Spacecraft Dynamics and Control – ASEN 5010

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Particle Kinematics

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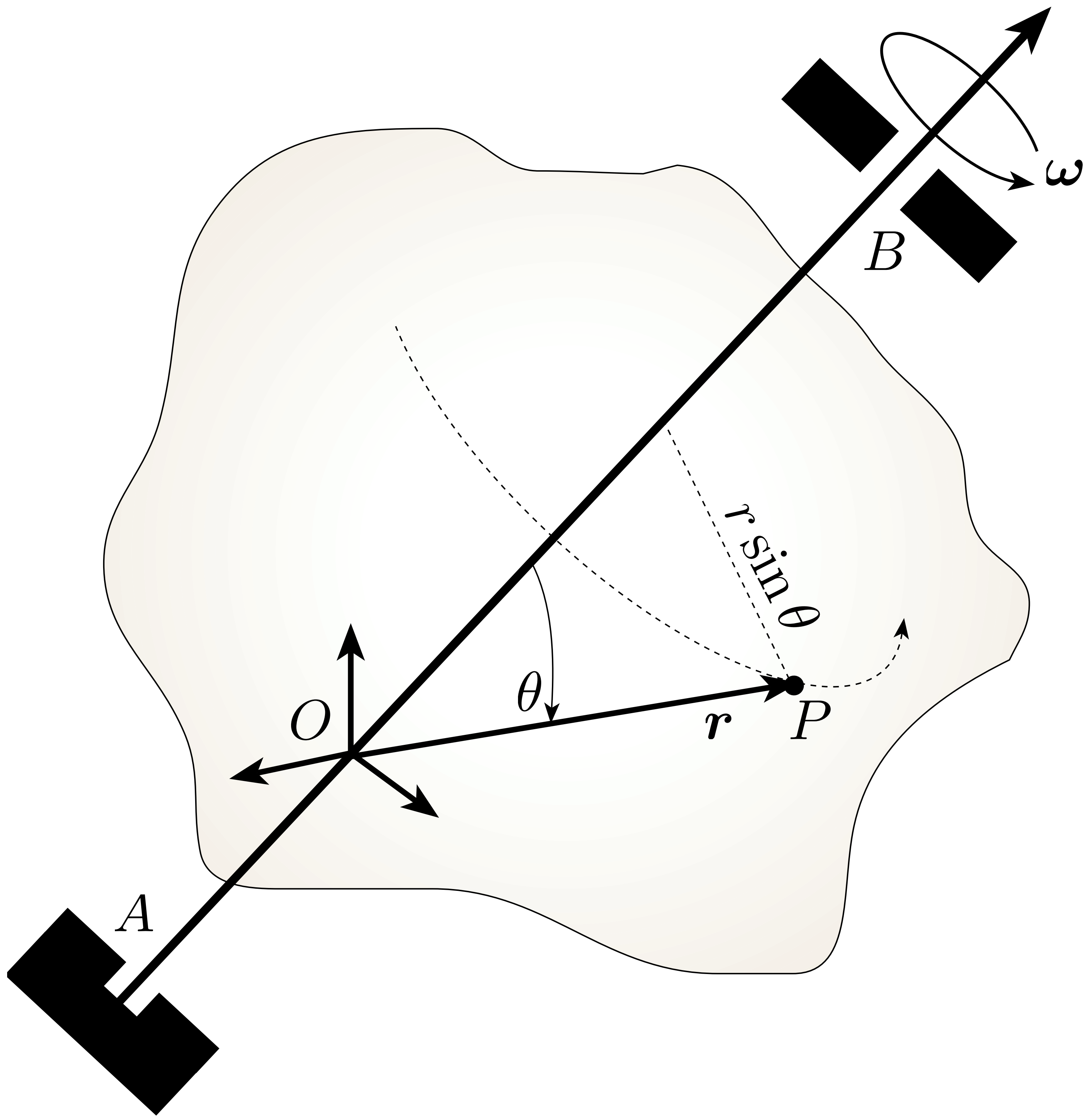
Outline

- Vector Notation
- Vector Differentiation
- Lots of brushing up on this material on your own!



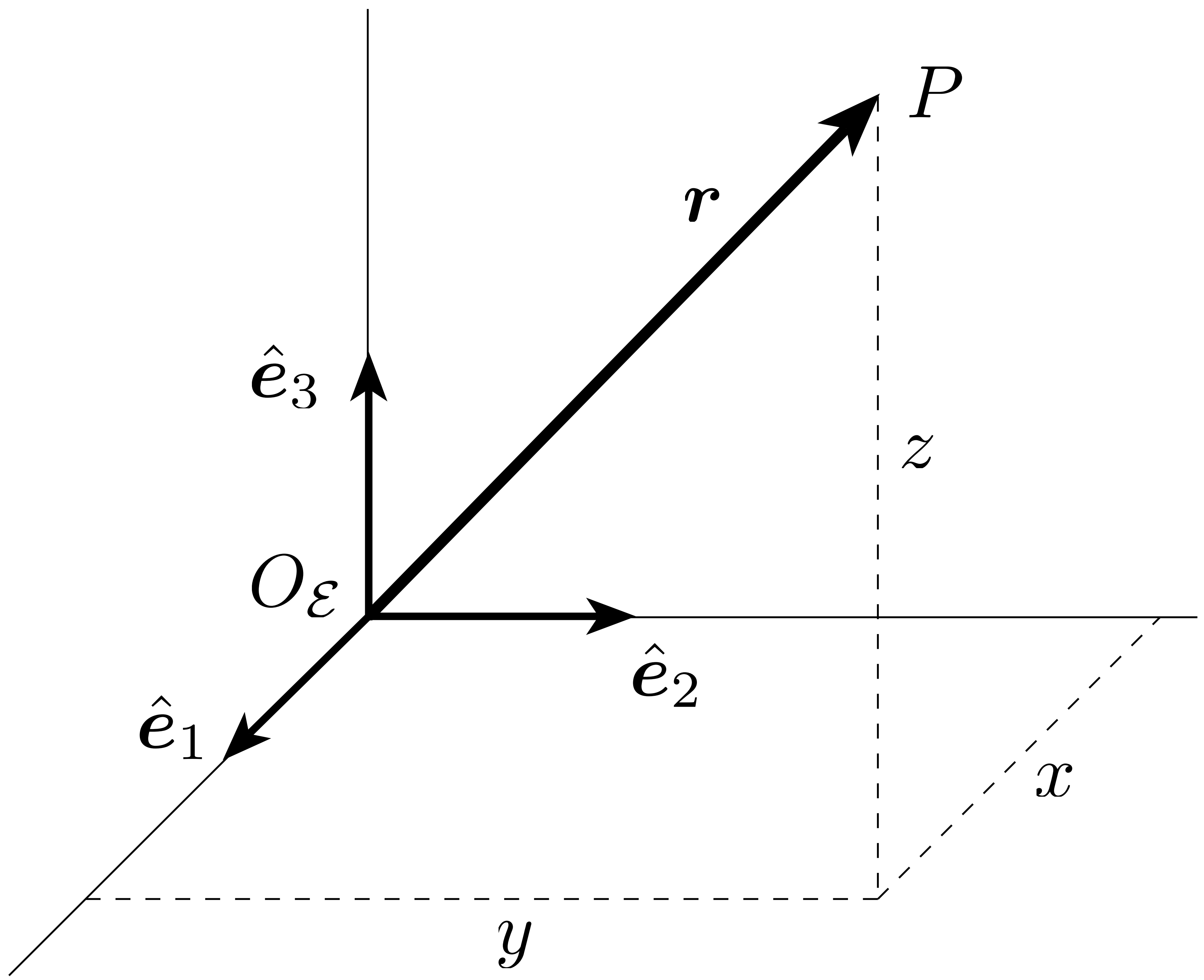
Vector Notation

Hopefully a boring topic for you by now...

What is a vector?

- Something with a direction and magnitude.
- A vector can be written as

$$
\begin{aligned}
\boldsymbol{r} &= x\hat{\boldsymbol{e}}_1 + y\hat{\boldsymbol{e}}_2 + z\hat{\boldsymbol{e}}_3 \\
&= r\hat{\boldsymbol{e}}_r \\
&= {}^{\mathcal{E}}\!\begin{pmatrix} x \\ y \\ z \end{pmatrix}
\end{aligned}
$$



Vector Addition

$$\boldsymbol{q} = \boldsymbol{r} + \boldsymbol{p} \quad \Rightarrow \quad \text{True}$$

$$^{\mathcal{E}}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = {}^{\mathcal{E}}\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} + {}^{\mathcal{B}}\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \Rightarrow \begin{aligned} q_1 &= r_1 + p_1 \\ q_2 &= r_2 + p_2 \\ q_3 &= r_3 + p_3 \end{aligned} \quad \Rightarrow \quad \text{False}$$

$$^{\mathcal{E}}\boldsymbol{q} = {}^{\mathcal{E}}\boldsymbol{r} + {}^{\mathcal{B}}\boldsymbol{p} \quad \Rightarrow \quad \text{False}$$

Coordinate frame

- Let a coordinate frame *B* be defined through the three unit orthogonal vectors:

$$\hat{\boldsymbol{b}}_1 \qquad \hat{\boldsymbol{b}}_2 \qquad \hat{\boldsymbol{b}}_3$$

- Let the origin of this frame be given by

$$\mathcal{O}_{\mathcal{B}}$$

- The frame is then defined through

$$\mathcal{B} : \{\mathcal{O}_{\mathcal{B}}, \hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2, \hat{\boldsymbol{b}}_3\}$$

- If we can ignore the frame origin, then we often use the shorthand notation

$$\mathcal{B} : \{\hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2, \hat{\boldsymbol{b}}_3\}$$

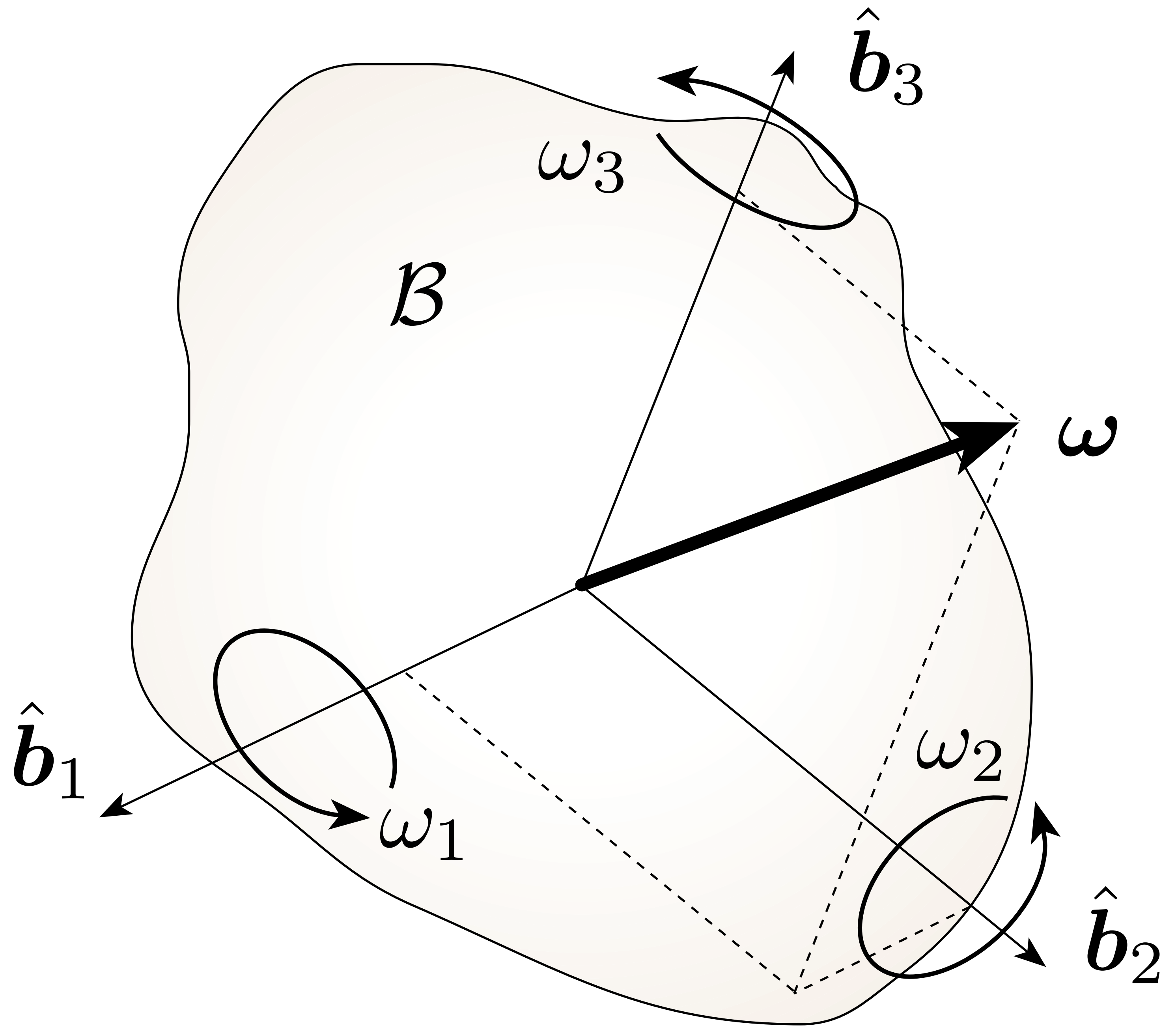
Angular Velocity Vector

- Angular velocity vector can be expressed as

$$\boldsymbol{\omega} = \omega_1 \hat{\boldsymbol{b}}_1 + \omega_2 \hat{\boldsymbol{b}}_2 + \omega_3 \hat{\boldsymbol{b}}_3$$

$${}^{\mathcal{B}}\boldsymbol{\omega} = {}^{\mathcal{B}}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

- ω_i are instantaneous body rates about the orthogonal $\hat{\boldsymbol{b}}_i$ axes.



Vector Differentiation

A crucial ability for attitude dynamics research...

Fixed Axis Rotation

- The rigid body is rotating about a fixed axis.
- The speed of *P* is given by

$$|\dot{\boldsymbol{r}}| = (r\sin\theta)\,\omega$$

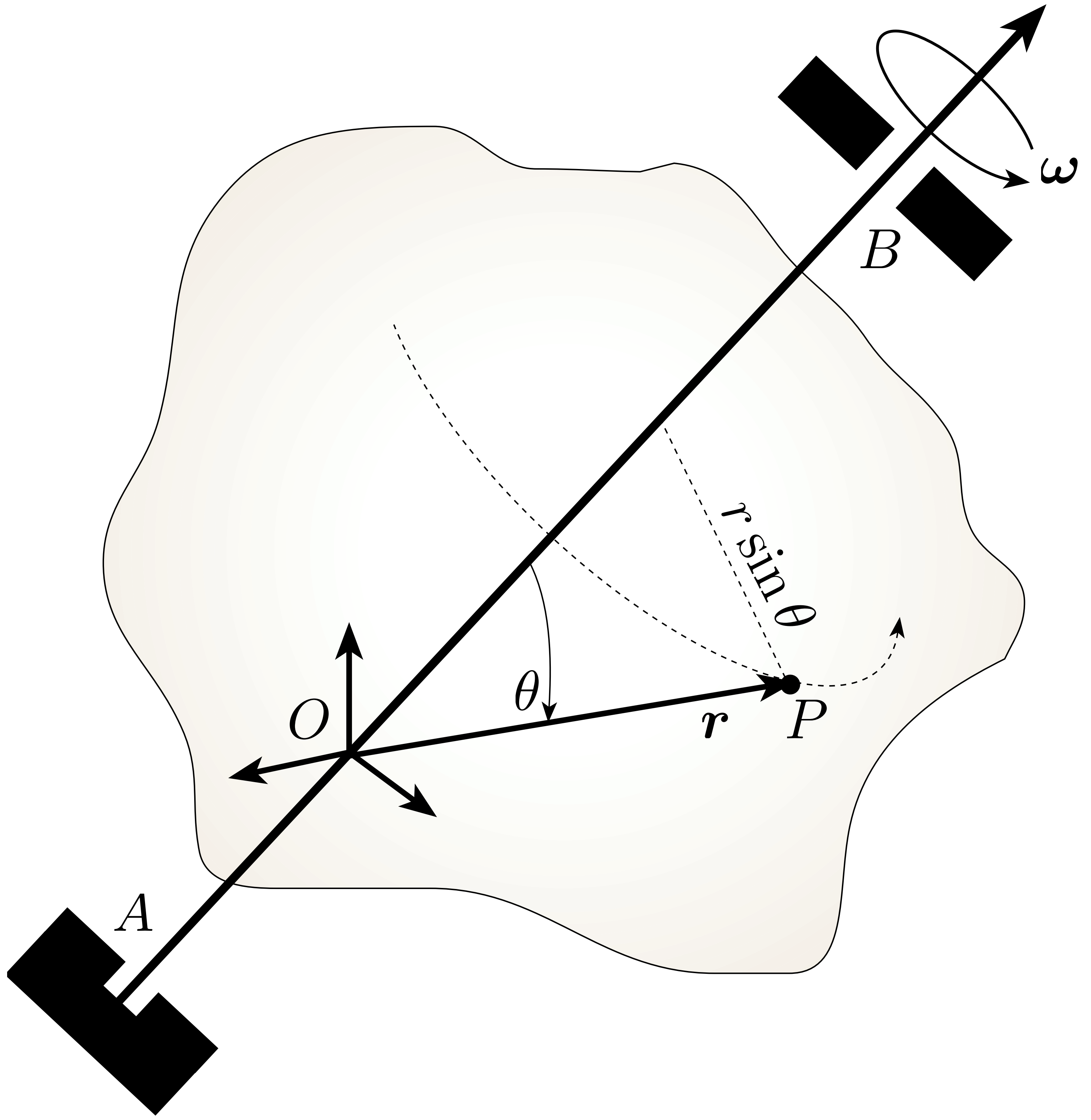
- note that

$$\dot{\boldsymbol{r}} = (r\sin\theta)\,\omega\left(\frac{\boldsymbol{\omega}\times\boldsymbol{r}}{|\boldsymbol{\omega}\times\boldsymbol{r}|}\right)$$

- thus the transport velocity is

$$|\boldsymbol{\omega}\times\boldsymbol{r}| = \omega r\sin\theta$$

$$\boxed{\dot{\boldsymbol{r}} = \boldsymbol{\omega}\times\boldsymbol{r}}$$



Transport Theorem

- Let a position vector be written as

$$\boldsymbol{r} = r_1 \hat{\boldsymbol{b}}_1 + r_2 \hat{\boldsymbol{b}}_2 + r_3 \hat{\boldsymbol{b}}_3$$

while the angular velocity vector is written as

$$\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = \omega_1 \hat{\boldsymbol{b}}_1 + \omega_2 \hat{\boldsymbol{b}}_2 + \omega_3 \hat{\boldsymbol{b}}_3$$

- The derivative of a vector with respect to the $\mathcal{B}$ frame is written as

$$\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}) = \dot{r}_1 \hat{\boldsymbol{b}}_1 + \dot{r}_2 \hat{\boldsymbol{b}}_2 + \dot{r}_3 \hat{\boldsymbol{b}}_3$$

since

$$\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\left(\hat{\boldsymbol{b}}_i\right) = 0$$

Transport Theorem

- The inertial derivative of the position vector is

$$\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}) = \dot{r}_1\hat{\boldsymbol{b}}_1 + \dot{r}_2\hat{\boldsymbol{b}}_2 + \dot{r}_3\hat{\boldsymbol{b}}_3 + r_1\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}\left(\hat{\boldsymbol{b}}_1\right) + r_2\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}\left(\hat{\boldsymbol{b}}_2\right) + r_3\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}\left(\hat{\boldsymbol{b}}_3\right)$$

- Note that $\hat{\boldsymbol{b}}_i$ are body fixed vectors, thus we find

$$\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}\left(\hat{\boldsymbol{b}}_i\right) = \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \hat{\boldsymbol{b}}_i$$

- This allows us to write the inertial derivative of the position vector as

$$\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}) = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}) + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}$$

Transport Theorem

$$\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}) = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}) + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}$$

Learn to be one with this equation, and three-dimensional rotations will never haunt you again!

Comments

- Another noted otherwise, the following short-hand notation is used to denote inertial vector derivatives:

$$\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{x}\right) \equiv \dot{\boldsymbol{x}}$$

- Note that we can analytically differentiate vectors, without first assigning specific coordinate frame. In fact, it is typically easier to wait until the very last steps before specifying a vectors through the vector components.