

UNIFIED STOCHASTIC FRAMEWORK FOR NEURAL NETWORK QUANTIZATION AND PRUNING

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ABSTRACT. Quantization and pruning are two essential techniques for compressing neural networks, yet they are often treated independently, with limited theoretical analysis connecting them. This paper introduces a unified framework for post-training quantization and pruning using stochastic path-following algorithms. Our approach builds on the Stochastic Path Following Quantization (SPFQ) method, extending its applicability to pruning and low-bit quantization, including challenging 1-bit regimes. By incorporating a scaling parameter and generalizing the stochastic operator, the proposed method achieves robust error correction and yields rigorous theoretical error bounds for both quantization and pruning as well as their combination.

1. INTRODUCTION

Modern deep neural networks (DNNs) have achieved significant success but require substantial memory and computation due to their large number of parameters. Model compression techniques, including quantization, pruning, knowledge distillation, and low-rank decomposition, mitigate these issues. We focus on quantization and pruning.

Quantization replaces the weights in a DNN with elements from a finite set, allowing them to be represented with fewer bits and consequently not only compresses the network but also accelerates inference. DNN quantization methods include quantization-aware training and post-training quantization. Quantization-aware training (e.g., [3, 4, 7, 14, 26, 28, 33]) modifies standard neural network training to account for weight quantization, often requiring retraining and hyperparameter tuning, which can be computationally expensive, especially if one needs to repeat the process for multiple bit depths. In contrast, post-training quantization (e.g., [5, 6, 10, 13, 18, 20, 21, 27, 30, 32]) operates on pre-trained models, using relatively little data to replace floating-point parameters with quantized counterparts. Post-training quantization methods are generally much less demanding than the original training process, allowing for easy re-quantization of a base model at various bit depths.

Meanwhile, pruning techniques aim to set as many of a DNN’s weights to zero as possible. Pruning techniques can also be categorized into those that operate pre-training, during training, or post-training. Pre-training pruning methods (e.g., [15, 23, 24, 25]) prune the model before training and often rely significantly on good initialization. Methods that prune during training (e.g., [8, 12, 11, 17, 31]) attempt to identify and remove unnecessary parameters during training, and can increase overall training cost. Post-training pruning methods

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(e.g., [2, 9, 16, 19]) prune already trained models and fine-tune them if necessary. Similar to their quantization counterparts, post-training methods allow models to be pruned to various sparsity levels, enabling implementation on platforms with different memory, power, and computational resources.

In this theory-focused paper, we extend post-training algorithms for quantization and pruning by building on the work presented in [29], which introduced Stochastic Path Following Quantization (SPFQ). SPFQ is a stochastic error-correcting algorithm that sequentially quantizes a neuron’s weights to minimize accumulated error, with an accompanying theoretical upper bound on the Euclidean error that scales as $\sqrt{\min\{m, N\}} \cdot \text{polylog}(N)$, where N is the dimension of the neuron and m is the number of calibration data points. This scaling compares favorably to the N scaling associated with the most common Round-to-Nearest (RTN) method. However, SPFQ’s focus on minimizing accumulated error without constraints limits its applicability to extremely low-bit quantizers, as it cannot guarantee rigorous error bounds in such cases. Additionally, while SPFQ includes an analysis for finite-bit quantization when the number of bits is not too small, its framework does not readily extend to pruning.

We provide some important generalizations to SPFQ and its analysis. We show that one can replace the quantization operator in SPFQ by any stochastic operator that satisfies certain properties, and we introduce a scaling parameter into the argument of this operator. Loosely speaking, this scaling parameter controls the amount of error correction in the algorithm, and is critical to allowing us to handle the low-bit (even 1-bit) quantization regime. These modifications make our approach versatile for various tasks, including pruning and the afore-mentioned quantization, and we prove rigorous error bounds for these applications.

This paper is organized as follows. Section 2 introduces notation. In Section 3, we present our algorithm and the intuition behind it. Section 4 states and proves the main theoretical result. Sections 5, 6, and 7 discuss applications—quantization, pruning, and combined quantization with pruning—and apply the main theorem to derive error bounds, demonstrating the algorithm’s versatility for low-bit quantization and sparsification.

2. NOTATION

We define an L -layer perceptron $\Phi : \mathbb{R}^{N_0} \rightarrow \mathbb{R}^{N_L}$ via its action

$$\Phi(x) = \rho^{(L)} \circ A^{(L)} \circ \rho^{(L-1)} \circ A^{(L-1)} \circ \dots \circ \rho^{(1)} \circ A^{(1)}(x),$$

where $\rho^{(i)} : \mathbb{R}^{N_i} \rightarrow \mathbb{R}^{N_i}$ is an entry-wise nonlinear activation function, and $A^{(i)} : \mathbb{R}^{N_{i-1}} \rightarrow \mathbb{R}^{N_i}$ is an affine map defined by $A^{(i)}(x) = W^{(i)\top} x + b^{(i)}$. Here, $W^{(i)} \in \mathbb{R}^{N_{i-1} \times N_i}$ is the *weight* matrix and $b^{(i)} \in \mathbb{R}^{N_i}$ is the *bias* vector in the i -th layer. It is worth noting that $W^{(i)\top} x + b^{(i)}$ can be rewritten as $(W^{(i)\top} b^{(i)})(x^\top 1)^\top$. Therefore, we can absorb the bias vector into the weight matrix by extending the data x to $(x^\top 1)^\top$. As a result, we assume throughout this paper without loss of generality, that the bias vector $b^{(i)} = 0$. Additionally, we assume $\max_{i,j,k} |W_{jk}^{(i)}| \leq K$.

We denote by $\Phi^{(i)}$ the composition of the first i layers of the model, so that

$$\Phi^{(i)}(x) = \rho^{(i)} \circ A^{(i)} \circ \rho^{(i-1)} \circ A^{(i-1)} \circ \dots \circ \rho^{(1)} \circ A^{(1)}(x)$$

represents the *activations* at layer i . Similarly, we denote the compressed model by $\tilde{\Phi}$, and its first i layers by $\tilde{\Phi}^{(i)}$. $X \in \mathbb{R}^{m \times N_0}$ represents the data used for compressing the network,

where each row is a data point in \mathbb{R}^{N_0} . We extend notation so $\Phi^{(i)}(X) \in \mathbb{R}^{m \times N_i}$ denotes the application of $\Phi^{(i)}$ to each row of X .

For a matrix $M \in \mathbb{R}^{m \times n}$, $M_j \in \mathbb{R}^m$ represents the j -th column of M , M_{ij} represents the (i, j) -th entry of M , and for a vector $v \in \mathbb{R}^n$, $v_j \in \mathbb{R}$ represents the j -th entry of v . If A and B are two symmetric matrices, $A \preceq B$ means $B - A$ is positive semi-definite. For vectors $x \in \mathbb{R}^d$, we require the ℓ_2 and ℓ_∞ norms $\|x\|_\infty := \max_i |x_i|$, $\|x\| := \sqrt{\sum_i x_i^2}$. Meanwhile for matrices X , $\|X\|_F = \sqrt{\sum_{i,j} X_{ij}^2}$ denotes its Frobenius norm. We use $\mathcal{N}(\mu, \Sigma)$ to denote a Gaussian random vector with mean μ and covariance matrix Σ , and $U[a, b]$ to denote a random variable drawn uniformly on the interval $[a, b]$. Finally, P_v , where v is a vector, denotes the orthogonal projection onto $\text{span}(v)$.

3. STOCHASTIC PATH FOLLOWING COMPRESSION

We focus on the L -layer perceptron¹ Φ defined above. The aim is to compress the weights of Φ through quantization or pruning to obtain a network $\tilde{\Phi}$ with $\Phi(X) \approx \tilde{\Phi}(X)$. We accomplish this via Algorithm 1 below which we now briefly explain.

Algorithm 1 STOCHASTIC PATH FOLLOWING COMPRESSION($\mathcal{T}, \theta, C, \Phi, X$)

Parameters: Stochastic operator \mathcal{T} , constant $C \geq 1$.

Input: L -layer perceptron Φ with weight matrices $W^{(i)} \in \mathbb{R}^{N_{i-1} \times N_i}$, data $X \in \mathbb{R}^{m \times N_0}$.

Output: Compressed neural network $\tilde{\Phi}$.

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1: for  $i = 1$  to  $L$  do
2:   Set  $X^{(i-1)} = \Phi^{(i-1)}(X) \in \mathbb{R}^{m \times N_{i-1}}$ .
3:   Set  $\tilde{X}^{(i-1)} = \tilde{\Phi}^{(i-1)}(X) \in \mathbb{R}^{m \times N_{i-1}}$ .
4:   repeat
5:     Choose a column  $w \in \mathbb{R}^{N_{i-1}}$  of  $W^{(i)}$ .
6:     Run Algorithm 2 to obtain “compressed” weights  $q$ :
      
$$q = \text{COMPRESSION}(\mathcal{T}, C, w, X, \tilde{X}).$$

7:   until All columns of  $W^{(i)}$  are compressed into  $Q^{(i)}$ .
8:   Set:  $\tilde{\Phi}^{(i)}(\cdot) = \rho^{(i)} \circ \tilde{A}^{(i)} \circ \rho^{(i-1)} \circ \dots \circ \rho^{(1)} \circ \tilde{A}^{(1)}(\cdot)$ , where  $\tilde{A}^{(i)}(x) = Q^{(i)\top} x$ .
9: end for

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Algorithm 1 uses m data points in \mathbb{R}^{N_0} represented by $X \in \mathbb{R}^{m \times N_0}$ to guide the compression. The basic idea is to perform the compression one layer at a time, starting with the first. Thus, at layer i , we have access to $X^{(i-1)} := \Phi^{(i-1)}(X)$ and $\tilde{X}^{(i-1)} := \tilde{\Phi}^{(i-1)}(X)$, both in $\mathbb{R}^{m \times N_{i-1}}$, and the goal is to compress $W^{(i)} \in \mathbb{R}^{N_{i-1} \times N_i}$ into $Q^{(i)} \in \mathbb{R}^{N_{i-1} \times N_i}$ such that $X^{(i-1)}W^{(i)} \approx \tilde{X}^{(i-1)}Q^{(i)}$. To accomplish this, we use Algorithm 2 to independently, and possibly in parallel, replace each column (neuron) $w \in \mathbb{R}^{N_{i-1}}$ of $W^{(i)}$, entry by entry, with a new neuron q such that $X^{(i-1)}w \approx \tilde{X}^{(i-1)}q$. In Algorithm 2, to ensure $X^{(i-1)}w \approx \tilde{X}^{(i-1)}q$, we proceed sequentially by computing, for $t = 1, 2, \dots, N_{i-1}$, the quantized or pruned coefficient q_t that aligns the sums $\sum_{j=1}^t w_j X_j^{(i-1)}$ and $\sum_{j=1}^t q_j \tilde{X}_j^{(i-1)}$. This step forms the core of our

¹Our algorithm and theory are also applicable to convolutional neural networks, but for simplicity we consider only multi-layer perceptrons (see, e.g., [30])

method. It balances two competing requirements, namely reducing the error while ensuring that q_t lies within the quantization alphabet or has a high likelihood of being set to zero, depending on the application.

Denote the accumulated error at step $t - 1$ by $u_{t-1} := \sum_{j=1}^{t-1} w_j X_j^{(i-1)} - \sum_{j=1}^{t-1} q_j \tilde{X}_j^{(i-1)}$. To balance these requirements, we choose q_t such that $\frac{1}{C} \cdot u_{t-1} + w_t X_t^{(i-1)} - q_t \tilde{X}_t^{(i-1)} \approx 0$, where $C \geq 1$ controls the trade-off between correcting the prior error and only approximating w_t . If we were picking q_t to minimize $\|\frac{1}{C} \cdot u_{t-1} + w_t X_t^{(i-1)} - q_t \tilde{X}_t^{(i-1)}\|$ over \mathbb{R} , in the absence of quantization or sparsity constraints, q_t is best chosen by projecting $\frac{1}{C} \cdot u_{t-1} + w_t X_t^{(i-1)}$ onto $\tilde{X}_t^{(i-1)}$, yielding $q_t = \frac{\langle h_t, \tilde{X}_t^{(i-1)} \rangle}{C \|\tilde{X}_t^{(i-1)}\|^2}$, where $h_t = C w_t X_t^{(i-1)} + u_{t-1}$. However, since our goal is quantization or pruning, we instead set $q_t = \mathcal{T}\left(\frac{\langle h_t, \tilde{X}_t^{(i-1)} \rangle}{C \|\tilde{X}_t^{(i-1)}\|^2}\right)$, where \mathcal{T} is a stochastic operator that promotes sparsity and/or quantizes its argument. Each neuron w can be handled this way in parallel to compress the entire layer.

In the case of quantization, if the ratio $\frac{\langle h_t, \tilde{X}_t^{(i-1)} \rangle}{C \|\tilde{X}_t^{(i-1)}\|^2}$ falls outside the range of the chosen alphabet, the resulting error can become unbounded. In [29], where $C = 1$, this issue is avoided by enlarging the alphabet to include more quantization levels. In contrast, to work with a fixed 1-bit alphabet, we allow a scaling constant $C \geq 1$ to ensure the quantization remains controlled. In Theorem 1, we show that choosing $C \sim \log(N_0 N_1)$, where N_0 and N_1 are the input and output channel dimensions respectively, guarantees that the quantized weights remain within the 1-bit alphabet, while still achieving a logarithmic reconstruction error bound on $\|XW - XQ\|_\infty$ with high probability..

In the remainder of this paper, we show that these modifications allow us to control the error with high probability on the stochasticity of the operator \mathcal{T} .

Algorithm 2 COMPRESSION($\mathcal{T}, C, w, X, \tilde{X}$)

Parameters: Stochastic operator \mathcal{T} , constant $C \geq 1$.

Input: Neuron $w = (w_1, w_2, \dots, w_N)^\top \in \mathbb{R}^N$, data (activations) $X = (X_1, X_2, \dots, X_N) \in \mathbb{R}^{m \times N}$, activations from quantized network $\tilde{X} = (\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_N) \in \mathbb{R}^{m \times N}$.

Output: Compressed weights $q = (q_1, \dots, q_N)^\top$.

- 1: Initialize accumulated error $u_0 = 0$.
 - 2: **for** $t = 1$ to N **do**
 - 3: Compute $h_t = C w_t X_t + u_{t-1}$
 - 4: Compute $v_t = \frac{\langle h_t, \tilde{X}_t \rangle}{C \|\tilde{X}_t\|^2}$
 - 5: Compute $q_t = \mathcal{T}(v_t)$
 - 6: Update accumulated error: $u_t = u_{t-1} + w_t X_t - q_t \tilde{X}_t$
 - 7: **end for**
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4. THEORETICAL ANALYSIS

The primary focus of our theoretical analysis of Algorithm 1 is the error analysis for a single layer of the network. Specifically, we aim to bound the error introduced when compressing the weight matrix of a single layer. Consider the first layer of a neural network, characterized

by the weight matrix $W = (W_1, W_2, \dots, W_{N_1}) \in \mathbb{R}^{N_0 \times N_1}$, where each W_i represents a neuron (column) of W ². Given the input data $X \in \mathbb{R}^{m \times N_0}$, Algorithm 1 compresses W into $Q = (Q_1, Q_2, \dots, Q_{N_1})$ and the error of this compression is measured in the Frobenius norm,

$$\|XW - XQ\|_F^2 = \sum_{i=1}^{N_1} \|X(W_i - Q_i)\|_2^2.$$

Since this function is separable with respect to Q_1, \dots, Q_{N_1} , Algorithm 1 processes each column W_1, W_2, \dots, W_{N_1} independently to produce Q_1, Q_2, \dots, Q_{N_1} . To simplify notation, we represent a single column by lowercase letters w and q , denoting the uncompressed and compressed versions, respectively. The error analysis for the entire layer thus reduces to bounding $\|X(w - q)\|_2^2$, which corresponds to the error from Algorithm 2. The remainder of this section is dedicated to proving Theorem 4.2, which provides explicit bounds on the accumulated error at each step t . Its proof requires the concept of convex ordering which we will first introduce. Then, we state and prove Theorem 4.2 using a series of technical lemmas, which we also subsequently prove.

4.1. Main Theorem. We now define *convex ordering* [22], a notion that will allow us to control random vectors generated by our algorithm by replacing them with Gaussian random vectors.

Definition 4.1. Let X, Y be n -dimensional random vectors such that

$$\mathbb{E}f(X) \leq \mathbb{E}f(Y)$$

holds for all convex functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$, provided the expectations exist. Then X is said to be less than Y in the *convex order*, denoted by $X \prec_{cx} Y$.

Recall that we initialize the accumulated error as $u_0 = 0$ and note that at each step t , we compute $h_t = Cw_t X_t + u_{t-1}$. Then we use h_t to compute $v_t = \frac{\langle h_t, X_t \rangle}{C\|X_t\|^2}$ and compute the compressed entry $q_t = \mathcal{T}(v_t)$. Finally, we update the accumulated error $u_t = u_{t-1} + w_t X_t - q_t X_t$ and move on to the next step.

Using convex ordering, we can prove that u_i is dominated by a Gaussian random vector. This result, stated in the theorem below, allows us to control the errors and probabilities for the quantization and pruning applications in Sections 5, 6, and 7. Here we adapt the proof technique from [1].

Theorem 4.2. Let $C \geq 1$. Assume the weight vector satisfies $\|w\|_\infty < K$. Further, assume that the operator \mathcal{T} satisfies $\mathbb{E}[\mathcal{T}(v)] = v$ and $|\mathcal{T}(v) - v| \leq M$ for all $v \in \mathbb{R}$. Then, by running Algorithm 2 with $w \in \mathbb{R}^{N_0}$ and $X \in \mathbb{R}^{m \times N_0}$, the resulting u_t satisfies

$$u_t \prec_{cx} \mathcal{N}(0, \beta_t I), \quad \text{for } 1 \leq t \leq N,$$

where β_t is given by

$$\beta_t = \frac{C\pi M^2}{2} \max_{1 \leq i \leq t} \|X_i\|^2.$$

Proof. First by Lemma 4.4, we deduce $u_t \prec_{cx} \mathcal{N}(0, \Sigma_t)$, where Σ_t is defined as in Lemma 4.4. Further by Lemma 4.5, we have $\Sigma_t \preceq \beta_t I$. We then apply Lemmas 4.3, c and a to complete the proof. \square

²One could also consider any arbitrary layer and work with X being either the corresponding activations of the compressed or original network

4.2. Technical Lemmas. In this subsection, we present the lemmas used for the proof of Theorem 4.2. We start with a lemma collecting various results on convex ordering, with the proofs in [1, 22].

Lemma 4.3. *The following all hold:*

- a. Let X, Y and Z be n -dimensional random vectors. If $X \prec_{cx} Y$ and $Y \prec_{cx} Z$, then $X \prec_{cx} Z$. (Lemma 2.3 in [1])
- b. Let X, Y be n -dimensional random vectors. If $X \prec_{cx} Y$, then for any matrix $M \in \mathbb{R}^{n \times n}$, we have $MX \prec_{cx} MY$. (Lemma A.3 in [29])
- c. If A and B are two $n \times n$ positive semi-definite matrices and $A \preceq B$, then $\mathcal{N}(0, A) \prec_{cx} \mathcal{N}(0, B)$. (Lemma A.2 in [29])
- d. Let X, Y, W , and Z be n -dimensional random vectors and suppose that X and Y live on the same probability space. Let W and Z be independent and suppose that $X \prec_{cx} W$ and $(Y - X)|X \prec_{cx} Z$. Then $Y \prec_{cx} W + Z$. (Lemma 2.5 in [1])
- e. Let X be a real-valued random variable with $\mathbb{E}X = 0$ and $|X| \leq C$. Then $X \prec_{cx} \mathcal{N}(0, \frac{\pi C^2}{2})$. (Lemma 2.6 in [1])
- f. Let X be an n -dimensional random vector such that $X \prec_{cx} \mathcal{N}(\mu, \sigma^2 I)$, and let $\alpha > 0$. Then

$$\mathbb{P}(\|X - \mu\|_\infty \leq \alpha) \geq 1 - \sqrt{2}ne^{-\frac{\alpha^2}{4\sigma^2}}.$$

In particular, if $\alpha = 2\sigma\sqrt{\log(\frac{\sqrt{2}n}{\gamma})}$ with $\gamma \in (0, 1]$, we have

$$\mathbb{P}(\|X - \mu\|_\infty \leq 2\sigma\sqrt{\log(\frac{\sqrt{2}n}{\gamma})}) \geq 1 - \gamma.$$

(Lemma B.2 in [29])

Using Lemma 4.3 allows to deduce the following result on the distribution of u_j . We use a proof technique similar to that in [1].

Lemma 4.4. *Under the same assumptions as in Theorem 4.2, by running Algorithm 2 with $w \in \mathbb{R}^{N_0}$ and $X \in \mathbb{R}^{m \times N_0}$, we obtain u_t satisfying*

$$u_t \prec_{cx} \mathcal{N}(0, \Sigma_t) \text{ for } 1 \leq t \leq N,$$

where $\Sigma_0 = 0$ and $\Sigma_t, t \geq 1$ is defined recursively as

$$\Sigma_t = \left(I - \frac{P_{X_t}}{C}\right)\Sigma_{t-1}\left(I - \frac{P_{X_t}}{C}\right) + \frac{\pi M^2}{2}X_t X_t^\top.$$

Proof. We use induction. For the base case, we have $u_0 = 0$, so the statement is trivially true. Now, assume $u_{t-1} \prec_{cx} \mathcal{N}(0, \Sigma_{t-1})$. By the update formula of q_t in Algorithm 2:

$$\begin{aligned} u_t &= u_{t-1} + w_t X_t - q_t X_t \\ &= u_{t-1} + w_t X_t - v_t X_t + (v_t - q_t) X_t \\ &= \left(I - \frac{P_{X_t}}{C}\right)u_{t-1} + (v_t - \mathcal{T}(v_t))X_t. \end{aligned}$$

The third equality is due to the definition of v_t and q_t . By the assumption on \mathcal{T} , and conditioning on u_{t-1} , $(v_t - \mathcal{T}(v_t) | u_{t-1})$ is mean zero and $|v_t - \mathcal{T}(v_t)| \leq M$. We then use

e from Lemma 4.3, to deduce $(v_t - \mathcal{T}(v_t) \mid u_{t-1}) \prec_{cx} \mathcal{N}(0, \frac{\pi M^2}{2})$. Further applying Lemma 4.3, b, we obtain

$$((v_t - \mathcal{T}(v_t))X_t \mid u_{t-1}) \prec_{cx} \mathcal{N}\left(0, \frac{\pi M^2}{2} X_t X_t^\top\right).$$

Since $u_{t-1} \prec_{cx} \mathcal{N}(0, \Sigma_{t-1})$ by the induction hypothesis, we similarly have

$$\left(I - \frac{P_{X_t}}{C}\right)u_{t-1} \prec_{cx} \mathcal{N}\left(0, \left(I - \frac{P_{X_t}}{C}\right)\Sigma_{t-1}\left(I - \frac{P_{X_t}}{C}\right)\right).$$

Finally, by Lemma 4.3, d with $\left(I - \frac{P_{X_t}}{C}\right)u_{t-1}$ in place of X , $(v_t - \mathcal{T}(v_t))X_t \mid u_{t-1}$ in place of $Y - X \mid X$ and $\mathcal{N}\left(0, \left(I - \frac{P_{X_t}}{C}\right)\Sigma_{t-1}\left(I - \frac{P_{X_t}}{C}\right)\right)$ in place of W , with $\mathcal{N}\left(0, \frac{\pi M^2}{2} X_t X_t^\top\right)$ as Z , where W and Z are chosen independently, we conclude

$$u_t \prec_{cx} \mathcal{N}\left(0, \left(I - \frac{P_{X_t}}{C}\right)\Sigma_{t-1}\left(I - \frac{P_{X_t}}{C}\right) + \frac{\pi M^2}{2} X_t X_t^\top\right).$$

This completes the induction. \square

Having controlled our object of interest by a Gaussian random vector with a known covariance matrix, Σ_t defined in Lemma 4.4, we next control the covariance matrix itself.

Lemma 4.5. *Let $C \geq 1$, then for the above defined Σ_t , we have*

$$\Sigma_t \preceq \beta_t I,$$

where $\beta_0 = 0$ and $\beta_t = \frac{C\pi M^2}{2} \max_{1 \leq i \leq t} \|X_i\|^2$, when $t \geq 1$.

Proof. The base case is obvious. Let us now proceed by induction. Suppose $\Sigma_{t-1} \preceq \beta_{t-1} I$. Then

$$\begin{aligned} \Sigma_t &= \left(I - \frac{P_{X_t}}{C}\right)\Sigma_{t-1}\left(I - \frac{P_{X_t}}{C}\right) + \frac{\pi M^2 X_t X_t^\top}{2} \\ &\preceq \beta_{t-1} \left(I - \frac{P_{X_t}}{C}\right)^2 + \frac{\pi M^2 X_t X_t^\top}{2} \\ &= \beta_{t-1} \left(I + \frac{1}{C^2} \frac{X_t X_t^\top}{\|X_t\|^2} - 2 \frac{X_t X_t^\top}{C \|X_t\|^2}\right) + \frac{\pi M^2 X_t X_t^\top}{2} \\ &\preceq \beta_t \left(I + \frac{1}{C^2} \frac{X_t X_t^\top}{\|X_t\|^2} - 2 \frac{X_t X_t^\top}{C \|X_t\|^2}\right) + \frac{\pi M^2 X_t X_t^\top}{2} \\ &= \beta_t I + \frac{1}{\|X_t\|^2} \left(\beta_t \left(\frac{1}{C^2} - \frac{2}{C}\right) + \frac{\pi M^2}{2} \|X_t\|^2\right) X_t X_t^\top \\ &= \beta_t I + \frac{\pi M^2}{\|X_t\|^2} \left(\frac{C}{2} \max_{1 \leq i \leq t} \|X_i\|^2 \left(\frac{1 - 2C}{C^2}\right) + \frac{1}{2} \|X_t\|^2\right) X_t X_t^\top. \end{aligned}$$

Since $C \geq 1$, we know $1 - 2C \leq 0$, then we can further deduce

$$\begin{aligned}\Sigma_t &\preceq \beta_t I + \frac{\pi M^2}{\|X_t\|^2} \left(\frac{C}{2} \|X_t\|^2 \left(\frac{1-2C}{C^2} \right) + \frac{1}{2} \|X_t\|^2 \right) X_t X_t^\top \\ &= \beta_t I + \left(\frac{\pi M^2}{2} \left(\frac{1}{C} - 2 \right) + \frac{\pi M^2}{2} \right) X_t X_t^\top \\ &\preceq \beta_t I + \left(-\frac{\pi M^2}{2} + \frac{\pi M^2}{2} \right) X_t X_t^\top \\ &= \beta_t I.\end{aligned}$$

This completes the induction. \square

5. ONE-BIT QUANTIZATION

We are now ready to first describe the selection of \mathcal{T} for one-bit quantization and then provide the corresponding error analysis for running Algorithm 1 on a single layer of a neural network. Notably, our analysis extends with minimal additional effort to finite multi-bit quantization, a consideration we leave to the interested reader.

5.1. Choice of \mathcal{T} for one-bit quantization. Suppose we want to quantize an L -layer perceptron which has all its weights strictly bounded in absolute value by $K > 0$, and our goal is to achieve one-bit quantization. We begin by considering an infinite alphabet $\mathcal{A} = \{\dots, -10K, -6K, -2K, 2K, 6K, 10K, \dots\}$. Then, we choose the operator \mathcal{T} to be the stochastic scalar quantizer $\mathcal{Q} : \mathbb{R} \rightarrow \mathcal{A}$, with

$$\mathcal{Q}(z) = \begin{cases} (\lfloor \frac{z}{4K} \rfloor) \cdot 4K & \text{with probability } p_z \\ (\lceil \frac{z}{4K} \rceil) \cdot 4K & \text{with probability } 1 - p_z, \end{cases}$$

where $p_z = \lceil \frac{z}{2K} \rceil - \frac{z}{2K}$.

It is straightforward to verify that the operator \mathcal{Q} satisfies the assumptions in Theorem 4.2 with $M = 4K$. We will show that with this choice of operator, with high probability, Algorithm 1 gives logarithmic reconstruction error with respect to the input dimension, and, crucially, only ever uses two values from the alphabet: $-2K$ and $2K$. That is, although the quantizer is defined over an infinite alphabet, the output of the algorithm effectively requires only 1-bit quantization with high probability, while maintaining small errors.

5.2. Error analysis. Consider one layer of a neural-network, and denote its action by $\Phi(x) = \rho(W^\top x)$, where $x \in \mathbb{R}^{N_0}$ represents the input data or activations, $W \in \mathbb{R}^{N_0 \times N_1}$ is the weight matrix and ρ is the ReLU activation function, or any (say) 1-Lipschitz function. Now, we use Algorithm 1 to quantize Φ using $X \in \mathbb{R}^{m \times N_0}$, where each row represents a data point. The following holds.

Proposition 1. *Let $C \geq 1$ and fix any $p \geq 1$. Suppose that Q is the quantized weight matrix resulting from Algorithm 1, applied to Φ with the stochastic quantizer \mathcal{Q} . Then, the resulting error satisfies*

$$\max_{i,j} |\rho(XW)_{ij} - \rho(XQ)_{ij}| \leq 4K \sqrt{2\pi Cp \log N_0} \max_{1 \leq i \leq N_0} \|X_i\|$$

and

$$\max_{i,j} |Q_{i,j}| \leq 2K$$

with probability greater than

$$1 - N_1 \sum_{t=1}^{N_0} \sqrt{2} \exp\left(-\frac{C\|X_t\|^2}{32\pi \max_{1 \leq i \leq t-1} \|X_i\|^2}\right) - \sqrt{2}mN_1N_0^{-p},$$

where X_t represents the t -th column of X .

Proof. Let $\kappa := 4K\sqrt{2\pi C p \log N_0} \max_{1 \leq i \leq N} \|X_i\|$. Consider any neuron (column) $w \in \mathbb{R}^{N_0}$ from W . Let the quantized neuron q and accumulated error u_t be as defined in Subsection 4.1. By Theorem 4.2, we have $u_t \prec_{cx} \mathcal{N}(0, \beta_t I)$, where β_t is defined in Lemma 4.5. Recall that at each step in Algorithm 1, the quantized weight is $q_t = \mathcal{Q}\left(w_t + \frac{\langle u_{t-1}, X_t \rangle}{C\|X_t\|^2}\right)$. Let B_t be the event $\{|q_t| > 2K\}$. We know $B_t \subseteq A_t = \left\{ \left| \frac{\langle u_{t-1}, X_t \rangle}{C\|X_t\|^2} \right| > K \right\}$ since $|w_t| < K$. To obtain the result in the statement, it suffices to bound the following failure probability

$$\mathbb{P}(\{\|u_{N_0}\|_\infty > \kappa\} \cup A_1 \cup \dots \cup A_{N_0}).$$

We now bound the probability of each set individually. First, $\mathbb{P}(A_1)$ is 0 since $u_0 = 0$. We then control $\mathbb{P}(A_t)$ for $t = 2, \dots, N_0$. By Theorem 4.2 and Lemma 4.3, b, we have

$$\frac{\langle u_{t-1}, X_t \rangle}{C\|X_t\|^2} \prec_{cx} \mathcal{N}\left(0, \frac{\beta_{t-1}}{C^2\|X_t\|^2}\right),$$

where $\beta_{t-1} = 8C\pi K^2 \max_{1 \leq i \leq t-1} \|X_i\|^2$. Applying Lemma 4.3, f, we obtain

$$\mathbb{P}\left(\left|\frac{\langle u_{t-1}, X_t \rangle}{C\|X_t\|^2}\right| > K\right) \leq \sqrt{2} \exp\left(-\frac{C\|X_t\|^2}{32\pi \max_{1 \leq i \leq t-1} \|X_i\|^2}\right). \quad (1)$$

For $\mathbb{P}(\{\|u_{N_0}\|_\infty \geq \kappa\})$, we use the second part of Lemma 4.3, f with $\gamma = \sqrt{2}mN_0^{-p}$ to get

$$\mathbb{P}(\|u_{N_0}\|_\infty > \kappa) \leq \sqrt{2}mN_0^{-p}. \quad (2)$$

Then we use apply a union bound on the failure probability to obtain an upper bound

$$\mathbb{P}(\{\|u_{N_0}\|_\infty > \kappa\} \cup A_1 \cup \dots \cup A_{N_0}) \leq \sqrt{2}mN_0^{-p} + \sum_{t=2}^{N_0} \sqrt{2} \exp\left(-\frac{C\|X_t\|^2}{32\pi \max_{1 \leq i \leq t-1} \|X_i\|^2}\right).$$

This implies that

$$\begin{aligned} & \mathbb{P}(\|\rho(Xw) - \rho(Xq)\|_\infty \leq \kappa \text{ and } \|q\|_\infty \leq 2K) \\ & \geq 1 - \sum_{t=2}^{N_0} \sqrt{2} \exp\left(-\frac{C\|X_t\|^2}{32\pi \max_{1 \leq i \leq t-1} \|X_i\|^2}\right) - \sqrt{2}mN_0^{-p}, \end{aligned} \quad (3)$$

by the definition of u_{N_0} and the Lipschitz property of the activation function ρ . Inequality (3) holds true for each column of $W \in \mathbb{R}^{m \times N_1}$. Taking a union bound over all the columns completes the proof. \square

Remark 1. This proposition ensures that each entry of $|Q|$ is bounded by $2K$ with high probability. Moreover, since Q takes values in the discrete alphabet \mathcal{A} and the only elements of \mathcal{A} within $[-2K, 2K]$ are $-2K$ and $2K$, this implies that each entry of Q must be either $-2K$ or $2K$. Hence, although the quantizer is defined over a larger alphabet, the quantized weights are effectively supported on just two values, realizing 1-bit quantization in practice.

Remark 2. For multi-layer analysis, we can use the ‘data alignment’ technique introduced in [29], which would allow establishing an error bound for the entire neural network. We leave this to the reader.

This result demonstrates that, for one neuron w , by choosing C proportional to $\log(N_0)$, the Euclidean norm error $\|Xw - Xq\|$ associated with quantization scales as $K\sqrt{m} \cdot \text{polylog } N_0 \cdot \max_i \|X_i\|$ with high probability, where m is the number of data points. In the case when $m > N$, one can modify the argument in the proof of Proposition 1 via an SVD of the data matrix $X = U\Sigma V^\top$. To be more specific, one can prove that running Algorithm 2 with X is equivalent to running it with ΣV^\top . Moreover, $\|X(w - q)\| = \|\Sigma V^\top(w - q)\|$, thereby yielding an error of order $K\sqrt{r} \cdot \text{polylog } N_0 \cdot \max_i \|X_i\|$, where $r \leq N_0$ is the rank of X . In summary, the error scales like $K\sqrt{\min\{m, N_0\}} \cdot \text{polylog } N_0 \cdot \max_i \|X_i\|$. This stands in sharp contrast to the Euclidean norm error from the naive round-to-nearest (RTN) algorithm, where each entry of w is rounded to the nearest element of \mathcal{A} . In that case, the only error bound, in general, is via triangle inequalities and leads to a Euclidean norm error $\|X(w - q)\| \leq \sum_{i=1}^{N_0} \|X_i\| |w_i - q_i| \leq 2K \cdot N_0 \cdot \max_i \|X_i\|$.

6. PRUNING

In this section, we specify the choice of \mathcal{T} used for pruning, and we present the error analysis of applying Algorithm 1 to a single layer of a neural network.

6.1. Choice of \mathcal{T} for Pruning. Suppose our layer weights are bounded by $K > 0$ in absolute value. Given $c > 0$, our goal is to encourage weights below cK to be set to 0 while keeping weights above cK unchanged. To achieve this, we set \mathcal{T} to the stochastic operator \mathcal{S} , with

$$\mathcal{S}(z) = \begin{cases} z & \text{if } |z| > cK, \\ \xi_z & \text{if } |z| \leq cK. \end{cases} \quad (4)$$

Here ξ_z is a random variable that is set to 0 with probability $1 - \frac{2}{(c+1)K}|z|$ and is sampled from $\text{sgn}(z) \cdot U[cK, K]$ with probability $\frac{2}{(c+1)K}|z|$. It is straightforward to verify that the operator \mathcal{S} satisfies the assumptions in Theorem 4.2 with M being K .

6.2. Error analysis. Consider again $\Phi(x) = \rho(W^\top x)$, where $x \in \mathbb{R}^{N_0}$ is the input data or activation, $W \in \mathbb{R}^{N_0 \times N_1}$ is the weight matrix and ρ is the ReLU (or any 1-Lipschitz) activation function. We apply Algorithm 1 to prune Φ using $X \in \mathbb{R}^{m \times N_0}$, where each row represents a data point. We can then deduce the following result.

Proposition 2. *Let $C \geq 1$ and fix any $p \geq 1$. Suppose that Q is the pruned weight matrix resulting from Algorithm 1, applied to Φ with the stochastic pruning operator \mathcal{S} . Then, the resulting error satisfies*

$$\max_{i,j} |\rho(XW)_{ij} - \rho(XQ)_{ij}| \leq K\sqrt{2C\pi p \log N_0} \max_{1 \leq i \leq N_0} \|X_i\|$$

with probability greater than $1 - \sqrt{2mN_1N_0^{-p}}$, where X_t represents the t -th column of X .

Proof. Consider any neuron (column) $w \in \mathbb{R}^{N_0}$ from W . Let q and u_i be defined as in Subsection 4.1. We run Algorithm 2 independently for each w .

It is straightforward to verify that the assumptions in Theorem 4.2 are satisfied with $M = K$. Thus, from Theorem 4.2, we deduce the accumulated error satisfies

$$u_t \prec_{cx} \mathcal{N}(0, \beta_t I) \text{ for } 1 \leq t \leq N_0,$$

where $\beta_t = \frac{C\pi K^2}{2} \max_{1 \leq i \leq t} \|X_i\|^2$.

Applying Lemma 4.3, f with $\gamma = \sqrt{2mN_0^{-p}}$, we obtain

$$\mathbb{P}(\|u_{N_0}\|_\infty > K\sqrt{2C\pi p \log N_0} \max_{1 \leq i \leq N} \|X_i\|) \leq \sqrt{2mN_0^{-p}}.$$

This implies

$$\mathbb{P}(\|\rho(Xw) - \rho(Xq)\|_\infty > K\sqrt{2C\pi p \log N_0} \max_{1 \leq i \leq N} \|X_i\|) \leq \sqrt{2mN_0^{-p}}. \quad (5)$$

The inequality (5) holds for any column w of $W \in \mathbb{R}^{N_0 \times N_1}$. Applying a union bound over all the columns completes the proof. \square

7. QUANTIZATION WITH PRUNING

In this section, we show that our methods can handle combinations of operators, and provide the choices for \mathcal{T} used in joint quantization with pruning, followed by an error analysis for applying Algorithm 1 to a single layer of a neural network.

7.1. Choice of \mathcal{T} for Quantization with Pruning. Consider an L -layer perceptron with all weights strictly bounded by $K > 0$ in absolute value. We again begin with an infinite alphabet $\mathcal{A} = \{\dots, -6K, -4K, -2K, 0, 2K, 4K, 6K, \dots\}$. Let \mathcal{S} be the pruning operator defined in (4), with $c > 0$, and let $\mathcal{Q} : \mathbb{R} \rightarrow \mathcal{A}$ be the stochastic scalar quantizer with

$$\mathcal{Q}(z) = \begin{cases} (\lfloor \frac{z}{2K} \rfloor) \cdot 2K & \text{with probability } p_z \\ (\lceil \frac{z}{2K} \rceil) \cdot 2K & \text{with probability } 1 - p_z, \end{cases}$$

where $p_z = \lceil \frac{z}{2K} \rceil - \frac{z}{2K}$. Our goal is to encourage weights below cK to be set to 0 and then quantize the weights using \mathcal{A} . To that end, we define $\mathcal{T} = \mathcal{Q} \circ \mathcal{S}$.

7.2. Error analysis. With the same notation as before, we can deduce the following result.

Proposition 3. *Let $C \geq 1$, and fix any $p \geq 1$. Suppose that Q is the pruned and quantized weight matrix resulting from Algorithm 1, applied to Φ with the stochastic operator $\mathcal{T} = \mathcal{Q} \circ \mathcal{S}$. Then, the resulting error satisfies*

$$\max_{i,j} |\rho(XW)_{ij} - \rho(XQ)_{ij}| \leq 2K\sqrt{2\pi Cp \log N_0} \max_{1 \leq i \leq N_0} \|X_i\|$$

and

$$\max_{i,j} |Q|_{ij} \leq 2K$$

with probability greater than

$$1 - N_1 \sum_{t=1}^{N_0} \sqrt{2} \exp\left(-\frac{C \|X_t\|^2}{8\pi \max_{1 \leq i \leq t-1} \|X_i\|^2}\right) - \sqrt{2}mN_1N_0^{-p},$$

where X_t represents the t -th column of X .

Proof. First, we verify that the operator \mathcal{T} satisfies the assumptions of Theorem 4.2 with $M = 2K$.

We begin by showing that for any $z \in \mathbb{R}$, the error $|\mathcal{T}(z) - z|$ is bounded by $2K$. Indeed, if $|z| > cK$, then $\mathcal{S}(z) = z$, so $\mathcal{T}(z) = \mathcal{Q}(z)$, which rounds z up or down to an adjacent alphabet element; since the grid size is $2K$, the rounding error is at most $2K$. If $|z| \leq cK$, then $\mathcal{S}(z)$ maps z to a value in $[-K, K]$ with the same sign. Without loss of generality, suppose $z \in [0, K]$, so $\mathcal{S}(z) \in [0, K]$ and $\mathcal{Q}(\mathcal{S}(z)) \in [0, 2K]$. Hence, both z and $\mathcal{T}(z)$ lie in $[0, 2K]$, and the total deviation remains at most $2K$.

For the expectation, we consider two independent probability spaces $(\Omega_1, \mathcal{E}_1, \mathbb{P}_1)$ and $(\Omega_2, \mathcal{E}_2, \mathbb{P}_2)$, which govern the randomness of the operators \mathcal{Q}_{ω_1} and \mathcal{S}_{ω_2} respectively, with $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$. Jointly, the probability space for \mathcal{T} is the direct product of these two spaces. Then, for any $z \in \mathbb{R}$,

$$\begin{aligned} \mathbb{E}[\mathcal{Q}_{\omega_1} \circ \mathcal{S}_{\omega_2}(z)] &= \int_{\Omega_2} \int_{\Omega_1} \mathcal{Q}_{\omega_1}(\mathcal{S}_{\omega_2}(z)) d\mathbb{P}_1(\omega_1) d\mathbb{P}_2(\omega_2) \\ &= \int_{\Omega_2} \mathbb{E}_{\omega_1}[\mathcal{Q}_{\omega_1}(\mathcal{S}_{\omega_2}(z))] d\mathbb{P}_2(\omega_2) \\ &= \int_{\Omega_2} \mathcal{S}_{\omega_2}(z) d\mathbb{P}_2(\omega_2) \\ &= \mathbb{E}_{\omega_2}[\mathcal{S}_{\omega_2}(z)] = z. \end{aligned}$$

The third equality follows from the unbiasedness of \mathcal{Q} , and the final equality from the unbiasedness of \mathcal{S} . The rest of the proof proceeds identically to that of Proposition 1, so we omit the details. \square

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