Modified Rodrigues Parameters (MRPs)

The “cool” new attitude coordinates...

MRP Definitions

Euler parameter relationship:

$$\sigma_i = \frac{\beta_i}{1+\beta_0} \qquad i = 1,2,3$$

Singular if -1
(±360° case)

$$\beta_0 = \frac{1-\sigma^2}{1+\sigma^2}$$

$$\beta_i = \frac{2\sigma_i}{1+\sigma^2} \qquad i = 1,2,3$$

Singular if ∞
(±360° case)

PRV relationship:

$$\boldsymbol{\sigma} = \tan\frac{\Phi}{4}\hat{\boldsymbol{e}}$$

Singular for ±360°

$$\boldsymbol{\sigma} \approx \frac{\Phi}{4}\hat{\boldsymbol{e}}$$

Linearizes to
angles over 4.

(Show Mathematica Example)

CRP relationship:

$$\boldsymbol{q} = \frac{2\boldsymbol{\sigma}}{1-\sigma^2}$$

$$\boldsymbol{\sigma} = \frac{\boldsymbol{q}}{1+\sqrt{1+\boldsymbol{q}^T\boldsymbol{q}}}$$

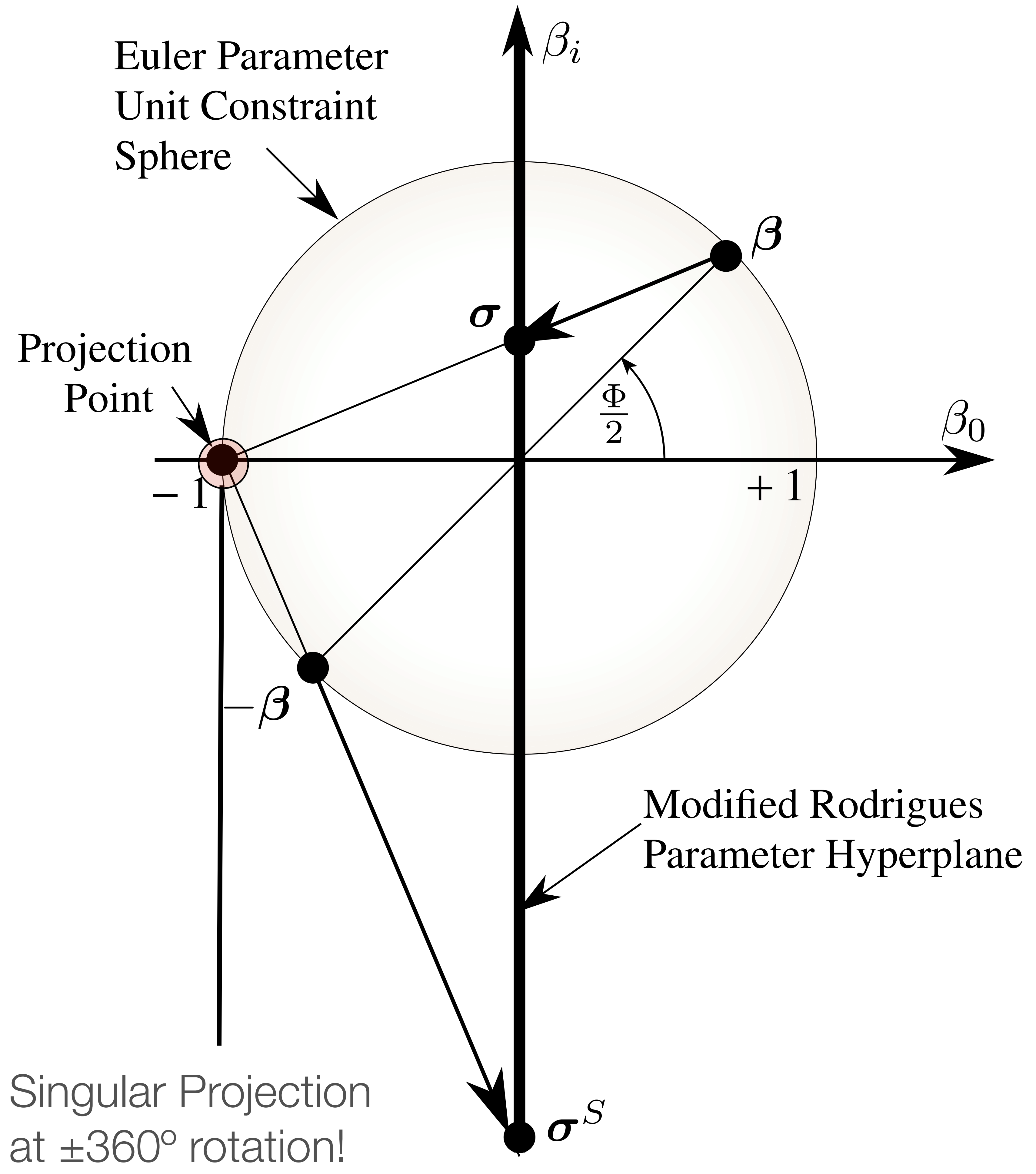
MRP Definitions

Relationship to DCM:

$$[\tilde{\boldsymbol{\sigma}}] = \frac{[C]^T - [C]}{\zeta(\zeta+2)} \qquad \zeta = \sqrt{\text{trace}([C]) + 1} = \beta_0/2$$

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} = \frac{1}{\zeta(\zeta+2)} \begin{pmatrix} C_{23} - C_{32} \\ C_{31} - C_{13} \\ C_{12} - C_{21} \end{pmatrix}$$

Stereographic Projection

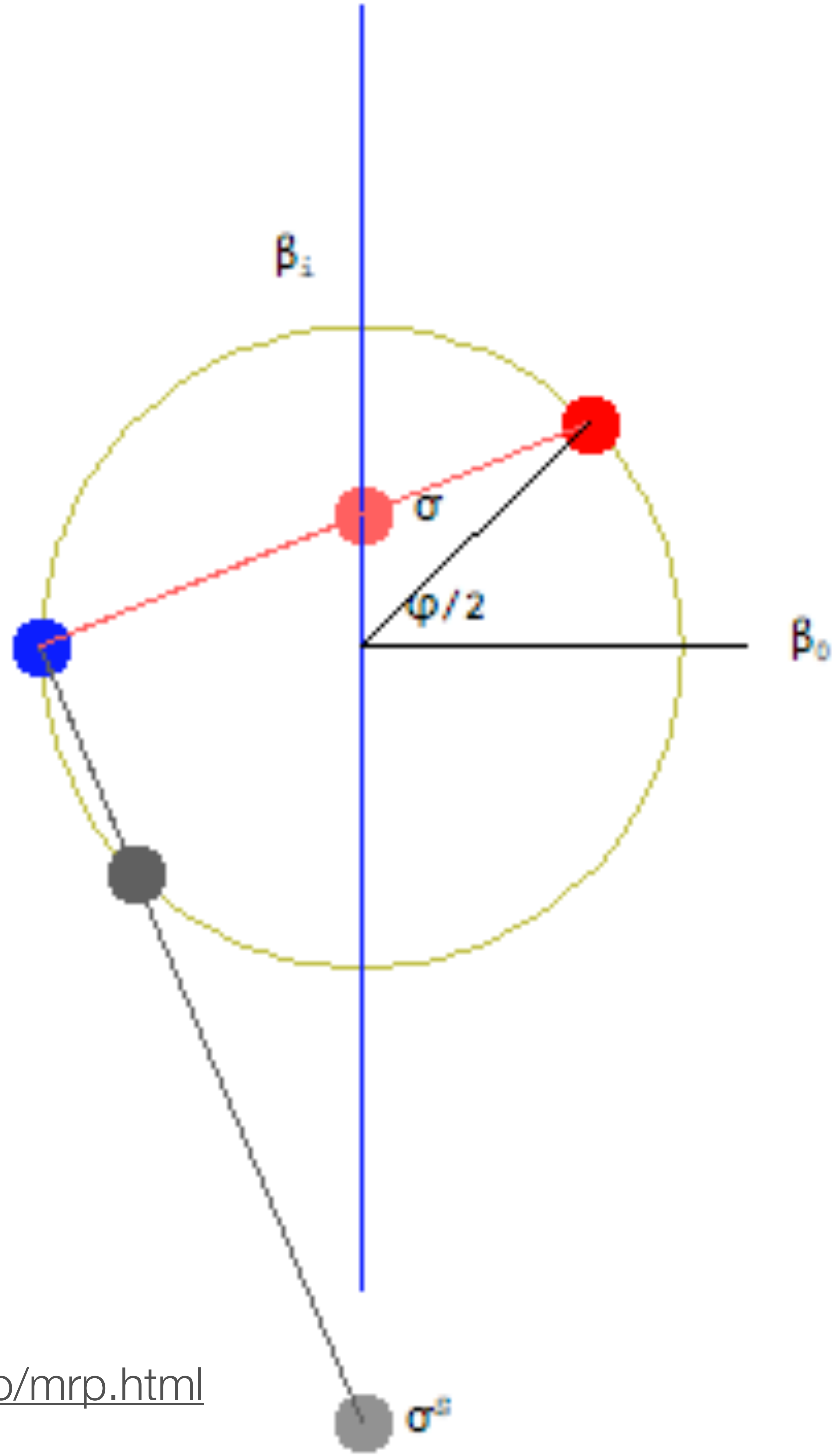


Projection Point: $(-1, 0, 0, 0)$

Projection Plane: $\beta_0 = 0$

Any attitude (surface point) is projected onto the hyper-plane to form the modified Rodrigues parameters.

The two EP sets yield *distinct* MRP coordinate values with different singular behaviors.



http://hanspeterschaub.info/mrp.html

Shadow MRP Set

- Using the alternate set of Euler parameters, we can find the "shadow" set of MRP parameters:

$$\sigma_i^S = \frac{-\beta_i}{1-\beta_0} = \frac{-\sigma_i}{\sigma^2} \qquad i = 1, 2, 3$$

Unique MRP Parameters

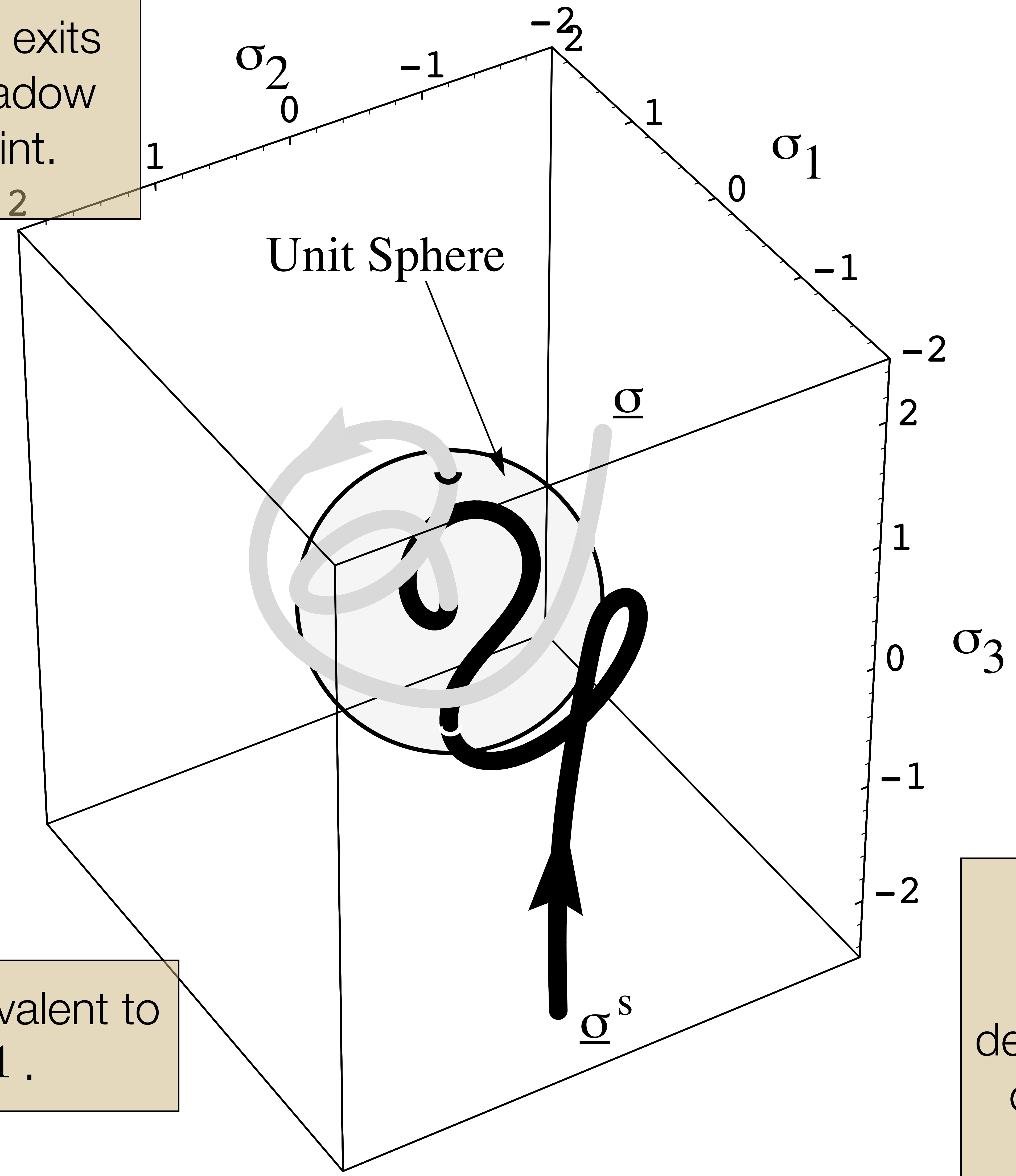
A common switching surface is $\sigma^2 = \boldsymbol{\sigma} \cdot \boldsymbol{\sigma} = 1$. Note that

$$|\boldsymbol{\sigma}| \leq 1 \quad \text{if} \quad \Phi \leq 180^{\circ}$$
$$|\boldsymbol{\sigma}| \geq 1 \quad \text{if} \quad \Phi \geq 180^{\circ}$$
$$|\boldsymbol{\sigma}| = 1 \quad \text{if} \quad \Phi = 180^{\circ}$$

$$\boldsymbol{\sigma}^S = \tan\left(\frac{\Phi - 2\pi}{4}\right)\hat{\boldsymbol{e}}$$

$$\boldsymbol{\sigma}^S = \tan\left(\frac{\Phi'}{4}\right)\hat{\boldsymbol{e}}$$

As one set of MRP coordinates exits the unit sphere surface, the shadow set enters at the opposite point.



Setting $\beta_0 \geq 0$ is equivalent to enforcing $|\boldsymbol{\sigma}| \leq 1$.

The original shadow set of MRPs are convenient to describe tumbling bodies. The coordinates always point to the zero attitude along the shortest rotational path

Direction Cosine Matrix

Matrix components:

$$[C] = \frac{1}{(1+\sigma^2)^2} \begin{bmatrix} 4\left(\sigma_1^2 - \sigma_2^2 - \sigma_3^2\right) + (1-\sigma^2)^2 & 8\sigma_1\sigma_2 + 4\sigma_3(1-\sigma^2) & \cdots \\ 8\sigma_2\sigma_1 - 4\sigma_3(1-\sigma^2) & 4\left(-\sigma_1^2 + \sigma_2^2 - \sigma_3^2\right) + (1-\sigma^2)^2 & \cdots \\ 8\sigma_3\sigma_1 + 4\sigma_2(1-\sigma^2) & 8\sigma_3\sigma_2 - 4\sigma_1(1-\sigma^2) \end{bmatrix}$$

$$\cdots \begin{bmatrix} 8\sigma_1\sigma_3 - 4\sigma_2(1-\sigma^2) \\ 8\sigma_2\sigma_3 + 4\sigma_1(1-\sigma^2) \\ 4\left(-\sigma_1^2 - \sigma_2^2 + \sigma_3^2\right) + (1-\sigma^2)^2 \end{bmatrix}$$

Vector computation:

$$[C] = [I_{3\times3}] + \frac{8[\tilde{\boldsymbol{\sigma}}]^2 - 4\left(1-\sigma^2\right)[\tilde{\boldsymbol{\sigma}}]}{\left(1+\sigma^2\right)^2}$$

Interesting property:

$$[C(\boldsymbol{\sigma})]^{-1} = [C(\boldsymbol{\sigma})]^T = [C(-\boldsymbol{\sigma})]$$

Attitude Addition/Subtraction

- DCM method:

$$[FN(\boldsymbol{\sigma})] = [FB(\boldsymbol{\sigma}'')][BN(\boldsymbol{\sigma}')]$$

- Direct method:

$$\boldsymbol{\sigma} = \frac{(1 - |\boldsymbol{\sigma}'|^2)\boldsymbol{\sigma}'' + (1 - |\boldsymbol{\sigma}''|^2)\boldsymbol{\sigma}' - 2\boldsymbol{\sigma}'' \times \boldsymbol{\sigma}'}{1 + |\boldsymbol{\sigma}'|^2|\boldsymbol{\sigma}''|^2 - 2\boldsymbol{\sigma}' \cdot \boldsymbol{\sigma}''}$$

Attitude Addition

$$\boldsymbol{\sigma}'' = \frac{(1 - |\boldsymbol{\sigma}'|^2)\boldsymbol{\sigma} - (1 - |\boldsymbol{\sigma}|^2)\boldsymbol{\sigma}' + 2\boldsymbol{\sigma} \times \boldsymbol{\sigma}'}{1 + |\boldsymbol{\sigma}'|^2|\boldsymbol{\sigma}|^2 + 2\boldsymbol{\sigma}' \cdot \boldsymbol{\sigma}}$$

Relative Attitude (Subtraction)

Differential Kinematic Equations

Matrix components:

$$\dot{\boldsymbol{\sigma}} = \frac{1}{4}\begin{bmatrix} 1-\sigma^2+2\sigma_1^2 & 2\left(\sigma_1\sigma_2-\sigma_3\right) & 2\left(\sigma_1\sigma_3+\sigma_2\right) \\ 2\left(\sigma_2\sigma_1+\sigma_3\right) & 1-\sigma^2+2\sigma_2^2 & 2\left(\sigma_2\sigma_3-\sigma_1\right) \\ 2\left(\sigma_3\sigma_1-\sigma_2\right) & 2\left(\sigma_3\sigma_2+\sigma_1\right) & 1-\sigma^2+2\sigma_3^2 \end{bmatrix} {}^{\mathcal{B}}\!\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix}$$

Vector computation:

$$\dot{\boldsymbol{\sigma}} = \frac{1}{4}\left[\left(1-\sigma^2\right)[I_{3\times 3}] + 2[\tilde{\boldsymbol{\sigma}}] + 2\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right] {}^{\mathcal{B}}\boldsymbol{\omega} = \frac{1}{4}[B(\boldsymbol{\sigma})]\, {}^{\mathcal{B}}\boldsymbol{\omega}$$

Note: Only contains quadratic nonlinearities, but is singular for $\Phi = \pm 360^{\circ}$.

- Now, let's invert the differential kinematic equation and find:

$$\boldsymbol{\omega} = 4[B]^{-1}\dot{\boldsymbol{\sigma}}$$

- Note the near-orthogonal property of the [*B*] matrix:

$$[B]^{-1} = \frac{1}{\left(1+\sigma^2\right)^2}[B]^T$$

  You can proof this by investigating [*B*][*B*]^T.

- This leads to the elegant inverse transformation

$$\boldsymbol{\omega} = \frac{4}{\left(1+\sigma^2\right)^2}[B]^T\dot{\boldsymbol{\sigma}}$$

$$\boldsymbol{\omega} = \frac{4}{\left(1+\sigma^2\right)^2}\left[\left(1-\sigma^2\right)[I_{3\times3}] - 2[\tilde{\boldsymbol{\sigma}}] + 2\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right]\dot{\boldsymbol{\sigma}}$$

Cayley Transform

- Let [*S*] be a skew-symmetric matrix, [*C*] be a proper orthogonal matrix, and [*I*] be a identity matrix. These matrices can be of any dimension *N*. The **extended Cayley Transform** is then defined as:

$$\boxed{[C] = ([I] - [S])^2([I] + [S])^{-2} = ([I] + [S])^{-2}([I] - [S])^2}$$

Unfortunately no equivalent inverse transformation exists. Instead, we define [*W*] to be the "square root" of [*C*]:

$$[C] = [W][W]$$

$$[C] = [V][D][V]^*$$

Adjoint Operator

- The "matrix square root" can then be defined as

$$[W] = [V] \begin{bmatrix} \ddots & & 0 \\ & \sqrt{[D]_{ii}} & \\ 0 & & \ddots \end{bmatrix} [V]^*$$

$$[W] = [V] \begin{bmatrix} e^{+i\frac{\theta_1}{2}} & 0 & \cdots & & & 0 \\ 0 & e^{-i\frac{\theta_1}{2}} & \cdots & & & 0 \\ \vdots & \vdots & \ddots & & & 0 \\ & & & e^{+i\frac{\theta_{N-1}}{2}} & 0 & 0 \\ & & & 0 & e^{-i\frac{\theta_{N-1}}{2}} & 0 \\ & & & 0 & 0 & +1 \end{bmatrix} [V]^*$$ Odd dimension

$$[W] = [V] \begin{bmatrix} e^{+i\frac{\theta_1}{2}} & 0 & \cdots & & 0 \\ 0 & e^{-i\frac{\theta_1}{2}} & \cdots & & 0 \\ \vdots & \vdots & \ddots & & 0 \\ & & & e^{+i\frac{\theta_{N-1}}{2}} & 0 \\ & & & 0 & e^{-i\frac{\theta_{N-1}}{2}} \end{bmatrix} [V]^*$$ Even dimension

- The standard Cayley transform can now be used to between between the skew-symmetric [*S*] matrix and the orthogonal [*W*] matrix:

$$[W] = ([I] - [S])([I] + [S])^{-1} \qquad = ([I] + [S])^{-1}([I] - [S])$$
$$[S] = ([I] - [W])([I] + [W])^{-1} \qquad = ([I] + [W])^{-1}([I] - [W])$$

- As with the CRP coordinates, for the 3D case the [*S*] matrix elements are MRP attitude coordinates. For higher dimensional cases, this allows us to parameterize *N*-dimensional proper orthogonal matrices using higher dimensional MRP coordinates.

- Recall that regardless of the dimensionality of the orthogonal matrix [$W(t)$], it must evolve according

$$[\dot{W}] = -[\tilde{\boldsymbol{\Omega}}][W]$$

These higher-dimensional "body angular velocities" can be related to the higher dimensional MRPs using:

$$[\tilde{\boldsymbol{\omega}}] = [\tilde{\boldsymbol{\Omega}}] + [W][\tilde{\boldsymbol{\Omega}}][W]^T$$

$$[\dot{S}] = \frac{1}{2}\left([I] + [S]\right)[\tilde{\boldsymbol{\Omega}}]\left([I] - [S]\right)$$

- This parameterization is singular whenever a principal rotation of 360º is performed.

- If these higher dimensional MRPs are singular for ±360º rotations, can this singularity be avoided by switching to “higher-dimensional shadow” set?

- This question was raised by some structures engineers trying to apply this extended Cayley transform to parameterize a proper orthogonal matrix in their problem.

- This is still an unsolved problem, is waiting to be investigated by some enterprising graduate student...