Exercise. 6.1

①

Condition:

$\langle u,v\rangle = 2u_1v_1 + 3u_2v_2$

$u=(1,1)$, $v=(3,2)$, $w=(0,-1)$ and $k=3$.

a) $\langle u,v\rangle$

Solution,

$\langle u,v\rangle = 2u_1v_1 + 3u_2v_2$

$= 2(1)(3)+3(1)(2)$

$= 6+6 \Rightarrow 12$

b) $\langle kv,w\rangle$

$\langle kv,w\rangle = 2(kv)(w)+3(kv)(w)$

$kv = (3)(3,2) \Rightarrow (9,6)$

$= 2(9)(0)+3(6)(-1)$

$= 0-18 \Rightarrow -18$

c) $\langle u+v,w\rangle$

$\langle u+v,w\rangle = 2(u+v)(w)+3(u+v)(w)$

$u+v = (1,1)+(3,2) = (4,3)$

$= 2(4)(0)+3(3)(-1)$

$= 0-9 \Rightarrow -9$

d) $\|v\|$

$\|v\| = \sqrt{2(v_1)^2+2(v_2)^2}$

$\|v\| = \sqrt{2(3)^2+3(2)^2}$

$\|v\| = \sqrt{2(9)+3(4)}$

$\|v\| = \sqrt{18+12}$

$\|v\| = \sqrt{30}$

e) $d(u,v)$

$d(u,v) = \|u-v\|$

$u-v = (1,1)-(3,2)$

$u-v = (-2,-1)$

$\|u-v\| = \sqrt{2(u_1-v_1)^2+3(u_2-v_2)^2}$

$= \sqrt{2(-2)^2+3(-1)^2}$

$= \sqrt{8+3}$

$= \sqrt{11}$

f) $\|u-kv\|$

$kv = (3)(3,2)$

$kv = (9,6)$

$\|u-kv\| =$

$= \sqrt{2(u_1-kv)^2+3(u-kv)^2}$

$u-kv = (1,1)-(9,6)$

$u-kv = (-8,-5)$

$= \sqrt{2(-8)^2+3(-5)^2}$

$= \sqrt{128+75}$

$= \sqrt{203}$

② Condition:

$\langle u,v\rangle = \frac{1}{2}u_1v_1 + 5u_2v_2$

values are same as in ①

a) $\langle u,v\rangle$

$\langle u,v\rangle = \frac{1}{2}(1)(3)+5(1)(2)$

$= \frac{3}{2}+10$

$= \frac{23}{2}$

b) $\langle kv,w\rangle$

$kv = (9,6)$

$\langle kv,w\rangle = \frac{1}{2}(9)(0)+5(6)(-1)$

$= 0+(-30)$

$= -30$

c) $\langle u+v,w\rangle$

$u+v = (4,3)$

$\langle u+v,w\rangle = \frac{1}{2}(4)(0)+5(3)(-1)$

$= 0+(-15)$

$= -15$

d) $\|v\|$

$\|v\| = \sqrt{\frac{1}{2}(v_1)^2+5(v_2)^2}$

$= \sqrt{\frac{1}{2}(3)^2+5(2)^2}$

$= \sqrt{9/2+20}$

$= \sqrt{\frac{49}{4}}$

$= \frac{7}{\sqrt{2}}$

e) $d(u,v)$

$\|u-v\|$

$u-v = (-2,-1)$

$\sqrt{\frac{1}{2}(\cancel{4}\,2)+5(1)}$

$\sqrt{7}$

f) $\|u-kv\|$

$u-kv = (-8,-5)$

$\sqrt{\frac{1}{2}(64)+5(25)}$

$\sqrt{32+\cancel{50}\,125}$

$\sqrt{157}$

③ Just Replace condition with

a) $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

values are same as ①:

a) $\langle u,v \rangle$

$\langle u,v \rangle = A \cdot u \cdot Av$

$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} 3 \\ 2 \end{bmatrix}\begin{bmatrix} 8 \\ 5 \end{bmatrix}$

so,

$= 3\times 8 + 2\times 5$

$= 34$

b) $\langle kv, w \rangle$

$kv = \begin{bmatrix} 9 \\ 6 \end{bmatrix}$

$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 9 \\ 6 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} 24 \\ 15 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$= 24\times -1 + 15\times -1$

$= -24 - 15$

$= -39$

c) $(u+v, w)$

$u+v = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} 11 \\ 7 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$= 11\times -1 + 7\times -1$

$= -11 - 7 \Rightarrow -18$

c) $\|v\|$

$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} 8 \\ 5 \end{bmatrix}\begin{bmatrix} 8 \\ 5 \end{bmatrix}$

$= 8\times 8 + 5\times 5$

$= 89 = \sqrt{89}$

e) $d(u,v)$

$u - v = (-2, -1) = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} -5 \\ -3 \end{bmatrix}\begin{bmatrix} -5 \\ -3 \end{bmatrix}$

$= -5\times -5 + (-3)(-3)$

$= 34 \Rightarrow 34 \Rightarrow \sqrt{34}$

f) $\|u - kv\|$

$kv = \begin{bmatrix} 9 \\ 6 \end{bmatrix}$

$u - kv = \begin{bmatrix} -8 \\ -5 \end{bmatrix}$

$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -8 \\ -5 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -8 \\ -5 \end{bmatrix}$

$= \begin{bmatrix} -21 \\ -13 \end{bmatrix}\begin{bmatrix} -21 \\ -13 \end{bmatrix}$

$= (-21)(-21) + (-13)(-13)$

$= \sqrt{610}$

→ Same condition applies on part ⓑ (4)

⑤ $\langle u,v \rangle = 2u_1v_1 + 3u_2v_2$

As eucliden inner product will be:

$(u,v) = u_1v_1w_1 + u_2v_2w_2 \ldots$

in matrix form:

$$\begin{bmatrix} \sqrt{w_1} & 0 & \cdots & 0 & 0 \\ 0 & \sqrt{w_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \sqrt{w_3} & & \vdots \\ 0 & 0 & & \cdots & \sqrt{w_n} \end{bmatrix}$$

So, the matrix for given condition will be:

$$\boxed{A = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \end{bmatrix}}$$

⑥ $\langle u,v \rangle = \frac{1}{2}u_1v_1 + 5u_2v_2$

Same as previous so, here matrix is.

$$\boxed{A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \sqrt{5} \end{bmatrix}}$$

⑦ $u = (0,-3)$ $v = (6,2)$

$A = \begin{bmatrix} 4 & 1 \\ 2 & -3 \end{bmatrix}$ find $\langle u,v \rangle$

$\langle u,v \rangle = Au \cdot Av$

$= \begin{bmatrix} 4 & 1 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ -3 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 6 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} -3 \\ 9 \end{bmatrix}\begin{bmatrix} 26 \\ 6 \end{bmatrix}$

$= -3\times 26 + 9\times 6 \Rightarrow -24$

⑧ $u = (0,-3)$, $v = (6,2)$

$A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$ find $\langle u,v \rangle$

$= \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ -3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 6 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} -3 \\ -9 \end{bmatrix}\begin{bmatrix} 14 \\ 0 \end{bmatrix}$

$= -3\times 14 + 0\times -9$

$= -42$

⑨ $u = \begin{bmatrix} 3 & -2 \\ 4 & 8 \end{bmatrix}$, $v = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$

$\langle u,v \rangle = \begin{bmatrix} 3 & -2 \\ 4 & 8 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$

$= 3(-1) + (-2)3 + (1) + 8(1)$

$= -3 - 6 + 4 + 8$

$= 3$

(10)

$U = \begin{bmatrix} 1 & 2 \\ -3 & 5 \end{bmatrix}, V = \begin{bmatrix} 4 & 6 \\ 0 & 8 \end{bmatrix}$

Solution:

$\langle u, v \rangle = \begin{bmatrix} 1 & 2 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 4 & 6 \\ 0 & 8 \end{bmatrix}$

$= 1(4) + 2(6) + (-3)(0) + (5)(8)$

$= 4 + 12 + 0 + 40$

$= 56$

(11) $p = -2 + x + 3x^2$

$q = 4 - 7x^2$

$\langle p, q \rangle = ?$

$\langle p, q \rangle = (-2)(4) + (1)(0) + (3)(-7)$

$= -8 + 0 - 21$

$= -29$

(12) $p = -5 + 2x + x^2$

$q = 3 + 2x - 4x^2$

$\langle p, q \rangle = ?$

$\langle p, q \rangle = (-5)(3) + (2)(2) + (1)(-4)$

$= (-15) + \cancel{4} - \cancel{4}$

$= -15$.

(13) $\langle u, v \rangle = 3u_1v_1 + 5u_2v_2$

Its answer will

As we know, matrix will,

$\begin{bmatrix} \sqrt{c_1} & 0 \\ 0 & \sqrt{c_2} \end{bmatrix}$

c = co-efficient

$\boxed{\begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{5} \end{bmatrix}}$

(14) $\langle u, v \rangle = 4u_1v_1 + 6u_2v_2$

So,

$\begin{bmatrix} \sqrt{4} & 0 \\ 0 & \sqrt{6} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & \sqrt{6} \end{bmatrix}$

(15): $p = x + x^3$

$q = 1 + x^2$

(15) $x_0 = -2, x_1 = -1,$

$x_2 = 0, x_3 = 1$

$\langle p, q \rangle =$

$\cancel{p = -1 + 1}$

$\cancel{q = 1 + 0}$

So,

$= \cancel{(-1)(1) + (1)(0)}$

$= \cancel{-1 + 0}$

$= -1$.

$p(x_0) = -2 + (-2)^3 = -10$

$p(x_1) = -1 + (-1)^3 = -2$

$p(x_2) = 0 + 0 = 0$

$p(x_3) = 1 + (1)^3 = 2$

$q(x_0) = 1 + (-2)^2 = 5$

$q(x_1) = 1 + (-1)^2 = 2$

$q(x_2) = 1 + (0) = 1$

$q(x_3) = 1 + (1)^2 = 2$

So,

$\langle p, q \rangle = p(x_0)q(x_0) + p(x_1)q(x_1) + p(x_2)q(x_2) + p(x_3)q(x_3)$

$\langle p, q \rangle = (-10)(5) + (-2)(2) + (0)(1) + (2)(2)$

$\langle p, q \rangle = -50 - \cancel{4} + 0 + \cancel{4}$

$\langle p, q \rangle = -50$

(16) $x_0 = -1, x_1 = 0,$ same as earlier.

(17) find $\|u\|$ and $d(u, v)$.

$\langle u, v \rangle = 2u_1v_1 + 3u_2v_2$

$u = (-3, 2), v = (1, 7)$

$\|u\| = \sqrt{2(-3)^2 + 3(2)^2}$

$= \sqrt{18 + 12}$

$\sqrt{30}$.

$d(u, v) = \|u - v\|$

$u - v = (-3, 2) - (1, 7)$

$= (-4, -5)$

$\|u - v\| = \sqrt{2(-4)^2 + 3(-5)^2}$

$= \sqrt{32 + 75}$

$= \sqrt{107}$

(18) $u = (-1, 2), v = (2, 5)$

Condition are same as (17).

$\|u\| = \sqrt{2(-1)^2 + 3(2)^2}$

$= \sqrt{2 + 12}$

$= \sqrt{14}$

$d(u, v) = \|u - v\|$

$u - v = (-1, 2) - (2, 5)$

$= (-3, -3)$

$= \sqrt{2(-3)^2 + 3(-3)^2}$

$= \sqrt{18 + 27}$

$= \sqrt{45}$.

(19) find $\|p\|$ and $d(p,q)$.

$p = -2 + x + 3x^2$

$q = 4 - 7x^2$

$\|p\| = \sqrt{\langle p, p\rangle^2} = \sqrt{(-2)^2 + (1)^2 + (3)^2}$

$\sqrt{4+1+9}$

$\sqrt{14}$

$d(p,q) = \|p-q\|$

$p - q = (-2 + x + 3x^2) - (4 - 7x^2)$

$= (-6 + x + 10x^2)$

$\|p-q\| = \sqrt{(-6)^2 + (1)^2 + (10)^2}$

$= \sqrt{136 + 1 + 100}$

$= \sqrt{137}$.

(20) $p = -5 + 2x + x^2$

$q = 3 + 2x - 4x^2$

$\|p\| = \sqrt{(-5)^2 + (2)^2 + (1)^2}$

$= \sqrt{30}$

$d(p,q) = \|p-q\|$

$p - q = (-5 + 2x + x^2) - (3 + 2x - 4x^2)$

$p - q = (-8 + 5x^2)$

$\|p-q\| = \sqrt{(-8)^2 + (5)^2}$

$= \sqrt{64 + 25}$

$= \sqrt{89}$

(21) $U = \begin{bmatrix} 3 & -2 \\ 4 & 8 \end{bmatrix}, V = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$

$\|U\| = \begin{bmatrix} 3 & -2 \\ 4 & 8 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 4 & 8 \end{bmatrix}$

$\|U\| = \begin{bmatrix} \cancel{9} & +4 \\ 16 & 64 \end{bmatrix}$

$\|U\| = \sqrt{U^T U}$

$= \begin{bmatrix} 3 & 4 \\ -2 & 8 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 4 & 8 \end{bmatrix}$

$= \begin{bmatrix} 25 & 26 \\ 26 & 68 \end{bmatrix}$

$= 25 + 68 = 93$

$\|U\| = \sqrt{93}$.

$d(u,v) = \|u-v\|$

$u - v = \begin{bmatrix} 4 & -5 \\ 3 & 7 \end{bmatrix}$

$= \sqrt{(U-V)^t (u-v)}$

$= \begin{bmatrix} 4 & 3 \\ -5 & 7 \end{bmatrix}\begin{bmatrix} 4 & -5 \\ 3 & 7 \end{bmatrix}$

$= \begin{bmatrix} 25 & 1 \\ 1 & 74 \end{bmatrix}$

$= 25 + 74 = 99$

$= \sqrt{99}$.

(22) $U = \begin{bmatrix} 1 & 2 \\ -3 & 5 \end{bmatrix}, V = \begin{bmatrix} 4 & 6 \\ 0 & 8 \end{bmatrix}$

$\|U\| = \sqrt{U^T U}$

$= \begin{bmatrix} 1 & -3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & 5 \end{bmatrix}$

$= \sqrt{1^2 + 2^2 + (-3)^2 + 5^2}$

$= \sqrt{39}$.

$d(u,v) = \|u-v\|$

$u - v = \begin{bmatrix} -3 & -4 \\ -3 & -3 \end{bmatrix}$

$\|u-v\| = \sqrt{(-3)^2 + (-4)^2 + (-3)^2 + (-3)^2}$

$= \sqrt{43}$.

(23) $p = x + x^3$

$q = 1 + x^2$

$\|p\| = ?\ d(p,q)$.

$x_0 = -2, x_1 = -1, x_2 = 0, x_3 = 1$

$p(x_0) = -2$.

Question 23, 24 & 15, 16 are same.

(25) $u = (-1, 2), v = (2, 5)$

$A = \begin{bmatrix} 4 & 0 \\ 3 & 5 \end{bmatrix}$, find $\|u\|, d(u,v)$.

$\|u\| = \sqrt{A(u)^2 \cdot A(u)^2}$

$Au = \begin{bmatrix} 4 & 0 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} -4 \\ -7 \end{bmatrix}$

$\|u\| = \sqrt{(-4)^2 + (-7)^2}$

$= \sqrt{16 + 49}$

$= \sqrt{65}$.

$d(u,v) = \|u-v\|$

$u - v = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$

$= \begin{bmatrix} 4 & 0 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} -3 \\ -3 \end{bmatrix}$

$= \begin{bmatrix} -12 \\ -24 \end{bmatrix}$

$= \sqrt{(-12)^2 + (-24)^2}$

$= \sqrt{144 + 576}$

$= \sqrt{720}$.

6.1 Examples:

Example # 9

$x_0 = -2,\ x_1 = 0,\ x_2 = 2$

$p = x^2,\ q = 1 + x.$

find $\langle p, q \rangle,\ \|p\| = ?$

$p(x_0) = (-2)^2 = 4$

$p(x_1) = (0)^2 = 0$

$p(x_2) = (2)^2 = 4$

$q(x_0) = 1 + (-2) = -1$

$q(x_1) = 1 + 0 = 1$

$q(x_2) = 1 + 2 = 3$

$$\langle p, q \rangle = p(x_0)q(x_0) + p(x_1)q(x_1) + p(x_2)q(x_2)$$

$$= (4)(-1) + (0)(1) + (4)(3)$$

$$= -4 + 12$$

$$= 8.$$

$$\|p\| = \sqrt{4^2 + 0^2 + 4^2}$$

$$= \sqrt{16 + 16}$$

$$= \sqrt{32} = \sqrt{16 \times 2}$$

$$4\sqrt{2}.$$