# Direction Cosine Matrix

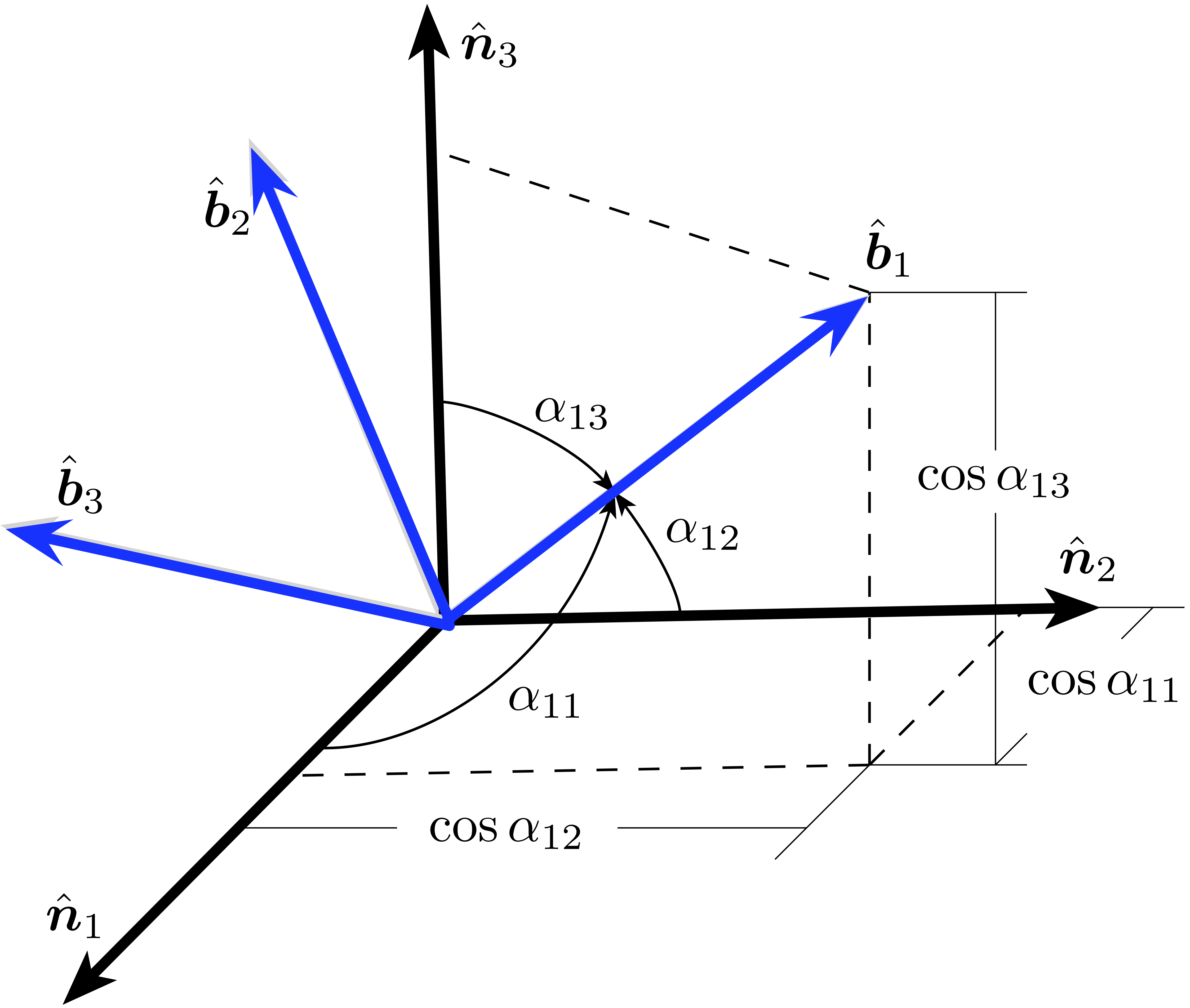
The mother of all attitude parameterizations...

# Coordinate Frames

- A vectrix is a matrix of vectors.

$$\{\hat{\boldsymbol{n}}\} \equiv \begin{bmatrix} \hat{\boldsymbol{n}}_1 \\ \hat{\boldsymbol{n}}_2 \\ \hat{\boldsymbol{n}}_3 \end{bmatrix}$$

$$\{\hat{\boldsymbol{b}}\} \equiv \begin{bmatrix} \hat{\boldsymbol{b}}_1 \\ \hat{\boldsymbol{b}}_2 \\ \hat{\boldsymbol{b}}_3 \end{bmatrix}$$



# Coordinate Frames

Frame base vectors are related through:

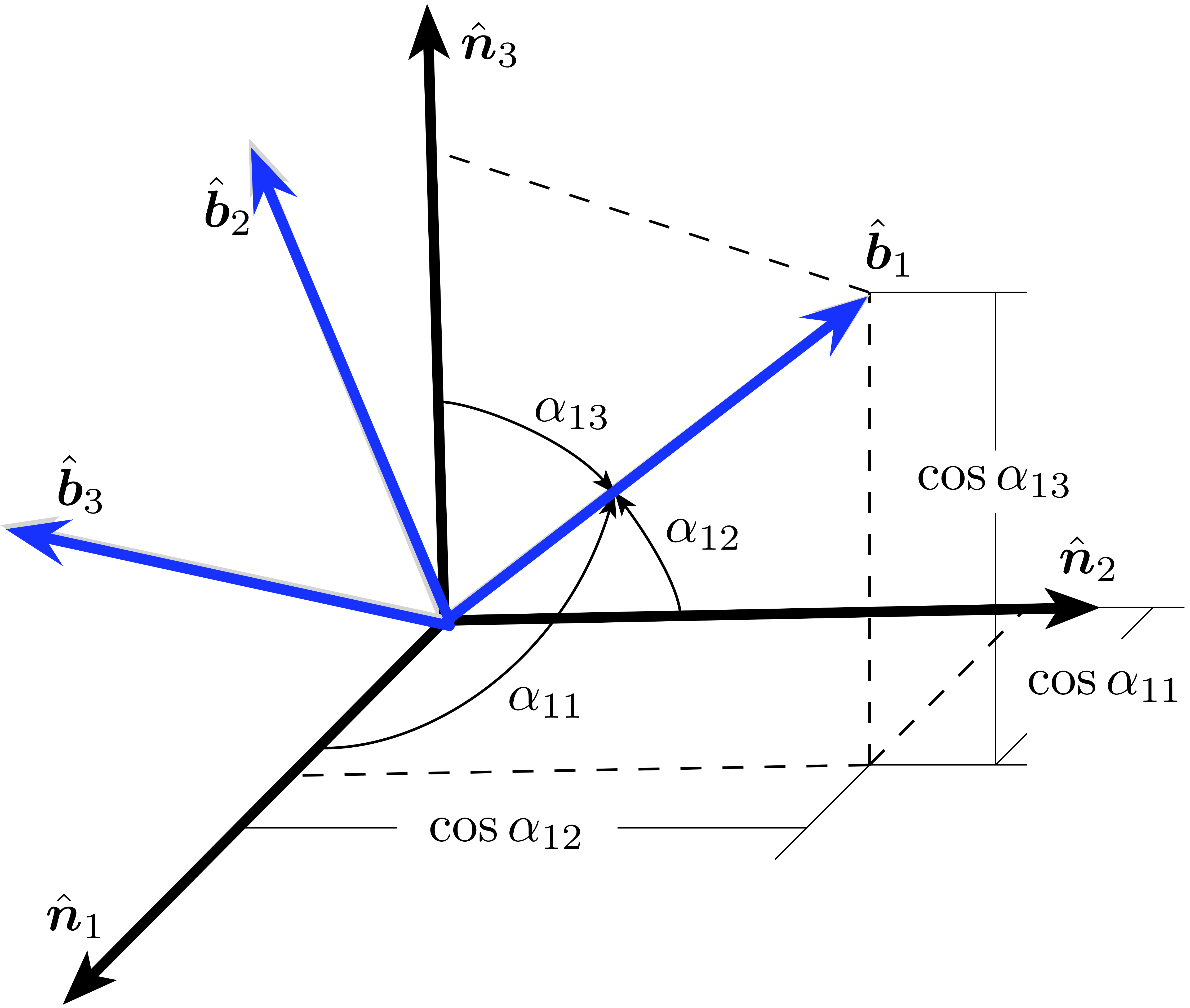
$$\hat{\boldsymbol{b}}_1 = \cos\alpha_{11}\hat{\boldsymbol{n}}_1 + \cos\alpha_{12}\hat{\boldsymbol{n}}_2 + \cos\alpha_{13}\hat{\boldsymbol{n}}_3$$
$$\hat{\boldsymbol{b}}_2 = \cos\alpha_{21}\hat{\boldsymbol{n}}_1 + \cos\alpha_{22}\hat{\boldsymbol{n}}_2 + \cos\alpha_{23}\hat{\boldsymbol{n}}_3$$
$$\hat{\boldsymbol{b}}_3 = \cos\alpha_{31}\hat{\boldsymbol{n}}_1 + \cos\alpha_{32}\hat{\boldsymbol{n}}_2 + \cos\alpha_{33}\hat{\boldsymbol{n}}_3$$

$$\{\hat{\boldsymbol{b}}\} = \begin{bmatrix} \cos\alpha_{11} & \cos\alpha_{12} & \cos\alpha_{13} \\ \cos\alpha_{21} & \cos\alpha_{22} & \cos\alpha_{23} \\ \cos\alpha_{31} & \cos\alpha_{32} & \cos\alpha_{33} \end{bmatrix} \{\hat{\boldsymbol{n}}\} = [C]\{\hat{\boldsymbol{n}}\}$$

Note that: $C_{ij} = \cos(\angle\, \hat{\boldsymbol{b}}_i, \hat{\boldsymbol{n}}_j) = \hat{\boldsymbol{b}}_i \cdot \hat{\boldsymbol{n}}_j$

Analogously, we can find:

$$\{\hat{\boldsymbol{n}}\} = \begin{bmatrix} \cos\alpha_{11} & \cos\alpha_{21} & \cos\alpha_{31} \\ \cos\alpha_{12} & \cos\alpha_{22} & \cos\alpha_{32} \\ \cos\alpha_{13} & \cos\alpha_{23} & \cos\alpha_{33} \end{bmatrix} \{\hat{\boldsymbol{b}}\} = [C]^T\{\hat{\boldsymbol{b}}\}$$



# Matrix Inverse

Combining these two results, we find

$$\{\hat{\boldsymbol{b}}\} = [C][C]^T\{\hat{\boldsymbol{b}}\} \quad \longrightarrow \quad [C][C]^T = [I_{3\times3}]$$

$$\{\hat{\boldsymbol{n}}\} = [C]^T[C]\{\hat{\boldsymbol{n}}\} \quad \longrightarrow \quad [C]^T[C] = [I_{3\times3}]$$

Therefore, the inverse of a direction cosine matrix is simply the transpose operation.

$$\boxed{[C]^{-1} = [C]^T}$$

# DCM Determinant

- Let's find the determinant of the [$C$] by first evaluating

$$\det\left(CC^T\right) = \det\left([I_{3\times3}]\right) = 1$$

- Since [$C$] is a square matrix, we find that

$$\det(C)\det(C^T) = 1$$

- Because det([$C$]) is the same as det([$C$]$^T$), this is further reduced to

$$\left(\det(C)\right)^2 = 1 \Longleftrightarrow \det(C) = \pm 1$$

- Note that this is true for any orthogonal matrix.

- For a proper rotation matrix with right-handed coordinate system, then det($C$) = +1.

# Coordinate Frame Transformation

- Let a vector have its components taken in the body frame *B* or the inertial frame *N*:

$$\boldsymbol{v} = v_{b_1}\hat{\boldsymbol{b}}_1 + v_{b_2}\hat{\boldsymbol{b}}_2 + v_{b_3}\hat{\boldsymbol{b}}_3 = \{v_b\}^T\{\hat{\boldsymbol{b}}\}$$
$$\boldsymbol{v} = v_{n_1}\hat{\boldsymbol{n}}_1 + v_{n_2}\hat{\boldsymbol{n}}_2 + v_{n_3}\hat{\boldsymbol{n}}_3 = \{v_n\}^T\{\hat{\boldsymbol{n}}\}$$

- we can now rearrange the vector expression as

$$\boldsymbol{v} = \{v_n\}^T\{\hat{\boldsymbol{n}}\} = \{v_n\}^T[C]^T\{\hat{\boldsymbol{b}}\} = \{v_b\}^T\{\hat{\boldsymbol{b}}\}$$

- Equating components, we find that the two vector component sets must be related through

$$\boldsymbol{v}_b = [C]\boldsymbol{v}_n \qquad \boldsymbol{v}_n = [C]^T\boldsymbol{v}_b$$

- From here on, we will make use of the short-hand notation:

$${}^{\mathcal{B}}\boldsymbol{v} \equiv \boldsymbol{v}_b \qquad {}^{\mathcal{N}}\boldsymbol{v} \equiv \boldsymbol{v}_n$$

# Adding DCM's

- Assume three coordinate frames given: $\mathcal{N}:\{\hat{\boldsymbol{n}}\}$ $\mathcal{B}:\{\hat{\boldsymbol{b}}\}$ $\mathcal{R}:\{\hat{\boldsymbol{r}}\}$
- Let *N* and *B* frame orientation be related through $\{\hat{\boldsymbol{b}}\} = [C]\{\hat{\boldsymbol{n}}\}$
- Let *R* and *B* frame orientation be related through $\{\hat{\boldsymbol{r}}\} = [C']\{\hat{\boldsymbol{b}}\}$
- Then the *R* and *N* frame orientation are directly related through

$$\{\hat{\boldsymbol{r}}\} = [C'][C]\{\hat{\boldsymbol{n}}\} = [C'']\{\hat{\boldsymbol{n}}\}$$

- Let us introduce the two-letter DCM notation [*NB*] as mapping from *B* to *N* frame, then the DCM addition is

$$\boxed{[RN] = [RB][BN]}$$

# Kinematic Differential Equation

- What does this mean??
  - kinematic ➠ position description
  - differential equation ➠ time rate equation

➠ what is $\boxed{[\dot{C}] = \frac{\mathrm{d}}{\mathrm{d}t}[C]}$

- How does the [*C*] direction cosine matrix evolve over time. The rotation rate of a rigid body is expressed through the body angular velocity vector:

$$\boldsymbol{\omega} = \omega_1 \hat{\boldsymbol{b}}_1 + \omega_2 \hat{\boldsymbol{b}}_2 + \omega_3 \hat{\boldsymbol{b}}_3$$

- This vector determines how a body will rotate, and thus also how the DCM describing the orientation will evolve.

# Kinematic Differential Equation

- Let's study how the body frame vectors will evolve over time as seen by the inertial frame. To do so, we differentiate the vectrix of body frame orientation vectors.

$$\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}\{\hat{\boldsymbol{b}}_i\} = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\{\hat{\boldsymbol{b}}_i\} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \{\hat{\boldsymbol{b}}_i\}$$

- Let us introduce the matrix cross-product operator:

$$[\tilde{\boldsymbol{x}}] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

where $\boldsymbol{x} \times \boldsymbol{y} \equiv [\tilde{\boldsymbol{x}}]\boldsymbol{y}$

and $[\tilde{\boldsymbol{x}}]^T = -[\tilde{\boldsymbol{x}}]$

- The body frame vectrix differential equation is then simply

$$\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}\{\hat{\boldsymbol{b}}\} = -[\tilde{\boldsymbol{\omega}}]\{\hat{\boldsymbol{b}}\}$$

# Kinematic Differential Equation

- Next take the inertial derivative of $$\{\hat{\boldsymbol{b}}\} = [C]\{\hat{\boldsymbol{n}}\}$$

$$\frac{{}^{N}\mathrm{d}}{\mathrm{d}t}\{\hat{\boldsymbol{b}}\} = \frac{{}^{N}\mathrm{d}}{\mathrm{d}t}\left([C]\{\hat{\boldsymbol{n}}\}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left([C]\right)\{\hat{\boldsymbol{n}}\} + [C]\frac{{}^{N}\mathrm{d}}{\mathrm{d}t}\left(\{\hat{\boldsymbol{n}}\}\right) = [\dot{C}]\{\hat{\boldsymbol{n}}\}$$

- This leads to

$$\frac{{}^{N}\mathrm{d}}{\mathrm{d}t}\{\hat{\boldsymbol{b}}\} = -[\tilde{\boldsymbol{\omega}}]\{\hat{\boldsymbol{b}}\} = -[\tilde{\boldsymbol{\omega}}][C]\{\hat{\boldsymbol{n}}\} = [\dot{C}]\{\hat{\boldsymbol{n}}\}$$

$$\left([\dot{C}] + [\tilde{\boldsymbol{\omega}}][C]\right)\{\hat{\boldsymbol{n}}\} = 0$$

- Since this must be true for any *N* frame orientation, we find

$$\boxed{[\dot{C}] = -[\tilde{\boldsymbol{\omega}}][C]}$$

# Kinematic Differential Equation

- An interesting fact is that this matrix differential equation holds for *any NxN orthogonal matrix*!

$$\frac{\mathrm{d}}{\mathrm{d}t}\left([C][C]^T\right) = [\dot{C}][C]^T + [C][\dot{C}]^T = 0$$

using the differential equation $[\dot{C}] = -[\tilde{\boldsymbol{\omega}}][C]$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left([C][C]^T\right) = -[\tilde{\boldsymbol{\omega}}][C][C]^T - [C][C]^T[\tilde{\boldsymbol{\omega}}]^T$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left([C][C]^T\right) = -[\tilde{\boldsymbol{\omega}}] + [\tilde{\boldsymbol{\omega}}] = 0$$