Lyapunov's Direct Method

Powerful method to prove nonlinear stability using energy methods...

More Definitions...

Positive (Negative) Definite Function: A scalar continuous function *V*(***x***) is said to be locally positive (negative) definite about ***x****ᵣ* if

$$\boldsymbol{x} = \boldsymbol{x}_r \implies V(\boldsymbol{x}) = 0$$

and there exists a δ>0 such that

$$\forall\ \boldsymbol{x} \in B_\delta(\boldsymbol{x}_r) \implies V(\boldsymbol{x}) > 0 \quad (V(\boldsymbol{x}) < 0)$$

Positive (Negative) Semi-Definite Function: A scalar continuous function *V*(***x***) is said to be locally positive (negative) semi-definite about $\boldsymbol{x}_r$ if

$$\boldsymbol{x} = \boldsymbol{x}_r \implies V(\boldsymbol{x}) = 0$$

and there exists a δ>0 such that

$$\forall\ \boldsymbol{x} \in B_\delta(\boldsymbol{x}_r) \implies V(\boldsymbol{x}) \geq 0 \quad (V(\boldsymbol{x}) \leq 0)$$

Examples: $$V(x, \dot{x}) = \frac{1}{2}x^2 + \frac{1}{2}\dot{x}^2$$

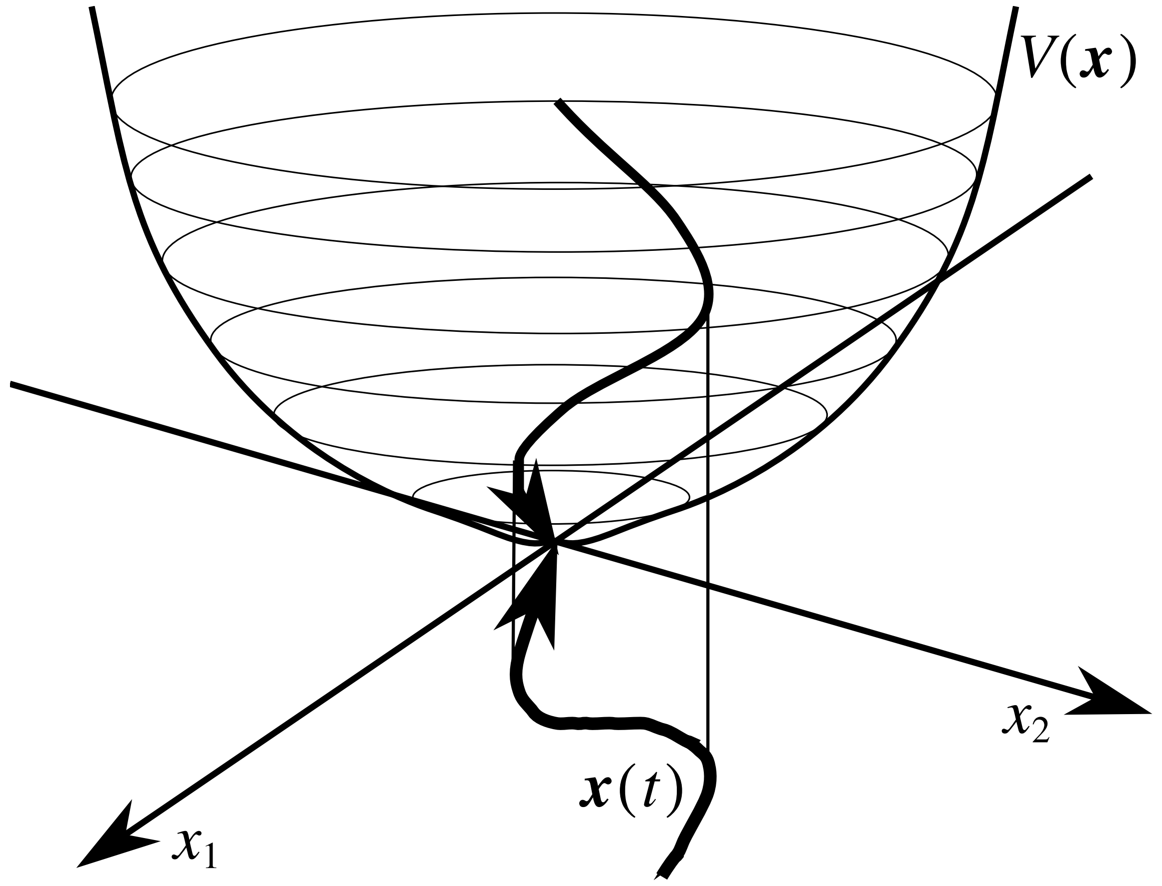
A matrix [*K*] is said to be positive or negative (semi-) definite if for every state vector ***x***:

$$\boldsymbol{x}^T[K]\boldsymbol{x} \begin{cases} > 0 & \Rightarrow \text{positive definite} \\ \geq 0 & \Rightarrow \text{positive semi-definite} \\ < 0 & \Rightarrow \text{negative definite} \\ \leq 0 & \Rightarrow \text{negative semi-definite} \end{cases}$$

Lyapunov Function

Lyapunov Function: The scalar function $V(\boldsymbol{x})$ is a Lyapunov function for the dynamical system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ if it is continuous and there exists a δ>0 such that for any $\boldsymbol{x} \in B_\delta(\boldsymbol{x}_r)$

1) $V(\boldsymbol{x})$ is a positive definite function about $\boldsymbol{x}_r$
2) $V(\boldsymbol{x})$ has continuous partial derivatives
3) $\dot{V}(\boldsymbol{x})$ is negative semi-definite



$$\dot{V} = \frac{\partial V}{\partial \boldsymbol{x}}^T \dot{\boldsymbol{x}} = \frac{\partial V}{\partial \boldsymbol{x}}^T \boldsymbol{f}(\boldsymbol{x}) \leq 0$$

All projections of the dynamical motion on to the Lyapunov function surface must point toward the reference state $\boldsymbol{x}_r$.

Lyapunov Stability: If a Lyapunov function $V(\boldsymbol{x})$ exists for the dynamical system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, then this system is stable about the origin.

Example: Consider the spring-mass system

$$m\ddot{x} + kx = 0$$

Let us use the total system energy as a candidate Lyapunov function.

$$V(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

The Lyapunov rate is then expressed as

$$\dot{V}(x, \dot{x}) = (m\ddot{x} + kx)\,\dot{x} = 0 \leq 0$$

Asymptotic Stability: Assume $V(\boldsymbol{x})$ is a Lyapunov function about $\boldsymbol{x}_r(t)$ for the dynamical system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$; then the system is asymptotically stable if

1) the system is stable about $\boldsymbol{x}_r$
2) $\dot{V}(\boldsymbol{x})$ is negative definite about $\boldsymbol{x}_r$

Example: Consider the spring-mass-damper system:

$$m\ddot{x} + c\dot{x} + kx = 0$$

with the Lyapunov function

$$V(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

Taking the derivative we only determine stability.

$$\dot{V}(x, \dot{x}) = (m\ddot{x} + kx)\,\dot{x} = -c\dot{x}^2 \leq 0$$

Evaluating the higher derivatives on the set $\dot{x} = 0$ yields:

$$\ddot{V}(\dot{x} = 0) = -2c\ddot{x}\dot{x} = 2\frac{c}{m}\,(c\dot{x} + kx)\,\dot{x} = 0$$

$$\dddot{V} = -2\frac{c}{m^2}\left((c\dot{x} + kx)^2 + c^2\dot{x}^2 + ckx\dot{x} - k\dot{x}^2\right) \Rightarrow \dddot{V}(\dot{x} = 0) = -2\frac{ck^2}{m^2}x^2 < 0$$

Theorem:* Assume there exists a Lyapunov function $V(\boldsymbol{x})$ of the dynamical system $\dot{\boldsymbol{x}}=\boldsymbol{f}(\boldsymbol{x})$. Let Ω be the non-empty set of state vectors such that

$$\boldsymbol{x} \in \Omega \implies \dot{V}(\boldsymbol{x}) = 0$$

If the first k-1 derivatives of $V(\boldsymbol{x})$, evaluated on the set Ω, are zero

$$\frac{\mathrm{d}^i V(\boldsymbol{x})}{\mathrm{d}t^i} = 0 \quad \forall\ \boldsymbol{x} \in \Omega \qquad i = 1, 2, \ldots, k-1$$

and the k^{th} derivative is negative definite on the set Ω

$$\frac{\mathrm{d}^k V(\boldsymbol{x})}{\mathrm{d}t^k} < 0 \quad \forall\ \boldsymbol{x} \in \Omega$$

then the system $\boldsymbol{x}(t)$ is asymptotically stable if k is an odd number.

*R. Mukherjee and D. Chen, "Asymptotic Stability Theorem for Autonomous Systems," *Journal of Guidance, Control and Dynamics*, Vol. 16, Sept. –Oct. 1993, pp. 961–963.

Lyapunov Stability of Linear System

- Assume that the dynamical system is of the linear form:

$$\dot{\boldsymbol{x}} = [A]\boldsymbol{x}$$

- Let [*P*]>0 be a symmetric, p.d. matrix, then we define

$$V(\boldsymbol{x}) = \boldsymbol{x}^T[P]\boldsymbol{x}$$

$$\Downarrow$$

$$\dot{V} = \dot{\boldsymbol{x}}^T[P]\boldsymbol{x} + \boldsymbol{x}^T[P]\dot{\boldsymbol{x}}$$

$$\Downarrow$$

$$\dot{V} = \boldsymbol{x}^T\left([A]^T[P] + [P][A]\right)\boldsymbol{x} < 0 \quad \checkmark$$

is this negative definite?

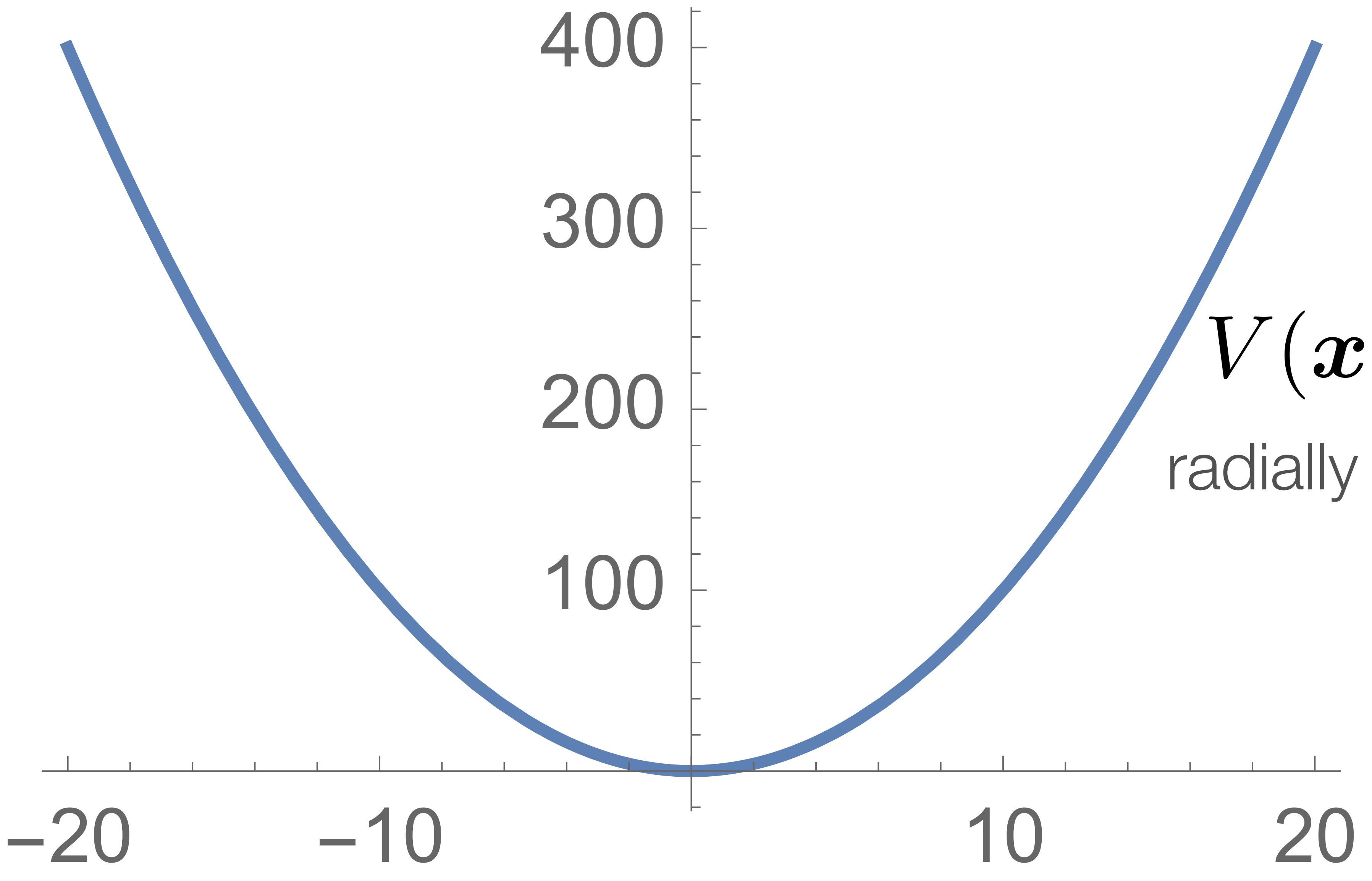
Theorem: An autonomous linear system $\dot{\boldsymbol{x}} = [A]\boldsymbol{x}$ is stable if and only if for any symmetric, positive definite [*R*] there exists a corresponding symmetric, positive definite [*P*] such that

$$[A]^T[P] + [P][A] = -[R]$$

algebraic Lyapunov equation

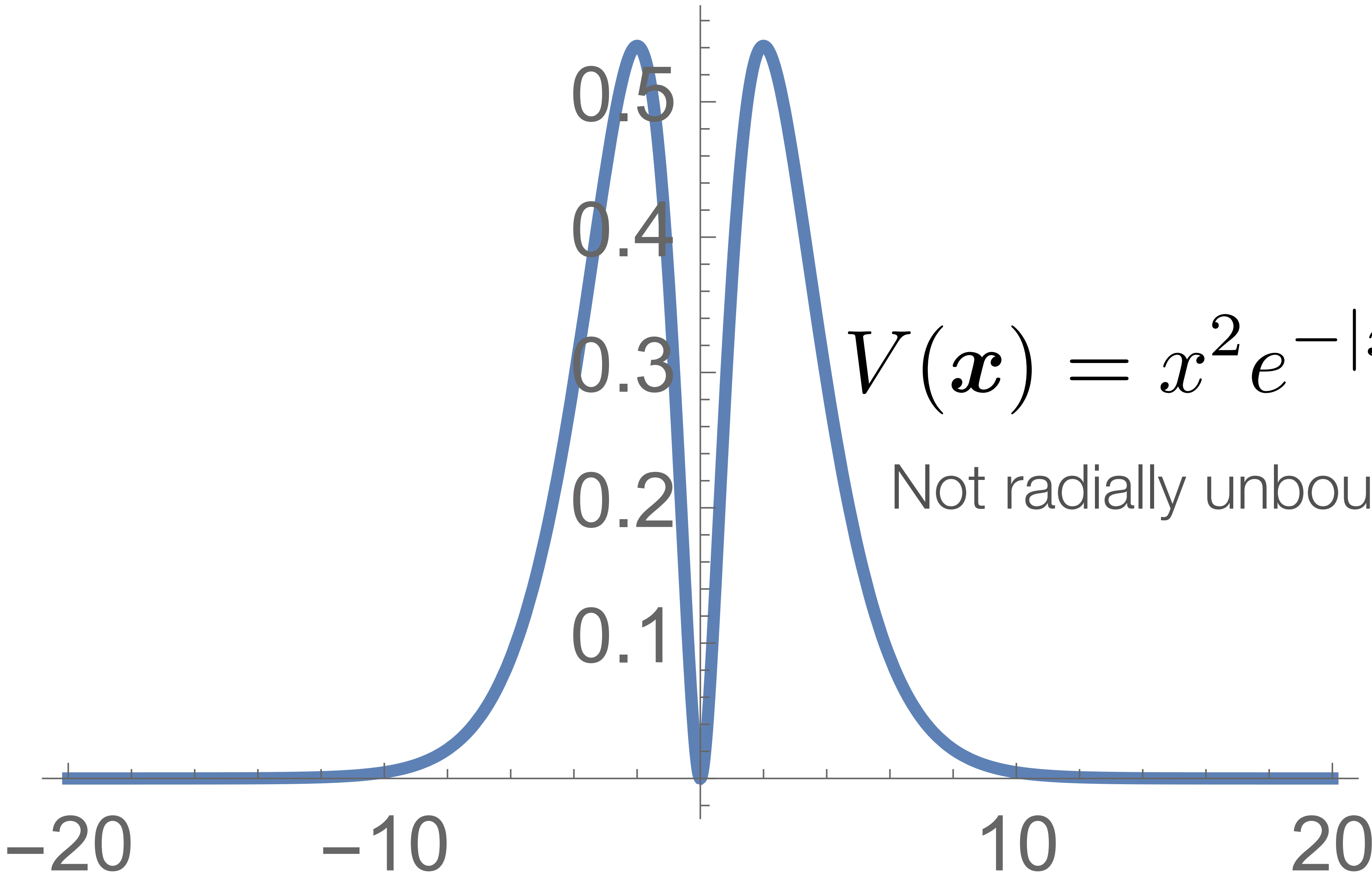
Global Stability

- The stability argument holds for any initial conditions
- The V(**x**) function is radially unbounded $\lim_{|\boldsymbol{x}|\to\infty} V(\boldsymbol{x}) = \infty$



$V(\boldsymbol{x}) = x^2$

radially unbounded



$V(\boldsymbol{x}) = x^2 e^{-|x|}$

Not radially unbounded

Lyapunov Functions

Elegant energy functions to make the control design/analysis simpler...

Elemental Velocity-Based Lyapunov Functions

Finding proper Lyapunov functions can be a difficult task for many systems.

We will break up this search into rate and position based Lyapunov functions.

Goal: drive only the state rates to zero

$$\dot{\boldsymbol{q}} \rightarrow 0$$

General Mechanical System

State Vector: $(\boldsymbol{q}, \dot{\boldsymbol{q}})$

Goal: $\dot{\boldsymbol{q}} \to 0$

Kinetic Energy: $\boxed{T = \frac{1}{2}\dot{\boldsymbol{q}}^T[M]\dot{\boldsymbol{q}}}$ True for any natural mechanical system

with $[M] = [M]^T > 0$

EOM: $\boxed{[M(\boldsymbol{q})]\ddot{\boldsymbol{q}}} = -[\dot{M}(\boldsymbol{q}, \dot{\boldsymbol{q}})]\dot{\boldsymbol{q}} + \frac{1}{2}\dot{\boldsymbol{q}}^T[M_q(\boldsymbol{q})]\dot{\boldsymbol{q}} + \boxed{\boldsymbol{Q}}$ with $\dot{\boldsymbol{q}}^T[M_{\boldsymbol{q}}(\boldsymbol{q})]\dot{\boldsymbol{q}} \equiv \begin{pmatrix} \dot{\boldsymbol{q}}^T \left[\frac{\partial M}{\partial q_1}\right] \dot{\boldsymbol{q}} \\ \vdots \\ \dot{\boldsymbol{q}}^T \left[\frac{\partial M}{\partial q_N}\right] \dot{\boldsymbol{q}} \end{pmatrix}$

Generalized Force Vector

Lyapunov Function: $\boxed{V(\dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}}^T[M(\boldsymbol{q})]\dot{\boldsymbol{q}}}$

Let's chose to use the kinetic energy function as our Lyapunov function!

Lyapunov Rate:

$$\dot{V} = \dot{\boldsymbol{q}}^T\boxed{[M]\ddot{\boldsymbol{q}}} + \frac{1}{2}\dot{\boldsymbol{q}}^T[\dot{M}]\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}^T\left(-\frac{1}{2}[\dot{M}]\dot{\boldsymbol{q}} + \frac{1}{2}\dot{\boldsymbol{q}}^T[M_{\boldsymbol{q}}]\dot{\boldsymbol{q}} + \boldsymbol{Q}\right)$$

Note that $\dot{\boldsymbol{q}}^T\left(\dot{\boldsymbol{q}}^T[M_{\boldsymbol{q}}]\dot{\boldsymbol{q}}\right) = \sum_{i=1}^{N} \dot{q}_i \left(\dot{\boldsymbol{q}}^T[M_{q_i}]\dot{\boldsymbol{q}}\right) = \dot{\boldsymbol{q}}^T[\dot{M}]\dot{\boldsymbol{q}}$

$\boxed{\dot{V} = \dot{\boldsymbol{q}}^T\boldsymbol{Q}}$ This is the generalized work/energy equation! ⇨ $\boldsymbol{Q} = -[P]\dot{\boldsymbol{q}}$ ⇨ $\dot{V} = -\dot{\boldsymbol{q}}^T[P]\dot{\boldsymbol{q}} < 0$ Globally asymptotically stabilizing

- Next, let us consider the tracking problem of a generalized mechanical system:

Reference States: $(\boldsymbol{q}_r, \dot{\boldsymbol{q}}_r)$

State Vector: $(\boldsymbol{q}, \dot{\boldsymbol{q}})$

Tracking Error: $\delta\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}_r$

Goal: $\delta\dot{\boldsymbol{q}} \to 0$

Lyapunov Function: $$V(\dot{\boldsymbol{q}}) = \frac{1}{2}\delta\dot{\boldsymbol{q}}^T[M(\boldsymbol{q})]\delta\dot{\boldsymbol{q}}$$

Energy-function-like positive definite measure of tracking error.

Lyapunov Rate: $$\dot{V} = \delta\dot{\boldsymbol{q}}^T\left(-\frac{1}{2}[\dot{M}]\left(\dot{\boldsymbol{q}} + \dot{\boldsymbol{q}}_r\right) + \frac{1}{2}\dot{\boldsymbol{q}}^T[M_q]\dot{\boldsymbol{q}} - [M]\ddot{\boldsymbol{q}}_r + \boldsymbol{Q}\right)$$

Note that the work/energy principle doesn't hold with these non-mechanical energy function, and the Lyapunov rate is no longer the simply the power equation.

Proposed Control: $$\boldsymbol{Q} = \frac{1}{2}[\dot{M}]\left(\dot{\boldsymbol{q}} + \dot{\boldsymbol{q}}_r\right) - \frac{1}{2}\dot{\boldsymbol{q}}^T[M_q]\dot{\boldsymbol{q}} + [M]\ddot{\boldsymbol{q}}_r - [P]\delta\dot{\boldsymbol{q}}$$

Feedback linearization

Feedforward compensation

Proportional Feedback

$$\Rightarrow \quad \dot{V}(\delta\dot{\boldsymbol{q}}) = -\delta\dot{\boldsymbol{q}}^T[P]\delta\dot{\boldsymbol{q}} < 0$$

Globally asymptotically stabilizing

- Example of Mechanical System Stabilization: (Ex: 8.8 in S&J)

State vector: $\boldsymbol{q} = (\theta_1, \theta_2, \theta_3)^T$

Goal: $\dot{\boldsymbol{q}} \rightarrow 0$

EOM:

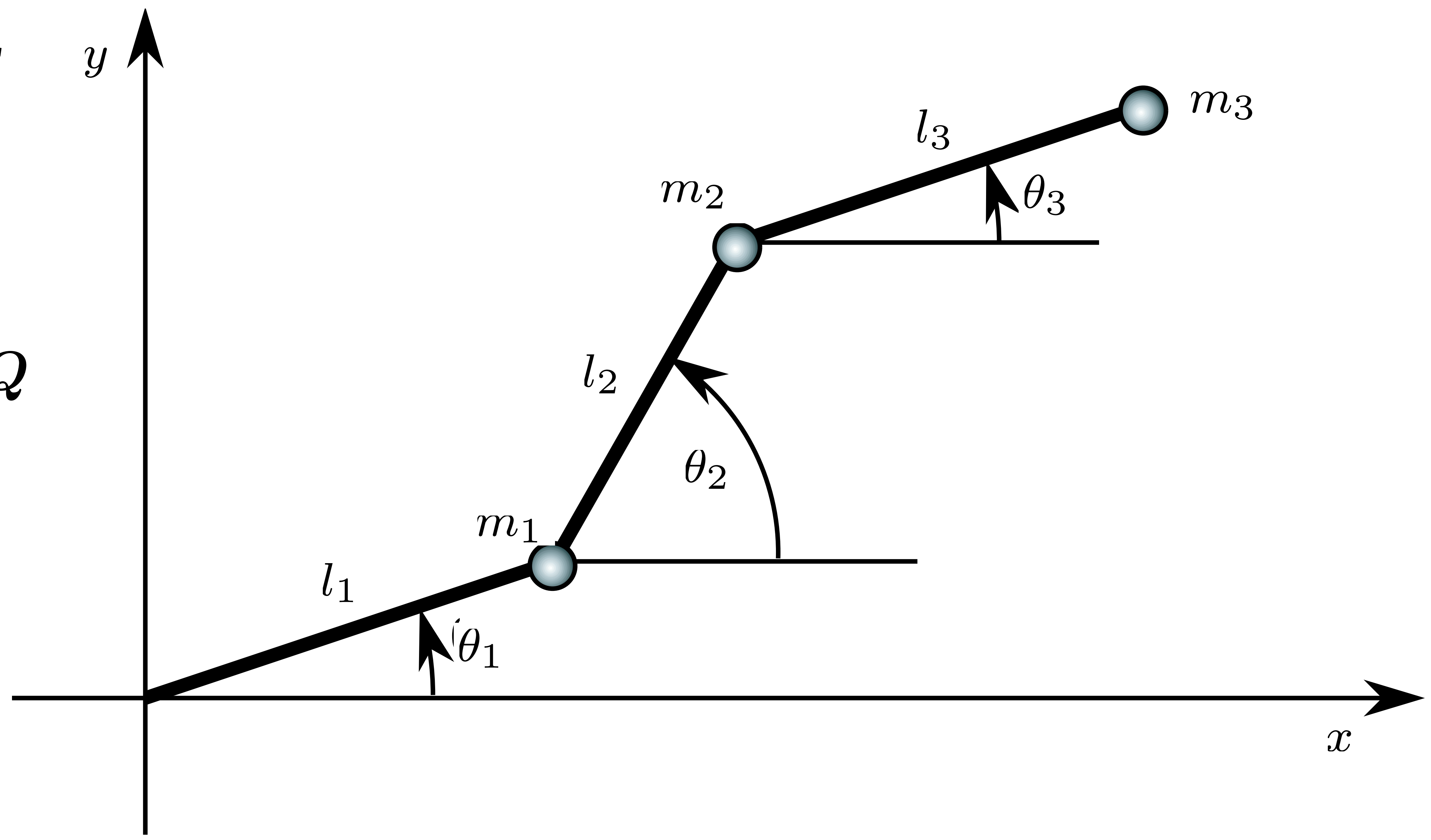
$$[M]\ddot{\boldsymbol{q}} + [\dot{M}]\dot{\boldsymbol{q}} - \frac{1}{2}\dot{\boldsymbol{q}}^T[M_{\boldsymbol{q}}]\dot{\boldsymbol{q}} = \boldsymbol{Q}$$

Lyapunov Function:

$$V(\dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}}^T[M(\boldsymbol{q})]\dot{\boldsymbol{q}}$$

Lyapunov Rate:

$$\dot{V} = \dot{\boldsymbol{q}}^T\boldsymbol{Q}$$



Control:

$$\boxed{\boldsymbol{Q}_1 = -P_1\dot{\boldsymbol{q}}}$$ Rate Feedback

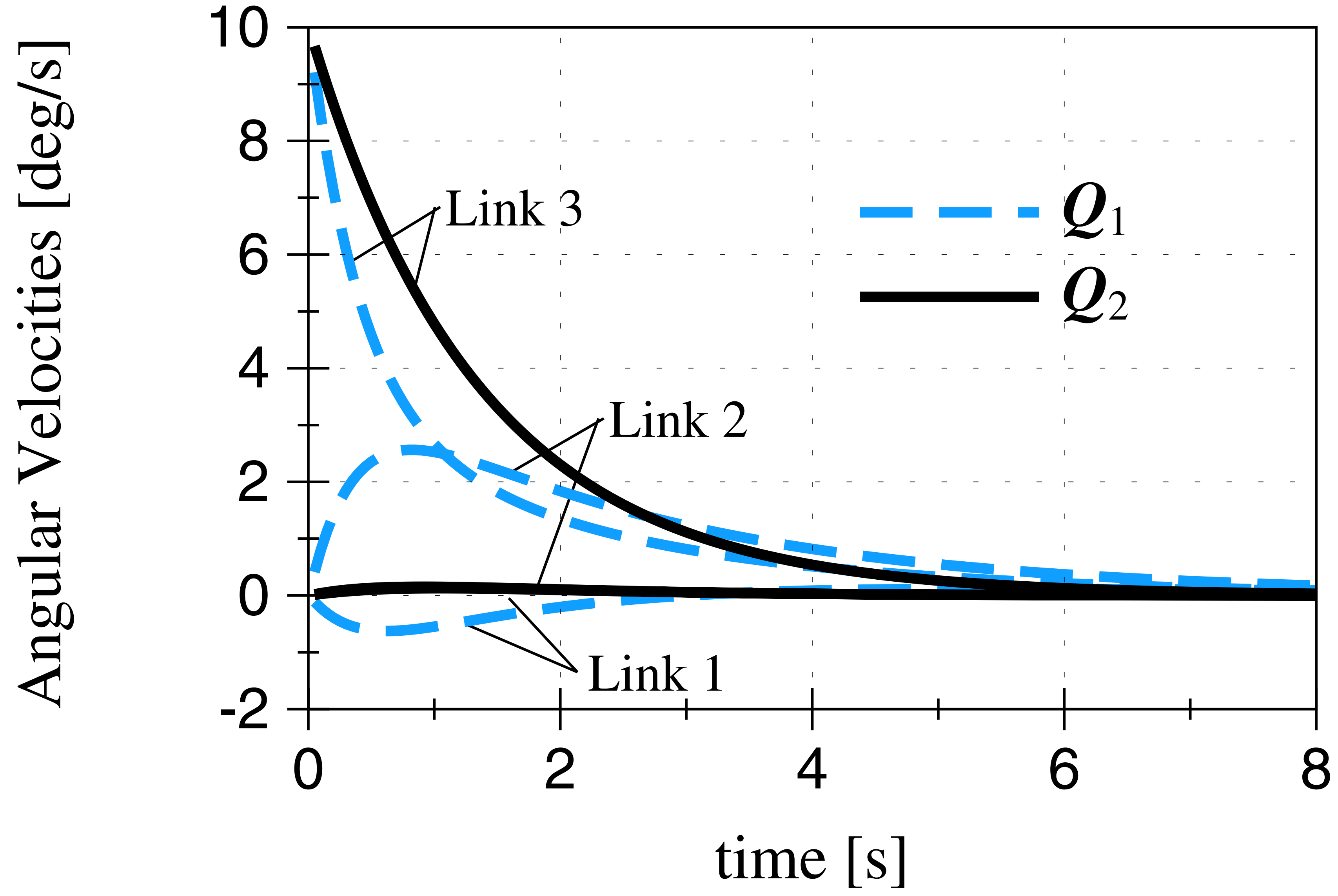
$$\boxed{\boldsymbol{Q}_2 = -P_2[M(\boldsymbol{q})]\dot{\boldsymbol{q}}}$$ "Momentum" Feedback

Symmetric, positive definite Mass Matrix:

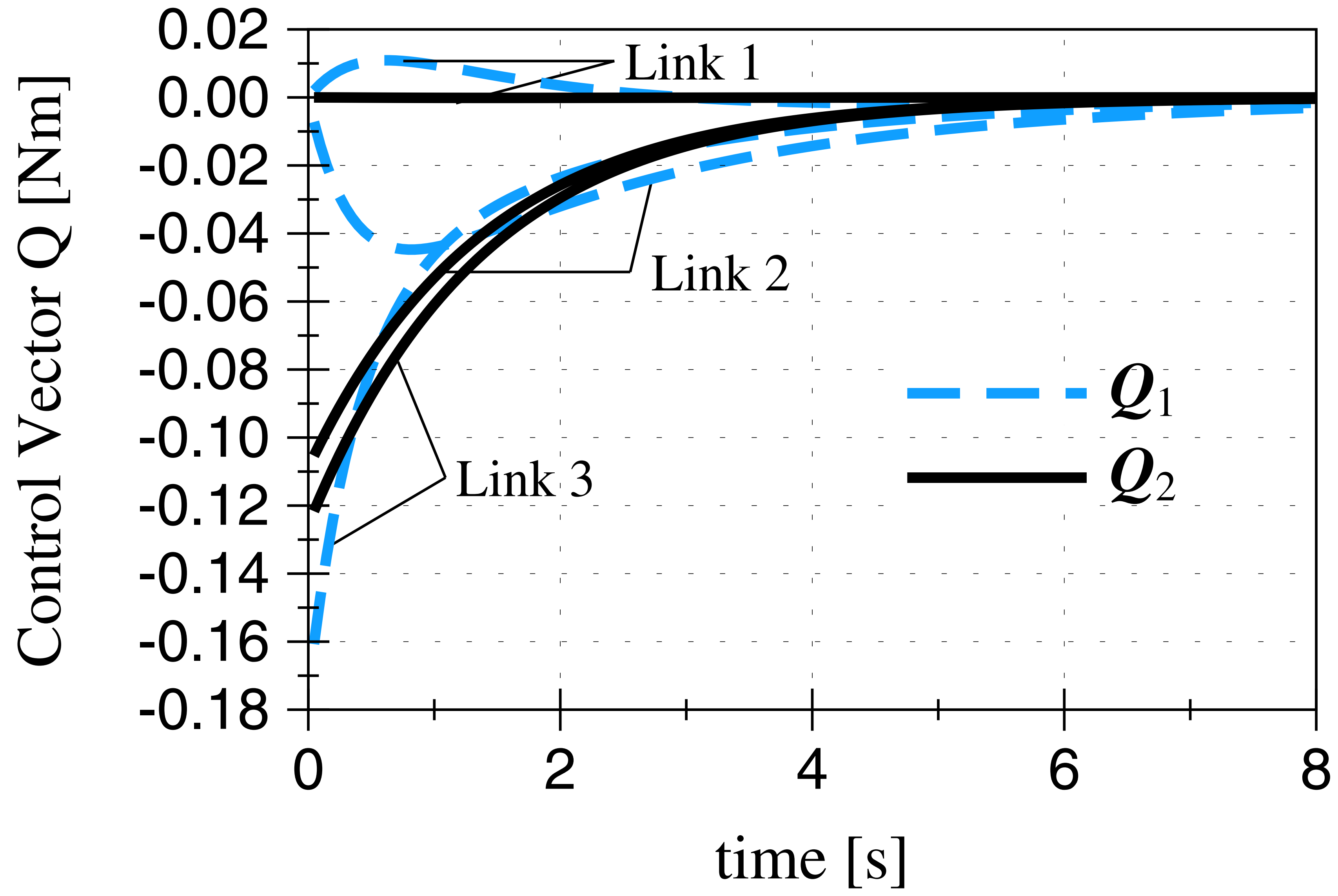
$$M(\boldsymbol{q}) = \begin{bmatrix} (m_1+m_2+m_3)l_1^2 & (m_2+m_3)l_1 l_2\cos(\theta_2-\theta_1) & m_3 l_1 l_3 \cos(\theta_3-\theta_1) \\ (m_2+m_3)l_1 l_2\cos(\theta_2-\theta_1) & (m_2+m_3)l_2^2 & m_3 l_1 l_3\cos(\theta_3-\theta_2) \\ m_3 l_1 l_3\cos(\theta_3-\theta_1) & m_3 l_2 l_3\cos(\theta_3-\theta_2) & m_3 l_3^2 \end{bmatrix}$$

Simulation Parameters

Parameter	Value	Units
l_i	1	m
m_i	1.0	kg
P_1	1.0	kg-m^2/sec
P_2	0.72	kg-m^2/sec
$\boldsymbol{x}(t_0)$	$[-90\ 30\ 0]$	deg
$\dot{\boldsymbol{x}}(t_0)$	$[0.0\ 0.0\ 10]$	deg/sec



While both controls are asymptotically stabilizing, the 2nd control solution is actually exponentially stabilizing, and successfully isolates the motion of the third link from the first link.



Rigid Body Detumbling

State Vector: $\boldsymbol{\omega}$ Goal: $\boldsymbol{\omega} \rightarrow 0$

EOM: $[I]\dot{\boldsymbol{\omega}} = -[\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} + \boldsymbol{Q}$

External control torque

Lyapunov Function:

$$V(\boldsymbol{\omega}) = T = \frac{1}{2}\boldsymbol{\omega}^T[I]\boldsymbol{\omega}$$

Constant in Body frame components

Lyapunov Rate:

$$\dot{V} = \boldsymbol{\omega}^T([I]\dot{\boldsymbol{\omega}}) = \boldsymbol{\omega}^T(-[\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} + \boldsymbol{Q})$$

$$= \boldsymbol{\omega}^T\boldsymbol{Q}$$

Power form of work/ energy equation

Control: $\boxed{\boldsymbol{Q} = -[P]\boldsymbol{\omega}}$ with $[P] = [P]^T > 0$

$$\dot{V}(\boldsymbol{\omega}) = -\boldsymbol{\omega}^T[P]\boldsymbol{\omega} < 0$$

Globally asymptotically stabilizing

Note: This control result does not require any knowledge of the inertia matrix! It is very robust to inertia modeling errors.

Reference: $\boldsymbol{\omega}_r$ Goal: $\delta\boldsymbol{\omega} = \boldsymbol{\omega} - \boldsymbol{\omega}_r \rightarrow 0$

Note: ${}^{\mathcal{B}}\delta\boldsymbol{\omega} = {}^{\mathcal{B}}\boldsymbol{\omega} - [BR]^{\mathcal{R}}\boldsymbol{\omega}_r$

Lyapunov Function: $V(\delta\boldsymbol{\omega}) = \frac{1}{2}\delta\boldsymbol{\omega}^T[I]\delta\boldsymbol{\omega}$

Lyapunov Rate: $\dot{V} = \delta\boldsymbol{\omega}^T[I]\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\delta\boldsymbol{\omega})$

Note: $\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\delta\boldsymbol{\omega}) = \dot{\boldsymbol{\omega}} - \dot{\boldsymbol{\omega}}_r + \boldsymbol{\omega} \times \boldsymbol{\omega}_r$

$$\dot{V} = \delta\boldsymbol{\omega}^T(-[\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} + [I]\boldsymbol{\omega} \times \boldsymbol{\omega}_r - [I]\dot{\boldsymbol{\omega}}_r + \boldsymbol{Q})$$

Control:

$$\boxed{\boldsymbol{Q} = [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} - [I][\tilde{\boldsymbol{\omega}}]\boldsymbol{\omega}_r + [I]\dot{\boldsymbol{\omega}}_r - [P]\delta\boldsymbol{\omega}}$$

$$\dot{V}(\delta\boldsymbol{\omega}) = -\delta\boldsymbol{\omega}^T[P]\delta\boldsymbol{\omega} < 0$$

Globally asymptotically stabilizing

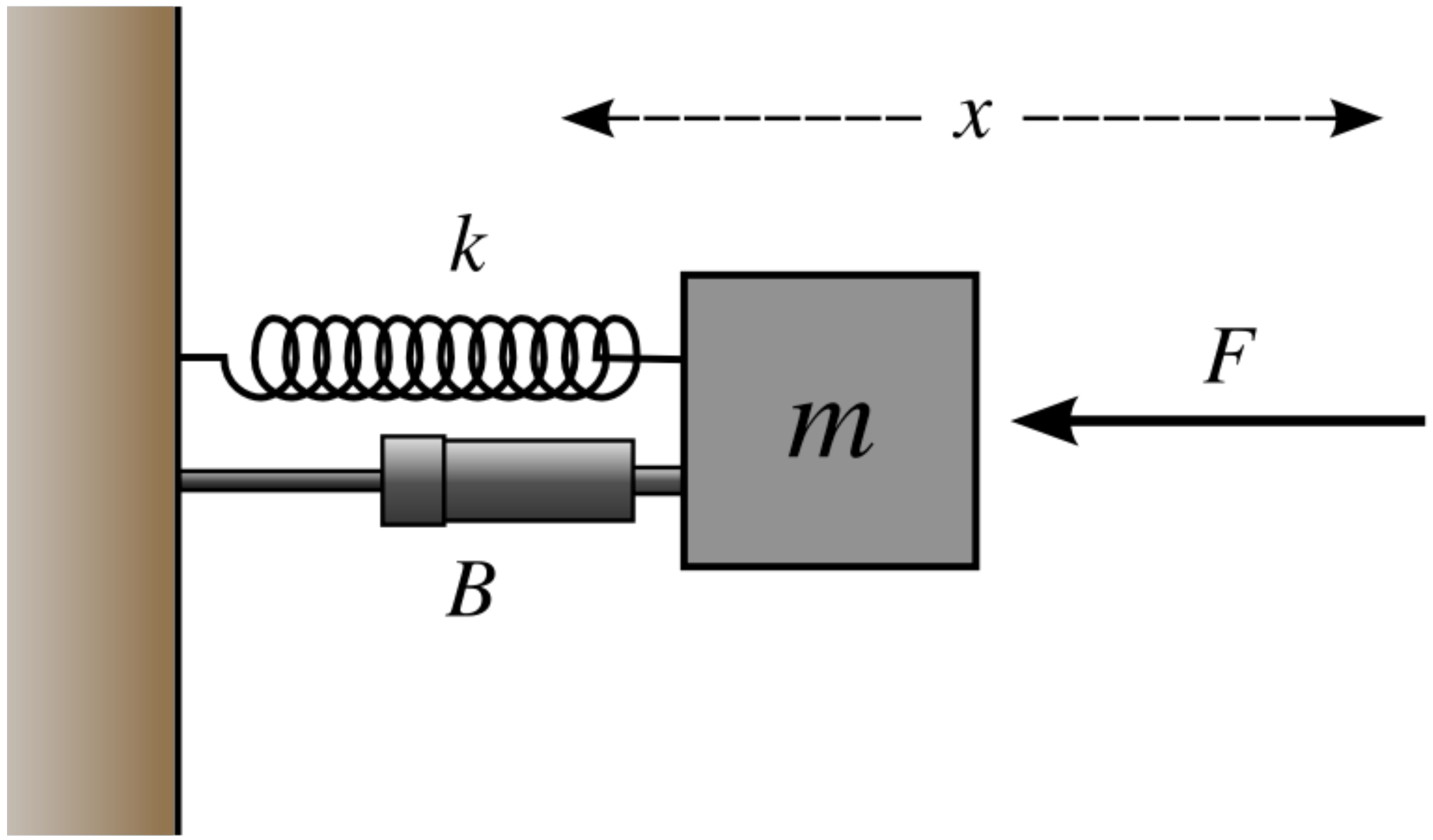
Elemental Position-Based Lyapunov Functions

Physical Motivation is the linear spring energy with stiffness k:

$$T = \frac{1}{2} k x^2$$

We will seek similar energy-like functions which provide positive definite error measures of the position-errors.

The state rate $\dot{x}$ is treated as a control variable in this discussion. This is typically the case in robotic control where a lower level servo loop implements the required x.



Euler Angle Potential Functions

State Vector: $$\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)^T$$

Lyapunov Function: $$V(\boldsymbol{\theta}) = \frac{1}{2}\boldsymbol{\theta}^T[K]\boldsymbol{\theta} \quad \text{with} \quad [K] = [K]^T > 0$$

Kinematic Differential Equations: $$\dot{\boldsymbol{\theta}} = [B(\boldsymbol{\theta})]\boldsymbol{\omega}$$

Lyapunov Rate: $$\dot{V} = \boldsymbol{\omega}^T \left([B(\boldsymbol{\theta})]^T[K]\boldsymbol{\theta}\right)$$

This function can later on be used in Lyapunov position and rate feedback law developments.

Attitude Tracking Error between *B* and reference frame *R*: $$\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)^T$$

Kinematic Differential Equations: $$\dot{\boldsymbol{\theta}} = [B(\boldsymbol{\theta})]\delta\boldsymbol{\omega} \quad \text{with} \quad \delta\boldsymbol{\omega} = \boldsymbol{\omega} - \boldsymbol{\omega}_r$$

Lyapunov Function: $$V(\boldsymbol{\theta}) = \frac{1}{2}\boldsymbol{\theta}^T[K]\boldsymbol{\theta}$$

There is no algebraic distinction between position based regulator and tracking Lyapunov functions, and we won't distinguish the two from remainder of the position-based Lyapunov function discussion.

Classical Rodrigues Parameters

A brute force approach would define a candidate Lyapunov function as spring-mass energy-like form:

$$V(\boldsymbol{q}) = \boldsymbol{q}^T[K]\boldsymbol{q}$$

Taking the derivative we find

$$\dot{V} = \boldsymbol{\omega}^T \left(\left(I - [\tilde{\boldsymbol{q}}] + \boldsymbol{q}\boldsymbol{q}^T \right) [K]\boldsymbol{q} \right)$$

which gets reduced to

$$\dot{V} = \boldsymbol{\omega}^T \left(K \left(1 + q^2 \right) \boldsymbol{q} \right)$$

if the gain matrix [*K*] is a scalar *K* and $q^2 = \boldsymbol{q}^T\boldsymbol{q}$.

This term may lead to nonlinear feedback laws.

A more elegant Gibbs-vector Lyapunov function is given by:

$$V(\boldsymbol{q}) = K \ln\left(1 + \boldsymbol{q}^T\boldsymbol{q}\right)$$

Taking the derivative, and substituting the differential kinematic equations, a surprisingly simple form is found:

$$\boxed{\dot{V} = \boldsymbol{\omega}^T (K\boldsymbol{q})}$$

leads to linear attitude feedback!

Let's develop an attitude servo law, we define

$$V(\boldsymbol{q}) = \ln\left(1 + \boldsymbol{q}^T\boldsymbol{q}\right) \quad \text{with} \quad \dot{V} = \boldsymbol{\omega}^T\boldsymbol{q}$$

The body rate control vector is then defined as

$$\boldsymbol{\omega} = -[K]\boldsymbol{q} \Rightarrow \dot{V}(\boldsymbol{q}) = -\boldsymbol{q}^T[K]\boldsymbol{q} < 0$$

Other Parameters

Modified Rodrigues Parameters:

Lyapunov Function:

$$V(\boldsymbol{\sigma}) = 2K \ln\left(1 + \boldsymbol{\sigma}^T \boldsymbol{\sigma}\right)$$

If you switch to the shadow MRP set on the $\sigma^2=1$ surface, then this Lyapunov function is continuous.

Lyapunov Rate:

$$\dot{V} = \boldsymbol{\omega}^T \left(K\boldsymbol{\sigma}\right)$$

This leads to elegant linear attitude feedback laws which are globally stabilizing by switching between the original and shadow MRP set.

Euler Parameters:

Ideal Attitude: $\hat{\boldsymbol{\beta}} = \begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}^T$

Lyapunov Function:

$$V(\boldsymbol{\beta}) = K\left(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}\right)^T\left(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}\right)$$

Lyapunov Rate:

$$\dot{V} = K\boldsymbol{\omega}^T[B(\boldsymbol{\beta})]^T\left(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}\right)$$

recall that $[B(\boldsymbol{\beta})]^T\boldsymbol{\beta} = 0$

This leads to

$$\dot{V} = K\boldsymbol{\omega}^T\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix} = \boldsymbol{\omega}^T\left(K\boldsymbol{\epsilon}\right)$$

Note that will will stabilize the attitude to $\beta_0 = \pm 1$, which is the same attitude. However, no guarantee is made if the long or short rotational path is used.