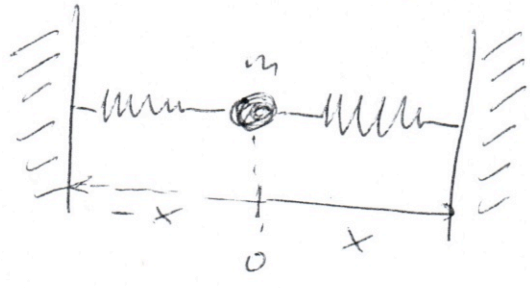
2. Übung

Lösen Sie die Schwingungsgleichung für einen Federschwinger



unter der Annahme, daß die Federkraft proportional zur Auslenkung ist

a) ohne Reibung $\quad m\ddot{x} = -kx$

b) mit Reibung $\quad m\ddot{x} + b\dot{x} + kx = 0$

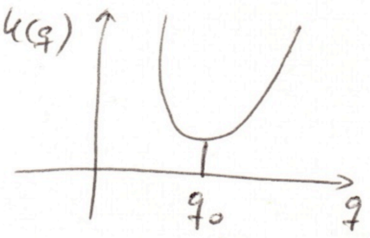
mit $\dfrac{k}{m} = \omega_0^2$, $\dfrac{b}{m} = 2h$

1. Seminar

Pendelschwingungen um eine Gleichgewichtslage:

kin. Energie: $T = \frac{1}{2} m v^2 = \frac{1}{2} m \dot{x}^2$

pot. Energie: $U = \frac{k}{2} x^2$



Lagrange-Fkt: $L(x, \dot{x}) = \frac{m}{2} \dot{x}^2 - \frac{k}{2} x^2$

$x = (q - q_0)$

$U(q) - U(q_0) \cong \frac{k}{2} (q - q_0)^2$

Entw. f. kleine Auslenkungen

Bewegungsgl.:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \quad \Rightarrow \quad \ddot{x} + \frac{k}{m} x = 0$$

Berücksichtigung der Reibung: Rayleighsche Dissipationsfkt. $F = \gamma m \dot{x}^2$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} + \frac{\partial F}{\partial \dot{x}} = 0$$

$$\Rightarrow \quad m\ddot{x} + kx - 2\gamma m \dot{x} = 0$$

$$\ddot{x} + 2\gamma \dot{x} + \frac{k}{m} x = 0$$

Rückführung auf Dgl. 1. Ord.

$$z_1 = x \qquad z_2 = \dot{x}$$

$$\dot{z}_1 = z_2$$

$$\dot{z}_2 = -2\gamma z_2 - \frac{k}{m} z_1$$

$$\dot{z} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -2\gamma \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

1.) Ist das System dissipativ?

$\dot{z} = f(z) \rightarrow$ suche div f!

$$\operatorname{div} f = \Lambda = \frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2} = 0 - 2\gamma = -2\gamma$$

$\gamma > 0$ Dämpfung $\rightarrow$ dann div $f < 0$ dissipativ

$\gamma = 0$ keine Dämpfung $\rightarrow$ System konservativ

2.) Bew. im System

Ansatz: $z \sim e^{\lambda t} y$

$$\dot{z} = \lambda e^{\lambda t} y$$

$$\lambda \begin{pmatrix} y_1 e^{\lambda t} \\ y_2 e^{\lambda t} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -2\gamma \end{pmatrix} \begin{pmatrix} y_1 e^{\lambda t} \\ y_2 e^{\lambda t} \end{pmatrix}$$

$$\begin{pmatrix} -\lambda & 1 \\ -\frac{k}{m} & -2\gamma - \lambda \end{pmatrix} \begin{pmatrix} y_1 e^{\lambda t} \\ y_2 e^{\lambda t} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

lin. Gl. system. Hat nur dann eine nichttriviale Lsg., wenn

$$\det \begin{pmatrix} -\lambda & 1 \\ -\frac{k}{m} & -2\gamma - \lambda \end{pmatrix} = 0$$

$$\begin{vmatrix} -\lambda & 1 \\ -\frac{k}{m} & -2\gamma - \lambda \end{vmatrix} = \lambda^2 + 2\gamma\lambda + \frac{k}{m} = P(\lambda) = 0$$

charakteristisches Polynom

$$\lambda_{1,2} = -\gamma \pm \sqrt{\gamma^2 - \frac{k}{m}} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

$\omega_0^2 = \frac{k}{m}$ Eigenfrequenz

a) ungedämpft

$$\gamma = 0 \Rightarrow \lambda_{1,2} = \pm i\sqrt{\frac{k}{m}} = \pm i\omega_0$$

$$z = e^{i\omega_0 t} y^{(1)} + e^{-i\omega_0 t} y^{(2)}$$

$$\begin{pmatrix} -i\omega_0 & 1 \\ -\omega_0^2 & -i\omega_0 \end{pmatrix} \begin{pmatrix} y_1^{(1)} \\ y_2^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-i\omega_0 y_1^{(1)} + y_2^{(1)} = 0 \Rightarrow y_2^{(1)} = i\omega_0 y_1^{(1)}$$

$$-\omega_0^2 y_1^{(1)} - i\omega_0 y_2^{(1)} = 0$$

$y_1^{(1)}$ beliebig, wähle $y_1^{(1)} = 1$
$y_2^{(1)} = i\omega_0$

$$\begin{pmatrix} +i\omega_0 & 1 \\ -\omega_0^2 & +i\omega_0 \end{pmatrix} \begin{pmatrix} y_1^{(2)} \\ y_2^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$i\omega_0 y_1^{(2)} + y_2^{(2)} = 0 \qquad y_2^{(1)} = -i\omega_0 y_1^{(2)}$$

$$y_2^{(1)} = -i\omega_0 \qquad y_1^{(2)} = 1$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = C_1 e^{i\omega_0 t} \begin{pmatrix} 1 \\ i\omega_0 \end{pmatrix} + C_2 e^{-i\omega_0 t} \begin{pmatrix} 1 \\ -i\omega_0 \end{pmatrix}$$

Anfangsbed. $z_1(0) = z_1^0 = C_1 + C_2$
$z_2(0) = z_2^0 = i\omega_0 C_1 - i\omega_0 C_2$

oder:
$$z_1 = C_1 \cos\omega_0 t + C_1 i \sin\omega_0 t + C_2 \cos\omega_0 t - iC_2 \sin\omega_0 t$$
$$= (C_1 + C_2)\cos\omega_0 t + i(C_1 - C_2)\sin\omega_0 t$$

$$z_2 = i\omega_0 C_1 \cos\omega_0 t + i\omega_0 C_1 i \sin\omega_0 t - C_2 i\omega_0 \cos\omega_0 t - C_2 i\omega_0 (-i)\sin\omega_0 t$$
$$= i\omega_0 (C_1 - C_2)\cos\omega_0 t + \omega_0(-C_1 - C_2)\sin\omega_0 t$$

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$$z_1^0 = c_1 + c_2 \implies c_1 = z_1^0 - c_2$$

$$z_2^0 = i\omega_0(z_1^0 - c_2) - i\omega_0 c_2 = i\omega_0 z_1^0 - 2i\omega_0 c_2$$

$$2i\omega_0 c_2 = i\omega_0 z_1^0 - z_2^0$$

$$\left.\begin{aligned} c_2 &= \frac{z_1^0}{2} - \frac{z_2^0}{2i\omega_0} \\ c_1 &= \frac{z_1^0}{2} + \frac{z_2^0}{2i\omega_0} \end{aligned}\right\} \quad \begin{aligned} c_1 + c_2 &= z_1^0 \\ c_1 - c_2 &= \frac{z_2^0}{i\omega_0} \end{aligned}$$

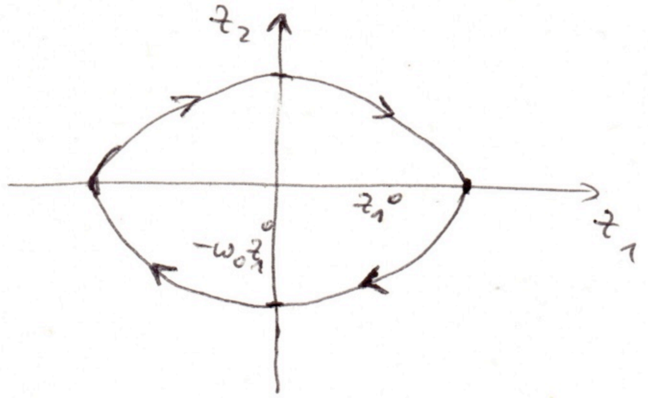
$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \left(\frac{z_1^0}{2} + \frac{z_2^0}{2i\omega_0}\right) e^{i\omega_0 t} \begin{pmatrix} 1 \\ i\omega_0 \end{pmatrix} + \left(\frac{z_1^0}{2} - \frac{z_2^0}{2i\omega_0}\right) e^{-i\omega_0 t} \begin{pmatrix} 1 \\ -i\omega_0 \end{pmatrix}$$

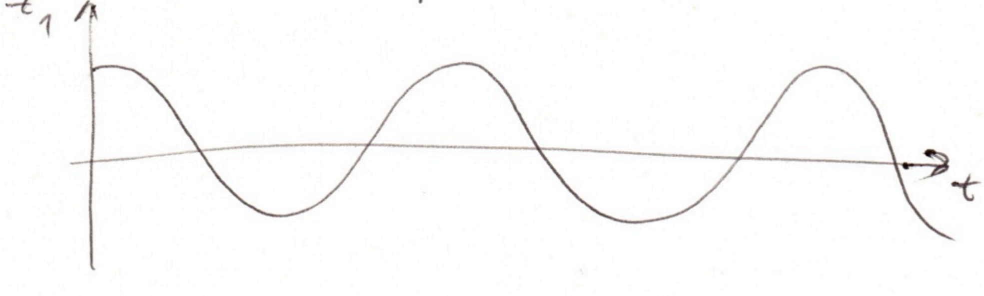
Oder

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ i\omega_0(c_1 - c_2) \end{pmatrix} \cos\omega_0 t + \begin{pmatrix} i(c_1 - c_2) \\ -\omega_0(c_1 + c_2) \end{pmatrix} \sin\omega_0 t$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1^0 \\ z_2^0 \end{pmatrix} \cos\omega_0 t + \begin{pmatrix} \frac{z_2^0}{\omega_0} \\ -\omega_0 z_1^0 \end{pmatrix} \sin\omega_0 t$$

$$\dot{z}_1 = -\omega_0 z_1^0 \sin\omega t + z_2^0 \cos\omega_0 t$$





b) gedämpft $\gamma \neq 0$

$$\lambda_{1,2} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

1 schwache Dämpfung $\gamma^2 < \omega_0^2 \Rightarrow \lambda_{1,2}$ komplex

mit $\omega^2 = \omega_0^2 - \gamma^2 \Rightarrow \lambda = -\gamma \pm i\omega$

$$z = c_1 e^{-\gamma t} e^{i\omega t} y^{(1)} + c_2 e^{-\gamma t} e^{-i\omega t} y^{(2)}$$

$$\begin{pmatrix} \gamma - i\omega & 1 \\ -\omega_0^2 & -2\gamma + \gamma - i\omega \end{pmatrix} \begin{pmatrix} y_1^{(1)} \\ y_2^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} \gamma + i\omega & 1 \\ -\omega_0^2 & -2\gamma + \gamma + i\omega \end{pmatrix} \begin{pmatrix} y_1^{(2)} \\ y_2^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$(\gamma - i\omega) y_1^{(1)} + y_2^{(1)} = 0$

$y_2^{(1)} = (-\gamma + i\omega) y_1^{(1)}$

$y_1^{(1)} = 1$

$y_2^{(1)} = -\gamma + i\omega$

$(\gamma + i\omega) y_1^{(2)} + y_2^{(2)} = 0$

$y_2^{(2)} = (-\gamma - i\omega) y_1^{(2)}$

$y_1^{(2)} = 1$

$y_2^{(2)} = -\gamma - i\omega$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = c_1 e^{-\gamma t} e^{i\omega t} \begin{pmatrix} 1 \\ -\gamma + i\omega \end{pmatrix} + c_2 e^{-\gamma t} e^{-i\omega t} \begin{pmatrix} 1 \\ -\gamma - i\omega \end{pmatrix}$$

$z_1^0 = c_1 + c_2$

$z_2^0 = c_1(-\gamma + i\omega) + c_2(-\gamma - i\omega)$

$c_1 = z_1^0 - c_2$

$z_2^0 = (z_1^0 - c_2)(-\gamma + i\omega) + c_2(-\gamma - i\omega) = (-\gamma + i\omega) z_1^0 - 2i\omega c_2$

$$\left.\begin{aligned} c_2 &= -\frac{\gamma}{2i\omega} z_1^0 + \frac{z_1^0}{2} - \frac{z_2^0}{2i\omega} \\ c_1 &= \frac{z_1^0}{2} + \frac{\gamma}{2i\omega} z_1^0 + \frac{z_2^0}{2i\omega} \end{aligned}\right\}$$

$c_1 + c_2 = z_1^0$

$c_1 - c_2 = \frac{\gamma}{i\omega} z_1^0 + \frac{z_2^0}{i\omega}$

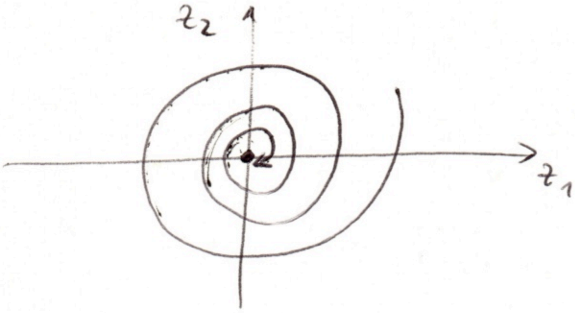
oder

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{matrix} (c_1+c_2)e^{-\nu t}\cos\omega t + i(c_1-c_2)e^{-\nu t}\sin\omega t \\ (c_1+c_2)(-\nu)\cos\omega t + i\omega(c_1-c_2)\cos\omega t \\ + i(c_1-c_2)(-\nu)\sin\omega t + i\omega \cdot i(c_1+c_2)\sin\omega t \end{matrix}$$

$$[\, i c_1(-\nu) + i c_1 \cdot i\omega + - i c_2 \cdot (-\nu) - i c_2(-i\omega)\,]\sin\omega t$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1^0 \\ \cancel{-\nu z_1^0} \cancel{+\nu z_1^0} + z_2^0 \end{pmatrix} e^{-\nu t}\cos\omega t + \begin{pmatrix} \frac{\nu}{\omega} z_1^0 + \frac{z_2^0}{\omega} \\ (-\nu)\left(\frac{\nu}{\omega} z_1^0 + \frac{z_2^0}{\omega}\right) - \omega z_1^0 \end{pmatrix} e^{-\nu t}\sin\omega t$$

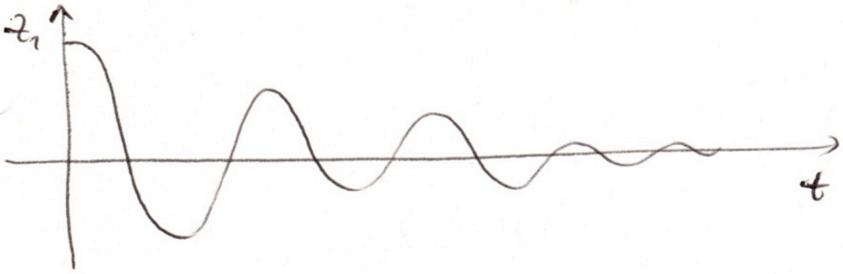
$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1^0 \\ z_2^0 \end{pmatrix} e^{-\nu t}\cos\omega t + \begin{pmatrix} \frac{\nu z_1^0 + z_2^0}{\omega} \\ (-\nu)\,\frac{\nu z_1^0 + z_2^0}{\omega} - \omega z_1^0 \end{pmatrix} e^{-\nu t}\sin\omega t$$



Für $t \to \infty$

$z_1 \to 0, \quad z_2 \to 0$

Pendel strebt gegen Ruhelage



gedämpfte Schwingung

(2) starke Dämpfung

$$\nu^2 > \omega_0^2 \Rightarrow \lambda_{1,2} = -\nu \pm \sqrt{\nu^2 - \omega_0^2} \quad \text{reell} \quad \lambda_1, \lambda_2 < 0$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = c_1 e^{-\nu t} e^{\sqrt{\nu^2-\omega_0^2}\,t} y^{(1)} + c_2 e^{-\nu t} e^{-\sqrt{\nu^2-\omega_0^2}\,t} y^{(2)}$$

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$$\begin{pmatrix} \gamma - \sqrt{\ } & +1 \\ -\omega_0^2 & -2\gamma + \gamma - \sqrt{\ } \end{pmatrix} \begin{pmatrix} y_1^{(1)} \\ y_2^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} \gamma + \sqrt{\ } & +1 \\ -\omega_0^2 & -2\gamma + \gamma + \sqrt{\ } \end{pmatrix} \begin{pmatrix} y_1^{(2)} \\ y_2^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(\gamma - \sqrt{\ })\, y_1^{(1)} + y_2^{(1)} = 0$$

$$y_2^{(1)} = (-\gamma + \sqrt{\ })\, y_1^{(1)}$$

$$y_1^{(1)} = 1$$

$$y_2^{(1)} = -\gamma + \sqrt{\gamma^2 - \omega_0^2}$$

$$(\gamma + \sqrt{\ })\, y_1^{(2)} + y_2^{(2)} = 0$$

$$y_2^{(2)} = (-\gamma - \sqrt{\ })\, y_1^{(2)}$$

$$y_1^{(2)} = 1$$

$$y_2^{(2)} = -\gamma - \sqrt{\gamma^2 - \omega_0^2}$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = c_1\, e^{-\gamma t} e^{\sqrt{\gamma^2 - \omega_0^2}\, t} \begin{pmatrix} 1 \\ -\gamma + \sqrt{\gamma^2 - \omega_0^2} \end{pmatrix} + c_2\, e^{-\gamma t} e^{-\sqrt{\gamma^2 - \omega_0^2}\, t} \begin{pmatrix} 1 \\ -\gamma - \sqrt{\gamma^2 - \omega_0^2} \end{pmatrix}$$

$$z_1^0 = c_1 + c_2$$

$$z_2^0 = c_1(-\gamma + \sqrt{\ }) + c_2(-\gamma - \sqrt{\ })$$

$$c_1 = z_1^0 - c_2$$

$$z_2^0 = (z_1^0 - c_2)(-\gamma + \sqrt{\ }) + c_2(-\gamma - \sqrt{\ })$$

$$= z_1^0(-\gamma + \sqrt{\ }) + \cancel{c_2 \gamma} - c_2\sqrt{\ } - \cancel{c_2 \gamma} - c_2\sqrt{\ }$$

$$= z_1^0(-\gamma + \sqrt{\ }) - 2c_2\sqrt{\ }$$

$$c_2 = \frac{z_1^0}{2} - \frac{\gamma z_1^0}{2\sqrt{\ }} - \frac{z_2^0}{2\sqrt{\ }}$$

$$c_1 = \frac{z_1^0}{2} + \frac{1}{2\sqrt{\ }}(+\gamma z_1^0 + z_2^0)$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \frac{z_1^0}{2} + \frac{\gamma z_1^0 + z_2^0}{2\sqrt{\gamma^2 - \omega_0^2}} \\ \frac{z_1^0}{2} - \frac{\gamma z_1^0 + z_2^0}{2\sqrt{\gamma^2 - \omega_0^2}} \end{pmatrix}$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \left(\frac{z_1^0}{2} + \frac{\gamma z_1^0 + z_2^0}{2\sqrt{\gamma^2-\omega_0^2}} \right) e^{-\gamma t} e^{\sqrt{\gamma^2-\omega_0^2}\, t} \begin{pmatrix} 1 \\ -\gamma + \sqrt{\gamma^2-\omega_0^2} \end{pmatrix}$$

$$+ \left(\frac{z_1^0}{2} - \frac{\gamma z_1^0 + z_2^0}{2\sqrt{\gamma^2-\omega_0^2}} \right) e^{-\gamma t} e^{-\sqrt{\gamma^2-\omega_0^2}\, t} \begin{pmatrix} 1 \\ -\gamma - \sqrt{\gamma^2-\omega_0^2} \end{pmatrix}$$

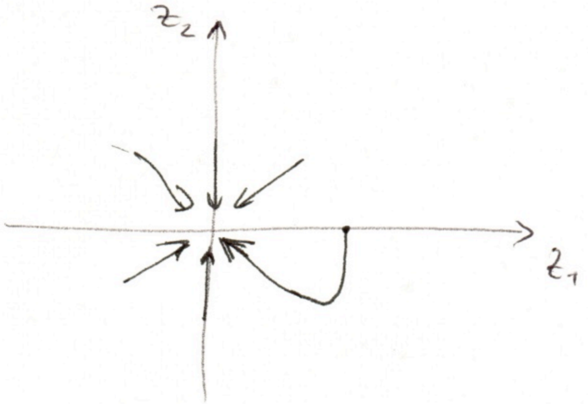
$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{matrix} z_1^0 e^{-\gamma t} \frac{1}{2}\left(e^{\sqrt{\gamma^2-\omega_0^2}\, t} + e^{-\sqrt{\gamma^2-\omega_0^2}\, t}\right) + \frac{\gamma z_1^0 + z_2^0}{\cancel{2}\sqrt{\gamma^2-\omega_0^2}} e^{-\gamma t} \left(\frac{1}{2}\left[e^{\Gamma t} - e^{-\Gamma t}\right] \right) \\ (-\gamma) z_1^0 e^{-\gamma t} \frac{1}{2}\left(e^{\Gamma t} + e^{-\Gamma t}\right) + z_1^0 \sqrt{\gamma^2-\omega_0^2}\, e^{-\gamma t} \frac{1}{2}\left(e^{\Gamma t} - e^{-\Gamma t}\right) \\ (-\gamma) \frac{\gamma z_1^0 + z_2^0}{\cancel{2}\sqrt{\gamma^2-\omega_0^2}} e^{-\gamma t} \frac{1}{2}\left(e^{\Gamma t} - e^{-\Gamma t}\right) + \frac{\gamma z_1^0 + z_2^0}{\cancel{2}\sqrt{\gamma^2-\omega_0^2}} e^{-\gamma t} \sqrt{\gamma^2-\omega_0^2} \left(\frac{1}{2}\left[e^{\Gamma t} + e^{-\Gamma t}\right] \right) \end{matrix}$$

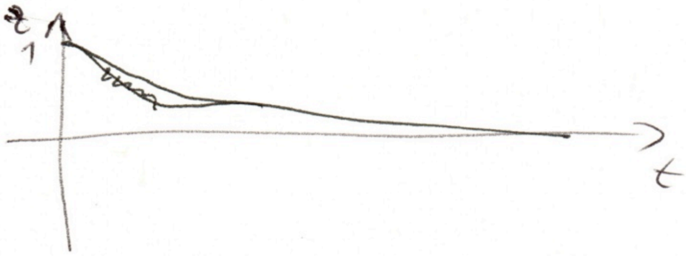
$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1^0 \\ (-\gamma) z_1^0 + \frac{\gamma z_1^0 + z_2^0}{\sqrt{\gamma^2-\omega_0^2}} \sqrt{\gamma^2-\omega_0^2} \end{pmatrix} e^{-\gamma t} \cosh \sqrt{\gamma^2-\omega_0^2}\, t$$

$$+ \begin{pmatrix} \frac{\gamma z_1^0 + z_2^0}{\sqrt{\gamma^2-\omega_0^2}} \\ z_1^0 \sqrt{\gamma^2-\omega_0^2} + (-\gamma) \frac{\gamma z_1^0 + z_2^0}{\sqrt{\gamma^2-\omega_0^2}} \end{pmatrix} e^{-\gamma t} \sinh \sqrt{\gamma^2-\omega_0^2}\, t$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1^0 \\ z_2^0 \end{pmatrix} e^{-\gamma t} \cosh \sqrt{\gamma^2-\omega_0^2}\, t$$

$$\begin{pmatrix} \frac{\gamma z_1^0 + z_2^0}{\sqrt{\gamma^2-\omega_0^2}} \\ z_1^0 \sqrt{\gamma^2-\omega_0^2} - \gamma \frac{\gamma z_1^0 + z_2^0}{\sqrt{\gamma^2-\omega_0^2}} \end{pmatrix} e^{-\gamma t} \sinh \sqrt{\gamma^2-\omega_0^2}\, t$$





c) aperiodischer Grenzfall

$\gamma = \omega_0^2 \qquad \lambda = -\gamma$ doppelter Eigenwert

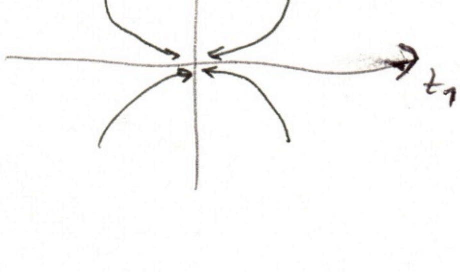
$$z_1 = e^{-\gamma t}(c_1 + c_2 t)$$

$$z_1 = e^{-\gamma t}\left(z_1^0 + (\gamma z_1^0 + z_2^0)t\right)$$

$$z_2 = \left(z_1^0 + [\gamma z_1^0 + z_2^0]t\right)(-\gamma)\, e^{-\gamma t} + e^{-\gamma t}(\gamma z_1^0 + z_2^0)$$

$$z_2 = e^{-\gamma t}\left[z_2^0 - \gamma(\gamma z_1^0 + z_2^0)t\right]$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1^0 \\ z_2^0 \end{pmatrix} e^{-\gamma t} + \begin{pmatrix} \gamma z_1^0 + z_2^0 \\ -\gamma(\gamma z_1^0 + z_2^0) \end{pmatrix} t\, e^{-\gamma t}$$



Lineare Pendelschwingungen

$$\dot z_1 = z_2 \qquad\qquad z_1(0) = z_1^0$$
$$\dot z_2 = -2\rho z_2 - \frac{k}{m} z_1 = -2\rho z_2 - \omega_0^2 z_1 \qquad\qquad z_2(0) = z_2^0$$

Lösungen:

1.) ungedämpft $\rho = 0$ $\quad \lambda = \pm i\omega_0$ $\qquad \omega_0^2 = \frac{k}{m}$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1^0 \\ z_2^0 \end{pmatrix} \cos\omega_0 t + \begin{pmatrix} \frac{z_2^0}{\omega_0} \\ -\omega_0 z_1^0 \end{pmatrix} \sin\omega_0 t$$

2.) gedämpft $\lambda = -\rho \pm \sqrt{\rho^2 - \omega_0^2}$ komplex $\quad \rho^2 < \omega_0^2$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1^0 \\ z_2^0 \end{pmatrix} e^{-\rho t} \cos\omega_0 t + \begin{pmatrix} \frac{\rho z_1^0 + z_2^0}{\omega} \\ -\rho \frac{\rho z_1^0 + z_2^0}{\omega} - \omega z_1^0 \end{pmatrix} e^{-\rho t} \sin\omega t$$

$$\omega^2 = \omega_0^2 - \rho^2$$

3.) gedämpft $\lambda = -\rho \pm \sqrt{\rho^2 - \omega_0^2}$ reell $\quad \rho^2 > \omega_0^2$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1^0 \\ z_2^0 \end{pmatrix} e^{-\rho t} \cosh\sqrt{\rho^2 - \omega_0^2}\, t$$

$$+ \begin{pmatrix} \frac{\rho z_1^0 + z_2^0}{\sqrt{\rho^2 - \omega_0^2}} \\ z_1^0 \sqrt{\rho^2 - \omega_0^2} - \rho \frac{\rho z_1^0 + z_2^0}{\sqrt{\rho^2 - \omega_0^2}} \end{pmatrix} e^{-\rho t} \sinh\sqrt{\rho^2 - \omega_0^2}\, t$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$