Principal Rotation Vector

The building block of many advanced attitude coordinates...

Theorem 3.1 (Euler's Principal Rotation): A rigid body or coordinate reference frame can be brought from an arbitrary initial orientation to an arbitrary final orientation by a single rigid rotation through a principal angle Φ about the principal axis $\hat{e}$; the principal axis is a judicious axis fixed in both the initial and final orientation.

That's great!! But, what does this mean???

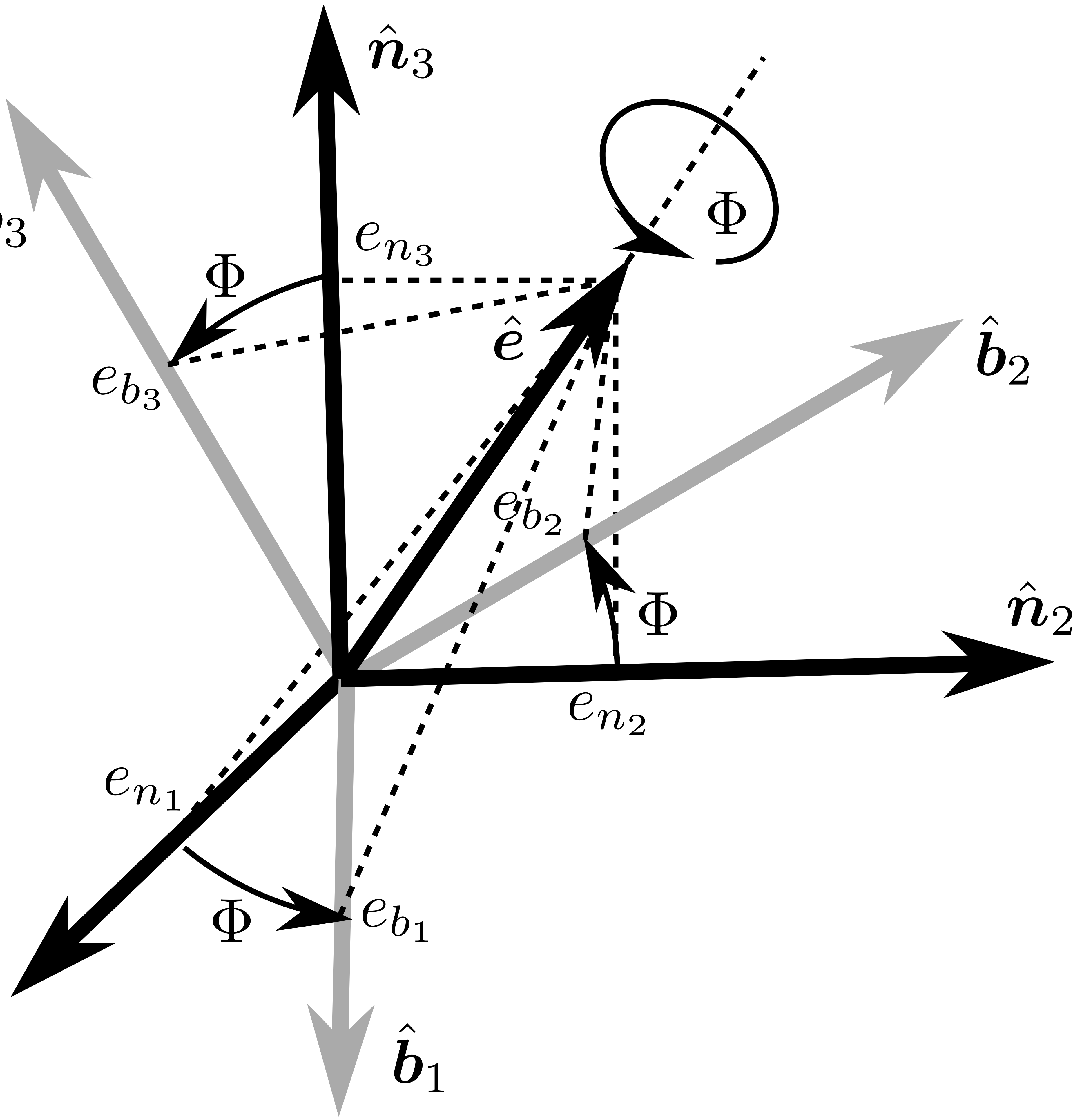
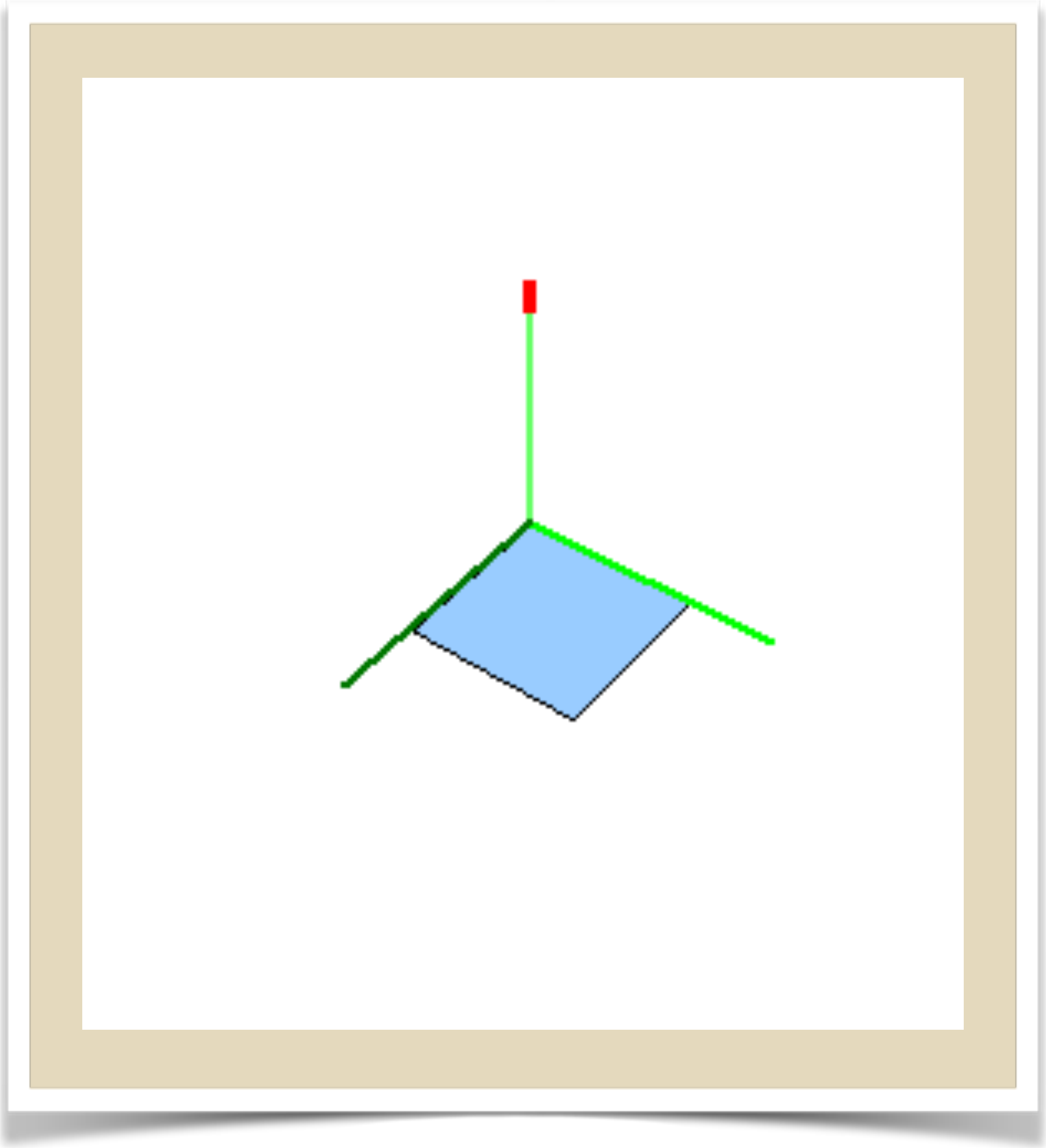
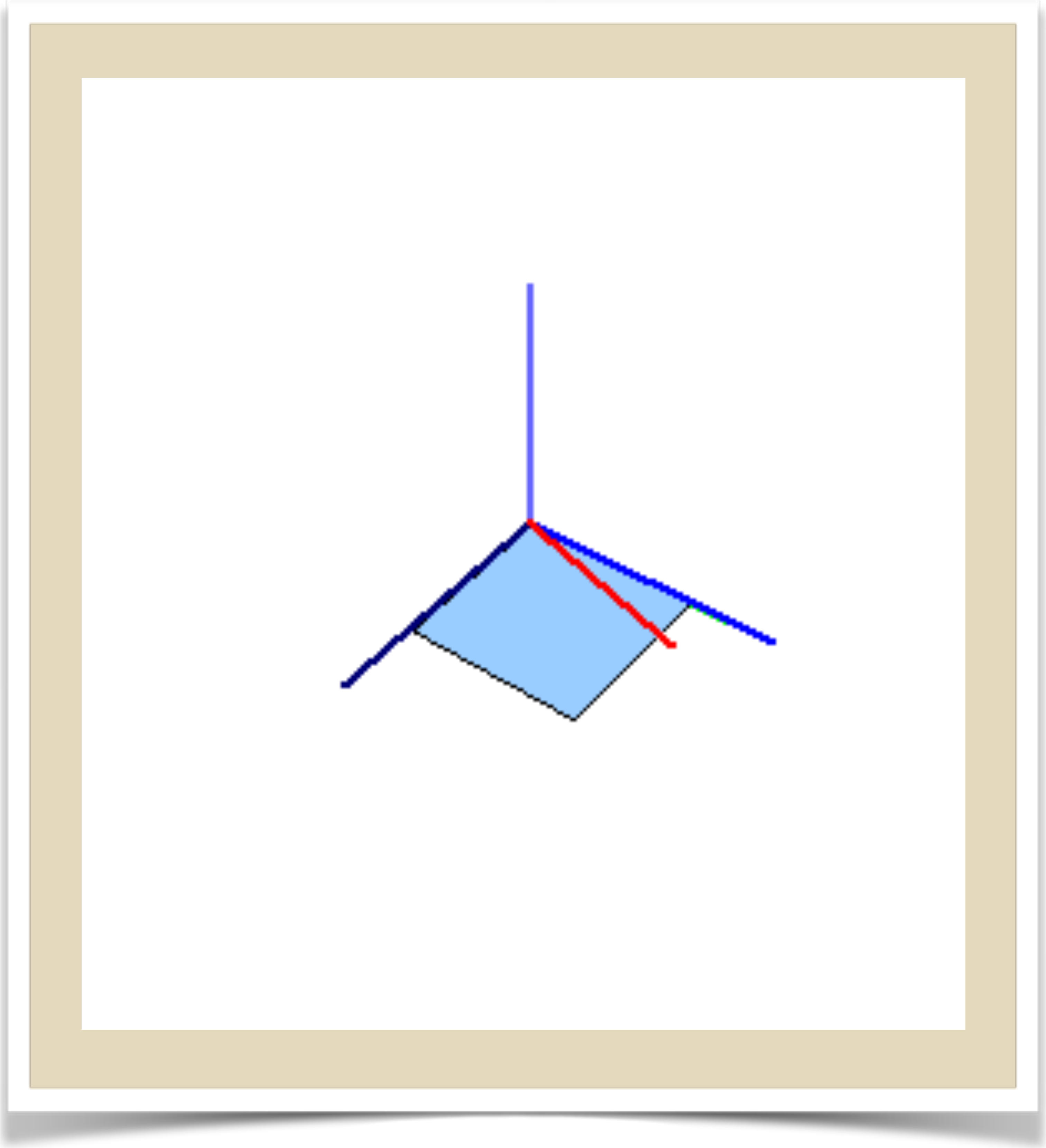


Illustration of Euler's Principal Rotation Theorem



(3-2-1) Euler Angles
(60,50,70) Degrees



Principal Rotation Vector

$$\Phi = 80.3385^o$$

$$\hat{\boldsymbol{e}} = (0.429577, 0.867729, 0.250019)^T$$

- Let's study the last statement of this theorem first: "the principal axis is a judicious axis fixed in both the initial and final orientation"

- This means that the principal axis unit vector will have the same vector components in the initial (i.e. inertial) and the final frame (i.e. body frame)

$$\hat{\boldsymbol{e}} = e_{b_1}\hat{\boldsymbol{b}}_1 + e_{b_2}\hat{\boldsymbol{b}}_2 + e_{b_3}\hat{\boldsymbol{b}}_3$$

$$\hat{\boldsymbol{e}} = e_{n_1}\hat{\boldsymbol{n}}_1 + e_{n_2}\hat{\boldsymbol{n}}_2 + e_{n_3}\hat{\boldsymbol{n}}_3$$

$$\Rightarrow \quad e_{b_i} = e_{n_i} = e_i$$

- Using the rotation matrix [*C*], the $\hat{\boldsymbol{e}}$ frame vector components in *B* and *N* frame can be related through

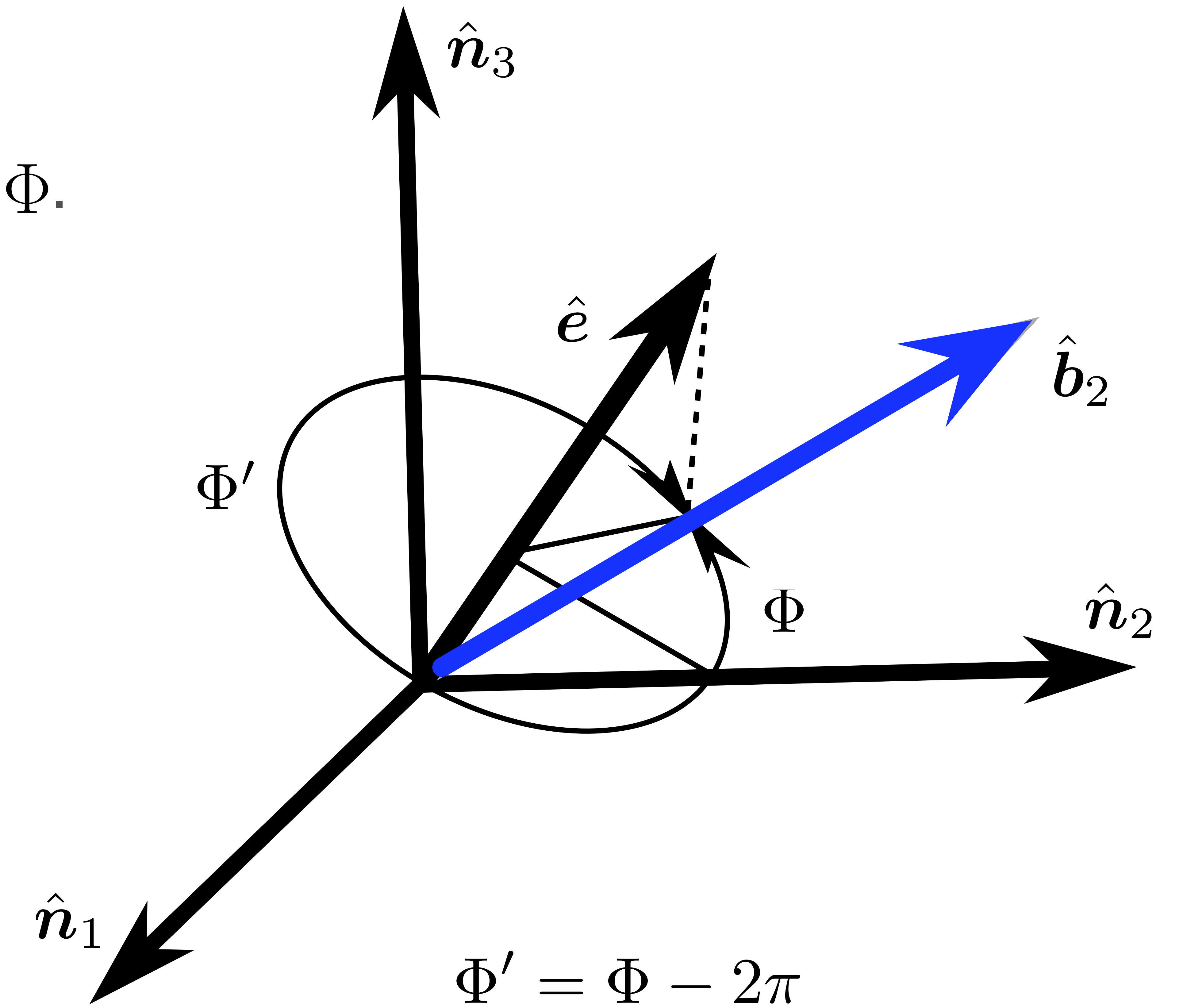
$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = [C] \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

- From this last equation, it is evident that $\hat{\boldsymbol{e}}$ must be an eigenvector of [*C*] with an eigenvalue of +1.

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = [C] \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

- This eigenvector is unique to within a sign of $\hat{\boldsymbol{e}}$ or Φ.
- The $\hat{\boldsymbol{e}}$ vector is not defined for a zero rotation!
- There are four possible principal rotations:

$$\begin{gathered} (\hat{\boldsymbol{e}}, \Phi) \\ (-\hat{\boldsymbol{e}}, -\Phi) \\ (\hat{\boldsymbol{e}}, \Phi') \\ (-\hat{\boldsymbol{e}}, -\Phi') \end{gathered}$$



$$\Phi' = \Phi - 2\pi$$

Relationship to DCM

- We can express the [C] matrix in terms of PRV components as

$$[C] = \begin{bmatrix} e_1^2\Sigma + c\Phi & e_1e_2\Sigma + e_3 s\Phi & e_1e_3\Sigma - e_2 s\Phi \\ e_2e_1\Sigma - e_3 s\Phi & e_2^2\Sigma + c\Phi & e_2e_3\Sigma + e_1 s\Phi \\ e_3e_1\Sigma + e_2 s\Phi & e_3e_2\Sigma - e_1 s\Phi & e_3^2\Sigma + c\Phi \end{bmatrix}$$

$$\Sigma = 1 - \mathrm{c}\Phi$$

- The inverse transformation from [C] to PRV is found by inspecting the matrix structure:

$$\cos\Phi = \frac{1}{2}(C_{11} + C_{22} + C_{33} - 1) \qquad \Phi' = \Phi - 2\pi$$

$$\hat{\boldsymbol{e}} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \frac{1}{2\sin\Phi}\begin{pmatrix} C_{23} - C_{32} \\ C_{31} - C_{13} \\ C_{12} - C_{21} \end{pmatrix}$$

PRV Addition

- DCM method:

$$[FN(\Phi, \hat{\boldsymbol{e}})] = [FB(\Phi_2, \hat{\boldsymbol{e}}_2)][BN(\Phi_1, \hat{\boldsymbol{e}}_1)]$$

- Direct method:

$$\Phi = 2\cos^{-1}\left(\cos\frac{\Phi_1}{2}\cos\frac{\Phi_2}{2} - \sin\frac{\Phi_1}{2}\sin\frac{\Phi_2}{2}\hat{\boldsymbol{e}}_1 \cdot \hat{\boldsymbol{e}}_2\right)$$

$$\hat{\boldsymbol{e}} = \frac{\cos\frac{\Phi_2}{2}\sin\frac{\Phi_1}{2}\hat{\boldsymbol{e}}_1 + \cos\frac{\Phi_1}{2}\sin\frac{\Phi_2}{2}\hat{\boldsymbol{e}}_2 + \sin\frac{\Phi_1}{2}\sin\frac{\Phi_2}{2}\hat{\boldsymbol{e}}_1 \times \hat{\boldsymbol{e}}_2}{\sin\frac{\Phi}{2}}$$

PRV Subtraction

- DCM method:

$$[FB(\Phi_2, \hat{\boldsymbol{e}}_2)] = [FN(\Phi, \hat{\boldsymbol{e}})][BN(\Phi_1, \hat{\boldsymbol{e}}_1)]^T$$

- Direct method:

$$\Phi_2 = 2\cos^{-1}\left(\cos\frac{\Phi}{2}\cos\frac{\Phi_1}{2} + \sin\frac{\Phi}{2}\sin\frac{\Phi_1}{2}\hat{\boldsymbol{e}}\cdot\hat{\boldsymbol{e}}_1\right)$$

$$\hat{\boldsymbol{e}}_2 = \frac{\cos\frac{\Phi_1}{2}\sin\frac{\Phi}{2}\hat{\boldsymbol{e}} - \cos\frac{\Phi}{2}\sin\frac{\Phi_1}{2}\hat{\boldsymbol{e}}_1 + \sin\frac{\Phi}{2}\sin\frac{\Phi_1}{2}\hat{\boldsymbol{e}}\times\hat{\boldsymbol{e}}_1}{\sin\frac{\Phi_2}{2}}$$

PRV Differential Kinematic Equation

- Mapping from body angular velocity vector to PRV rates:

$$\dot{\gamma} = \left[[I_{3\times3}] + \frac{1}{2}[\tilde{\gamma}] + \frac{1}{\Phi^2}\left(1 - \frac{\Phi}{2}\cot\left(\frac{\Phi}{2}\right)\right)[\tilde{\gamma}]^2 \right] {}^{\mathcal{B}}\boldsymbol{\omega}$$

- Mapping from PRV rates to body angular velocity vector:

$${}^{\mathcal{B}}\boldsymbol{\omega} = \left[[I_{3\times3}] - \left(\frac{1-\cos\Phi}{\Phi^2}\right)[\tilde{\gamma}] + \left(\frac{\Phi - \sin\Phi}{\Phi^3}\right)[\tilde{\gamma}]^2 \right] \dot{\gamma}$$

Conclusion

- PRV is based on a very fundamental rotation/orientation property called Euler's principal rotation theorem
- Singular for zero-rotation
- PRVs form the basis for many other attitude coordinates which are very useful for large angle rotations