculus

Definition. *A vector function $\vec{r}(t)$ is a function whose domain is a set of real numbers and whose range is a set of vectors. Any vector $\vec{r}(t)$ can be expressed in the component form*

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$$

Where f, g and h are three scalar functions of t.

4.1.1 Limit

The way we define limits of vector-valued functions is similar to the way we define limits of real-valued functions.

Definition. *Let $\vec{r}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$ be a vector function and $\boldsymbol{L}$ is a vector. We say $\vec{r}$ has limit $\boldsymbol{L}$ as t approach and write*

$$\lim_{t \to t_o} \vec{r}(t) = \boldsymbol{L}$$

The equation

$$\lim_{t\to t_o} \vec{r}(t) = \lim_{t\to t_o} f(t)\hat{\imath} + \lim_{t\to t_o} g(t)\hat{\jmath} + \lim_{t\to t_o} h(t)\hat{k}$$

provides a practical way to calculate limits of a vector functions.

Example 4.1. *Find the limit of the following vector-valued functions at the indicated value of t*

1.
$$\lim_{t\to\frac{\pi}{4}} \left(\cos(t)\hat{\imath} + \sin(t)\hat{\jmath} + t\hat{k}\right)$$

Solution.

$$\begin{aligned}\lim_{t\to\frac{\pi}{4}} \vec{r}(t) &= \lim_{t\to\frac{\pi}{4}} \left(\cos(t)\hat{\imath} + \sin(t)\hat{\jmath} + t\hat{k}\right)\\ &= \lim_{t\to\frac{\pi}{4}} \cos(t)\hat{\imath} + \lim_{t\to\frac{\pi}{4}} \sin(t)\hat{\jmath} + \lim_{t\to\frac{\pi}{4}} t\hat{k}\\ &= \frac{\sqrt{2}}{2}\hat{\imath} + \frac{\sqrt{2}}{2}\hat{\jmath} + \frac{\pi}{4}\hat{k}\end{aligned}$$

2.
$$\lim_{t\to 3} \left(\frac{2t-4}{t+1}\hat{\imath} + \frac{t}{t^2+1}\hat{\jmath} + (4t-3)\hat{k}\right)$$

Solution.

$$\begin{aligned}\lim_{t\to 3} \vec{r}(t) &= \lim_{t\to 3} \left(\frac{2t-4}{t+1}\hat{\imath} + \frac{t}{t^2+1}\hat{\jmath} + (4t-3)\hat{k}\right)\\ &= \lim_{t\to 3} \frac{2t-4}{t+1}\hat{\imath} + \lim_{t\to 3} \frac{t}{t^2+1}\hat{\jmath} + \lim_{t\to 3}(4t-3)\hat{k}\\ &= \frac{1}{2}\hat{\imath} + \frac{3}{10}\hat{\jmath} + 9\hat{k}\end{aligned}$$

Exercise. *Calculate* $\lim_{t\to -2} r(t)$ *for the function*

$$r(t) = \sqrt{t^2-3t-1}\hat{\imath} + (4t+3)\hat{\jmath} + \frac{\sin(t+1)\pi}{2}\hat{k}$$

4.1.2 Vector Differentiation

Differentiating a vector function is a simple extension of differentiating scalar quantities.

Definition. *The derivative of a vector-valued function $\hat{r}(t)$ is*

$$\frac{d\boldsymbol{r}}{dt} = \lim_{\triangle t \to 0} \frac{\boldsymbol{r}(t+\triangle t) - \boldsymbol{r}(t)}{\triangle t}$$

provided the limit exists. If $r'(t)$ exists, then r is differentiable at t. If $r'(t)$ exists for all t in an open interval $(a,\ b)$, then r is differentiable over the interval $(a,\ b)$.

Some Results on Differentiation

If $\vec{r}(t)$ and $\vec{s}(t)$ are vector valued functions, then

1. $\frac{d}{dt}\{c\vec{r}\} = c\frac{d\vec{r}}{dt}$ where c is constant.

2. $\frac{d}{dt}\{\vec{r}+\vec{s}\} = \frac{d\vec{r}}{dt} + \frac{d\vec{s}}{dt}$

3. $\frac{d}{dt}\{\vec{r}\cdot\vec{s}\} = \vec{s}\cdot\frac{d\vec{r}}{dt} + \vec{r}\cdot\frac{d\vec{s}}{dt}$

4. $\frac{d}{dt}\{\vec{r}\times\vec{s}\} = \vec{s}\times\frac{d\vec{r}}{dt} + \vec{r}\times\frac{d\vec{s}}{dt}$

Example 4.2. *Use the definition to calculate the derivative of the function*

$$r(t) = (3t+4)\hat{i} + (t^2-4t+3)\hat{j}$$

Solution.

$$\begin{aligned}
r'(t) &= \lim_{\triangle t\to 0}\frac{r(t+\triangle t)-r(t)}{\triangle t}\\
&= \lim_{\triangle t\to 0}\frac{\Big[(3(t+\triangle t)+4)\,\hat{i} + \big((t+\triangle t)^2 - 4(t+\triangle t)+3\big)\,\hat{j}\Big] - \Big[(3t+4)\hat{i} + (t^2-4t+3)\hat{j}\Big]}{t}\\
&= \lim_{\triangle t\to 0}\frac{(3t+3\triangle t+4)\hat{i} - (3t+4)\hat{i} + \big(t^2+2t\triangle t+(\triangle t)^2-4t-4\triangle t+3\big)\,\hat{j} - \big(t^2-4t+3\big)\,\hat{j}}{\triangle t}\\
&= \lim_{\triangle t\to 0}\frac{(3\triangle t)\hat{i} + \big(2t\triangle t+(\triangle t)^2-4\triangle t\big)\,\hat{j}}{\triangle t}\\
&= \lim_{\triangle t\to 0}(3\hat{i} + (2t+\triangle t-4)\hat{j})\\
&= 3\hat{i} + (2t-4)\hat{j}.
\end{aligned}$$

Velocity:

$$\vec{v} = \frac{d\vec{r}}{dt}$$

Acceleration:

$$a = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

Example 4.3. *If* $\vec{r}(t) = \sin(t)\hat{\imath} + e^{-t}\hat{\jmath} + 3\hat{k}$, *find* $\frac{d\vec{r}}{dt}$

Solution. Given $\vec{r}(t) = \sin(t)\hat{\imath} + e^{-t}\hat{\jmath} + 3\hat{k}$. Hence

$$\begin{aligned}\frac{d\vec{r}}{dt} &= \frac{d}{dt}\left(\sin(t)\hat{\imath} + e^{-t}\hat{\jmath} + 3\hat{k}\right)\\ &= \frac{d}{dt}(\sin(t))\hat{\imath} + \frac{d}{dt}(e^{-t})\hat{\jmath} + \frac{d}{dt}(3)\hat{k}\\ &= \cos(t)\hat{\imath} - e^{-t}\hat{\jmath}\end{aligned}$$

Example 4.4. *A particle moves along the curve*

$$f(t) = 4\cos(t), \quad g(t) = 4\sin(t), \quad h(t) = 6t$$

Find the velocity and acceleration at time t= 0.

Solution.
First we can write the position as a vector $\vec{r}$

$$\vec{r}(t) = 4\cos(t)\hat{\imath} + 4\sin(t)\hat{\jmath} + 6t\hat{k}$$

Then,

$$\vec{v} = \frac{d\vec{r}}{dt} = -4\sin(t)\hat{\imath} + 4\cos(t)\hat{\jmath} + 6\hat{k}$$

at t=0, $\vec{v} = 4\hat{\jmath} + 6\hat{k}$

$$a = \frac{d^2\vec{r}}{dt^2} = -4\cos(t)\hat{\imath} - 4\sin(t)\hat{\jmath}$$

at t=0, $a = -4\hat{\imath}$

4.1.3 Integrals of Vector-Valued Functions

Definition. *Let f, g, and h be integrable real-valued functions over the closed interval [a, b].*

1. *The indefinite integral of a vector-valued function $r(t) = f(t)\hat{i} + g(t)\hat{j}$ is*
$$\int [f(t)\hat{i} + g(t)\hat{j}]dt = \left[\int f(t)dt\right]\hat{i} + \left[\int g(t)dt\right]\hat{j}.$$
The definite integral of a vector-valued function is
$$\int_a^b [f(t)\hat{i} + g(t)\hat{j}]dt = \left[\int_a^b f(t)dt\right]\hat{i} + \left[\int_a^b g(t)dt\right]\hat{j}.$$

2. *The indefinite integral of a vector-valued function*
$$r(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$$
is
$$\int [f(t)\hat{i}+g(t)\hat{j}+h(t)\hat{k}]dt = \left[\int f(t)dt\right]\hat{i}+\left[\int g(t)dt\right]\hat{j}+\left[\int h(t)dt\right]\hat{k}.$$
The definite integral of a vector-valued function is
$$\int_a^b [f(t)\hat{i}+g(t)\hat{j}+h(t)\hat{k}]dt = \left[\int_a^b f(t)dt\right]\hat{i}+\left[\int_a^b g(t)dt\right]\hat{j}+\left[\int_a^b h(t)dt\right]\hat{k}.$$

Example 4.5. *Calculate each of the following integrals:*

1. $$\int \left[(3t^2 + 2t)\hat{i} + (3t - 6)\hat{j} + (6t^3 + 5t^2 - 4)\hat{k}\right] dt$$

Solution.
$$\begin{aligned}
&\int \left[(3t^2 + 2t)\hat{i} + (3t - 6)\hat{j} + (6t^3 + 5t^2 - 4)\hat{k}\right] dt \\
&= \int \left[(3t^2 + 2t)dt\right]\hat{i} + \int \left[(3t - 6)dt\right]\hat{j} + \left[(6t^3 + 5t^2 - 4)dt\right]\hat{k} \\
&= (t^3 + t^2)\hat{i} + \left(\frac{3}{2}t^2 - 6t\right)\hat{j} + \left(\frac{3}{2}t^4 + \frac{5}{3}t^3 - 4t\right)\hat{k} + C
\end{aligned}$$

2.
$$\int_0^{\frac{\pi}{3}} \left[\sin 2t\hat{i} + \tan t\hat{j} + e^{-2t}\hat{k}\right] dt$$

Solution.

$$\begin{aligned}
&\int_0^{\frac{\pi}{3}} \left[\sin 2t\hat{i} + \tan t\hat{j} + e^{-2t}\hat{k}\right] dt \\
&= \int_0^{\frac{\pi}{3}} [\sin 2tdt]\,\hat{i} + \int_0^{\frac{\pi}{3}} [\tan tdt]\,\hat{j} + \int_0^{\frac{\pi}{3}} [e^{-2t}dt]\,\hat{k} \\
&= \left[\frac{-1}{2}\cos t\right]_0^{\frac{\pi}{3}} \hat{i} + [\ln(\cos t)]_0^{\frac{\pi}{3}}\,\hat{j} + \left[\frac{-1}{2}e^{-2t}\right]_0^{\frac{\pi}{3}} \hat{k} \\
&= \frac{3}{4}\hat{i} + \ln 2\hat{j} + \left(\frac{1}{2} - \frac{1}{2}e^{\frac{-2\pi}{3}}\right)\hat{k}
\end{aligned}$$

4.2 Curves and Their Length

Definition. *A space* ***curve*** *is a range of a continuous vector - valued function on an interval of real numbers.*

When a particle moves through space during a time interval I, then the particles' coordinates can be considered as function defined on I. $x = f(t)$, $y = g(t)$, $z = h(t)$

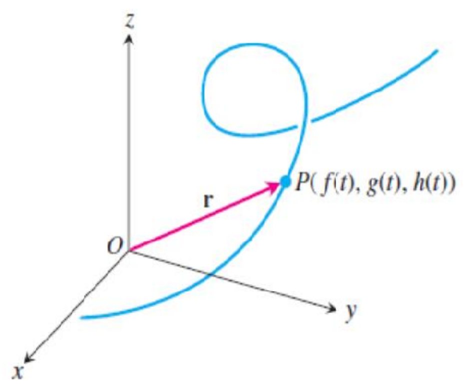

Figure 4.1: A parametric representation of a curve

We denoted a curve by C, and the vector-valued function whose range is the Curve by $\vec{r}$. Here we say that C is **parameterized** by $\vec{r}$, or that r is the **parametrization** of C.

Examples of Parameterizations of some Curves:

1. A straight line L through a point A with position vector $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ in the direction of a constant vector $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ can be represented by in the form

$$\vec{r}(t) = \vec{a} + t\vec{b} = (a_1 + tb_1)\hat{i} + (a_2 + tb_2)\hat{j} + (a_3 + tb_3)\hat{k}$$

2. The vector fuction $\vec{r}(t) = a\cos(t)\hat{i} + a\sin(t)\hat{j} + ct\hat{k}$ is defined for all real values of t. The curve traced by $\vec{r}$ is a **circular helix**(from an old Greek word for "spiral") that winds around the circular cylinder $x^2 + y^2 = 1$"(Fig 2). The curve lies on the cylinder because the $\hat{i}$ and $\hat{j}$ components of $\vec{r}$, being the x and y coordinates of the tip of $\vec{r}$, satisfy the cylinder's equation: $x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1$

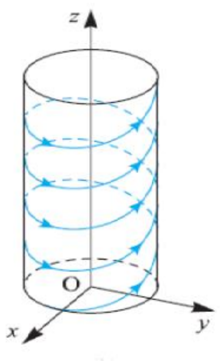

Figure 4.2: circular helix

3. The vector function $\vec{r}(t) = (x_0 + a\cos t)\hat{i} + (y_0 + b\sin t)\hat{j}$ represents an **ellipse** in the xy-plane with center (x_0, y_0) and the principal axis in the direction of the x and y axis. If $a = b$, then $\vec{r}(t) = (x_0 + a\cos t)\hat{i} + (y_0 + a\sin t)\hat{j}$ is an equation of **circle** centered at(x_0, y_0) with radius a.

Definition. *A **plane curve** is a curve that lies in a plane. A curve that is not a plane curve is called a **twisted curve**.*

Example 4.6. *A graph of continuous function of one variable $y = f(x)$ is a plane curve; whereas a circular helix is a twisted curve.*

Definition. *A curve C is **closed** if and only if it has a parametrization $\vec{r}$ on [a, b] such that $\vec{r}(a) = \vec{r}(b)$. In other words a curve is **closed** on [a, b] iff its initial and terminal points coincide.*

Example 4.7. *The unit circle $C : \vec{r}(t) = \cos(t)\hat{\imath} + \sin(t)\hat{\jmath}$, for $0 \le t \le 2\pi$ is closed since $\vec{r}(0) = \vec{r}(2\pi)$ whereas line-segments, cycloids, circular helix, etc are not closed.*

Definition. *The curve traced by $\vec{r}$ is **smooth** if $\frac{d\vec{r}}{dt}$ is continuous and never 0, i.e if f, g, and h have continuous first derivatives that are not simultaneously 0.*

Definition. *The curve,C, traced by a continuous vector-valued function $\vec{r}$ defined on an interval I is said to be **piece-wise smooth** iff I is composed of finite number of sub intervals on each of which C is smooth and $\vec{r}$ has one-sided derivative at each interior point of I.*

Example 4.8. *The circular helix $\vec{r}(t) = \cos(t)\hat{\imath} + \sin(t)\hat{\jmath} + t\hat{k}$ is smooth for $-\infty < t < \infty$ since $\frac{d\vec{r}}{dt} = -\sin(t)\hat{\imath} + \cos(t)\hat{\jmath} + \hat{k} \neq 0$ for all $t \in \mathbb{R} = (-\infty, \infty)$.*

4.2.1 Arc Length of a Curve

One of the special features of smooth space curves is that they have a measurable length.

Definition. *Suppose C is a piece-wise smooth curve with parametrization of $\vec{r}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$ on [a, b]. The length L of C is defined by*

$$L = \int_a^b \left\| \frac{d\vec{r}}{dt} \right\|$$

Example 4.9.

1. Calculate the arc length for each of the following vector-valued functions:

   (a) $\vec{r}(t) = t\hat{i} + \frac{\sqrt{6}t^2}{2}\hat{j} + t^3\hat{k}, \quad for \quad -1 \leq t \leq 1$

   Solution.
   The vector function is $\vec{r}(t) = t\hat{i} + \frac{\sqrt{6}t^2}{2}\hat{j} + t^3\hat{k}$.
   So
   $$\frac{d\vec{r}}{dt} = \hat{i} + \sqrt{6}t\hat{j} + 3t^2\hat{k}$$
   Thus the arc length is
   $$\begin{aligned} L &= \int_a^b \left\| \frac{d\vec{r}}{dt} \right\| \\ &= \int_{-1}^{1} \sqrt{1 + 6t^2 + 9t^4} dt \\ &= \int_{-1}^{1} (3t^2 + 1) dt \\ &= t^3 + 1|_{-1}^{1} = 4 \end{aligned}$$

   (b) $r(t) = (3t - 2)\hat{i} + (4t + 5)\hat{j}$, $1 \leq t \leq 5$

   Solution. Using the definition,
   $$r'(t) = 3\hat{i} + 4\hat{j}$$
   So,
   $$\begin{aligned} L &= \int_a^b \left\| \frac{d\vec{r}}{dt} \right\| \\ &= \int_{1}^{5} \sqrt{3^2 + 4^2} dt \\ &= \int_{1}^{5} 5 dt \\ &= 5t|_1^5 = 20 \end{aligned}$$

2. Find the length of one turn of the helix

$$\vec{r}(t) = \cos(t)\hat{\imath} + \sin(t)\hat{\jmath} + t\hat{k}$$

Solution.
The helix makes one full turn as L runs from 0 to 2π.

$$\frac{d\vec{r}}{dt} = -\sin(t)\hat{\imath} + \cos(t)\hat{\jmath} + \hat{k}$$

Using the length formula, the length of this portion of this curve is:

$$\begin{aligned}
L &= \int_0^{2\pi} \left\| \frac{d\vec{r}}{dt} \right\| \\
&= \int_0^{2\pi} \sqrt{(-\sin(t))^2 + (\cos(t))^2 + 1dt} \\
&= \int_0^{2\pi} \sqrt{\sin^2(t) + \cos^2(t) + 1dt} \\
&= \int_0^{2\pi} \sqrt{2}dt \\
&= 2\sqrt{2}\pi units
\end{aligned}$$

Exercise. *Find the arc length for each of the following vector-valued functions:*

1. $r(t) = \langle t\cos t, t\sin t, 2t\rangle$, $0 \leq t \leq 2\pi$
2. $r(t) = \langle 2t^2 + 1, 2t^2 - 1, t^3\rangle$, $0 \leq t \leq 3$

Arc-Length Parameterizations

Definition. *Let C be a smooth curve parameterized by*

$$\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$$

on an interval I, and let $a \in I$ be fixed number. The ***arc length function****, s, is defined by*

$$s(t) = \int_a^t \left\| \frac{d\vec{r}}{du} \right\| du = \int_a^t \sqrt{(x'(u))^2 + (y'(u))^2 + (z'(u))^2}du$$

Note: The arc length parametrization possess some advantages. By the fundamental theorem of calculus, we then have

$$\frac{ds}{dt} = \left\|\frac{d\vec{r}}{dt}\right\|$$

Example 4.10. *Find the arc length function for the following vector-valued functions*

1. $\vec{r}(t) = (3-3t)\hat{\imath} + 4t\hat{\jmath}$, $0 \leq t \leq 1$

   Solution. *First*

$$\frac{d\vec{r}}{dt} = -3\hat{\imath} + 4\hat{\jmath}$$
$$\left\|\frac{d\vec{r}}{dt}\right\| = \sqrt{(-3)^2 + (4)^2} = 5$$

   Then,

$$\begin{aligned} s &= \int_0^t \left\|\frac{d\vec{r}}{du}\right\| du \\ &= \int_0^t 5du \\ &= 5t \end{aligned}$$

2. $r(t) = 4\cos t\hat{\imath} + 4\sin t\hat{\jmath}$, $t \geq 0$

   Solution. *First*

$$\frac{dr}{dt} = -4\sin t\hat{\imath} + 4\cos t\hat{\jmath}$$
$$\left\|\frac{dr}{dt}\right\| = \sqrt{(-4\sin t)^2 + (4\cos t)^2} = 4$$

   Then,

$$\begin{aligned} s(t) &= \int_0^t \left\|\frac{dr}{du}\right\| du \\ &= \int_0^t 4du \\ &= 4t \end{aligned}$$

Example 4.11. *Reparametrize the curves*

1.
$$\vec{r}(t) = \frac{t^2}{2}\hat{i} + \frac{t^3}{3}\hat{k}, \qquad 0 \le t \le 2$$

Solution.

$$\frac{d\vec{r}}{dt} = t\hat{i} + t^2\hat{k}$$

$$\begin{aligned} s(t) &= \int_0^t \left\| \frac{d\vec{r}}{du} \right\| du \\ &= \int_0^t \sqrt{u^2 + u^4} \\ &= \frac{(t^2+1)^{\frac{3}{2}} - 1}{3} \end{aligned}$$

Inverting this produces

$$t = [(3s+1)^{\frac{2}{3}} - 1]^{\frac{1}{2}}$$

and the new parametrization is

$$\vec{r}(s) = \frac{(3s+1)^{\frac{2}{3}} - 1}{2}\hat{i} + \frac{[(3s+1)^{\frac{2}{3}} - 1]^{\frac{3}{2}}}{3}\hat{k}$$

2.
$$\vec{r}(t) = (2\cos t)\hat{i} + (2\sin t)\hat{j} \qquad 0 \le t \le 2\pi$$

Solution. *Exercise!!!*

Note: If we take s the parameter in place of t then the magnitude of the tangent vector, i.e., $\left\|\frac{d\vec{r}}{ds}\right\| = 1$.

4.3 Tangent, Curvature and Torsion

Tangent to a Curve

Definition. *Let C be a curve traces by $\vec{r}$ defined on an interval I. If the vector $\frac{d\vec{r}}{dt} \neq 0$, we call $\frac{d\vec{r}}{dt}$ is a **tangent vector** to the space curve C. The corresponding unit vector is **unit tangent vector** denoted by, $\vec{T}$, is given by*

$$\vec{T}(t) = \frac{d\vec{r}/dt}{\|d\vec{r}/dt\|}$$

Example 4.12.

1. Find the unit tangent vector of the helix

$$\vec{r}(t) = \cos t\hat{i} + \sin t\hat{j} + t\hat{k}$$

   .

   Solution.
   The vector equation of the curve is

$$\vec{r}(t) = \cos t\hat{i} + \sin t\hat{j} + t\hat{k}$$

$$d\vec{r}/dt = -\sin t\hat{i} + \cos t\hat{j} + \hat{k}$$

   $\Rightarrow \|d\vec{r}/dt\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2}$
   Then,

$$\begin{aligned}\vec{T}(t) &= \frac{d\vec{r}/dt}{\|d\vec{r}/dt\|} \\ &= \frac{-\sin t}{\sqrt{2}}\hat{i} + \frac{\cos t}{\sqrt{2}}\hat{j} + \frac{1}{\sqrt{2}}\hat{k}\end{aligned}$$

2. Determine the unit tangent vector at the point (2, 4, 7) for the curve with parametric equations

$$x = 2t \quad , y = t^2 + 3, \quad z = 2t^2 + 5$$

   . **Solution.**
   First we see that the point (2, 4, 7) corresponds to t=1.
   The vector equation of the curve is

$$\vec{r}(t) = 2t\hat{i} + (t^2 + 3)\hat{j} + (2t^2 + 5)\hat{k}$$

Therefore,

$$d\vec{r}/dt = 2\hat{i} + 2t\hat{j} + 4t\hat{k}$$

and at t=1,

$$d\vec{r}/dt = 2\hat{i} + 2\hat{j} + 4\hat{k}$$

Hence

$$\|d\vec{r}/dt\| = \{4 + 4 + 16\}^{\frac{1}{2}} = 2\sqrt{6}$$

Then

$$\begin{aligned}\vec{T}(t) &= \frac{d\vec{r}/dt}{\|d\vec{r}/dt\|}\\ &= \frac{2\hat{i} + 2\hat{j} + 4\hat{k}}{2\sqrt{6}}\\ &= \frac{1}{\sqrt{6}}\{\hat{i} + \hat{j} + \hat{k}\}\end{aligned}$$

Exercise.

1. Find the unit tangent vector of the curve

$$\vec{r}(t) = (\cos t + t\sin t)\hat{i} + (\sin t - t\cos t)\hat{j}, \quad t > 0$$

.

2. Find the unit tangent vector at the point $(2, 0, \pi)$ for the curve with parametric equations

$$x = 2\sin\theta, \quad y = 3\cos\theta, \quad z = 2\theta$$

Note:

1. **Principal Normal.** If $d\vec{T}/dt$ is not the zero vector, we define a unit vector $\mathbf{N}$ to be $d\vec{T}/dt$ divided by its own magnitude,

$$\mathbf{N} = \frac{d\vec{T}/dt}{\left\|d\vec{T}/dt\right\|}$$

This vector is called the *principal normal.*

2. **Binormal.** A third unit vector B defined by **B** = **T** x **N**, is called the *binormal.* Since **T** and **N** are unit vectors, **B** is also a unit vector perpendicular to both **T** and **N**.

Example 4.13. *Find* $\boldsymbol{T}$ *and* $\boldsymbol{N}$ *for the circular motion*

$$\vec{r}(t) = \cos 2t\hat{i} + \sin 2t\hat{j} \quad (0 \leq t \leq 2\pi)$$

Solution.
To find $\boldsymbol{T}(t)$ *and* $\boldsymbol{N}(t)$*, first we must compute* $d\vec{r}/dt$ *and* $\|d\vec{r}/dt\|$*. Doing so, we obtain*

$$d\vec{r}/dt = -2\sin 2t\hat{i} + 2\cos 2t\hat{j}$$

and

$$\|d\vec{r}/dt\| = \sqrt{(-2\sin 2t)^2 + (2\cos 2t)^2} = 2$$

so,

$$\begin{aligned}\boldsymbol{T} &= \frac{d\vec{r}/dt}{\|d\vec{r}/dt\|} \\ &= \frac{-2\sin 2t\hat{i} + 2\cos 2t\hat{j}}{2} \\ &= -\sin 2t\hat{i} + \cos 2t\hat{j}\end{aligned}$$

From this we find

$$d\boldsymbol{T}/dt = -2\cos 2t\hat{i} - 2\sin 2t\hat{j}$$

and

$$\|d\boldsymbol{T}/dt\| = \sqrt{(-2\cos 2t)^2 + (-2\sin 2t)^2} = 2$$

So,

$$\begin{aligned}\boldsymbol{N} &= \frac{d\boldsymbol{T}/dt}{\|d\boldsymbol{T}/dt\|} \\ &= \frac{-2\cos 2t\hat{i} - 2\sin 2t\hat{j}}{2} \\ &= -\cos 2t\hat{i} - \sin 2t\hat{j}\end{aligned}$$

Definition. *Let C be a curve that is parameterized by*

$$\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}.$$

Then,
(1) *The* ***curvature****, κ, is given by*

$$\kappa = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3} = \frac{\|T'(t)\|}{\|r'(t)\|}$$

(2) *The* ***torsion****, τ is given by,*

$$\tau = \frac{\begin{vmatrix} f' & g' & h' \\ f'' & g'' & h'' \\ f''' & g''' & h''' \end{vmatrix}}{\|r'(t) \times r''(t)\|^2}$$

Example 4.14. *Find the curvature and torsion of the curve*

$$x = a\cos t, \quad y = a\sin t \quad z = ct$$

.

Solution. The vector equation of the curve is $\vec{r}(t) = a\cos t\hat{i} + a\sin t\hat{j} + ct\hat{k}$ Therefore,

$$dr/dt = -a\sin t\hat{i} + a\cos t\hat{j} + c\hat{k}$$

and

$$r''(t) = -a\cos t\hat{i} - a\sin t\hat{j}$$

So,

$$r'(t) \times r''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin t & a\cos t & c \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = ac\sin t\hat{i} + ac\cos t\hat{j} + a^2\hat{k}$$

Then,

$$\|r'(t) \times r''(t)\| = \sqrt{a^2c^2 + a^4} = a\sqrt{a^2 + c^2}$$

and

$$\|r'(t)\| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + c^2} = \sqrt{a^2 + c^2}$$

Therefore,

$$\kappa = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3} = \frac{a\sqrt{a^2 + c^2}}{(\sqrt{a^2 + c^2})^3} = \frac{a}{\sqrt{a^2 + c^2}}$$

Also

$$\tau = \frac{\begin{vmatrix} f' & g' & h' \\ f'' & g'' & h'' \\ f''' & g''' & h''' \end{vmatrix}}{\|r'(t) \times r''(t)\|^2}$$
$$= \frac{\begin{vmatrix} -\sin t & a\cos t & c \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix}}{a^2(a^2+c^2)} = \frac{c}{a^2+c^2}$$

Exercise

1. Find **T**, **N**, **B**, κ and τ for $\vec{r}(t) = 6\sin 2t\hat{\imath} + 6\cos 2t\hat{\jmath} + 5t\hat{k}$

2. Find the torsion of C:$\vec{r}(t) = \langle t, t^2, t^3 \rangle$

3. Show that the helix $\vec{r}(t) = \langle a\cos t, a\sin t, ct \rangle$ can be represented by $\langle a\cos(s/k), a\sin(s/k), cs/k \rangle$, where $k = \sqrt{a^2+c^2}$ and s is the arc length. Show that $\tau = c/k^2$.

4.4 Scalar and Vector Fields

1. If to each point P of a region **R** in space there corresponds a definite scalar denoted by f(P), then f(P) is called a **scalar point function** in **R**. The region **R** so defined is called a **scalar field**.

   The temperature at any instant, density of a body and potential due to gravitational matter are all examples of scalar point functions.

2. If to each point P of a region **R** in space there corresponds a definite vector denoted by **F**(P), then it is called the **vector point function** in **R**. The region **R** so defined is called a **vector field**.

   The velocity of a moving fluid at any instant, the gravitational intensity of force are examples of vector point functions.

Definition. *A vector field **F** in $\mathbb{R}^2$ is an assignment of a two-dimensional vector **F**(x, y) to each point (x, y) of a subset D of $\mathbb{R}^2$. The subset D is the domain of the vector field.*

*A vector field **F** in $\mathbb{R}^3$ is an assignment of a three-dimensional vector **F**(x, y, z) to each point (x, y, z) of a subset D of $\mathbb{R}^3$. The subset D is the domain of the vector field.*

- **Vector Fields in $\mathbb{R}^2$**
  A vector field in $\mathbb{R}^2$ can be represented in either of two equivalent ways. The first way is to use a vector with components that are two-variable functions:
  $$\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$$
  The second way is to use the standard unit vectors:
  $$\mathbf{F}(x, y) = P(x, y)\hat{\imath} + Q(x, y)\hat{\jmath}.$$
  A vector field is said to be continuous if its component functions are continuous.
- **Vector Fields in $\mathbb{R}^3$**
  We can represent vector fields in $\mathbb{R}^3$ with component functions. We simply need an extra component function for the extra dimension. We write either
  $$\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$$
  or
  $$\mathbf{F}(x, y, z) = P(x, y, z)\hat{\imath} + Q(x, y, z)\hat{\jmath} + R(x, y, z)\hat{k}$$

4.5 Gradients of Scalar Fields

Definition. *The vector function ∇f is defined as the gradient of the scalar point function f(x, y, z) and is written as grad f.*

$$Thus \qquad grad f = \nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}$$

It should be noted that ∇f is a vector whose three components are $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$. Thus, if f is a scalar point function, then ∇f is a vector point function.

Definition. *Let f be a differentiable function of x and y and $\vec{u} = u_1\hat{\imath} + u_2\hat{\jmath}$ be a unit vector. Then the directional derivative of f at (x_0, y_0) in the direction of $\vec{u}$, denoted by $D_u f(x, y)$, is defined by*

$$D_u f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h}$$

provided that this limit exists.

Theorem. *Let f be differentiable at* (x_0, y_0)*. Then f has a directional derivative at* (x_0, y_0) *in every direction. More over, if* $\vec{u} = a_1\hat{i} + a_2\hat{j}$ *is a unit vector, then*

$$D_u f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)a_1 + \frac{\partial f}{\partial y}(x_0, y_0)a_2$$

Hence the directional derivative in the direction of any unit vector $\vec{u}$ is

$$D_u f(x, y) = \nabla f.\vec{u}$$

Note: The gradient is not merely a notational device to simplify the formula for the directional derivative; we will see that the length and direction of the gradient ∇f provide important information about the function f and the surface $z = f(x, y)$

Theorem. *Let f be a function of two variables that is differentiable at (x,y). Suppose that* $\nabla f \neq 0$ *and* $D_u f(x, y) = \nabla f(x, y).\vec{u} = \|\nabla f(x, y)\| \, \|\vec{u}\| \cos\theta = \|\nabla f(x, y)\| \cos\theta$*, where* θ *is the angle between* $\nabla f(x, y)$ *and* $\vec{u}$*.*

1. *The maximum value of* $D_u f(x, y)$ *is* $\|\nabla f(x, y)\|$*, and this maximum occurs when* $\theta = 0$*, that is, when* $\vec{u}$ *is in the direction of* $\nabla f(x, y)$*.*

2. *The minimum value of* $D_u f(x, y)$ *is* $-\|\nabla f(x, y)\|$*, and this minimum occurs when* $\theta = \pi$*, that is, when* $\vec{u}$ *is in the oppositely directed to* $\nabla f(x, y)$*.*

Theorem. *If f is differentiable at* (x_0, y_0) *and if* $\nabla f(x_), y_0) \neq 0$*, then* $\nabla f(x_0, y_0)$ *is normal to the level surface of f through* (x_0, y_0)*.*

Example 4.15. *If* $f = 3x^2y – y^3z^2$*, find grad f at the point (1, –2, –1).*

Solution.

$$\begin{aligned} grad f = \nabla f &= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(3x^2y – y^3z^2) \\ &= \hat{i}\frac{\partial}{\partial x}(3x^2y – y^3z^2) + \hat{j}\frac{\partial}{\partial y}(3x^2y – y^3z^2) + \hat{k}\frac{\partial}{\partial z}(3x^2y – y^3z^2) \\ &= 6xy\hat{i} + (3x^2 – 3y^2z^2)\hat{j} – 2y^3z\hat{k} \end{aligned}$$

$$\begin{aligned} grad\ \ f(1, –2, –1) &= 6(1)(-2)\hat{i} + (3(1) - 3(4)(1)\hat{j} - 2(-8)(-1)\hat{k} \\ &= -12\hat{i} - 9\hat{j} - 16\hat{k} \end{aligned}$$

Example 4.16. *Find the directional derivative of* $f = x^y + y^2z + z^2x$ *at the point (1, -1, 2) in the direction of the vector* $4\hat{i} + 2\hat{j} - 5\hat{k}$.

Solution. Because $f = x^2y + y^2z + z^2x$

$$\nabla f = (2xy + z^2)\hat{i} + (x^2 + 2yz)\hat{j} + (y^2 + 2xz)\hat{k}$$

At (1, -1, 2),

$$\nabla f = 2\hat{i} - 3\hat{j} + 5\hat{k}$$

$$\begin{aligned} \vec{a} = 4\hat{i} + 2\hat{j} - 5\hat{k} &\Longrightarrow \|\vec{a}\| = \sqrt{16+4+25} \\ &\Longrightarrow 3\sqrt{5} \end{aligned}$$

Let $\vec{u}$ a be the unit vector in the given direction.
Then

$$\vec{u} = \frac{4\hat{i} + 2\hat{j} - 5\hat{k}}{3\sqrt{5}}$$

$\therefore$ Directional derivative

$$\begin{aligned} D_u f(1,-1,2) &= \nabla f(1,-1,2).\vec{u} \\ &= (2\hat{i} - 3\hat{j} + 5\hat{k}).\frac{1}{3\sqrt{5}}(4\hat{i} + 2\hat{j} - 5\hat{k}) \\ &= \frac{1}{3\sqrt{5}}(8 - 6 - 25) \\ &= \frac{-23}{3\sqrt{5}} \end{aligned}$$

Example 4.17. *Let* $f(x,y) = x^2e^y$. *Find the maximum value of a directional derivative at (-2, 0), and find the unit vector in the direction in which the maximum value occurs.*

Solution. Since

$$\nabla f(x,y) = f_x(x,y)\hat{i} + f_y(x,y)\hat{j} = 2xe^y\hat{i} + x^2e^y\hat{j}$$

the gradient at $(-2, 0$ is

$$\nabla f(-2,0) = -4\hat{i} + 4\hat{j}$$

By theorem above, the maximum value of the directional derivative is

$$\|\nabla f(-2,0)\| = \sqrt{(-4)^2 + 4^2} = \sqrt{32} = 4\sqrt{2}$$

This maximum occurs in the direction of $\nabla f(-2,0)$. The unit vector in this direction is

$$\begin{aligned} \vec{u} &= \frac{1}{4\sqrt{2}}(-4\hat{i} + 4\hat{j} \\ &= -\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} \end{aligned}$$

Example 4.18. *Find the unit normal to the surface $x^2y + 2xz = 4$ at the point (2, –2, 3).*

Solution. Since $f(x, y, z) = x^2y + 2xz$, and

$$\begin{aligned}\nabla f(x, y, z) &= f_x(x, y, z)\hat{\imath} + f_y(x, y, z)\hat{\jmath} + f_z(x, y, z)\hat{k} \\ &= (2xy + 2z)\hat{\imath} + x^2\hat{\jmath} + 2x\hat{k}\end{aligned}$$

the gradient at $(2, -2, 3)$ is

$$\nabla f(2, -2, 3) = -2\hat{\imath} + 4\hat{\jmath} + 4\hat{k}$$

This vector is normal to the level surface $x^2y + 2xz = 4$. So, a unit vector normal to the surface

$$\begin{aligned}&= \frac{\nabla f(2, -2, 3)}{\|\nabla f(2, -2, 3)\|} \\ &= \frac{-2\hat{\imath} + 4\hat{\jmath} + 4\hat{k}}{\sqrt{4 + 16 + 16}} \\ &= \frac{-2\hat{\imath} + 4\hat{\jmath} + 4\hat{k}}{6} = \frac{1}{3}(-\hat{\imath} + 2\hat{\jmath} + 2\hat{k})\end{aligned}$$

Exercise

1. Find ∇f when $f = (x^2 + y^2 + z^2)e^{-\sqrt{x^2+y^2+z^2}}$
2. Find the directional derivative of $f = x^2yz + 4xz^2$ at $(1, –2, –1)$ in the direction $2\hat{\imath}–\hat{\jmath}–2\hat{k}$. In what direction the directional derivative will be maximum and what is its magnitude? Also find a unit normal to the surface $x^2yz + 4xz^2 = 6$ at the point $(1, –2, –1)$.

4.6 Divergence and Curl of Vector Fields

We will now define two important operations on vector fields in 3-space the *divergence* and the *curl* of the field. These names originate in the study of fluid flow, in which case the divergence relates to the way in which fluid flows towards to or away from a point and the curl relates to the rotational properties of the fluid at a point.

Definition. *Let* $\boldsymbol{F}(x,y,z) = f(x,y,z)\hat{\imath} + g(x,y,z)\hat{\jmath} + h(x,y,z)\hat{k}$ *be a continuously differentiable vector point function. Then*

1. *The divergence of* $\boldsymbol{F}$ *is denoted by div* $\boldsymbol{F}$ *is a scalar function and is defined by the equation*

$$\begin{aligned} div\boldsymbol{F} = \nabla.\boldsymbol{F} &= \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right) \boldsymbol{.} \left(f(x,y,z)\hat{\imath} + g(x,y,z)\hat{\jmath} + h(x,y,z)\hat{k}\right) \\ &= \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \end{aligned}$$

2. *The curl of* $\boldsymbol{F}$ *is denoted by curl* $\boldsymbol{F}$ *is a vector function and is defined by the equation*

$$\begin{aligned} curl\boldsymbol{F} = \nabla \times \boldsymbol{F} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f & g & h \end{vmatrix} \\ &= \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\hat{\imath} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\hat{k} \end{aligned}$$

Example 4.19. *If* $\boldsymbol{F} = xy^2\hat{\imath} + 2x^2yz\hat{\jmath}–3yz^2\hat{k}$ *then find div* $\boldsymbol{F}$ *and curl* $\boldsymbol{F}$ *at the point (1, – 1, 1).*

Solution. We have $\mathbf{F} = xy^2\hat{\imath} + 2x^2yz\hat{\jmath}–3yz^2\hat{k}$

$$\begin{aligned} div\mathbf{F} &= \frac{\partial}{\partial x}\left(xy^2\right) + \frac{\partial}{\partial y}\left(2x^2yz\right) + \frac{\partial}{\partial z}\left(-3yz^2\right) \\ &= y^2 + 2x^2z - 6yz \end{aligned}$$

So, $div\mathbf{F}(1,-1,1) = (-1)^2 + 2(1)^2 1 - 6(-1)1 = 9$
Again,

$$\begin{aligned} curl\mathbf{F} &= curl[xy^2\hat{\imath} + 2x^2yz\hat{\jmath}–3yz^2\hat{k}] \\ &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & –3yz^2 \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(2x^2yz)\right)\hat{\imath} + \left(\frac{\partial}{\partial z}(xy^2) - \frac{\partial}{\partial x}(-3yz^2)\right)\hat{\jmath} + \left(\frac{\partial}{\partial x}(2x^2yz) - \frac{\partial}{\partial y}(xy^2)\right)\hat{k} \\ &= (–3z^2–2x^2y)\hat{\imath} + (0–0)\hat{\jmath} + (4xyz–2xy)\hat{k} \\ &= (–3z^2–2x^2y)\hat{\imath} + (4xyz–2xy)\hat{k} \end{aligned}$$

So, curl $\mathbf{F}$(1,-1,1)$= -\hat{\imath} - 2\hat{k}$

Exercise

1. Find the divergence and curl of the vector $(x^2 – y^2)\hat{\imath} + 2xy\hat{\jmath} + (y^2 - xy)\hat{k}$.

2. Find div **F** and curl F, where $\mathbf{F} = grad(x^3 + y^3 + z^3 - 3xyz)$.

Definition. *Given a vector field* $\boldsymbol{F}$.

1. $\boldsymbol{F}$ *is said to be* ***divergence free*** *or* ***Solenoidal****, if* $\nabla.\boldsymbol{F} = 0$.
2. $\boldsymbol{F}$ *is said to be* ***irrotational****, if* $\nabla \times \boldsymbol{F} = 0$

Example 4.20.

1. If vector $\mathbf{F} = 3x\hat{\imath} + (x + y)\hat{\jmath} - az\hat{k}$ is solenoidal. Find a. (**Ans.** a=4)

2. Find the constants a, b, c so that $\mathbf{F} = (x + 2y + az)\hat{\imath} + (bx - 3y - z)\hat{\jmath} + (4x + cy + 2z)\hat{k}$ is irrotational. (**Ans.** a=4, b=2, c=-1)

4.7 Physical Interpretation

4.7.1 Physical Interpretation of Divergence

A revealing interpretation of the divergence of a vector field arises from the study of fluid flowing through a region. Suppose $\vec{V}$ represents the velocity field of fluid, for instance air flowing through a surface, such as a screen. Then in physical terms, $\nabla.\vec{V}(x, y, z)$ represents the rate (with respect to time) of mass flow per unit volume of the fluid from the point (x,y,z).
Note.

1. If $\nabla.\vec{V}(x, y, z) > 0$, then the point (x,y,z) is called a **source** of the fluid.
2. If $\nabla.\vec{V}(x, y, z) < 0$, then the point (x,y,z) is called a **sink** of the fluid.
3. If $\nabla.\vec{V}(x, y, z) = 0$, then there are neither sources nor sinks in the region. A fluid whose velocity field is divergence free is called **incompressible**

Example 4.21. *Let* $\vec{V}(x, y, z) = x^3yz^2\hat{\imath} + x^2y^2z^2\hat{\jmath} + x^2yz^3\hat{k}$ *be the velocity field of a fluid at point (x,y,z). Determine which points in space are sources, sinks? Where the fluid is incompressible?*

Ans.
• Source: $\forall_y > 0$ and $\forall_{x,y} \neq 0$;
• Sink: $\forall_y < 0$ and $\forall_{x,y} \neq 0$;
• Incompressible: $\forall_{x,y,z} = 0$

4.7.2 Physical Interpretation of curl

Suppose the $\vec{V}(x, y, z)$ represents the velocity of a fluid flowing through a solid region. Then it turns out that curl of $\vec{V}(x, y, z)$, i.e., $\nabla \times \vec{V}(x, y, z)$ measures the tendency of the fluid to curl or rotate about an axis. More specifically particles in the fluid tend to rotate about an axis in the direction of $\nabla \times \vec{V}(x, y, z)$ and the length of $\nabla \times \vec{V}(x, y, z)$ measures the swiftness of the motion of the particles around the axis.