## Exercise # 4.9

**Q1) a)**

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -3 & 3 \\ 4 & 8 & -4 & 4 \end{bmatrix}$$

**Solution:**

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -3 & 3 \\ 4 & 8 & -4 & 4 \end{bmatrix}$$

$R_2 - 2R_1,\ R_3 - 3R_1$
$R_4 - 4R_1$

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{Rank}(A) = 1$

Nullity(N) = No. of Row − Rank(A)

Nullity $= 4 - 1 = 3$

**b)**

$$A = \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ -3 & 6 & -1 & 1 & -7 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

**Solution:**

$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ -3 & 6 & -1 & 1 & -7 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

$R_2 + 3R_1$ ; $R_3 + 3R_1$

$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 5 & 10 & -10 \\ 0 & 0 & 1 & 2 & -2 \end{bmatrix}$$

$R_3$ Swap $R_2$

$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 5 & 10 & -10 \end{bmatrix}$$

$R_3 - 5R_2$

$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boxed{\text{Rank} = 2}$ ; Nullity $= 5 - 2 = 3$

**Q2) a):**

$$A = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 1 & 3 & 0 & -4 \end{bmatrix}$$

**Solution:**

$$\begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 1 & 3 & 0 & -4 \end{bmatrix}$$

$R_3 + 2R_1$

$$\begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ 0 & -1 & -3 & 1 & 3 \\ 0 & 1 & 3 & 0 & -4 \end{bmatrix}$$

$-R_2$

$$\begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 3 & -1 & -3 \\ 0 & -1 & -3 & 1 & 3 \\ 0 & 1 & 3 & 0 & -4 \end{bmatrix}$$

$R_3 + R_2,\ R_4 - R_2$

$$\begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 3 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

$R_4$ Swap $R_3$.

$$\begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 3 & -1 & -3 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus,

$\text{Rank}(A) = 3$

Nullity = Total − Rank(A)
$= 5 - 3$

$\boxed{\text{Nullity}(A) = 2}$

**b)**

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & +1 \\ 2 & 0 & -4 & -2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix}$$

$R_3 + 3R_1,\ R_4 - 3R_1,\ R_5 - 2$

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 9 & 9 & 8 \\ 0 & -5 & -5 & -8 \\ 0 & -6 & -6 & -8 \end{bmatrix}$$

$R_3 - 9R_2,\ R_4 + 5R_2,$
$R_5 + 6R_2.$

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & -8 \\ 0 & 0 & 0 & -8 \end{bmatrix}$$

$R_1 - 3R_2$

$$A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & -8 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

$R_3/8,\ R_4 + R_3,$
$R_5 - R_3.$

$$A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(A) Rank $= 3$

Nullity (A) = Total − Rank
$= 4 - 3$

Nullity(A) $= 1$

**Example # 1**

$A = [-1$

on next page.

*Note: we have to find Nullity as shown in example.

## * Example # 1 VIIP IMP

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

$-R_1$

$$A = \begin{bmatrix} 1 & -2 & 0 & -4 & -5 & 3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

$R_2 - 3R_1$, $R_3 - 2R_1$, $R_4 - 4R_1$

$$A = \begin{bmatrix} 1 & -2 & 0 & -4 & -5 & 3 \\ 0 & -1 & 2 & 12 & 16 & -5 \\ 0 & -1 & 2 & 12 & 16 & -5 \\ 0 & -18 & 2 & 12 & 16 & -5 \end{bmatrix}$$

$-R_2$

$$A = \begin{bmatrix} 1 & -2 & 0 & -4 & -5 & 3 \\ 0 & 1 & -2 & -12 & -16 & +5 \\ 0 & -1 & 2 & 12 & 16 & -5 \\ 0 & -1 & 2 & 12 & 16 & -5 \end{bmatrix}$$

$R_3 + R_2$, $R_4 + R_2$

$$A = \begin{bmatrix} 1 & -2 & 0 & -4 & -5 & 3 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xRightarrow{R_1 + 2R_2} \boxed{\begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

So, The $\boxed{\text{Rank}(A) = 2}$

Now,

$$x_1 - 0x_2 - 4x_3 - 28x_4 - 37x_5 + 13x_6 = 0$$

$$x_2 - 2x_3 - 12x_4 - 16x_5 + 5x_6 = 0$$

So,

$$\boxed{x_2 = 2x_3 + 12x_4 + 16x_5 - 5x_6}$$

and

$$x_1 = 0x_2 + 2x_3 + 28x_4 + 37x_5 - 13x_6$$

$$\boxed{x_1 = 2x_3 + 28x_4 + 37x_5 - 13x_6}.$$

let ⇒ $x_3 = r$, $x_4 = s$, $x_5 = t$, $x_6 = u$.

So,

$$x_1 = 2r + 28s + 37t - 13u.$$

$$x_2 = 2r + 12s + 16t - 5u.$$

$$x_3 = r$$

$$x_4 = s$$

$$x_5 = t$$

$$x_6 = u.$$

So, in column:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

So, the Nullity$(A) = 4$

Moreover can find by:

Nullity = Total − Rank

= 6 − 2

$\boxed{\text{Nullity} = 4}$

## Exercise # 4.6:

① $x_1+x_2-x_3=0$

$-2x_1-x_2+2x_3=0$

$-x_1+x_3=0$

Solution:

$$\begin{bmatrix}1&1&-1\\-2&-1&2\\-1&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}1&1&-1&0\\-2&-1&2&0\\-1&0&1&0\end{array}\right]$$

$R_2+2R_1$ & $R_3+R_1$

$$\left[\begin{array}{ccc|c}1&1&-1&0\\0&1&0&0\\0&1&0&0\end{array}\right]$$

$R_1-R_2$, $R_3-R_2$

$$\left[\begin{array}{ccc|c}1&0&-1&0\\0&1&0&0\\0&0&0&0\end{array}\right]$$

$x_1-x_3=0$

$\boxed{x_2=0}$

let $\Rightarrow \boxed{x_3=t}$

So, $\boxed{x_1=t}$.

So,

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}t\\0\\t\end{bmatrix}=t\begin{bmatrix}1\\0\\1\end{bmatrix}$$

Basis is above & Dimension is (1).

② $3x_1+x_2+x_3+x_4=0$

$5x_1-x_2+x_3-x_4=0$

$$\left[\begin{array}{cccc|c}3&1&1&1&0\\5&-1&1&-1&0\end{array}\right]$$

$R_1\times\frac{1}{3}$

$$\left[\begin{array}{cccc|c}1&1/3&1/3&1/3&0\\5&-1&1&-1&0\end{array}\right]$$

$R_2-5R_1$

$$\left[\begin{array}{cccc|c}1&1/3&1/3&1/3&0\\0&-8/3&8/3&-8/3&0\end{array}\right]$$

$R_2\times -3/8$

$$\left[\begin{array}{cccc|c}1&1/3&1/3&1/3&0\\0&1&-1&-1&0\end{array}\right]$$

$R_1-\frac{1}{3}R_2$

$$\left[\begin{array}{cccc|c}1&0&2/3&2/3&0\\0&1&-1&-1&0\end{array}\right]$$

$x_1+0x_2+\frac{2}{3}x_3+\frac{2}{3}x_4=0$

$0x_1+x_2-x_3-x_4=0$

$$\left[\begin{array}{cccc|c}1&0&1/4&0&0\\0&1&1/4&1&0\end{array}\right]$$

$x_1+\frac{1}{4}x_3=0$

$x_2+\frac{1}{4}x_3+x_4=0$

$x_3=s$, $x_4=t$

$x_1=-\frac{1}{4}s$

$x_2=-\frac{1}{4}s-t$

So,

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=t\begin{bmatrix}0\\-1\\0\\1\end{bmatrix}+s\begin{bmatrix}-1/4\\-1/4\\1\\0\end{bmatrix}$$

So, Dimension = 2.

③ $2x_1+x_2+3x_3=0$

$x_1+5x_3=0$

$x_2+x_3=0$

Solution:

$$\left[\begin{array}{ccc|c}2&1&3&0\\1&0&5&0\\0&1&1&0\end{array}\right]$$

$R_2$ swap $R_1$

$$\left[\begin{array}{ccc|c}1&0&5&0\\2&1&3&0\\0&1&1&0\end{array}\right]$$

$R_2-2R_1$

$$\left[\begin{array}{ccc|c}1&0&5&0\\0&1&-7&0\\0&1&1&0\end{array}\right]$$

$$\left[\begin{array}{ccc|c}1&0&5&0\\0&1&-7&0\\0&1&1&0\end{array}\right]$$

$R_3-R_2$

$$\left[\begin{array}{ccc|c}1&0&5&0\\0&1&-7&0\\0&0&8&0\end{array}\right]$$

$R_3/8$.

$$\left[\begin{array}{ccc|c}1&0&5&0\\0&1&-7&0\\0&0&1&0\end{array}\right]$$

$R_1-5R_3$, $R_2+7R_3$.

$$\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right]$$

So, $\boxed{x_1=0}$, $\boxed{x_2=0}$, $\boxed{x_3=0}$

Dimension = $\boxed{\text{zero}}$

④ $x_1-4x_2+3x_3-x_4=0$

$2x_1-8x_2+6x_3-2x_4=0$

Solution:

$$\left[\begin{array}{cccc|c}1&-4&3&-1&0\\2&-8&6&-2&0\end{array}\right]$$

$R_2-2R_1$

$$\left[\begin{array}{cccc|c}1&-4&3&-1&0\\0&0&0&0&0\end{array}\right]$$

So, $x_1-4x_2+3x_3-x_4=0$

let: $x_2=s$, $x_3=t$, $x_4=u$.

$x_1=4s-3t+u$.

So,

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=s\begin{bmatrix}4\\1\\0\\0\end{bmatrix}+t\begin{bmatrix}-3\\0\\1\\0\end{bmatrix}+u\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$$

So, so the Dimension = 3.

(5)

$$x_1 - 3x_2 + x_3 = 0$$
$$2x_1 - 6x_2 + 2x_3 = 0$$
$$3x_1 - 9x_2 + 3x_3 = 0$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 2 & -6 & 2 & 0 \\ 3 & -9 & 3 & 0 \end{array}\right]$$

$R_2 - 2R_1$, $R_3 - 3R_1$

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So,

$$x_1 - 3x_2 + x_3 = 0$$
$$x_1 = 3x_2 - x_3$$
$$x_2 = s, \ x_3 = t$$

So,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

So, Dimension = 2.

(6)

$$x + y + z = 0$$
$$3x + 2y - 2z = 0$$
$$4x + 3y - z = 0$$
$$6x + 5y + z = 0$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 3 & 2 & -2 & 0 \\ 4 & 3 & -1 & 0 \\ 6 & 5 & 1 & 0 \end{array}\right]$$

$R_2 - 3R_1$, $R_3 - 4R_1$, $R_4 - 6R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & -5 & 0 \\ 0 & -1 & -5 & 0 \\ 0 & -1 & -5 & 0 \end{array}\right]$$

$-R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & -1 & -5 & 0 \\ 0 & -1 & -5 & 0 \end{array}\right]$$

$R_1 - R_2$, $R_3 + R_2$, $R_4 + R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So,

$$x_1 - 4x_3 = 0$$
$$x_2 + 5x_3 = 0$$
$$x_1 = 4x_3$$
$$x_2 = -5x_3$$
$$x_3 = s, \ \cancel{x_4 = t}$$

So,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cancel{x_4} \end{bmatrix} = s\begin{bmatrix} 4 \\ -5 \\ 1 \\ \cancel{0} \end{bmatrix} + \cancel{t\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix}$$

Dimension = 1.

So, There is no example of this exercise.