IOP Publishing | Royal Swedish Academy of Sciences Physica Scripta

Phys. Scr. **89** (2014) 075202 (10pp) doi:10.1088/0031-8949/89/7/075202

A new analytical approach for limit cycles and quasi-periodic solutions of nonlinear oscillators: the example of the forced Van der Pol Duffing oscillator

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Received 18 November 2013, revised 20 March 2014
Accepted for publication 2 April 2014
Published 8 May 2014

Abstract

In this paper we propose a technique to obtain limit cycles and quasi-periodic solutions of forced nonlinear oscillators. We apply this technique to the forced Van der Pol oscillator and the forced Van der Pol Duffing oscillator and obtain for the first time their limit cycles (periodic) and quasi-periodic solutions analytically. We introduce a modification of the homotopy analysis method to obtain these solutions. We minimize the square residual error to obtain accurate approximations to these solutions. The obtained analytical solutions are convergent and agree well with numerical solutions even at large times. Time trajectories of the solution, its first derivative and phase plots are presented to confirm the validity of the proposed approach. We also provide rough criteria for the determination of parameter regimes which lead to limit cycle or quasi-periodic behaviour.

Keywords: nonlinear oscillators, modified HAM, limit Cycle solutions, quasi-periodic solutions

1. Introduction

The study of nonlinear oscillators has attracted appreciable attention in the literature because of their applications in a wide variety of scientific and engineering disciplines [1–4]. Nonlinear oscillators show very complex behaviour like limit cycles (periodic), quasi-periodicity and chaos with respect to different parameter values and different initial conditions. Determining the structure of the phase space of such systems is difficult through numerical techniques alone. Hence it is advantageous to develop approximate analytical techniques which can provide at least a rough guide to the structure of the phase space. Solutions of some nonlinear oscillators have been obtained analytically by a number of authors [5–16] by using different approaches like the peturbation technique, the homotopy perturbation method, harmonic balance method, the homotopy analysis method (HAM), etc. The asymptotic perturbation method based on harmonic balance and the perturbation method [17], the method of multiple scales [18] and the asymptotic method based on the harmonic balance method and the method of multiple scales [19] have been applied to analyse the periodic and quasi-periodic behaviours of nonlinear oscillators. It is clear that in all the above three methods we need a perturbation parameter to develop the approximate analytical solution which is a disadvantage of these techniques. An analytical technique free from this restriction would hence be valuable.

The Van der Pol oscillator was first proposed by Balthasar van der Pol [20]. It has many applications in areas such as electrical circuits [21], biology [22], seismology [23], etc. Perturbation solutions for the Van der Pol equation (without forcing) have been given by Buonomo [24]. He developed an explicit form for the frequency of the limit cycle and the damping parameter was taken as a perturbation parameter. Chen and Liu [10] have developed analytical solutions for the same problem by the homotopy analysis method without taking any parameter present in the problem as the perturbation parameter because HAM can be applied whether a perturbation parameter is present or not. Therefore, the solution expressions developed by Chen and Liu are more general than those developed by Buonomo. Uniform limit cycle solutions for the Van der Pol Duffing oscillator (without forcing) have been developed by the same authors [11] using HAM. They have developed an explicit form for the frequency of the limit cycle and have shown the variation of the frequency of the limit cycle with respect to the damping parameter. In all the above cases the forcing term is absent. As is well known, problem complexity increases if we add external forcing to the Van der Pol and the Van der Pol Duffing oscillators. To the best of our knowledge in the literature only one attempt has been made by Qaisi [25] to develop analytical solutions for limit cycles for the case of forcing without damping. He has obtained the solution expressions by using the power series method. Recently, Shukla *et al* [26] have analysed the limit cycle solutions of period one for the forced Van der Pol Duffing oscillator by HAM. They obtained the frequency of the limit cycle under the assumption that the frequency of the nonlinear oscillator is the same as the external frequency. The present work deals with the case where both the frequencies are not the same and in fact we are free to choose the values of these frequencies. This assumption makes the problem more complex.

In fact, it is difficult to find parameter regimes and initial conditions which lead to limit cycle or quasi-periodic behaviour numerically. Hence it is important to develop techniques to determine such regimes analytically and the technique used should ideally be independent of the existence of a perturbation parameter. It is well known that a quasi-periodic solution has a minimum of two frequencies present in the solution which are not rationally related and in a limit cycle the frequencies present in the solution are rationally related.

In this paper our interest is to propose such a technique to obtain limit cycle solutions as well as quasi-periodic solutions for nonlinear oscillators. To test our approach we consider the forced Van der Pol oscillator and the forced Van der Pol Duffing oscillator. To the best of our knowledge no attempt has made so far for the determination of limit cycles and quasi-periodic solutions, analytically for the problems stated above. Kimiaeifar *et al* [27] have given approximate analytical solutions for the forced Van der Pol Duffing oscillator but they have neither discussed the limit cycle nor the quasi-periodic case, possibly because of the base function chosen.

The role of base functions is important for a particular behaviour of the system considered in HAM. The homotopy analysis method [5, 28–31] given by Liao in 1992, provides us great freedom to express our solution in terms of different base functions and one can adjust and control the rate of convergence of the solution series. The accuracy of the HAM solutions can be enhanced by the optimal homotopy approach [32–37] where one minimizes the square residual error to choose the optimal value of the convergence control parameter present in the frame of HAM. We validate our results by showing a comparison between the analytical solution and the numerical solution. Numerical solutions have been obtained by using the NDSolve scheme in Mathematica.

2. Problem description

The forced Van der Pol Duffing oscillator which has been used in different areas of science and engineering is

$$x'' - \mu\left(1 - x^2\right)x' + \alpha x + \beta x^3 = g\cos\left(\omega_f t\right), \quad \mu > 0 \quad (1)$$

where x denotes displacement from the equilibrium position, the prime(s) denote the derivatives of x with respect to t, $\mu > 0$ is the damping parameter, ω_f is the external frequency, g is the amplitude of the external forcing and α, β are constants. For $\beta \neq 0$ the Duffing oscillator can be interpreted as a forced oscillator with a spring which is either a hardening or a softening spring. In this paper we have only considered the case of a hardening spring [1].

The forced Van der Pol oscillator can be obtained by putting $\beta = 0$ in equation (1). Solutions of equation (1) at $\alpha = 1$, $\beta = 0$, $g = 0$ and $\alpha = 1$, $\beta \neq 0$ and $g = 0$ have been obtained analytically by Chen and Liu in [10] and [11] respectively by HAM.

3. Application of the homotopy analysis method, modification and the square residual error

We follow the same methodology as discussed in [11] and [26]. With the transformation $\tau = \omega t$ and $x(t) = x(\tau)$, equation (1) reduces to

$$\omega^2 x'' - \omega\mu\left(1 - x^2\right)x' + \alpha x(\tau) + \beta x^3 = g\cos\left(\frac{\omega_f}{\omega}\tau\right). \quad (2)$$

Choose the initial conditions as

$$x(0) = a \quad x'(0) = 0. \quad (3)$$

Limit cycle solutions of period one of equation (2) with respect to the initial conditions (3) have been obtained in [26], when $\omega_f = \omega$.

In order to determine the limit cycle and quasi-periodic solutions of (2) we choose the set of base functions as

$$\left\{\sin(m\tau)\cos\left(n\frac{\omega_f}{\omega}\tau\right) \quad : \quad m, n \geqslant 0\right\} \quad (4)$$

where ω and ω_f are rationally or irrationally related when the solution is periodic or quasi-periodic, respectively. We note that this base function is different from the base function taken in [27]. We use the auxiliary linear operator as

$$L\left[\phi(\tau; q)\right] = \frac{\partial^2\left[\phi(\tau; q)\right]}{\partial\tau^2} + \phi(\tau; q). \quad (5)$$

Let the initial guess be $x_0(\tau) = a_0\cos(\tau)$. Here a_0 is the initial approximation of a. We also choose ω_0 as the initial approximation of ω.

In view of equation (2) the nonlinear operator is of the form:

$$N\left[\phi(\tau;q),\Omega(q),A(q)\right]=\phi''(\tau;q)\Omega^2(q)\\ -\mu\Omega(q)\phi'(\tau;q)+\mu\Omega(q)\phi^2(\tau;q)\phi'(\tau;q)\\ +\alpha\phi(\tau;q)+\beta\phi^3(\tau;q)-g\cos\left(\frac{\omega_f}{\Omega(q)}\tau\right) \tag{6}$$

where, $q\in[0,1]$ is the embedding parameter. The so-called zero order deformation equation is

$$(1-q)L\left[\phi(\tau;q)-x_0(\tau)\right]\\ =qhH(\tau)N\left[\phi(\tau;q),\Omega(q),A(q)\right] \tag{7}$$

with the initial conditions:

$$\phi(0;q)=A(q),\quad \left.\frac{\partial\phi(\tau;q)}{\partial\tau}\right|_{\tau=0}=0. \tag{8}$$

As the embedding parameter q varies from 0 to 1, $\phi(\tau;q),\Omega(q)$ and $A(q)$ vary from the initial guesses $x_0(\tau)$, ω_0 and a_0 to $x(\tau)$, ω and a respectively. Obviously, at $q=0$ we get $\phi(\tau;0)=x_0(\tau)$ from (7). Similarly from (7) at $q=1$ equations (2) and (3) are the same as equations (6) and (8) and we obtain:

$$\phi(\tau;1)=x(\tau),\quad A(1)=a,\quad \Omega(1)=\omega \tag{9}$$

respectively. On expanding $\phi(t;q),\Omega(q)$ and $A(q)$ in a Taylor's series with respect to q, we have

$$\phi(\tau;q)=\sum_{m=0}^{\infty}x_m(\tau)q^m \tag{10}$$

$$\Omega(q)=\sum_{m=0}^{\infty}\omega_m q^m \tag{11}$$

$$A(q)=\sum_{m=0}^{\infty}a_m q^m \tag{12}$$

where

$$x_m(\tau)=\left.\frac{1}{m!}\frac{\partial^m\phi(\tau;q)}{\partial q^m}\right|_{q=0} \tag{13}$$

$$\omega_m=\left.\frac{1}{m!}\frac{\partial^m\Omega(q)}{\partial q^m}\right|_{q=0} \tag{14}$$

$$a_m=\left.\frac{1}{m!}\frac{\partial^m A(q)}{\partial q^m}\right|_{q=0}. \tag{15}$$

We introduce a new form for the homotopy series $\Omega(q)$ which is present in the denominator in equation (6). We use the following form for this particular series which is associated with the forcing term as:

$$\Omega(q)=\omega_0+\sum_{m=1}^{\infty}\omega_m q^{(m+r)} \tag{16}$$

where r is a non-negative positive integer. We choose $r\geqslant(m+1)$ for the mth order HAM approximation.

Table 1. The square residual error (Δ_m) and the convergence control parameter (h) at different order of HAM approximations for $\alpha=4$, $\beta=0$, $g=1$, $\mu=0.1$ and $\omega_f=3$.

h	Δ_m at 5th order	Δ_m at 10th order	Δ_m at 15th order
−0.15	1.746×10^{-3}	1.158×10^{-6}	5.302×10^{-7}
−0.16	8.223×10^{-4}	6.968×10^{-7}	5.295×10^{-7}
−0.17	3.615×10^{-4}	5.657×10^{-7}	5.297×10^{-7}
−0.18	1.460×10^{-4}	5.317×10^{-7}	5.300×10^{-7}
−0.19	5.328×10^{-5}	5.261×10^{-7}	5.300×10^{-7}
−0.20	1.761×10^{-5}	5.274×10^{-7}	5.301×10^{-7}
−0.21	5.637×10^{-6}	5.291×10^{-7}	5.301×10^{-7}
−0.22	2.053×10^{-6}	5.300×10^{-7}	5.301×10^{-7}
−0.23	1.003×10^{-6}	5.300×10^{-7}	5.301×10^{-7}
−0.24	1.121×10^{-6}	5.299×10^{-7}	5.301×10^{-7}

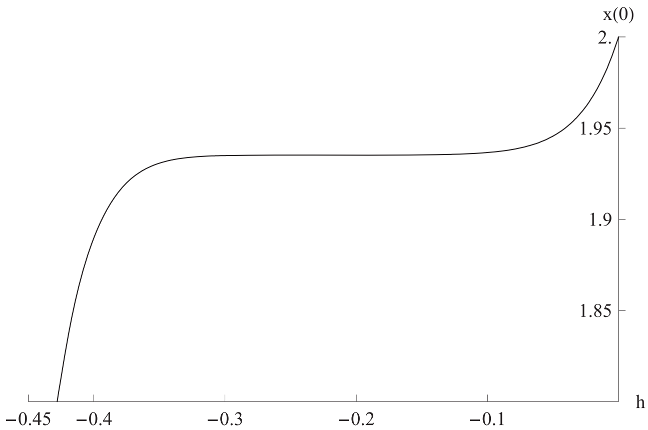

Figure 1. h-curve: $x(0)$ versus h for $\alpha=1$, $\beta=1$, $g=0.001$, $\mu=0.1$ and $\omega_f=2$ at tenth order of HAM approximation.

If the initial guess, the auxiliary linear operator, the auxiliary parameter, and the auxiliary function are properly chosen, such that the series (10), (11) and (12) converge at $q=1$, we obtain the approximate solution as:

$$x(\tau)=\sum_{m=0}^{\infty}x_m(\tau) \tag{17}$$

$$\omega=\sum_{m=0}^{\infty}\omega_m \tag{18}$$

$$a=\sum_{m=0}^{\infty}a_m. \tag{19}$$

Differentiating equations (7) and (8) m-times with respect to q and then setting $q=0$ and finally dividing by $m!$, the mth order deformation equation is:

$$L\left[x_m(\tau)-\chi_m x_{m-1}(\tau)\right]=hH(\tau)R_m(x_{m-1}) \tag{20}$$

subject to the conditions

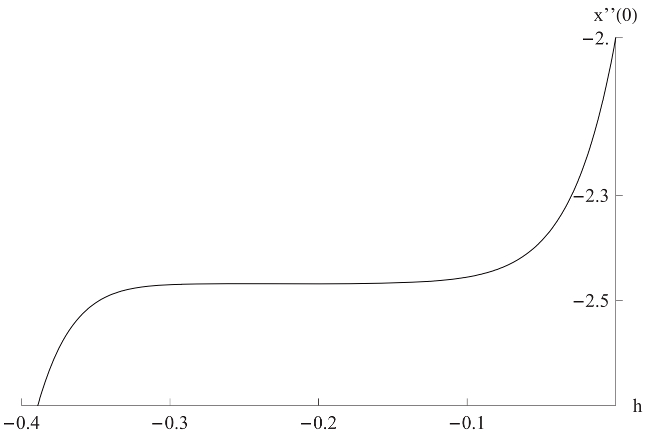

Figure 2. h-curve: $x''(0)$ versus h for $\alpha = 1$, $\beta = 1$, $g = 0.001$, $\mu = 0.1$ and $\omega_f = 2$ at tenth order of HAM approximation.

Table 2. The convergence of series (18) and (19) for ω and a respectively, the square residual error (Δ_m) and the optimal convergence control parameter (h) at different order of HAM approximations for $\alpha = 4$, $\beta = 0$, $g = 1$, $\mu = 0.1$ and $\omega_f = 3$.

Order of approximation	ω	a	h	Δ_m
5	1.99972	1.78180	−0.23	1.003×10^{-6}
10	1.99972	1.78168	−0.19	5.261×10^{-7}
15	1.99972	1.78168	−0.159	5.295×10^{-7}

$$x_m(0) = a_m, \quad x'_m(0) = 0 \tag{21}$$

where

$$\chi_m = \begin{cases} 0 & \text{for } m = 1 \\ 1 & \text{for } m > 1 \end{cases} \tag{22}$$

and

$$\begin{aligned} R_m(x_{m-1}) = {} & \sum_{k=0}^{m-1} x''_{m-k-1}(\tau) \sum_{n=0}^{k} \omega_n \omega_{k-n} \\ & + \mu \sum_{k=0}^{m-1} \omega_{m-k-1} \sum_{n=0}^{k} x'_{k-n}(\tau) \sum_{j=0}^{n} x_j x_{n-j} \\ & - \mu \sum_{k=0}^{m-1} \omega_k x'_{m-k-1} + \alpha x_{m-1} \\ & + \beta \sum_{k=0}^{m-1} x_{m-k-1} \sum_{n=0}^{k} x_n x_{k-n} - r_1. \end{aligned} \tag{23}$$

Since $r \geqslant (m+1)$ the last term of (23) at the mth order deformation will be

$$r_1 = \begin{cases} g \cos\left(\dfrac{\omega_f}{\omega_0}\tau\right) & \text{for } m = 1 \\ 0 & \text{for } m > 1. \end{cases} \tag{24}$$

If we choose $r = 0$ (then series (16) is the same as series (11)), the last term then produces secular terms from the second order deformation equation onwards and we can not link these terms to other secular terms that occur in the expansion because the frequency is different. In other words the rule of solution expression [5] does not allow us to keep these terms in the solution. If these terms are retained our solutions will diverge and we will not achieve our goal of limit cycles and quasi-periodic solutions. To avoid the occurence of these secular terms we choose $r \geqslant (m+1)$ for the mth order HAM approximation, so that the occurence of secular terms can be avoided. In particular, if the order of HAM approximation is 10 then the value of r will be 11 or greater. The validity of this approach can be seen from the comparison presented between analytical and numerical solutions later in this work.

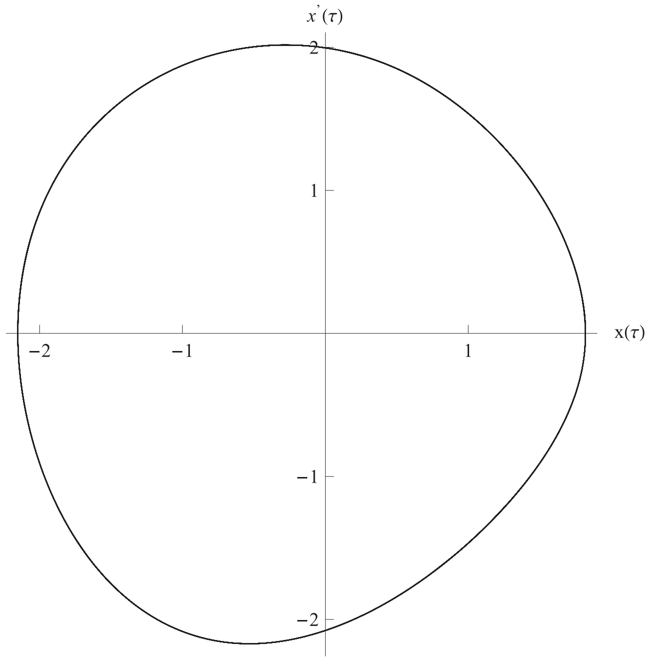

Figure 3. HAM phase plot (limit cycle of period one): $x'(\tau)$ versus $x(\tau)$ for $\alpha = 1$, $\beta = 0$, $g = \frac{1}{2}$, $\mu = 0.1$ and $\omega_f = 2$ at tenth order of HAM approximation with $h = -0.78$.

3.1. The square residual error

The minimization of the square residual error has been demonstrated in [32] and [38]. Let Δ_m be the square residual error of equation (2) at the mth order HAM approximation, then

$$\Delta_m = \int_0^{2\pi\omega_0/\omega_f} \left(N\left[\sum_{i=0}^{m} x_i(\xi) \right] \right)^2 \mathrm{d}\xi. \tag{25}$$

We can determine the convergence control parameter h by solving the following algebraic equation:

$$\frac{\partial \Delta_m}{\partial h} = 0. \tag{26}$$

The method used in this paper is to plot Δ_m versus h and choose the h which corresponds to the minimum of the square residual error [39, 40].

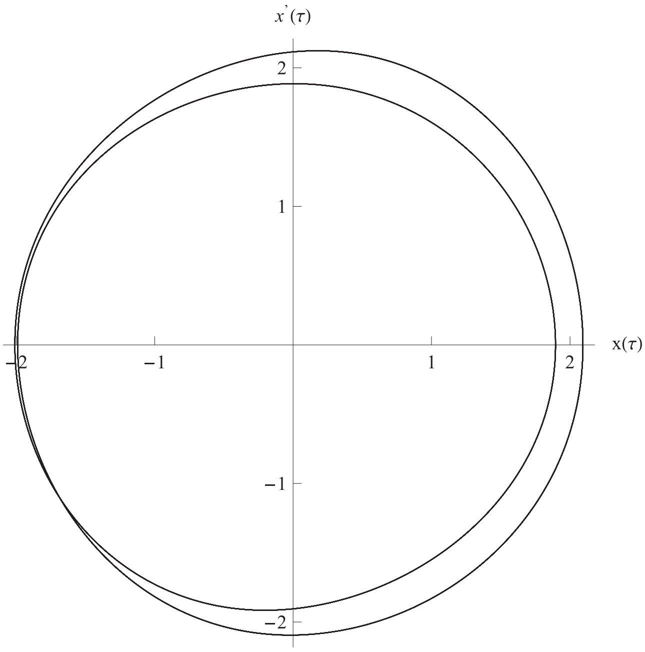

Figure 4. HAM phase plot (limit cycle of period two): $x'(\tau)$ versus $x(\tau)$ for $\alpha = 4$, $\beta = 0$, $g = \frac{1}{2}$, $\mu = 0.1$ and $\omega_f = 3$ at tenth order of HAM approximation with $h = -0.21$.

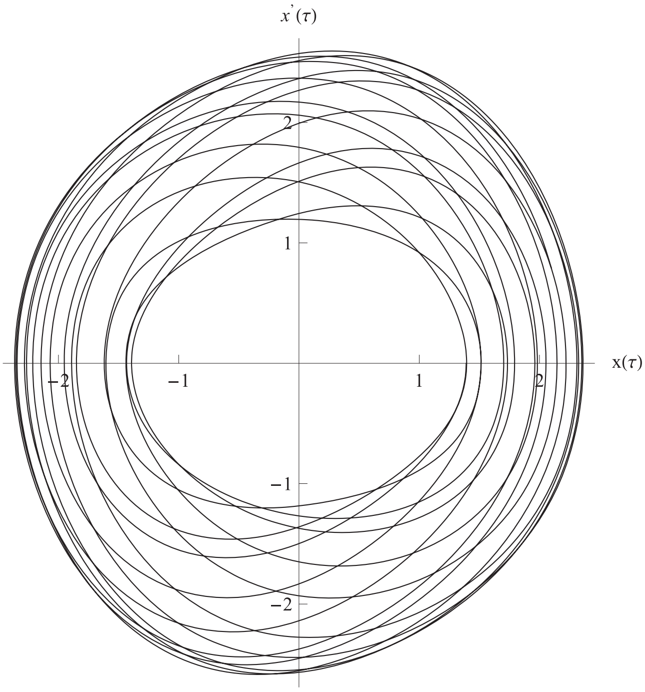

Figure 5. HAM phase plote (quasi-periodic): $x'(\tau)$ versus $x(\tau)$ at $\alpha = 1$, $\beta = 0$, $g = \frac{1}{2}$, $\mu = 0.1$ and $\omega_f = \sqrt{2}$ at tenth order of HAM approximation with $h = -0.87$.

4. Solution procedure

Following the rule of solution expression discussed by Liao in [5], we remove the secular terms by setting the coefficients of $\sin(\tau)$ and $\cos(\tau)$ to zero. Thus, we obtain the following

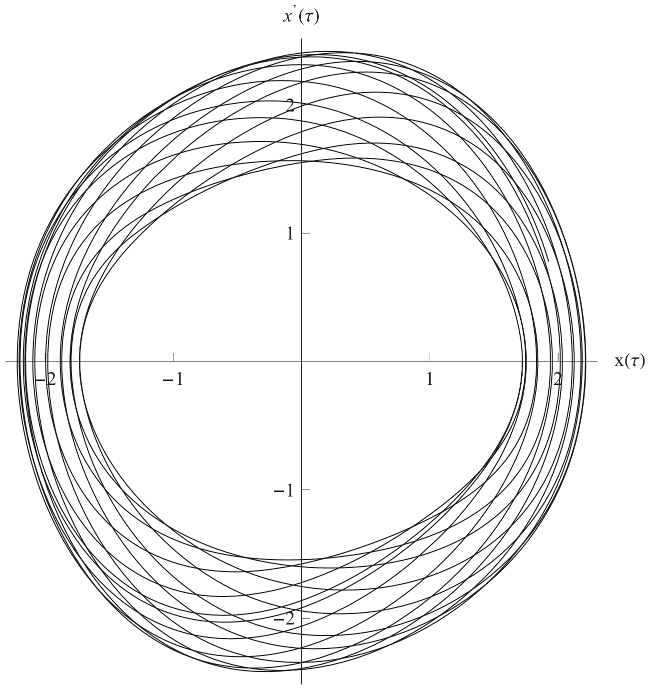

Figure 6. HAM phase plot (quasi-periodic): $x'(\tau)$ versus $x(\tau)$ for $\alpha = 1$, $\beta = 0$, $g = \frac{1}{2}$, $\mu = 0.1$ and $\omega_f = \sqrt{3}$ at tenth order of HAM approximation with $h = -0.68$; for numerical solution $\omega = 0.999\,41$ and $a = 1.721\,99$.

equations for the first order deformation equation by putting $m = 1$ in equation (23)

$$\alpha a_0 + \frac{3}{4}\beta a_0^3 - a_0\omega_0^2 = 0, \quad \mu a_0\omega_0 - \frac{1}{4}\mu a_0^3\omega_0 = 0. \quad (27)$$

We obtain the values of a_0 and ω_0 by solving the above algebraic equations. On solving (27) we obtain for $\mu \neq 0$, $\omega_0 \neq 0$ and $a_0 \neq 0$

$$a_0 = \pm 2 \quad \text{and} \quad \omega_0 = \pm\sqrt{\alpha + 3\beta}. \quad (28)$$

Similarly on increasing the order of HAM approximation (i.e, $m > 1$ in equation (23)) we obtain the other terms a_m and ω_m for each m. After obtaining the mth order solution, expressions (17–19) contain the auxiliary parameter h. We use equation (25) to obtain the optimal value of h by plotting Δ_m versus h. We fix $\mu = 0.1$ throughout the paper.

4.1. Convergence

Convergence is important for any approximate analytical solution. As pointed out by Liao, in the frame of HAM, we can control the convergence of the approximate analytical solutions by choosing either the proper value of the convergence control parameter h through h-curves [5] or by the optimal values of h through the minimization of the square residual error [30–32]. We obtain the convergence mainly through the optimal approach for all the set of parameter values but for one set of parameter values we obtain the h-curve also.

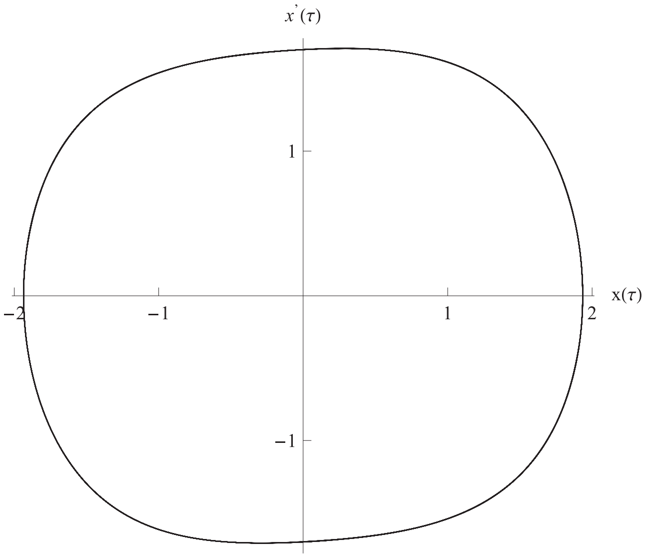

Figure 7. HAM phase plot (limit cycle of period one): $x'(\tau)$ versus $x(\tau)$ for $\alpha = 1$, $\beta = 1$, $g = 0.001$, $\mu = 0.1$ and $\omega_f = 2$ at tenth order of HAM approximation with $h = -0.24$.

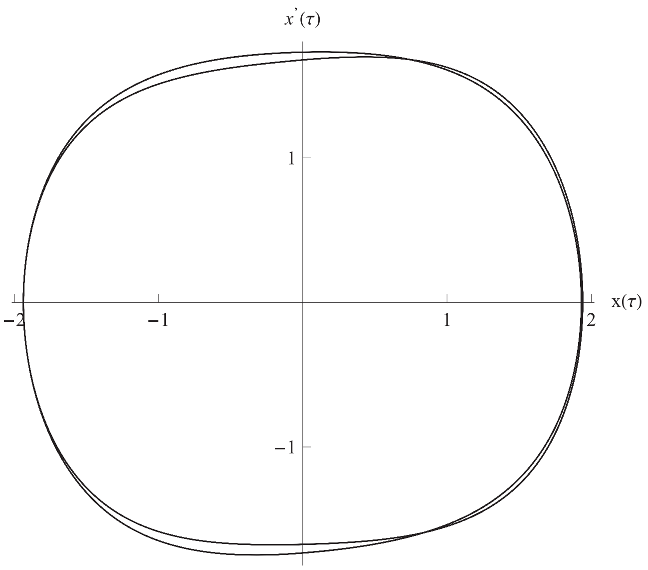

Figure 8. HAM phase plot (limit cycle of period two): $x'(\tau)$ versus $x(\tau)$ for $\alpha = 1$, $\beta = 1$, $g = \frac{1}{4}$, $\mu = 0.1$ and $\omega_f = 5$ at tenth order of HAM approximation with $h = -0.11$.

For $\alpha = 4$, $\beta = 0$, $g = 1$, $\mu = 0.1$ and $\omega_f = 3$ we obtain the square residual error at different order of approximations for varying h in the range between -0.15 to -0.24. It is clear from table 1 that the analytical solution series (17) obtained by the proposed approach converges and the square residual error is minimun. The corresponding series (18) and (19) for ω and a, respectively also converge, as can be seen from table 2 on increasing the order of HAM approximations and the square residual error is about 10^{-7}. At $\alpha = 1$, $\beta = 1$, $g = 0.001$, $\mu = 0.1$ and $\omega_f = 2$ we obtain the optimal value $h = -0.24$, as given in table 4. For this case we obtain the h-curves as shown in figures 1, 2 and obtain a common flat region as the region of convergence. The optimal value of $h = -0.24$ lies in the obtained region of convergence, which proves that both the criteria yield the correct value of h.

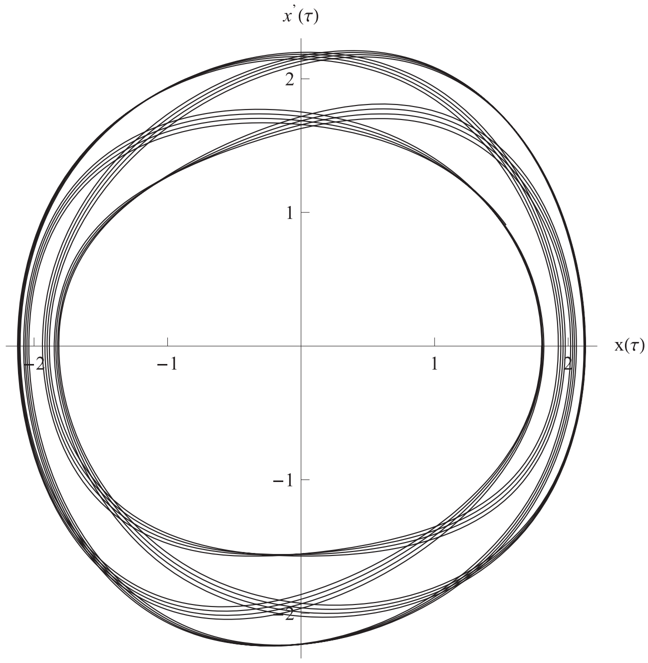

Figure 9. HAM phase plot (quasi-periodic): $x'(\tau)$ versus $x(\tau)$ at $\alpha = 1$, $\beta = 0.1$, $g = \frac{1}{2}$, $\mu = 0.1$ and $\omega_f = 2$ at fifth order of HAM approximation with $h = -0.68$.

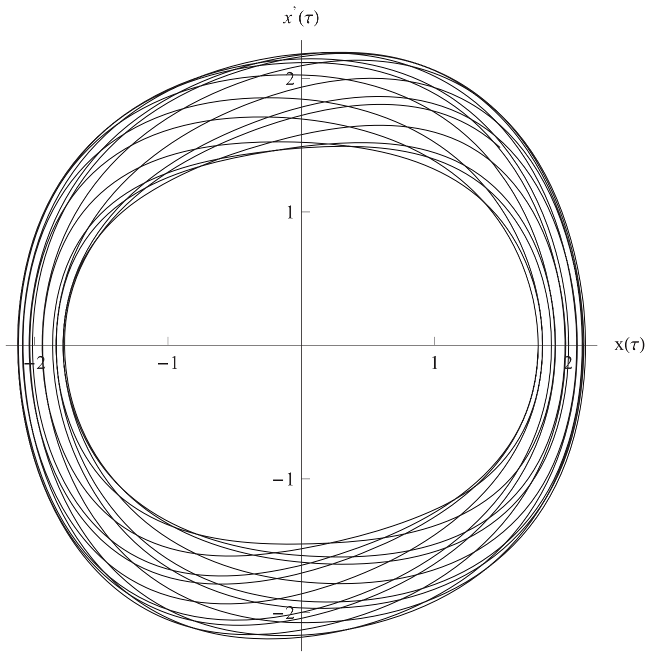

Figure 10. HAM phase plot (quasi-periodic): $x'(\tau)$ versus $x(\tau)$ at $\alpha = 1$, $\beta = 0.2$, $g = \frac{1}{2}$, $\mu = 0.1$ and $\omega_f = 2$ at fifth order of HAM approximation with $h = -0.60$; for numerical solution $\omega = 1.254\,25$ and $a = 1.777\,47$.

Table 3. Parameter values, the optimal ω and a, the square residual error, the optimal h and the nature of the dynamics for the forced Van der Pol oscillator at $\mu = 0.1$ and $\beta = 0$.

α	g	ω_f	ω	a	Δ_m	h	Nature of the dynamics
1	1/2	2	0.99937	1.82060	1.98×10^{-6}	−0.78	Period one
4	1/2	3	1.99970	1.89598	1.54×10^{-7}	−0.21	Period two
1	1/2	$\sqrt{2}$	0.99983	1.39028	2.06×10^{-7}	−0.87	Quasi-periodic
1	1/2	$\sqrt{3}$	0.99941	1.72199	1.86×10^{-6}	−0.68	Quasi-periodic
4	1	3	1.99972	1.78168	5.26×10^{-7}	−0.19	Period two

Table 4. Parameter values, the optimal ω and a, the square residual error, the optimal h and the nature of the dynamics for the forced Van der Pol Duffing oscillator at $\mu = 0.1$ and $\alpha = 1$.

β	g	ω_f	ω	a	Δ_m	h	Nature of the dynamics
1	0.001	2	1.92861	1.93512	3.05×10^{-6}	−0.24	Period one
1	1/4	5	1.92958	1.93149	7.72×10^{-4}	−0.11	Period two
0.1	1/2	2	1.13602	1.80118	6.82×10^{-5}	−0.68	Quasi-periodic
0.2	1/2	2	1.25425	1.77747	2.73×10^{-4}	−0.60	Quasi-periodic

4.2. *Case 1: The forced Van der Pol oscillator*

For this case we put $\beta = 0$ in all the equations described above. We fix $g = \frac{1}{2}$ for this case unless otherwise stated.

4.2.1. Limit cycle solutions of period one and two. In order to obtain limit cycle solutions of period one we take $\alpha = 1$ and $\omega_f = 2$, similarly for limit cycle solutions of period two we choose $\alpha = 4$ and $\omega_f = 3$. The corresponding phase plots are presented in figures 3 and 4.

4.2.2. Quasi-periodic solution. In order to obtain quasi-periodic solutions we choose $\alpha = 1$ and $\omega_f = \sqrt{2}$, the corresponding phase plot is shown in figure 5. For other choices like $\alpha = 1$, $g = \frac{1}{2}$ and $\omega_f = \sqrt{3}$ we again obtain a quasi-periodic phase plot figure 6.

4.3. *Case 2: The forced Van der Pol Duffing oscillator*

For this case β will be non-zero. We fix $\alpha = 1$ for this case.

4.3.1. Limit cycle solutions of period one and two. In order to obtain limit cycle solutions of period one we take $\beta = 1$, $g = 0.001$ and $\omega_f = 2$, similarly, for limit cycle solution of period two we choose $\beta = 1$, $g = \frac{1}{4}$ and $\omega_f = 5$. The corresponding phase plots are presented in figures 7 and 8.

4.3.2. Quasi-periodic solution. In order to obtain quasi-periodic solutions we choose $\beta = 0.1, 0.2$, $g = \frac{1}{2}$ and $\omega_f = 2$. Phase plots are shown in figures 9 and 10.

5. Discussion

The comparison between the analytical and numerical solutions of the time trajectories of the forced Van der Pol and the forced Van der Pol Duffing oscillators for the set of parameter values considered is shown in figures 11–16. The corresponding comparison of the first derivative of the time trajectories is shown in figures figures 17–22. As can be seen from the figures there is good agreement between the analytical and numerical approach, thus confirming the validity of the proposed approach. The square residual error, the values of ω and a for the optimal value of h and the nature of the dynamics for some set of parameter values are presented in tables 3 and 4. The value of the minimum square residual error confirms the accuracy of the analytical solutions obtained by the proposed approach. In our computation, we note that whenever ω_0 (given in equation (28)) and ω_f are rationally related and integral multiples of each other we obtain limit cycle solutions of period one and when they are not an integral multiple we obtain the limit cycle solutions of period 2 i.e, both ω_0 and ω_f are rationally related but one is not an integral multiple of the other. On the other hand if ω_0 and ω_f are irrationally related we obtain quasi-periodic solutions. This can be explained from the choice of base functions. There are two time scales in the base functions we have chosen. When ω_f and ω_0 are not rationally related, the two time scales namely, '1' and 'ω_f/ω_0' are irrationally related and if these series converge the results will be a quasi-periodic solution if the secular terms are removed. If ω_f is a multiple of ω_0 then the resultant frequencies in the problem will be harmonics and hence only a limit cycle solution of period one will be obtained when the secular terms are removed. Hence, we can conclude roughly that a proper choice of ω_0 and ω_f leads to limit cycle or quasi-periodic behaviour. Therefore, we can determine *a priori*, parameter values that may lead to limit cycle or quasi-periodic behaviours. This is an advantage of the proposed technique over numerical techiniques. In regard to the physics,

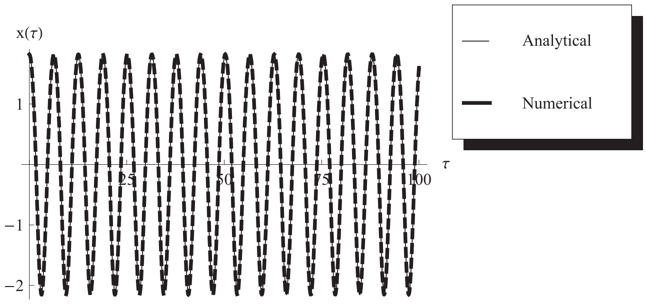

Figure 11. Comparison of time trajectories between the analytical and numerical solution (period one): $x(\tau)$ versus τ for $\alpha = 1$, $\beta = 0$, $g = \frac{1}{2}$, $\mu = 0.1$ and $\omega_f = 2$ at tenth order of HAM approximation with $h = -0.78$; for numerical solution $\omega = 0.999\,37$ and $a = 1.820\,60$.

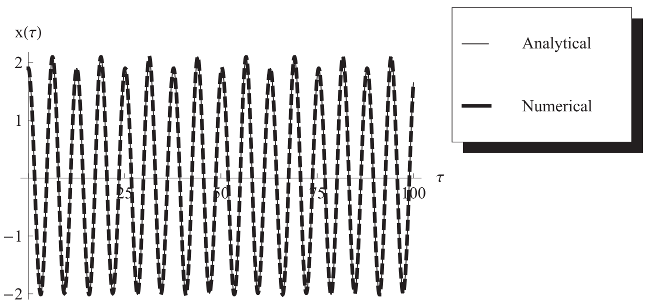

Figure 12. Comparison of time trajectories between the analytical and numerical solution (period two): $x(\tau)$ versus τ for $\alpha = 4$, $\beta = 0$, $g = \frac{1}{2}$, $\mu = 0.1$ and $\omega_f = 3$ at tenth order of HAM approximation with $h = -0.21$; for numerical solution $\omega = 1.999\,70$ and $a = 1.895\,98$.

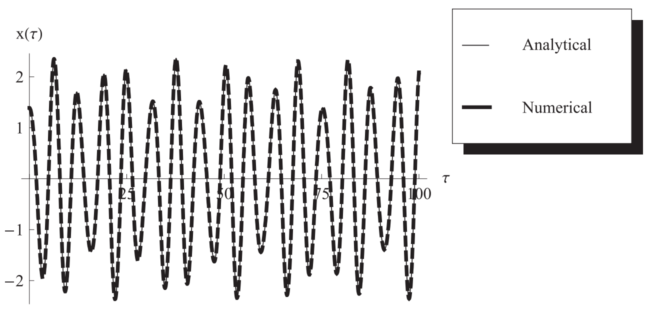

Figure 13. Comparison of time trajectories between the analytical and numerical solution (quasi-periodic): $x(\tau)$ versus τ for $\alpha = 1$, $\beta = 0$, $g = \frac{1}{2}$, $\mu = 0.1$ and $\omega_f = \sqrt{2}$ at tenth order of HAM approximation with $h = -0.87$; for numerical solution $\omega = 0.999\,83$ and $a = 1.390\,28$.

limit cycle behaviour implies an isolated periodic orbit and quasi-periodic behaviour implies an aperiodic bounded orbit which is dense in a subset of the phase space. For each set of parameter values, at first we obtain the optimal ω and a from the proposed approach after minimizing the square residual error and then we use these values in equations (2) and (3) to develop numerical solutions by using the NDSolve command in Mathematica. The square residual error is minimized in the time interval between 0

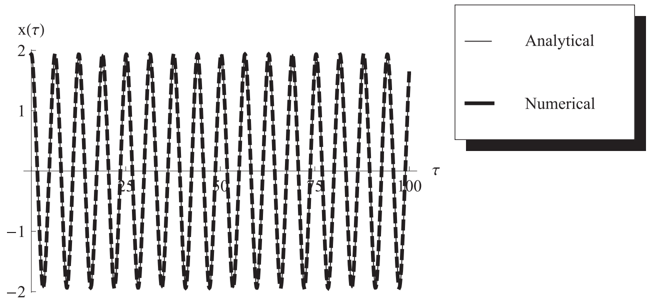

Figure 14. Comparison of time trajectories between the analytical and numerical solution (period one): $x(\tau)$ versus τ for $\alpha = 1$, $\beta = 1$, $g = 0.001$, $\mu = 0.1$ and $\omega_f = 2$ at tenth order of HAM approximation with $h = -0.24$; for numerical solution $\omega = 1.928\,61$ and $a = 1.935\,12$.

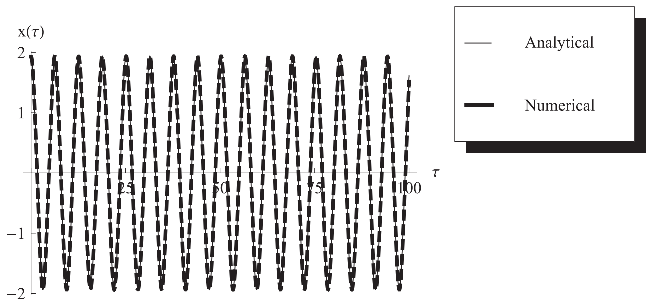

Figure 15. Comparison of time trajectories between the analytical and numerical solution (period two): $x(\tau)$ versus τ for $\alpha = 1$, $\beta = 1$, $g = \frac{1}{4}$, $\mu = 0.1$ and $\omega_f = 5$ at tenth order of HAM approximation with $h = -0.11$; for numerical solution $\omega = 1.929\,58$ and $a = 1.931\,49$.

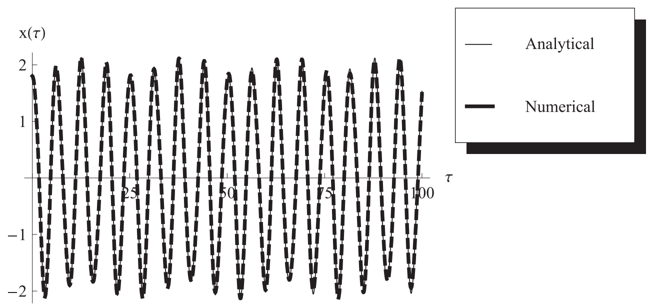

Figure 16. Comparison of time trajectories between the analytical and numerical solution (quasi-periodic): $x(\tau)$ versus τ for $\alpha = 1$, $\beta = 0.1$, $g = \frac{1}{2}$, $\mu = 0.1$ and $\omega_f = 2$ at fifth order of HAM approximation with $h = -0.68$; for numerical solution $\omega = 1.136\,02$ and $a = 1.801\,18$.

to $\frac{2\pi\omega_0}{\omega_f}$, as given in equation (25). We use Mathematica to develop the analytical solutions.

6. Conclusion

Limit cycles and quasi-periodic solutions are obtained for the first time by a modified homotopy approach for the forced Van der Pol and the forced Van der Pol Duffing oscillators.

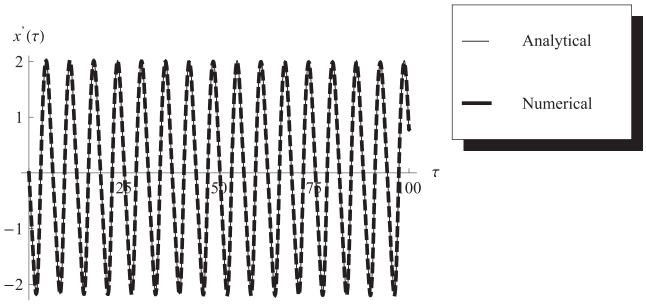

Figure 17. Comparison of time trajectories of the first derivatives between the analytical and numerical solution (period one): $x'(\tau)$ versus τ for $\alpha = 1$, $\beta = 0$, $g = \frac{1}{2}$, $\mu = 0.1$ and $\omega_f = 2$ at tenth order of HAM approximation with $h = -0.78$; for numerical solution $\omega = 0.999\,37$ and $a = 1.820\,60$.

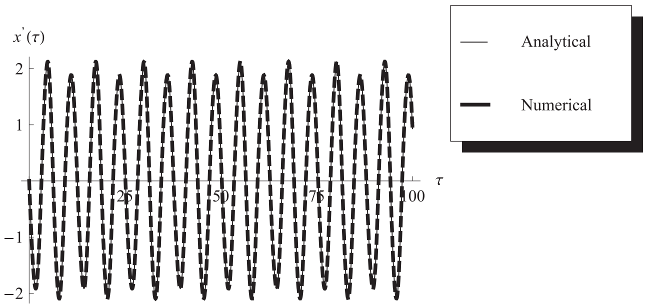

Figure 18. Comparison of time trajectories of the first derivatives between the analytical and numerical solution (period two): $x'(\tau)$ versus τ for $\alpha = 4$, $\beta = 0$, $g = \frac{1}{2}$, $\mu = 0.1$ and $\omega_f = 3$ at tenth order of HAM approximation with $h = -0.21$; for numerical solution $\omega = 1.999\,70$ and $a = 1.895\,98$.

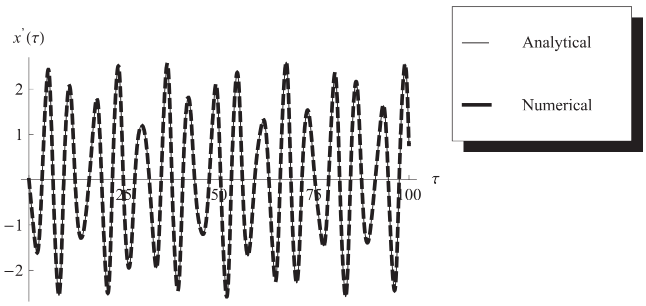

Figure 19. Comparison of time trajectories of the first derivatives between the analytical and numerical solution (quasi-periodic): $x'(\tau)$ versus τ for $\alpha = 1$, $\beta = 0$, $g = \frac{1}{2}$, $\mu = 0.1$ and $\omega_f = \sqrt{2}$ at tenth order of HAM approximation with $h = -0.87$; for numerical solution $\omega = 0.999\,83$ and $a = 1.390\,28$.

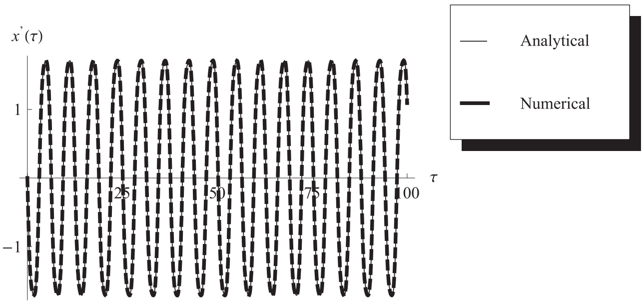

Figure 20. Comparison of time trajectories of the first derivatives between the analytical and numerical solution (period one): $x'(\tau)$ versus τ for $\alpha = 1$, $\beta = 1$, $g = 0.001$, $\mu = 0.1$ and $\omega_f = 2$ at tenth order of HAM approximation with $h = -0.24$; for numerical solution $\omega = 1.928\,61$ and $a = 1.935\,12$.

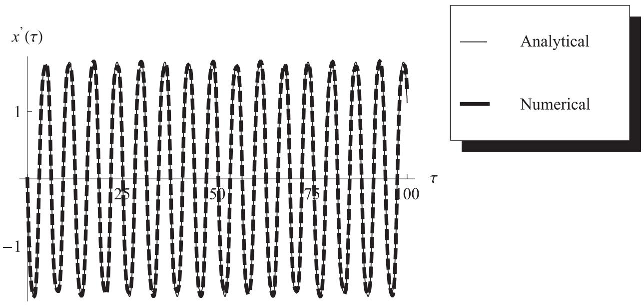

Figure 21. Comparison of time trajectories of the first derivatives between the analytical and numerical solution (period two): $x'(\tau)$ versus τ for $\alpha = 1$, $\beta = 1$, $g = \frac{1}{4}$, $\mu = 0.1$ and $\omega_f = 5$ at tenth order of HAM approximation with $h = -0.11$; for numerical solution $\omega = 1.929\,58$ and $a = 1.931\,49$.

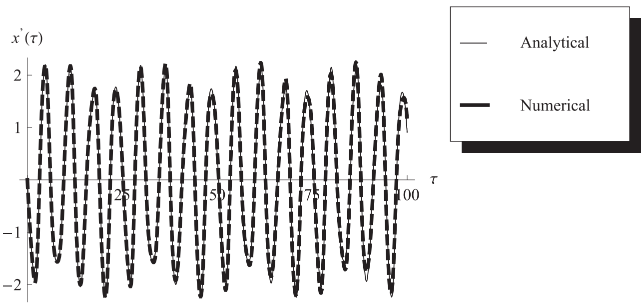

Figure 22. Comparison of time trajectories of the first derivatives between the analytical and numerical solution (quasi-periodic): $x'(\tau)$ versus τ for $\alpha = 1$, $\beta = 0.1$, $g = \frac{1}{2}$, $\mu = 0.1$ and $\omega_f = 2$ at fifth order of HAM approximation with $h = -0.68$; for numerical solution $\omega = 1.136\,02$ and $a = 1.801\,18$.

Comparison between the analytical solutions and the numerical solutions is good. The convergence of the analytical solution is demonstrated and the accuracy of the analytical solutions is confirmed by minimizing the square residual error. Therefore the proposed approach seems to be promising for the development of analytical solutions of nonlinear oscillators specially for limit cycles and quasi-periodic solutions. Thus, we have shown that a proper choice of parameter values can lead to limit cycles or quasi-periodic solutions for the forced Van der Pol and the forced Van der Pol Duffing oscillators. However, many more phenomena remain to be considered. The advantage of the proposed technique over numerical techniques is that we can determine *a priori* the parameter values that may lead to limit cycles or quasi-periodic solutions. As is well known *a priori* determination of this numerically is quite difficult. This non-perturbative approach can be extended to other nonlinear systems also.

Acknowledgments

The authors wish to thank the Council of Scientific and Industrial Research (CSIR), India, for financial support and Head, CSIR Fourth Paradigm Institute, for constant encouragement. We also wish to acknowledge the referees for valuable comments.

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