Kalman Filter using MATLAB
9: Kalman estimator - steady state analysis

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Overview

1. Steady-state analysis
    - Observability
2. Kalman estimators in steady-state

Steady-state analysis

Steady-state analysis

- Steady-state analysis studies the behaviour of the system in steady-state, i.e., as $k \to \infty$.
- In steady-state analysis of Kalman estimators we focus on the convergence of the Riccati matrices and estimator gains as k increases.
- A sufficient requirement for the steady-state convergence of estimators is the observability of the system.

Observability

- Consider the discrete-time linear system

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k \\ \mathbf{y}_k &= \mathbf{C}\mathbf{x}_k \end{aligned} \tag{1}$$

- Observability: Can we recover the state from outputs?

$$\begin{aligned} \mathbf{y}_0 &= \mathbf{C}\mathbf{x}_0 \\ \mathbf{y}_1 &= \mathbf{C}\mathbf{x}_1 = \mathbf{C}\mathbf{A}\mathbf{x}_0 + \mathbf{C}\mathbf{B}\mathbf{u}_0 \\ &\vdots \\ \mathbf{y}_{n-1} &= \mathbf{C}\mathbf{A}^{n-1}\mathbf{x}_0 + \mathbf{C}\Sigma_{k=0}^{n-2}\mathbf{A}^{(n-2)-k}\mathbf{B}\mathbf{u}_k \end{aligned} \tag{2}$$

- Which can be compactly written as

$$\begin{bmatrix} \mathbf{y}_0 \\ \mathbf{y}_1 - \mathbf{C}\mathbf{B}\mathbf{u}_0 \\ \vdots \\ \mathbf{y}_{n-1} - \mathbf{C}\Sigma_{k=0}^{n-2}\mathbf{A}^{(n-2)-k}\mathbf{B}\mathbf{u}_k \end{bmatrix} = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \vdots \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix} \mathbf{x}_0 \tag{3}$$

Observability

- By defining $\mathbf{O}_x = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}$ and $\mathbf{Y}_n = \begin{bmatrix} \mathbf{y}_0 \\ \mathbf{y}_1 - \mathbf{CBu}_0 \\ \vdots \\ \mathbf{y}_{n-1} - \mathbf{C}\Sigma_{k=0}^{n-2}\mathbf{A}^{(n-2)-k}\mathbf{Bu}_k \end{bmatrix}$ we can rewrite the last equation as

$$\mathbf{O}_x\mathbf{x}_0 = \mathbf{Y}_n \tag{4}$$

- The above equation gives a unique solution $\mathbf{x}_0$ if rank of $\mathbf{O}_x = n$.
- The linear system $[\mathbf{A}, \mathbf{C}]$ is said to be observable, if the observability matrix $\mathbf{O}_x = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}$ *has rank* $= n$.

Kalman estimators in steady-state

Steady state analysis - Kalman predictor

- For LTI systems, if $[\mathbf{A}, \mathbf{C}]$ is observable and $\mathbf{Q} > 0$, the Kalman predictor DRE with $\mathbf{P}_0 > 0$ converges to a unique positive definite solution $\mathbf{P}$ of the Algebraic Riccati Equation (ARE)

$$\mathbf{P} = [\mathbf{A} - \mathbf{LC}]\mathbf{P}[\mathbf{A} - \mathbf{LC}]^T + \mathbf{Q} + \mathbf{LRL}^T \tag{5}$$

- This results in the unique estimator gain

$$\mathbf{L} = \mathbf{APC}^T[\mathbf{CPC}^T + \mathbf{R}]^{-1} \tag{6}$$

such that all the eigenvalues of $\mathbf{A} - \mathbf{LC}$ lies inside the unit disk.

Steady state analysis - Kalman filter

- For LTI systems, if $[\mathbf{A}, \mathbf{C}]$ is observable and $\mathbf{Q} > 0$, the Kalman filter DRE with $\mathbf{P}_0 > 0$ converges to a unique positive definite solution $\mathbf{P}$ of the Algebraic Riccati Equation (ARE)

$$\mathbf{P} = [\mathbf{I} - \mathbf{L}\mathbf{C}]\left[\mathbf{A}\mathbf{P}\mathbf{A}^T + \mathbf{Q}\right][\mathbf{I} - \mathbf{L}\mathbf{C}]^T + \mathbf{L}\mathbf{R}\mathbf{L}^T \tag{7}$$

- This results in the unique estimator gain

$$\mathbf{L} = \left[[\mathbf{A}\mathbf{P}\mathbf{A}^T + \mathbf{Q}]\mathbf{C}^T\right]\left[\mathbf{C}[\mathbf{A}\mathbf{P}\mathbf{A}^T + \mathbf{Q}]\mathbf{C}^T + \mathbf{R}\right]^{-1} \tag{8}$$

such that all the eigenvalues of $\mathbf{A} - \mathbf{L}\mathbf{C}$ lies inside the unit disk.

Kalman estimators in steady-state

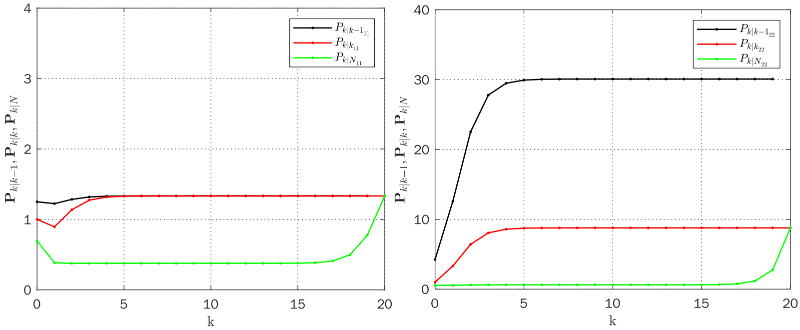

Figure 1: Kalman estimators in steady-state

Kalman estimators in steady-state

- Among the three estimators, the steady-state variance is minimum for the Kalman smoother and maximum for the Kalman predictor.
- The Kalman smoother gives more reliable estimate of the states.
- The Kalman smoother cannot estimate the states in real time, i.e., there is a delay associated with the Kalman smoother estimate, hence not suitable for implementing state feedback.

Thank you