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Exercise (8.3) (1-34)

Using Trigonometric Substitution (1-14)

① $\int \frac{dx}{\sqrt{9+x^2}}$

It's first condition

$\int \frac{dx}{\sqrt{(3)^2+(x)^2}}$ — i)

first condition:

$1+\tan^2\theta = \sec^2\theta$

$x = a\tan\theta$

$\sqrt{a^2+x^2} = \underset{\text{variable}}{a^2+x^2}$

So,

$\boxed{x = 3\tan\theta}$

$\boxed{dx = 3\sec^2\theta\, d\theta}$

Replacing in eq - i)

$\int \frac{3\sec^2\theta\, d\theta}{\sqrt{(3)^2+(3\tan\theta)^2}}$

$\int \frac{\cancel{3}\sec^2\theta\, d\theta}{\cancel{3}\sqrt{1+\tan^2\theta}}$

$\int \frac{\sec^2\theta\, d\theta}{\sqrt{\sec^2\theta}}$

$\int \sec\theta\, d\theta$

$\int \frac{1}{\cos\theta}\, d\theta$

$\boxed{\ln|\sec\theta+\tan\theta|+c}$

So,

$x = 3\tan\theta$

$\boxed{\tan\theta = \frac{x}{3}}$

For value of $\sec\theta$:

$1+\tan^2\theta = \sec^2\theta$

$1+\left(\frac{x}{3}\right)^2 = \sec^2\theta$

$1+\frac{x^2}{9} = \sec^2\theta$

$\frac{9+x^2}{9} = \sec^2\theta$

$\sec\theta = \sqrt{\frac{9+x^2}{9}}$

$\boxed{\sec\theta = \frac{\sqrt{9+x^2}}{3}}$

$\ln\left|\frac{\sqrt{9+x^2}}{3}+\frac{x}{3}\right|+c$

$\boxed{\ln|\sqrt{9+x^2}+x|+c}$

② $\int \frac{3dx}{\sqrt{1+9x^2}}$

Again first condition.

$\int \frac{3dx}{\sqrt{(1)^2+(3x)^2}}$ — i)

So, ~~$x = \tan\theta$~~

~~$dx =$~~ So, first we let:

$u = 3x$

then

$\int \frac{du}{\sqrt{(1)^2+u^2}}$ — i)

$u = \tan\theta$

$du = \sec^2\theta\, d\theta$

So,

$\int \frac{\sec^2\theta\, d\theta}{\sqrt{1+\tan^2\theta}}$

$\int \frac{\sec^2\theta\, d\theta}{\sec\theta}$

$\int \sec\theta\, d\theta$

$\ln|\sec\theta+\tan\theta|+c \Rightarrow \ln|\sqrt{u^2+1}+u|+c$

$\boxed{\ln|\sqrt{1+9x^2}+3x|+c}$

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for $\sin^{-1}x$

$$\int \frac{1}{\sqrt{1-x^2}} = \sin^{-1}\frac{x}{a}$$

Used in (3) Formula: for tan

$$\int \frac{1}{a+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+C$$

for $\sec^{-1}x$

$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}\frac{x}{a}+C$$

(3) $$\int_{-2}^{2} \frac{dx}{4+x^2}$$

$$\int_{-2}^{2} \frac{dx}{(2)^2+(x)^2}$$

So,

$\cancel{x = 2\tan\theta}$

$\cancel{dx = 2\sec^2\theta\, d\theta}$

So,

$$\cancel{\int_{-2}^{2} \frac{2\sec^2\theta\, d\theta}{(2)^2+}}$$

As we know,

$$\tan^{-1}x = \frac{1}{1+x^2}$$

So,

$$\left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_{-2}^{2}$$

$$\left[\frac{1}{2}\tan^{-1}\left(\frac{2}{2}\right) - \frac{1}{2}\tan^{-1}\left(-\frac{2}{2}\right)\right]$$

$$\left[\frac{1}{2}\tan^{-1}(1) - \frac{1}{2}\tan^{-1}(-1)\right]$$

$$\frac{1}{2}\left(\frac{\pi}{4}\right) - \frac{1}{2}\left(-\frac{\pi}{4}\right)$$

$$\frac{\pi}{8}+\frac{\pi}{8}$$

$$\frac{2\pi}{8} \Rightarrow \boxed{\frac{\pi}{4}}$$

(4) $$\int_{0}^{2} \frac{dx}{8+2x^2}$$

$$\int_{0}^{2} \frac{dx}{2(4+x^2)}$$

$$\frac{1}{2}\int_{0}^{2} \frac{dx}{4+x^2}$$

$$\frac{1}{2}\int_{0}^{2} \frac{dx}{(2)^2+(x)^2}$$

$$\frac{1}{2}\left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^2$$

$$\frac{1}{2}\left[\frac{1}{2}\tan^{-1}\left(\frac{2}{2}\right)\right]$$

$$\frac{1}{2}\left[\frac{1}{2}\tan^{-1}(1)\right]$$

$$\frac{1}{2}\left[\frac{1}{2}\left(\frac{\pi}{4}\right)\right]$$

$$\frac{1}{2}\boxed{\frac{\pi}{8}} = \boxed{\frac{\pi}{16}}$$

(5) $$\int_{0}^{3/2} \frac{dx}{\sqrt{9-x^2}}$$

$$\int_{0}^{3/2} \frac{dx}{\sqrt{(3)^2-(x)^2}}$$

$$\left[\sin^{-1}\frac{x}{3}\right]_0^{3/2}$$

$$\left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)\right]$$

$$\sin^{-1}\left(\frac{1}{2}\right).$$

$$\boxed{\frac{\pi}{6}}.$$

(6) $$\int_{0}^{1/2\sqrt{2}} \frac{2dx}{\sqrt{1-4x^2}}$$

$$\int_{0}^{1/2\sqrt{2}} \frac{2dx}{\sqrt{1-(2x)^2}}$$

Let, $t = 2x$

$dt = 2dx$

$$\int_{0}^{1/2\sqrt{2}} \frac{dt}{\sqrt{1-(t)^2}}$$

$$\left[\sin^{-1}\frac{t}{1}\right]_0^{1/2\sqrt{2}}$$

$$\sin^{-1}\frac{1}{\sqrt{2}} - \sin^{-1}0$$

$$\frac{\pi}{4} - 0 \Rightarrow \boxed{\frac{\pi}{4}}.$$

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⑦ $\int \sqrt{25-t^2}\, dt.$

$\int \sqrt{(5)^2-t^2}\, dt$

2nd condition $\quad t = 5\sin\theta$

$dt = 5\cos\theta\, d\theta.$

$\int \sqrt{25-(5\sin\theta)}\ (5\cos\theta d\theta)$

$\int 5\sqrt{1-\sin^2\theta}\ (5\cos\theta d\theta$

$\int 5\sqrt{\cos^2\theta}\ (5\cos\theta d\theta$

$25\int \cos 2\theta\, d\theta$

$\therefore \cos 2\theta = \frac{\cos 2\theta + 1}{2}$

$\frac{25}{2}\int (1+\cos 2\theta)\, d\theta.$

$\frac{25}{2}\theta + \frac{25\sin 2\theta}{4}$

$25\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right) + c$

$25\left(\frac{\theta}{2} + \frac{\cancel{2}\sin\theta\cos\theta}{\cancel{4}\,2}\right) + c$

$\boxed{\frac{25}{2}(\theta + \sin\theta\cos\theta) + c}$

⑧ $\int \sqrt{1-9t^2}\, dt$

$\int \sqrt{1-(3t)^2}\, dt.$

$3t = \sin\theta$

$dt = \frac{1}{3}\cos\theta\, d\theta$

So,

$\int \sqrt{1-(\sin\theta)^2}\ (\tfrac{1}{3}\cos\theta\, d\theta.$

$\frac{1}{3}\int \sqrt{1-\sin^2\theta}\ \cos\theta\, d\theta$

$\frac{1}{3}\int \sqrt{\cos^2\theta}\ \cos\theta\, d\theta$

$\frac{1}{3}\int \cos 2\theta\, d\theta$

$\therefore \cos 2\theta = \frac{\cos 2\theta + 1}{2}$

$\frac{1}{3}\int \frac{1+\cos 2\theta}{2}\, d\theta$

$\frac{1}{3}\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right] + c$

$\frac{1}{3}\left[\frac{\theta}{2} + \frac{\cancel{12}\sin\theta\cos\theta}{\cancel{4}2}\right] + c$

$\frac{1}{3}\left[\frac{\theta}{2} + \frac{\sin\theta\cos\theta}{2}\right] + c$

$\frac{1}{6}(\theta + \sin\theta\cos\theta) + c$

Putting values.

$3t = \sin\theta$

$\theta = \sin^{-1} 3t$

$\cos\theta = \sqrt{1-9t^2}$

↳ we use, $1-\sin^2\theta = \cos^2$

$\boxed{\frac{1}{6}\left(\sin^{-1}3t + \frac{3t}{\theta}\sqrt{1+9t^2}\right) + c}$

⑨ $\int \frac{dx}{\sqrt{4x^2-49}}$

$\int \frac{dx}{\sqrt{(2x)^2-(7)^2}}$

$x = a\sec\theta$

$2x = 7\sec\theta$

$x = \frac{7}{2}\sec\theta$

$dx = \frac{7}{2}\sec\theta\tan\theta\, d\theta$

Putting values,

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$$\int \frac{dx}{\sqrt{(2x)^2-(7)^2}}$$

$$\int \frac{7\sec\theta\tan\theta\, d\theta}{2\sqrt{(7\sec\theta)^2-(7)^2}}$$

$$\int \frac{7\sec\theta\tan\theta\, d\theta}{2\cdot 7\sqrt{\sec^2\theta-1}}$$

$$\int \frac{\cancel{7}\sec\theta\,\cancel{\tan\theta}\, d\theta}{2\cdot\cancel{7}\,\cancel{\tan\theta}}$$

$$\frac{1}{2}\int \sec\theta\, d\theta$$

$$\frac{1}{2}\ln|\sec\theta+\tan\theta|+c$$

$$\sec\theta = \frac{2x}{7}$$

$$\sqrt{\sec^2\theta-1} = \tan\theta$$

$$\sqrt{\left(\frac{2x}{7}\right)^2-1} = \tan\theta$$

$$\frac{\sqrt{4x^2-49}}{7} = \tan\theta$$

So,

$$\boxed{\frac{1}{2}\ln\left|\frac{2x}{7}+\frac{\sqrt{4x^2-49}}{7}\right|+c}$$

(10) $$\int \frac{5dx}{\sqrt{25x^2-9}}, \quad x > \frac{3}{5}$$

$$\int \frac{5dx}{\sqrt{(5x)^2-(3)^2}}$$

$$5x = 3\sec\theta$$

$$x = \frac{3}{5}\sec\theta$$

$$5\,dx = \frac{3}{\cancel{5}}\sec\theta\tan\theta\, d\theta.$$

Putting values,

$$\int \frac{3\sec\theta\tan\theta\, d\theta}{\sqrt{(3\sec\theta)^2-(3)^2}}$$

$$\int \frac{3\sec\theta\tan\theta\, d\theta}{\sqrt{9\sec^2\theta-9}}$$

$$\int \frac{3\sec\theta\tan\theta\, d\theta}{3\sqrt{\sec^2\theta-1}}$$

$$\because \tan^2\theta = \sec^2\theta-1$$

$$\int \frac{\cancel{3}\sec\theta\,\cancel{\tan\theta}\, d\theta}{\cancel{3}\,\cancel{\tan\theta}}$$

$$\int \sec\theta\, d\theta$$

$$\boxed{\ln|\sec\theta+\tan\theta|+c}$$

$$\sec\theta = \frac{5x}{3}$$

$$\tan\theta = \sqrt{\sec^2\theta-1}$$

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$$\tan\theta = \sqrt{\left(\frac{5x}{3}\right)^2 - 1}$$

$$\tan\theta = \sqrt{\frac{25x^2 - 9}{9}}$$

$$\tan\theta = \frac{\sqrt{25x^2-9}}{3}$$

So,

$$\boxed{\ln\left|\frac{5x}{3} + \frac{\sqrt{25x^2-9}}{3}\right| + c}$$

(11) $\displaystyle\int \frac{\sqrt{y^2-49}}{y}\,dy, \; y > 7$

$$\int \frac{\sqrt{(y)^2-(7)^2}}{y}\,dy$$

$$y = 7\sec\theta$$

$$dy = 7\sec\theta\tan\theta\,d\theta$$

$$\int \frac{\sqrt{(7\sec\theta)^2-(7)^2}}{7\sec\theta}(7\sec\theta\tan\theta\,d\theta)$$

$$\int \frac{7\sqrt{\sec^2\theta-1}}{\cancel{7\sec\theta}}\cancel{7\sec\theta}\tan\theta\,d\theta$$

$$\int 7\tan\theta\tan\theta\,d\theta$$

$$7\int \tan^2\theta\,d\theta$$

$$7\int (\sec^2\theta - 1)\,d\theta$$

$$\boxed{7(\tan\theta - \theta) + c}$$

$$\theta = \sec^{-1}\left(\frac{y}{7}\right)$$

$$\sqrt{\sec^2\theta - 1} = \tan\theta$$

$$\tan\theta = \sqrt{\left(\frac{y}{7}\right)^2 - 1}$$

$$\tan\theta = \frac{\sqrt{y^2-49}}{7}$$

So,

$$\boxed{7\left(\frac{\sqrt{y^2-49}}{7}\right) - \left(\sec^{-1}\left(\frac{y}{7}\right)\right) + c}$$

(12) $\displaystyle\int \frac{\sqrt{y^2-25}}{y^3}\,dy, \; y > 5$

$$\int \frac{\sqrt{(y)^2-(5)^2}}{y^3}\,dy$$

$$y = 5\sec\theta$$

$$dy = 5\sec\theta\tan\theta\,d\theta$$

$$5\sqrt{\sec^2\theta - 1} \Rightarrow 5\tan\theta$$

$$\int \frac{5\tan\theta}{(5\sec\theta)^2}\cancel{5}\sec\theta\tan\theta\,d\theta$$

$$\int \frac{\cancel{5}\tan\theta}{5\cancel{25}\sec\theta}\tan\theta\,d\theta$$

$$\frac{1}{5}\int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta.$$

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$$\frac{1}{5}\int \frac{\sin^2\theta}{\cancel{\cos^2\theta}} \times \cancel{\cos^2\theta}\, d\theta$$

$$\frac{1}{5}\int \sin^2\theta\, d\theta$$

$$\frac{1}{10}\int (1-\cos 2\theta)\, d\theta$$

$$\frac{1}{10}(\theta - \sin\theta\cos\theta) + c$$

$$\boxed{\frac{1}{10}\left(\sec^{-1}\left(\frac{y}{5}\right) - \left(\frac{\sqrt{y^2-25}}{y}\right)\left(\frac{5}{y}\right)\right) + c}$$

(13) $$\int \frac{dx}{x^2\sqrt{x^2-1}}$$

$$\int \frac{dx}{x^2\sqrt{(x)^2-(1)^2}}$$

$x = \sec\theta$

$dx = \sec\theta\tan\theta\, d\theta$

$\sqrt{\sec^2\theta - 1} = \tan\theta$

So,

$$\int \frac{\cancel{\sec\theta}\cancel{\tan\theta}\, d\theta}{\sec^{\cancel{2}}\theta\,\cancel{\tan\theta}}$$

$$\int \frac{1}{\sec\theta}\, d\theta$$

$$\int \cos\theta\, d\theta \Rightarrow \cancel{\sec\theta}\ \boxed{\sin\theta + c}$$

(14) $$\int \frac{2dx}{x^3\sqrt{x^2-1}}, \quad x > 1$$

$$\int \frac{2dx}{x^3\sqrt{(x)^2-(1)^2}}$$

$x = \sec\theta$

$dx = \sec\theta\tan\theta\, d\theta$

$\sqrt{\sec^2\theta - 1} = \tan\theta$

$$\int \frac{2\cancel{\sec\theta}\cancel{\tan\theta}\, d\theta}{\sec^3\theta\,\cancel{\tan\theta}}$$

$$2\int \frac{1}{\sec^2\theta}\, d\theta \Rightarrow 2\int \cos^2\theta\, d\theta.$$

$$\cancel{2}\int \left(\frac{1+\cos 2\theta}{\cancel{2}}\right) d\theta \Rightarrow \boxed{\theta + \sin\theta\cos\theta + c}$$

(15) $$\int \frac{x}{\sqrt{9-x^2}}\, dx \qquad u = 9 - x^2$$

$$\cancel{\int \frac{x}{\sqrt{(3)^2-(x)^2}}}$$

$\cancel{u = 3\sec\theta}$

$du = -2x\, dx$

$-\frac{1}{2}\, du = x\, dx$

So,

$$-\frac{1}{2}\int \frac{1}{\sqrt{u}}\, du = -\frac{1}{\cancel{2}} \times \cancel{2}\sqrt{u} + c$$

$$\sqrt{u} + c \Rightarrow \boxed{\sqrt{9-x^2} + c}$$

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(16) $\int \frac{x^2}{4+x^2}\,dx$

$\int \frac{x^2}{(2)^2+(x)^2}\,dx$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

$(2)^2 + (2\tan\theta)^2 \Rightarrow 4\sec^2\theta$

So,

$\int \frac{(\cancel{4}\tan^2\theta)(2\cancel{\sec^2\theta})}{(\cancel{4\sec^2\theta})}\,d\theta$

$2\int \tan^2\theta\,d\theta$

$2\int(\sec^2\theta - 1)\,d\theta$

$2\int \sec^2\theta\,d\theta - 2\int d\theta$

$2\tan\theta - 2\theta + C$

$\theta = \tan^{-1}\left(\frac{x}{2}\right)$

$\sqrt{\sec^2\theta - 1} = \tan^2\theta$

$\cancel{\sec}\ \frac{\cancel{2\tan\theta}\,x}{\cancel{2\tan\theta}} - 2\tan^{-1}\left(\frac{x}{2}\right) + C$

$\boxed{x - 2\tan^{-1}\left(\frac{x}{2}\right) + C}$

(17) $\int \frac{x^3\,dx}{\sqrt{x^2+4}}$

$\int \frac{x^3\,dx}{\sqrt{(x)^2+(2)^2}}$

$u = 2\,\tan\theta$ ~~$\sec\theta$~~

$du = 2\sec^2\theta\ \cancel{\tan\theta}\,d\theta$

~~$\sqrt{\sec^2\theta + }$~~

$du = \frac{2\,d\theta}{\cos^2\theta}$

$\sqrt{x^2+4} \Rightarrow \left(\frac{2}{\cos^2\theta}\right)^2 + 4 \Rightarrow \frac{2}{\cos\theta}.$