Introduction To Differential Equation:

Def: An equation containing the derivatives of one or more ~~than~~ dependent variables, with respect to one or more independent variables, is said to be a differential equation.

e.g (i) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 3$

(ii) $LI'' + RI' + \frac{1}{C}I = E$ (RLC circuit)

(iii) $mv' = mg - bv^2$ (Parachutist)

$\frac{d^2y}{dx^2}$ can also be written as y''.

in eq (i) y is dependent variable and x is independent variable.

Types of Differential Equations

ODE : Ordinary differential equation
PDE : Partial differential equation

examples: (i), (ii), (iii) are ODEs

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Classification by order

highest order

$$\frac{d^2 y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x$$

2nd order ODE

highest order

$$2\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0$$

4th order PDE

Differential form (for 1st order ODE)

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$y - x + 4x\frac{dy}{dx} = 0$$

can be expressed in differential form as

$$(y-x)\,dx + 4x\,dy = 0$$

general form of an nth order ODE

$$F(x, y', y'', ---, y^{(n)}) = 0$$

normal form

$$\frac{d^n y}{dx^n} = f(x, y, y', --, y^{(n-1)}).$$

Solution of an ODE

A function ϕ defined on I and possessing at least n derivatives that are continuous on I, which when substituted into an nth order differential equation reduces the equation to an identity, is said to be a solution.

In other words, a solution of nth-order ODE is a function ϕ that possesses at least n derivatives and

$$F(x, \phi(x), \phi'(x), \cdots \phi^{(n)}(x)) = 0$$

for all x in I.

Linearity of DE

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

is linear if "F" is linear in $y, y', \ldots, y^{(n)}$

Linearity of an operator

$$T(x_1 + x_2) = T(x_1) + T(x_2)$$

$$T(\alpha x) = \alpha T(x)$$

A solut... for ... in...

A Solution of a DE is an expression for the dependent variable in terms of independent variables which satisfies the relation.

→ There are two types of solutions

(*) General solution (includes all possible solutions)

(**) particular solution (A solution without arbitrary constants / functions)

→ IVP (initial value problem)

If x_0 is an initial point

solve : $\dfrac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)})$

subject to : $y(x_0) = y_0, y'(x_0) = y_1, \ldots$

$y^{(n-1)}(x_0) = y_{n-1}$.

$$\frac{dy}{dx} = f(x, y)$$

$$y(x_0) = y_0$$

if a D.E is linear it must possesses following two properties

1) $y, y', \ldots, y^{(n)}$ all have power 1.

2) The coefficients $a_0, a_1, \ldots, a_n$ of $y, y', \ldots, y^{(n)}$ dependents at most on the independent variable x.

Linear	non linear
① $(y-x)dx + 4x\,dy = 0$ $\Rightarrow y - x + 4x\frac{dy}{dx} = 0$ $\Rightarrow \boxed{4x\frac{dy}{dx} + y = x}$	① $(1-y)y' + 2y = e^x$
② $y'' - 2y' + y = 0$	② $\frac{d^2y}{dx^2} + \sin y = 0$
③ $x^3\frac{d^3y}{dx^3} + x\frac{dy}{dx} - 5y = e^x$	③ $\frac{d^4y}{dx^4} + y^2 = 0$

→ General form of a linear equations

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

→ DE :
An equation containing the derivative of one or more unknow functions w.r.t one or more independent variables is said to be a DE.

→ Types of D.E

① ODE $\quad \frac{dy}{dx} + 5y = e^x \quad , \quad \frac{dx}{dt} + \frac{dy}{dt} = 2x+y$

② PDE $\quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

→ Order of a D.E

is the highest derivative in the equation

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x$$

2nd order ODE.

→ Linearity of DE.

$$F(x, y, y', \dots, y^{(n)}) = 0$$

is linear if "F is linear in $y, y', \dots y^{(n)}$"

i.e $\left(\begin{aligned} T(v_1+v_2) &= T(v_1) + T(v_2) \\ T(\alpha v) &= \alpha T(v) \end{aligned} \right)$

Separable Equation

A first order differential equation of the form

$$\frac{dy}{dx} = g(x)\,h(y)$$

is said to be separable.

Practically we have $g(y)\,y' = f(x)$

$\Rightarrow \int g(y)\,dy = \int f(x)\,dx + C$

Example

$$(1+x)\,dy - y\,dx = 0$$

Exercise 2-2 (1-28)

Exp# 1, 2, 7, 10
12, 14, 17, 19
22, 24, 27.

practice questions

17

$$\frac{dP}{dt} = P - P^2$$

$$\Rightarrow \frac{1}{P(1-P)}\, dP = dt$$

$$\frac{1}{P(1-P)} = \frac{A}{P} + \frac{B}{1-P}$$

$$1 = A(1-P) + BP$$

$$P = 0 \Rightarrow \boxed{A = 1}$$

$$P = 1 \Rightarrow \boxed{B = 1}$$

$$\Rightarrow \left[\frac{1}{P} + \frac{1}{1-P}\right] dP = dt$$

$$\Rightarrow \int \frac{dP}{P} + \int \frac{dP}{1-P} = \int 1\, dt$$

$$\ln|P| - \ln|1-P| = t + C$$

$$\Rightarrow \ln\left|\frac{P}{1-P}\right| = t + c$$

$$\Rightarrow \frac{P}{1-P} = e^{t+c}$$

$$\Rightarrow \frac{P}{1-P} = c_1 e^t$$

$$\Rightarrow P = c_1 e^t (1-P)$$

$$\Rightarrow P(1 + c_1 e^t) = c_1 e^t$$

$$\Rightarrow P = \frac{c_1 e^t}{1 + c_1 e^t}.$$

Q19 $$\frac{dy}{dx} = \frac{xy + 3x - y - 3}{xy - 2x + 4y - 8} = \frac{x(y+3) - (y+3)}{x(y-2) + 4(y-2)}$$

Ex

$$\frac{dy}{dx} = \frac{(x-1)(y+3)}{(x+4)(y-2)} \Rightarrow \frac{y-2}{y+3}\, dy = \frac{x-1}{x+4}\, dx$$

$$\Rightarrow \left(1 - \frac{5}{y+3}\right) dy = \left(1 - \frac{5}{x+4}\right) d$$

Q23

$$(e^x+e^{-x})\frac{dy}{dx} = y^2$$

$$\frac{1}{y^2}\,dy = (e^x+e^{-x})^{-1}dx$$

$$\int y^{-2}dy = \int\left(\frac{e^x}{(e^x)^2+1}\right)dx$$

$$-\frac{1}{y} = \tan^{-1}(e^x)+c$$

$$\Rightarrow y = -\frac{1}{\tan^{-1}(e^x)+c}.$$

Q24

$$\frac{dy}{dx} = \frac{y^2-1}{x^2-1}, \quad y(2)=2$$

$$\Rightarrow \frac{1}{y^2-1}\,dy = \frac{1}{x^2-1}\,dx$$

$$\frac{1}{(y-1)(y+1)}\,dy = \frac{1}{(x-1)(x+1)}\,dx$$

$$\frac{1}{2}\left[\frac{1}{y-1}dy + \frac{1}{y+1}dy\right] = \frac{1}{2}\left[\frac{1}{x-1}dx + \frac{dx}{x+1}\right]$$

$$\ln|y-1| + \ln|y+1| = \ln|x-1| + \ln|x+1| + c$$

$$\ln\left|\frac{y-1}{y+1}\right| = \ln\left|c\left(\frac{x-1}{x+1}\right)\right|$$

$$\Rightarrow \frac{y-1}{y+1} = c\left(\frac{x-1}{x+1}\right)$$

$$y(2)=2 \Rightarrow \frac{1}{3} = c\left(\frac{1}{3}\right) \Rightarrow c=1$$

$$\frac{y-1}{y+1} = \frac{x-1}{x+1}$$

$$\Rightarrow \quad y-1 = \frac{x-1}{x+1}(y+1)$$

$$\Rightarrow \quad y - \left(\frac{x-1}{x+1}\right)y = 1 + \frac{x-1}{x+1}$$

$$\Rightarrow \quad y\left(1 - \frac{x-1}{x+1}\right) = \frac{x+\cancel{1}+x-\cancel{1}}{x+1} = \frac{2x}{x+1}$$

$$\Rightarrow \quad y\left(\frac{\cancel{x}+1-\cancel{x}+1}{\cancel{x+1}}\right) = \frac{2x}{\cancel{x+1}}$$

$$\Rightarrow \quad y = x$$

Q27 $\sqrt{1-y^2}\,dx - \sqrt{1-x^2}\,dy = 0$ $\qquad y(0) = \frac{\sqrt{3}}{2}$

$$\Rightarrow \quad \int \frac{1}{\sqrt{1-y^2}}\,dy = \int \frac{1}{\sqrt{1-x^2}}\,dx$$

$$\Rightarrow \quad -\sin^{-1}(y) = -\sin^{-1}x + C$$

$$\Rightarrow \quad -\sin^{-1}(y) + \sin^{-1}x = C$$

$$\Rightarrow \quad -\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - 0 = C$$

$$\Rightarrow \quad -\frac{\pi}{3} = C$$

$$\Rightarrow \quad y = \sin\left(\sin^{-1}x + \frac{\pi}{3}\right).$$

Linear Equation

A first order differential equation of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$ is

said to be linear eq in var. y.

→ standard form

$$\frac{dy}{dx} + P(x)\,y = f(x) \quad -①$$

I-F $\quad \mu = e^{\int P(x)dx}$

multiply both sides of ① with μ

$$e^{\int P(x)dx}\frac{dy}{dx} + e^{\int P(x)dx}P(x)\,y = e^{\int P(x)dx}f(x)$$

$$\frac{d}{dx}\left[e^{\int P(x)dx}\cdot y\right] = e^{\int P(x)dx}f(x)$$

now integrating both of the sides w.r.t x, we have

$$e^{\int P(x)dx}\,y = \int e^{\int P(x)dx}f(x)\,dx$$

Exp 1 $\quad \frac{dy}{dx} - 3y = 0$

it is already in standard form

$$\mu = e^{\int P(x)dx} = e^{\int -3dx} = e^{-3x}$$

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = 0$$

$$\Rightarrow \frac{d}{dx}\left(e^{-3x}\cdot y\right) = 0$$

$$\Rightarrow \quad e^{-3x} \cdot y = c$$

$$\Rightarrow \quad y = c\,e^{3x}.$$

Exp2

$$\frac{dy}{dx} - 3y = 6$$

$$\mu = e^{\int -3\,dx} = e^{-3x}$$

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = 6e^{-3x}$$

$$\frac{d}{dx}\left(e^{-3x} \cdot y\right) = 6e^{-3x}$$

$$e^{-3x}y = 6\int e^{-3x}dx$$

$$e^{-3x}y = -2e^{-3x} + c$$

$$\Rightarrow y = -2 + ce^{3x}.$$

Exp3

$$x\frac{dy}{dx} - 4y = x^6 e^x \Rightarrow \frac{dy}{dx} - \frac{4}{x}y = x^5 e^x$$

$$\text{I.F} = \mu = e^{\int(-\frac{4}{x})dx} = e^{-4\ln|x|} = e^{\ln|x|^{-4}} = x^{-4}$$

$$\frac{d}{dx}\left(e^{\int P(x)dx} \cdot y\right) = e^{\int P(x)dx} \cdot f(x)$$

$$x^{-4}y = \int x^{-4} \cdot x^5 e^x$$

$$= \int x e^x dx$$

$$= xe^x - e^x + c$$

$$\Rightarrow \quad y = x^5e^x - x^4e^x + cx^4.$$

Exercise (2.3) (1–36)

2, 4, 5, 7, 11, 12
14, 16, 18, 20, 22
24, 25, 27, 30
31, 33, 35, 36

transient term will vanish when $t \to \infty$ while remaining term is called steady state

Q5

$$y' + 3x^2 y = x^2$$

$$\text{I.F} = \mu = e^{\int p(x)dx} = e^{\int 3x^2 dx} = e^{x^3}$$

$$\frac{d}{dx}(e^{x^3} y) = e^{x^3} \cdot x^2$$

$$e^{x^3} y = \int x^2 e^{x^3} dx$$

$$e^{x^3} y = \frac{1}{3} e^{x^3} + C$$

$$y = \frac{1}{3} + \underline{Ce^{-x^3}}$$

↳ transient term

Q7

$$x^2 y' + xy = 1$$

$$\Rightarrow y' + \frac{1}{x} y = x^{-2} \quad : \quad \mu = e^{\int p(x)dx} = e^{\int \frac{1}{x}dx} = x$$

Thus $\frac{d}{dx}(x \cdot y) = x \cdot x^{-2}$

$$\Rightarrow x \cdot y = \int \frac{1}{x} dx$$

$$\Rightarrow xy = \ln|x| + C$$

$$\Rightarrow y = \underline{\frac{1}{x} \ln|x| + Cx^{-1}}$$

↳ transient.

Q12

$$(1+x)\frac{dy}{dx} - xy = x + x^2$$

$$\frac{dy}{dx} - \frac{x}{1+x} y = \frac{x(1+x)}{1+x} = x$$

$$\mu = e^{-\int \frac{x}{1+x}dx} = e^{-\int\left(1-\frac{1}{1+x}\right)dx}$$

$$= e^{-x \cdot \ln|1+x|} = e^{-x}(1+x)$$

$$\frac{d}{dx}\left(e^{-x}(1+x)\cdot y\right) = x(1+x)e^{-x}$$

$$\Rightarrow (1+x)e^{-x}y = \int x(1+x)e^{-x}dx$$

$$\Rightarrow (1+x)e^{-x}y = \int xe^{-x}dx + \int x^2e^{-x}dx$$

$$= \int xe^{-x}dx + x^2e^{-x} + 2\int xe^{-x}dx$$

$$= -x^2e^{-x} + 3xe^{-x} + 3e^{-x} + c$$

$$= -x^2e^{-x} - xe^{-x} - 2xe^{-x} + 3e^{-x} + c$$

$$= -x(x+1)e^{-x} - (2x-3)e^{-x} + c$$

$$\Rightarrow y = -x + \frac{(-2x+3)}{1+x} + \frac{ce^x}{1+x}$$

Q14

$$xy' + (1+x)y = e^{-x}\sin 2x$$

$$y' + \frac{(1+x)}{x}y = \frac{1}{x}e^{-x}\sin 2x$$

$$\mu = e^{\int\left(\frac{1}{x}+1\right)dx} = x e^x$$

$$\Rightarrow \frac{d}{dx}\left(xe^x y\right) = xe^x \cdot \frac{1}{x}e^{-x}\sin 2x$$

$$\Rightarrow \quad x e^{x} y = \int \sin 2x \, dx$$

$$\Rightarrow \quad x e^{x} y = -\frac{\cos 2x}{2} + c$$

$$\Rightarrow \quad y = \underbrace{-\frac{1}{2}\frac{\cos 2x}{x e^{x}} + c\frac{e^{-x}}{x}}_{\text{transient}}$$

Q20 $$(x+2)^2 \frac{dy}{dx} = 5 - 8y - 4xy$$

$$\Rightarrow \quad (x+2)^2 \frac{dy}{dx} + 4(2+x)y = 5$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{4}{x+2} y = 5(x+2)^{-2}$$

$$\mu = e^{4\int \frac{dx}{x+2}} = e^{4\ln|x+2|} = e^{\ln|x+2|^4} = (x+2)^4$$

$$\frac{d}{dx}\left((x+2)^4 \cdot y\right) = 5(x+2)^2$$

$$(x+2)^4 \cdot y = \int 5(x+2)^2 dx$$

$$(x+2)^4 y = \frac{5}{3}(x+2)^3 + c$$

$$\Rightarrow y = \frac{5}{3}(x+2)^{-1} + c(x+2)^{-4}$$

Q 25 ✓ $$\frac{dy}{dx} = x + 5y \quad , \quad y(0) = 3$$

$$\frac{dy}{dx} - 5y = x \qquad \mu = e^{-5x}$$

$$\frac{d}{dx}\left(e^{-5x} \cdot y\right) = x e^{-5x}$$

$$e^{-5x} y = \int x e^{-5x} dx$$

$$e^{-5x} y = \int x e^{-5x} dx$$

$$= x \frac{e^{-5x}}{-5} + \frac{1}{5} \int e^{-5x} dx$$

$$= \frac{1}{5}\left(-x e^{-5x} - \frac{1}{5} e^{-5x}\right) + c$$

$$y = -\frac{1}{5} x - \frac{1}{25} + c e^{5x} \quad \text{— (*)}$$

$$y(0) = 3$$

(*) becomes $\Longrightarrow$ $3 = 0 - \frac{1}{25} + c$

$$\Rightarrow \quad c = 3 + \frac{1}{25} = \frac{76}{25}$$

Thus the ~~general~~ solution is

$$y = -\frac{1}{5} x - \frac{1}{25} + \frac{76}{25} e^{5x}.$$

Q30 $\quad \frac{dT}{dt} = k(T - T_m) \quad , \ T(0) = T_0$

$$\frac{dT}{dt} - kT = -kT_m$$

$$\mu = e^{\int (-k) dt} = e^{-kt}$$

$$\Rightarrow \quad \frac{d}{dt}\left(e^{-kt} \cdot T\right) = -k e^{-kt} T_m$$

$$e^{-kt} \cdot T = -k T_m \int e^{-kt} dt$$

$$= -k T_m \frac{e^{-kt}}{-k} + c$$

$$e^{-kt} \cdot T = T_m e^{-kt} + c$$

$$T = T_m + c e^{kt} \quad \text{— (*)}$$

$$T(0) = T_0$$

(*) becomes

$$T_0 = T_m + c e^0$$

$$\Rightarrow \quad c = T_0 - T_m$$

Thus we have ~~general~~ solution.

$$T = T_m + (T_0 - T_m) e^{kt}.$$

Q36 ✓

$$y' + (\tan x) y = \cos^2 x \quad , \; y(0) = -1$$

$$\mu = e^{\int p(x)dx} = e^{\int \tan x \, dx} = e^{\int \frac{\sin x}{\cos x} dx}$$

$$= e^{-\ln|\cos x|} = e^{\ln|\sec x|} = \sec x.$$

multiplying by I.F we have

$$\frac{d}{dx}(\sec x \cdot y) = \sec x \cdot \cos^2 x$$

$$\Rightarrow \frac{d}{dx}(\sec x \cdot y) = \cos x$$

$$\sec x \cdot y = \int \cos x \, dx$$

$$\Rightarrow$$

$$\Rightarrow \quad \sec x \cdot y = \sin x + c$$

$$\Rightarrow \quad y = \cos x \sin x + c \cos x \quad \text{—(*)}$$

$$y(0) = -1$$

$$(*) \Rightarrow \quad -1 = c$$

Thus we have

$$y = \cos x \sin x - \cos x.$$

Def: A differential expression

$$M(x,y)dx + N(x,y)dy$$ is an

exact differential in a region R of the xy-plane if it corresponds to the differential of some function $f(x,y)$ defined in R.

$$\left(\text{i.e } df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = M(x,y)dx + N(x,y)dy \right)$$

- A first order differential equation of the form

$$M(x,y)dx + N(x,y)dy = 0$$

is said to be an exact equation if the expression on the left hand side is an exact differential. (i.e $d(f(x,y)) = 0$ and the solution will be $f(x,y) = c$)

Q: What is the criterion for a DE to be an exact differential equation.

A: 1) M, N should be continuous and have continuous first partial derivatives in region R

2) $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Second method

if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then the Solution is given by

$$\int M\,dx + \int (\text{terms of } N \text{ independent of } \cancel{\text{free from}}\ x)\,dy = \text{Constant.}$$

Exp 1 $\quad 2xy\,dx + (x^2-1)\,dy = 0$

$$M = 2xy \quad , \quad N = x^2 - 1$$

$$\frac{\partial M}{\partial y} = 2x \quad , \quad \frac{\partial N}{\partial x} = 2x$$

Thus $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ hold

So, the general solution is

$$\int M\,dx + \int (\text{terms of } N \text{ Independent of } \cancel{\text{free from}}\ x)\,dy = C$$

$$\int (2xy)\,dx + \int (-1)\,dy = C$$

$$x^2y - y = C$$

Exp 2 $\quad (3x^2y+2)\,dx + (x^3+y)\,dy = 0$

$$M = 3x^2y + 2 \quad , \quad N = x^3 + y$$

$$\frac{\partial M}{\partial y} = 3x^2 \ \text{—}(*) \quad , \quad \frac{\partial N}{\partial x} = 3x^2 \ \text{—}(**)$$

from (*) and (**) we have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Thus we have the solution.

$$\int M\,dx + \int (\text{terms of } N \text{ independent of } x)\,dy = c$$

$$\int (3x^2y+2)\,dx + \int y\,dy = C$$

$$x^3y + 2x + \frac{y^2}{2} = C$$

$$\Rightarrow 2x^3y + 4x + y^2 = 2C$$

Exp3 $(e^{2y} - y\cos xy)\,dx + (2xe^{2y} - x\cos xy + 2y)\,dy = 0$

$$M = e^{2y} - y\cos xy$$

$$\frac{\partial M}{\partial y} = 2e^{2y} - \cos xy - y(-\sin xy\,(x))$$

$$= 2e^{2y} - \cos xy + xy\sin xy \quad \text{---} (*)$$

$$N = 2xe^{2y} - x\cos xy + 2y$$

$$\frac{\partial N}{\partial x} = 2e^{2y} - \cos xy - x(-\sin xy\,(y))$$

$$= 2e^{2y} - \cos xy + xy\sin xy \quad \text{---} (**)$$

from (*) and (**) we have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Thus the solution is

$$\int (e^{2y} - y\cos xy)\,dx + \int 2y\,dy = c$$

$$\Rightarrow \quad xe^{2y} - y\frac{\sin xy}{y} + y^2 = c$$

$$\Rightarrow \quad xe^{2y} - \sin xy + y^2 = c.$$

Exercise 2.4 (1–38)

2, 4, 7, 8, 11
13, 16, 17, 20
22, 24, 26, 28
31, 34, 36

Q8

$$\left(1+\ln x + \frac{y}{x}\right)dx = (1-\ln x)\,dy$$

$$\Rightarrow \left(1+\ln x + \frac{y}{x}\right)dx - (1-\ln x)\,dy = 0$$

$$M = 1+\ln x + \frac{y}{x}, \qquad N = -1+\ln x$$

$$\frac{\partial M}{\partial y} = \frac{1}{x}, \qquad \frac{\partial N}{\partial x} = \frac{1}{x}$$

solution is

$$\int M\,dx + \int (\text{terms of } N \text{ independent of } \cancel{\text{free from}}\ x)\,dy = c$$

$$\int \left(1+\ln x + \frac{y}{x}\right)dx + \int (-1)\,dy = c$$

$$x + x\ln x - x + y\ln x + (-y) = c$$

$$\Rightarrow \quad x\ln x + y\ln x - y = c$$

Q13 $\quad x\frac{dy}{dx} = 2xe^x - y + 6x^2$

$\Rightarrow \quad x\,dy = (2xe^x - y + 6x^2)\,dx$

$\Rightarrow \quad -(2xe^x - y + 6x^2)\,dx + x\,dy = 0$

$M = -2xe^x + y - 6x^2$ | $N = x$

$\frac{\partial M}{\partial y} = 1$ | $\frac{\partial N}{\partial x} = 1$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

So, solution is

$\int M\,dx + \int$ (terms of N ~~free from~~ Independent of x) $dy = c$

$$\int(-2xe^x + y - 6x^2)\,dx + \int 0\,dy = c$$

$$\Rightarrow -2(xe^x - e^x) + xy - 2x^3 + c_1 = c$$

$$\Rightarrow \quad -2xe^x + 2e^x + xy - 2x^3 = c.$$

$(e^x\ xy$

M

the

$$(e^x + y)\,dx + (2 + x + ye^y)\,dy = 0$$

$$y(0) = 1$$

$M = e^x + y$ | $N = 2 + x + ye^y$

$\dfrac{\partial M}{\partial y} = 1$ | $\dfrac{\partial N}{\partial x} = 1$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

So, Solution is

$$\int (e^x + y)\,dx + \int (2 + ye^y)\,dy = c$$

$$e^x + xy + 2y + ye^y - e^y = c$$

$$y(0) = 1 \Rightarrow 1 + 0 + 2 + 1 - 1 = c$$

$$3 = c$$

$$\Rightarrow$$

$$\therefore\ e^x + xy + 2y + ye^y - e^y = 3$$

Q28

$$\left(6xy^3 + \cos y\right)dy + \left(2Kx^2y^2 - x\sin y\right)d$$

find K ?

$M = 6xy^3 + \cos y$ | $N = 2Kx^2y^2 - x\sin y$

$\frac{\partial M}{\partial y} = 16xy^2 - \sin y$ | $\frac{\partial N}{\partial x} = 4Kxy^2 - \sin y$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\Rightarrow \quad 16xy^2 - \sin y = 4Kxy^2 - \sin y$

$\Rightarrow \quad 16 = 4K$

$\Rightarrow \quad \boxed{K = 4}$

t

Non exact Differential equation

If we have a nonexact differential equation

$$M(x,y)dx + N(x,y)\,dy = 0.$$

Then, it is sometimes possible to find an integrating factor $\mu(x,y)$ sothat, The left hand side of

$$\mu M dx + \mu N dy = 0$$ is an exact.

What will be the integrating factor.

(1) If $(M_y - N_x)/N$ is a function of x alone Then

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}$$

(2) If $(N_x - M_y)/M$ is a function of y alone then

$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}.$$

Exp

$$xy\,dx + (2x^2 + 3y^2 - 20)dy = 0$$

$$M = xy \quad , \quad N = 2x^2 + 3y^2 - 20$$

$$M_y = \frac{\partial M}{\partial y} = x \quad , \quad N_x = \frac{\partial N}{\partial x} = 4x$$

So, given DE is not exact.

Now first we will compute

$$\frac{M_y - N_x}{N} = \frac{x - 4x}{2x^2 + 3y^2 - 20}$$

$$= \frac{-3x}{2x^2 + 3y^2 - 20}.$$

since above is not the function of x alone, so we will try

$$\frac{N_x - M_y}{M} = \frac{4x - x}{xy}$$

$$= \frac{3x}{xy} = \frac{3}{y}$$

Thus

$$\mu = e^{\int \frac{3}{y} dy} = e^{3\ln|y|} = y^3.$$

multiplying μ with given DE we get (24)

$$xy^4\,dx + (2x^2y^3 + 3y^5 - 20y^3)\,dy = 0 \quad \text{— (ii)}$$

Now $M = xy^4$, $N = 2x^2y^3 + 3y^5 - 20y^3$

$$M_y = \frac{\partial M}{\partial y} = 4xy^3, \quad N_x = \frac{\partial N}{\partial x} = 4xy^3$$

Since $M_y = N_x$ therefore (ii) is an exact and the solution is

$$\int xy^4\,dx + \int (3y^5 - 20y^3)\,dy = C$$

$$\frac{x^2}{2}y^4 + \frac{3y^6}{6} - 20\frac{y^4}{4} = C$$

$$\Rightarrow \frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 = C$$

Q33 $\quad 6xy\,dx + (4y + 9x^2)\,dy = 0 \quad$ — (i)

$M = 6xy$, $N = 4y + 9x^2$

$M_y = 6x$, $N_x = 18x$

$M_y \neq N_x$

Now

$$\frac{N_x - M_y}{M} = \frac{18x - 6x}{6xy} = \frac{12x}{6xy} = \frac{2}{y}$$

$$\text{I.F} = \mu = e^{\int \frac{2}{y} dy} = e^{2\ln|y|} = e^{\ln|y|^2} = y^2$$

Now

multiplying μ with ① we have

$$6xy^3 dx + (4y^3 + 9x^2y^2) dy = 0$$

$$\boxed{M = 6xy^3 \quad , \quad N = 4y^3 + 9x^2y^2}$$

$M_y = N_x$

Solution is

$$\int (6xy^3) dx + \int (4y^3) dy = 0$$

$$\cancel{3 \cancel{6} y^3 \left(\frac{x^2}{\cancel{2}}\right) + \cancel{4}\left(\frac{y^4}{\cancel{4}}\right)}$$

$$\Rightarrow \cancel{\frac{6xy^4}{4} + y^4 = 0}$$

$$\Rightarrow \cancel{\frac{3}{2}xy^4 + y^4 = c}$$

$$\cancel{3x^2y^3 + y^4 = 0}$$

$$3x^2y^3 + y^4 = c$$

Q34

$$\cos x \, dx + \left(1 + \frac{2}{y}\right) \sin x \, dy = 0 \quad -①.$$

$$M = \cos x \quad , \quad N = \left(1 + \frac{2}{y}\right) \sin x$$

$$M_y = 0 \quad , \quad N_x = \left(1 + \frac{2}{y}\right) \cos x$$

Now

$$\frac{M_y - N_x}{N} = \frac{0 - \left(1 + \frac{2}{y}\right) \cos x}{\left(1 + \frac{2}{y}\right) \sin x}$$

$$= -\frac{\cos x}{\sin x} = -\cot x.$$

$$\text{I.F} = \mu = e^{-\int \frac{\cos x}{\sin x} dx} = e^{-\ln|\sin x|} = e^{\ln \operatorname{cosec} x} = \operatorname{cosec} x.$$

so ① becomes $\cos x \cdot \operatorname{cosec} x \, dx + \left(1 + \frac{2}{y}\right) dy = 0$

solution

Now $M = \cos x \csc x$, $N = 1+\frac{2}{y}$

$M_y = 0$, $N_x = 0$

$M_y = N_x$.

so (11) is an exact.

Thus solution is

$$\int \cos x \csc x \, dx + \int \left(1+\frac{2}{y}\right) dy = c$$

$$\int \frac{\cos x}{\sin x} dx + y + 2\ln|y| = c$$

$$\ln|\sin x| + y + \ln y^2 = c$$

Homogeneous Equation

If a function possesses the property

$f(tx, ty) = t^{\alpha} f(x, y)$ for some real α,

Then f is said to be a homogeneous function of degree α.

For example $f(x, y) = x^3 + y^3$ is a homogeneous function of degree 3 because

$$f(tx, ty) = (tx)^3 + (ty)^3 = t^3(x^3 + y^3)$$
$$= t^3 f(x, y).$$

* A first order DE in differential form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$ is

said to be homogeneous if both M and N are homogeneous functions of same degree.

i.e $$M(tx, ty) = t^{\alpha} M(x, y)$$
$$N(tx, ty) = t^{\alpha} N(x, y).$$

Example (P72)

$$(x^2+y^2)\,dx + (x^2-xy)\,dy = 0 \quad \text{--- (1)}$$

$M = x^2+y^2$, $N = x^2-xy$

$M(tx,ty) = t^2 M(x,y)$, $N(tx,ty) = t^2 N(x,y)$

Note: Homogeneous if in each term sum of powers of x, y is same. here in This example sum of powers in each term is 2.

let $y = ux$

$\Rightarrow dy = d(ux)$

$dy = x\,du + u\,dx$

So (1) becomes

$$(x^2+u^2x^2)\,dx + (x^2 - x(ux))(x\,du + u\,dx) = 0$$

$$\Rightarrow (1+u^2)x^2\,dx + x^2(1-u)(x\,du + u\,dx) = 0$$

$$\Rightarrow (1+u^2)x^2\,dx + x^3(1-u)\,du + u(1-u)x^2\,dx = 0$$

dividing by x^2 we have

$$(1+u^2)\,dx + x(1-u)\,du + u(1-u)\,dx = 0$$

$$\Rightarrow (1+u^2+u(1-u))\,dx + x(1-u)\,du = 0$$

$$\Rightarrow (1+u)\,dx + x(1-u)\,du = 0$$

$$\Rightarrow x(u-1)\,du = (1+u)\,dx$$

$$\Rightarrow \quad \left(\frac{u-1}{u+1}\right) du = \frac{dx}{x}$$

Now integrate it

$$\int \left(\frac{u-1}{u+1}\right) du = \int \frac{dx}{x}$$

$$\Rightarrow \int \left(1 - \frac{2}{u+1}\right) du = \ln|x| + \ln c$$

$$\Rightarrow \quad u - 2\ln|u+1| = \ln|x| + \ln c$$

$$\Rightarrow \quad u + \ln(u+1)^{-2} = \ln|x| + \ln c \quad \text{(ii)}$$

Now by $y = ux \Rightarrow u = y/x$ in (ii)

we have

$$y/x + \ln\left(\frac{y}{x}+1\right)^{-2} = \ln|x| + \ln c$$

$$\frac{y}{x} + \ln\left(\frac{y}{x}+1\right)^{-2} = \ln|cx|$$

$$\cancel{\Rightarrow y + \ln\left(\frac{y}{x}+1\right)^{-2} = x}$$

$$\cancel{\Rightarrow \ln\left(\frac{y}{x}+1\right)^{-2} - \ln(cx) = \frac{y}{x}}$$

$$\frac{y}{x} = \ln|cx| - \ln\left(\frac{y}{x}+1\right)^{-2}$$

$$= \ln\left(\frac{cx}{\left(\frac{y}{x}+1\right)^{-2}}\right)$$

$$= \ln\left(\frac{cx}{x^2 (y+x)^{-2}}\right)$$

$$= \ln\left(\frac{c\,(y+x)^2}{x}\right)$$

$$\Rightarrow \quad e^{y/x} = \frac{c}{x}(y+x)^2$$

$$\Rightarrow \quad (x+y)^2 = c' x\, e^{y/x}$$

Exercise 2.5

(1 – 14 homogeneous
15 – 22 Bernoulli
3, 5, 8, 10, 12
14, 16, 19, 20
22)

Q5 $(y^2+xy)dx - x^2 dy = 0$

it is homogenous of degree 2.

let $y = ux \Rightarrow dy = u\,dx + x\,du$

① becomes

$$(u^2x^2 + x(ux))\,dx - x^2(u\,dx + x\,du) = 0$$

$$x^2(u^2+u)\,dx - x^2(u\,dx + x\,du) = 0$$

$$\Rightarrow \quad (u^2+u)\,dx - (u\,dx + x\,du) = 0$$

$$\Rightarrow \quad u^2\,dx + x\,du = 0$$

let $x = vy$

$\Rightarrow dx = v\,dy + y\,dv$

put it in ①

$$y(v\,dy + y\,dv) + vy(\ln vy - \ln y - 1)\,dy = 0$$

$$vy\,dy + y^2\,dv + vy(\ln v - 1)\,dy = 0$$

$$\Rightarrow \quad y^2\,dv + vy\ln v\,dy = 0$$

$$\Rightarrow \quad y\,dv + v\ln v\,dy = 0$$

$$\Rightarrow \quad v\ln v\,dy = -y\,dv$$

$$\Rightarrow \quad -\int \frac{dy}{y} = \int \frac{1}{v\ln v}\,dv$$

$$\Rightarrow \quad -\ln|y| = \int \frac{1/v}{\ln v}\,dv$$

$$-\ln|y| = \ln[\ln|v|] + \ln c$$

$$+\ln|y|^{-1} = \ln(c\,|\ln v|)$$

$$\Rightarrow \quad \frac{1}{y} = c\ln|v|$$

$v = \frac{x}{y}$ implies

$$\Rightarrow \quad \frac{1}{y} = c\ln\left|\frac{x}{y}\right|$$

Now apply the initial condition and get the soluti

$$u^2 dx = -x du$$

$$\Rightarrow -\frac{du}{u^2} = \frac{dx}{x}$$

$$\Rightarrow -\int \frac{du}{u^2} = \ln|x| + \ln c$$

$$\Rightarrow -\frac{u^{-2+1}}{-2+1} = \ln|cx|$$

$$\Rightarrow \frac{1}{u^{-1}} = \ln|cx| \quad -(11)$$

$y = ux \Rightarrow u = \frac{y}{x} \Rightarrow$ (11) becomes

$$\left(\frac{x}{y}\right) = \ln|cx|$$

$$\Rightarrow e^{y^3/x^3} = cx$$

Q14

$$y\,dx + x(\ln x - \ln y - 1)\,dy = 0 \quad (1) \qquad y(1) = e$$

$M = y, \qquad N = x(\ln x - \ln y - 1)$

$M(tx, ty) = ty$

$= tM(x, y)$

$N(tx, ty) = tx(\ln(tx) - \ln(ty) - 1)$

$= tx(\ln x + \cancel{\ln t} - \cancel{\ln t} - \ln y - 1)$

$= tx(\ln x - \ln y - 1)$

$= t\,N(x, y)$

So, it is homogeneous.

$$\Rightarrow u^2 dx + x du = 0$$

Bernoulli's Differential equation

The differential equation

$$\frac{dy}{dx} + P(x)\,y = f(x)\,y^n \quad \text{---} (1)$$

where n is any real number, is called Bernoulli's equation. If $n=0,1$ Then above equation is linear.

* For $n \neq 0,1$, the substitution $u = y^{1-n}$ reduces any equation of the form (1) to a linear equation.

__Example__

$$x\frac{dy}{dx} + y = x^2 y^2$$

$n=2$, ~~therefor~~

first divide by x we have

$$\frac{dy}{dx} + \frac{1}{x}y = x y^2 \quad \text{---} (1)$$

now let $u = y^{1-n} = y^{1-2} = y^{-1}$

$$\Rightarrow \boxed{y = u^{-1}} \text{ and } \boxed{\frac{dy}{dx} = \frac{d(u^{-1})}{dx} = -u^{-2}\frac{du}{dx}}$$

By substituting these in eq (1) we have

$$-u^{-2}\frac{du}{dx} + u^{-1} = x u^{-2}$$

$$\Rightarrow \frac{du}{dx} - \frac{1}{x}u = -x \quad \text{—(ii)}$$

$$I.F = \mu = e^{\int p(x)dx} = e^{-\int \frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$$

Thus (ii) becomes

$$\frac{d}{dx}(x^{-1}u) = -1$$

Integrating $\Rightarrow \quad x^{-1}u = -\int 1 dx$

$$\Rightarrow \quad x^{-1}u = -x + C$$

$$\Rightarrow \quad u = -x^2 + Cx$$

now put $u = y^{-1}$ we have

$$y^{-1} = -x^2 + Cx$$

$$\Rightarrow \quad y = \frac{1}{-x^2 + Cx}.$$

Q19

$$t^2 \frac{dy}{dt} + y^2 = ty$$

$$\Rightarrow \frac{dy}{dt} - \frac{1}{t}y = -\frac{1}{t^2}y^2 \quad \text{—(i)}$$

$n = 2$ let $u = y^{1-n} = y^{1-2} = y^{-1}$

$$\Rightarrow \boxed{y = u^{-1}} \Rightarrow \frac{dy}{dt} = -u^{-2}\frac{du}{dt}$$

Substitute in eq (i) we have

$$-u^{-2}\frac{du}{dt} - \frac{1}{t}u^{-1} = -\frac{1}{t^2}u^{-2}$$

$$\Rightarrow \frac{du}{dt} + \frac{1}{t}u = \frac{1}{t^2}$$

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I.F $\quad u = e^{\int p(t)\,dt} = e^{\int \frac{1}{t}\,dt} = t$

multiplying I.F we have.

$$\frac{d}{dt}(t \cdot u) = t^{-1}$$

By integrating we have

$$t \cdot u = \int t^{-1}\,dt$$

$$t \cdot u = \ln|t| + \ln|c|$$

$$t \cdot u = \ln(tc)$$

$$\Rightarrow e^{tu} = t \cdot c$$

$$u = y^{-1} \Rightarrow e^{t/y} = t \cdot c$$

(22) $\quad y^{1/2}\dfrac{dy}{dx} + y^{3/2} = 1 \qquad y(0) = 4$

$$\Rightarrow \frac{dy}{dx} + y^{3/2 - 1/2} = y^{-1/2}$$

$$\Rightarrow \frac{dy}{dx} + y = y^{-1/2} \quad —(1)$$

$n = -1/2 \qquad$ let $u = y^{1-n} = y^{1+1/2} = y^{3/2}$

$$\Rightarrow y = u^{2/3} \Rightarrow \frac{dy}{dx} = \frac{2}{3}u^{-\frac{1}{3}}\frac{du}{dx}$$

so (1) $\Rightarrow \dfrac{2}{3}u^{-1/3}\dfrac{du}{dx} + u^{2/3} = u^{-1/3}$

$\Rightarrow \dfrac{du}{dx} + \dfrac{3}{2}u = \dfrac{3}{2}$ $\quad$ remaing (do yourself)