# Kalman Filter using MATLAB
# 1: Introduction

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# Overview

1. Introduction
   - State estimation: Basic idea
   - Types of dynamical systems
2. State estimation
   - Types of estimation problems
   - Deterministic estimator
   - Stochastic estimator

# Introduction

# State estimation: Basic idea

- State estimation problem deals with estimating the states of a dynamical system using the available information.
- States are the dynamic variables of a system that captures the dynamics of the system.
- Knowing the full state information has many practical significance in control systems, guidance and navigation, etc.
- In general measuring all the states using sensors is impractical.
- **Kalman estimator**: is an optimal linear estimator.

# Notations

- $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ : Set of natural numbers, integers and real numbers
- $\mathbb{R}^n$ : $n-$ dimensional Euclidean space
- $\mathbb{R}^{m\times n}$ : Space of $m \times n$ real matrices
- $X, x$ : Scalar $X, x$
- $\mathbf{X}, \mathbf{x}$ : Matrix/vector $\mathbf{X}, \mathbf{x}$
- $\mathbb{X}$ : Set $\mathbb{X}$
- $\mathbf{P} > 0$ : Real symmetric positive definite matrix $\mathbf{P}$
- $\mathbf{P} \geq 0$ : Real symmetric positive semidefinite matrix $\mathbf{P}$
- $\mathbf{I}, \mathbf{0}$ : Identity matrix and zero matrix.

# Notations

- $k$ : Discrete time instant $k \in \{0, 1, ..., N\}$
- $\mathbf{x}_k$ : State vector (actual state), $\mathbf{x}_k \in \mathbb{R}^n$
- $\mathbf{u}_k$ : Control input vector, $\mathbf{u}_k \in \mathbb{R}^m$
- $\mathbf{y}_k$ : Output vector, $\mathbf{y}_k \in \mathbb{R}^p$
- $\mathbf{d}_k$ : Disturbance vector, $\mathbf{d}_k \in \mathbb{R}^n$
- $\mathbf{v}_k$ : Noise vector, $\mathbf{v}_k \in \mathbb{R}^p$
- $\mathbf{A}$ : System matrix, $\mathbf{A} \in \mathbb{R}^{n \times n}$
- $\mathbf{B}$ : Input matrix, $\mathbf{B} \in \mathbb{R}^{n \times m}$
- $\mathbf{C}$ : Output matrix, $\mathbf{C} \in \mathbb{R}^{p \times n}$
- $\hat{\mathbf{x}}_k$ : Estimated state vector
- $\hat{\mathbf{x}}_{k|i}$ : Estimate of the state at time instant $k$ computed using the measurement information at time instant $i$.

# Types of dynamical systems

- In general dynamical systems can be classified as
  1. **Deterministic system**: in which the states can be exactly predicted using the model given the initial state and control input. Eg. Deterministic linear time invariant (LTI) system:

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k \\ \mathbf{y}_k &= \mathbf{C}\mathbf{x}_k \end{aligned} \tag{1}$$

  2. **Uncertain system**: has some uncertain parameters/variables in the system model and the model predicts the state approximately. Eg. Stochastic linear time invariant (LTI) system:

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{d}_k \\ \mathbf{y}_k &= \mathbf{C}\mathbf{x}_k + \mathbf{v}_k \end{aligned} \tag{2}$$

# Types of dynamical systems

- Deterministic system: Example

$$x_{k+1} = 0.5x_k + u_k \tag{3}$$

- Stochastic system: Example

$$x_{k+1} = 0.5x_k + u_k + d_k \tag{4}$$

- Simulation parameters

$$x_0 = 5, \quad u_k = -0.2x_k, \quad d_k = 0.25r_1, \quad N = 25 \tag{5}$$

**Algorithm 1** (Forward simulation)

1: Initialize $x_0 = 5$
2: **for** $k = 0$ *to* $N - 1$ **do**
3: $\quad u_k = -0.2x_k$
4: $\quad d_k = 0.25r_1, \qquad d_k = 0 \ (for\ deterministic\ system)$
5: $\quad x_{k+1} = 0.5x_k + u_k + d_k$
6: **end for**

# Types of dynamical systems

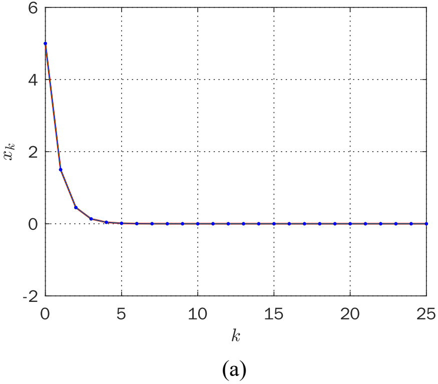


Figure 1: (a) Deterministic system (b) Stochastic system

# State estimation

# State estimator: Block diagram

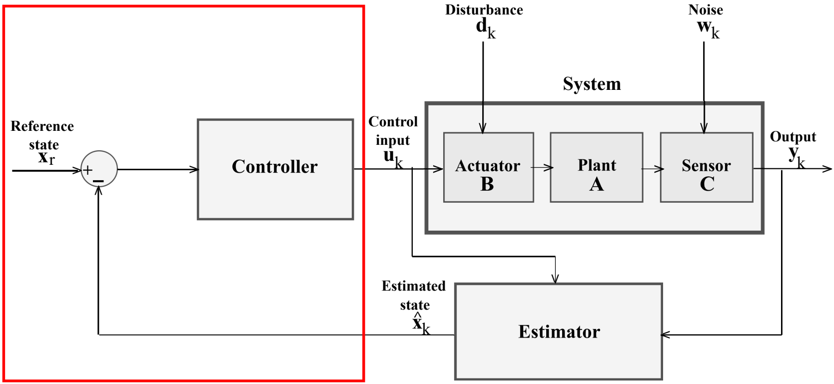


Figure 2: State estimator with a state feedback controller

# Types of estimation problems

- Let $l$ denotes the time instant up to which the output/measurement data is available, and suppose the estimate of the state: $\hat{\mathbf{x}}_k$ at time instant $k$ is made using the measurement up to time instant $l$, i.e., $\{\mathbf{y}_0, \mathbf{y}_1, ..., \mathbf{y}_l\}$.
- Based on $l$, the estimation problem can be classified as follows
  1. **Prediction problem**: in which $l < k$ and the estimator is called predictor.
  2. **Filtering problem**: in which $l = k$ and the estimator is called filter.
  3. **Smoothing problem**: in which $l > k$ and the estimator is called smoother.
- Based on nature of the system, estimation problems are grouped as
  1. **Deterministic estimation**: which deals with the state estimation of a deterministic system.
  2. **Stochastic estimation**: which deals with the state estimation of a stochastic system.

# Deterministic estimator

- One of the popular deterministic estimator is the Luenberger observer
- For the deterministic LTI system

$$
\begin{aligned}
\mathbf{x}_{k+1} &= \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k \\
\mathbf{y}_k &= \mathbf{C}\mathbf{x}_k
\end{aligned} \tag{6}
$$

- The Luenberger observer is defined as

$$
\hat{\mathbf{x}}_{k+1} = \mathbf{A}\hat{\mathbf{x}}_k + \mathbf{B}\mathbf{u}_k + \mathbf{L}[\mathbf{y}_k - \hat{\mathbf{y}}_k]. \tag{7}
$$

- Define the estimation error vector $\mathbf{x}_{e_k} = \mathbf{x}_k - \hat{\mathbf{x}}_k$, and the error dynamics becomes

$$
\begin{aligned}
\mathbf{x}_{e_{k+1}} &= \mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k - \mathbf{A}\hat{\mathbf{x}}_k - \mathbf{B}\mathbf{u}_k - \mathbf{L}[\mathbf{C}\mathbf{x}_k - \mathbf{C}\hat{\mathbf{x}}_k] \\
&= [\mathbf{A} - \mathbf{L}\mathbf{C}]\mathbf{x}_{e_k}
\end{aligned} \tag{8}
$$

- For which the solution is

$$
\mathbf{x}_{e_k} = [\mathbf{A} - \mathbf{L}\mathbf{C}]\mathbf{x}_{e_{k-1}} = [\mathbf{A} - \mathbf{L}\mathbf{C}]^2\mathbf{x}_{e_{k-2}} = \ldots = [\mathbf{A} - \mathbf{L}\mathbf{C}]^k\mathbf{x}_{e_0} \tag{9}
$$

# Stochastic estimator

- For the stochastic LTI system

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{d}_k \\ \mathbf{y}_k &= \mathbf{C}\mathbf{x}_k + \mathbf{v}_k \end{aligned} \tag{10}$$

- The Luenberger observer is

$$\hat{\mathbf{x}}_{k+1} = \mathbf{A}\hat{\mathbf{x}}_k + \mathbf{B}\mathbf{u}_k + \mathbf{L}[\mathbf{y}_k - \hat{\mathbf{y}}_k]. \tag{11}$$

- Define the error vector $\mathbf{x}_{e_k} = \mathbf{x}_k - \hat{\mathbf{x}}_k$, and the error dynamics becomes

$$\begin{aligned} \mathbf{x}_{e_{k+1}} &= \mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1} \\ &= \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{d}_k - \mathbf{A}\hat{\mathbf{x}}_k - \mathbf{B}\mathbf{u}_k - \mathbf{L}[\mathbf{C}\mathbf{x}_k + \mathbf{v}_k - \mathbf{C}\hat{\mathbf{x}}_k] \\ &= [\mathbf{A} - \mathbf{L}\mathbf{C}]\mathbf{x}_{e_k} + \mathbf{d}_k - \mathbf{L}\mathbf{v}_k \end{aligned} \tag{12}$$

- For faster error convergence the observer gain is to be large, and for noise rejection, the observer gain is to be small.
- For stochastic systems, a Luenberger observer with fixed gain $\mathbf{L}$ may not be sufficient, and one can go for the Kalman estimator, which is an optimal state estimator with time-varying gain $\mathbf{L}_k$.

Thank you