Multistability and Transient Dynamics on Networked Systems

Der Fakultät für Mathematik & Naturwissenschaften
der Carl von Ossietzky Universität Oldenburg
zur Erlangung des Grades und Titels
Doktor der Naturwissenschaften (Dr. rer. nat.)
angenommene Dissertation

von

Kalel Luiz Rossi

geboren am 13.10.1997 in Curitiba, Parana, Brasilien

Gutachter: Prof. Dr. Ulrike Feudel

Gutachter: Prof. Dr. Klaus Lehnertz

Tag der Disputation: 15.01.2024

Contents

List of Publications

This dissertation is based on the following publications:

Chapter 3: Kalel L. Rossi, Roberto C. Budzinski, Bruno R. R. Boaretto, Lyle E. Muller, and Ulrike Feudel. Small changes at single nodes can shift global network dynamics. *Physical Review Research* 5, 013220 (2023).

Chapter 4: Kalel L. Rossi, Everton S. Medeiros, Peter Ashwin and Ulrike Feudel. Transients versus network interactions.

Chapter 5: Kalel L. Rossi, Roberto C. Budzinski, Bruno R. R. Boaretto, Lyle E. Muller, and Ulrike Feudel. Dynamical properties and mechanisms of metastability: a perspective in neuroscience.

On top of these main thesis papers, I have also collaborated in other works, which resulted in three further publications, with me as a co-author.

- George Datseris, Kalel L. Rossi, and Alexandre Wagemakers. Framework for global stability analysis of dynamical systems. *Chaos* 33, 073151 (2023).
- Bruno R. R. Boaretto, Roberto C. Budzinski, Kalel L. Rossi, Thiago L. Prado, Sergio R. Lopes and Cristina Masoller. Temporal Correlations in Time Series Using Permutation Entropy, Ordinal Probabilities and Machine Learning. *Entropy* 23, 1025 (2021).
- Bruno R.R. Boaretto, Roberto C. Budzinski, Kalel L. Rossi, Cristina Masoller, Elbert E.N. Macau. Spatial permutation entropy distinguishes resting brain states. *Chaos, Solitons and Fractals* 171, 113453 (2023).

Abstract

All work and no play makes Jack a dull boy. All work and no play makes Jack a dull boy.

Zusammenfassung

Arbeiten ohne Vergnügen macht Jack zu einem langweiligen Jungen Arbeiten ohne Vergnügen macht Jack zu einem langweiligen Jungen

Chapter 1

Introduction

1.1 Networks

Several natural and artificial systems are composed of separate entities that interact together, forming networks - or, at least, they can be approximately modelled as networks. Often, these interactions generate complex behaviors, which would not exist without the interactions. For instance, neural circuits blabla.

A major area of research today is to understand precisely how this large-scale complex dynamics emerges from the interactions between units in networks. This ranges from setting up experiments XX, modelling to a high precision, and also building the basic theory that aims to describe the fundamental aspects of these networks. For this thesis we have focused on the latter case, aiming to study the fundamental behavior of simple, non-specific, networks. To do this, we have relied on a second layer of abstraction: often, networks can be modelled as dynamical systems following ODEs of the general form:

$$\dot{x}_i = f(x_i) + g(x) \tag{1.1}$$

where XX. Introduce topology, connections. Examples? power grids and neural networks and kuramoto?

This abstraction is quite helpful, because systems of this form can be studied using techniques from dynamical systems theory. XX?

Among the plethora of important dynamics arising in networks, we have in this thesis focused on three particular behaviors: malleability, multistability and metastability. Although separate, they are intrisically related, as we will see. For the first two, we have focused on how they are controlled by the network's topology.

The first behavior we have studied is *dynamical malleability*, which refers to the capacity of a network to change its dynamics when the individual parameters of units or connections are changed. We studied this behavior in Kuramoto oscillator networks of the form

$$\dot{\theta}... \tag{1.2}$$

These networks serve as paradigmatic models to understand emergent behavior - in particular, synchronization - in complex networks. They can be derived as an approximation for generic coupled limit cycle oscillators under weak coupling []. Although it does not model a particular real-world system, it has been used as a simple model for large-scale brain networks [] and power grids [] ?. Originally used to describe chemical oscillations XX.

1.2 Multistability

1.3 Metastability

usefulness for computations?

Chapter 2

Methodology

2.1 Some fundamental aspects of dynamical systems theory

2.1.1 Our dynamical systems and the uniqueness and existence of their solutions

In this thesis we study dynamical systems described by a state variable $x = (x_1, x_2, \ldots, x_n)^T \in M$, where $M \subseteq \mathbb{R}^n$ is the state space, and T denotes the transpose operation. The state variable is a point in this n-dimensional state space. In a continuous-time dynamical system, the state evolves according to the equation:

$$\dot{x}(t) = f(x(t)) \tag{2.1}$$

where $f : M \to M$. Systems obeying Eq. 2.1 are deterministic: there is no randomness, no stochasticity, no noise. This means that, starting from one single state at time t, we can in principle describe the whole past and future evolution of the system. Furthermore, there is a lack of an explicit time dependence in f - i.e., $\partial f_i/\partial t = 0$ for $i = 1, \ldots, n$. In this case, the dynamical system is said to be autonomous.

To obtain solutions to system 2.1 we need to provide one state, which we typically call an initial condition $x_0 = x(0) \in \mathbb{R}^n$. The combination of $\dot{x} = f(x)$ with $x(0) = x_0$ defines an initial value problem. A fundamental theorem makes our lives studying this problem much easier. This is the theorem of existence and uniqueness of solutions. For $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$, it requires that f is continuous and that all of its partial derivatives $\frac{\partial f_i}{\partial x_j}$, for $i, j = 1 \ldots n$ are continuous in some open connected set $D \subset \mathbb{R}^n$. This basically means that it requires our function f to be sufficiently smooth. Then, for initial conditions $x_0 \in D$, the initial value problem has a solution $x(t)$ on some time interval $(-\tau, \tau)$ about $t = 0$, and the solution is unique! [13]

In state space, each solution describes a trajectory, a path, that goes through its initial condition x_0. The uniqueness of solutions implies that, within this time interval $(-\tau, \tau)$, different trajectories do not intersect in state space. This is a crucial property underlying all systems we study.

A useful notation for the evolution of a continuous dynamical system is through the evolution operator $\Phi^t(x)$, which, informally defined, evolves the point x forward t time units. That is, $\Phi^t(x(0)) = x(t)$.

2.1.2 The fate of linear dynamical systems

Although trajectories do not cross, they can share the same fate, meaning they can converge to the region in state space. We can introduce this notion with a very simple mathematical example of a linear system. It has the form

$$\dot{x}(t) = Ax(t) \tag{2.2}$$

where A is a constant $(n \times n)$ matrix.

If the eigenvalues $\lambda_i \in \mathbb{C}$ of A are all unique, its eigenvectors $v_i \in \mathbb{R}^n$ are linearly independent. Then, the general solution to this system can be written as Ref. [13]:

$$x(t) = \sum_{i=1}^{n} C_i e^{\lambda_i t} v_i. \tag{2.3}$$

Then, each initial condition determines the constant coefficients $C_i \in \mathbb{R}$. From Eq. 2.3 we can already notice that the origin of the system, $o = (0, \ldots, 0)^T$, is a solution. In fact, it is an equilibrium: $\dot{x} = f(o) = 0$. A trajectory on the origin does not change over time.

As we see from Eq. 2.3, the behavior of trajectories depends on the eigenvalues λ_i of the matrix A. We can classify the equilibrium at the origin based on these eigenvalues, as shown in Fig. 2.1. If the real parts of all the eigenvalues are negative, then all trajectories in state space converge to the origin as $t \to \infty$. In this case, the origin is said to be a stable equilibrium (Figs. 2.1A-B). If at least one eigenvalue is negative, the trajectories diverge from the origin, which is then an unstable equilibrium (Figs. 2.1C-E). Stability here refers to the behavior of trajectories near the equilibrium. If it stable, nearby trajectories converge to the equilibrium - or, equivalently, small perturbations that take a trajectory away from the equilibrium will eventually go back to the equilibrium. If it is unstable, then nearby trajectories diverge from it.

Stable equilibria are the only attracting solution, or attractor, of linear systems. In this case, although different trajectories cannot not intersect, they all converge to the origin as $t \to \infty$. In summary, the ultimate fate of linear systems is kind of boring: either trajectories end up at the origin or they diverge off to infinity. But the journey, the path that trajectories take before before the end, the *transient dynamics*, is more interesting. As shown in Fig. 2.1, this is dictated by the constellation of eigenvalues λ_i. For more details, the reader can refer to standard books on linear/nonlinear dynamics, such as Ref. [13].

2.1.3 The fate of nonlinear dynamical systems I: attractors

As just seen, stable equilibria are the only possible attractors in linear systems. Going beyond Eq.2.2, nonlinear systems can have more interesting and complicated long-term dynamics (Fig. 2.2). Stable equilibria are still possible, as shown in Figs.2.2A-B. The system here is a conductance-based neuronal model following equations [7]

$$\begin{aligned}
\dot{x} &= \big(I - g_L(x_i - E_L) - g_{Na} m_\infty(x_i)(x_i - E_{Na}) - g_K y_i (x_i - E_K)\big)/C, \\
\dot{y} &= (n_\infty(x) - y_i)/\tau,
\end{aligned} \tag{2.4}$$

with all parameters and functions defined in detail in Chapter **??**. The input current I is chosen to be $I = 2.0$ so the system has excitable dynamics. Its state space is composed of a stable equilibrium, the only attractor, and two unstable equilibria, which create excitable dynamics. Excitability is a type of transient different than seen for linear systems. Some trajectories are forced to go on long excursions (excitations) before converging to the stable equilibrium. We study more about this again in Chapter **??**.

Besides equilibria, nonlinear systems can also have periodic solutions, also called limit cycles. They vary in time with a certain period T (Fig. 2.2C) and correspond to closed curves in state space (Fig. 2.2). The system used in this example is still the neuronal model of Eq.2.4, but with a different parameter $I = 6$, which leads to the system now having a stable limit cycle. We see in this figure again an example of a long transient, with the trajectory initially going on a long excursion before converging to the limit cycle.

Not all curves in state space are closed, however. One can have quasiperiodic dynamics, in which trajectories never repeat exactly, although they might almost repeat. This is seen in Figs. 2.2E-F. Simulating the trajectory for longer times would fill up the figure more and more. Further, note the varying amplitude of the time series. The system in this example is the forced

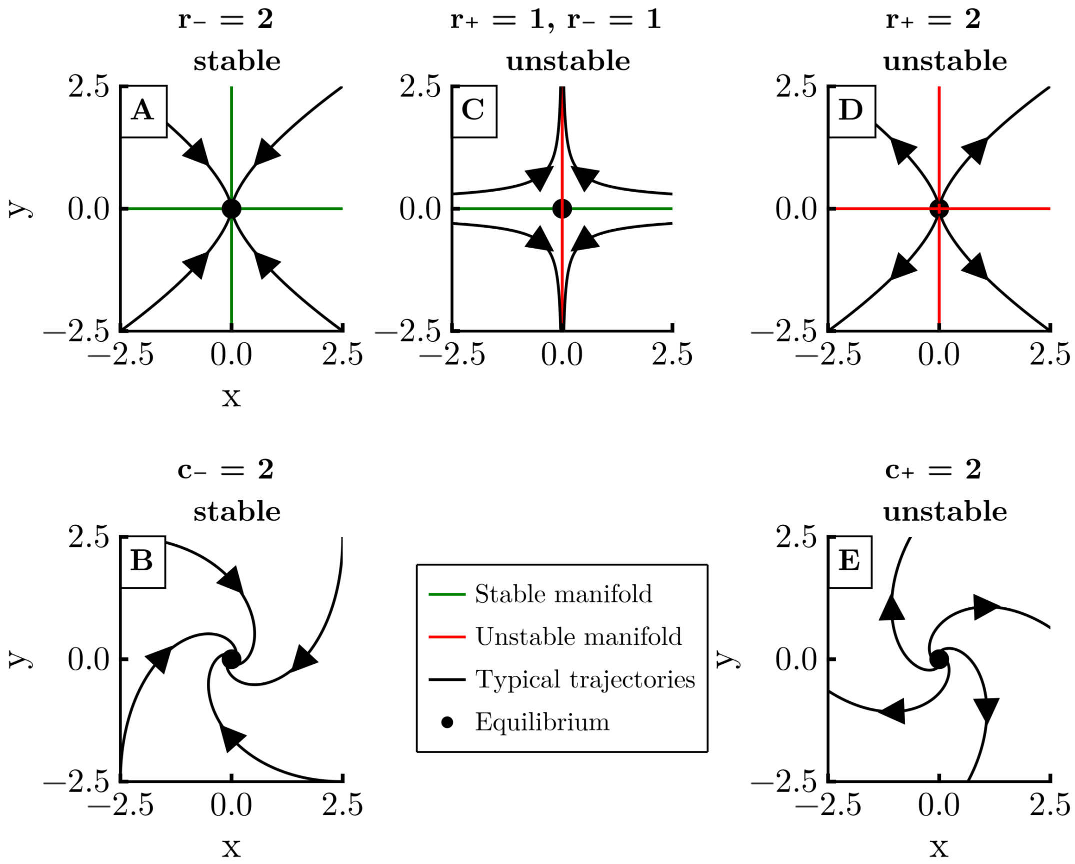

Figure 2.1: Hyperbolic equilibria in 2D linear systems. The title specifies the number of eigenvalues that are purely real negative r_- or positive r_+, or that are complex with real part negative c_- or positive c_+. The first row shows equilibria whose eigenvalues are purely real, while the second one shows equilibria with complex eigenvalues. In the first column, the equilibria are stable - they are the two possible attractors in linear systems. In the second column, we have a saddle-point for purely real eigenvalues. In the third column, the equilibria are completely unstable, known as repellers.

Van der Pol oscillator,

$$\dot{x} = v \tag{2.5}$$
$$\dot{v} = \mu(1 - x^2)v - \alpha x + g\cos(\omega_f t), , \tag{2.6}$$

with parameters $\mu = 0.1$, $\alpha = 1.0$, $g = 0.5$, $\omega_f = \sqrt{3}$ taken from Ref.[11].

Finally, one can also have chaotic attractors (Figs.2.2G-H). These solutions have a wild behavior that nearby trajectories tend to diverge at an exponential rate []. Despite this local divergence, however, the solutions remain bounded in space. In other words, systems with chaotic attractors are very sensitive to the initial conditions - small changes in initial conditions lead to trajectories that can look very different. The system used to generate is shown as the Lorenz system, with equations

$$\dot{x} = \sigma.(y - x) \tag{2.7}$$
$$\dot{y} = x(\rho - z) - y \tag{2.8}$$
$$\dot{z} = x * y - \beta * z, \tag{2.9}$$

and $\sigma = 10$, $\rho = 28$, $\beta = 8/3$. This chaotic attractor in particular has a shape that resembles a butterfly, with trajectories spending some time on one wing before switching to the other wing [1].

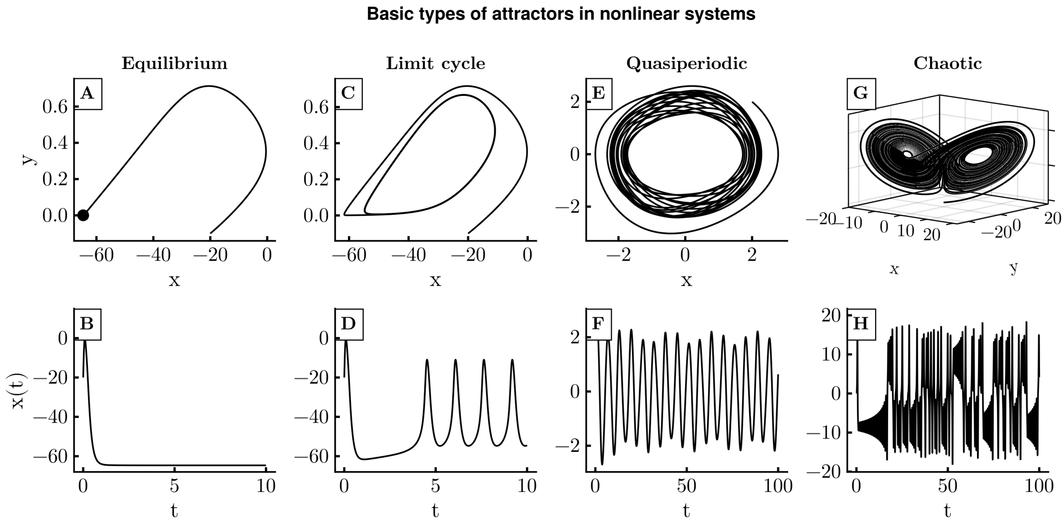

Figure 2.2: Basic types of attractors in nonlinear dynamical systems. Each column shows respectively the state space and a time-series of a typical trajectory converging to a type of attractor. The first column corresponds to the neuronal model of Eq.2.4 with $I = 2.0$, which has excitable dynamics, converging to a stable equilibrium. The second column shows again the neuronal system of Eq.2.4 but with $I = 6.0$, when the attractor is now a stable limit cycle. The third column shows the system defined in Eqs.2.6, with a quasiperiodic attractor Finally, column four has an example of a chaotic trajectory on the Lorenz system (Eq. 2.9).

Given now these examples, let us now define the terms we have used a bit more properly.

2.1.4 Formalizing attractors and basins

We have just presented examples of attractors, sets of points in state space to which trajectories eventually converge, and their basins of attraction, the regions containing those converging trajectories. Since in this thesis we will deal a lot with these concepts, we provide now an attempt at formalizing. The idea is to have the concepts clear in mind for later. In practice, we will only use the informal definition we just gave. In particular, the definition of attractor can vary considerably in the literature. Without attempting to claim any superiority, we attempt here to provide a definition that suits our studies.

First, we define an omega limit set $w(x)$ of a point $x_0 \in M$ as [9]:

$$\omega(x_0) = \{x : \forall T \, \forall \epsilon > 0 \text{ there exists } t > T \text{ such that } |f(x_0, t) - x| < \epsilon\}. \tag{2.10}$$

Consider a point $x \in \omega(x_0)$ in the ω limit set of x_0. Then, by definition, a trajectory that passes through x_0 comes arbitrarily close to x infinitely often as t increases.

From this, we can define the *basin of attraction* of a set A as $\mathcal{B}(A) = \{x \in M : \omega(x) \subset A\}$. This only looks at the long-term behavior of trajectories; the transient dynamics could be anything, including the case that trajectories go very far from A, as long as they go back to it and stay there eventually.

Now to define an attractor, we first define a weaker (or, on the more optimistic side, a more general) version, called the *Milnor attractor*. It is a useful concept when dealing with metastability. A set A is a Milnor attractor if:

1. Its basin of attraction $\mathcal{B}(A)$ has strictly positive measure (i.e., if $m(\mathcal{B}(A)) > 0$), where $m(S)$ denotes a measure equivalent to the Lebesgue measure of set S [9]. This condition says that there is some probability that a randomly chosen point will be attracted to A [9].

2. For any closed proper subset $A' \subset A$, the set difference $\mathcal{B}(A) \setminus \mathcal{B}(A')$ also has strictly positive measure. This ensures that every part of A plays an essential role - one cannot decompose A into an attracting part and another part that does not attract [9, 14]. A closed set here means that it contains all its limit points. And proper means its non-empty.

Furthermore, the Milnor attractor does not have to attract all the points in its neighborhood, and there can also be orbits that transiently go very far from the attractor, even if initially close, before eventually getting close to it. Further, it can in principle be composed into the union of two smaller Milnor attractors. To avoid these cases, we call a set A an *attractor* if

1. A is a Milnor attractor.

2. A contains an orbit that is dense in A. Basically, this means that the there is an orbit in A that passes arbitrarily close to every point in A. This condition ensures that the attractor is not the union of two smaller attracting sets [14].

3. There are arbitrarily small neighborhoods U of A such that $\forall x \in U$ one has $\Phi^t(x) \subset U \; \forall t > 0$ and such that $\forall y \in U$ one has $\omega(y) \subset \omega(x)$. That is, there are arbitrarily small neighborhoods around the attractor in which points inside stay inside and converge to A. This criterion is given in Ref. [2].

2.1.5 Invariant manifolds: structures that organize state space

In Sec. 2.1.2 we only considered the case when all the eigenvalues of the matrix A in the linear system $\dot{x} = Ax$ were positive. If one eigenvalue λ_k is positive, then trajectories will diverge to infinity following the corresponding eigenvector v_k. When some eigenvalues are positive, and some are negative, the origin is a saddle-point. If all eigenvalues are positive, it is called a repeller. Figure 2.1 shows examples of equilibria in 2D linear systems. Note that typical trajectories approach the saddle-point along the y-axis and then diverge along the x-axis. That is, for $t \to -\infty$, trajectories converge to the y-axis and for $t \to \infty$ they converge to the x-axis. The y-axis is called the stable manifold $\mathbb{W}^s(o)$ of the origin o and the x-axis is the unstable manifold $\mathbb{W}^u(o)$ of the origin. We can define these manifolds

$$\mathbb{W}^s(o) = \{x \in M : \Phi^t(x) \to o \text{ as } t \to \infty\}, \quad \mathbb{W}^u(o) = \{x \in M : \Phi^t(x) \to o \text{ as } t \to -\infty\}. \tag{2.11}$$

Let us separate the eigenvectors v_i into two parts: the ones with negative eigenvalues $v_1^-, \ldots, v_{n_s}^-$ and the ones with positive eigenvalues $v_1^+, \ldots, v_{n_u}^+$. Then we can define the stable and unstable subspaces, respectively, as

$$\mathbb{E}^s = \text{span}(v_1^-, \ldots, v_{n_s}^-) \qquad \mathbb{E}^u = \text{span}(v_1^+, \ldots, v_{n_u}^+) \tag{2.12}$$

For a linear system, the stable manifold of the origin coincides with the stable space $\mathbb{E}^s$ and the unstable manifold coincides with the unstable space. In general, as in the example of the saddle-point, these manifolds act to organize the behavior of trajectories in state space.

These concepts can be extended for nonlinear systems. To do this, the first step is to think about the linearization of the nonlinear system. Suppose our nonlinear system of interest has an equilibrium $x^\star \in M$. It turns out that the behavior sufficiently close to this equilibrium is linear, despite the system globally being nonlinear [8, 6]! To see this, we first move the origin of our system to $x^\star$ by defining a new variable $y(t) = x(t) - x^\star$. Then,

$$\dot{y} = \dot{x} = f(y + x^\star) \equiv g(y) \tag{2.13}$$

where we define a convenience function $g(y)$. Expanding $g(y)$ around $y = 0$ (i.e., around the equilibrium $x(t) = x^\star$) gives us

$$\dot{y} = g(0) + J_g(0)y + \mathcal{O}(y^2), \tag{2.14}$$

where $J_g(y) = \frac{\partial g_i(y)}{\partial y_j}$ is the Jacobian of g. It is related to the Jacobian of f by $J_g(y) = J_f(x)$, so $J_g(y = 0) = J_f(x = x^\star)$. Since $g(0) = f(x^\star) = 0$, then if we are sufficiently close to the origin we can also ignore the terms $\mathcal{O}(y^2)$ and therefore we get

$$\dot{y} = J_g(0)y. \tag{2.15}$$

That is, the behavior of the nonlinear system sufficiently close to the equilibrium is linear, with the constant matrix function being the Jacobian evaluated at the equilibrium!

But the good news don't stop here! There is the Hartman-Grobman theorem, which basically shows that the state space near a hyperbolic equilibrium to the state space of the linearization. An equilibrium is hyperbolic if the eigenvalues of the Jacobian evaluated on it are all nonzero, i.e., if $\lambda_i \neq 0 \forall i = 1, \ldots, n$. *Topologically equivalent* means that the linearized state space and the local state space near the equilibrium are distorted versions of each other. They can be bended and warped, but not ripped. In particular, closed orbits have to remain closed, and connections between saddle points have to remain [13]. Mathematically, topologically equivalent means there is a *homeomorphism* (continuous deformation with continuous inverse) from one state space into the other; trajectories can be mapped from one to the other, and the direction of time is the same [13].

Stating the theorem more formally, suppose a hyperbolic equilibrium $x^\star \in M$ such that $f(x^\star) = 0$ and such that all its eigenvalues are nonzero. Then, there is a neighborhood N of $x^\star$ and a homeomorphism $h : N \to M$ such that [1]

- $h(x^\star) = 0$
- the flow $\dot{x} = f(x)$ in N is topologically conjugate to the flow of the linearization $\dot{y} = Ay$ by the continuous map $y = h(x)$. Topologically conjugate basically meaning a change of coordinates in a topological sense.

This guarantees that the stability of the equilibrium is the same in both cases, so we can use the linearization to gain important insights about the stability of equilibria in the nonlinear system!

What about the stable and unstable manifolds? In analogy to the linear case, we can define local stable and unstable sets near a neighborhood U of an equilibrium $x^\star$ for the nonlinear system [1]:

$$\mathbb{W}^s_{\text{loc}}(x^\star) = \{x \in M : \Phi^t(x) \to o \text{ as } t \to +\infty \text{ and } \Phi^t(x) \in U\ \forall t \geq 0\}, \tag{2.16}$$

$$\mathbb{W}^u_{\text{loc}}(x^\star) = \{x \in M : \Phi^t(x) \to o \text{ as } t \to -\infty \text{ and } \Phi^t(x) \in U\ \forall t \leq 0\}. \tag{2.17}$$

Herein comes the stable manifold theorem. It states that, for a hyperbolic equilibrium $x^\star$:

- The local stable set $\mathbb{W}^s_{\text{loc}}(x^\star)$ is a smooth manifold whose tangent space has the same dimension n_s as the stable space $\mathbb{E}^s$ of the linearization of f at $x^\star$. $\mathbb{W}^s_{\text{loc}}(x^\star)$ is also tangent to $\mathbb{E}^s$ at $x^\star$.
- The local unstable set $\mathbb{W}^u_{\text{loc}}(x^\star)$ is a smooth manifold whose tangent space has the same dimension n_u as the unstable space $\mathbb{E}^u$ of the linearization of f at $x^\star$. $\mathbb{W}^u_{\text{loc}}(x^\star)$ is also tangent to $\mathbb{E}^u$ at $x^\star$.

The homeomorphism guaranteed by the Hartman-Grobman theorem maps $\mathbb{W}^s_{\text{loc}}(x^\star)$ into $\mathbb{E}^s$ and $\mathbb{W}^u_{\text{loc}}(x^\star)$ into $\mathbb{E}^u$ one-to-one, as shown in Fig. XX. Further, the stable manifold theorem guarantees that $\mathbb{E}^s$ and $\mathbb{E}^u$ actually approximate the local manifolds $\mathbb{W}^s_{\text{loc}}(x^\star)$ and $\mathbb{W}^u_{\text{loc}}(x^\star)$, respectively [1]. As a consequence, we get the behavior illustrated in Fig. 2.3

The manifolds we just looked at are defined for a local neighborhood U around the equilibrium. We can extend them towards the whole of state space by defining global manifolds as:

$$\mathbb{W}^s(x^\star) = \bigcup_{t \leq 0} \Phi^t(\mathbb{W}^s_{\text{loc}}(x^\star)) \tag{2.18}$$

$$\mathbb{W}^u(x^\star) = \bigcup_{t \geq 0} \Phi^t(\mathbb{W}^u_{\text{loc}}(x^\star)) \tag{2.19}$$

That is, the global stable manifold is obtained by integrating the local stable manifold backwards, looking at where the trajectories on it came from. For the unstable manifold, we integrate the local unstable manifold forwards, to see where it goes to.

An important fact about the local and global manifolds that follows from their definitions is that they are invariant: trajectories starting on these manifolds stay on them forever [1]. Furthermore, the uniqueness of solutions prohibits certain crossings of manifolds: stable manifolds of two distinct equilibria cannot cross, unstable manifolds of two distinct equilibria also cannot, and the same manifold cannot cross itself - otherwise, where the crossing points would have to obey two distinct paths! Meanwhile, stable and unstable manifolds, either of the same equilibrium or of two different equilibria can cross.

As mentioned before, these manifolds usually play a big role in organizing state space. As we will see in Chapter **??**, they can organize the transient dynamics of systems. There, we study a dynamical system wherein certain trajectories are forced to go on long excursions before converging to the stable equilibrium, the only attractor in state space (see Figs.2.2A-B). As explained there, this long excursion is generated by the arrangement of the invariant manifolds of the saddle-point that exists in state space. The invariant manifolds can also organize the long-term behavior of systems: the next section briefly shows how stable manifolds of unstable equilibria can act as the boundary separating two basins of attraction.

2.1.6 The fate of nonlinear dynamical systems II: multistability and basins of attraction

In Sec. 2.1.3 we saw that the ultimate fate of nonlinear systems, their attractors, can be much more complicated than that of linear ones. Not only are the attractors themselves complicated, but they can also coexist in state space. If there are two coexisting attractors, this means that the state space will be separated into three regions: the basin of attraction of attractor one, the basin of attractor two, and the boundary between them. Usually, the basin boundary is formed by stable manifolds of saddle-type objects: saddle-points, saddle-limit-cycles, and even chaotic saddles! [10]. Figure 2.4 illustrates this for a relatively simple system with two stable equilibria, where the basin boundary is the stable manifold of the saddle-point in the middle. This system is known as the Duffing oscillator:

$$\dot{x} = v \tag{2.20}$$

$$\dot{v} = -(-kx + cv + lx^3)/m, \tag{2.21}$$

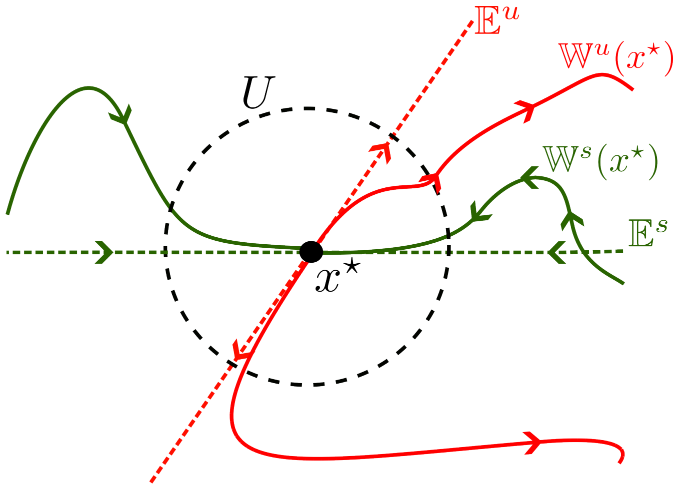

Figure 2.3: Invariant manifolds of saddle point $x^\star$. The local stable $\mathbb{W}^s_{\text{loc}}(x^\star)$ and unstable $\mathbb{W}^u_{\text{loc}}(x^\star)$ manifolds of the saddle point $x^\star$ respectively can be associated with the stable $\mathbb{E}^s$ and unstable $\mathbb{E}^u$ subspaces and become tangent to them near the saddle. This follows from the Hartman-Grobman and the stable manifold theorems. The global stable $\mathbb{W}^s(x^\star)$ and unstable $\mathbb{W}^u(x^\star)$ manifolds extend the definition of the local manifolds beyond the neigborhood U. Figure is inspired by Fig. 6.2.4 from Ref. [1].

with $k = 1$, $c = 0.5$, $l = 1$, $m = 1$. This system represents a ball of mass m rolling downhill at position x and velocity v on a quartic potential landscape of the form $U(x) = -lx^4/4 - kx^2/2$ with a friction term $-cv$. Following the definition of global manifolds in Eq.2.19, these global manifolds are essentially obtained by integrating trajectories starting on the local manifolds of the saddle-point.

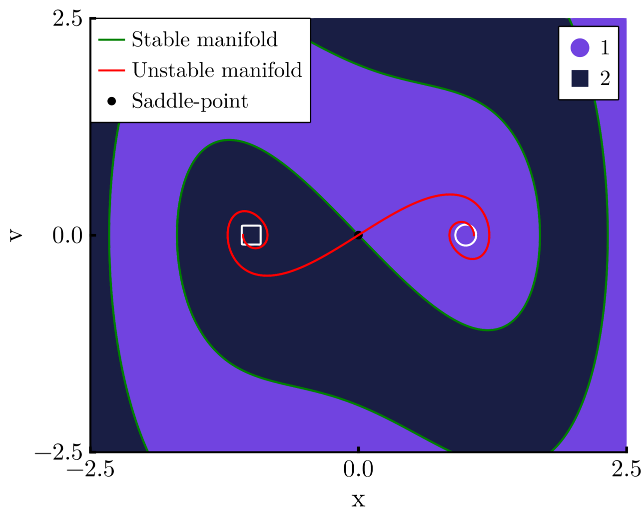

Figure 2.4: Bistability in Duffing model. Two stable equilibria (white square and circle) are shown with their respective basins of attraction in two shades of purple. The global stable and unstable manifolds of the saddle-point (black point) in the middle are also shown as green and red lines respectively. The global stable manifold of the saddle coincides with the boundary between the basins.

In this thesis we study two examples of multistability occuring in networked systems. In Chapter **??** we study networks of Kuramoto units, and see there the coexistence of multiple attractors depending on how strongly the units are interacting. We also see how this multistability impacts the sensitivity of the system to small changes in parameters of the units. Later, in Chapter **??** we study how multistability arises when two excitable neurons are coupled together diffusively. Both studies require that we find the attractors in the systems. This is what we deal with in the next section.

2.1.7 How to find attractors

Finding all the attractors of a given dynamical system is not necessarily a trivial task. For equilibria, one can find all the roots of the system function, i.e., $f(x^\star) = 0$ and then check their stability through the eigenvalues of the Jacobian evaluated on them. However, for more complicated attractors the problem becomes more complicated. To start off, simply proving that a set is an attractor, following the criteria given in Sec. **??**, is usually not possible. Instead, in practice we use the looser definition of an attractor as the long-term dynamics of trajectories. Numerically, this means a brute-force approach of simulating several trajectories in state space for long integration times and seeing where they converge.

This comes with two problems. First, it does not rule out the possibility that a certain set is just a very long transient. To remedy this, we usually integrate trajectories on the set for very long and check if there is any escape. Second, some attractors might have very small basins of attraction, such that randomly chosen intiial conditions are unlikely to end on them, so it is unlikely that we find those attractors. So far, however, this brute force approach is the best we have for general systems []. Within this approach, there are two main methods in the literature for finding attractors. They differ in how they check converge to attractors.

The first approach was proposed in Ref. [] and implemented with improvements in Ref. [4]. Given a trajectory, it XX.

The second approach has been proposed by Ref. [5] and soon thereafter also by Ref. [12]. Later it was implemented efficiently with improvements in Ref. [3].

Both methods can be applied across a parameter range and used in a continuation fashion. Developing this method, along with implementing, improving, and maintaining the featurizing method, was one of my contributions in my PhD. This led to the publication in Ref. [3]. The continuation method works by XX

2.2 Bifurcations

What happens to the attractors - and, in general, to the state space structures - of a dynamical system when we vary its parameters? In terms of the qualitative properties, there are two possibilties: either they stay similar or they change drastically. We can be a bit more rigorosu. Two systems are qualitatively similar if they are topologically equivalent. The notion of topological equivalence was already mentioned in Sec. 2.1.5. As a reminder, two systems are topologically equivalent if the state space of one can be obtained by a continuous transformation of the other [**?**]. Mathematically, this means that they are topologically equivalent if there is a homeomorphism $h : M \to M$ mapping orbits of the first system onto orbits of the second, preserving the direction of time.

As the parameters of a system are varied, we obtain different dynamical systems that are usually topologically equivalent. The attractors, for instance, may move, but they retain their stability. At some point, however, there may be a drastic change, and the new system may no longer be equivalent. The attractor may have disappeared, or lost its stability. Or a new attractor may have emerged. These drastic qualitative changes in the behavior of a dynamical system are called bifurcations. A bit more rigorously, a bifurcation is a change in the topological type of a system as its parameters pass through a critical (bifurcation) value [**?**]. There are many different types of bifurcations, and one can literally write a whole book about this [**?**]. For this thesis we focus briefly on just a few bifurcations that will be relevant for later.

2.2.1 Saddle-node bifurcation (or equilibria and limit cycles)

mention ghosts

2.2.2 Hopf bifurcation

2.2.3 Homoclinic bifurcation

Acknowledgments

Write your acknowledgments here.

Bibliography

[1] John H. Argyris, Gunter Faust, Maria Haase, and Rudolf Friedrich. *An Exploration of Dynamical Systems and Chaos.* Springer Berlin, Heidelberg, 2 edition, 2015.

[2] Henk Broer and Floris Takens. *Dynamical Systems and Chaos.* Applied Mathematical Sciences. 2011.

[3] George Datseris, Kalel Luiz Rossi, and Alexandre Wagemakers. Framework for global stability analysis of dynamical systems. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 33(7):073151, 2023.

[4] George Datseris and Alexandre Wagemakers. Effortless estimation of basins of attraction. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 32(2):023104, 2022.

[5] Maximilian Gelbrecht, Jürgen Kurths, and Frank Hellmann. Monte Carlo basin bifurcation analysis. *New Journal of Physics*, 22(3):033032, 2020.

[6] Paul Glendinning. *Stability, Instability And Chaos: An Introduction To The Theory Of Nonlinear Differential Equations (cambridge Texts In Applied Mathematics).* Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge [England], 1 edition, 1994.

[7] Eugene M Izhikevich. *Dynamical systems in neuroscience.* MIT press, 2007.

[8] Jorge V. José and Eugene J. Saletan. *Classical Dynamics: A Contemporary Approach.* Cambridge University Press, Cambridge, 1998.

[9] John Milnor. On the concept of attractor. *Communications in Mathematical Physics*, 99(2):177–195, 1985.

[10] Alexander N. Pisarchik and Alexander E. Hramov. Multistability in Physical and Living Systems, Characterization and Applications. *Springer Series in Synergetics*, 2022.

[11] Anant Kant Shukla, T R Ramamohan, and S Srinivas. A new analytical approach for limit cycles and quasi-periodic solutions of nonlinear oscillators: the example of the forced Van der Pol Duffing oscillator. *Physica Scripta*, 89(7):075202, 2014.

[12] Merten Stender and Norbert Hoffmann. bSTAB: an open-source software for computing the basin stability of multi-stable dynamical systems. *Nonlinear Dynamics*, pages 1–18, 2021.

[13] Strogatz. *Nonlinear Dynamics and Chaos.* Studies in nonlinearity. Westview, 2002.

[14] Robert L.V. Taylor. Attractors: Nonstrange to Chaotic. *SIAM Undergraduate Research Online*, 4:72–80, 2011. Publisher: Society for Industrial & Applied Mathematics (SIAM).