6.1 Examples:

Example # 9

$x_0 = -2,\ x_1 = 0,\ x_2 = 2$

$p = x^2,\ q = 1 + x.$

find $\langle p, q\rangle$, $\|p\| = ?$

$p(x_0) = (-2)^2 = 4$

$p(x_1) = (0)^2 = 0$

$p(x_2) = (2)^2 = 4$

$q(x_0) = 1 + (-2) = -1$

$q(x_1) = 1 + 0 = 1$

$q(x_2) = 1 + 2 = 3$

$$\langle p, q\rangle = p(x_0)q(x_0) + p(x_1)q(x_1) + p(x_2)q(x_2)$$

$$= (4)(-1) + (0)(1) + (4)(3)$$

$$= -4 + 12$$

$$= 8.$$

$$\|p\| = \sqrt{4^2 + 0^2 + 4^2}$$

$$= \sqrt{16+16}$$

$$= \sqrt{32} = \sqrt{16\times 2}$$

$$4\sqrt{2}.$$

④ $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix};\ x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Solution: $Ax = \lambda x$

$$Ax = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$Ax = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda = 0$$

So, x is eigan value.

CHAPTER 5.

Ex - 5.1.

①: $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix};\ x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Solution

$Ax = \lambda x$

$$Ax = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$Ax = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So,

$\lambda = -1$

and x is eigan value.

② $A = \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix};\ x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Solution:

$Ax = \lambda x$

$$Ax = \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$Ax = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So,

$\lambda = 4$

and x is eigan value

③ $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix},\ x = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Solution

$A\cancel{x} = \lambda x$

$$Ax = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$Ax = \begin{bmatrix} 5 \\ 10 \\ 5 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$\lambda = 5$

and x is eigan value.

⑤ ⓐ $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

Solution:

$$\lambda I - A = \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda-1 & -4 \\ -2 & \lambda-3 \end{bmatrix} \quad -a)$$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & -4 \\ -2 & \lambda-3 \end{vmatrix}$$

$$\det(\lambda I - A) = (\lambda-1)(\lambda-3) - 8$$

$$'' = \lambda^2 - 3\lambda - \lambda + 3 - 8$$

$$'' = \lambda^2 - 4\lambda - 5$$

$$= \lambda^2 - 5\lambda + \lambda - 5$$

$$= \lambda(\lambda-5) + 1(\lambda-5)$$

$$= (\lambda+1)(\lambda-5)$$

So, $\boxed{\lambda = -1}$; $\boxed{\lambda = 5}$

when $\lambda = -1$: in eq - a)

$$\begin{bmatrix} \lambda-1 & -4 \\ -2 & \lambda-3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -4 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-2x - 4y = 0$

$-2x - 4y = 0$

So,

$-2x - 4y = 0$

$-2(x + 2y) = 0$

$x + 2y = 0$

$x = -2y$

let $(y = t)$ So,

$x = 2t$

$y = t$

basis:

for $\underline{\lambda = -1}$ $\begin{bmatrix} -2t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

When $\lambda = 5$
in eq - a)

$$\begin{bmatrix} \lambda-1 & -4 \\ -2 & \lambda-3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5-1 & -4 \\ -2 & 5-3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -4 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$4x - 4y = 0$
$-2x + 2y = 0$

let:
$4x - 4y = 0$
$4(x-y) = 0$
$x - y = 0$
$x = y$
let $y = t$
So,
$$\boxed{\begin{matrix} x = t \\ y = t \end{matrix}}$$

So, basis for $\lambda = 5$

$$\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t$$

(b) $\begin{bmatrix} -2 & -7 \\ 1 & 2 \end{bmatrix}$

$$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -2 & -7 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda+2 & +7 \\ -1 & \lambda \mp 2 \end{bmatrix} \text{— i)}$$

$$\det(\lambda I - A) = \begin{bmatrix} \lambda-2 & 7 \\ -1 & \lambda+2 \end{bmatrix}$$

$= (\lambda-2)(\lambda+2) + 7$

$= \lambda^2 - 4 + 7$

$= \lambda^2 + 3$

Since $\lambda^2 + 3 = 0$ has no real root we have all that A has no eigan value.

So, although it has two complex solutions:

$\lambda^2 + 3 = 0$

$\lambda^2 = -3$

$\boxed{\lambda = \pm\sqrt{3}}$

So,
when $\lambda = -\sqrt{3}$
in eq - i)

$$\begin{bmatrix} \lambda+2 & 7 \\ -1 & \lambda \mp 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\sqrt{3}+2 & +7 \\ -1 & -\sqrt{3} \mp 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(-\sqrt{3}+2)x + 7y = 0$
$-x + (-\sqrt{3} \mp 2)y = 0$

So,
$-x + (-\sqrt{3} - 2)y = 0$

$x = (-\sqrt{3} - 2)y$

So, $y = t$

$x = (-\sqrt{3} - 2)t$
$y = t$

basis for $\lambda = -\sqrt{3}$

$$\begin{bmatrix} (-\sqrt{3}-2)t \\ t \end{bmatrix} = t\begin{bmatrix} (-\sqrt{3}-2) \\ 1 \end{bmatrix}$$

When $\lambda = \sqrt{3}$
in eq - i)

$$\begin{bmatrix} \lambda+2 & 7 \\ -1 & \lambda-2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{3}+2 & 7 \\ -1 & \sqrt{3}-2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(\sqrt{3}+2)x + 7y = 0$
$-x + (\sqrt{3}-2)y = 0$

So,
$-x + (\sqrt{3}-2)y = 0$

$x = (\sqrt{3}-2)y$

let $y = t$

$\boxed{x = (\sqrt{3}-2)t}$ $y = t$

basis for $\lambda = \sqrt{3}$

$$\begin{bmatrix} (\sqrt{3}-2)\cancel{y} \\ t \end{bmatrix} = t\begin{bmatrix} (\sqrt{3}-2) \\ 1 \end{bmatrix}$$

(c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Solution:

$$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda-1 & 0 \\ 0 & \lambda-1 \end{bmatrix}$$

$$\det(\lambda I - A) = \begin{bmatrix} \lambda-1 & 0 \\ 0 & \lambda-1 \end{bmatrix}$$

$= (\lambda-1)(\lambda-1) + 0$

$= \lambda^2 - \lambda - \lambda + 1$

$= \lambda^2 - 2\lambda + 1$

Thus, only one eigan value of A. As.
$(\lambda-1)^2 = \boxed{\lambda = 1}$.

$$\begin{bmatrix} \lambda-1 & 0 \\ 0 & \lambda-1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus, $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ are the basis of eigan space correspondig to value $\boxed{\lambda = 1}$.

(d) $\cancel{\begin{bmatrix} -2 & -7 \\ 1 & 2 \end{bmatrix}} \to \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

Solution:

$$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} +1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda - 1 & 2 \\ 0 & \lambda - 1 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda-1 & 2 \\ 0 & \lambda-1 \end{bmatrix}$$

$\det = (\lambda-1)(\lambda-1) + 0$
$= \lambda^2 - 2\lambda + 1$

Done same as previous. (c) Part.
Answer will be $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(6) (a) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$= \begin{bmatrix} \lambda-2 & -1 \\ -1 & \lambda-2 \end{bmatrix}$ —(i)

$\det = (\lambda-2)(\lambda-2) - 1$

$= \lambda^2 - 4\lambda + 4 - 1$

$= \lambda^2 - 4\lambda + 3$

$= \lambda^2 - 3\lambda - \lambda + 3$

$= \lambda(\lambda-3) - 1(\lambda-3)$

$(\lambda-1)(\lambda-3)$

$\boxed{\lambda = 1}$; $\boxed{\lambda = 3}$.

when $\lambda = 1$ eq-(i)

$\begin{bmatrix} 1-2 & -1 \\ -1 & 1-2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-x - y = 0$
$-x - y = 0$

So, $-x - y = 0$

$x = -y$

let $y = t$

$x = -t$

$y = t$

So, basis for $\lambda = 1$,

$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

when $\lambda = 3$ in eq-(i)

$\begin{bmatrix} 3-2 & -1 \\ -1 & 3-2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x - y = 0$
$-x + y = 0$

So, $x = y$

$y = t$ So,

So, basis of $\lambda = 3$

$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$

$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$

$= \begin{bmatrix} \lambda-2 & +3 \\ 0 & \lambda-2 \end{bmatrix}$

$\det = (\lambda-2)(\lambda-2) - 0$

$= \lambda^2 - 4\lambda + 4$

$(\lambda-2)^2 = 0$

$\boxed{\lambda = 2}$.

This have only one eigen value

$\begin{bmatrix} 2-2 & 3 \\ 0 & 2-2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$3y = 0$ let $x = 1$, $y = 0$

So; $y = 0$ So, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$= \begin{bmatrix} \lambda-2 & 0 \\ 0 & \lambda-2 \end{bmatrix}$

$\det = (\lambda-2)^2 = \lambda = 2$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

So, let $x = t$, $y = s$

$= \begin{bmatrix} t \\ s \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

basis will be:

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

(d) $\begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$

$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$

$= \begin{bmatrix} \lambda-1 & -2 \\ 2 & \lambda+1 \end{bmatrix}$ —(i)

$\det = (\lambda-1)(\lambda+1) - (-4)$

$= \lambda^2 - 1 + 4$

$= \lambda^2 + 3$

$\lambda^2 + 3 = 0$

$\lambda^2 = -3$

$\lambda = \pm\sqrt{3}$

So, when $\lambda = -\sqrt{3}$ the eq-(i)

$\begin{bmatrix} -\sqrt{3}-1 & -2 \\ 2 & -\sqrt{3}+1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$(-\sqrt{3}-1)x - 2y = 0$

$2x + (-\sqrt{3}+1)y = 0$

So, $2x + (-\sqrt{3}+1)y = 0$

$x = \dfrac{(-1+\sqrt{3})y}{2}$

$y = t$

$x = \dfrac{(-1+\sqrt{3})t}{2}$

$y = t$

So, basis of $-\sqrt{3}$,

$\begin{bmatrix} \frac{(-1+\sqrt{3})t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2}(-1+\sqrt{3}) \\ 1 \end{bmatrix}$

$\lambda = \sqrt{3}$, the eq-(i)

$(\sqrt{3}-1)x - 2y = 0$

$2x + (\sqrt{3}+1)y = 0$

So, $x = \dfrac{(-1-\sqrt{3})y}{2}$

$y = t$

$x = \dfrac{(-1-\sqrt{3})t}{2}$

So basis are

$\begin{bmatrix} \frac{(-1-\sqrt{3})t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2}(-1-\sqrt{3}) \\ 1 \end{bmatrix}$

Ans.

taking eq-i) and iii) | x_1 - $-x_2$ x_3

(7) $\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

So,

$$\lambda I - A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda-4 & 0 & -1 \\ 2 & \lambda-1 & 0 \\ 2 & 0 & \lambda-1 \end{bmatrix} \text{ — a)}$$

$= (\lambda-4)(\lambda-1)(\lambda-1) + 1(2(\lambda-1))$

$= (\lambda-4)(\lambda^2-\lambda-\lambda+1) + 2\lambda \mp 2$

$= \lambda^3 - 2\lambda^2 + \lambda - 4\lambda^2 + 8\lambda - 4 + 2\lambda \mp 2$

$= \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$ — Characteristics of equation.

So, let $x = 1, 2, 3$. (can be calculated by calculator)

when $\boxed{\lambda = 1}$ in eq-a)

$$\begin{bmatrix} -3 & 0 & -1 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-3x + 0y - z = 0$ — i)

$2x + 0y + 0z = 0$ — ii)

$2x + 0y + 0z = 0$ — iii)

So, from eq-i) and ii)

$$\frac{x}{\begin{vmatrix} 0 & -1 \\ 0 & 0 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -3 & -1 \\ 2 & 0 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -3 & 0 \\ 2 & 0 \end{vmatrix}}$$

$$\frac{x}{0} = \frac{-y}{2} = \frac{z}{0}$$

(x2)

$$\frac{x}{0} = \frac{-y}{1} = \frac{z}{0}$$

So, basis of $\lambda = 1$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

So, when $\lambda = 2$ in eq-a)

$$\begin{bmatrix} \lambda-4 & 0 & -1 \\ 2 & \lambda-1 & 0 \\ 2 & 0 & \lambda-1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & -1 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x + 0y - z = 0$ — i)

$2x + y + 0z = 0$ — ii)

$2x + 0y + z = 0$

Taking eq-i) and ii)

$$\frac{x}{\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -2 & -1 \\ 2 & 0 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & 0 \\ 2 & 1 \end{vmatrix}}$$

$$\frac{x}{1} = \frac{-y}{2} = \frac{z}{-2}$$

(x2)

$$\frac{x}{1/2} = \frac{y}{-1} = \frac{z}{-1}$$

eq-i)

$-2x - z = 0$

$x = -\frac{1}{2} z$

$\boxed{z = t}$

$\boxed{x = -\frac{1}{2} t}$

$2(-\frac{1}{2} t) + y = 0$

$-t + y = 0$

$\boxed{y = t}$

So, basis are

$$\begin{bmatrix} -\frac{1}{2} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

when $\boxed{\lambda = 3}$

$$\begin{bmatrix} \lambda-4 & 0 & -1 \\ 2 & \lambda-1 & 0 \\ 2 & 0 & \lambda-1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & -1 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-1x + 0y - 1z = 0$

$2x + 2y + 0z = 0$

$2x + 0y + 2z = 0$

let,

$-1x - 1z = 0$

$x = -z$

$\boxed{z = t}$.

$\boxed{x = -t}$.

So,

$2x + 2y = 0$

$2(-t) + 2y = 0$

$-t + y = 0$

$\boxed{y = t}$.

So, basis are

$$\begin{bmatrix} -t \\ t \\ t \end{bmatrix} = \boxed{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} t}$$

(8) $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix}$

$$|\lambda I - A| = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda-1 & 0 & 2 \\ 0 & \lambda & 0 \\ 2 & 0 & \lambda-4 \end{bmatrix}$$

$= (\lambda-1)(\lambda)(\lambda-4) \mp 2(2\lambda)$

$= (\lambda-1)(\lambda^2 - 4\lambda) \mp 4\lambda$

$\lambda^3 - 4\lambda^2 - \lambda^2 + \cancel{4\lambda} \mp \cancel{4\lambda}$

$\lambda^3 - 5\lambda^2 + \cancel{8\lambda}$

$\boxed{\lambda^3 - 5\lambda^2 + \cancel{8\lambda} = 0}$

~~$x_1 = 0,\ x_2 = \frac{5}{2} - \frac{i}{2}\sqrt{7}$~~

~~$x_3 = \frac{5}{2} + \frac{i}{2}\sqrt{7}$~~

~~$x_1 = 0,\ x_2 = 5,$~~

$\lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = 5$

when $\lambda = 0$,

$$\begin{bmatrix} \lambda-1 & 0 & 2 \\ 0 & \lambda & 0 \\ 2 & 0 & \lambda-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1 + 0x_2 + 2x_3 = 0$

$0x_1 + 0x_2 + 0x_3 = 0$

$2x_1 + 0x_2 - 4x_3 = 0$

Examples: (5.1)

Example # 1:

$x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$

$\lambda = ?$

$Ax = \lambda A$

$Ax = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$Ax = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

$Ax = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$Ax = 3x$

So, $\boxed{\lambda = 3}$.

Example #2:

$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$

$|\lambda I - A| = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$

$= \begin{bmatrix} \lambda - 3 & 0 \\ -8 & \lambda + 1 \end{bmatrix}$

$(\lambda - 3)(x + 1) + 0$

$\boxed{\lambda = 3}$, $\boxed{\lambda = -1}$.

Example # 3:

$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$

$|\lambda I - A| = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$

$= \begin{bmatrix} \lambda - 0 & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda - 8 \end{bmatrix}$

$= \cancel{\lambda(\lambda)(\lambda - 8) + 1(4)}$

$= \cancel{\lambda(\lambda^2 - 8\lambda) + 4}$

$= \cancel{\lambda^3 - 8\lambda^2}$

$\lambda^3 - P\lambda^2 + Q\lambda - |A| = 0$

$\boxed{P = 8}$ $Q = \begin{vmatrix} 0 & 1 \\ -17 & 8 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 4 & 8 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$

$Q = 17 + 0 + 0 \Rightarrow \boxed{Q = 17}$.

$\boxed{|A| = 4}$

So,

$\boxed{\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0}$

$\boxed{\lambda_1 = 4}$ $\boxed{\lambda_2 = 2 - \sqrt{3}}$ $\boxed{\lambda_3 = 2 + \sqrt{3}}$

Example #6

$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$

$|\lambda I - A| = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$

$= \begin{bmatrix} \lambda + 1 & -3 \\ -2 & \lambda - 0 \end{bmatrix}$

$\det(\lambda I - A) = (\lambda + 1)(\lambda) - 6$

$= \lambda^2 + \lambda - 6$

$= \lambda^2 + 3\lambda - 2\lambda - 6$

$= \lambda(\lambda + 3) - 2(\lambda + 3)$

$(\lambda + 3)(\lambda - 2)$

$\boxed{\lambda = -3}$ $\boxed{\lambda = 2}$.

if $\lambda = 2 \Rightarrow$

$\begin{bmatrix} \lambda + 1 & -3 \\ -2 & \lambda - 0 \end{bmatrix}$

$\begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$3x_1 - 3x_2 = 0$

$-2x_1 + 2x_2 = 0$

let $3x_1 - 3x_2 = 0$

$x_1 - x_2 = 0$

$x_1 = x_2$

$\boxed{x_2 = t}$

$\boxed{x_1 = t}$.

So, $\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t$

if $\boxed{\lambda = -3}$

$\begin{bmatrix} -2 & -3 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-2x_1 - 3x_2 = 0$

$-2x_1 - 3x_2 = 0$

$-2x_1 - 3x_2 = 0$

$+2x_1 = -3x_2$

$x_1 = -\frac{3}{2}x_2$

$\boxed{x_2 = t}$

$\boxed{x_1 = -\frac{3}{2}t}$

So, $\begin{bmatrix} -3/2\, t \\ t \end{bmatrix} = t\begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$

Example # 7

$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

$|\lambda I - A| = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

$= \begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda - 2 & -1 \\ -1 & 0 & \lambda - 3 \end{bmatrix}$

$\det(\lambda I - A) = \lambda^3 \mp P\lambda^2 + Q\lambda \mp |A| = 0$

$\boxed{P = 5}$, $Q = \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 0 & -2 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix}$

$Q = 6 + 2 + 0 \Rightarrow \boxed{Q = 8}$.

$\boxed{|A| = 4}$.

So,

$\lambda^3 - P\lambda^2 + Q\lambda - |A| = 0$

$\boxed{\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0}$

eigen value,

$\boxed{\lambda_1 = 1}$, $\boxed{\lambda_2 = 2}$,

when $\boxed{\lambda_1 = 1}$.

$\begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$1x_1 + 0x_2 + 2x_3 = 0$

$-1x_1 - 1x_2 - 1x_3 = 0$

$-1x_1 + 0x_2 - 2x_3 = 0$

$1x_1 + 0x_2 + 2x_3 = 0$
$-1x_1 - 1x_2 - 1x_3 = 0$
$-1x_1 + 0x_2 - 2x_3 = 0$

eq - i)

$x_1 + 2x_3 = 0$

$x_1 = -2x_3$

$\boxed{x_3 = t}$

$\boxed{x_1 = -2t}$

in eq - ii)

$-x_1 - x_2 - x_3 = 0$

$-(-2t) - x_2 - t = 0$

$2t - t - x_2 = 0$

$\boxed{x_2 = +t}$

So,
$$\begin{bmatrix} -2t \\ +t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ +1 \\ 1 \end{bmatrix}$$

OR: 2nd method:

taking eq - i) and ii)

$$\frac{x_1}{\begin{vmatrix} 0 & 2 \\ -1 & -1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 2 \\ -1 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 0 \\ -1 & -1 \end{vmatrix}}$$

$$\frac{x_1}{2} = \frac{-x_2}{1} = \frac{x_3}{-1}$$

So,
$$\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = -\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

when $\boxed{\lambda = 2}$

$$\begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x_1 + 0x_2 + 2x_3 = 0$
$-1x_1 + 0x_2 - 1x_3 = 0$
$-1x_1 + 0x_2 - 1x_3 = 0$

taking eq - i)

$2x_1 + 2x_3 = 0$

$x_1 + x_3 = 0$

$x_1 = -x_3$

$\boxed{x_3 = t}$.

$\boxed{x_1 = -t}$.

So, $\boxed{x_2 = s}$.

As we know,

$$\begin{bmatrix} -t \\ s \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \cdot s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

OR eq - i) iii)

$$\frac{x_1}{\begin{vmatrix} 0 & 2 \\ 0 & -1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 2 & 2 \\ -1 & +1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 2 & 0 \\ -1 & 0 \end{vmatrix}}$$

$$\frac{x_1}{0} = \frac{-x_2}{0} = \frac{x_3}{0}$$

So, It is clear that 2nd method does not work if any one value of x is completely zero.

Exercise. 5.2:

5: $A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$

$$|\lambda I - A| = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda - 1 & 0 \\ -6 & \lambda + 1 \end{bmatrix}$$

$\lambda I - A/x = 0$

$$\begin{bmatrix} \lambda - 1 & 0 \\ -6 & \lambda + 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

As we know,
$\boxed{\lambda = 1, \lambda = -1}$.

So,
when $\lambda = 1$

$$\begin{bmatrix} 0 & 0 \\ -6 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$0x_1 + 0x_2 = 0$

$-6x_1 + 2x_2 = 0$

$-3x_1 + x_2 = 0$

$x_2 = 3x_1$

$\boxed{x_1 = t}$

$\boxed{x_2 = 3t}$.

So,
$$\begin{bmatrix} t \\ 3t \end{bmatrix} = t\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

when $\lambda = -1$

$$\begin{bmatrix} -2 & 0 \\ -6 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-2x_1 + 0x_2 = 0$
$-6x_1 + 0x_2 = 0$

$\boxed{x_2 = t}$.

$\boxed{x_1 = 0}$.

So,
$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$P = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$, $P^{-1} = \frac{1}{|P|} \text{Adj} P$

$$P^{-1} = \frac{1}{|P|}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

So, let
$PD = AP$

$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}$$

Hence proved.

⑥ $A = \begin{bmatrix} -14 & 12 \\ -20 & 17 \end{bmatrix}$

$$|\lambda I - A| = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -14 & 12 \\ -20 & 17 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda+14 & -12 \\ 20 & \lambda-17 \end{bmatrix}$$

$$\Rightarrow |\lambda I - A| x = 0$$

$$\begin{bmatrix} \lambda+14 & -12 \\ 20 & \lambda-17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

As we know:

$$= (\lambda+14)(\lambda-17) + 240$$

$$= \lambda^2 - 17\lambda + 14\lambda - 238 + 240$$

$$= \lambda^2 - 3\lambda + 2 = 0$$

$$\lambda^2 - 2\lambda - \lambda + 2 = 0$$

$$(\lambda-1)(\lambda-2) = 0$$

$\boxed{\lambda = 1}$; $\boxed{\lambda = 2}$.

when $(\lambda = 1) \Rightarrow$

$$\begin{bmatrix} 15 & -12 \\ 20 & -16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$15x_1 - 12x_2' = 0$$

$$20x_1 - 16x_2' = 0$$

$$15x_1 - 12x_2 = 0$$

$$5x_1 - 4x_2 = 0$$

$$5x_1 = 4x_2$$

$$x_1 = \frac{4}{5} x_2$$

$$\boxed{x_2 = t}$$

$$\boxed{x_1 = \frac{4}{5} t}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4t \\ 5t \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} t$$

when $\lambda = 2$

$$\begin{bmatrix} 16 & -12 \\ 20 & -15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$16x_1 - 12x_2 = 0$$

$$20x_1 - 15x_2 = 0$$

$$16x_1 - 12x_2 = 0$$

$$4(4x_1 - 3x_2) = 0$$

$$4x_1 - 3x_2 = 0$$

$$4x_1 = 3x_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$D = P^{-1}AP$$

$$P = \begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix}$$

$$P^{-1} = \frac{1}{|P|} \text{Adj } P$$

$$|P| = 16 - 15 = 1$$

$$P^{-1} = \frac{1}{1} \begin{bmatrix} 4 & -3 \\ -5 & 4 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 4 & -3 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} -14 & 12 \\ -20 & 17 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -3 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} 4 & 6 \\ 5 & 8 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$AP = DP$$

$$\begin{bmatrix} -14 & 12 \\ -20 & 17 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 6 \\ 5 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 5 & 8 \end{bmatrix}$$

Hence proved.

⑦