Topic: "Multivariate Distributions"

⇒ Bivariate Distributions.

Def: Let X and Y be two discrete random variables defined on the same sample space. Let the sets of possible values of X and Y be A and B, respectively. The function

$$p(x,y) = P(X=x,\ Y=y).$$

is called the joint probability mass function of X and Y.

NOTE: i) $\sum_x \sum_y p(x,y) = 1.$

ii) $p_X(x) = \sum_y p(x,y).$

iii) $p_Y(y) = \sum_x p(x,y).$

—

iv) $Cov(X,Y) = E(XY) - E(X)E(Y)$

v) $Var(X) = E(X^2) - [E(X)]^2$

vi) $Corr(X,Y) = \dfrac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$

Q-1: A small college has 90 male and 30 female professors. An Adhoc committee of five is selected at random to write the vision and mission of the college. Let X and Y be the number of men and women on this committee, respectively.

a) Find the joint probability mass function of X and Y.

b) Find P_X and P_Y, the marginal probability mass functions of X and Y.

c) find E(X) and E(Y).

d) Find E(XY)

e) Test Independence (if $P(X,Y) = P(X)P(Y)$)
$E(XY) = E(X)E(Y)$

f)

12-

a) $$P(x, y) = \frac{{}^{90}C_x \; {}^{30}C_y}{{}^{120}C_5} \qquad x = 0, 1, 2, 3, 4, 5$$

$y = 0, 1, 2, 3, 4, 5$

$x + y = 5$

$= 0$ otherwise

x \ y	0	1	2	3	4	5
0	–	–	–			${}^{90}C_0 \, {}^{30}C_5 \div {}^{120}C_5$
1					✓	
2				✓		
3			✓			
4		✓				
5	✓					

b) $$P(x) = \frac{{}^{90}C_x \; {}^{30}C_{5-x}}{{}^{120}C_5} \qquad x = 0, 1, 2, 3, 4, 5$$

$$P(y) = \frac{{}^{90}C_{5-y} \; {}^{30}C_y}{{}^{120}C_5} \qquad x = 0, 1, 2, 3, 4, 5.$$

c) $$E(x) = \sum_{x=0}^{5} x \, P(x)$$

$$E(y) = \sum_{y=0}^{5} y \, P(y).$$

Q-2. Let the joint probability mass function of random variables X and Y be given by

$$P(x,y) = \begin{cases} \frac{1}{70} x(x+y) & \text{if } x = 1,2,3, \quad y = 3,4. \\ 0 & \text{otherwise} \end{cases}$$

find $E(X)$, $E(Y)$, $E(X+Y)$

Sol:-

Method I:

Y \ X	1	2	3	P(Y)	YP(Y)
3	$\frac{1(1+3)}{70} = \frac{4}{70}$	$\frac{10}{70}$	$\frac{18}{70}$	$\frac{32}{70}$	$\frac{96}{70}$
4	$\frac{5}{70}$	$\frac{12}{70}$	$\frac{21}{70}$	$\frac{38}{70}$	$\frac{152}{70}$
P(x)	$\frac{9}{70}$	$\frac{22}{70}$	$\frac{39}{70}$	1	$\frac{248}{70}$
xP(x)	$\frac{9}{70}$	$\frac{44}{70}$	$\frac{117}{70}$	$\frac{170}{70}$	

$$\therefore E(X) = \Sigma\, x P(X) = \frac{170}{70} = 2.43$$

$$E(Y) = \Sigma\, Y P(Y) = \frac{248}{70} = 3.54$$

$$E(X+Y) = E(X) + E(Y)$$

$$= \frac{170}{70} + \frac{248}{70}$$

$$= \frac{418}{70} = 5.97$$

ction thod II :-

$$P(X) = P(x,3) + P(x,4).$$

$$= \frac{x(x+3)}{70} + \frac{x(x+4)}{70}$$

$$= \frac{x^2 + 3x + x^2 + 4x}{70}$$

$$= \frac{x^2}{35} + \frac{x}{10} \qquad x = 1, 2, 3.$$

$$\therefore E(X) = \sum_{x=1}^{3} x\,P(x)$$

$$= \sum_{x=1}^{3} x\left(\frac{x^2}{35} + \frac{x}{10}\right)$$

$$= \left(\frac{1}{35} + \frac{1}{10}\right) + 2\left(\frac{4}{35} + \frac{2}{10}\right) + 3\left(\frac{9}{35} + \frac{3}{10}\right)$$

$$= \frac{1+8+27}{35} + \frac{1+4+9}{10} = \frac{36}{35} + \frac{14}{10}$$

$$= 17/7 = 2.43.$$

$$p(y) = p(1, y) + p(2, y) + p(3, y)$$

$$= \frac{1+y}{70} + \frac{2(2+y)}{70} + \frac{3(3+y)}{70}$$

$$= \frac{1+4+9}{70} + \frac{y+2y+3y}{70}$$

$$= \frac{1}{5} + \frac{3y}{35} \quad , \quad y = 3, 4.$$

$$\therefore E(Y) = \sum_{y=3}^{4} y\, p(y)$$

$$= \sum_{y=3}^{4} y \left(\frac{1}{5} + \frac{3}{35} y \right)$$

$$= 3\left(\frac{1}{5} + \frac{3 \times 3}{35} \right) + 4\left(\frac{1}{5} + \frac{3 \times 4}{35} \right)$$

$$= \frac{3 * 4}{5} + \frac{27 + 48}{35}$$

$$= \frac{49}{35} + \frac{75}{35}$$

$$= \frac{124}{35} = 3.54.$$

For Continuous.

Q:- Let the joint density of Two r.Vs, X and Y be given by.

$$f(x,y) = \begin{cases} \frac{1}{4}(2x+y) & 0 \le x \le 1, \quad 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

a) find marginal density function of x and y respectively.

b) find $E(X)$, $E(X^2)$, $Var(X)$.

c) find $E(Y)$, $E(Y^2)$, $Var(Y)$.

d) find Conditional density of X given that Y. $f(x/y)$.

e) find $E(X/Y)$.

f) find unconditional $E(X)$.
(by multiplying density of y by $E(X/Y)$ and integrating over y)

g) find Conditional variance, i.e; by using $E(X^2/Y)$, find $Var(X/Y)$.

h) find expected value of Conditional variance. $E(Var(X/Y))$.

CamScanner

Sol:- a) Marginal density function of X;

E(+)

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_{0}^{2} \frac{1}{4}(2x+y)\,dy$$

$$= \frac{1}{4}\left[2xy + \frac{y^2}{2}\right]_0^2$$

$$= \frac{1}{4}\left[4x+2\right)$$

$$= \frac{1}{2}(2x+1) \qquad 0 \le x \le 1$$

Marginal density function of Y;

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_{0}^{1} \frac{1}{4}(2x+y)\,dx$$

$$= \frac{1}{4}\left[2\frac{x^2}{2} + xy\right|_0^1$$

$$= \frac{1}{4}\left[1+y\right] \qquad 0 \le y \le 2.$$

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_0^1 x \cdot \frac{1}{2}(2x+1)\, dx.$$

$$= \frac{1}{2}\int_0^1 (2x^2 + x)\, dx.$$

$$= \frac{1}{2}\left[\frac{2x^3}{3} + \frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{2}{3} + \frac{1}{2}\right]$$

$$= \frac{1}{2} \times \frac{7}{6} = 7/12.$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_0^1 x^2 \cdot \frac{1}{2}(2x+1)\, dx.$$

$$= \frac{1}{2}\int_0^1 (2x^3 + x^2)\, dx$$

$$= \frac{1}{2}\left[\frac{2x^4}{4} + \frac{x^3}{3}\right|_0^1$$

$$= \frac{1}{2}\left[\frac{2}{4} + \frac{1}{3}\right]$$

$$= \frac{1}{2} \times \frac{10}{12} = 5/12$$

$$Var(X) = E(X^2) - [E(X)]^2$$ e)

$$= \frac{5}{12} - \left(\frac{7}{12}\right)^2$$

$$= \frac{5}{12} - \frac{49}{144}$$

$$= \frac{11}{144}$$

c) $E(Y) =$

$E(Y^2) =$

$Var(Y) =$.

d) Conditional density of X given that Y.

$$f(X/y) = \frac{f(x,y)}{f(y)}$$

$$= \frac{\frac{1}{4}(2x+y)}{\frac{1}{4}(1+y)}$$

$$= \frac{2x+y}{1+y}$$

e) $$E(X|Y) = \int_{-\infty}^{\infty} x\, f(x/y)\, dx$$

$$= \int_0^1 x \cdot \frac{(2x+y)}{1+y}\, dx.$$

$$= \frac{1}{1+y} \int_0^1 x(2x+y)\, dx.$$

$$= \frac{1}{1+y} \int_0^1 (2x^2 + xy)\, dx.$$

$$= \frac{1}{1+y} \left\{ \frac{2x^3}{3} + \frac{x^2}{2} y \right|_0^1$$

$$= \frac{1}{1+y} \left[\frac{2}{3} + \frac{1}{2} y \right]$$

$$= \frac{1}{1+y} \left[\frac{4+3y}{6} \right]$$

$$= \frac{4+3y}{6(1+y)}$$

CamScanner

f) Unconditional expected value of X?

i.e; E(X)

$$E(X) = E_y[E(X/Y)].$$

$$= \int_{-\infty}^{\infty} E(X/y)\, f(Y)\, dy.$$

$$= \int_0^2 \frac{(4+3y)}{6\cancel{(1+y)}} \cdot \frac{1}{4}\cancel{(y+1)}\, dy$$

$$= \frac{1}{24}\int_0^2 (4+3y)\, dy.$$

$$= \frac{1}{24}\left| 4y + \frac{3y^2}{2} \right|_0^2$$

$$= \frac{1}{24}\left| 8+6-0 \right|$$

$$= \frac{14}{24} = \frac{7}{12}.$$

CamScanner

g)

$$Var(X/Y) = ?$$

$$Var(X/Y) = E[X^2/Y] - \{E[X/Y]\}^2$$

$$\therefore E[X^2/Y] = \int_0^1 x^2 \cdot \frac{(2x+y)}{1+y}\, dx.$$

$$= \frac{1}{1+y} \int_0^1 (2x^3 + x^2 y)\, dx.$$

$$= \frac{1}{1+y} \left\{ \frac{\cancel{2} x^4}{\cancel{4}_2} + \frac{x^3}{3} y \right|_0^1$$

$$= \frac{1}{1+y} \left(\frac{1}{2} + \frac{y}{3} \right)$$

$$= \frac{1}{1+y} \left(\frac{3+2y}{6} \right)$$

$$= \frac{(3+2y)}{6(1+y)}.$$

$$\therefore Var(X/Y) = E(X^2/Y) - [E(X/Y)]^2.$$

$$= \frac{(3+2y)}{6(1+y)} - \left[\frac{4+3y}{6(1+y)}\right]^2$$

$$= \frac{6(3+2y)(1+y) - (4+3y)^2}{36(1+y)^2}$$

$$= \frac{18+30y+12y^2-16-9y^2-24y}{36(1+y)^2}.$$

$$= \frac{3y^2+6y+2}{36(1+y)^2}$$

If $Y=1$.

$$\Rightarrow Var(X/Y=1) = Var(X)$$

$$= \frac{3+6+2}{36(1+1)^2}$$

$$= \frac{11}{144}.$$

h) $$E(Var(X/Y)) = \int_0^2 Var(X/Y)\, f(Y)\, dy$$

$$= \int_0^2 \frac{(3y^2+6y+2)}{36(1+y)^2} \times \frac{1}{4}(1+y)\, dy.$$

- A program consists of two modules. The no. of error, X, in the first module and the no. of errors, Y, in the second module have the joint distribution, $P(0,0) = P(0,1) = P(1,0) = 0.2$, $P(1,1) = P(1,2) = P(1,3) = 0.1$, $P(0,2) = P(0,3) = 0.05$.

Find

a) marginal distributions of X and Y.

b) the probability of no errors in the first module,

c) the distribution of the total number of errors in the program.

d) if errors in the two modules occur independently.

sol:-

a)

X \ Y	0	1	2	3	P(X)
0	0.2	0.2	0.05	0.05	0.50
1	0.2	0.10	0.10	0.10	0.50
P(Y)	0.40	0.30	0.15	0.15	1

b) $P_x(0) = 0.50$

c) Let Z = Total no. of errors in the program.

$\Rightarrow \quad Z = X + Y$

$\therefore Z = 0, 1, 2, 3, 4$

CamScanner

$P_z(0) = P(0,0) = 0.20$

$P_z(1) = P(1,0) + P(1,0)$

$= 0.2 + 0.2 = 0.4$

$P_z(2) = P(0,2) + P(1,1)$

$= 0.05 + 0.10 = 0.15$

$P_z(3) = P(0,3) + P(1,2)$

$= 0.05 + 0.10 = 0.15$

$P_z(4) = P(1,3) = 0.10$.

Verification: $0.2 + 0.4 + 0.15 + 0.15 + 0.1 = 1$

d) To check independency;

$P(X,Y) = P(X)\,P(Y)$

$P(0,0) = P(0)\,P(0)$

$\Rightarrow 0.2 = 0.5 \times 0.4$ ✓

But $P(0,1) = 0.2$

although $P(0) \times P(1) = 0.5 \times 0.3$

$= 0.15$.

$\Rightarrow P(0,1) \neq P(0)\,P(1)$.

$\therefore$ the numbers of errors in Two modules are dependent.

$E(X) = 0.5$, $Var(X) = 0.25$, $E(Y) = 1.05$, $E(Y^2) = 2.25$, $Var(Y) = 1.1475$
$E(XY) = 0.6$, $cov(X,Y) = 0.075$, $\rho = 0.14$

Q- Two ballpoints are selected at random from a box that contains 3 blue pens, 2 Red pens and 3 green pens. If X is the no. of blue pens selected and Y is the no. of red pens selected, find joint p.f

$$f(x,y) = \frac{{}^{3}C_{x} \; {}^{2}C_{y} \; {}^{3}C_{2-x-y}}{{}^{8}C_{2}}$$

$x = 0, 1, 2$

$y = 0, 1, 2.$