Kalman Filter using MATLAB
7: Kalman smoother derivation

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Overview

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Kalman smoother: Basic idea

Kalman smoother: Basic idea

- In Kalman smoother, the measurements upto time instant N are used for computing the estimate $\hat{\mathbf{x}}_k$, i.e., $l = N$.
- Kalman smoother consists of a forward recursion (filtering) followed by a backward recursion (smoothing), i.e., the algorithm has two stages:
  1. **Forward pass**: in which the state $\mathbf{x}_k$ is estimated using the measurement information upto time instant k, i.e., computing $\hat{\mathbf{x}}_{k|k}$ and $\mathbf{P}_{k|k}$ using a forward recursion (Kalman Filter).
  2. **Backward pass**: in which the estimate $\hat{\mathbf{x}}_{k|k}$ is improved using the measurements: $\{\mathbf{y}_{k+1}, ..., \mathbf{y}_N\}$, i.e., computing $\hat{\mathbf{x}}_{k|N}$ and $\mathbf{P}_{k|N}$ using a backward recursion.

Kalman smoother derivation

Kalman smoother derivation

- Consider the stochastic linear system

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}_k\mathbf{x}_k + \mathbf{B}_k\mathbf{u}_k + \mathbf{d}_k \\ \mathbf{y}_k &= \mathbf{C}_k\mathbf{x}_k + \mathbf{v}_k \end{aligned} \tag{1}$$

- Assume that the disturbance, noise and initial state vector as Gaussian with mean $\mathbf{E}(\mathbf{d}_k) = \mathbf{0}, \mathbf{E}(\mathbf{v}_k) = \mathbf{0}$ and $\mathbf{E}(\mathbf{x}_0)$ is known.
- Also, the vectors $\mathbf{x}_k, \mathbf{d}_k, \mathbf{v}_k$ are assumed to be independent:

$$\mathbf{V}(\mathbf{x}_k, \mathbf{d}_k) = \mathbf{0}, \;\; \mathbf{V}(\mathbf{x}_k, \mathbf{v}_k) = \mathbf{0}, \;\; \mathbf{V}(\mathbf{d}_k, \mathbf{v}_k) = \mathbf{0} \tag{2}$$

- Let $\mathbf{x}$ and $\mathbf{d}$ are two independent random vectors such that $\mathbf{E}(\mathbf{x}) = \mathbf{0}$, $\mathbf{E}(\mathbf{d}) = \mathbf{0}$, then

$$\begin{aligned} \mathbf{E}(\mathbf{A}\mathbf{x} + \mathbf{d}) &= \mathbf{A}\mathbf{E}(\mathbf{x}) + \mathbf{E}(\mathbf{d}) = \mathbf{0} \\ \mathbf{V}(\mathbf{A}\mathbf{x} + \mathbf{d}) &= \mathbf{A}\mathbf{V}(\mathbf{x})\mathbf{A}^T + \mathbf{V}(\mathbf{d}) \end{aligned} \tag{3}$$

Forward pass (Kalman filter algorithm)

- In the forward pass, we compute the estimate $\hat{\mathbf{x}}_{k|k}$ using the Kalman filter equation

$$\begin{aligned}
\hat{\mathbf{x}}_{k|k-1} &= \mathbf{A}_{k-1}\hat{\mathbf{x}}_{k-1|k-1} + \mathbf{B}_{k-1}\mathbf{u}_{k-1} \\
\hat{\mathbf{x}}_{k|k} &= \hat{\mathbf{x}}_{k|k-1} + \mathbf{L}_k[\mathbf{y}_k - \hat{\mathbf{y}}_{k|k-1}].
\end{aligned} \tag{4}$$

- Here $\mathbf{P}_{k|k-1}, \mathbf{L}_k, \mathbf{P}_{k|k}$ is obtained as in the Kalman filter:

$$\begin{aligned}
\mathbf{P}_{k|k-1} &= \mathbf{A}_{k-1}\mathbf{P}_{k-1|k-1}\mathbf{A}_{k-1}^T + \mathbf{Q}_{k-1} \\
\mathbf{L}_k &= \mathbf{P}_{k|k-1}\mathbf{C}_k^T\left[\mathbf{C}_k\mathbf{P}_{k|k-1}\mathbf{C}_k^T + \mathbf{R}_k\right]^{-1} \\
\mathbf{P}_{k|k} &= [\mathbf{I} - \mathbf{L}_k\mathbf{C}_k]\mathbf{P}_{k|k-1}[\mathbf{I} - \mathbf{L}_k\mathbf{C}_k]^T + \mathbf{L}_k\mathbf{R}_k\mathbf{L}_k^T
\end{aligned} \tag{5}$$

Backward pass

- Using the conditional expectation equation, the expected value of $\mathbf{x}_k$ given $\mathbf{x}_{k+1}$ can be obtained as

$$\begin{aligned}\hat{\mathbf{x}}_{k|k+1} &= \mathbf{E}(\mathbf{x}_k|\mathbf{x}_{k+1}) = \mathbf{E}(\mathbf{x}_k) + \mathbf{L}[\mathbf{x}_{k+1} - \mathbf{E}(\mathbf{x}_{k+1})] \\ &= \hat{\mathbf{x}}_{k|k} + \mathbf{L}_{s_k}[\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|k}]\end{aligned} \tag{6}$$

  where $\mathbf{L} = \mathbf{L}_{s_k} \in \mathbb{R}^{n\times n}$ is the smoother gain.

- Define the error $\mathbf{x}_{e_{k|k+1}} = \mathbf{x}_k - \hat{\mathbf{x}}_{k|k+1}$, and the error dynamics

$$\begin{aligned}\mathbf{x}_{e_{k|k+1}} &= \mathbf{x}_k - \hat{\mathbf{x}}_{k|k+1} = \mathbf{x}_k - \hat{\mathbf{x}}_{k|k} - \mathbf{L}_{s_k}[\mathbf{A}_k\mathbf{x}_k + \mathbf{B}_k\mathbf{u}_k + \mathbf{d}_k - \mathbf{A}_k\hat{\mathbf{x}}_{k|k} - \mathbf{B}_k\mathbf{u}_k] \\ &= [\mathbf{I} - \mathbf{L}_{s_k}\mathbf{A}_k]\mathbf{x}_{e_{k|k}} - \mathbf{L}_{s_k}\mathbf{d}_k\end{aligned} \tag{7}$$

- For which the variance matrix $\mathbf{P}_{k|k+1} = \mathbf{V}(\mathbf{x}_{e_{k|k+1}})$ becomes

$$\mathbf{P}_{k|k+1} = [\mathbf{I} - \mathbf{L}_{s_k}\mathbf{A}_k]\mathbf{P}_{k|k}[\mathbf{I} - \mathbf{L}_{s_k}\mathbf{A}_k]^T + \mathbf{L}_{s_k}\mathbf{Q}_k\mathbf{L}_{s_k}^T. \tag{8}$$

Backward pass

- The cost function for Kalman smoother is defined as

$$J = E(\mathbf{x}_{e_{k|k+1}} \mathbf{x}_{e_{k|k+1}}^T) = Trace(\mathbf{P}_{k|k+1}) \tag{9}$$

- Then the smoother gain is obtained from $\frac{\partial Trace(\mathbf{P}_{k|k+1})}{\partial \mathbf{L}_{s_k}} = \mathbf{0}$, which results in:

$$\begin{aligned} &-2[\mathbf{I} - \mathbf{L}_{s_k}\mathbf{A}_k]\mathbf{P}_{k|k}\mathbf{A}_k^T + 2\mathbf{L}_{s_k}\mathbf{Q}_k = \mathbf{0} \\ &\implies \mathbf{L}_{s_k} = \mathbf{P}_{k|k}\mathbf{A}_k^T[\mathbf{A}_k\mathbf{P}_{k|k}\mathbf{A}_k^T + \mathbf{Q}_k]^{-1} = \mathbf{P}_{k|k}\mathbf{A}_k^T\mathbf{P}_{k+1|k}^{-1}. \end{aligned} \tag{10}$$

- The Kalman smoother gain can also derived from the conditional expectation formula,

$$\begin{aligned} \mathbf{L}_{s_k} &= \mathbf{V}(\mathbf{x}_k, \mathbf{x}_{k+1})\mathbf{V}(\mathbf{x}_{k+1})^{-1} \\ &= \mathbf{V}(\mathbf{x}_k, \mathbf{A}_k\mathbf{x}_k + \mathbf{B}_k\mathbf{u}_k + \mathbf{d}_k)\mathbf{V}(\mathbf{x}_{k+1})^{-1} \\ &= \mathbf{P}_{k|k}\mathbf{A}_k^T\mathbf{P}_{k+1|k}^{-1}. \end{aligned} \tag{11}$$

Backward pass

- **Law of iterated expectation**: for random vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$, the law of iterated expectation is $\mathbf{E}(\mathbf{x}|\mathbf{y}) = \mathbf{E}(\mathbf{E}(\mathbf{x}|\mathbf{z})|\mathbf{y})$.
- Now, for $\mathbf{x} = \mathbf{x}_k, \mathbf{y} = \mathbf{y}_N, \mathbf{z} = \mathbf{x}_{k+1}$, we obtain

$$\begin{aligned} \hat{\mathbf{x}}_{k|N} &= \mathbf{E}(\mathbf{x}_k|\mathbf{y}_N) = \mathbf{E}(\mathbf{E}(\mathbf{x}_k|\mathbf{x}_{k+1})|\mathbf{y}_N) \\ &= \mathbf{E}(\hat{\mathbf{x}}_{k|k} + \mathbf{L}_{s_k}[\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|k}]|\mathbf{y}_N) \\ &= \hat{\mathbf{x}}_{k|k} + \mathbf{L}_{s_k}[\hat{\mathbf{x}}_{k+1|N} - \hat{\mathbf{x}}_{k+1|k}]. \end{aligned} \tag{12}$$

- Define the error vector $\mathbf{x}_{e_{k|N}} = \mathbf{x}_k - \hat{\mathbf{x}}_{k|N}$, and the error dynamics

$$\begin{aligned} \mathbf{x}_{e_{k|N}} &= \mathbf{x}_k - \hat{\mathbf{x}}_{k|N} = \mathbf{x}_k - \hat{\mathbf{x}}_{k|k} - \mathbf{L}_{s_k}[\hat{\mathbf{x}}_{k+1|N} - \hat{\mathbf{x}}_{k+1|k} + \mathbf{x}_{k+1} - \mathbf{x}_{k+1}] \\ &= \mathbf{x}_{e_{k|k}} + \mathbf{L}_{s_k}[\mathbf{x}_{e_{k+1|N}} - \mathbf{x}_{e_{k+1|k}}] \end{aligned} \tag{13}$$

- For which we obtain the variance matrix $\mathbf{P}_{k|N} = \mathbf{V}(\mathbf{x}_{e_{k|N}})$ as

$$\mathbf{P}_{k|N} = \mathbf{P}_{k|k} + \mathbf{L}_{s_k}[\mathbf{P}_{k+1|N} - \mathbf{P}_{k+1|k}]\mathbf{L}_{s_k}^T. \tag{14}$$

Kalman smoother: Algorithm

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Algorithm 1 (Kalman smoother)

1: Compute and store $\hat{\mathbf{x}}_{k|k-1}$, $\hat{\mathbf{x}}_{k|k}$, $\mathbf{P}_{k|k-1}$, $\mathbf{P}_{k|k}$ for $k = 1, 2, ..., N$ using Kalman filter algorithm.
2: **for** $k = N-1$ *to* 0 **do**
3: $\quad \mathbf{A}_k = [\mathbb{A}]_{k+1}$
4: $\quad \mathbf{L}_{s_k} = \mathbf{P}_{k|k}\mathbf{A}_k^T\mathbf{P}_{k+1|k}^{-1}$
5: $\quad \hat{\mathbf{x}}_{k|N} = \hat{\mathbf{x}}_{k|k} + \mathbf{L}_{s_k}[\hat{\mathbf{x}}_{k+1|N} - \hat{\mathbf{x}}_{k+1|k}]$
6: $\quad \mathbf{P}_{k|N} = \mathbf{P}_{k|k} + \mathbf{L}_{s_k}[\mathbf{P}_{k+1|N} - \mathbf{P}_{k+1|k}]\mathbf{L}_{s_k}^T$
7: **end for**

Thank you