Example # 01

Growth and Decay

$$\frac{dx}{dt} \propto x$$

$$\frac{dx}{dt} = kx \quad ; x(t_0) = x_0$$

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we Replaced x by 'P'

$$\frac{dP}{dt} \propto P$$

$$\frac{dP}{dt} = kP$$

$$\Rightarrow \quad \frac{dP}{dt} - kP = 0$$

with $\quad t_0 = 0$ with initial Condition

$$P(t_0) = P_0$$

$$\Rightarrow \quad \boxed{P(0) = P_0}$$

Given that $\boxed{P(1) = \frac{3}{2} P_0}$

to determine constant of proportionality 'k'.

$$\frac{dP}{dt} - kP = 0$$

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$$\frac{dP}{dt} = KP$$

$$\frac{1}{P}\,dP = K\,dt$$

$$\int \frac{1}{P}\,dP = K\int dt$$

$$\ln P = Kt + c$$

$$\ln P - \ln c = Kt$$

$$\ln(P/c) = Kt$$

$$\Rightarrow P/c = e^{Kt}$$

$$\Rightarrow P = ce^{Kt}$$

$$\boxed{P(t) = ce^{Kt}}$$ required solution.

<u>For $t=0$</u>

$$P(0) = ce^{0}$$

$$P_0 = c$$

Then $$\boxed{P(t) = P_0 e^{Kt}}$$

<u>For $t=1$</u>

$$P(1) = P_0 e^{K(1)}$$

$$\frac{3}{2}P_0 = P_0 e^{K} \Rightarrow e^{K} = 3/2$$

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$$\ln e^{k} = \ln 3/2$$

$$k(1) = \ln \frac{3}{2}$$

$$\boxed{k = 0.4055}$$

So,

$$P(t) = P_0 e^{0.4055t}$$

Now

~~find for t=3~~, Bact

$$P(t) = 3P_0$$

Find $t = ?$

$$3\cancel{P_0} = \cancel{P_0} e^{0.4055t}$$

$$\ln 3 = \ln e^{0.4055t}$$

$$\ln 3 = 0.4055t(1)$$

$$\Rightarrow \quad t = \frac{\ln 3}{0.4055}$$

$$t \approx 2.71h$$

After 2.71h, the bacteria will be Tripled

Example # 02

Sol:- Let $A(t)$ denote the amount of Plutonium remaining at time 't'. The solution of IVP.

$$\frac{dA}{dt} = KA \quad , \quad A(0) = A_0$$

$$\frac{1}{A} dt = K dt$$

$$\ln A = Kt + c$$

$$\ln A - \ln c = Kt$$

$$\ln(A/c) = Kt$$

$$A/c = e^{Kt}$$

$$\boxed{A = ce^{Kt}}$$

$$A(t) = ce^{Kt}$$

For $A(0) = A_0$

$$A(0) = ce^{Kt}$$

$$A_0 = ce^{(0)} \Rightarrow \boxed{C = A_0}$$

$$\Rightarrow \boxed{A = A_0 e^{Kt}}$$

If 0.043% of the atoms of A_0 have

disintegrated, then 99.957% of the substance remains.

To find the decay constant k,

We used

$$A(t_0) = A_0$$

$$A(15) = 0.99957\, A_0$$

Hence

$$A(t) = A_0 e^{kt}$$

For t=15

$$A(15) = A_0 e^{15k}$$

$$0.99957 \cancel{A_0} = \cancel{A_0} e^{15k}$$

$$\ln 0.99957 = \ln e^{15k}$$

$$-0.0043 = 15k$$

$$k = \frac{-0.0043}{15}$$

$$k = -2.867 \times 10^{-5}$$

$$\boxed{k = -0.00002867}$$

Hence,

$$\boxed{A(t) = A_0 e^{-0.00002867t}}$$

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ow, the half life is the corresponding value of time at which

$$A(t) = \frac{1}{2} A_0$$

$$A(t) = A_0 e^{-0.00002867 t}$$

$$\frac{1}{2} \cancel{A_0} = \cancel{A_0} e^{-0.00002867t}$$

$$\ln\left(\frac{1}{2}\right) = (-0.00002867)t$$

$$\Rightarrow \quad t = \frac{\ln(1/2)}{-0.00002867}$$

$$\boxed{t \simeq 24{,}177 \text{ yr.}}$$

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Example # 03

$$\frac{dA}{dt} = KA$$

$$\Rightarrow \quad A(t) = A_0 e^{Kt} \quad \text{——} \quad (1)$$

To determine the value of the decay constent 'K' we use the fact that

$$A(5730) = \frac{1}{2} A_0$$

$\Rightarrow$ in (1)

$$A(5730) = A_0 e^{5730K}$$

$$\frac{1}{2} \cancel{A_0} = \cancel{A_0} e^{5730K}$$

$$\frac{1}{2} = e^{5730K}$$

$$\ln(1/2) = \ln e^{5730K}$$

$$\ln(1/2) = 5730K(1)$$

$$\Rightarrow \quad K = \frac{\ln(1/2)}{5730}$$

$$\boxed{K = -0.00012097}$$

Therefore,

$$A(t) = A_0 e^{-0.00012097t}$$

Given that

$$A(t) = 0.1\% A_0$$

$$A(t) = 0.001 A_0$$

So,

$$A(t) = A_0 e^{-0.00012097t}$$

$$0.001 \cancel{A_0} = \cancel{A_0} e^{-0.00012097t}$$

$$\ln(0.001) = -0.00012097t$$

$$\Rightarrow \quad t = \frac{\ln(0.001)}{-0.00012097t}$$

$$\Rightarrow \quad \boxed{t = 57{,}100 \text{ years}}$$

3.1 (1–6, 13–16)

Date: Day:

(1) The Population of a community is known to increase at a rate proportional to number of People present at time t. intial population double in 5 years. how long it take to triples quadtriple.

Let the population of the community at t = P

$\frac{dP}{dt} \rightarrow$ rate of change

$$\frac{dP}{dt} \propto P$$

$$\frac{dP}{dt} = kP$$

$$\frac{1}{P} dP = k\, dt$$

Taking Integral on B·S

$$\int \frac{1}{P} dp = k \int dt$$

$$\ln P = kt + c$$

Taking e on B·S

$$e^{\ln P} = e^{kt+c}$$

Date: Day:

$$P = e^{kt} \cdot e^{c}$$

$$P = ce^{kt} \quad \text{—(i)}$$

putting $t = 0$, $P = P_0$

$$P_0 = ce^{0}$$

put $C = P_0$

Thus eq (i)

$$P = P_0 e^{kt} \quad \text{—(ii)}$$

To Find K value put $t = 5$ and $P = 2P_0$

$t = 5$, $P = 2P_0$

$$2P_0 = P_0 e^{k5}$$

$$2 = e^{5k}$$

Taking Log on B.S

$$\ln 2 = \ln e^{5k}$$

$$\ln 2 = 5k$$

$$k = \frac{1}{5} \ln 2$$

For triple we need to find time

$t = ?$, $P = 3P_0$

$$3\cancel{P_0} = \cancel{P_0} e^{\frac{1}{5}\ln 2 \cdot t}$$

$$\ln 3 = \frac{1}{5} \ln 2 \cdot t$$

Date: Day:

$$t = \frac{5 \ln 3}{\ln 2}$$

$$t = 8$$

Time required to get quadruple.

$t = ?$ $\quad P = 4P_0$

$$4P_0 = P_0 e^{\frac{1}{5}\ln 2 \cdot t}$$

$$\ln 4 = \frac{1}{5} \ln 2 \cdot t$$

$$\frac{5 \ln 4}{\ln 2} = t$$

$$t = 10$$

(ii) Suppose it is known that population of Community in Prob. 1 is 10000 after 3 years. What will be initial P_0. What will be the Population in 10 years. How fast is the population growing at $t = 10$.

$$\frac{dP}{dt} \propto P$$

$$\frac{dP}{dt} = kP$$

Date: Day:

$\frac{1}{P} dP = k dt$

Taking integral on B.S

$\int \frac{1}{P} dP = k \int dt$

$\ln P = kt + C$

$P = ce^{k \cdot t}$ —(i)

$P_0 = C$

$P = P_0 e^{kt}$ —(ii)

For finding 'k'

$t = 5, \quad P = 2P_0$

$2P_0 = P_0 e^{5k}$

$\ln 2 = 5k$

$k = \frac{1}{5} \ln 2$

Population is 10000, after 3 years. Find initial Population

$t = 3, \quad P = 10,000$

$P = P_0 e^{kt}$

$10000 = P_0 e^{\frac{1}{5} \ln 2 (3)}$

$10,000 = P_0 e^{\frac{3}{5} \ln 2}$

$\frac{10000}{e^{\frac{3}{5} \ln 2}} = P_0$

Date: ______ Day. ______

$$P_0 = 10000\, e^{-\frac{3}{5}\ln 2}$$
$$= 10000\,(0.65975)$$
$$P_0 = 6597.54$$

What will be the population after 10 year.

$$t = 10, \quad P = ?$$
$$P = P_0 e^{kt}$$
$$= 6597.54\, e^{\frac{1}{5}\ln 2 \cdot 10}$$
$$= 6597.54\, e^{2\ln 2}$$
$$= 6597.54\,(4)$$
$$P = 26390.16$$

How fast population will grow at t=10 year.

$$P' = kP$$
$$= \frac{1}{5}(\ln 2) \cdot 26390.16$$
$$P' = 3658 \text{ person/year}$$

(3) The population of a town grows at a rate proportional to the population present at time t. The initial population of 500 increase by 15% in year what will the population in 30 years

Day:

How fast is the growing rate $t = 30$.

Let the population of Town = P

$\frac{dP}{dt} \to P$

$\frac{dP}{dt} \propto P$

$\frac{dP}{dt} = kP$

$\frac{1}{P} dP = k\,dt$

$\int \frac{1}{P} dP = k \int dt$

$\ln P = kt + c$

$P = ce^{kt}$

$P_0 = c$

$P = P_0 e^{kt}$

To Find k. initial population is 500 and increased 15%

$P_0 = 500$ $\qquad$ $15\%(500) = 500 \times 0.15$

$P = 575$ $\qquad$ $= 75$

$P = P_0 e^{kt}$

$575 = 500\, e^{10k}$

$\frac{575}{500} = e^{10k}$

Date: Day:

$$\ln\left(\frac{23}{20}\right) = 10k$$

$$\frac{0.1398}{10} = k$$

$$k = 0.01398$$

Calculating population after 30 years

$t = 30 \qquad P = ?$

$$P = P_0 e^{kt}$$

$$= 500 e^{0.01398(30)}$$

$$= 500(1.5209)$$

$$P = 760.4375$$

Growth rate of population in 30 years

$$P' = kP$$

$$= (0.01398)(760.4375)$$

$$= 10.6$$

$$P' \simeq 11 \text{ person / year}$$

4 The population ~~of~~ Bacteria in a culture growth at a rate proportional to number of bacteria Present at time t. After 3 hrs it observed

Day:

that 400 bacteria are present. After 10 hrs 2000. what was the initial number of bacteria.

Let the population of a bacteria in a culture as P

$\frac{dP}{dt} \to P$

$\frac{dP}{dt} \propto P$

$\frac{dP}{dt} = kP$

$\frac{1}{P} dP = k dt$

$\int \frac{1}{P} dP = k \int dt$

$\ln P = kt$

$P = ce^{kt}$ —(i)

when $t = 3$, 400 bacteria are present.

$t = 3$, $P = 400$

$P = ce^{kt}$

$400 = ce^{3k}$

$= \frac{400}{e^{3k}}$

Date. Day.

$c = 400e^{-3k}$

put in (i)

$P = 400e^{-3k} \cdot e^{kt}$

$= 400(e^{kt} \cdot e^{-3k})$

$= 400(e^{kt-3k})$

$P = 400e^{k(t-3)}$

when time is 10hr, 2000 bacteria were present

$t = 10, \; P = 2000$

$P = 400e^{k(t-3)}$

$2000 = 400e^{k(t-3)}$

$\frac{2000}{400} = e^{k(t-3)}$

$5 = e^{k(t-3)}$

$5 = e^{k(7)}$

$5 = e^{7k}$

$\ln 5 = 7k$

$\frac{1}{7}\ln 5 = k$

$k = 0.22992$

Find the initial number of Bacteria.

$t = 0$

Date: Day:

$$P = 400\, e^{k(t-3)}$$

$$P = 400\, e^{0.22992\,(0-3)}$$

$$P = 400\, e^{0.22992(-3)}$$

$$= 400\, e^{-0.68976}$$

$$= 400 \times 0.5017$$

$$= 200.678$$

$$P \approx 201$$

At $t=0$, No. of bacteria $= 201$

5 The radioactive isotope of lead, Pb-209 decay rate proportional to the amount present at time t and has half life of 3.3 hrs. If 1 gram of this isotope is present initially, how long will it take for 90% of the lead to decay.

Let P be the amount of radio active isotope at time t.

$$\frac{dP}{dt} \to -P$$

$$\frac{dP}{dt} \propto -P$$

$$\frac{dP}{dt} = -kP$$

Date. Day.

$$\int \frac{1}{P} dP = -k \int dt$$

$$\ln P = -kt + c$$

$$P = e^{-kt} \cdot e^{c}$$

$$P = ce^{-kt}$$

Initially we have 1 gram at time = 0

$t = 0$, $P = 1$

$$1 = ce^{0}$$

$$c = 1$$

$$P = e^{-kt}$$

After 3.3 hr we have 0.5 or half life $\frac{1}{2}$ present

$t = 3.3$, $P = 0.5$

$$P = e^{-kt}$$

$$0.5 = e^{-k(3.3)}$$

$$\ln(0.5) = -3.3k$$

$$\frac{-0.693}{-3.3} = k$$

$$k = 0.21$$

Time it takes to decay 90%

$P = 10/100$

$P = 0.1$

$$P = e^{-kt}$$

$$0.1 = e^{-(0.21)t}$$

$$\ln(0.1) = -0.21\, t$$

$$\frac{-2.3}{-0.21} = t$$

$$t = 10.96 \text{ hrs}$$

6 Initially 100 miligram of a radioactive substance was present. After 6 hrs the mass had decreased by 3%. If the rate of decay is proportional to the amount of the substance present at time t. Find the amount remaining after 24 hrs.

Let we have amount of substance as P

$$\frac{dP}{dt} \rightarrow -P$$

$$\frac{dP}{dt} \propto -P$$

$$\frac{dP}{dt} = -kP$$

$$\frac{1}{P}\, dP = -k\, dt$$

Date. Day.

$$\int \frac{1}{P} dP = -k \int dt$$

$$\ln P = -kt + c$$

$$P = e^{-kt} \cdot e^{c}$$

$$P = c e^{-kt}$$

At initial time we have 100 milligram radioactive substance.

$$t = 0 \quad , \quad P = 100$$

$$100 = c e^{0}$$

$$c = 100$$

$$P = 100 e^{-kt}$$

After 6hr the mass has decreased by 3% mean 100% − 3%

$$t = 6 \quad , \quad P = 97$$

$$P = 100 e^{-kt}$$

$$97 = 100 e^{-k(6)}$$

$$\frac{97}{100} = e^{-6k}$$

$$0.97 = e^{-6k}$$

$$\ln(0.97) = -6k$$

$$\frac{-0.0305}{-6} = k$$

$$k = 0.0051$$

Day:

$$P = 100e^{-0.0051t}$$

Find remaining amount of substance after 24 hrs

$t = 24$, $P = ?$

$$P = 100e^{-0.0051(24)}$$

$$= 100e^{-0.12184}$$

$$= 100 \times 0.8853$$

$$= 88.53 \text{ milligrams}$$

2B A thermometer is removed from a room where the temperature is $70°F$ and is taken outside, where the air temperature is $10°F$. After one half minute the thermometer read $50°F$. What is the reading of thermometer at $t=1$? How long it take to reach $15°F$.

Let 'T' be the temperature at time t

$$\frac{dT}{dt} \propto (T - T_m)$$

$$\frac{dT}{dt} = k(T - T_m)$$