Assignment 7: seeing the butterfly in chaos

Task: Lorenz system

These exercises help us gain some intuition about chaotic dynamics: what is it exactly, how it is structured in state space, and why it matters. For this, we study the Lorenz system, a simple model derived to describe air circulation (convection) in the atmosphere, and which can also model other systems, like lasers. We study it here because it is a classical in nonlinear dynamics theory, because of its history and importance for the field (Lorenz was one of the first to show that deterministic dynamics can be unpredictable and irregular), and because it is a rather nice system. Its chaotic attractor form a butterfly shape that is quite beautiful!

The variables (x, y, z) of the Lorenz system are described by:

$$\dot{x} = -\sigma x + \sigma y$$
$$\dot{y} = -xz + rx - y$$
$$\dot{z} = xy - bz$$

The standard parameter values are $\sigma = 10$, $r = 28$, and $b = \frac{8}{3}$. To gain intuition about this model and about chaos, work on the following tasks:

1. **Check if the system is dissipative.** This is a crucial first step, as attractors (such as the chaotic attractor) can only occur in dissipative systems.
   1. For this, compute the mean dissipation rate Q in two ways:
   1. through the divergence $\nabla.\vec{F}$ of the equations. The notation means $\vec{F} := (\dot{x}, \dot{y}, \dot{z})$ and $\nabla.\vec{F} := \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z}$.
   2. through the Lyapunov exponents, which are $(\lambda_1, \lambda_2, \lambda_3) = (0.90, 0, -14.57)$ for the standard parameters.
   2. Next, discuss if the two ways are consistent, and discuss what they mean. Do the results make sense to you?
2. **Visualize the chaotic attractor:**
   1. Integrate a trajectory starting at some initial condition. Most initial conditions will work fine. Can you think of any that will not?
   2. Get the butterfly: plot the attractor. Notice that it is bounded and structured.
   3. Visualize the erratic jumping between hemispheres: plot the time series of each variable ($x(t)$, $y(t)$ and $z(t)$).
   4. Visualize the sensitivity to initial conditions: integrate two distinct, but nearby, initial conditions. Say the first initial condition is (x_0, y_0, z_0). The second initial condition can be, for instance, $(x_0 + 10^{-8}, y_0, z_0)$. Plot the attractor for both.
      1. Are the two plots similar? Are the two structures the same attractor?
      2. What happens to the distance between the two trajectories?
      3. For an easier visualization, compare the time series $x(t)$ of the two initial conditions. How long do the trajectories stay near each other, roughly?
      4. If you have time, calculate the distance between the two trajectories over time and plot it.
3. **Understand the system a bit better:**
   1. Compute the fixed points of the system. When do they exist, where are they in space? For the standard parameters, plot them along with the attractor.
   2. Allowing now r to vary, keeping b and σ at the standard values, show that there is a pitchfork bifurcation occurring at $r = 1$.
   3. Similarly to 2., show that there is a Hopf bifurcation at $r = 24.74$.

If you are curious about the system and want to know more, I recommend checking out the book "An Exploration of Dynamical Systems" by John Argyris. There is a lot more cool things going on in the chaotic attractor!