Exercise 5.2

Q17 Compute matrix A^{10} where

$$A = \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix}$$

Solution

$$|\lambda I - A| = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix}$$

$$|\lambda I - A| = \begin{bmatrix} \lambda & -3 \\ -2 & \lambda+1 \end{bmatrix} \text{— i)}$$

$$\det|\lambda I - A| = (\lambda)(\lambda+1) - 6$$

$$= \lambda^2 + \lambda - 6$$

$$\lambda^2 + 3\lambda - 2\lambda - 6 = 0$$

$$\lambda(\lambda+3) - 2(\lambda+3) = 0$$

$$(\lambda+3)(\lambda-2)$$

$\boxed{\lambda = -3}$ | $\boxed{\lambda = +2}$.

So, when $\lambda = -3$. in eq-i)

$$\begin{bmatrix} -3 & -3 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-3x_1 - 3x_2 = 0$$

$$-2x_1 - 2x_2 = 0$$

$$+x_1 + x_2 = 0$$

$$x_2 = -x_1$$

So,

$\boxed{x_2 = -t}$

$\boxed{x_1 = t}$

basis are

$$t\begin{bmatrix} +1 \\ -1 \end{bmatrix} = P_1$$

when $\lambda = 2$ in eq-i)

$$\begin{bmatrix} 2 & -3 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 2 - 3x_2 = 0$$

$$-2x_1 + 3x_2 = 0$$

So,

$$x_2 = \frac{3x_1}{2}$$

So,

$$P_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

So, $P = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$

$$P^{-1} = \begin{bmatrix} \frac{2}{5} & -\frac{3}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

So,

$$D = P^{-1}AP$$

$$D = \begin{bmatrix} \frac{2}{5} & -\frac{3}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$$

$$D = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$$

Now, we A^{10},

$$A^{10} = PD^{10}P^{-1}$$

$$= \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -3^{10} & 0 \\ 0 & 2^{10} \end{bmatrix}\begin{bmatrix} 2/5 & -3/5 \\ 1/5 & 1/5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -59049 & 0 \\ 0 & 1024 \end{bmatrix}\begin{bmatrix} 2/5 & -3/5 \\ 1/5 & 1/5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -23619.6 & 35429.4 \\ 204.8 & 204.8 \end{bmatrix}$$

$$\begin{bmatrix} -23005.2 & 36043.8 \\ 24029.2 & -35019.8 \end{bmatrix}$$

Q18 $A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$

Solution,

$$|\lambda I - A| = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda-1 & 0 \\ 1 & \lambda-2 \end{bmatrix}$$

$$\det|\lambda I - A| = (\lambda-1)(\lambda-2) + 0$$

$\boxed{\lambda = 1}$ $\boxed{\lambda = 2}$.

when $\lambda = 1$ $|\lambda I - A|x = 0$

$$\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x \boxed{-y = 0}$ So, the basis are

$\boxed{\cancel{x = t}}$

$\boxed{y = t}$. $\quad t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

if $\lambda = 2$ then,

$$\begin{bmatrix} 2-1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\boxed{x = 0}$

$\boxed{y = t}$ So, $t\begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow$

$$P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{|P|} \text{Adj} P$$

$$P^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$D = P^{-1}AP$$

$$D = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Now, A^{10},

$$A^{10} = PD^{10}P^{-1}$$

$$A^{10} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1^{10} & 0 \\ 0 & 2^{10} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$A^{10} = \begin{bmatrix} 1 & 0 \\ 1 & 1024 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$A^{10} = \begin{bmatrix} 1 & 0 \\ -1023 & 1024 \end{bmatrix}$$

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Q19. Confirm that P diagonalize A, compute A^{11}.

$$A=\begin{bmatrix}-1 & 7 & -1\\ 0 & 1 & 0\\ 0 & 15 & -2\end{bmatrix} \; \& \; P=\begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 1\\ 1 & 0 & 5\end{bmatrix}$$

$$D=P^{-1}AP$$

$$P=\begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 1\\ 1 & 0 & 5\end{bmatrix}$$

$$P^{-1}=\frac{1}{|P|}\,Adj P$$

$$|P|=1$$

$$\cancel{Adj P}=\begin{bmatrix}\cancel{1} & \cancel{1} & \cancel{-1}\\ \cancel{0} & \cancel{0} & 1\\ \cancel{1} & \cancel{0} & \cancel{5}\end{bmatrix}$$

$\cancel{P^{-1}}$ by using calculator,

$$Adj P=\begin{bmatrix}0 & -5 & 1\\ 1 & 4 & -1\\ 0 & 1 & 0\end{bmatrix}$$

$$D=\begin{bmatrix}0 & -5 & 1\\ 1 & 4 & -1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}-1 & 7 & -1\\ 0 & 1 & 0\\ 0 & 15 & -2\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 1\\ 1 & 0 & 5\end{bmatrix}$$

$$D=\begin{bmatrix}-1 & 17 & -3\\ -1 & -4 & 1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 1\\ 1 & 0 & 5\end{bmatrix}$$

$$D=\begin{bmatrix}-4 & -1 & 1\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Some calculation error correct it.

$$D=\begin{bmatrix}-2 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

So, P Diagonalize A

As now A^{11}.

So,

$$A^{11}=PD^{11}P^{-1}$$

$$A^{11}=\begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 1\\ 1 & 0 & 5\end{bmatrix}\begin{bmatrix}\overbrace{-2^{11}}^{-2048} & 0 & 0\\ 0 & -1^{11} & 0\\ 0 & 0 & 1^{11}\end{bmatrix}\begin{bmatrix}0 & -5 & 1\\ 1 & 4 & -1\\ 0 & 1 & 0\end{bmatrix}$$

$$A^{11}=\begin{bmatrix}\cancel{-2048}\,-1 & 10237 & -2047\\ 0 & 1 & 0\\ 0 & 10245 & -2048\end{bmatrix}$$

Q20. a) $A=\begin{bmatrix}1 & -2 & 8\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}$ and $P=\begin{bmatrix}1 & -4 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$

$$P^{-1}=\frac{1}{|P|}\,Adj P$$

$$|P|=1.$$

$$Adj P=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & -1 & 4\end{bmatrix}$$

$$P^{-1}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & -1 & 4\end{bmatrix}$$

$$D=P^{-1}AP$$

$$D=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & -1 & 4\end{bmatrix}\begin{bmatrix}1 & -2 & 8\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}1 & -4 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$$

$$D=\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$A=PDP^{-1}$$

$$A=\begin{bmatrix}1 & -4 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & -1 & 4\end{bmatrix}$$

$$A=\begin{bmatrix}1 & -2 & 8\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}$$ ~~Hence P~~

$A^{1000} \Rightarrow$

$$A=PD^{1000}P^{-1}$$

$$=\begin{bmatrix}1 & -4 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & -1 & 4\end{bmatrix}$$

$$A^{1000}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$ Answer

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(b) A^{-1000}

Simply $\Rightarrow$ if $A^{1000} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$

So, $\boxed{A^{-1000} = -I}$.

(c) A^{2301}

$$A = PD^{2301}P^{-1}$$

$$A = \begin{bmatrix} 1 & -4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(d)

$$A^{-2301} = \boxed{-A}.$$

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