Spacecraft Dynamics and Control – ASEN 5010

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Particle Kinematics

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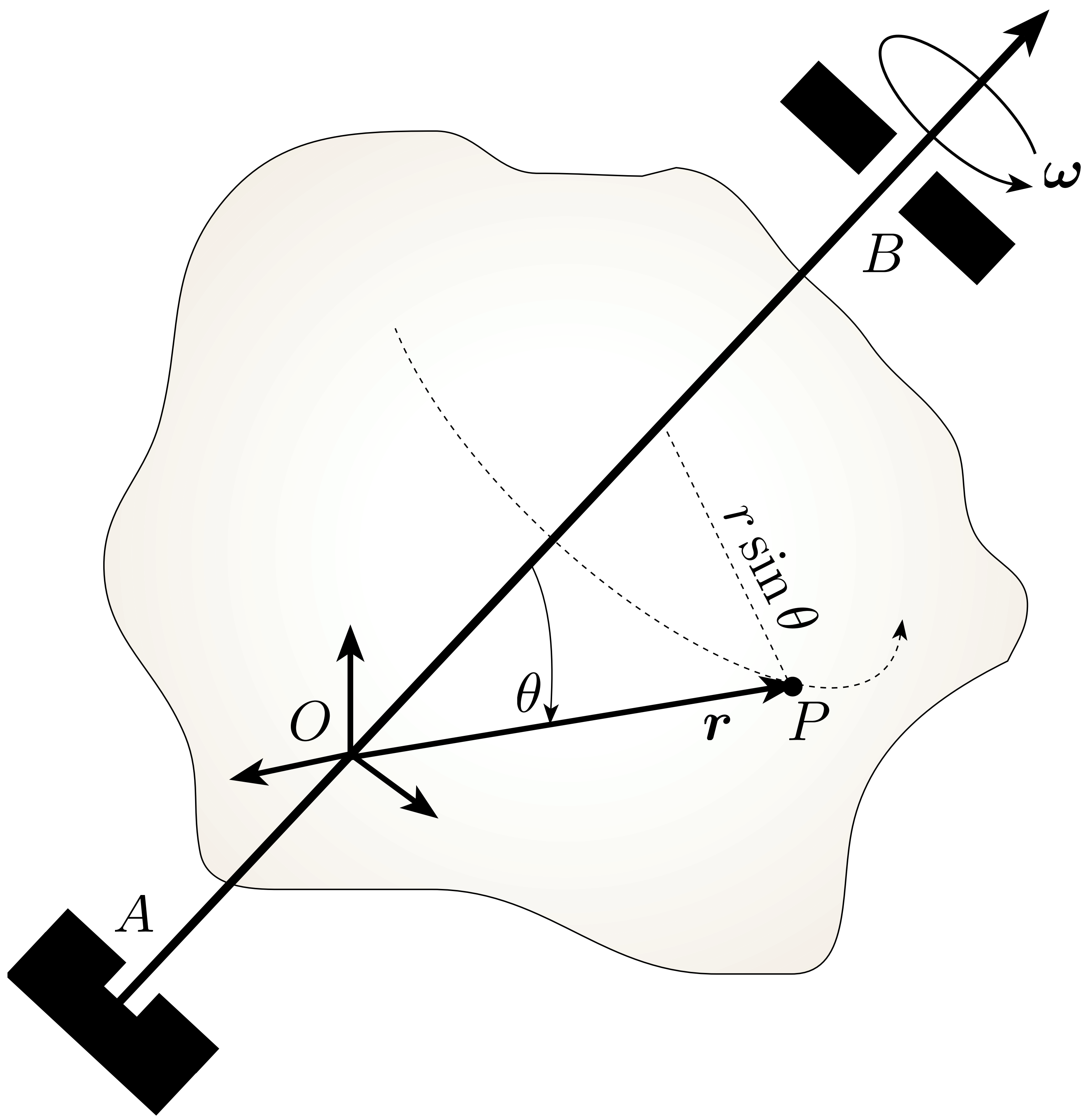
Outline

- Vector Notation
- Vector Differentiation
- Lots of brushing up on this material on your own!



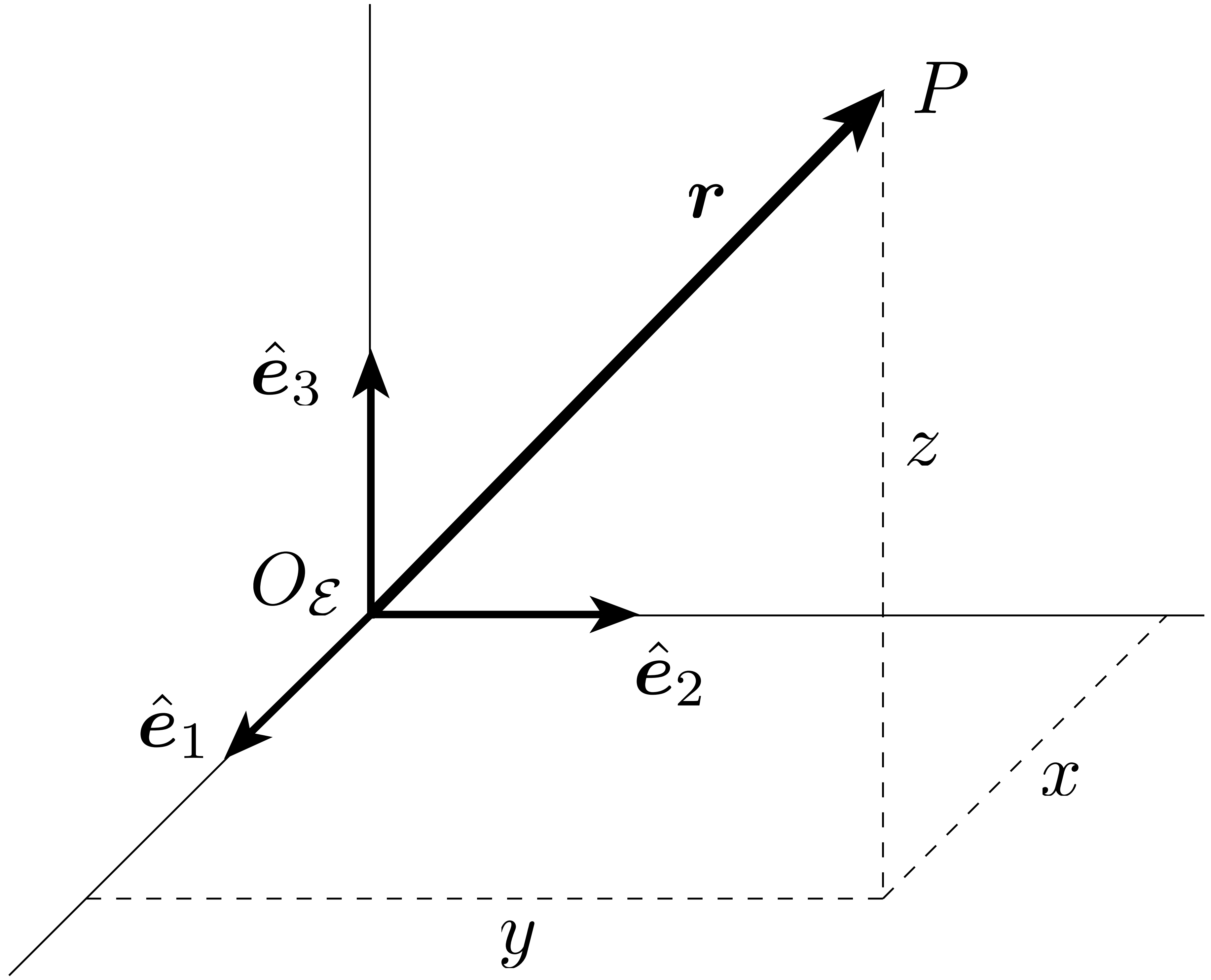
Vector Notation

Hopefully a boring topic for you by now...

What is a vector?

- Something with a direction and magnitude.
- A vector can be written as

$$
\begin{aligned}
\boldsymbol{r} &= x\hat{\boldsymbol{e}}_1 + y\hat{\boldsymbol{e}}_2 + z\hat{\boldsymbol{e}}_3 \\
&= r\hat{\boldsymbol{e}}_r \\
&= {}^{\mathcal{E}}\!\begin{pmatrix} x \\ y \\ z \end{pmatrix}
\end{aligned}
$$



Vector Addition

$$\boldsymbol{q} = \boldsymbol{r} + \boldsymbol{p} \quad \Rightarrow \quad \text{True}$$

$$^{\mathcal{E}}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = {}^{\mathcal{E}}\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} + {}^{\mathcal{B}}\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} q_1 &= r_1 + p_1 \\ q_2 &= r_2 + p_2 \\ q_3 &= r_3 + p_3 \end{aligned} \quad \Rightarrow \quad \text{False}$$

$$^{\mathcal{E}}\boldsymbol{q} = {}^{\mathcal{E}}\boldsymbol{r} + {}^{\mathcal{B}}\mathbf{p} \quad \Rightarrow \quad \text{False}$$

Coordinate frame

- Let a coordinate frame *B* be defined through the three unit orthogonal vectors:

$$\hat{\boldsymbol{b}}_1 \qquad \hat{\boldsymbol{b}}_2 \qquad \hat{\boldsymbol{b}}_3$$

- Let the origin of this frame be given by

$$\mathcal{O}_{\mathcal{B}}$$

- The frame is then defined through

$$\mathcal{B} : \{\mathcal{O}_{\mathcal{B}}, \hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2, \hat{\boldsymbol{b}}_3\}$$

- If we can ignore the frame origin, then we often use the shorthand notation

$$\mathcal{B} : \{\hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2, \hat{\boldsymbol{b}}_3\}$$

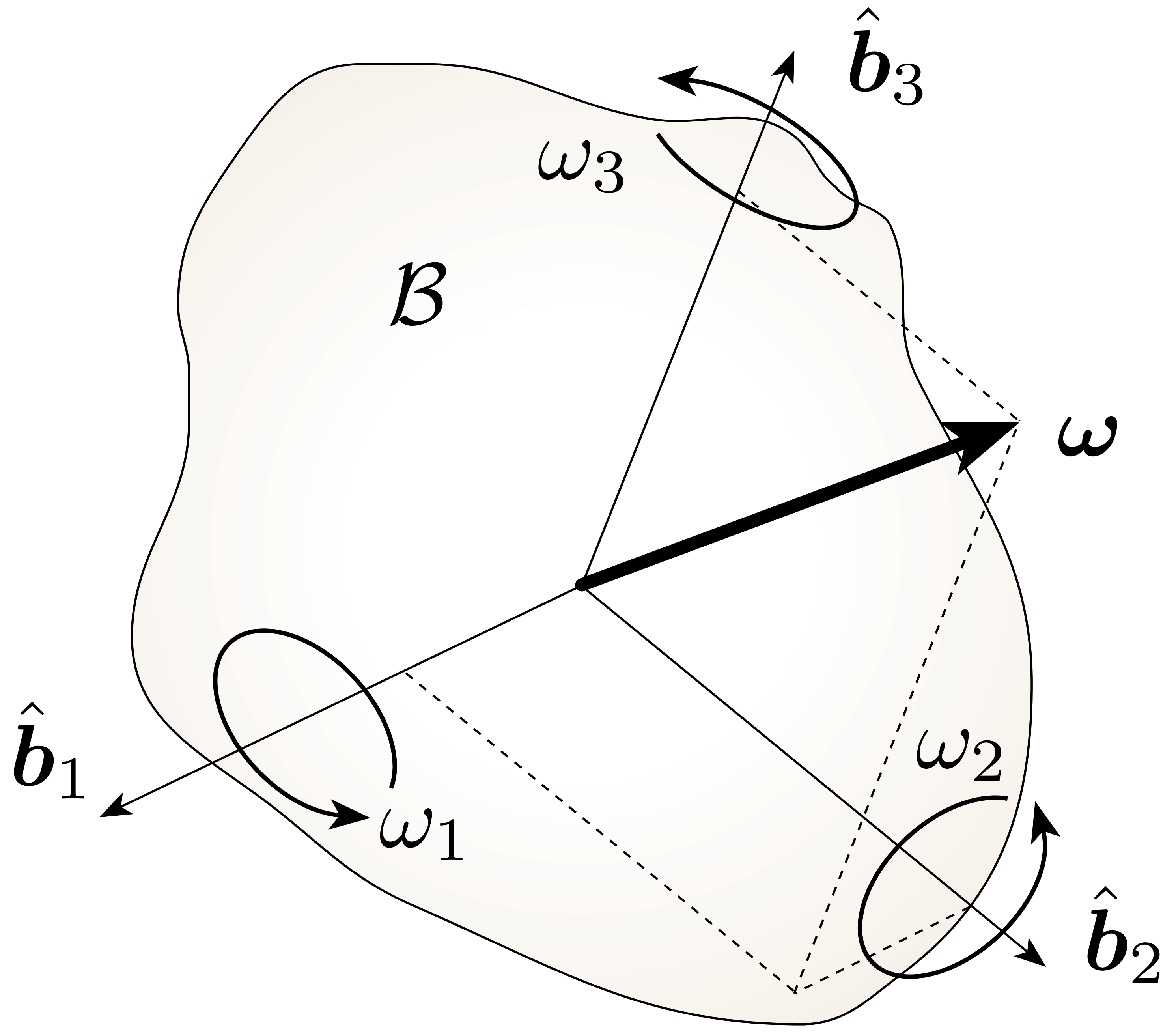
Angular Velocity Vector

- Angular velocity vector can be expressed as

$$\boldsymbol{\omega} = \omega_1 \hat{\boldsymbol{b}}_1 + \omega_2 \hat{\boldsymbol{b}}_2 + \omega_3 \hat{\boldsymbol{b}}_3$$

$${}^{\mathcal{B}}\boldsymbol{\omega} = {}^{\mathcal{B}}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

- ω_i are instantaneous body rates about the orthogonal $\hat{\boldsymbol{b}}_i$ axes.



Vector Differentiation

A crucial ability for attitude dynamics research...

Fixed Axis Rotation

- The rigid body is rotating about a fixed axis.
- The speed of *P* is given by

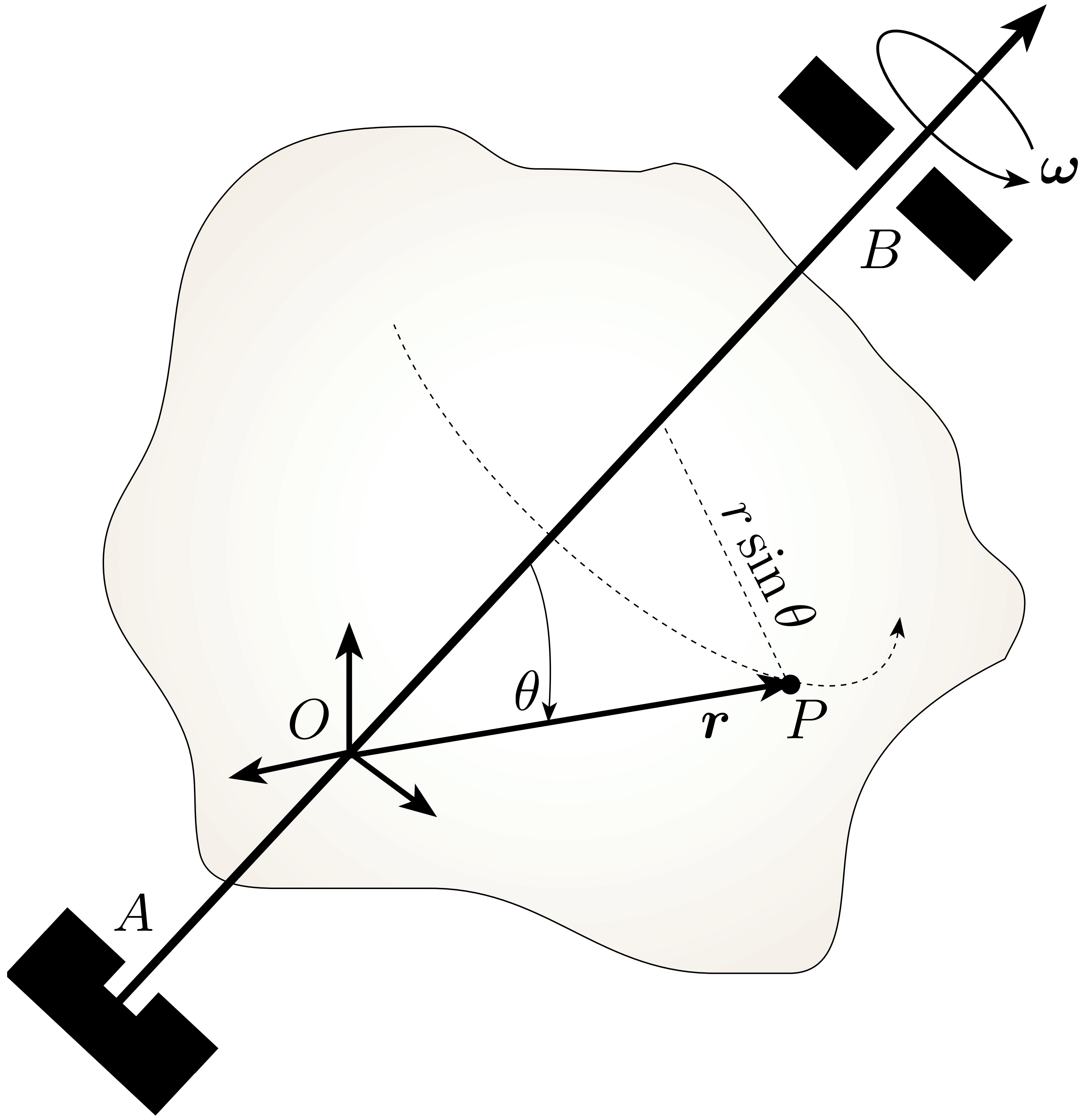
$$|\dot{\boldsymbol{r}}| = (r\sin\theta)\,\omega$$

- note that
$$\dot{\boldsymbol{r}} = (r\sin\theta)\,\omega\left(\frac{\boldsymbol{\omega}\times\boldsymbol{r}}{|\boldsymbol{\omega}\times\boldsymbol{r}|}\right)$$

- thus the transport velocity is

$$|\boldsymbol{\omega}\times\boldsymbol{r}| = \omega r\sin\theta$$

$$\boxed{\dot{\boldsymbol{r}} = \boldsymbol{\omega}\times\boldsymbol{r}}$$



Transport Theorem

- Let a position vector be written as

$$\boldsymbol{r} = r_1\hat{\boldsymbol{b}}_1 + r_2\hat{\boldsymbol{b}}_2 + r_3\hat{\boldsymbol{b}}_3$$

while the angular velocity vector is written as

$$\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = \omega_1\hat{\boldsymbol{b}}_1 + \omega_2\hat{\boldsymbol{b}}_2 + \omega_3\hat{\boldsymbol{b}}_3$$

- The derivative of a vector with respect to the $\mathcal{B}$ frame is written as

$$\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}) = \dot{r}_1\hat{\boldsymbol{b}}_1 + \dot{r}_2\hat{\boldsymbol{b}}_2 + \dot{r}_3\hat{\boldsymbol{b}}_3$$

since

$$\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\left(\hat{\boldsymbol{b}}_i\right) = 0$$

Transport Theorem

- The inertial derivative of the position vector is

$$\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}) = \dot{r}_1\hat{\boldsymbol{b}}_1 + \dot{r}_2\hat{\boldsymbol{b}}_2 + \dot{r}_3\hat{\boldsymbol{b}}_3 + r_1\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}\left(\hat{\boldsymbol{b}}_1\right) + r_2\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}\left(\hat{\boldsymbol{b}}_2\right) + r_3\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}\left(\hat{\boldsymbol{b}}_3\right)$$

- Note that $\hat{\boldsymbol{b}}_i$ are body fixed vectors, thus we find

$$\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}\left(\hat{\boldsymbol{b}}_i\right) = \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \hat{\boldsymbol{b}}_i$$

- This allows us to write the inertial derivative of the position vector as

$$\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}) = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}) + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}$$

Transport Theorem

$$\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}) = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}) + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r}$$

Learn to be one with this equation, and three-dimensional rotations will never haunt you again!

Comments

- Another noted otherwise, the following short-hand notation is used to denote inertial vector derivatives:

$$\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}(\boldsymbol{x}) \equiv \dot{\boldsymbol{x}}$$

- Note that we can analytically differentiate vectors, without first assigning specific coordinate frame. In fact, it is typically easier to wait until the very last steps before specifying a vectors through the vector components.

Direction Cosine Matrix

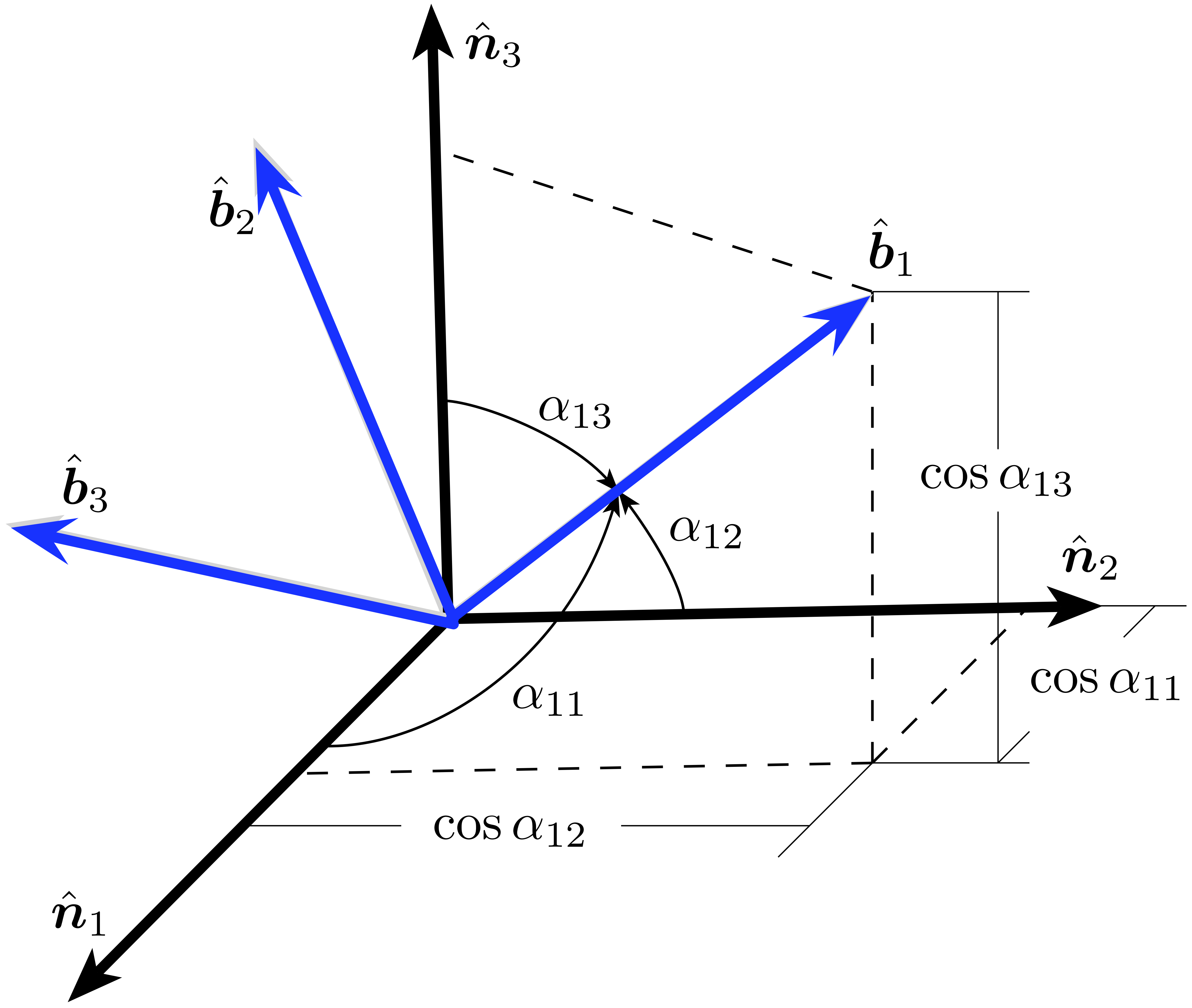
The mother of all attitude parameterizations...

Coordinate Frames

- A vectrix is a matrix of vectors.

$$\{\hat{\boldsymbol{n}}\} \equiv \begin{bmatrix} \hat{\boldsymbol{n}}_1 \\ \hat{\boldsymbol{n}}_2 \\ \hat{\boldsymbol{n}}_3 \end{bmatrix}$$

$$\{\hat{\boldsymbol{b}}\} \equiv \begin{bmatrix} \hat{\boldsymbol{b}}_1 \\ \hat{\boldsymbol{b}}_2 \\ \hat{\boldsymbol{b}}_3 \end{bmatrix}$$



Coordinate Frames

Frame base vectors are related through:

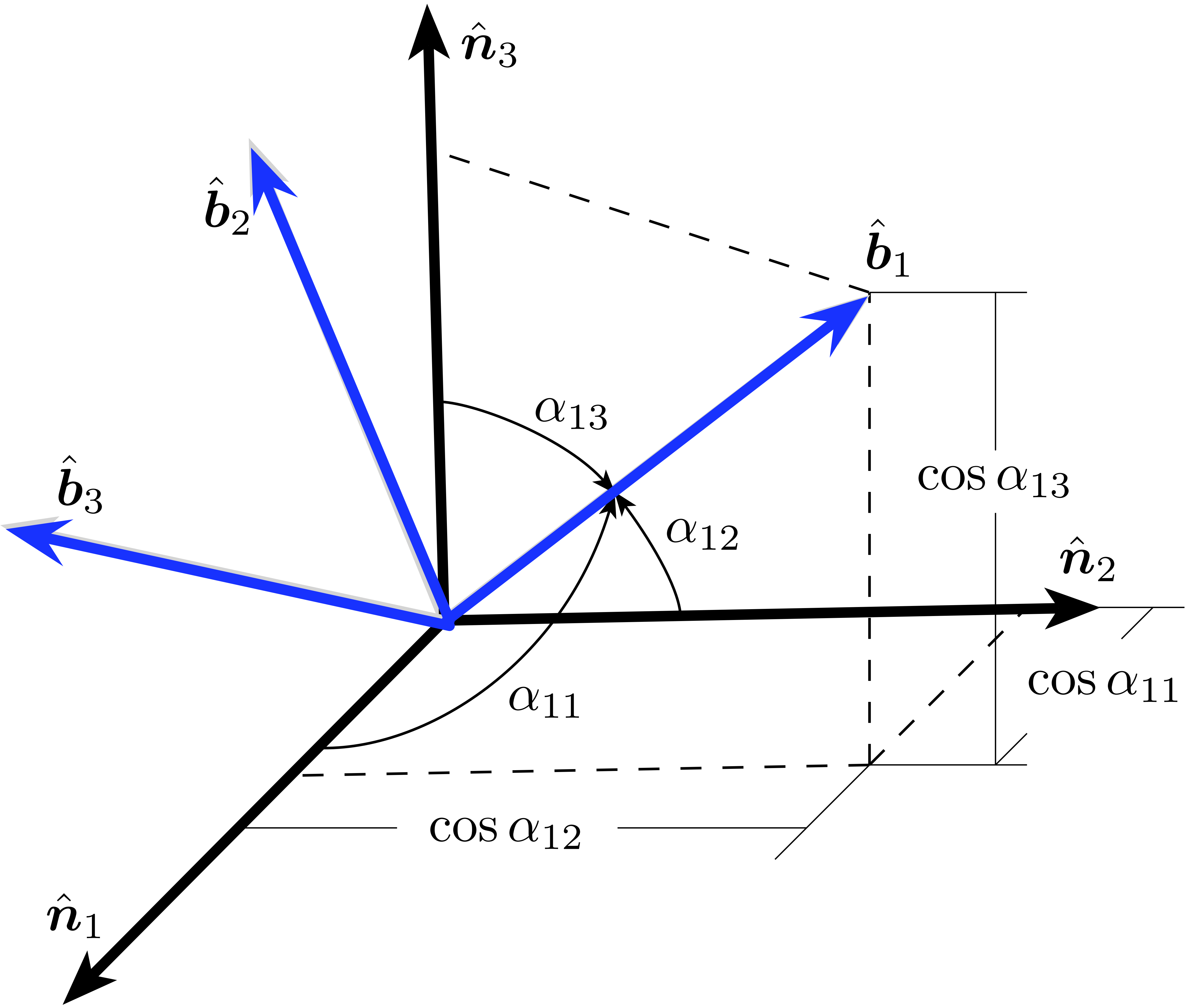
$$\hat{\boldsymbol{b}}_1 = \cos\alpha_{11}\hat{\boldsymbol{n}}_1 + \cos\alpha_{12}\hat{\boldsymbol{n}}_2 + \cos\alpha_{13}\hat{\boldsymbol{n}}_3$$
$$\hat{\boldsymbol{b}}_2 = \cos\alpha_{21}\hat{\boldsymbol{n}}_1 + \cos\alpha_{22}\hat{\boldsymbol{n}}_2 + \cos\alpha_{23}\hat{\boldsymbol{n}}_3$$
$$\hat{\boldsymbol{b}}_3 = \cos\alpha_{31}\hat{\boldsymbol{n}}_1 + \cos\alpha_{32}\hat{\boldsymbol{n}}_2 + \cos\alpha_{33}\hat{\boldsymbol{n}}_3$$

$$\{\hat{\boldsymbol{b}}\} = \begin{bmatrix} \cos\alpha_{11} & \cos\alpha_{12} & \cos\alpha_{13} \\ \cos\alpha_{21} & \cos\alpha_{22} & \cos\alpha_{23} \\ \cos\alpha_{31} & \cos\alpha_{32} & \cos\alpha_{33} \end{bmatrix} \{\hat{\boldsymbol{n}}\} = [C]\{\hat{\boldsymbol{n}}\}$$

Note that: $C_{ij} = \cos(\angle\, \hat{\boldsymbol{b}}_i, \hat{\boldsymbol{n}}_j) = \hat{\boldsymbol{b}}_i \cdot \hat{\boldsymbol{n}}_j$

Analogously, we can find:

$$\{\hat{\boldsymbol{n}}\} = \begin{bmatrix} \cos\alpha_{11} & \cos\alpha_{21} & \cos\alpha_{31} \\ \cos\alpha_{12} & \cos\alpha_{22} & \cos\alpha_{32} \\ \cos\alpha_{13} & \cos\alpha_{23} & \cos\alpha_{33} \end{bmatrix} \{\hat{\boldsymbol{b}}\} = [C]^T\{\hat{\boldsymbol{b}}\}$$



Matrix Inverse

Combining these two results, we find

$$\{\hat{\boldsymbol{b}}\} = [C][C]^T\{\hat{\boldsymbol{b}}\} \quad \longrightarrow \quad [C][C]^T = [I_{3\times 3}]$$

$$\{\hat{\boldsymbol{n}}\} = [C]^T[C]\{\hat{\boldsymbol{n}}\} \quad \longrightarrow \quad [C]^T[C] = [I_{3\times 3}]$$

Therefore, the inverse of a direction cosine matrix is simply the transpose operation.

$$\boxed{[C]^{-1} = [C]^T}$$

DCM Determinant

- Let's find the determinant of the [*C*] by first evaluating

$$\det\left(CC^T\right) = \det\left([I_{3\times3}]\right) = 1$$

- Since [*C*] is a square matrix, we find that

$$\det(C)\det(C^T) = 1$$

- Because det([*C*]) is the same as det([*C*]^T), this is further reduced to

$$\left(\det(C)\right)^2 = 1 \iff \det(C) = \pm 1$$

- Note that this is true for any orthogonal matrix.

- For a proper rotation matrix with right-handed coordinate system, then det(*C*) = +1.

Coordinate Frame Transformation

- Let a vector have its components taken in the body frame *B* or the inertial frame *N*:

$$\boldsymbol{v} = v_{b_1}\hat{\boldsymbol{b}}_1 + v_{b_2}\hat{\boldsymbol{b}}_2 + v_{b_3}\hat{\boldsymbol{b}}_3 = \{v_b\}^T\{\hat{\boldsymbol{b}}\}$$
$$\boldsymbol{v} = v_{n_1}\hat{\boldsymbol{n}}_1 + v_{n_2}\hat{\boldsymbol{n}}_2 + v_{n_3}\hat{\boldsymbol{n}}_3 = \{v_n\}^T\{\hat{\boldsymbol{n}}\}$$

- we can now rearrange the vector expression as

$$\boldsymbol{v} = \{v_n\}^T\{\hat{\boldsymbol{n}}\} = \{v_n\}^T[C]^T\{\hat{\boldsymbol{b}}\} = \{v_b\}^T\{\hat{\boldsymbol{b}}\}$$

- Equating components, we find that the two vector component sets must be related through

$$\boldsymbol{v}_b = [C]\boldsymbol{v}_n \qquad \boldsymbol{v}_n = [C]^T\boldsymbol{v}_b$$

- From here on, we will make use of the short-hand notation:

$$^{\mathcal{B}}\boldsymbol{v} \equiv \boldsymbol{v}_b \qquad ^{\mathcal{N}}\boldsymbol{v} \equiv \boldsymbol{v}_n$$

Adding DCM's

- Assume three coordinate frames given: $\mathcal{N} : \{\hat{\boldsymbol{n}}\}$ $\mathcal{B} : \{\hat{\boldsymbol{b}}\}$ $\mathcal{R} : \{\hat{\boldsymbol{r}}\}$
- Let *N* and *B* frame orientation be related through $\{\hat{\boldsymbol{b}}\} = [C]\{\hat{\boldsymbol{n}}\}$
- Let *R* and *B* frame orientation be related through $\{\hat{\boldsymbol{r}}\} = [C']\{\hat{\boldsymbol{b}}\}$
- Then the *R* and *N* frame orientation are directly related through

$$\{\hat{\boldsymbol{r}}\} = [C'][C]\{\hat{\boldsymbol{n}}\} = [C'']\{\hat{\boldsymbol{n}}\}$$

- Let us introduce the two-letter DCM notation [*NB*] as mapping from *B* to *N* frame, then the DCM addition is

$$\boxed{[RN] = [RB][BN]}$$

Kinematic Differential Equation

- What does this mean??
    - kinematic ➠ position description
    - differential equation ➠ time rate equation

what is $\boxed{[\dot{C}] = \frac{\mathrm{d}}{\mathrm{d}t}[C]}$

- How does the [*C*] direction cosine matrix evolve over time. The rotation rate of a rigid body is expressed through the body angular velocity vector:

$$\boldsymbol{\omega} = \omega_1 \hat{\boldsymbol{b}}_1 + \omega_2 \hat{\boldsymbol{b}}_2 + \omega_3 \hat{\boldsymbol{b}}_3$$

- This vector determines how a body will rotate, and thus also how the DCM describing the orientation will evolve.

Kinematic Differential Equation

- Let's study how the body frame vectors will evolve over time as seen by the inertial frame. To do so, we differentiate the vectrix of body frame orientation vectors.

$$\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}\{\hat{\boldsymbol{b}}_i\} = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\{\hat{\boldsymbol{b}}_i\} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \{\hat{\boldsymbol{b}}_i\}$$

- Let us introduce the matrix cross-product operator:

$$[\tilde{\boldsymbol{x}}] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

where $\boldsymbol{x} \times \boldsymbol{y} \equiv [\tilde{\boldsymbol{x}}]\boldsymbol{y}$

and $[\tilde{\boldsymbol{x}}]^T = -[\tilde{\boldsymbol{x}}]$

- The body frame vectrix differential equation is then simply

$$\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}\{\hat{\boldsymbol{b}}\} = -[\tilde{\boldsymbol{\omega}}]\{\hat{\boldsymbol{b}}\}$$

Kinematic Differential Equation

- Next take the inertial derivative of

$$\{\hat{\boldsymbol{b}}\} = [C]\{\hat{\boldsymbol{n}}\}$$

$$\frac{{}^N\mathrm{d}}{\mathrm{d}t}\{\hat{\boldsymbol{b}}\} = \frac{{}^N\mathrm{d}}{\mathrm{d}t}\left([C]\{\hat{\boldsymbol{n}}\}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left([C]\right)\{\hat{\boldsymbol{n}}\} + [C]\frac{{}^N\mathrm{d}}{\mathrm{d}t}\left(\{\hat{\boldsymbol{n}}\}\right) = [\dot{C}]\{\hat{\boldsymbol{n}}\}$$

- This leads to

$$\frac{{}^N\mathrm{d}}{\mathrm{d}t}\{\hat{\boldsymbol{b}}\} = -[\tilde{\boldsymbol{\omega}}]\{\hat{\boldsymbol{b}}\} = -[\tilde{\boldsymbol{\omega}}][C]\{\hat{\boldsymbol{n}}\} = [\dot{C}]\{\hat{\boldsymbol{n}}\}$$

$$\left([\dot{C}] + [\tilde{\boldsymbol{\omega}}][C]\right)\{\hat{\boldsymbol{n}}\} = 0$$

- Since this must be true for any *N* frame orientation, we find

$$\boxed{[\dot{C}] = -[\tilde{\boldsymbol{\omega}}][C]}$$

Kinematic Differential Equation

- An interesting fact is that this matrix differential equation holds for *any NxN orthogonal matrix*!

$$\frac{\mathrm{d}}{\mathrm{d}t}\left([C][C]^T\right) = [\dot{C}][C]^T + [C][\dot{C}]^T = 0$$

using the differential equation $[\dot{C}] = -[\tilde{\boldsymbol{\omega}}][C]$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left([C][C]^T\right) = -[\tilde{\boldsymbol{\omega}}][C][C]^T - [C][C]^T[\tilde{\boldsymbol{\omega}}]^T$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left([C][C]^T\right) = -[\tilde{\boldsymbol{\omega}}] + [\tilde{\boldsymbol{\omega}}] = 0$$

Principal Rotation Vector

The building block of many advanced attitude coordinates...

Theorem 3.1 (Euler's Principal Rotation): A rigid body or coordinate reference frame can be brought from an arbitrary initial orientation to an arbitrary final orientation by a single rigid rotation through a principal angle Φ about the principal axis $\hat{e}$; the principal axis is a judicious axis fixed in both the initial and final orientation.

That's great!! But, what does this mean???

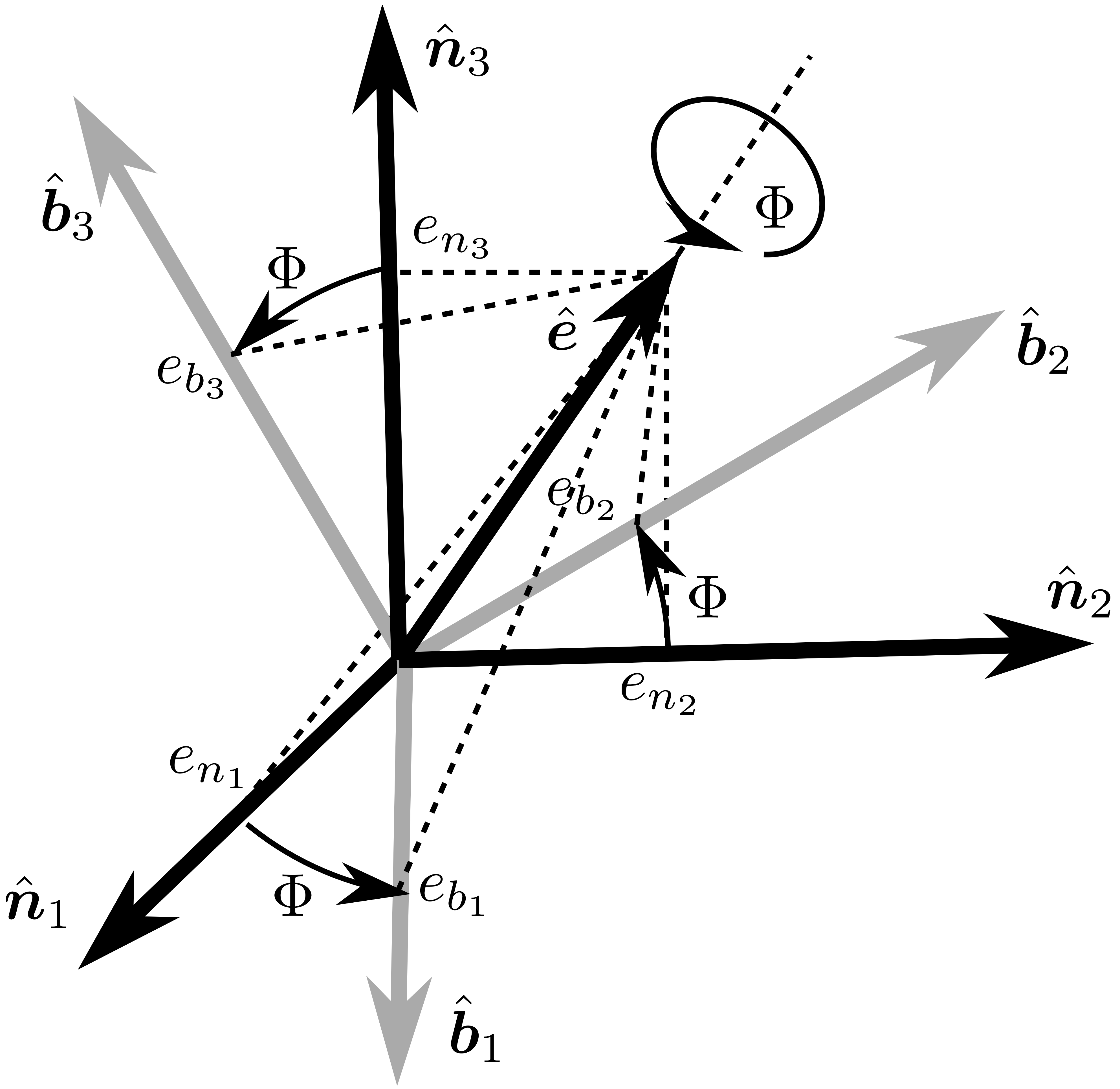
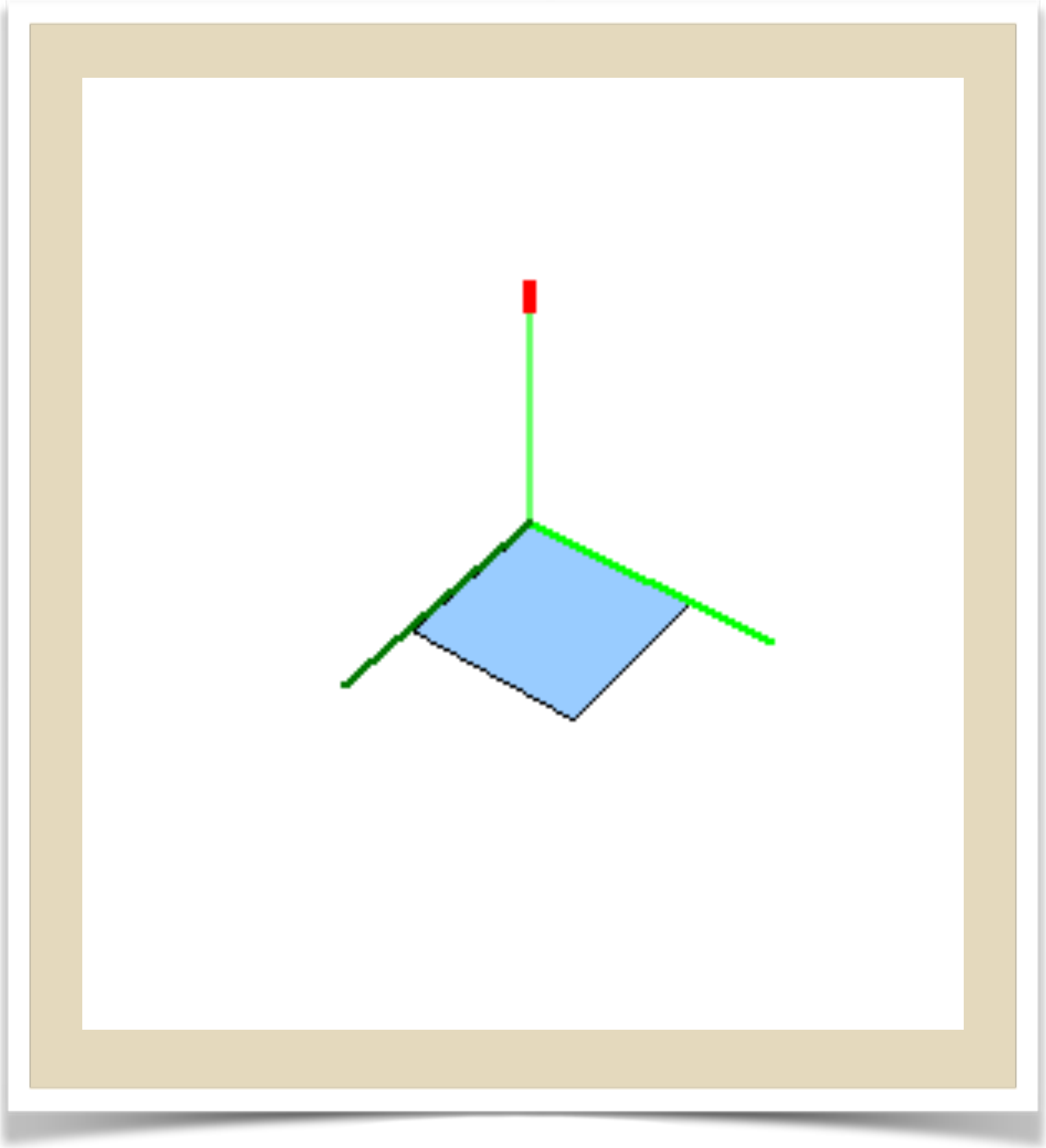
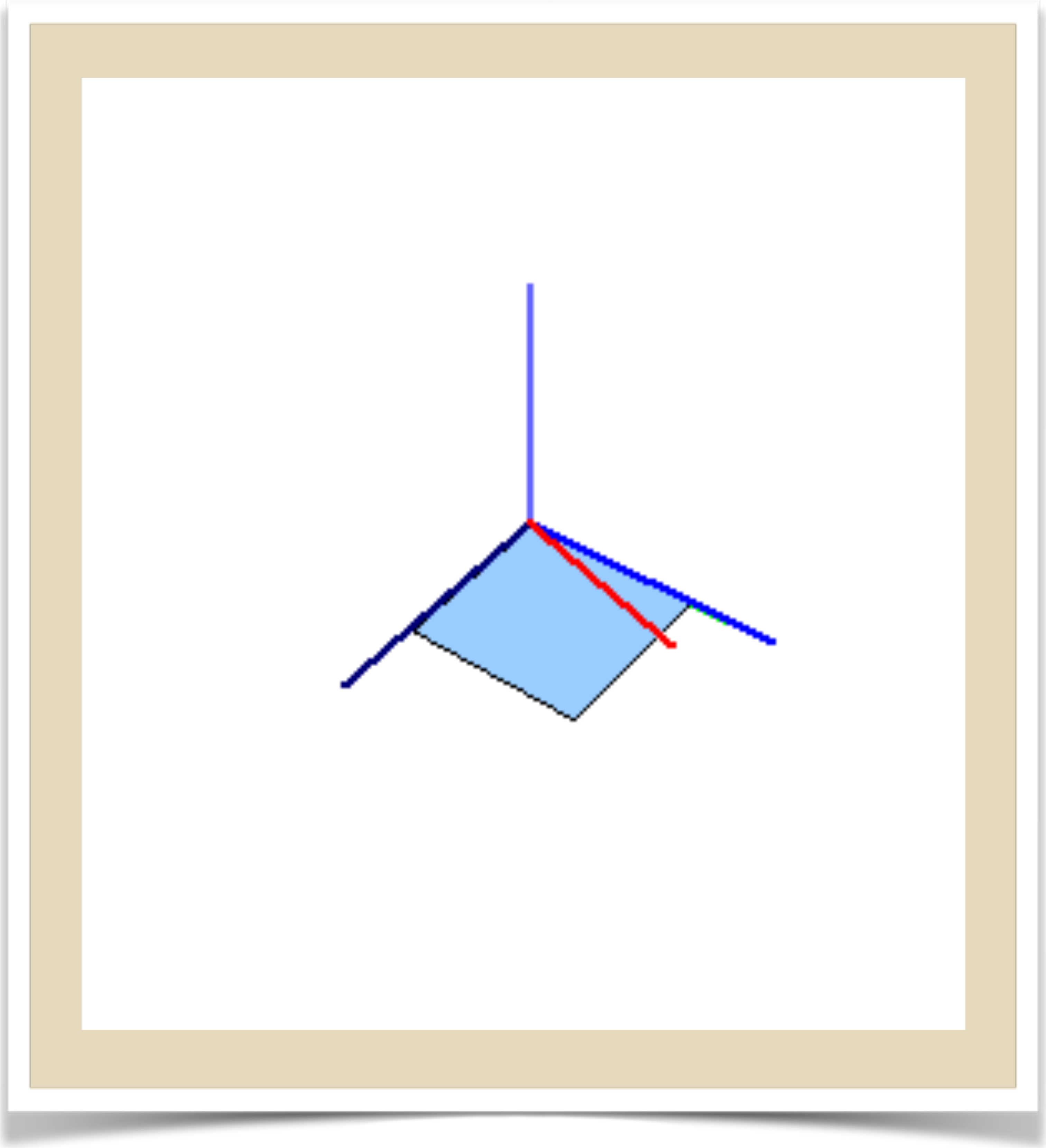


Illustration of Euler's Principal Rotation Theorem



(3-2-1) Euler Angles
(60,50,70) Degrees



Principal Rotation Vector

$$\Phi = 80.3385^o$$

$$\hat{\boldsymbol{e}} = (0.429577, 0.867729, 0.250019)^T$$

- Let's study the last statement of this theorem first: "the principal axis is a judicious axis fixed in both the initial and final orientation"

- This means that the principal axis unit vector will have the same vector components in the initial (i.e. inertial) and the final frame (i.e. body frame)

$$\hat{\boldsymbol{e}} = e_{b_1}\hat{\boldsymbol{b}}_1 + e_{b_2}\hat{\boldsymbol{b}}_2 + e_{b_3}\hat{\boldsymbol{b}}_3$$

$$\hat{\boldsymbol{e}} = e_{n_1}\hat{\boldsymbol{n}}_1 + e_{n_2}\hat{\boldsymbol{n}}_2 + e_{n_3}\hat{\boldsymbol{n}}_3$$

$$\Rightarrow \quad e_{b_i} = e_{n_i} = e_i$$

- Using the rotation matrix [*C*], the $\hat{\boldsymbol{e}}$ frame vector components in *B* and *N* frame can be related through

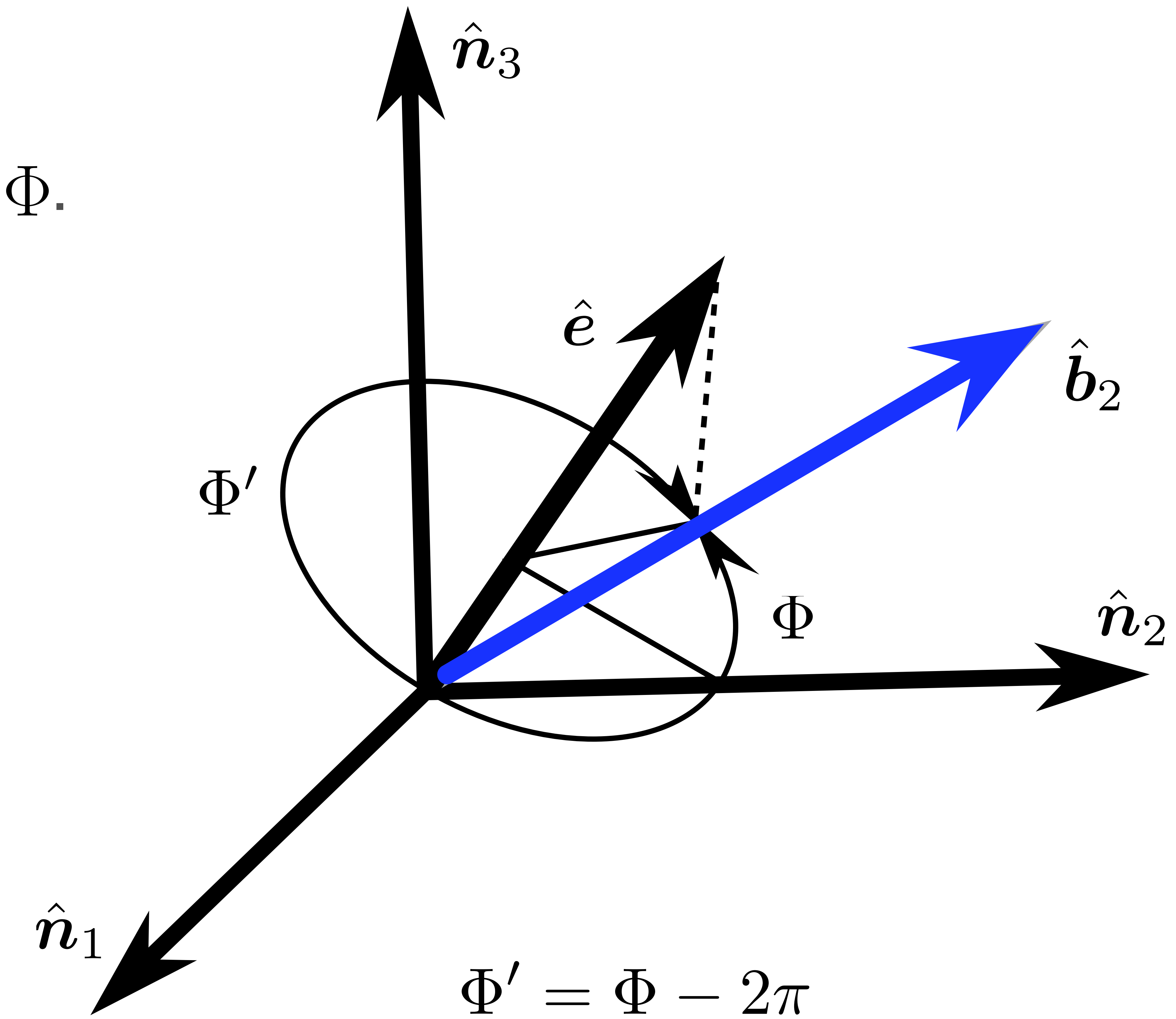
$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = [C] \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

- From this last equation, it is evident that $\hat{\boldsymbol{e}}$ must be an eigenvector of [C] with an eigenvalue of +1.

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = [C] \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

- This eigenvector is unique to within a sign of $\hat{\boldsymbol{e}}$ or Φ.
- The $\hat{\boldsymbol{e}}$ vector is not defined for a zero rotation!
- There are four possible principal rotations:

$$(\hat{\boldsymbol{e}}, \Phi)$$
$$(-\hat{\boldsymbol{e}}, -\Phi)$$
$$(\hat{\boldsymbol{e}}, \Phi')$$
$$(-\hat{\boldsymbol{e}}, -\Phi')$$



$$\Phi' = \Phi - 2\pi$$

Relationship to DCM

- We can express the [C] matrix in terms of PRV components as

$$[C] = \begin{bmatrix} e_1^2\Sigma + c\Phi & e_1e_2\Sigma + e_3 s\Phi & e_1e_3\Sigma - e_2 s\Phi \\ e_2e_1\Sigma - e_3 s\Phi & e_2^2\Sigma + c\Phi & e_2e_3\Sigma + e_1 s\Phi \\ e_3e_1\Sigma + e_2 s\Phi & e_3e_2\Sigma - e_1 s\Phi & e_3^2\Sigma + c\Phi \end{bmatrix}$$

$$\Sigma = 1 - \mathrm{c}\Phi$$

- The inverse transformation from [C] to PRV is found by inspecting the matrix structure:

$$\cos\Phi = \frac{1}{2}\left(C_{11} + C_{22} + C_{33} - 1\right) \qquad \Phi' = \Phi - 2\pi$$

$$\hat{\boldsymbol{e}} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \frac{1}{2\sin\Phi}\begin{pmatrix} C_{23} - C_{32} \\ C_{31} - C_{13} \\ C_{12} - C_{21} \end{pmatrix}$$

PRV Addition

- DCM method:

$$[FN(\Phi, \hat{\boldsymbol{e}})] = [FB(\Phi_2, \hat{\boldsymbol{e}}_2)][BN(\Phi_1, \hat{\boldsymbol{e}}_1)]$$

- Direct method:

$$\Phi = 2\cos^{-1}\left(\cos\frac{\Phi_1}{2}\cos\frac{\Phi_2}{2} - \sin\frac{\Phi_1}{2}\sin\frac{\Phi_2}{2}\hat{\boldsymbol{e}}_1 \cdot \hat{\boldsymbol{e}}_2\right)$$

$$\hat{\boldsymbol{e}} = \frac{\cos\frac{\Phi_2}{2}\sin\frac{\Phi_1}{2}\hat{\boldsymbol{e}}_1 + \cos\frac{\Phi_1}{2}\sin\frac{\Phi_2}{2}\hat{\boldsymbol{e}}_2 + \sin\frac{\Phi_1}{2}\sin\frac{\Phi_2}{2}\hat{\boldsymbol{e}}_1 \times \hat{\boldsymbol{e}}_2}{\sin\frac{\Phi}{2}}$$

PRV Subtraction

- DCM method:

$$[FB(\Phi_2, \hat{\boldsymbol{e}}_2)] = [FN(\Phi, \hat{\boldsymbol{e}})][BN(\Phi_1, \hat{\boldsymbol{e}}_1)]^T$$

- Direct method:

$$\Phi_2 = 2\cos^{-1}\left(\cos\frac{\Phi}{2}\cos\frac{\Phi_1}{2} + \sin\frac{\Phi}{2}\sin\frac{\Phi_1}{2}\hat{\boldsymbol{e}}\cdot\hat{\boldsymbol{e}}_1\right)$$

$$\hat{\boldsymbol{e}}_2 = \frac{\cos\frac{\Phi_1}{2}\sin\frac{\Phi}{2}\hat{\boldsymbol{e}} - \cos\frac{\Phi}{2}\sin\frac{\Phi_1}{2}\hat{\boldsymbol{e}}_1 + \sin\frac{\Phi}{2}\sin\frac{\Phi_1}{2}\hat{\boldsymbol{e}}\times\hat{\boldsymbol{e}}_1}{\sin\frac{\Phi_2}{2}}$$

PRV Differential Kinematic Equation

- Mapping from body angular velocity vector to PRV rates:

$$\dot{\gamma} = \left[[I_{3\times3}] + \frac{1}{2}[\tilde{\gamma}] + \frac{1}{\Phi^2}\left(1 - \frac{\Phi}{2}\cot\left(\frac{\Phi}{2}\right)\right)[\tilde{\gamma}]^2 \right] {}^{\mathcal{B}}\boldsymbol{\omega}$$

- Mapping from PRV rates to body angular velocity vector:

$${}^{\mathcal{B}}\boldsymbol{\omega} = \left[[I_{3\times3}] - \left(\frac{1 - \cos\Phi}{\Phi^2}\right)[\tilde{\gamma}] + \left(\frac{\Phi - \sin\Phi}{\Phi^3}\right)[\tilde{\gamma}]^2 \right] \dot{\gamma}$$

Conclusion

- PRV is based on a very fundamental rotation/orientation property called Euler's principal rotation theorem
- Singular for zero-rotation
- PRVs form the basis for many other attitude coordinates which are very useful for large angle rotations

Euler Parameters (Quaternions)

Voted most popular attitude coordinates in the non-singular category...

Introduction

- Very popular redundant set of attitude coordinates
- Are called either Euler Parameters (EPs) or quaternions
- Major benefits:
  - Non-singular attitude description
  - Linear differential kinematic equation
  - Works well for small and large rotations
- Drawbacks:
  - Constraint equation must be identified as all times
  - Not as simple to visualize

Definition of EP

- The redundant Euler Parameters are defined using the principal rotation components as

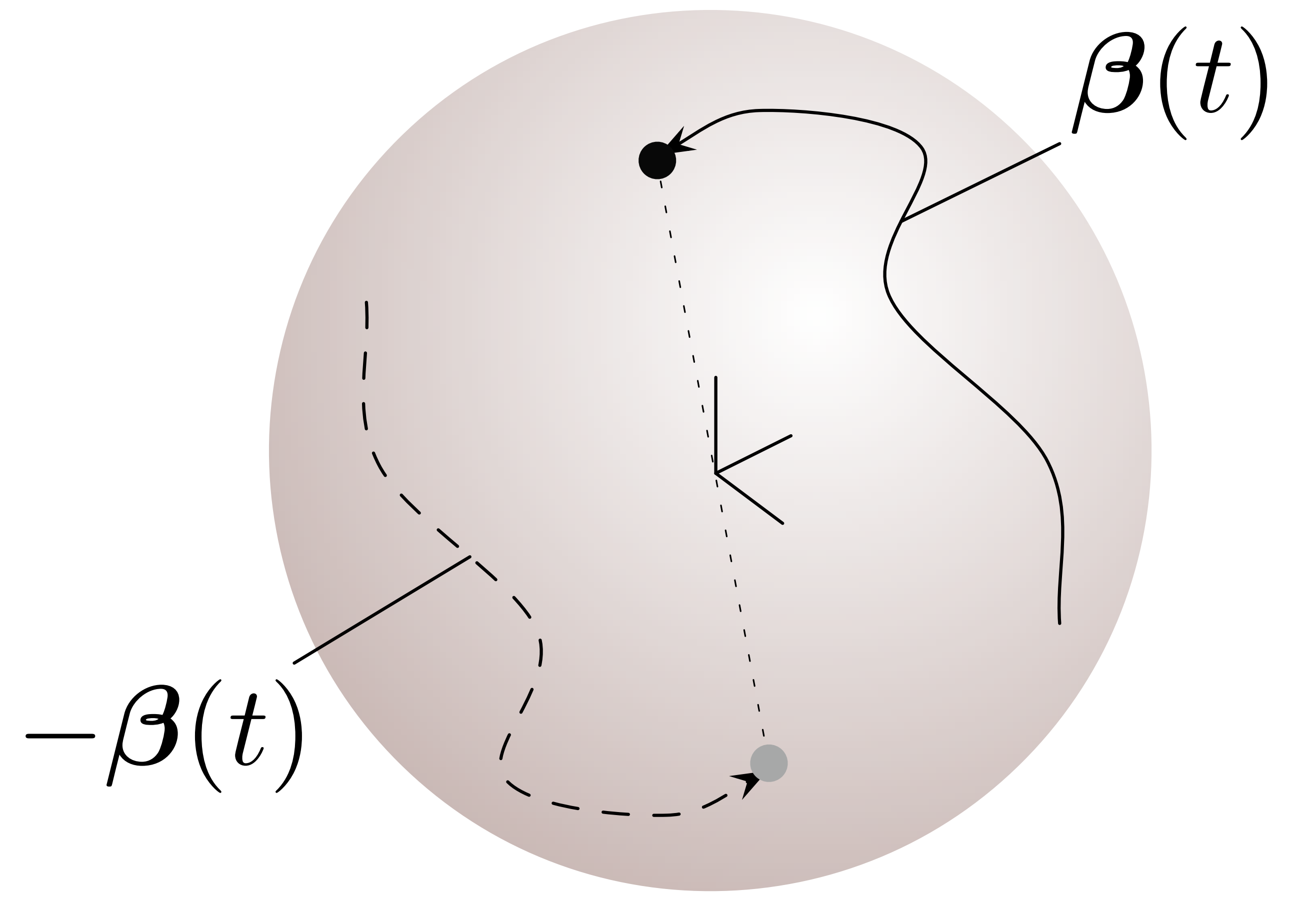
$$
\begin{aligned}
\beta_0 &= \cos(\Phi/2)\\
\beta_1 &= e_1 \sin(\Phi/2)\\
\beta_2 &= e_2 \sin(\Phi/2)\\
\beta_3 &= e_3 \sin(\Phi/2)
\end{aligned}
$$

Constraints:

$$e_1^2 + e_2^2 + e_3^2 = 1$$

$$\beta_0^2 + \beta_1^2 + \beta_2^2 + \beta_3^2 = 1$$

- Note that the 4-coordinate set has a single constraint equation! All EPs must lie on the three-dimensional surface of a 4-dimensional hypersphere.



Unit Hypersphere

- Since the PRV components are not unique, we find that the EP also isn't unique:

$$(-\hat{\boldsymbol{e}}, -\Phi) \qquad \beta_0' = \cos\left(-\frac{\Phi}{2}\right) = \cos\left(\frac{\Phi}{2}\right) = \beta_0$$

$$\beta_i' = -e_i \sin\left(-\frac{\Phi}{2}\right) = e_i \sin\left(\frac{\Phi}{2}\right) = \beta_i$$

$$(\hat{\boldsymbol{e}}, \Phi') \qquad \beta_0' = \cos\left(\frac{\Phi'}{2}\right) = \cos\left(\frac{\Phi}{2} - \pi\right) = -\cos\left(\frac{\Phi}{2}\right) = -\beta_0$$

$$\beta_i' = e_i \sin\left(\frac{\Phi'}{2}\right) = e_i \sin\left(\frac{\Phi'}{2} - \pi\right) = -e_i \sin\left(\frac{\Phi}{2}\right) = -\beta_i$$

- Note that the alternate EP set corresponds to performing the larger principle rotation angle (i.e., rotating the long way round)

Euler Parameter to DCM Relationship

- The rotation matrix can be expressed in terms or EPs as:

$$[C] = \begin{bmatrix} \beta_0^2+\beta_1^2-\beta_2^2-\beta_3^2 & 2(\beta_1\beta_2+\beta_0\beta_3) & 2(\beta_1\beta_3-\beta_0\beta_2) \\ 2(\beta_1\beta_2-\beta_0\beta_3) & \beta_0^2-\beta_1^2+\beta_2^2-\beta_3^2 & 2(\beta_2\beta_3+\beta_0\beta_1) \\ 2(\beta_1\beta_3+\beta_0\beta_2) & 2(\beta_2\beta_3-\beta_0\beta_1) & \beta_0^2-\beta_1^2-\beta_2^2+\beta_3^2 \end{bmatrix}$$

- The inverse relationship is found by inspection to be

$$\beta_0 = \pm\frac{1}{2}\sqrt{C_{11}+C_{22}+C_{33}+1}$$

$$\beta_1 = \frac{C_{23}-C_{32}}{4\beta_0}$$

$$\beta_2 = \frac{C_{31}-C_{13}}{4\beta_0}$$

$$\beta_3 = \frac{C_{12}-C_{21}}{4\beta_0}$$

Singular if: $\beta_0 \to 0$

- Sheppard's method is a robust method to compute the EP from a rotation matrix:

1st step: Find largest value of

$$\beta_0^2 = \frac{1}{4}\left(1 + \mathrm{trace}\left([C]\right)\right) \qquad \beta_2^2 = \frac{1}{4}\left(1 + 2C_{22} - \mathrm{trace}\left([C]\right)\right)$$

$$\beta_1^2 = \frac{1}{4}\left(1 + 2C_{11} - \mathrm{trace}\left([C]\right)\right) \qquad \beta_3^2 = \frac{1}{4}\left(1 + 2C_{33} - \mathrm{trace}\left([C]\right)\right)$$

2nd step: Compute the remaining EPs using

$$\beta_0\beta_1 = (C_{23} - C_{32})/4 \qquad \beta_1\beta_2 = (C_{12} + C_{21})/4$$

$$\beta_0\beta_2 = (C_{31} - C_{13})/4 \qquad \beta_3\beta_1 = (C_{31} + C_{13})/4$$

$$\beta_0\beta_3 = (C_{12} - C_{21})/4 \qquad \beta_2\beta_3 = (C_{23} + C_{32})/4$$

Adding Euler Parameters

- A very useful advantage of EPs is how you can add or subtract two orientations using them. Using DCMs, we can add two rotations using:

$$[FN(\boldsymbol{\beta})] = [FB(\boldsymbol{\beta}'')][BN(\boldsymbol{\beta}')]$$

- However, using EPs directly, we find the elegant result:

$$\begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{bmatrix} \beta_0'' & -\beta_1'' & -\beta_2'' & -\beta_3'' \\ \beta_1'' & \beta_0'' & \beta_3'' & -\beta_2'' \\ \beta_2'' & -\beta_3'' & \beta_0'' & \beta_1'' \\ \beta_3'' & \beta_2'' & -\beta_1'' & \beta_0'' \end{bmatrix} \begin{pmatrix} \beta_0' \\ \beta_1' \\ \beta_2' \\ \beta_3' \end{pmatrix}$$

- Note that this matrix is orthogonal!

- By reshuffling the terms (i.e. permutation) in the last EP addition equation, we can also write this as

$$\begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{bmatrix} \beta_0' & -\beta_1' & -\beta_2' & -\beta_3' \\ \beta_1' & \beta_0' & -\beta_3' & \beta_2' \\ \beta_2' & \beta_3' & \beta_0' & -\beta_1' \\ \beta_3' & -\beta_2' & \beta_1' & \beta_0' \end{bmatrix} \begin{pmatrix} \beta_0'' \\ \beta_1'' \\ \beta_2'' \\ \beta_3'' \end{pmatrix}$$

- To subtract two orientations described through EPs, we can use the last two equations and exploit the orthogonality property of the 4x4 matrix to invert it and solve for either $\boldsymbol{\beta}'$ or $\boldsymbol{\beta}''$.

Euler Parameter Differential Equation

- Using the differential equation of DCMs, and the relationship between EPs and DCMs, we can derive the differential kinematic equations of Euler parameters.
- However, this is a rather lengthy and algebraically complex task. The end result is the amazingly simple bi-linear result:

$$\begin{pmatrix} \dot{\beta}_0 \\ \dot{\beta}_1 \\ \dot{\beta}_2 \\ \dot{\beta}_3 \end{pmatrix} = \frac{1}{2} \begin{bmatrix} \beta_0 & -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_1 & \beta_0 & -\beta_3 & \beta_2 \\ \beta_2 & \beta_3 & \beta_0 & -\beta_1 \\ \beta_3 & -\beta_2 & \beta_1 & \beta_0 \end{bmatrix} \begin{pmatrix} 0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

- Rearranging the terms on the right hand side of this differential equation, we can also write this as

$$\begin{pmatrix} \dot{\beta}_0 \\ \dot{\beta}_1 \\ \dot{\beta}_2 \\ \dot{\beta}_3 \end{pmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{bmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$$

- Having the differential equation depend linearly on the EPs is important in estimation theory. This makes the EPs ideal coordinate candidates to be used in a Kalman filter (Spacecraft orientation estimator).

2nd Euler Parameter Differential Kinematic Eqs.

- The EP differential equations can also be written in the following convenient form for numerical integration:

$$\dot{\boldsymbol{\beta}} = \frac{1}{2}[B(\boldsymbol{\beta})]\boldsymbol{\omega} \qquad [B(\boldsymbol{\beta})] = \begin{bmatrix} -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_0 & -\beta_3 & \beta_2 \\ \beta_3 & \beta_0 & -\beta_1 \\ -\beta_2 & \beta_1 & \beta_0 \end{bmatrix}$$

- The [*B*] matrix satisfies the following useful identities:

$$[B(\boldsymbol{\beta})]^T \boldsymbol{\beta} = \mathbf{0}$$

$$[B(\boldsymbol{\beta})]^T \boldsymbol{\beta}' = -[B(\boldsymbol{\beta}')]^T \boldsymbol{\beta}$$

3rd Euler Parameter Differential Kinematic Eqs.

- In control applications, the scalar and vector components of the Euler parameters are sometimes treated separately.

Define: $$\boldsymbol{\epsilon} \equiv (\beta_1, \beta_2, \beta_3)^T$$

Define: $$[T(\beta_0, \boldsymbol{\epsilon})] = \beta_0 [I_{3\times 3}] + [\tilde{\boldsymbol{\epsilon}}]$$

Differential Equation:

$$\dot{\beta}_0 = -\frac{1}{2}\boldsymbol{\epsilon}^T \boldsymbol{\omega} = -\frac{1}{2}\boldsymbol{\omega}^T \boldsymbol{\epsilon}$$

$$\dot{\boldsymbol{\epsilon}} = \frac{1}{2}[T]\boldsymbol{\omega}$$

Conclusion

- Non-singular, redundant set of attitude coordinates
- Euler parameter vector must abide by the unit length constraint
- There are two sets of EPs that describe a particular orientation (short and long way round)
- Convenient method to add two EP vectors
- Linear differential kinematic equations

Classical Rodrigues Parameters (Gibbs Vector or CRPs)

Popular coordinates for large rotations and robotics....

CRP Definitions

Euler parameter relationship:

$$q_i = \frac{\beta_i}{\beta_0} \qquad i = 1, 2, 3$$

Singular if 0
(±180° case)

$$\beta_0 = \frac{1}{\sqrt{1 + \boldsymbol{q}^T \boldsymbol{q}}}$$

$$\beta_i = \frac{q_i}{\sqrt{1 + \boldsymbol{q}^T \boldsymbol{q}}} \qquad i = 1, 2, 3$$

Singular if ∞
(±180° case)

Principal rotation parameter relationship:

$$\boldsymbol{q} = \tan \frac{\Phi}{2} \hat{\boldsymbol{e}}$$

Singular for ±180°

$$\boldsymbol{q} \approx \frac{\Phi}{2} \hat{\boldsymbol{e}}$$

Linearizes to
angles over 2.

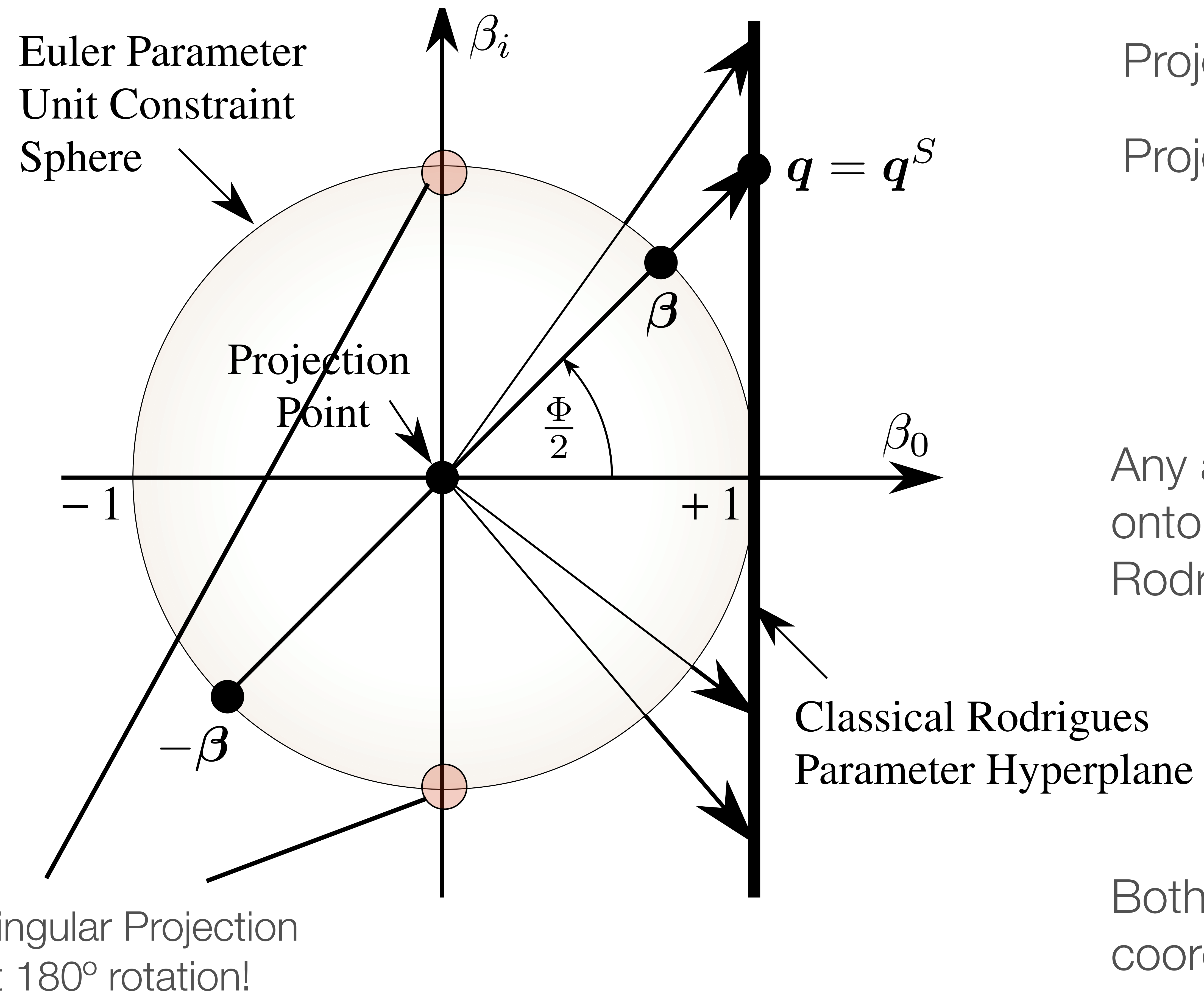
These parameters are much better suited for large spacecraft rotations than Euler angles, while remaining a minimal coordinate set.
Only the upside down description is singular.

CRP Definitions

Relationship to DCM: $$[\tilde{\boldsymbol{q}}] = \frac{[C]^T - [C]}{\zeta^2}$$ $$\zeta = \sqrt{\mathrm{trace}([C]) + 1} = 2\beta_0$$

$$\boldsymbol{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \frac{1}{\zeta^2} \begin{pmatrix} C_{23} - C_{32} \\ C_{31} - C_{13} \\ C_{12} - C_{21} \end{pmatrix}$$

Stereographic Projection

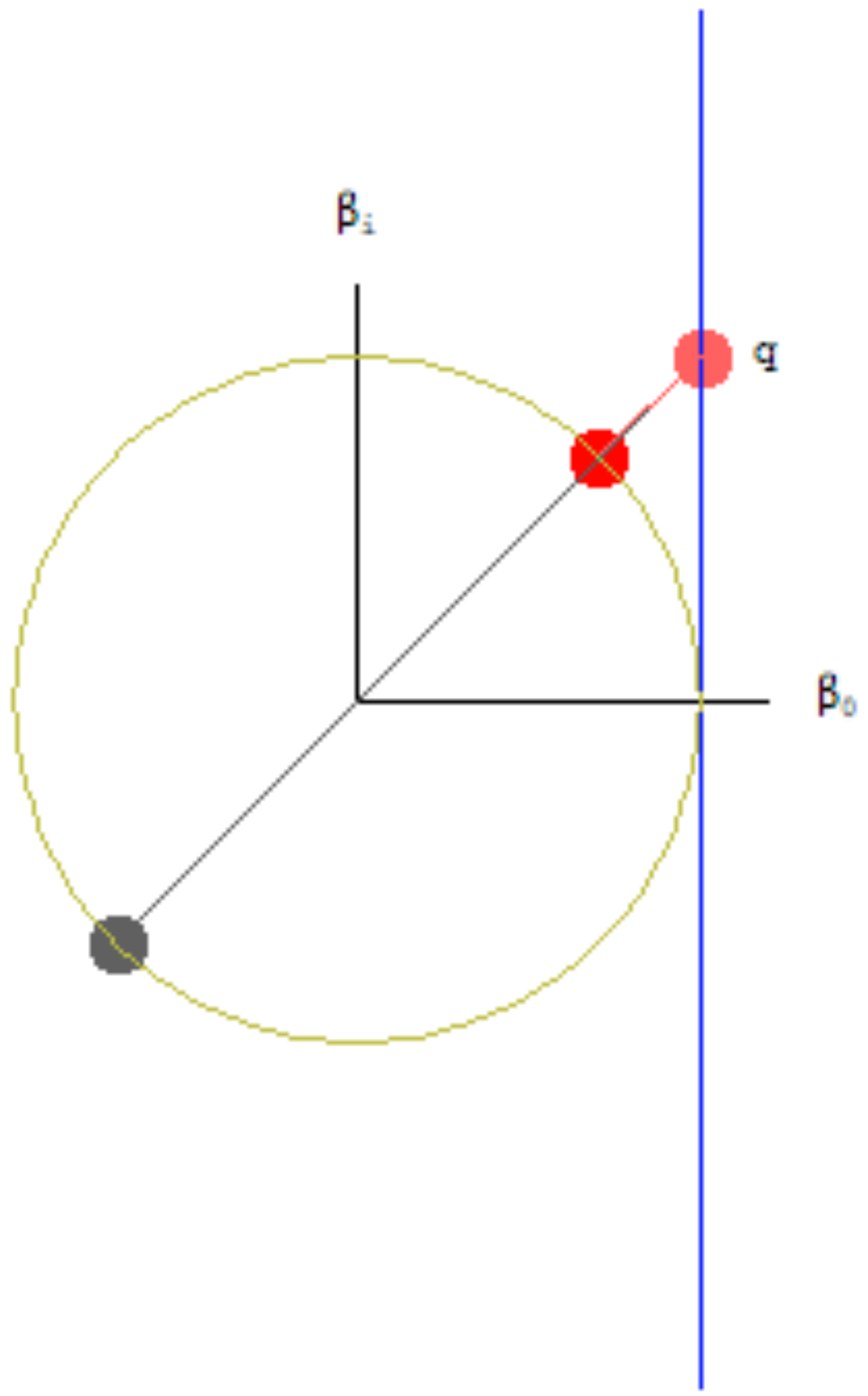


Projection Point: $(0,0,0,0)$

Projection Plane: $\beta_0 = +1$

Any attitude (surface point) is projected onto the hyper-plane to form the classical Rodrigues parameters.

Both EP sets yield the identical CRP coordinates.



http://hanspeterschaub.info/crp.html

Shadow CRP Set

- Using the alternate set of Euler parameters, we can find the "shadow" set of CRP parameters:

$$q_i^S = \frac{-\beta_i}{-\beta_0} = q_i$$

- For the case of CRPs, the shadow set and the original set of attitude parameters are identical. Thus, the shadow set cannot be used to avoid the 180º singularity.

Direction Cosine Matrix

Matrix components:

$$[C] = \frac{1}{1+\boldsymbol{q}^T\boldsymbol{q}}\begin{bmatrix} 1+q_1^2-q_2^2-q_3^2 & 2\left(q_1q_2+q_3\right) & 2\left(q_1q_3-q_2\right) \\ 2\left(q_2q_1-q_3\right) & 1-q_1^2+q_2^2-q_3^2 & 2\left(q_2q_3+q_1\right) \\ 2\left(q_3q_1+q_2\right) & 2\left(q_3q_2-q_1\right) & 1-q_1^2-q_2^2+q_3^2 \end{bmatrix}$$

$$[C] = \frac{1}{1+\boldsymbol{q}^T\boldsymbol{q}}\left(\left(1-\boldsymbol{q}^T\boldsymbol{q}\right)[I_{3\times 3}] + 2\boldsymbol{q}\boldsymbol{q}^T - 2[\tilde{\boldsymbol{q}}]\right)$$

$$[C(\boldsymbol{q})]^{-1} = [C(\boldsymbol{q})]^T = [C(-\boldsymbol{q})]$$

Attitude Addition/Subtraction

- DCM method:

$$[FN(\boldsymbol{q})] = [FB(\boldsymbol{q}'')][BN(\boldsymbol{q}')]$$

- Direct method:

$$\boldsymbol{q} = \frac{\boldsymbol{q}'' + \boldsymbol{q}' - \boldsymbol{q}'' \times \boldsymbol{q}'}{1 - \boldsymbol{q}'' \cdot \boldsymbol{q}'}$$

Attitude Addition

$$\boldsymbol{q}'' = \frac{\boldsymbol{q} - \boldsymbol{q}' + \boldsymbol{q} \times \boldsymbol{q}'}{1 + \boldsymbol{q} \cdot \boldsymbol{q}'}$$

Relative Attitude (Subtraction)

Note: Using $\delta\boldsymbol{q} = \boldsymbol{q} - \boldsymbol{q}'$ to compute the relative attitude, or attitude error, still yields a result that is a proper CRP attitude measure. However, also note that the approximation $\delta\boldsymbol{q} \approx \boldsymbol{q}''$only holds for small attitude differences.

Differential Kinematic Equations

Matrix components:

$$\dot{\boldsymbol{q}} = \frac{1}{2} \begin{bmatrix} 1+q_1^2 & q_1q_2 - q_3 & q_1q_3 + q_2 \\ q_2q_1 + q_3 & 1+q_2^2 & q_2q_3 - q_1 \\ q_3q_1 - q_2 & q_3q_2 + q_1 & 1+q_3^2 \end{bmatrix} {}^{\mathcal{B}}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

Vector computation:

$$\dot{\boldsymbol{q}} = \frac{1}{2} \left[[I_{3\times3}] + [\tilde{\boldsymbol{q}}] + \boldsymbol{q}\boldsymbol{q}^T \right] {}^{\mathcal{B}}\boldsymbol{\omega}$$

$${}^{\mathcal{B}}\boldsymbol{\omega} = \frac{2}{1 + \boldsymbol{q}^T\boldsymbol{q}} \left([I_{3\times3}] - [\tilde{\boldsymbol{q}}] \right) \dot{\boldsymbol{q}}$$

Note: Only contains quadratic nonlinearities, but is singular for $\Phi = 180°$.

Cayley Transform

- Amazingly elegant matrix transformation, that allows us to use attitude parameters in higher dimensional spaces. ☺️

- Let [Q] be a skew-symmetric matrix, [C] be a proper orthogonal matrix, and [*I*] be a identity matrix. These matrices can be of any dimension *N*. The Cayley Transform is then defined as:

$$\boxed{[C] = ([I] - [Q])([I] + [Q])^{-1} = ([I] + [Q])^{-1}([I] - [Q])}$$

$$[Q] = ([I] - [C])([I] + [C])^{-1} = ([I] + [C])^{-1}([I] - [C])$$

Note: Both the forward and backwards mapping between [Q] and [C] has the same algebraic form!

Example:

- For 3D space, the proper orthogonal [*C*] matrix is also a rotation or direction cosine matrix. In this case we find that

$$[Q] = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix}$$

where the unique matrix elements are the CRP!

$$[C] = \begin{bmatrix} 0.813797 & 0.296198 & -0.5 \\ 0.235888 & 0.617945 & 0.75 \\ 0.531121 & -0.728292 & 0.433012 \end{bmatrix}$$

$$\Downarrow$$

$$[Q] = \begin{bmatrix} 0 & -0.021052 & 0.359933 \\ 0.021052 & 0 & -0.516027 \\ -0.359933 & 0.516027 & 0 \end{bmatrix} \Rightarrow \boldsymbol{q} = \begin{pmatrix} 0.516027 \\ 0.359933 \\ 0.021052 \end{pmatrix}$$

CRP vector

- Higher Dimensional Example:

$$[C] = \begin{bmatrix} 0.505111 & -0.503201 & -0.215658 & 0.667191 \\ 0.563106 & -0.034033 & -0.538395 & -0.626006 \\ 0.560111 & 0.748062 & 0.272979 & 0.228387 \\ -0.337714 & 0.431315 & -0.767532 & 0.332884 \end{bmatrix}$$

$$\Downarrow$$

$$[Q] = \begin{bmatrix} 0 & 0.5 & 0.2 & -0.3 \\ -0.5 & 0 & 0.7 & -0.6 \\ -0.2 & -0.7 & 0 & -0.4 \\ 0.3 & -0.6 & 0.4 & 0 \end{bmatrix}$$

4D space CRP

> Note: The *N*-dimensional proper orthogonal matrices can be parameterized with higher dimensional attitude coordinates.

That's nice, but is there also a higher dimensional equivalent to the differential kinematic equations to solve [*Q(t)*]?

- Recall that regardless of the dimensionality of the orthogonal matrix [*C*(*t*)], it must evolve according to:

$$[\dot{C}] = -[\tilde{\boldsymbol{\omega}}][C]$$

- These higher-dimensional “body angular velocities” can be related to the higher dimensional CRPs using:

$$[\dot{Q}] = \frac{1}{2}\left([I] + [Q]\right)[\tilde{\boldsymbol{\omega}}]\left([I] - [Q]\right)$$
$$[\tilde{\boldsymbol{\omega}}] = 2\left([I] + [Q]\right)^{-1}[\dot{Q}]\left([I] - [Q]\right)^{-1}$$

- Thus, can solve for the [*C*(*t*)] using a reduced coordinate set.

- This parameterization is singular whenever a principal rotation of 180º is performed.

- **Physical Example:**

  Consider a typical mechanical system. The EOM can be written in the form

$$[M(\boldsymbol{x}, t)]\ddot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)$$

  To solve this system for the state accelerations, the system positive definite system mass matrix must be inverted, a numerically expensive operations for large dimensions.

  This inverse could be avoided by using the spectral decomposition:

$$[M] = [V][D][V]^T \qquad\qquad [M]^{-1} = [V]^T[D]^{-1}[V]$$

  where [*V*] is a proper orthogonal eigenvector matrix and [*D*] is a diagonal eigenvalue matrix. To determine [*V*(*t*)] the Cayley transform could be used to track a reduced parameter set:

$$[Q] = ([I] - [V])([I] + [V])^{-1}$$

Modified Rodrigues Parameters (MRPs)

The “cool” new attitude coordinates...

MRP Definitions

Euler parameter relationship:

$$\sigma_i = \frac{\beta_i}{1+\beta_0} \qquad i = 1, 2, 3$$

Singular if -1
(±360° case)

$$\beta_0 = \frac{1-\sigma^2}{1+\sigma^2}$$

$$\beta_i = \frac{2\sigma_i}{1+\sigma^2} \qquad i = 1, 2, 3$$

Singular if ∞
(±360° case)

PRV relationship:

$$\boldsymbol{\sigma} = \tan\frac{\Phi}{4}\hat{\boldsymbol{e}}$$

Singular for ±360°

$$\boldsymbol{\sigma} \approx \frac{\Phi}{4}\hat{\boldsymbol{e}}$$

Linearizes to angles over 4.

(Show Mathematica Example)

CRP relationship:

$$\boldsymbol{q} = \frac{2\boldsymbol{\sigma}}{1-\sigma^2}$$

$$\boldsymbol{\sigma} = \frac{\boldsymbol{q}}{1+\sqrt{1+\boldsymbol{q}^T\boldsymbol{q}}}$$

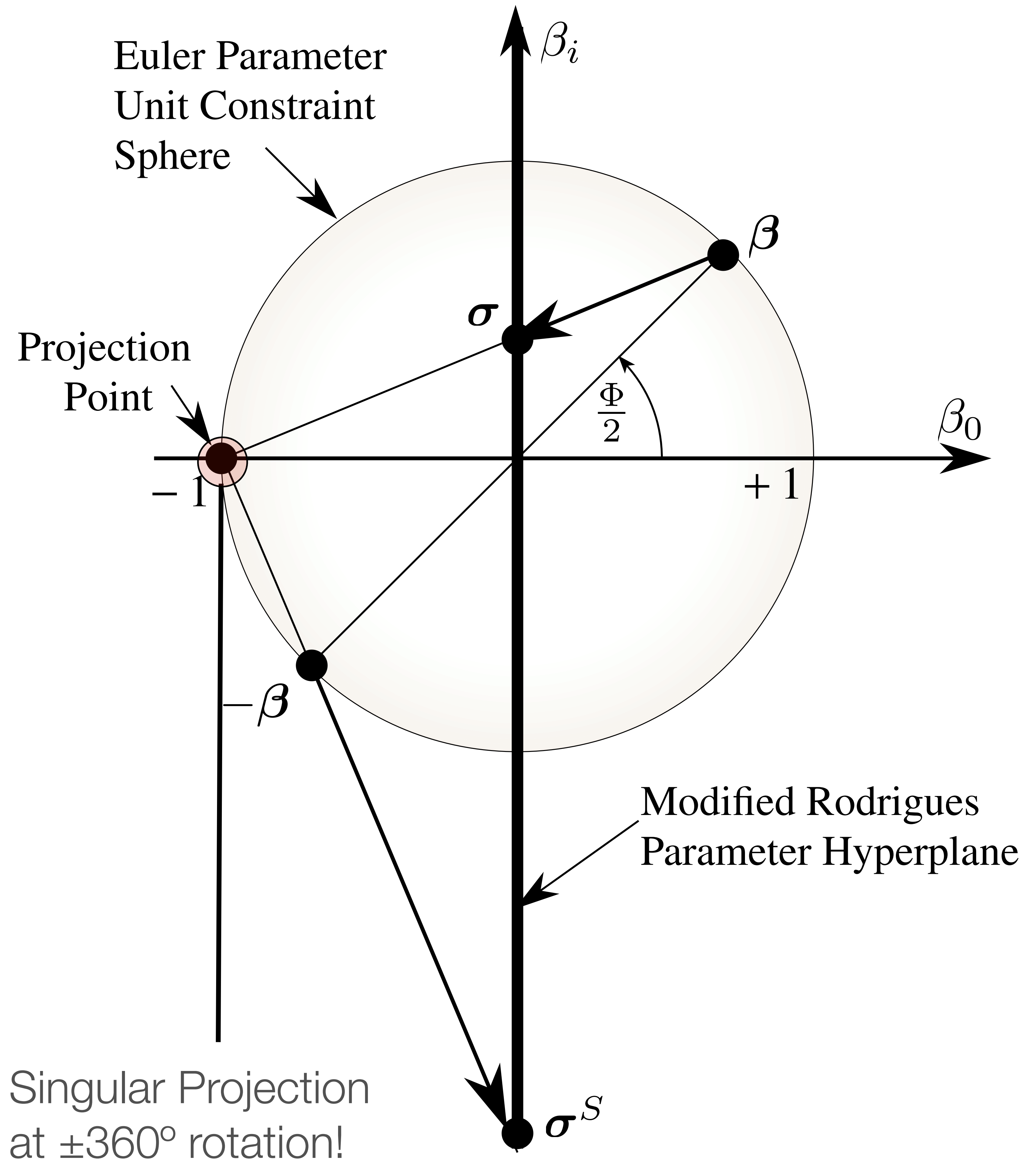
MRP Definitions

Relationship to DCM:

$$[\tilde{\boldsymbol{\sigma}}] = \frac{[C]^T - [C]}{\zeta(\zeta + 2)} \qquad \zeta = \sqrt{\text{trace}([C]) + 1} = \beta_0/2$$

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} = \frac{1}{\zeta(\zeta + 2)} \begin{pmatrix} C_{23} - C_{32} \\ C_{31} - C_{13} \\ C_{12} - C_{21} \end{pmatrix}$$

Stereographic Projection

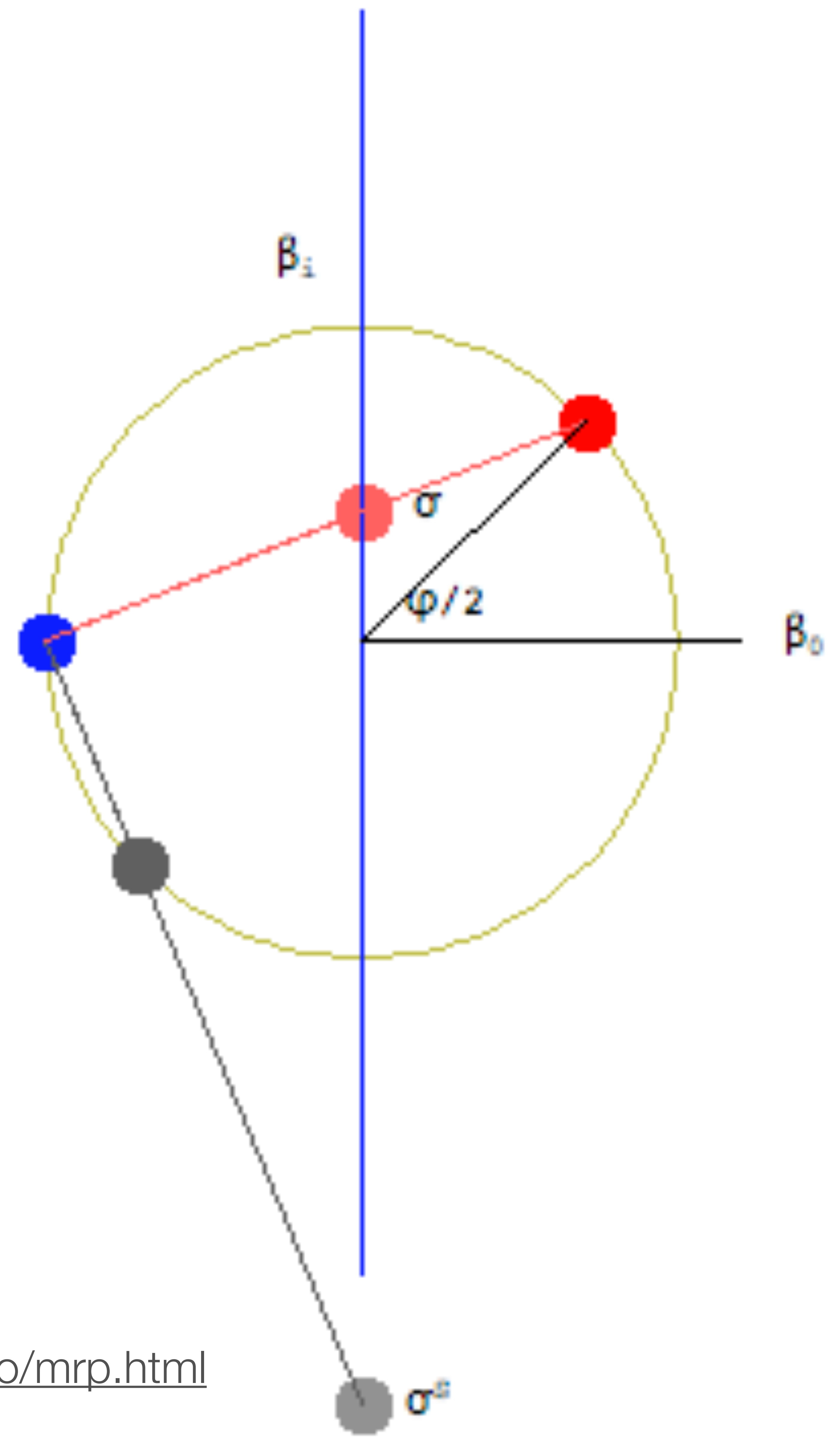


Projection Point: $(-1, 0, 0, 0)$

Projection Plane: $\beta_0 = 0$

Any attitude (surface point) is projected onto the hyper-plane to form the modified Rodrigues parameters.

The two EP sets yield *distinct* MRP coordinate values with different singular behaviors.



http://hanspeterschaub.info/mrp.html

Shadow MRP Set

- Using the alternate set of Euler parameters, we can find the "shadow" set of MRP parameters:

$$\sigma_i^S = \frac{-\beta_i}{1-\beta_0} = \frac{-\sigma_i}{\sigma^2} \qquad i = 1,2,3$$

Unique MRP Parameters

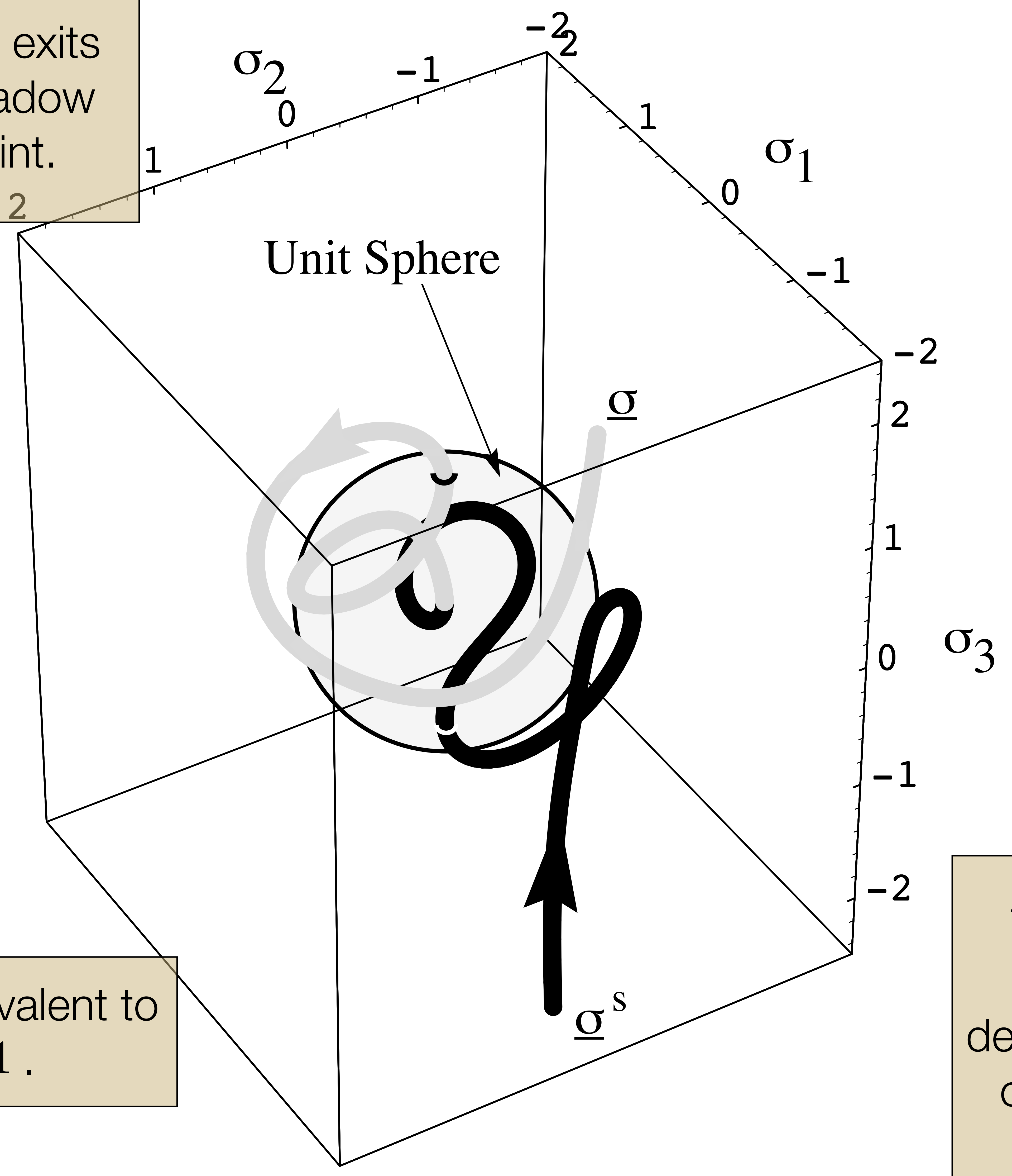
A common switching surface is $\sigma^2 = \boldsymbol{\sigma} \cdot \boldsymbol{\sigma} = 1$. Note that

$$|\boldsymbol{\sigma}| \leq 1 \quad \text{if} \quad \Phi \leq 180^{\circ}$$
$$|\boldsymbol{\sigma}| \geq 1 \quad \text{if} \quad \Phi \geq 180^{\circ}$$
$$|\boldsymbol{\sigma}| = 1 \quad \text{if} \quad \Phi = 180^{\circ}$$

$$\boldsymbol{\sigma}^S = \tan\left(\frac{\Phi - 2\pi}{4}\right)\hat{\boldsymbol{e}}$$

$$\boldsymbol{\sigma}^S = \tan\left(\frac{\Phi'}{4}\right)\hat{\boldsymbol{e}}$$

As one set of MRP coordinates exits the unit sphere surface, the shadow set enters at the opposite point.



Setting $\beta_0 \geq 0$ is equivalent to enforcing $|\boldsymbol{\sigma}| \leq 1$.

The original shadow set of MRPs are convenient to describe tumbling bodies. The coordinates always point to the zero attitude along the shortest rotational path

Direction Cosine Matrix

Matrix components:

$$[C] = \frac{1}{(1+\sigma^2)^2} \begin{bmatrix} 4\left(\sigma_1^2-\sigma_2^2-\sigma_3^2\right)+(1-\sigma^2)^2 & 8\sigma_1\sigma_2+4\sigma_3(1-\sigma^2) & \cdots \\ 8\sigma_2\sigma_1-4\sigma_3(1-\sigma^2) & 4\left(-\sigma_1^2+\sigma_2^2-\sigma_3^2\right)+(1-\sigma^2)^2 & \cdots \\ 8\sigma_3\sigma_1+4\sigma_2(1-\sigma^2) & 8\sigma_3\sigma_2-4\sigma_1(1-\sigma^2) & \cdots \end{bmatrix}$$

$$\cdots \begin{bmatrix} 8\sigma_1\sigma_3-4\sigma_2(1-\sigma^2) \\ 8\sigma_2\sigma_3+4\sigma_1(1-\sigma^2) \\ 4\left(-\sigma_1^2-\sigma_2^2+\sigma_3^2\right)+(1-\sigma^2)^2 \end{bmatrix}$$

Vector computation:

$$[C] = [I_{3\times 3}] + \frac{8[\tilde{\boldsymbol{\sigma}}]^2 - 4\left(1-\sigma^2\right)[\tilde{\boldsymbol{\sigma}}]}{\left(1+\sigma^2\right)^2}$$

Interesting property:

$$[C(\boldsymbol{\sigma})]^{-1} = [C(\boldsymbol{\sigma})]^T = [C(-\boldsymbol{\sigma})]$$

Attitude Addition/Subtraction

- DCM method:

$$[FN(\boldsymbol{\sigma})] = [FB(\boldsymbol{\sigma}'')][BN(\boldsymbol{\sigma}')]$$

- Direct method:

$$\boldsymbol{\sigma} = \frac{(1 - |\boldsymbol{\sigma}'|^2)\boldsymbol{\sigma}'' + (1 - |\boldsymbol{\sigma}''|^2)\boldsymbol{\sigma}' - 2\boldsymbol{\sigma}'' \times \boldsymbol{\sigma}'}{1 + |\boldsymbol{\sigma}'|^2|\boldsymbol{\sigma}''|^2 - 2\boldsymbol{\sigma}' \cdot \boldsymbol{\sigma}''}$$

Attitude Addition

$$\boldsymbol{\sigma}'' = \frac{(1 - |\boldsymbol{\sigma}'|^2)\boldsymbol{\sigma} - (1 - |\boldsymbol{\sigma}|^2)\boldsymbol{\sigma}' + 2\boldsymbol{\sigma} \times \boldsymbol{\sigma}'}{1 + |\boldsymbol{\sigma}'|^2|\boldsymbol{\sigma}|^2 + 2\boldsymbol{\sigma}' \cdot \boldsymbol{\sigma}}$$

Relative Attitude (Subtraction)

Differential Kinematic Equations

Matrix components:

$$\dot{\boldsymbol{\sigma}} = \frac{1}{4}\begin{bmatrix} 1-\sigma^2+2\sigma_1^2 & 2\left(\sigma_1\sigma_2-\sigma_3\right) & 2\left(\sigma_1\sigma_3+\sigma_2\right) \\ 2\left(\sigma_2\sigma_1+\sigma_3\right) & 1-\sigma^2+2\sigma_2^2 & 2\left(\sigma_2\sigma_3-\sigma_1\right) \\ 2\left(\sigma_3\sigma_1-\sigma_2\right) & 2\left(\sigma_3\sigma_2+\sigma_1\right) & 1-\sigma^2+2\sigma_3^2 \end{bmatrix} {}^{\mathcal{B}}\!\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

Vector computation:

$$\dot{\boldsymbol{\sigma}} = \frac{1}{4}\left[\left(1-\sigma^2\right)[I_{3\times 3}] + 2[\tilde{\boldsymbol{\sigma}}] + 2\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right] {}^{\mathcal{B}}\boldsymbol{\omega} = \frac{1}{4}[B(\boldsymbol{\sigma})]\, {}^{\mathcal{B}}\boldsymbol{\omega}$$

Note: Only contains quadratic nonlinearities, but is singular for $\Phi = \pm 360^{\circ}$.

- Now, let's invert the differential kinematic equation and find:

$$\boldsymbol{\omega} = 4[B]^{-1}\dot{\boldsymbol{\sigma}}$$

- Note the near-orthogonal property of the [*B*] matrix:

$$[B]^{-1} = \frac{1}{\left(1+\sigma^2\right)^2}[B]^T$$

You can proof this by investigating [*B*][*B*]^T.

- This leads to the elegant inverse transformation

$$\boldsymbol{\omega} = \frac{4}{\left(1+\sigma^2\right)^2}[B]^T\dot{\boldsymbol{\sigma}}$$

$$\boldsymbol{\omega} = \frac{4}{\left(1+\sigma^2\right)^2}\left[\left(1-\sigma^2\right)[I_{3\times3}] - 2[\tilde{\boldsymbol{\sigma}}] + 2\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right]\dot{\boldsymbol{\sigma}}$$

Cayley Transform

- Let [*S*] be a skew-symmetric matrix, [*C*] be a proper orthogonal matrix, and [*I*] be a identity matrix. These matrices can be of any dimension *N*. The **extended Cayley Transform** is then defined as:

$$\boxed{[C] = ([I]-[S])^2([I]+[S])^{-2} = ([I]+[S])^{-2}([I]-[S])^2}$$

Unfortunately no equivalent inverse transformation exists. Instead, we define [*W*] to be the “square root” of [*C*]:

$$[C] = [W][W]$$

$$[C] = [V][D][V]^*$$

Adjoint Operator

- The "matrix square root" can then be defined as

$$[W] = [V] \begin{bmatrix} \ddots & & 0 \\ & \sqrt{[D]_{ii}} & \\ 0 & & \ddots \end{bmatrix} [V]^*$$

$$[W] = [V] \begin{bmatrix} e^{+i\frac{\theta_1}{2}} & 0 & \cdots & & & 0 \\ 0 & e^{-i\frac{\theta_1}{2}} & \cdots & & & 0 \\ \vdots & \vdots & \ddots & & & 0 \\ & & & e^{+i\frac{\theta_{N-1}}{2}} & 0 & 0 \\ & & & 0 & e^{-i\frac{\theta_{N-1}}{2}} & 0 \\ & & & 0 & 0 & +1 \end{bmatrix} [V]^* \quad \text{Odd dimension}$$

$$[W] = [V] \begin{bmatrix} e^{+i\frac{\theta_1}{2}} & 0 & \cdots & & 0 \\ 0 & e^{-i\frac{\theta_1}{2}} & \cdots & & 0 \\ \vdots & \vdots & \ddots & & 0 \\ & & & e^{+i\frac{\theta_{N-1}}{2}} & 0 \\ & & & 0 & e^{-i\frac{\theta_{N-1}}{2}} \end{bmatrix} [V]^* \quad \text{Even dimension}$$

- The standard Cayley transform can now be used to between between the skew-symmetric [*S*] matrix and the orthogonal [*W*] matrix:

$$[W] = ([I] - [S])([I] + [S])^{-1} \qquad = ([I] + [S])^{-1}([I] - [S])$$

$$[S] = ([I] - [W])([I] + [W])^{-1} \qquad = ([I] + [W])^{-1}([I] - [W])$$

- As with the CRP coordinates, for the 3D case the [*S*] matrix elements are MRP attitude coordinates. For higher dimensional cases, this allows us to parameterize *N*-dimensional proper orthogonal matrices using higher dimensional MRP coordinates.

- Recall that regardless of the dimensionality of the orthogonal matrix [*W*(*t*)], it must evolve according

$$[\dot{W}] = -[\tilde{\boldsymbol{\Omega}}][W]$$

These higher-dimensional “body angular velocities” can be related to the higher dimensional MRPs using:

$$[\tilde{\boldsymbol{\omega}}] = [\tilde{\boldsymbol{\Omega}}] + [W][\tilde{\boldsymbol{\Omega}}][W]^T$$

$$[\dot{S}] = \frac{1}{2}\left([I] + [S]\right)[\tilde{\boldsymbol{\Omega}}]\left([I] - [S]\right)$$

- This parameterization is singular whenever a principal rotation of 360º is performed.

- If these higher dimensional MRPs are singular for ±360° rotations, can this singularity be avoided by switching to “higher-dimensional shadow” set?

- This question was raised by some structures engineers trying to apply this extended Cayley transform to parameterize a proper orthogonal matrix in their problem.

- This is still an unsolved problem, is waiting to be investigated by some enterprising graduate student...

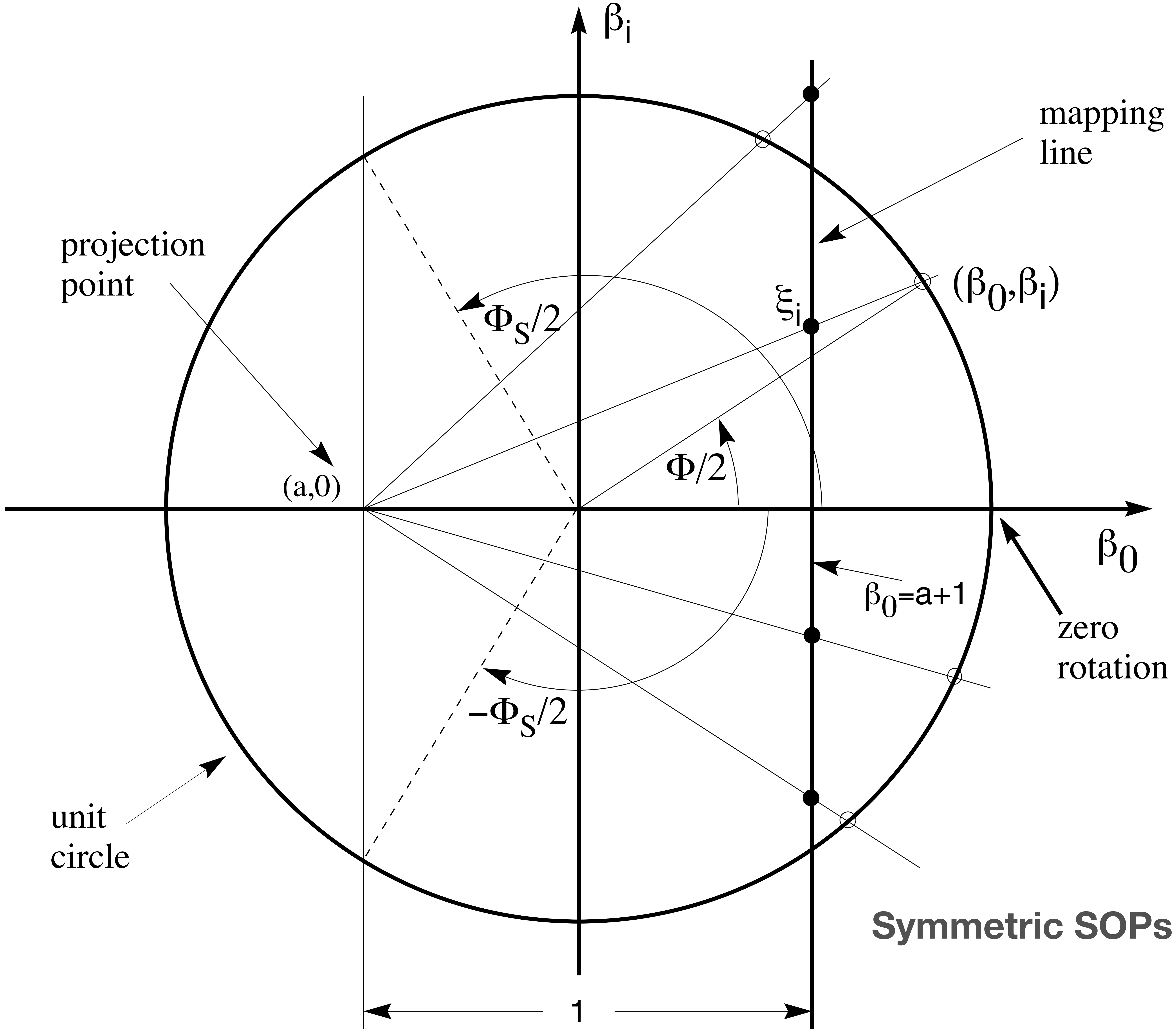
Stereographic Orientation Parameters (SOPs)

Elegant family of attitude coordinates...

Quick facts...

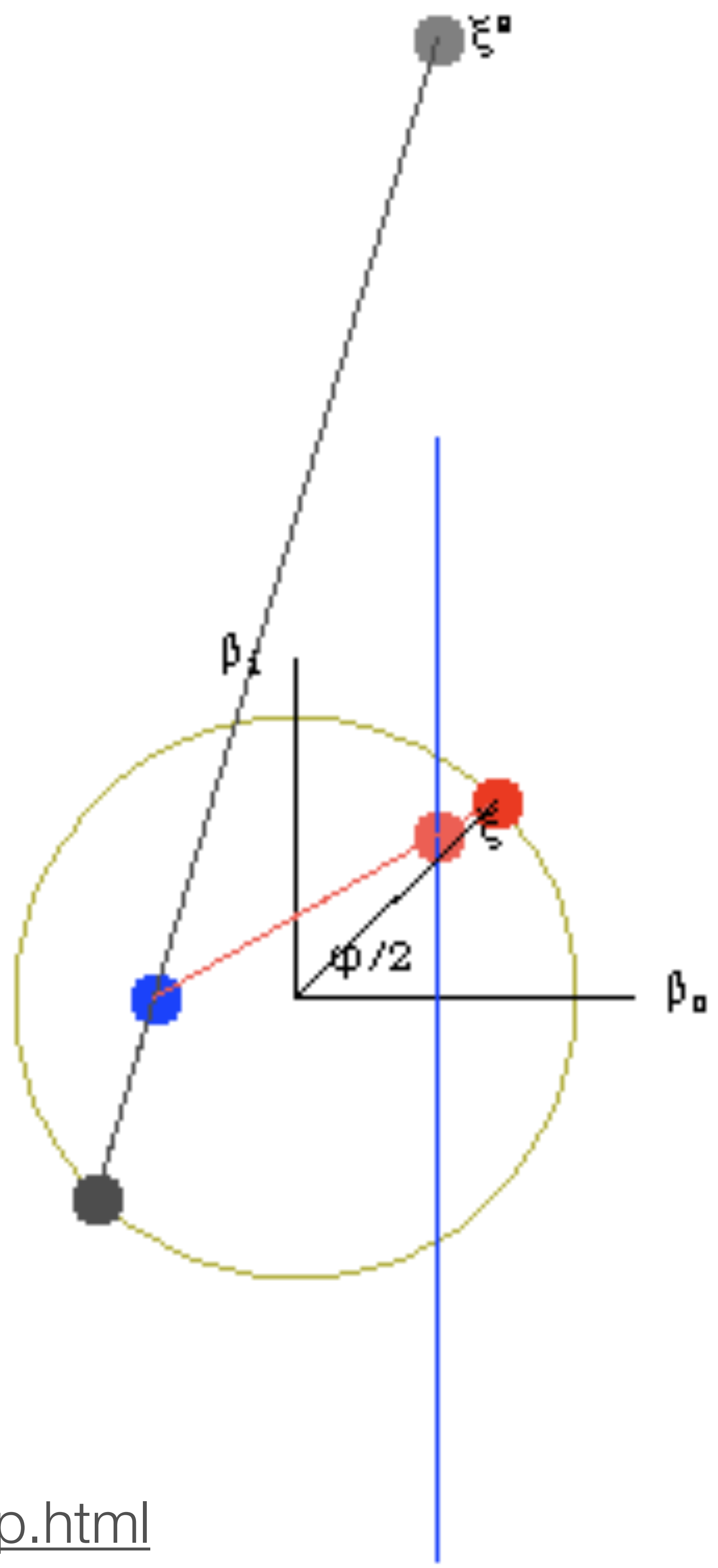
- The Stereographic Orientation Parameters are a class of attitude parameters that generalize the previously discussed classical and modified Rodrigues parameters.
- The are two types of SOPs:
  - Symmetric Set: Goes singular if a $\pm\Phi$ principal rotation is performed.
  - Asymmetric Set: Goes singular at either Φ_1 or Φ_2, and this rotation must be about a particular axis.
- References:

- H. Schaub and J.L. Junkins. ``Stereographic Orientation Parameters for Attitude Dynamics: A Generalization of the Rodrigues Parameters.'' *Journal of the Astronautical Sciences*, Vol. 44, No. 1, Jan.–Mar. 1996, pp. 1–19.
- C. M. Southward, J. Ellis and H. Schaub, "Spacecraft Attitude Control Using Symmetric Stereographic Orientation Parameters," *Journal of Astronautical Sciences*, Vol. 55, No. 3, July–September, 2007, pp. 389–405.

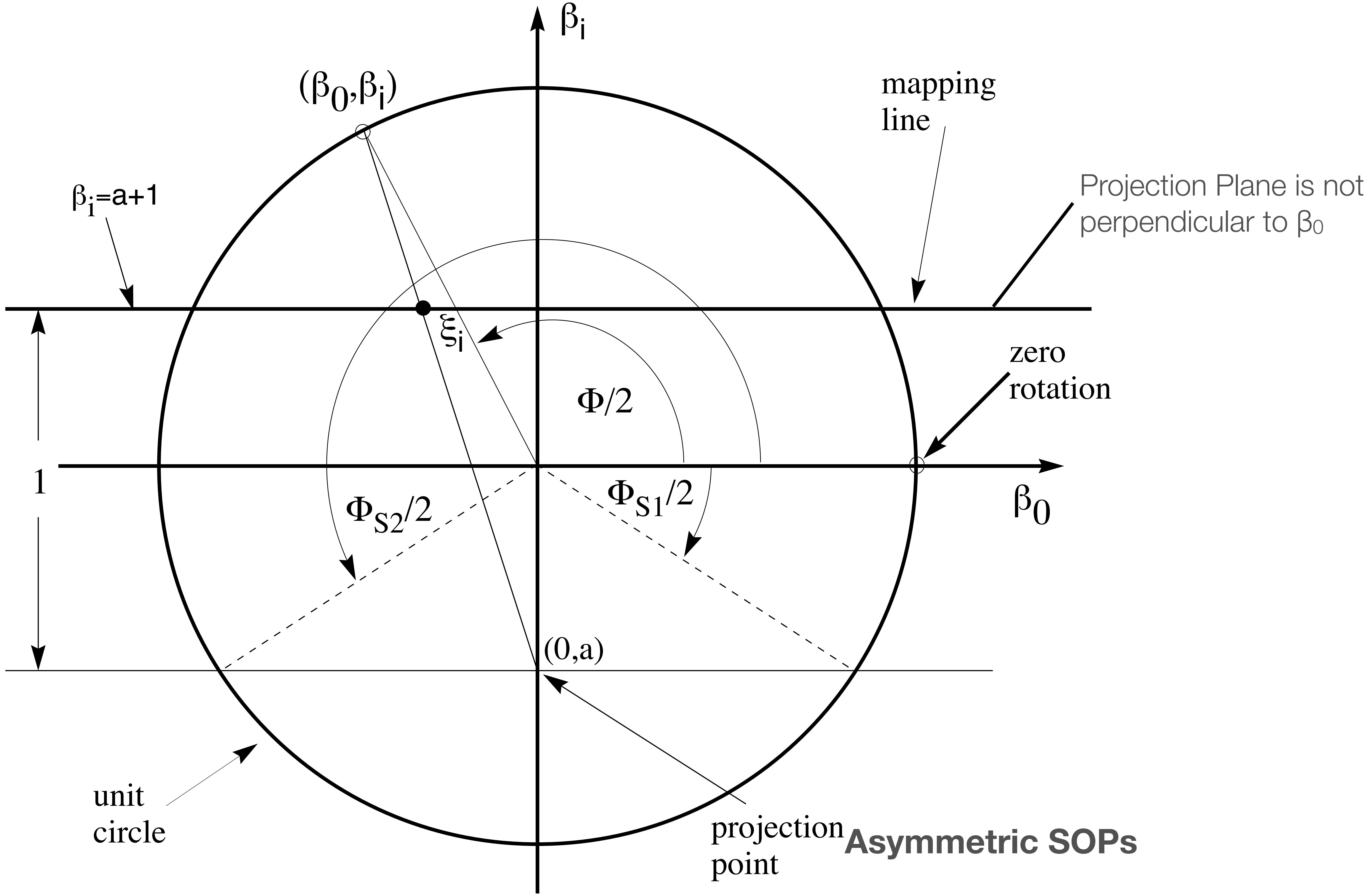


Symmetric SOPs

SSOP’s

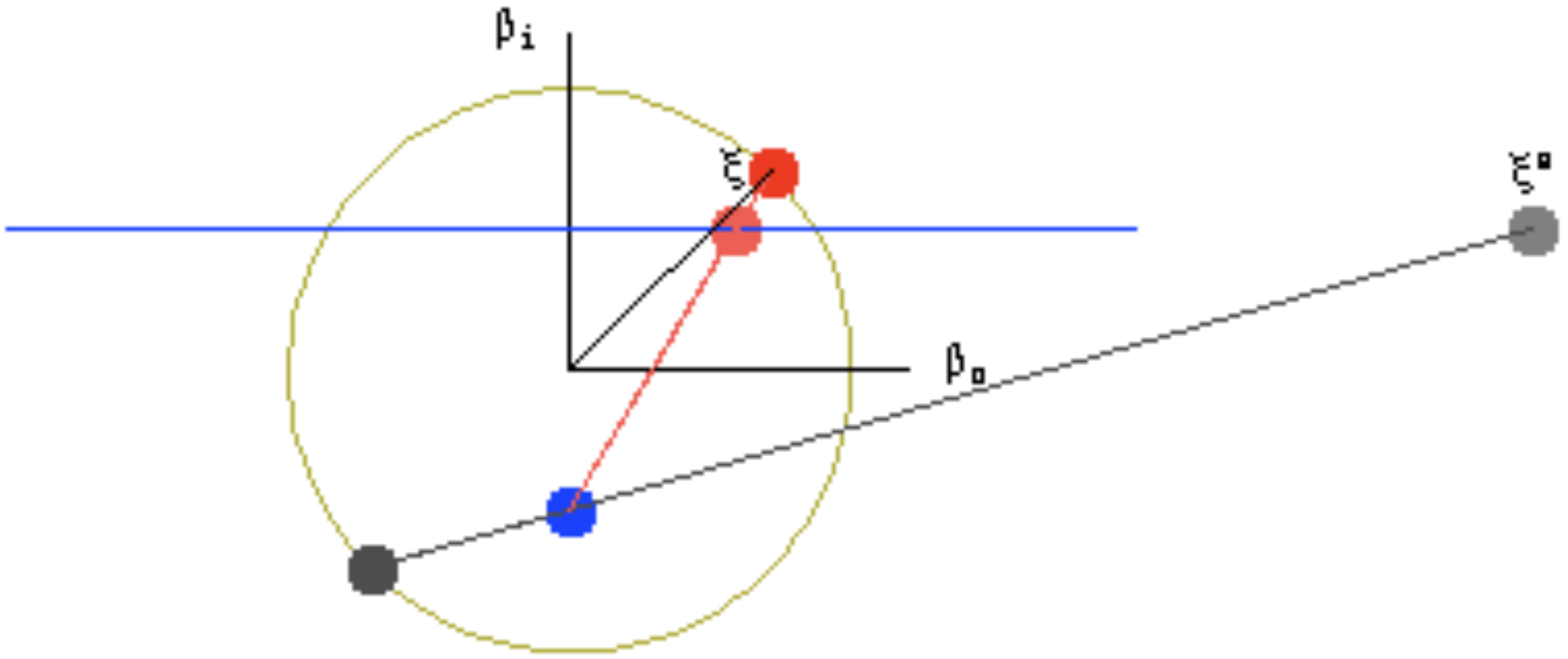


http://hanspeterschaub.info/ssop.html



Asymmetric SOPs

SOP's

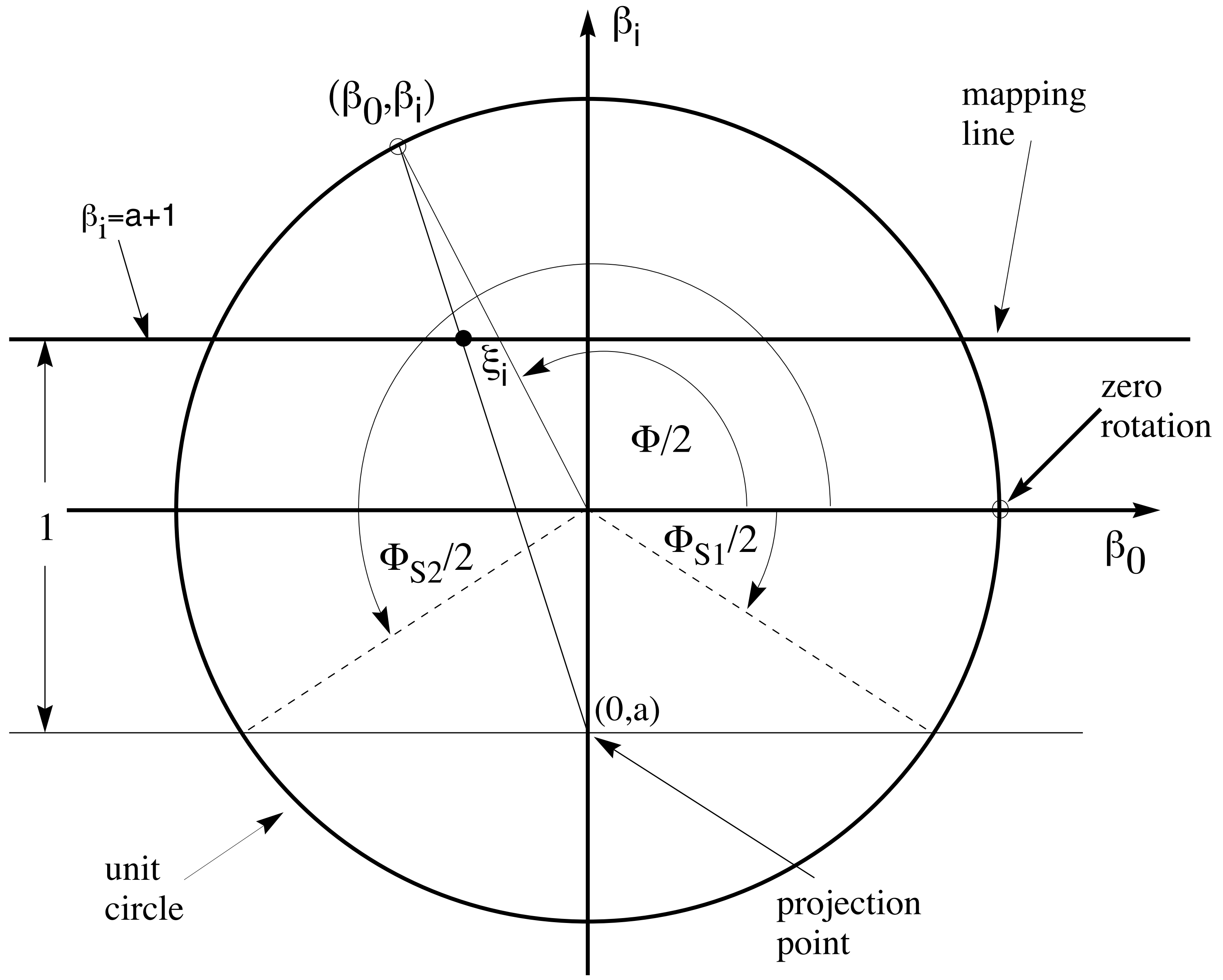


http://hanspeterschaub.info/assop.html

Example: asymmetric SOP

Projection plane: $\beta_1 = 0$

Projection point: $\beta_1 = -1$



Mapping from EP:

$$\eta_1 = \frac{\beta_0}{1+\beta_1} \quad \eta_2 = \frac{\beta_2}{1+\beta_1} \quad \eta_3 = \frac{\beta_3}{1+\beta_1}$$

Mapping to EP:

$$\beta_0 = \frac{2\eta_1}{1+\eta^2} \quad \beta_1 = \frac{1-\eta^2}{1+\eta^2} \quad \beta_2 = \frac{2\eta_2}{1+\eta^2} \quad \beta_3 = \frac{2\eta_3}{1+\eta^2}$$

$$\eta^2 = \boldsymbol{\eta}^T \boldsymbol{\eta}$$

Singular behavior:

$$\beta_1 \to -1 \quad \begin{cases} \Phi_1 = -180^{\circ} \\ \Phi_2 = +540^{\circ} \end{cases}$$

Shadow set:

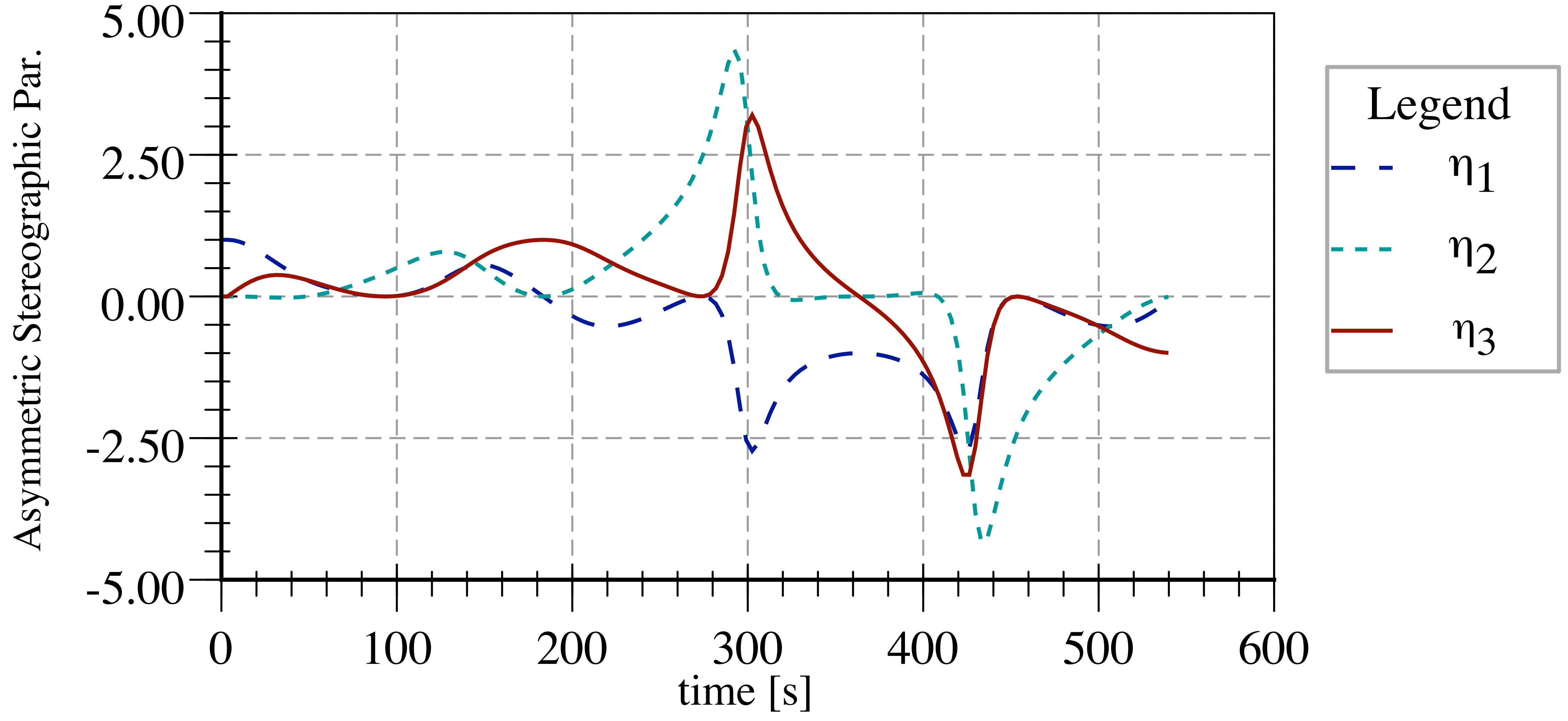
$$\boldsymbol{\eta}^S = -\frac{\boldsymbol{\eta}}{\boldsymbol{\eta}^T \boldsymbol{\eta}}$$

Prescribed 3-1-3 Euler Angle time histories:

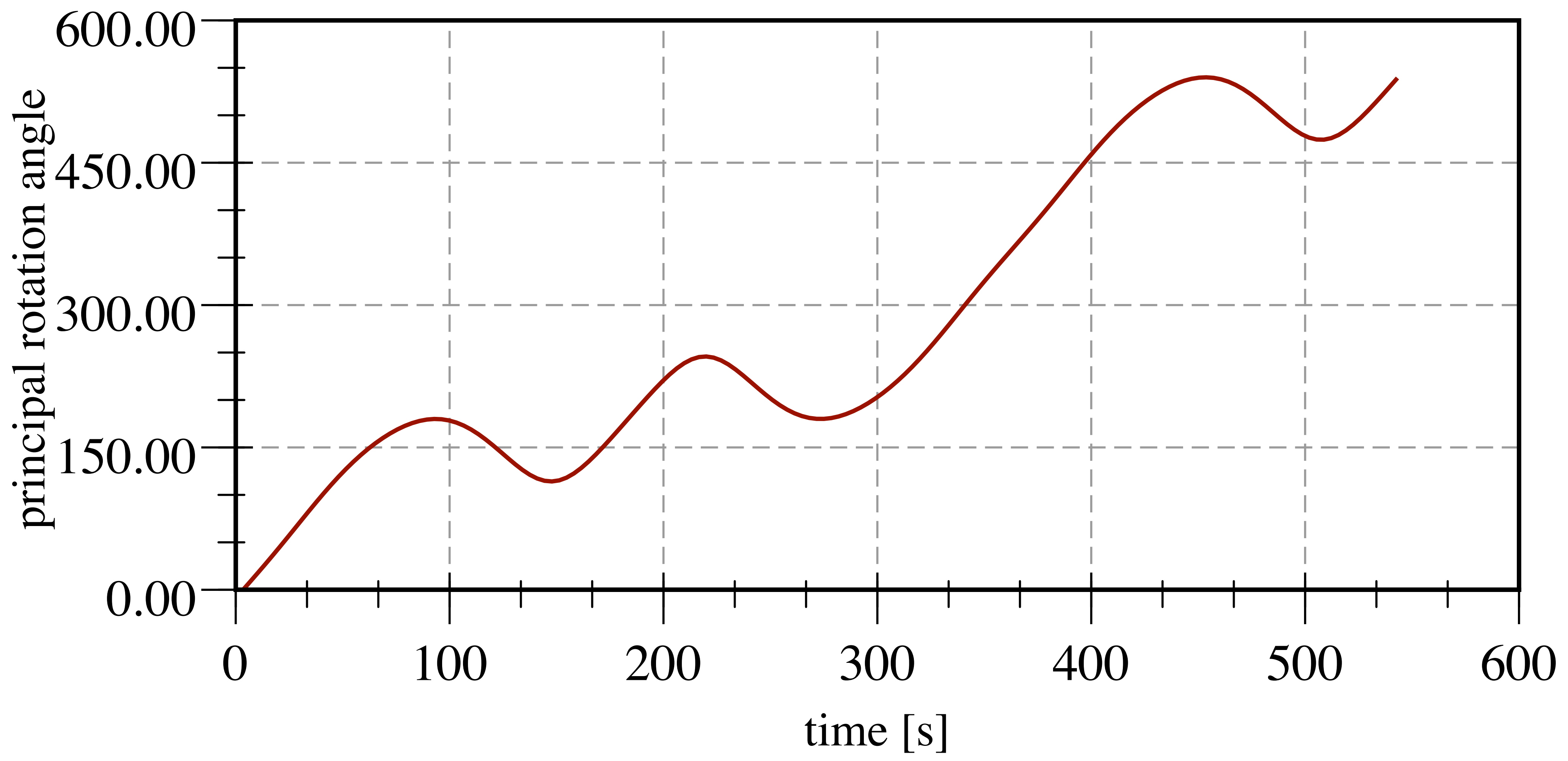
$$\theta_1(t) = t$$

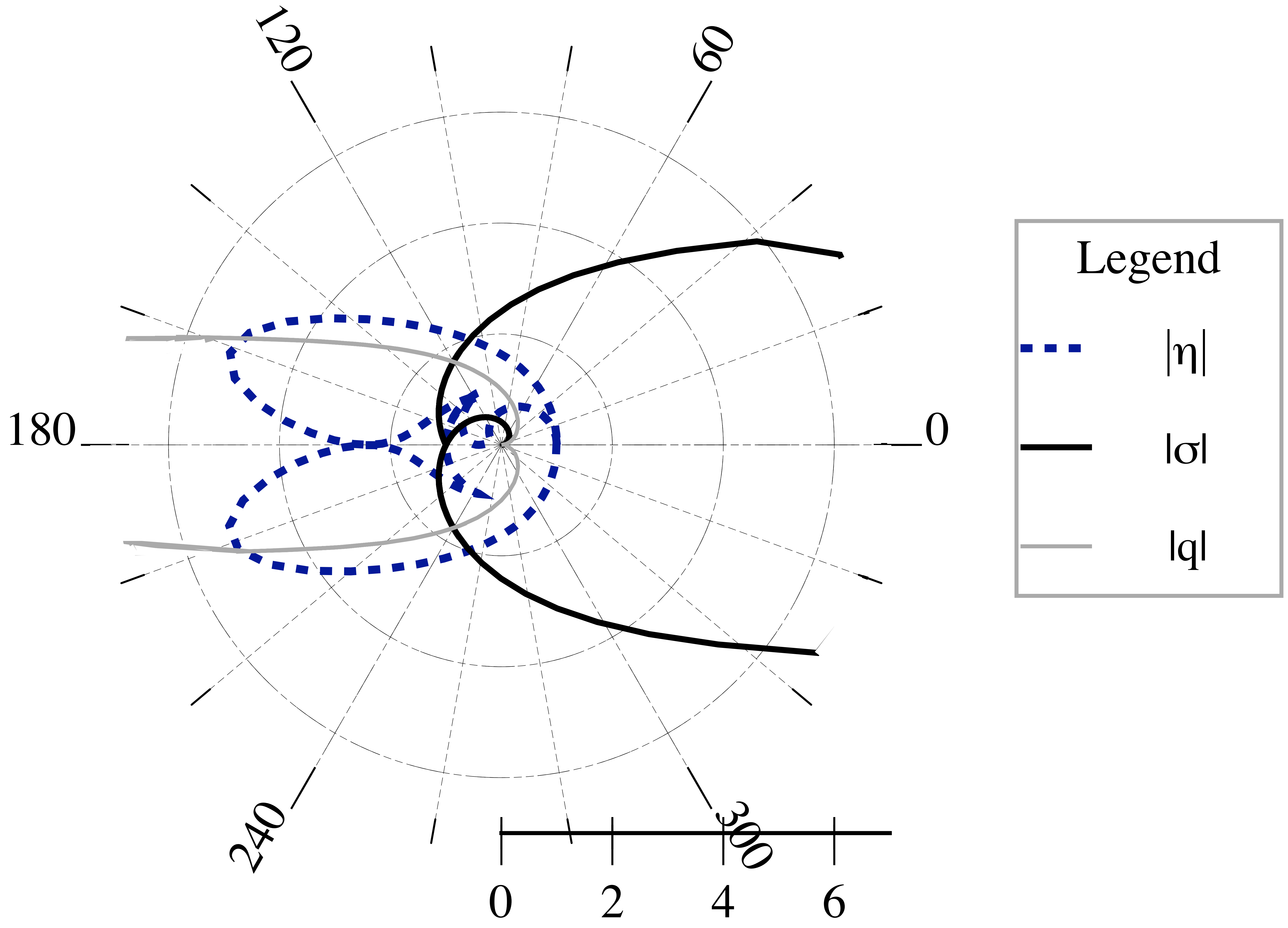
$$\theta_2(t) = (1 - \cos 2t)\frac{\pi}{2}$$

$$\theta_3(t) = (\sin 2t)\frac{\pi}{4}$$

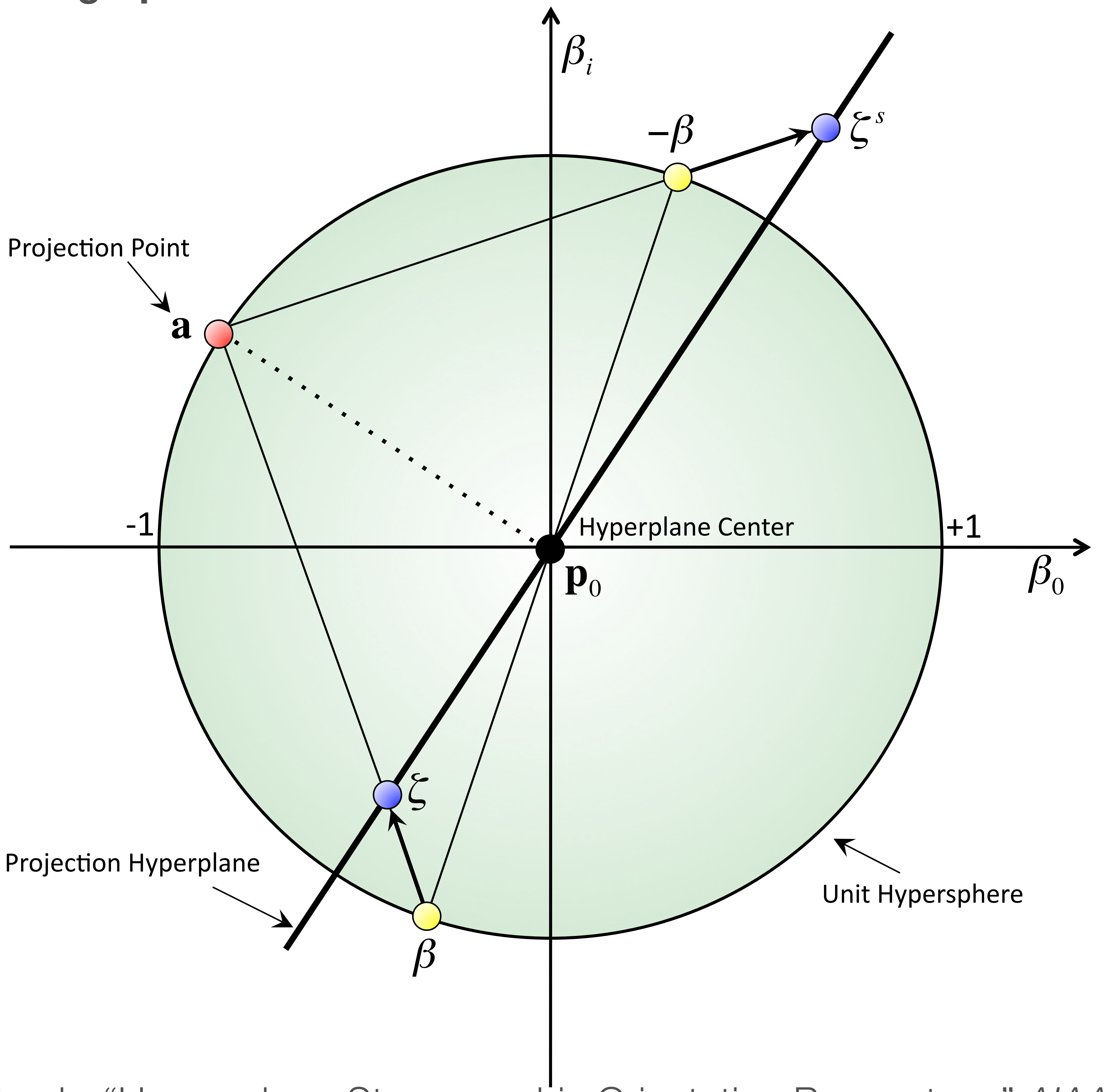


The body is essentially doing a tumble about the 1st body axis, while doing sinusoidal wobbles about the other axes.




120
60
180
0
240
300
0
2
4
6
Legend
|η|
|σ|
|q|

Hyper-Surface Stereographic Orientation Parameters



Mullen and H. Schaub, "Hypersphere Stereographic Orientation Parameters," *AIAA Journal of Guidance, Control and Dynamics*, Vol. 33, No. 1, Jan.–Feb., 2010, pp. 249–254. doi:10.2514/1.46783

Attitude Determination

ASEN 5010

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Introduction

- Attitude determination is broken up into two areas
  - **Static attitude determination**: All measurements are taken at the same time. Using this snap shot in time concept, the problem becomes up of optimally solving the geometry of the measurements
  - **Dynamic attitude determination**: Here measurements are taken over time. This is a much harder problem, in that attitude measurements are taken over time, along with some gyro (rotation rate) measurements, which then need to be optimally blended together (Kalman filter).

Basic Concept

- Consider the 2D attitude problem. How many direction measurements (unit direction vectors) does it take to determine your heading?

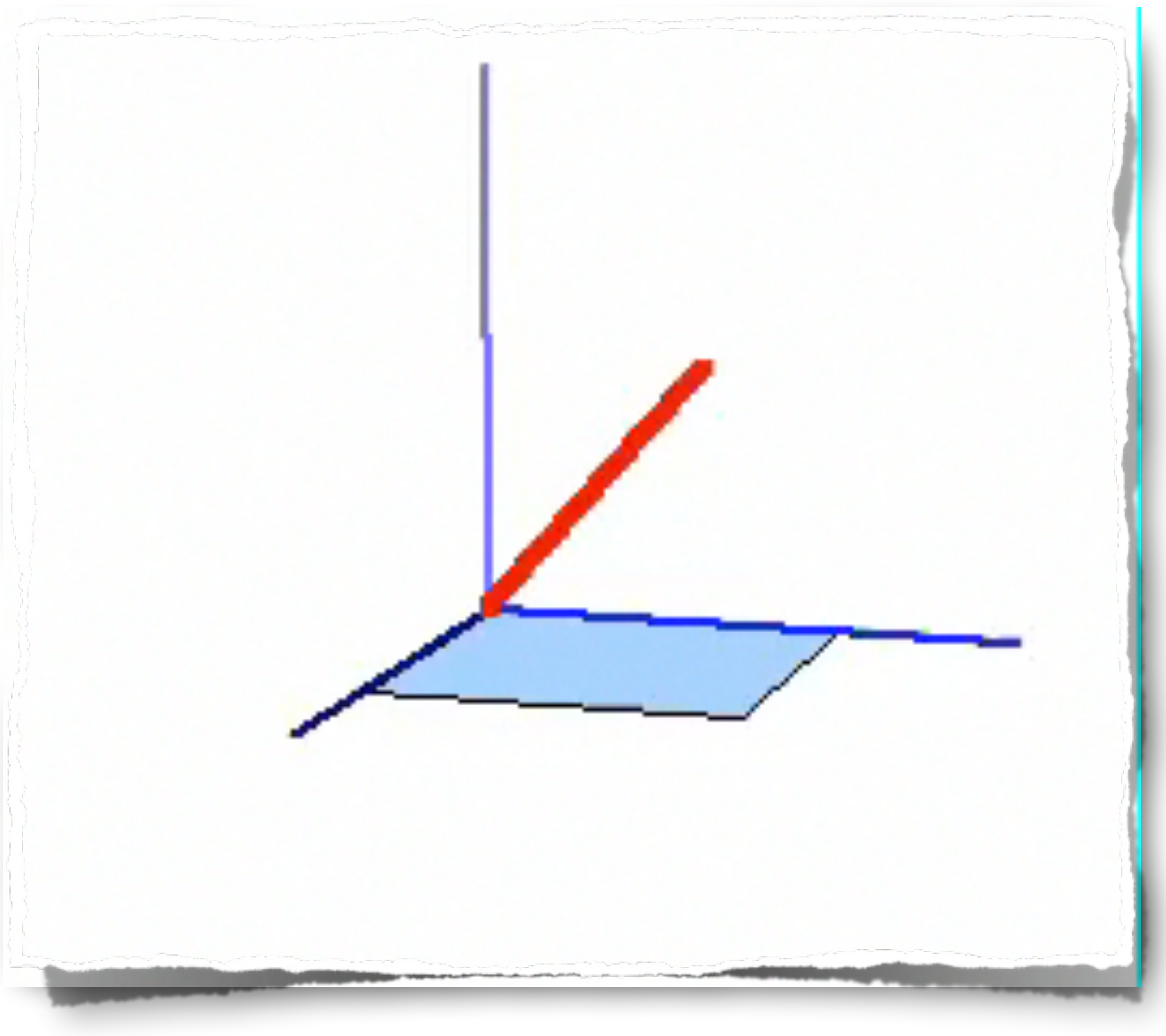
Answer: You only need one direction measurement for the 2D case.

Explanation: Headings in a 2D environment is a 1D measure. The unit direction vector (with the unit length constraint) provides all the required information.

- Next, let us consider the three dimensional orientation measurement. How many observation vectors (unit direction vectors) are required here?

Answer: You will need a minimum of two observation vectors.

Explanation: With only one measurement, you cannot sense rotations about this axis. Measuring a second direction will fix the complete three dimension orientation in space.



- To determine attitude, we assume you already know the inertial direction to certain objects (sun, Earth, magnetic field direction, stars, moon, etc.)
- Assume the sun direction is given by $\hat{\boldsymbol{s}}$ and the local magnetic field direction is given by $\hat{\boldsymbol{m}}$.
- If the vehicle has sensors on board that measure these directions, then these unit vectors are measured with components taken in the vehicle fixed body frame *B*.

Measured: ${}^{\mathcal{B}}\hat{\boldsymbol{m}}$ ${}^{\mathcal{B}}\hat{\boldsymbol{s}}$

Given: ${}^{\mathcal{N}}\hat{\boldsymbol{m}}$ ${}^{\mathcal{N}}\hat{\boldsymbol{s}}$

Mapping:

$$ {}^{\mathcal{B}}\hat{\boldsymbol{m}} = [\bar{B}N]^{\mathcal{N}}\hat{\boldsymbol{m}} $$

$$ {}^{\mathcal{B}}\hat{\boldsymbol{s}} = [\bar{B}N]^{\mathcal{N}}\hat{\boldsymbol{s}} $$

Challenge: How do we find [*BN*]?

Under or Over?

- Note that each observation vector (unit direction vector) contains two independent degrees of freedom.
- The 3D attitude problem is a three-degree of freedom problem.
- Thus, by measuring two observation directions, the attitude determination problem is always an over-determined problem!

Deterministic Attitude Estimation

Vector Triad Method

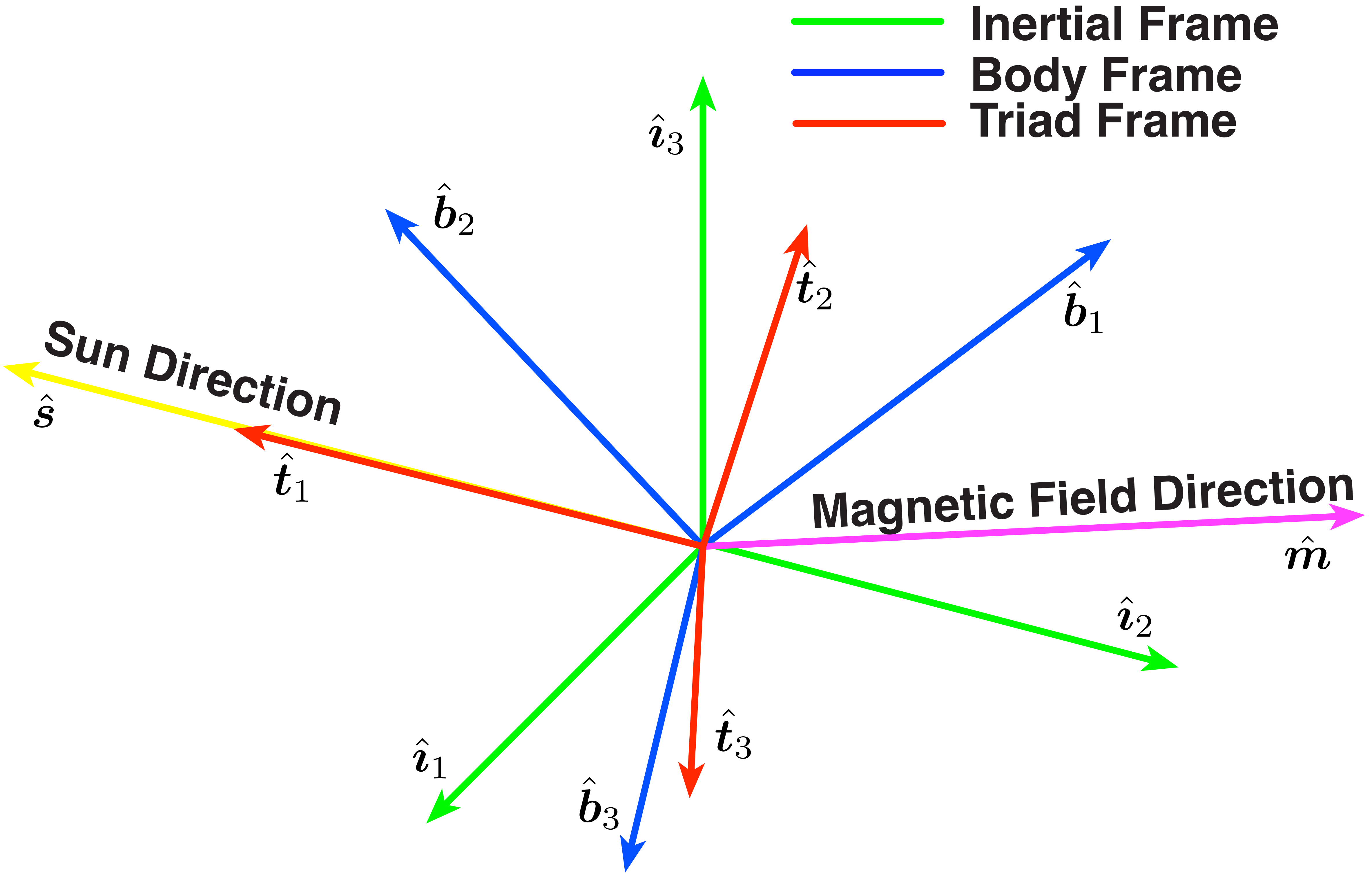
- To determine the desired [*BI*] matrix, we first introduce the triad coordinate frame *T*.

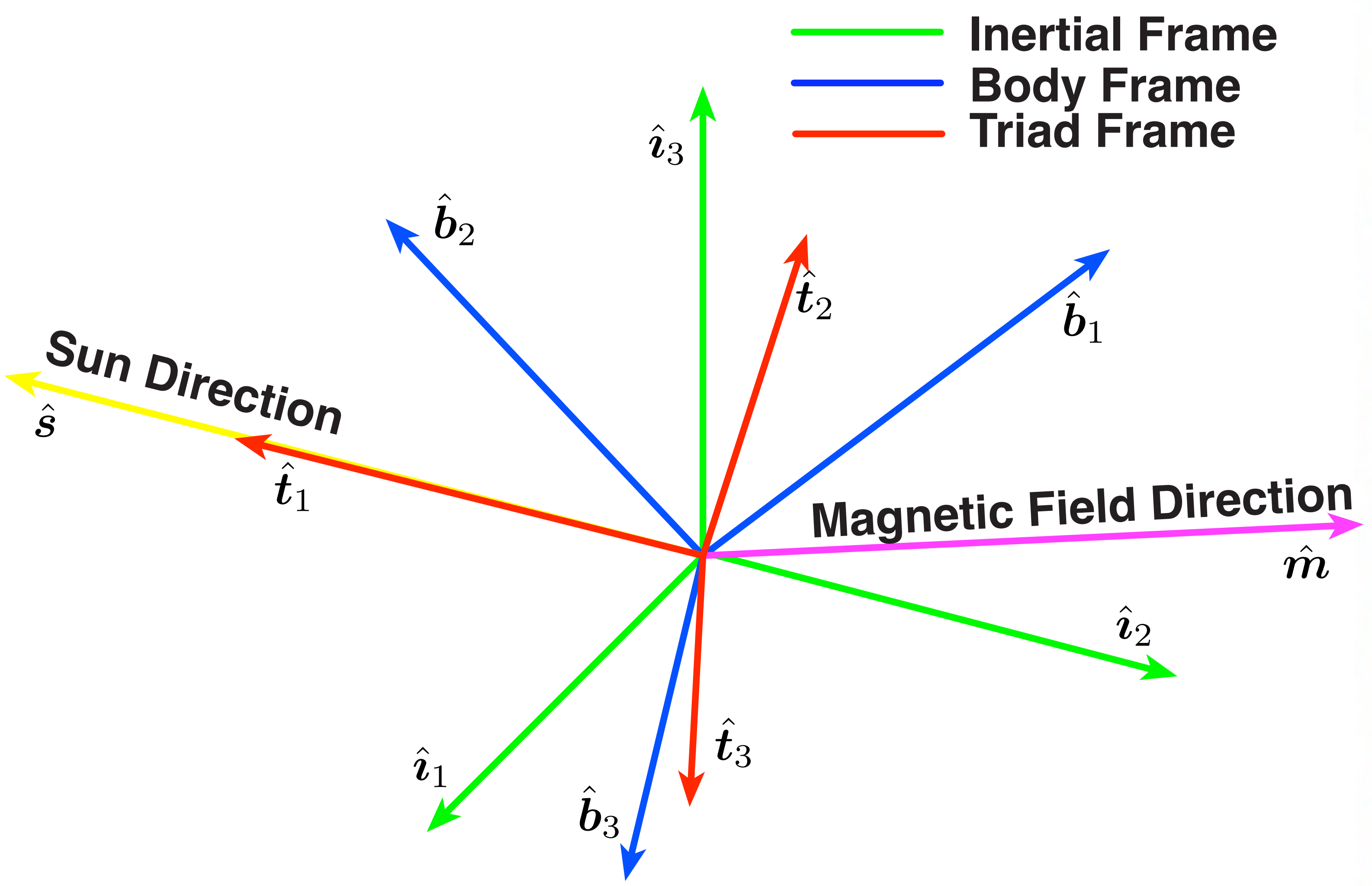
Assume: $\hat{\boldsymbol{t}}_1 = \hat{\boldsymbol{s}}$

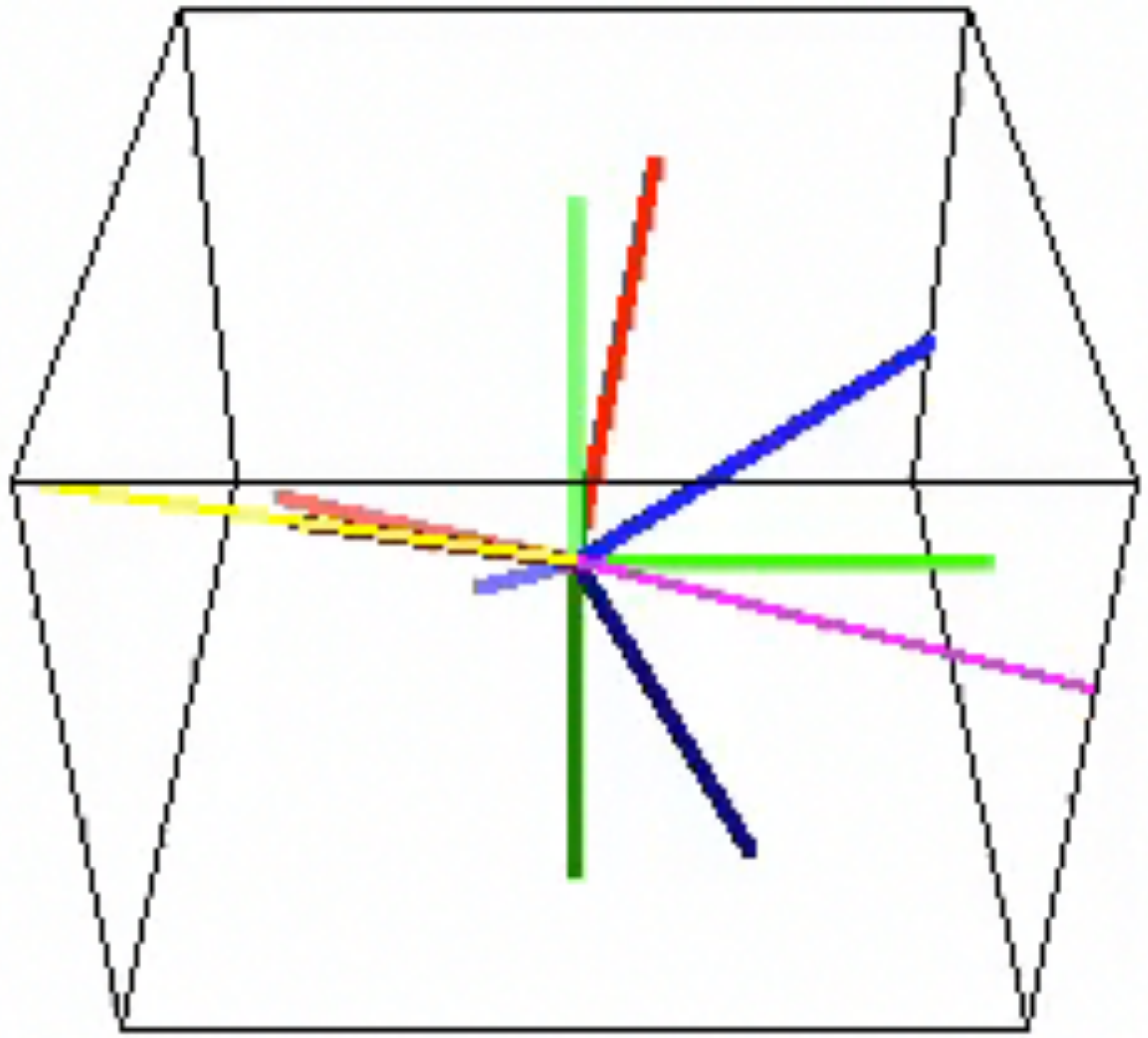
Then define:

$$\hat{\boldsymbol{t}}_2 = \frac{\hat{\boldsymbol{s}} \times \hat{\boldsymbol{m}}}{|\hat{\boldsymbol{s}} \times \hat{\boldsymbol{m}}|}$$

$$\hat{\boldsymbol{t}}_3 = \hat{\boldsymbol{t}}_1 \times \hat{\boldsymbol{t}}_2$$







3D Illustration of Triad Coordinate Frame

- We can compute the *T* frame direction axes using both *B* and I frame components using

$$
{}^{\mathcal{B}}\hat{\boldsymbol{t}}_1 = {}^{\mathcal{B}}\hat{\boldsymbol{s}}
$$

$$
{}^{\mathcal{B}}\hat{\boldsymbol{t}}_2 = \frac{({}^{\mathcal{B}}\hat{\boldsymbol{s}}) \times ({}^{\mathcal{B}}\hat{\boldsymbol{m}})}{|({}^{\mathcal{B}}\hat{\boldsymbol{s}}) \times ({}^{\mathcal{B}}\hat{\boldsymbol{m}})|}
$$

$$
{}^{\mathcal{B}}\hat{\boldsymbol{t}}_3 = ({}^{\mathcal{B}}\hat{\boldsymbol{t}}_1) \times ({}^{\mathcal{B}}\hat{\boldsymbol{t}}_2)
$$

Body Frame Triad Vectors

$$
{}^{\mathcal{N}}\hat{\boldsymbol{t}}_1 = {}^{\mathcal{N}}\hat{\boldsymbol{s}}
$$

$$
{}^{\mathcal{N}}\hat{\boldsymbol{t}}_2 = \frac{({}^{\mathcal{N}}\hat{\boldsymbol{s}}) \times ({}^{\mathcal{N}}\hat{\boldsymbol{m}})}{|({}^{\mathcal{N}}\hat{\boldsymbol{s}}) \times ({}^{\mathcal{N}}\hat{\boldsymbol{m}})|}
$$

$$
{}^{\mathcal{N}}\hat{\boldsymbol{t}}_3 = ({}^{\mathcal{N}}\hat{\boldsymbol{t}}_1) \times ({}^{\mathcal{N}}\hat{\boldsymbol{t}}_2)
$$

Inertial Frame Triad Vectors

- In the absence of measurement errors, both sets of Triad frame representations should be the same.

- We can write the various rotation matrices as

$$
[\bar{B}T] = \begin{bmatrix} {}^{\mathcal{B}}\hat{\boldsymbol{t}}_1 & {}^{\mathcal{B}}\hat{\boldsymbol{t}}_2 & {}^{\mathcal{B}}\hat{\boldsymbol{t}}_3 \end{bmatrix} \qquad [NT] = \begin{bmatrix} {}^{\mathcal{N}}\hat{\boldsymbol{t}}_1 & {}^{\mathcal{N}}\hat{\boldsymbol{t}}_2 & {}^{\mathcal{N}}\hat{\boldsymbol{t}}_3 \end{bmatrix}
$$

- Finally, we can compute the desired DCM matrix using

$$[\bar{B}N] = [\bar{B}T][NT]^T$$

- From the rotation matrix, we can now extract any desired set of attitude coordinates!

- Note that with this method we do not use the full magnetic field direction vector $\hat{\boldsymbol{m}}$. If this measurement were more accurate, then we could modify this method to define $\hat{\boldsymbol{t}}_1 = \hat{\boldsymbol{m}}$ instead.

See Mathematica Solution of Example 3.14

Statistical Attitude Determination

Wahba's Problem

- Assume we have N>1 observation measurements (i.e. measured directions to sun, magnetic field, stars, etc.), and we know the corresponding inertial vector directions. Then we can write attitude determination problem as

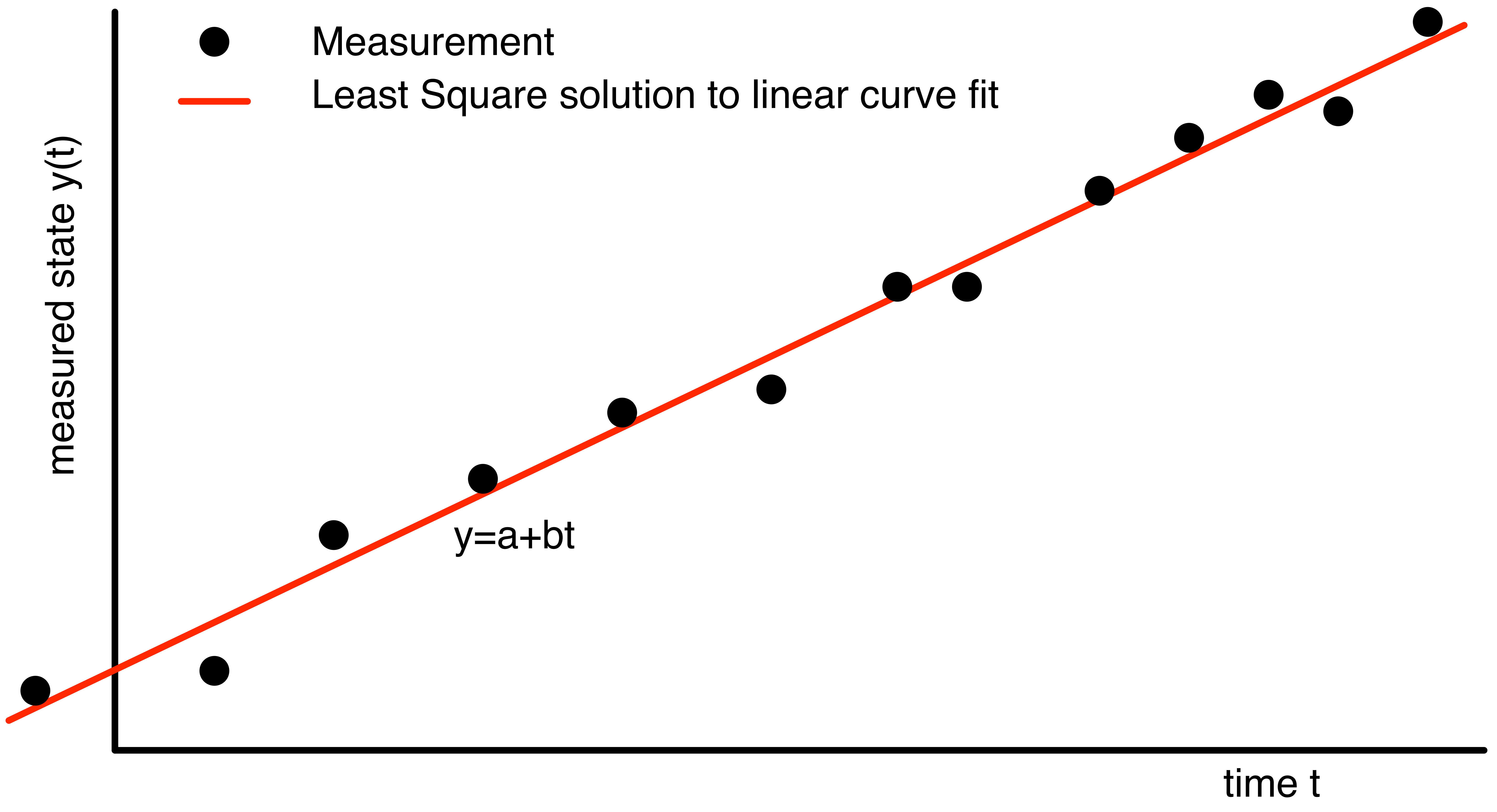
$$ {}^{\mathcal{B}}\hat{\boldsymbol{v}}_k = [\bar{B}N]\,{}^{\mathcal{N}}\hat{\boldsymbol{v}}_k \qquad \text{for } k = 1, \cdots, N $$

with the goal to find the rotation matrix [*BN*] such that the following loss function is minimized:

- If all measurements are perfect, then J = 0.

$$ J([\bar{B}N]) = \frac{1}{2} \sum_{k=1}^{N} w_k \left| {}^{\mathcal{B}}\hat{\boldsymbol{v}}_k - [\bar{B}N]\,{}^{\mathcal{N}}\hat{\boldsymbol{v}}_k \right|^2 $$

- Think of the cost function J as the error of the common least squares curve fitting problem:



Devenport's *q*-Method

- Let the 4-D Euler parameter (quaternion) vector be defined as

$$\bar{\boldsymbol{\beta}} = (\beta_0, \beta_1, \beta_2, \beta_3)^T$$

- The cost function can be rewritten

$$J = \frac{1}{2}\sum_{k=1}^{N} w_k \left({}^{\mathcal{B}}\hat{\boldsymbol{v}}_k - [\bar{B}N]\,{}^{\mathcal{N}}\hat{\boldsymbol{v}}_k\right)^T \left({}^{\mathcal{B}}\hat{\boldsymbol{v}}_k - [\bar{B}N]\,{}^{\mathcal{N}}\hat{\boldsymbol{v}}_k\right)$$

$$J = \frac{1}{2}\sum_{k=1}^{N} w_k \left({}^{\mathcal{B}}\hat{\boldsymbol{v}}_k^T\,{}^{\mathcal{B}}\hat{\boldsymbol{v}}_k + {}^{\mathcal{N}}\hat{\boldsymbol{v}}_k^T\,{}^{\mathcal{N}}\hat{\boldsymbol{v}}_k - 2\,{}^{\mathcal{B}}\hat{\boldsymbol{v}}_k^T[\bar{B}N]\,{}^{\mathcal{N}}\hat{\boldsymbol{v}}_k\right)$$

$$J = \sum_{k=1}^{N} w_k \left(1 - {}^{\mathcal{B}}\hat{\boldsymbol{v}}_k^T\,[\bar{B}N]\,{}^{\mathcal{N}}\hat{\boldsymbol{v}}_k\right)$$

- Minimizing *J* is equivalent to maximizing the gain function *g*:

$$g = \sum_{k=1}^{N} w_k \, {}^{\mathcal{B}}\hat{\boldsymbol{v}}_k^T \; [\bar{B}N] \; {}^{\mathcal{N}}\hat{\boldsymbol{v}}_k$$

- The rotation matrix can be written in terms of Euler parameters as

$$[\bar{B}N] = (\beta_0^2 - \boldsymbol{\epsilon}^T \boldsymbol{\epsilon})[I_{3\times3}] + 2\boldsymbol{\epsilon}\boldsymbol{\epsilon}^T - 2\beta_0[\tilde{\boldsymbol{\epsilon}}] \qquad \boldsymbol{\epsilon} = (\beta_1, \beta_2, \beta_3)$$

- This allows us to rewrite the gain function *g* using the 4x4 matrix [*K*]

$$\boxed{g(\bar{\boldsymbol{\beta}}) = \bar{\boldsymbol{\beta}}^T [K] \bar{\boldsymbol{\beta}}} \qquad [K] = \begin{bmatrix} \sigma & Z^T \\ Z & S - \sigma I_{3\times3} \end{bmatrix}$$

$$[B] = \sum_{k=1}^{N} w_k \, {}^{\mathcal{B}}\hat{\boldsymbol{v}}_k \, {}^{\mathcal{N}}\hat{\boldsymbol{v}}_k^T \qquad [S] = [B] + [B]^T$$

$$\sigma = \operatorname{tr}([B])$$

$$[Z] = \begin{bmatrix} B_{23} - B_{32} & B_{31} - B_{13} & B_{12} - B_{21} \end{bmatrix}^T$$

- However, since the Euler parameter vector must abide by the unit length constraint, we cannot solve this gain function directly. Instead, we use Lagrange multipliers to yield a new gain function *g*'

$$g'(\bar{\boldsymbol{\beta}}) = \bar{\boldsymbol{\beta}}^T [K] \bar{\boldsymbol{\beta}} - \lambda(\bar{\boldsymbol{\beta}}^T \bar{\boldsymbol{\beta}} - 1)$$

- We differentiate *g*' and set it equal to zero to find the extrema point of this function.

$$\frac{\mathrm{d}}{\mathrm{d}\bar{\boldsymbol{\beta}}}(g'(\bar{\boldsymbol{\beta}})) = 2[K]\bar{\boldsymbol{\beta}} - 2\lambda\bar{\boldsymbol{\beta}} = 0 \quad \Rightarrow \quad \boxed{[K]\bar{\boldsymbol{\beta}} = \lambda\bar{\boldsymbol{\beta}}}$$

- Clearly the desired Euler parameter vector is the eigenvector of the [*K*] matrix.

- To maximize the gain function, we need to choose the largest eigenvalue of the [*K*] matrix.

$$g(\bar{\boldsymbol{\beta}}) = \bar{\boldsymbol{\beta}}^T [K] \bar{\boldsymbol{\beta}} = \bar{\boldsymbol{\beta}}^T \lambda \bar{\boldsymbol{\beta}} = \lambda \bar{\boldsymbol{\beta}}^T \bar{\boldsymbol{\beta}} = \lambda$$

- In summary, to use the q-Method, we must
    - Compute the 4x4 matrix $[K]$
    - Find the eigenvalue and eigenvector of the $[K]$ matrix
    - Choose the largest eigenvalue and associated eigenvector.
    - This eigenvector is the Euler parameter vector
- Note that solving this eigenvalue, eigenvector problem is numerically rather intensive for real-time applications.

See Mathematica Solution of Example 3.15

QUEST

- Recall the cost function J and the gain function g

$$J = \sum_{k=1}^{N} w_k \left(1 - {}^{\mathcal{B}}\hat{\boldsymbol{v}}_k^T \; [\bar{B}N] \; {}^{\mathcal{N}}\hat{\boldsymbol{v}}_k\right)$$

$$g = \sum_{k=1}^{N} w_k \, {}^{\mathcal{B}}\hat{\boldsymbol{v}}_k^T \; [\bar{B}N] \; {}^{\mathcal{N}}\hat{\boldsymbol{v}}_k$$

- Further, we found that the optimal g will be

$$g(\bar{\boldsymbol{\beta}}) = \lambda_{\text{opt}}$$

- This can now be rewritten in the useful form

$$J = \sum_{k=1}^{N} w_k - g = \sum_{k=1}^{N} w_k - \lambda_{\text{opt}}$$

- Finally, the optimality condition can be written as

$$\lambda_{\text{opt}} = \sum_{k=1}^{N} w_k - J$$

- Note that *J* should be small for an optimal solution. This assumes that the measurement noise is reasonable small and Gaussian. The QUEST method then makes the elegant assumption that

$$\lambda_{\text{opt}} \approx \sum_{k=1}^{N} w_k$$

- This allows us to avoid the numerically intensive eigenvalue problem!

- However, we still need to find a solution for the eigenvector.

- The eigenvalues of [*K*] must satisfy the characteristic equation:

$$f(s) = \det([K] - s[I_{4\times4}]) = 0$$

- The desire root can be solve using a classic Newton-Raphson iteration method:

$$\lambda_0 = \sum_{k=1}^{N} w_k$$

$$\lambda_1 = \lambda_0 - \frac{f(\lambda_0)}{f'(\lambda_0)}$$

$$\vdots$$

$$\lambda_{\max} = \lambda_i = \lambda_{i-1} - \frac{f(\lambda_{i-1})}{f'(\lambda_{i-1})}$$

- Let use introduce the classical Rodrigues parameter vector $\boldsymbol{q}$

$$\boldsymbol{q} = \hat{\boldsymbol{e}} \tan\left(\frac{\Phi}{2}\right) = \frac{1}{\beta_0}\begin{bmatrix}\beta_1\\ \beta_2\\ \beta_3\end{bmatrix} = \frac{\boldsymbol{\epsilon}}{\beta_0}$$

- Note that

$$\frac{\bar{\boldsymbol{\beta}}}{\beta_0} = \begin{bmatrix}1\\ \bar{\boldsymbol{q}}\end{bmatrix}$$

- The eigenvector problem is now re-written as

$$[K]\frac{\bar{\boldsymbol{\beta}}}{\beta_0} = \lambda_{\text{opt}}\frac{\bar{\boldsymbol{\beta}}}{\beta_0}$$

$$\begin{bmatrix}\sigma & Z^T\\ Z & S - \sigma I_{3\times3}\end{bmatrix}\begin{bmatrix}1\\ \bar{\boldsymbol{q}}\end{bmatrix} = \lambda_{\text{opt}}\begin{bmatrix}1\\ \bar{\boldsymbol{q}}\end{bmatrix}$$

$$\left([S] - \sigma[I_{3\times3}]\right)\bar{\boldsymbol{q}} + [Z] = \lambda_{\text{opt}}\bar{\boldsymbol{q}}$$

- Finally, the classical Rodrigues parameter vector is found

$$\bar{\boldsymbol{q}} = \Big((\lambda_{\text{opt}} + \sigma)[I_{3\times 3}] - [S]\Big)^{-1}[Z]$$

- Note that we still have to take an inverse of a 3x3 matrix here. However, this is numerically a very fast process.

- To solve for the corresponding 4-D Euler parameter vector, we use

$$\bar{\boldsymbol{\beta}} = \frac{1}{\sqrt{1 + \bar{\boldsymbol{q}}^T \bar{\boldsymbol{q}}}} \begin{bmatrix} 1 \\ \bar{\boldsymbol{q}} \end{bmatrix}$$

See Mathematica Solution of Example 3.16

Optimal Linear Attitude Estimator (OLAE)

Cayley Transform

$$[\bar{B}N] = ([I_{3\times3}] + [\tilde{\bar{\boldsymbol{q}}}])^{-1}([I_{3\times3}] - [\tilde{\bar{\boldsymbol{q}}}])$$

$$^{\mathcal{B}}\hat{\boldsymbol{v}}_i = [\bar{B}N]^{\mathcal{N}}\hat{\boldsymbol{v}}_i$$

$$([I_{3\times3}] + [\tilde{\bar{\boldsymbol{q}}}])\ ^{\mathcal{B}}\hat{\boldsymbol{v}}_i = ([I_{3\times3}] - [\tilde{\bar{\boldsymbol{q}}}])\ ^{\mathcal{N}}\hat{\boldsymbol{v}}_i$$

$$^{\mathcal{B}}\hat{\boldsymbol{v}}_i - {}^{\mathcal{N}}\hat{\boldsymbol{v}}_i = -[\tilde{\bar{\boldsymbol{q}}}]({}^{\mathcal{B}}\hat{\boldsymbol{v}}_i + {}^{\mathcal{N}}\hat{\boldsymbol{v}}_i)$$

Define:

$$\boldsymbol{s}_i = {}^{\mathcal{B}}\hat{\boldsymbol{v}}_i + {}^{\mathcal{N}}\hat{\boldsymbol{v}}_i$$

$$\boldsymbol{d}_i = {}^{\mathcal{B}}\hat{\boldsymbol{v}}_i - {}^{\mathcal{N}}\hat{\boldsymbol{v}}_i$$

$$\boldsymbol{d}_i = [\tilde{\boldsymbol{s}}_i]\bar{\boldsymbol{q}}$$

$$\boldsymbol{d} = \begin{bmatrix} \boldsymbol{d}_1 \\ \cdots \\ \boldsymbol{d}_N \end{bmatrix} \qquad [S] = \begin{bmatrix} \tilde{\boldsymbol{s}}_1 \\ \cdots \\ \tilde{\boldsymbol{s}}_N \end{bmatrix}$$

$$[W] = \begin{bmatrix} w_1 I_{3\times3} & 0_{3\times3} & \ddots \\ 0_{3\times3} & \ddots & 0_{3\times3} \\ \ddots & 0_{3\times3} & w_N I_{3\times3} \end{bmatrix}$$

$$\boxed{\bar{\boldsymbol{q}} = \left([S]^T[W][S]\right)^{-1}[S]^T[W]\boldsymbol{d}}$$

See Mathematica Solution of Example 3.17

Continuous System

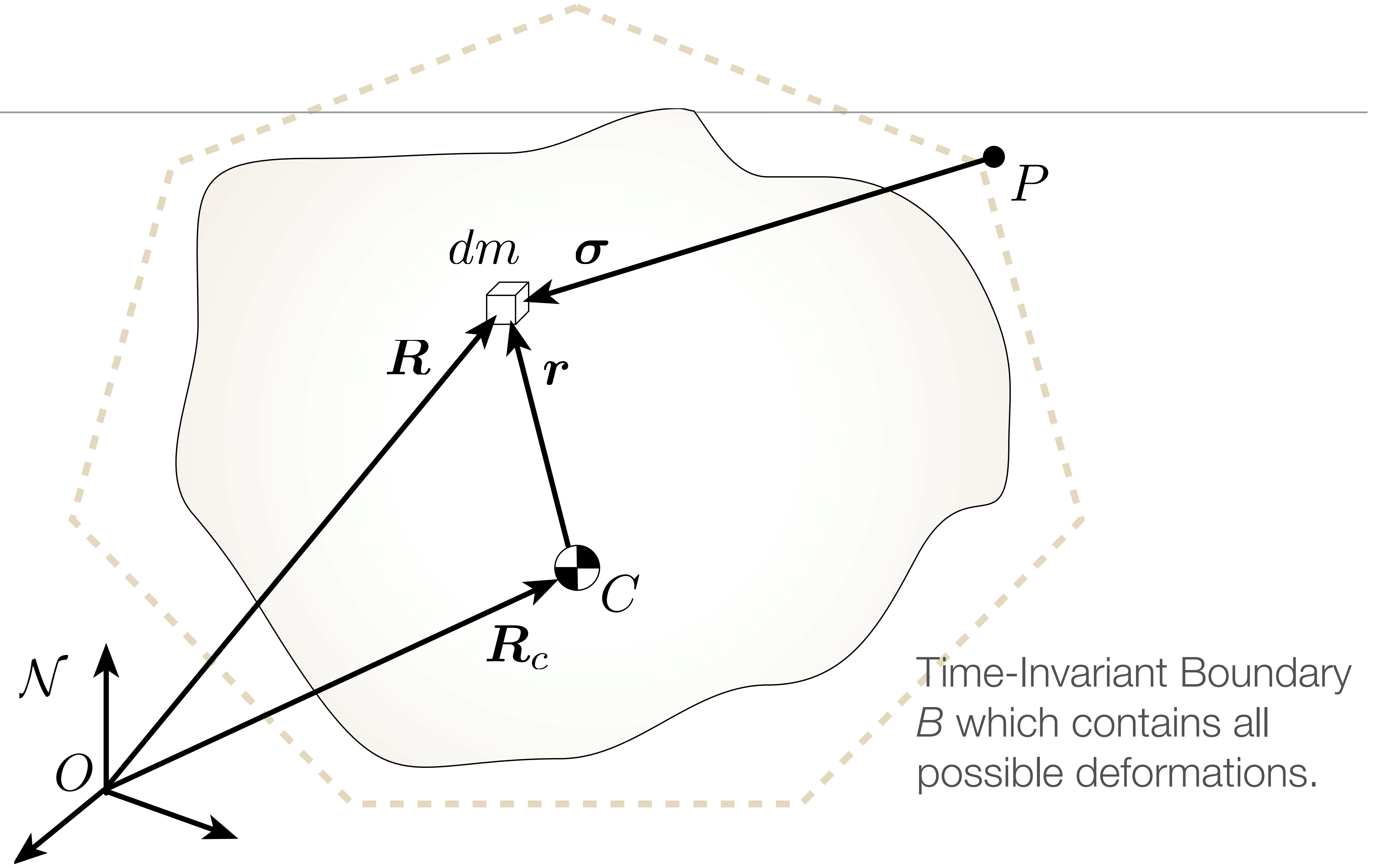
What does jello look like in space?

Equations of Motion

Newton's Law: $\mathrm{d}\boldsymbol{F} = \ddot{\boldsymbol{R}}\mathrm{d}m$

Force Vector: $\mathrm{d}\boldsymbol{F} = \mathrm{d}\boldsymbol{F}_E + \mathrm{d}\boldsymbol{F}_I$

Total Force acting on System: $\boldsymbol{F} = \int_B \mathrm{d}\boldsymbol{F} = \int_B \mathrm{d}\boldsymbol{F}_E$



Center of Mass: $M\boldsymbol{R}_c = \int_B \boldsymbol{R}\mathrm{d}m = \int_{\mathcal{B}} (\boldsymbol{R}_c + \boldsymbol{r})\mathrm{d}m \Rightarrow \int_{\mathcal{B}} \boldsymbol{r}\mathrm{d}m = \boldsymbol{0}$

$$M\ddot{\boldsymbol{R}}_c = \int_B \ddot{\boldsymbol{R}}\mathrm{d}m = \int_B \mathrm{d}\boldsymbol{F} \Rightarrow \boxed{M\ddot{\boldsymbol{R}}_c = \boldsymbol{F}}$$

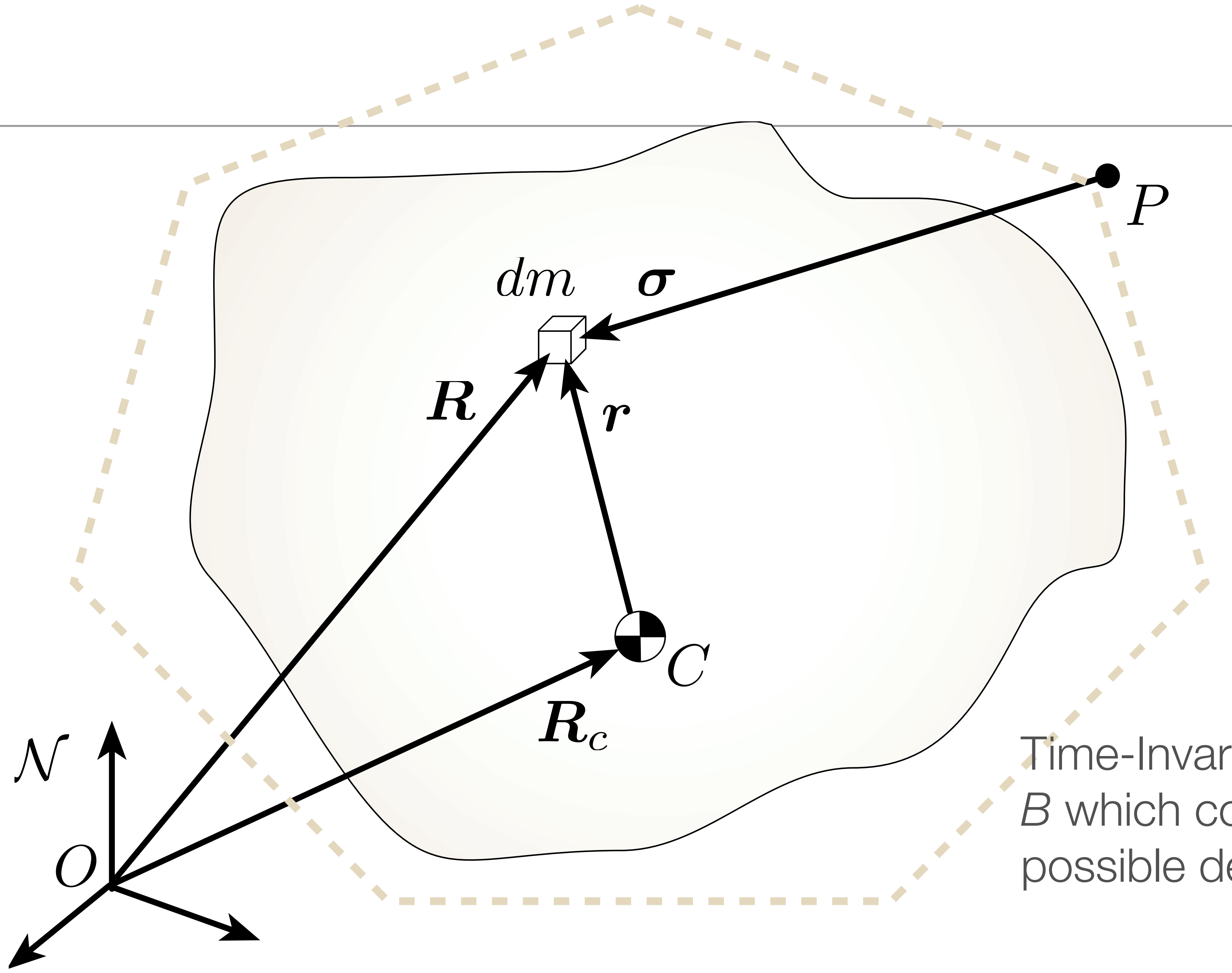
Super Particle Theorem

Kinetic Energy

Definition: $$T = \frac{1}{2}\int_B \dot{\boldsymbol{R}} \cdot \dot{\boldsymbol{R}} \mathrm{d}m$$

$$\Big\downarrow \; \dot{\boldsymbol{R}} = \dot{\boldsymbol{R}}_c + \dot{\boldsymbol{r}}$$

$$T = \frac{1}{2}\left(\int_B \mathrm{d}m\right)\dot{\boldsymbol{R}}_c \cdot \dot{\boldsymbol{R}}_c + \dot{\boldsymbol{R}}_c \cdot \cancel{\int_B \dot{\boldsymbol{r}}\mathrm{d}m} + \frac{1}{2}\int_B \dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}}\mathrm{d}m$$



$$\boxed{T = \frac{1}{2}M\dot{\boldsymbol{R}}_c \cdot \dot{\boldsymbol{R}}_c + \frac{1}{2}\int_B \dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}}\mathrm{d}m}$$

Energy of CM   Energy about CM

Work/Energy Principle

Differentiate Energy:

$$\frac{\mathrm{d}T}{\mathrm{d}t} = M\ddot{\boldsymbol{R}}_c \cdot \dot{\boldsymbol{R}}_c + \int_B \dot{\boldsymbol{r}} \cdot \ddot{\boldsymbol{r}} \mathrm{d}m$$

$$M\ddot{\boldsymbol{R}}_c = \boldsymbol{F} \qquad \ddot{\boldsymbol{r}} = \ddot{\boldsymbol{R}} - \ddot{\boldsymbol{R}}_c$$

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \boldsymbol{F} \cdot \dot{\boldsymbol{R}}_c + \int_B \left(\ddot{\boldsymbol{R}}\mathrm{d}m\right) \cdot \dot{\boldsymbol{r}} - \ddot{\boldsymbol{R}}_c \cdot \underbrace{\cancel{\int_B \dot{\boldsymbol{r}}\mathrm{d}m}}_{\text{C.M.}}$$

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \boldsymbol{F} \cdot \dot{\boldsymbol{R}}_c + \int_B \mathrm{d}\boldsymbol{F} \cdot \dot{\boldsymbol{r}}$$

$$\boxed{T(t_2) - T(t_1) = \int_{\boldsymbol{R}(t_1)}^{\boldsymbol{R}(t_2)} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{R}_c + \int_{\boldsymbol{r}(t_1)}^{\boldsymbol{r}(t_2)} \int_B \mathrm{d}\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}}$$

Work energy/principle for system of particles

Linear Momentum

Definition: $$\mathrm{d}\boldsymbol{p} = \dot{\boldsymbol{R}}\mathrm{d}m$$

$$\boldsymbol{p} = \int_{\mathcal{B}} \mathrm{d}\boldsymbol{p} = \int_{\mathcal{B}} \dot{\boldsymbol{R}}\mathrm{d}m = \int_{\mathcal{B}} (\dot{\boldsymbol{R}}_c + \dot{\boldsymbol{r}})\mathrm{d}m = \left(\int_{\mathcal{B}} \mathrm{d}m\right) \dot{\boldsymbol{R}}_c + \underbrace{\cancel{\int_{\mathcal{B}} \dot{\boldsymbol{r}}\mathrm{d}m}}_{\text{C.M.}}$$

$$\boldsymbol{p} = M\dot{\boldsymbol{R}}_c$$

Linear Momentum Rate: $$\dot{\boldsymbol{p}} = \int_{\mathcal{B}} \ddot{\boldsymbol{R}}\mathrm{d}m = \int_{\mathcal{B}} \mathrm{d}\boldsymbol{F} = \boldsymbol{F}$$

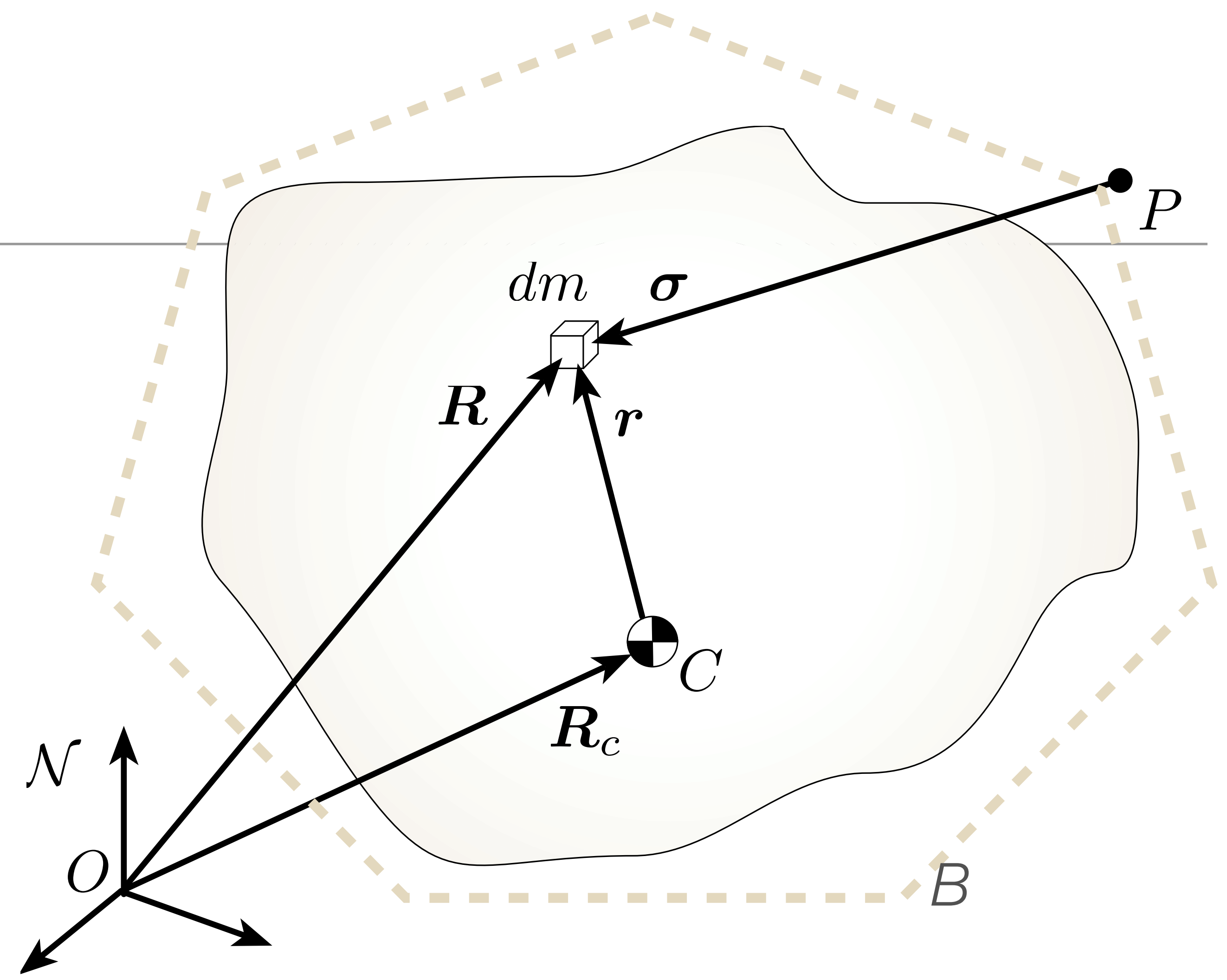
$$\Downarrow$$

$$\boldsymbol{F} = \frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}(\boldsymbol{p})$$

Angular Momentum

Ang. Momentum about P: $\boldsymbol{H}_P = \int_B \boldsymbol{\sigma} \times \dot{\boldsymbol{\sigma}} \mathrm{d}m$

$$\boldsymbol{\sigma} = \boldsymbol{R} - \boldsymbol{R}_P$$



Inertial Time Derivative: $\dot{\boldsymbol{H}}_P = \cancel{\int_B \dot{\boldsymbol{\sigma}} \times \dot{\boldsymbol{\sigma}} \mathrm{d}m} + \int_B \boldsymbol{\sigma} \times \boxed{\ddot{\boldsymbol{\sigma}}} \mathrm{d}m$

$$\dot{\boldsymbol{H}}_P = \boxed{\int_B \boldsymbol{\sigma} \times \ddot{\boldsymbol{R}} \mathrm{d}m} - \boxed{\left(\int_B \boldsymbol{\sigma} \mathrm{d}m\right)} \boxed{\times \ddot{\boldsymbol{R}}_P}$$

$$\int_B \boldsymbol{\sigma} \mathrm{d}m = \int_B \boldsymbol{R} \mathrm{d}m - \left(\int_B \mathrm{d}m\right) \boldsymbol{R}_P = \boxed{M(\boldsymbol{R}_c - \boldsymbol{R}_P)}$$

Torque about P: $\boldsymbol{L}_P = \boxed{\int_B \boldsymbol{\sigma} \times \ddot{\boldsymbol{R}} \mathrm{d}m} = \int_B \boldsymbol{\sigma} \times \mathrm{d}\boldsymbol{F}$

$$\dot{\boldsymbol{H}}_P = \boldsymbol{L}_P + M \ddot{\boldsymbol{R}}_P \times (\boldsymbol{R}_c - \boldsymbol{R}_P)$$

$$\boxed{\dot{\boldsymbol{H}}_P = \boldsymbol{L}_P}$$

If P is CM or Inertial

Rigid Body Dynamics

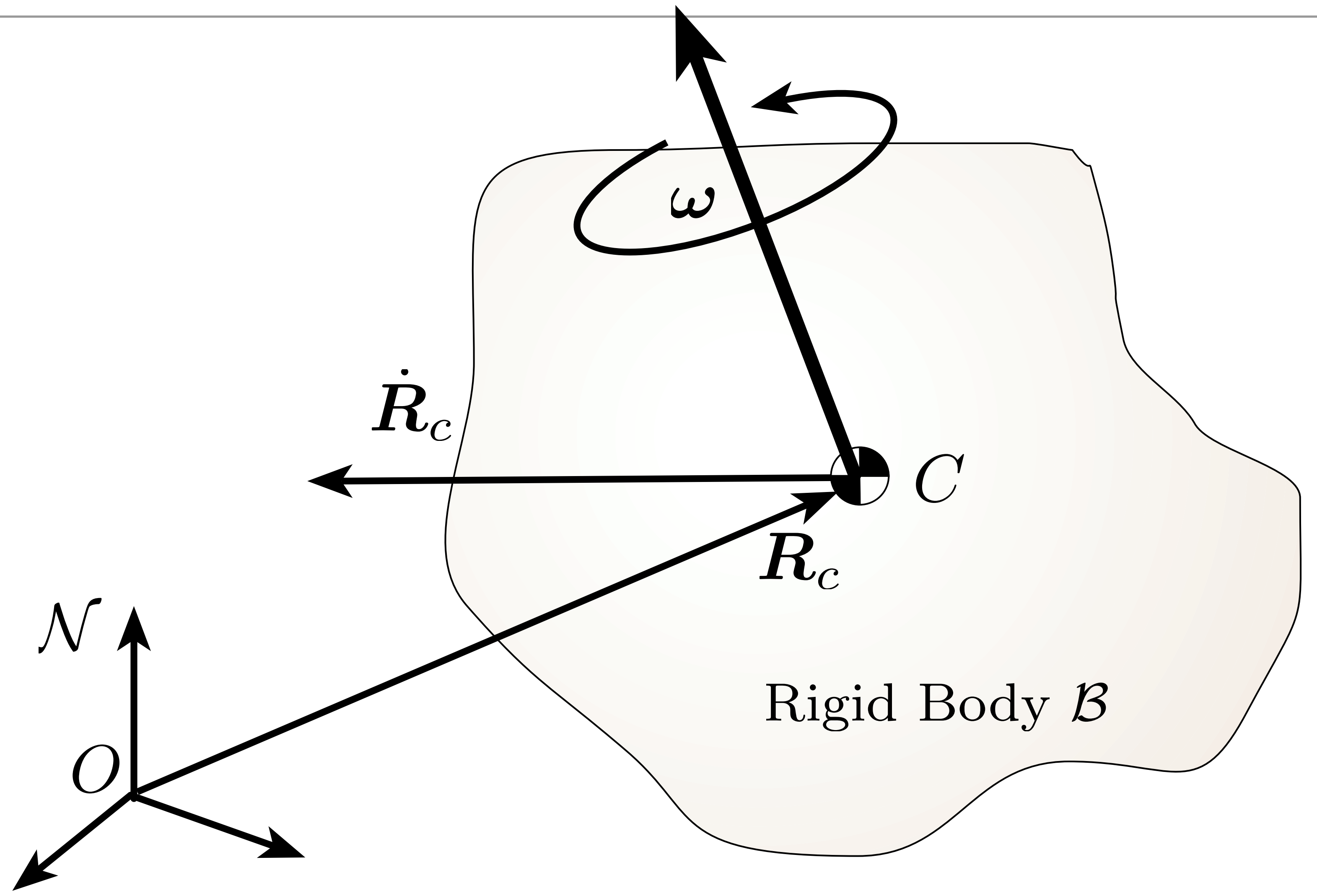
The 101 of spacecraft dynamics...

General Angular Momentum

Definition: $$\boldsymbol{H}_O = \int_B \boldsymbol{R} \times \dot{\boldsymbol{R}} \mathrm{d}m$$

or

$$\boldsymbol{H}_O = \underbrace{\boldsymbol{R}_c \times M\dot{\boldsymbol{R}}_c}_{\text{CM}} + \int_B \boldsymbol{r} \times \dot{\boldsymbol{r}} \mathrm{d}m$$



Momentum about CM:

$$\boldsymbol{H}_c = \int_B \boldsymbol{r} \times \dot{\boldsymbol{r}} \mathrm{d}m$$

$$\dot{\boldsymbol{r}} = \frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}) = \cancel{\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r})} + \boldsymbol{\omega} \times \boldsymbol{r} = \boldsymbol{\omega} \times \boldsymbol{r}$$

$$\boldsymbol{H}_c = \int_B \boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r})\,\mathrm{d}m = \left(\int_B -[\tilde{\boldsymbol{r}}][\tilde{\boldsymbol{r}}]\mathrm{d}m\right)\boldsymbol{\omega}$$

Inertia Tensor Properties

Definition:

$$ {}^{\mathcal{B}}[I_c] = \int_B -[\tilde{\boldsymbol{r}}][\tilde{\boldsymbol{r}}]\mathrm{d}m = \int_B {}^{\mathcal{B}}\begin{bmatrix} r_2^2 + r_3^2 & -r_1 r_2 & -r_1 r_3 \\ -r_1 r_2 & r_1^2 + r_3^2 & -r_2 r_3 \\ -r_1 r_3 & -r_2 r_3 & r_1^2 + r_2^2 \end{bmatrix} \mathrm{d}m $$

Angular Momentum Expression:

$$ \boldsymbol{H}_c = {}^{\mathcal{B}}\begin{pmatrix} H_{c_1} \\ H_{c_2} \\ H_{c_3} \end{pmatrix} = \int_B {}^{\mathcal{B}}\begin{bmatrix} r_2^2 + r_3^2 & -r_1 r_2 & -r_1 r_3 \\ -r_1 r_2 & r_1^2 + r_3^2 & -r_2 r_3 \\ -r_1 r_3 & -r_2 r_3 & r_1^2 + r_2^2 \end{bmatrix} {}^{\mathcal{B}}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \mathrm{d}m = [I_c]\boldsymbol{\omega} $$

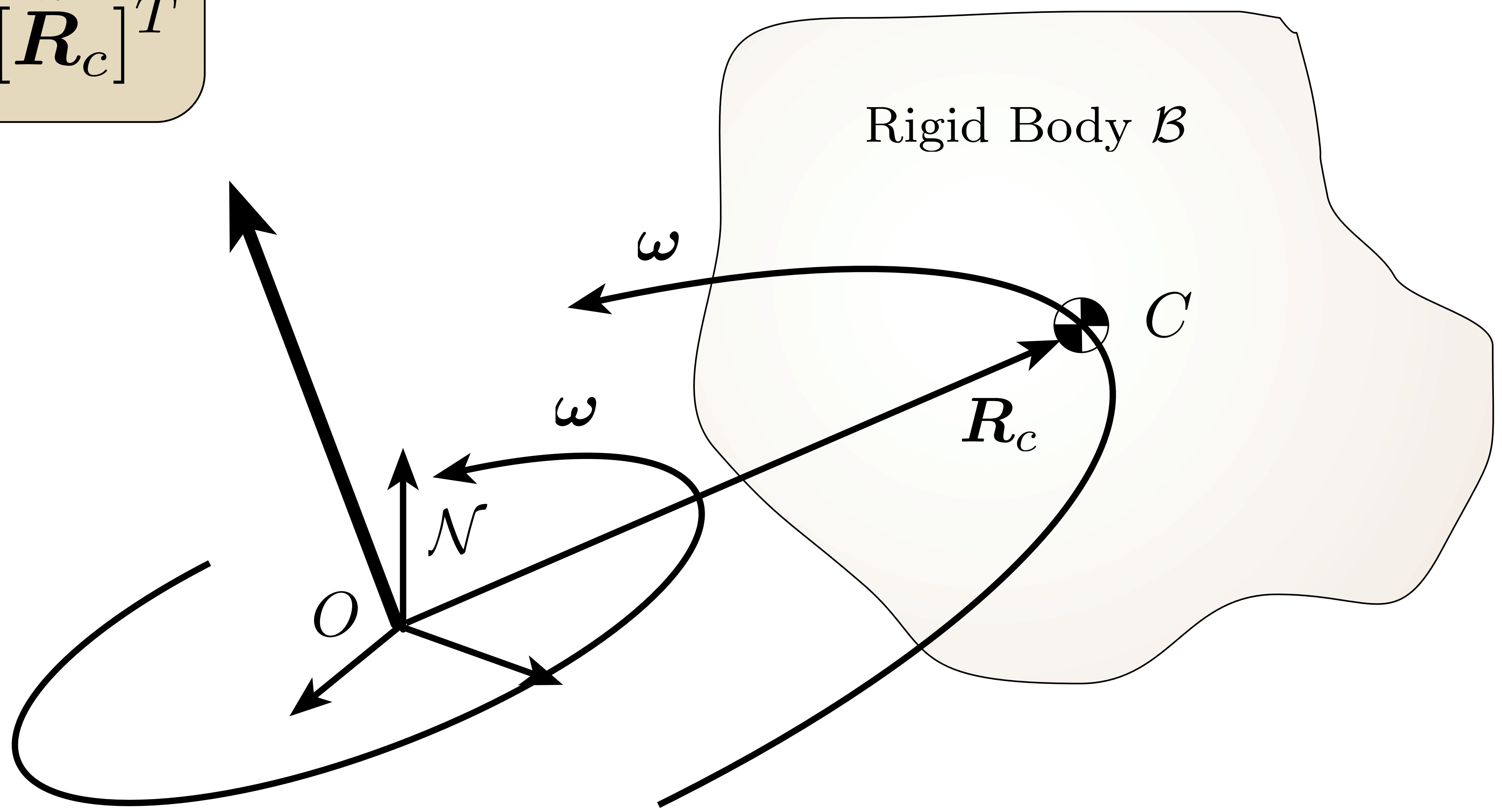
Parallel Axis Theorem

Total Mass

CM Offset

$$[I_O] = [I_c] + M[\tilde{\boldsymbol{R}}_c][\tilde{\boldsymbol{R}}_c]^T$$

Inertia about CM



Coordinate Transformation

$$^{\mathcal{F}}[I] = [FB]\,^{\mathcal{B}}[I]\,[FB]^T$$

B - Body Frame

F - 2nd Coordinate Frame

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} {}^{\mathcal{B}}\begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{12} & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{bmatrix} = {}^{\mathcal{F}}\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

Rotation matrix [*C*] contains the eigenvectors of [*I*]

Principal inertia matrix whose diagonal entries are the eigenvalues of [*I*]

$$[C] = [V]^T = \begin{bmatrix} \boldsymbol{v}_1^T \\ \boldsymbol{v}_2^T \\ \boldsymbol{v}_3^T \end{bmatrix}$$

Kinetic Energy

Total Energy: $$T = \frac{1}{2} M \dot{\boldsymbol{R}}_c \cdot \dot{\boldsymbol{R}}_c + \frac{1}{2} \int_B \dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} \mathrm{d}m = T_{\text{trans}} + T_{\text{rot}}$$

Rotational Energy: $$T_{\text{rot}} = \frac{1}{2} \int_B \dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} \mathrm{d}m = \frac{1}{2} \int_B (\boldsymbol{\omega} \times \boldsymbol{r}) \cdot (\boldsymbol{\omega} \times \boldsymbol{r}) \, \mathrm{d}m$$

$$T_{\text{rot}} = \frac{1}{2} \boldsymbol{\omega} \cdot \int_B \boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r}) \, \mathrm{d}m = \frac{1}{2} \boldsymbol{\omega} \cdot \boldsymbol{H}_c = \frac{1}{2} \boldsymbol{\omega}^T [I] \boldsymbol{\omega}$$

Energy Rate: $$\dot{T} = \boldsymbol{F} \cdot \dot{\boldsymbol{R}}_c + \boldsymbol{L}_c \cdot \boldsymbol{\omega}$$

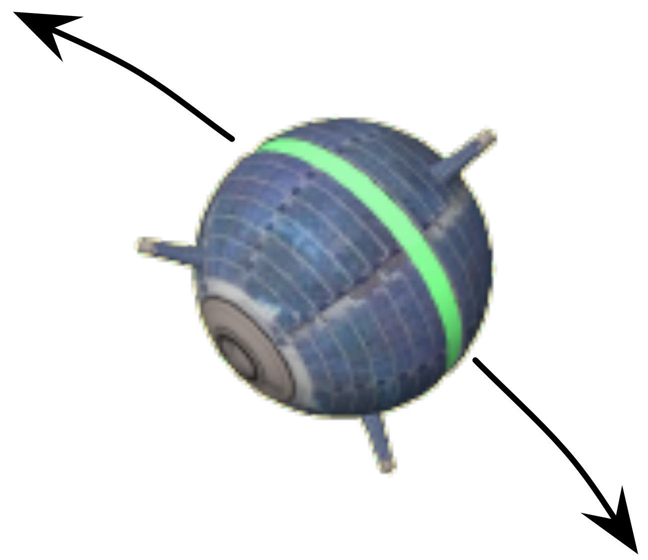
Work/Energy Principle: $$W = T(t_2) - T(t_1) = \int_{t_1}^{t_2} \boldsymbol{F} \cdot \dot{\boldsymbol{R}}_c \mathrm{d}t + \int_{t_1}^{t_2} \boldsymbol{L}_c \cdot \boldsymbol{\omega} \mathrm{d}t$$

Equations of Motion

Euler's equation:

$$\dot{\boldsymbol{H}}_c = \boxed{\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\boldsymbol{H}_c)} + \boldsymbol{\omega} \times \boldsymbol{H}_c = \boldsymbol{L}_c$$

$$\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\boldsymbol{H}_c) = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}([I])\,\boldsymbol{\omega} + [I]\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\omega}) = [I]\dot{\boldsymbol{\omega}}$$



Euler's rotational equations of motion:

$$[I]\dot{\boldsymbol{\omega}} = -[\tilde{\omega}][I]\boldsymbol{\omega} + \boldsymbol{L}_c$$

Principal axis version of rotational EOM:

$$I_{11}\dot{\omega}_1 = -(I_{33} - I_{22})\omega_2\omega_3 + L_1$$
$$I_{22}\dot{\omega}_2 = -(I_{11} - I_{33})\omega_3\omega_1 + L_2$$
$$I_{33}\dot{\omega}_3 = -(I_{22} - I_{11})\omega_1\omega_2 + L_3$$

Discuss how to integrate full EOM

Example: Slender Rod Falling

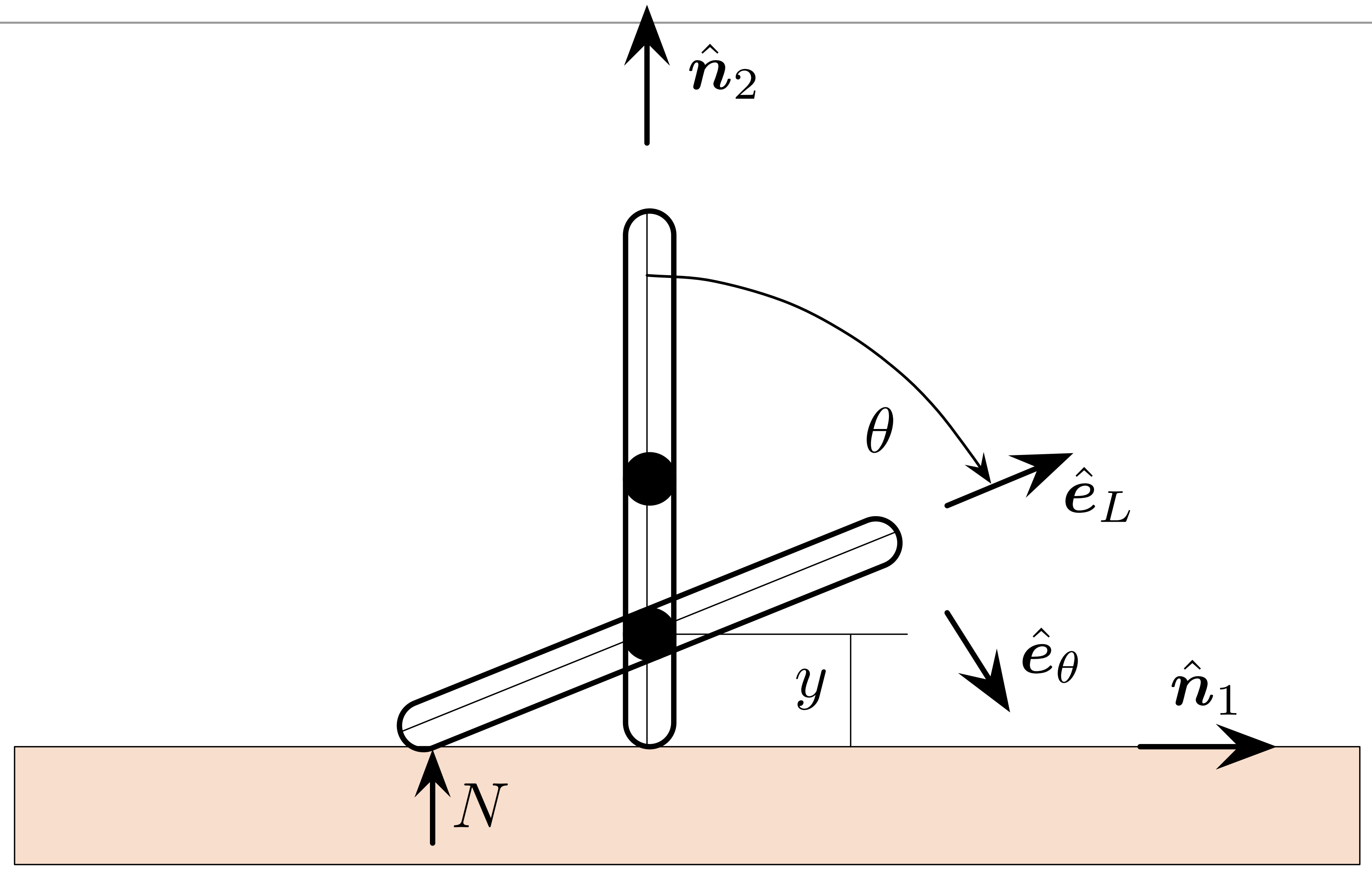
Rod Inertia about CM: $I_c = \frac{m}{12}L^2$

Momentum about CM: $\boldsymbol{H}_c = I_c\dot{\theta}\hat{\boldsymbol{e}}_3$

Torque: $\boldsymbol{L}_c = \left(-\frac{L}{2}\hat{\boldsymbol{e}}_L\right) \times N\hat{\boldsymbol{n}}_2 = \frac{L}{2}N\sin\theta\hat{\boldsymbol{e}}_3$

Euler's Eqn: $\dot{\boldsymbol{H}}_c = \boldsymbol{L}_c$

$$\frac{m}{12}L^2\ddot{\theta} - \frac{L}{2}N\sin\theta = 0$$



Newton's Eqn: $m\ddot{y}\hat{\boldsymbol{n}}_2 = (N - mg)\hat{\boldsymbol{n}}_2$

$$y = \frac{L}{2}\cos\theta \Rightarrow \ddot{y} = -\frac{L}{2}\ddot{\theta}\sin\theta - \frac{L}{2}\dot{\theta}^2\cos\theta$$

$$\Rightarrow N = mg - m\frac{L}{2}\ddot{\theta}\sin\theta - m\frac{L}{2}\dot{\theta}^2\cos\theta$$

EOM: $$\boxed{\frac{m}{12}L^2\ddot{\theta}(1 + 3\sin^2\theta) + \frac{m}{4}L^2\dot{\theta}^2\sin\theta\cos\theta - \frac{m}{2}Lg\sin\theta = 0}$$

Example: Slender Rod Falling

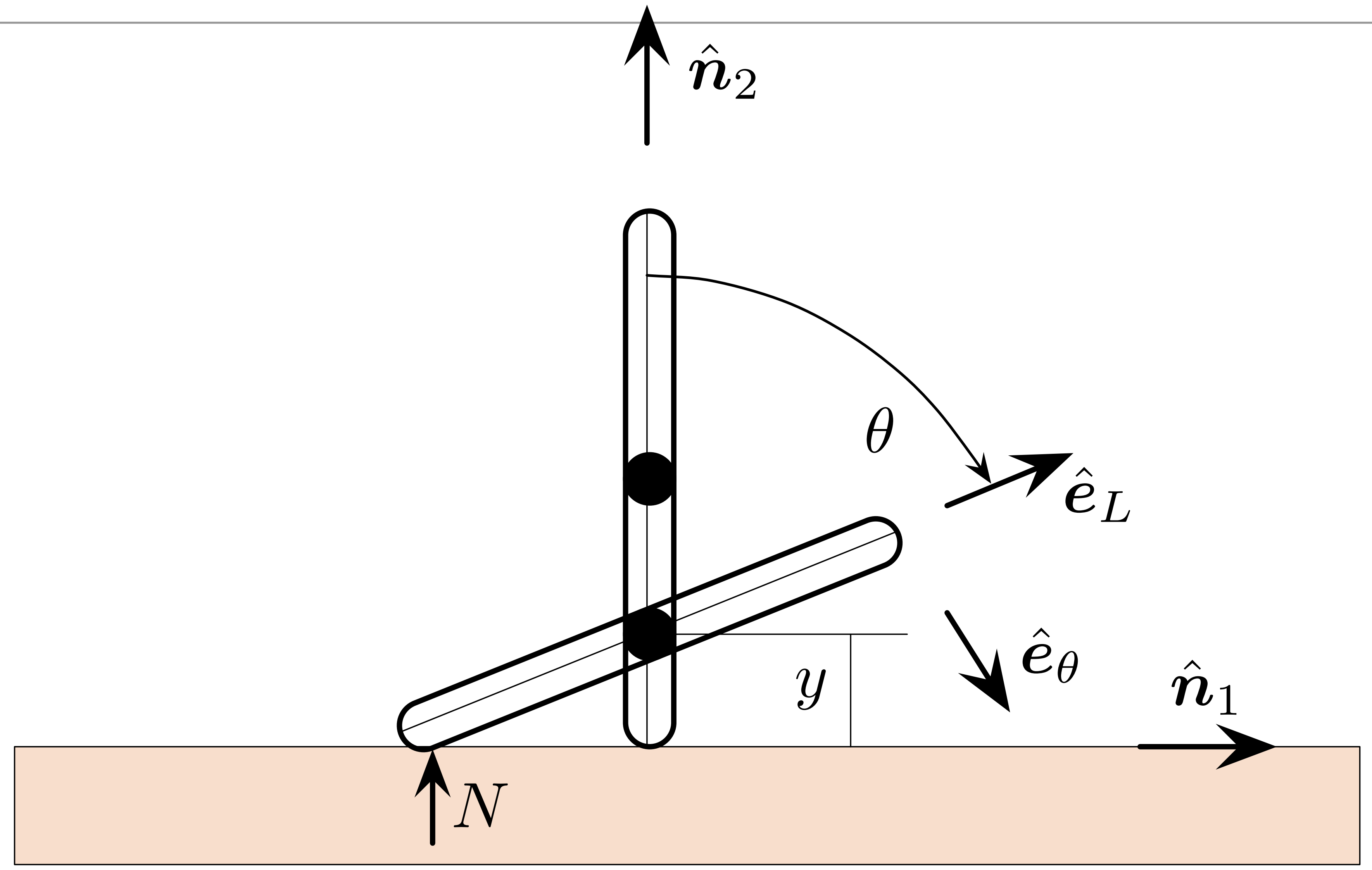
Energy functions:

$$V(\theta) = mgy = mg\frac{L}{2}\cos\theta$$

$$T(\theta,\dot\theta) = \frac{m}{2}\dot y^2 + \frac{I_c}{2}\dot\theta^2 = \frac{mL^2}{24}(1+3\sin^2\theta)\dot\theta^2$$

Initial energy level:

$$E(t_0) = T_0 + V_0 = mg\frac{L}{2}$$



$$T(\theta,\dot\theta) + V(\theta) = E(t_0)$$

Energy Conservation avoids having to solve ODE.

$$\Rightarrow \quad \boxed{\dot\theta^2 = \frac{12g(1-\cos\theta)}{L(1+3\sin^2\theta)}}$$

Momentum/Energy Surfaces

- Let's study a special class of rigid body motion where no external torque is acting on the body.
- In this case the kinetic energy and the angular momentum of the rigid body are conserved!
- In inertial frame vector components, we find

$$\boldsymbol{H} = {}^{\mathcal{N}}\boldsymbol{H} = [BN]^T \, {}^{\mathcal{B}}\boldsymbol{H}$$

$$\boldsymbol{H} = {}^{\mathcal{N}}\boldsymbol{H} = \begin{bmatrix} c\theta_2 c\theta_1 & c\theta_2 s\theta_1 & -s\theta_2 \\ s\theta_3 s\theta_2 c\theta_1 - c\theta_3 s\theta_1 & s\theta_3 s\theta_2 s\theta_1 + c\theta_3 c\theta_1 & s\theta_3 c\theta_2 \\ c\theta_3 s\theta_2 c\theta_1 + s\theta_3 s\theta_1 & c\theta_3 s\theta_2 s\theta_1 - s\theta_3 c\theta_1 & c\theta_3 c\theta_2 \end{bmatrix} \begin{pmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{pmatrix}$$

$$\dot{\boldsymbol{H}} = 0 = \boldsymbol{f}(\theta_1, \theta_2, \theta_3, \omega_1, \omega_2, \omega_3, \dot{\omega}_1, \dot{\omega}_2, \dot{\omega}_3)$$

- Notice that the constant angular momentum condition can become very complicated and difficult to study!
- Instead of writing ***H*** in inertial frame components, we chose to write it in the body frame where the inertia matrix is a constant for a rigid body.
- Assume the angular momentum vector ***H*** is written in body frame components, and that principal axes were chosen for the body frame *B*.

$$\boldsymbol{\omega} = \omega_1 \hat{\boldsymbol{b}}_1 + \omega_2 \hat{\boldsymbol{b}}_2 + \omega_3 \hat{\boldsymbol{b}}_3$$

$$\boldsymbol{H} = {}^{\mathcal{B}}\boldsymbol{H} = H_1 \hat{\boldsymbol{b}}_1 + H_2 \hat{\boldsymbol{b}}_2 + H_3 \hat{\boldsymbol{b}}_3 = I_1\omega_1 \hat{\boldsymbol{b}}_1 + I_2\omega_2 \hat{\boldsymbol{b}}_2 + I_3\omega_3 \hat{\boldsymbol{b}}_3$$

$$[I] = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

- This allows us to write ***H*** as:

$$\boldsymbol{H} = {}^{\mathcal{B}}\boldsymbol{H} = {}^{\mathcal{B}}\begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} = {}^{\mathcal{B}}\begin{pmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{pmatrix}$$

- Because ***H*** is constant, all possible rigid body angular velocities must lie on the surface of the following momentum ellipsoid:

$$H^2 = \boldsymbol{H}^T\boldsymbol{H} = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 = \text{constant}$$

- Similarly, since the kinetic energy is conserved, all possible rigid body angular velocities must also lie on the surface of the following energy ellipsoid:

$$T = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2$$

- The final admissible angular velocities will be on the intersection of these two ellipsoids.

- Using the momenta coordinates

$$H_1 = I_1\omega_1 \qquad H_2 = I_2\omega_2 \qquad H_3 = I_3\omega_3$$

- we can write the momentum magnitude constraint as

$$H^2 = H_1^2 + H_2^2 + H_3^2 \quad \Rightarrow \quad \text{Sphere}$$

- and the kinetic energy constraint as

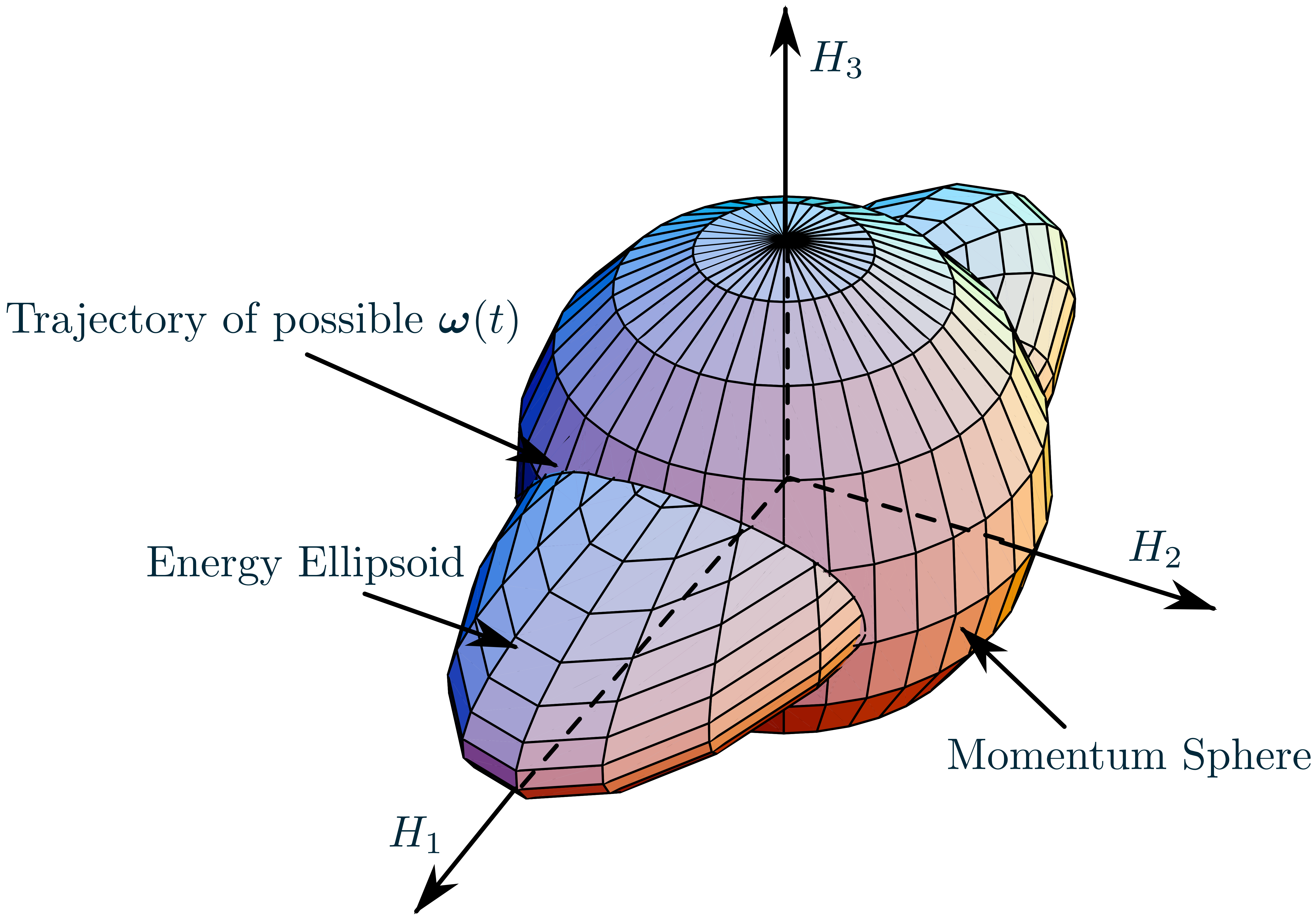
$$1 = \frac{H_1^2}{2I_1T} + \frac{H_2^2}{2I_2T} + \frac{H_3^2}{2I_3T}$$

Compare to ellipsoid equation:

$$1 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$$

$$\sqrt{2I_iT} \quad \Rightarrow \quad \text{semi-axes of ellipsoid}$$

- Clearly, for a given *H*, only a certain range of kinetic energies is possible.
- Let's assume the common notation: $I_1 \geq I_2 \geq I_3$



- Let's look at the Minimum Energy Case:
- The surfaces will only intersect at:

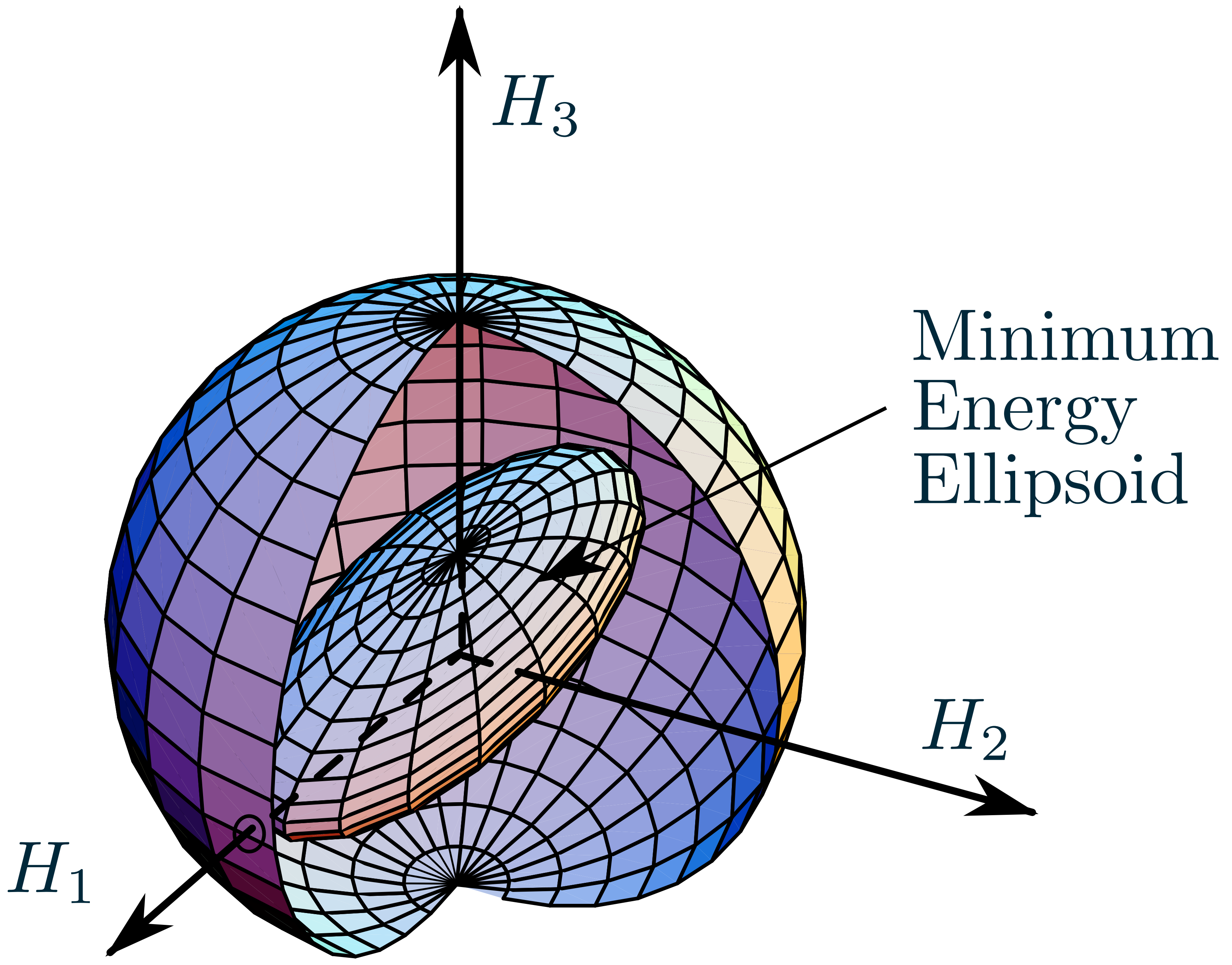
$$^{\mathcal{B}}\boldsymbol{H} = \pm H \hat{\boldsymbol{b}}_1$$

$$H_1 = H \qquad H_2 = H_3 = 0$$

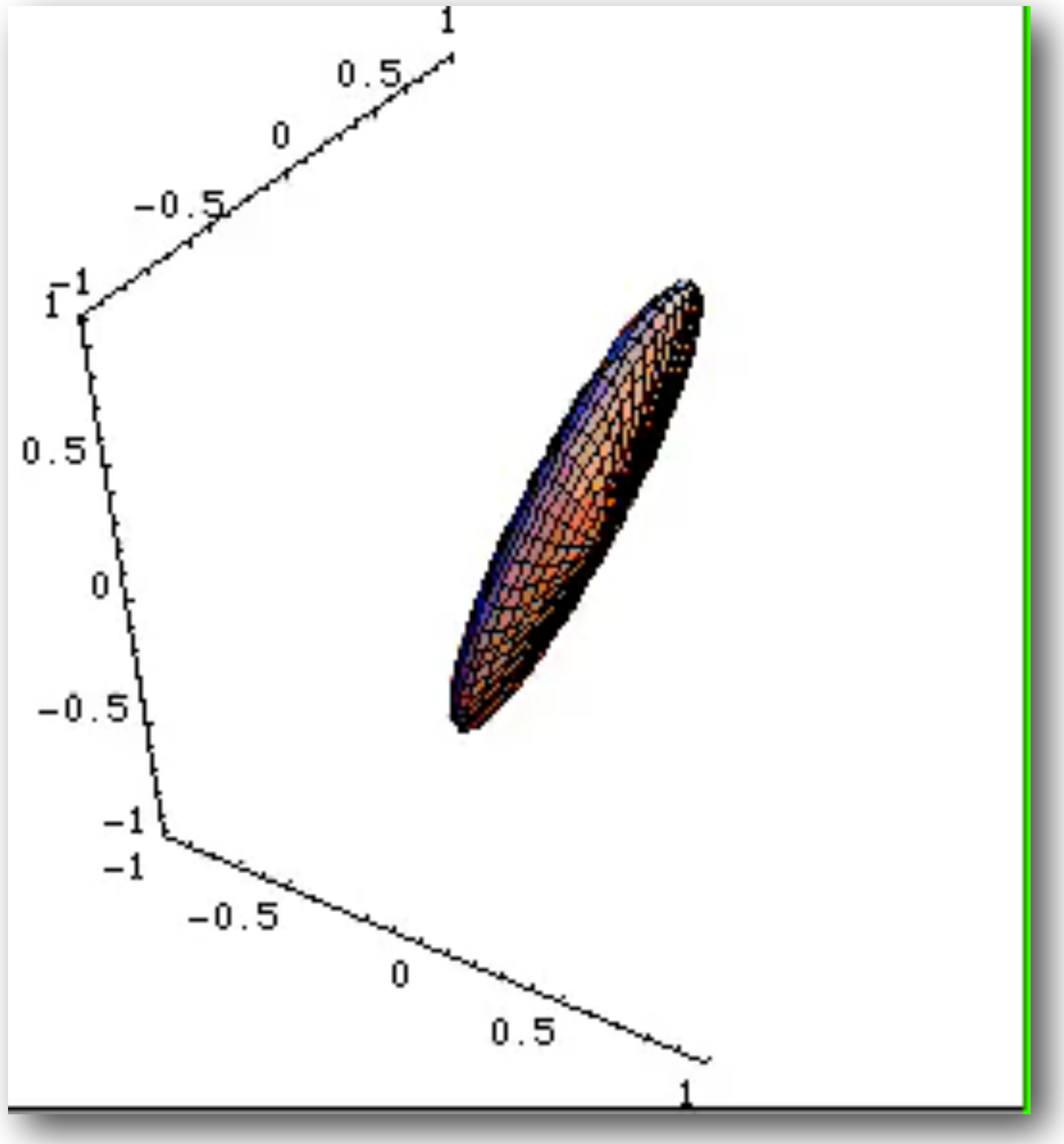
The kinetic energy is:

$$1 = \frac{H_1^2}{2I_1T} + \frac{H_2^2}{2I_2T} + \frac{H_3^2}{2I_3T}$$

$$\boxed{T_{\text{min}} = \frac{H^2}{2I_1}}$$

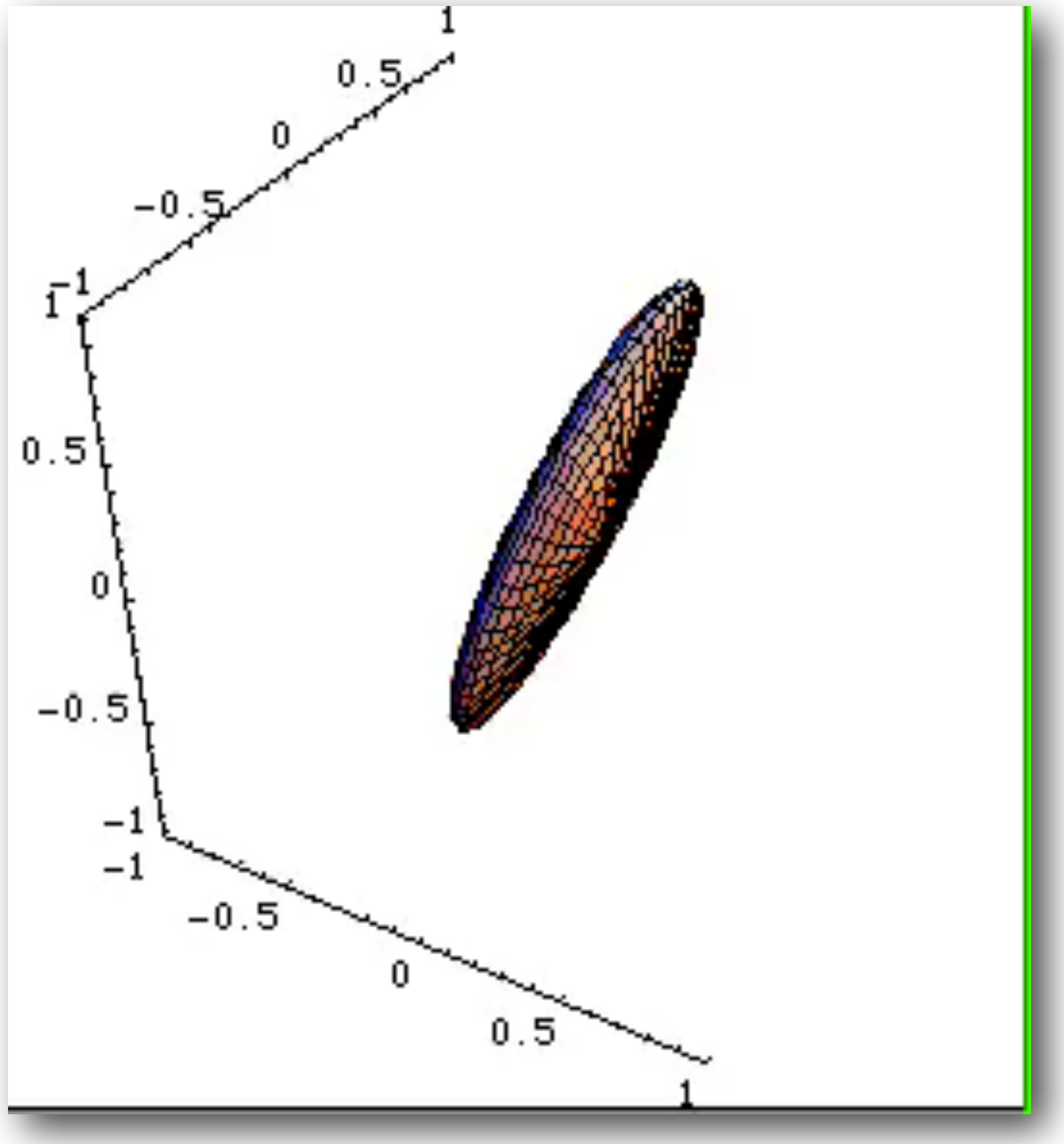


General Rotation of Rigid Body with $I_1 \geq I_2 \geq I_3$



Pure Spin

$\boldsymbol{\omega}_0 = (10^o, 0^o, 0^o)$ /s



Slightly Off Spin

$\boldsymbol{\omega}_0 = (10^o, 0.5^o, 0.5^o)$ /s

- Let's look at the Intermediate Energy Case:

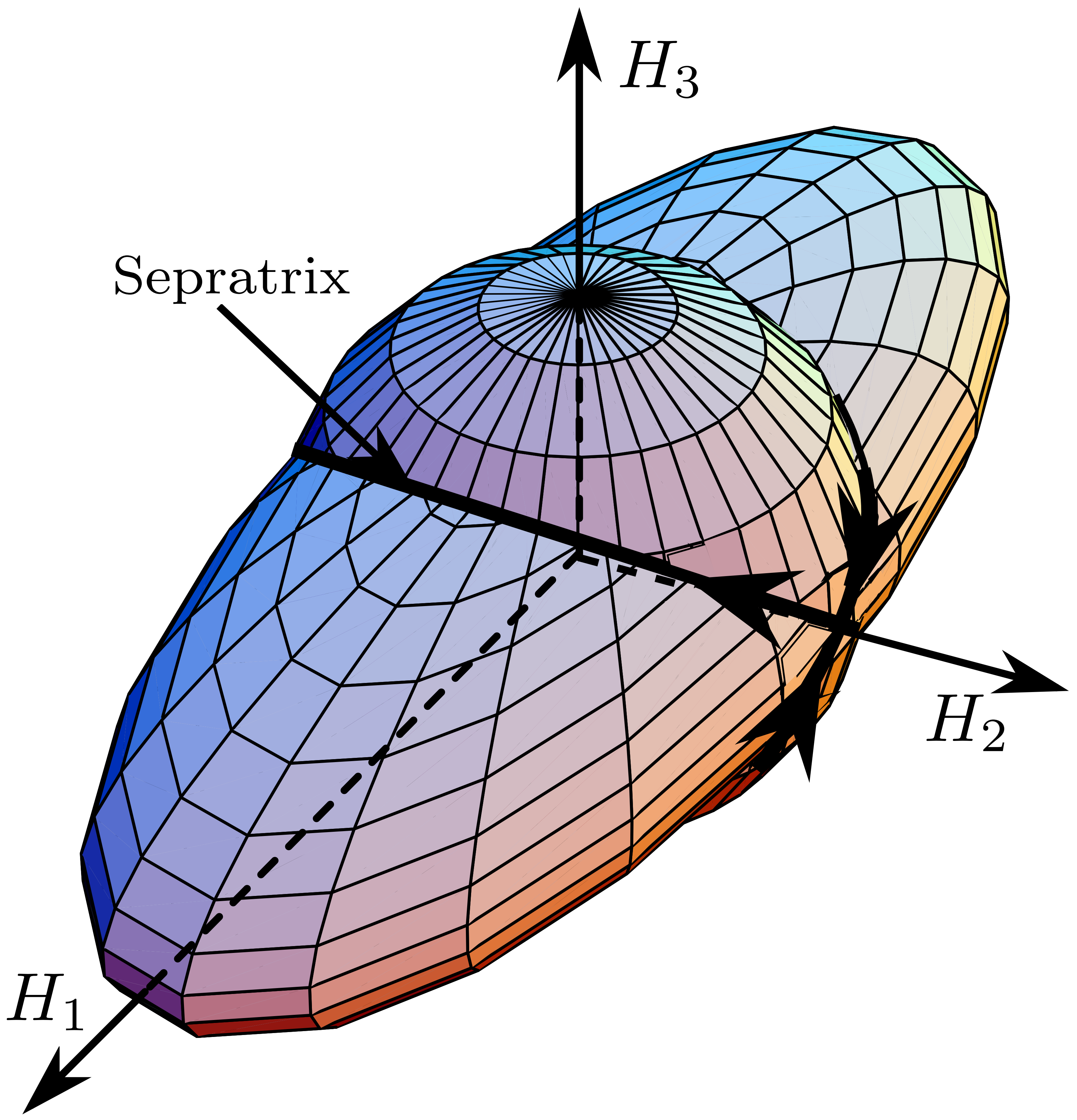
- The surfaces will only intersect at:

$$^{\mathcal{B}}\boldsymbol{H} = \pm H\hat{\boldsymbol{b}}_2$$

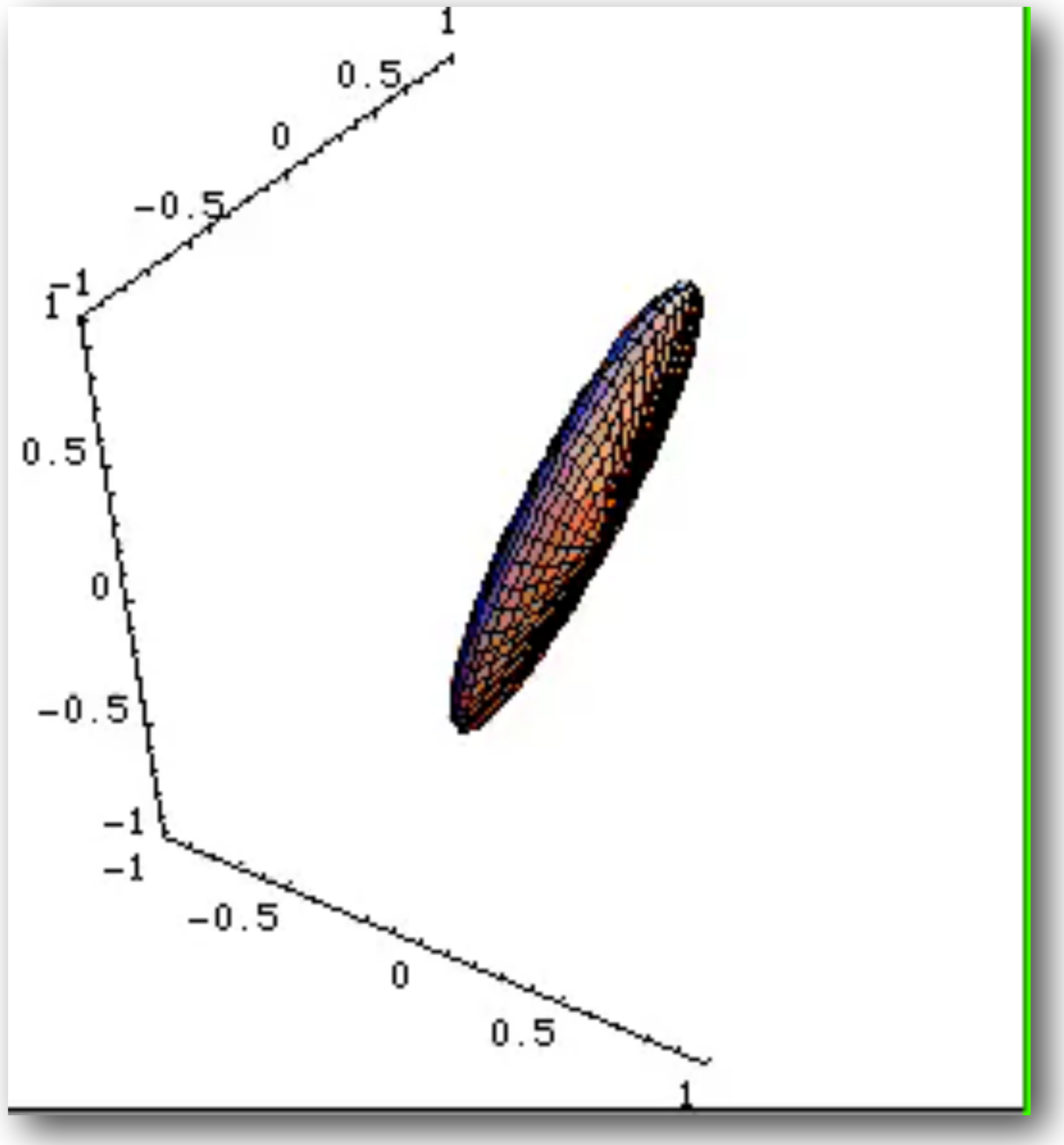
The kinetic energy is:

$$1 = \frac{H_1^2}{2I_1 T} + \frac{H_2^2}{2I_2 T} + \frac{H_3^2}{2I_3 T}$$

$$\boxed{T_{\text{int}} = \frac{H^2}{2I_2}}$$

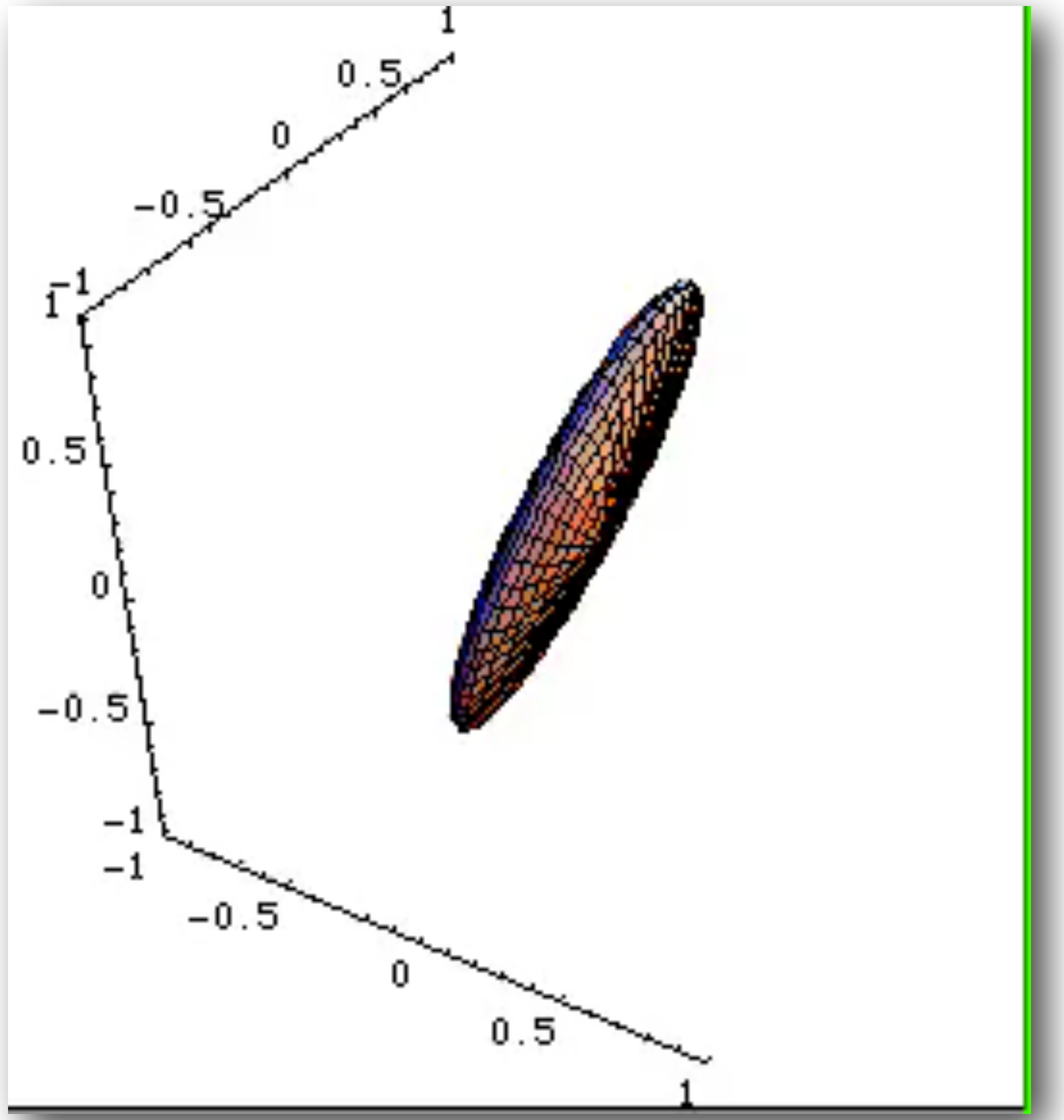


General Rotation of Rigid Body with $I_1 \geq I_2 \geq I_3$



Pure Spin

$$\boldsymbol{\omega}_0 = (0^o, 10^o, 0^o) \text{ /s}$$



Slightly Off Spin

$$\boldsymbol{\omega}_0 = (0.5^o, 10^o, 0.5^o) \text{ /s}$$

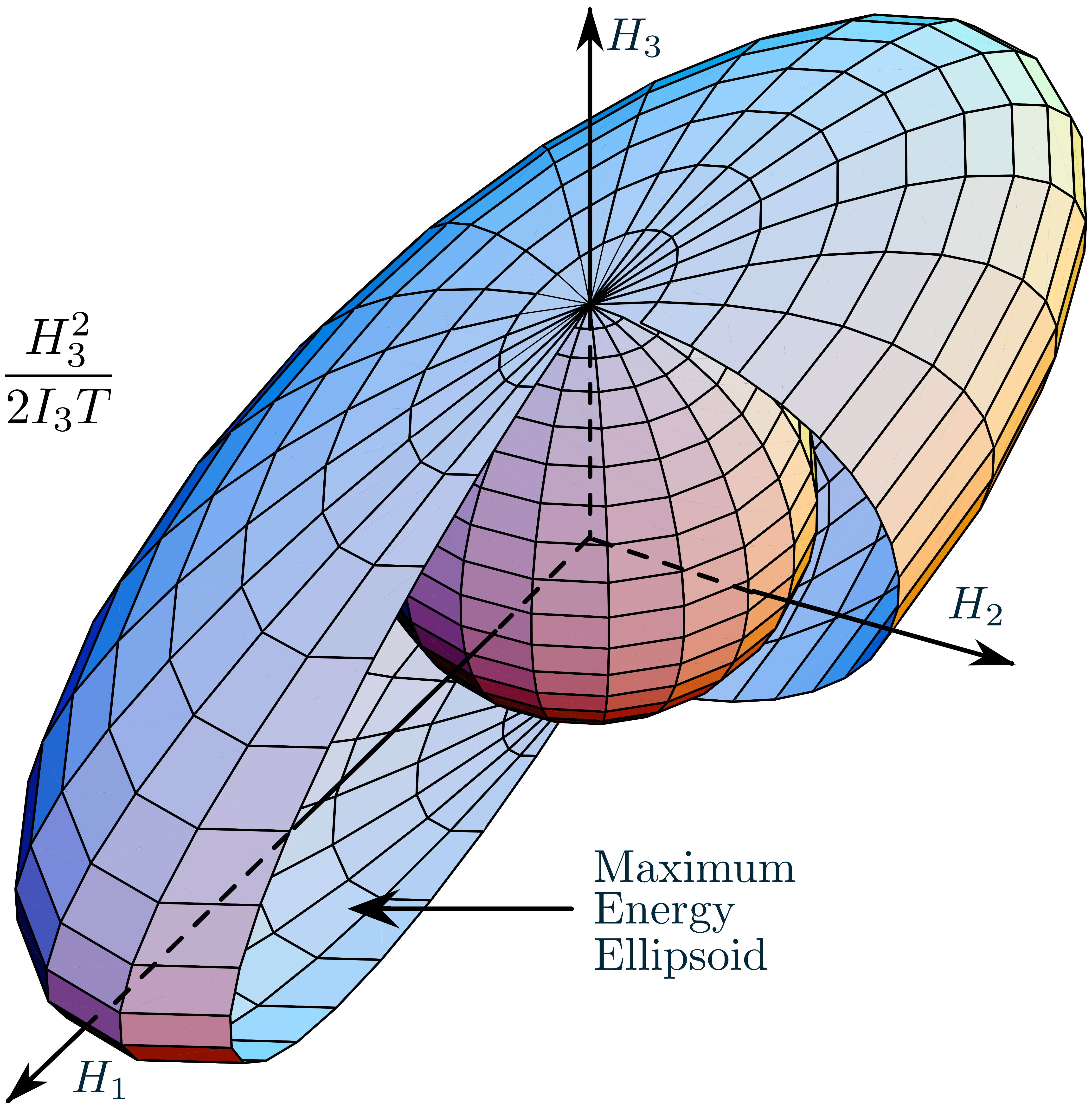
- Maximum Energy Case:

$$ {}^{\mathcal{B}}\boldsymbol{H} = H\hat{\boldsymbol{b}}_3 $$

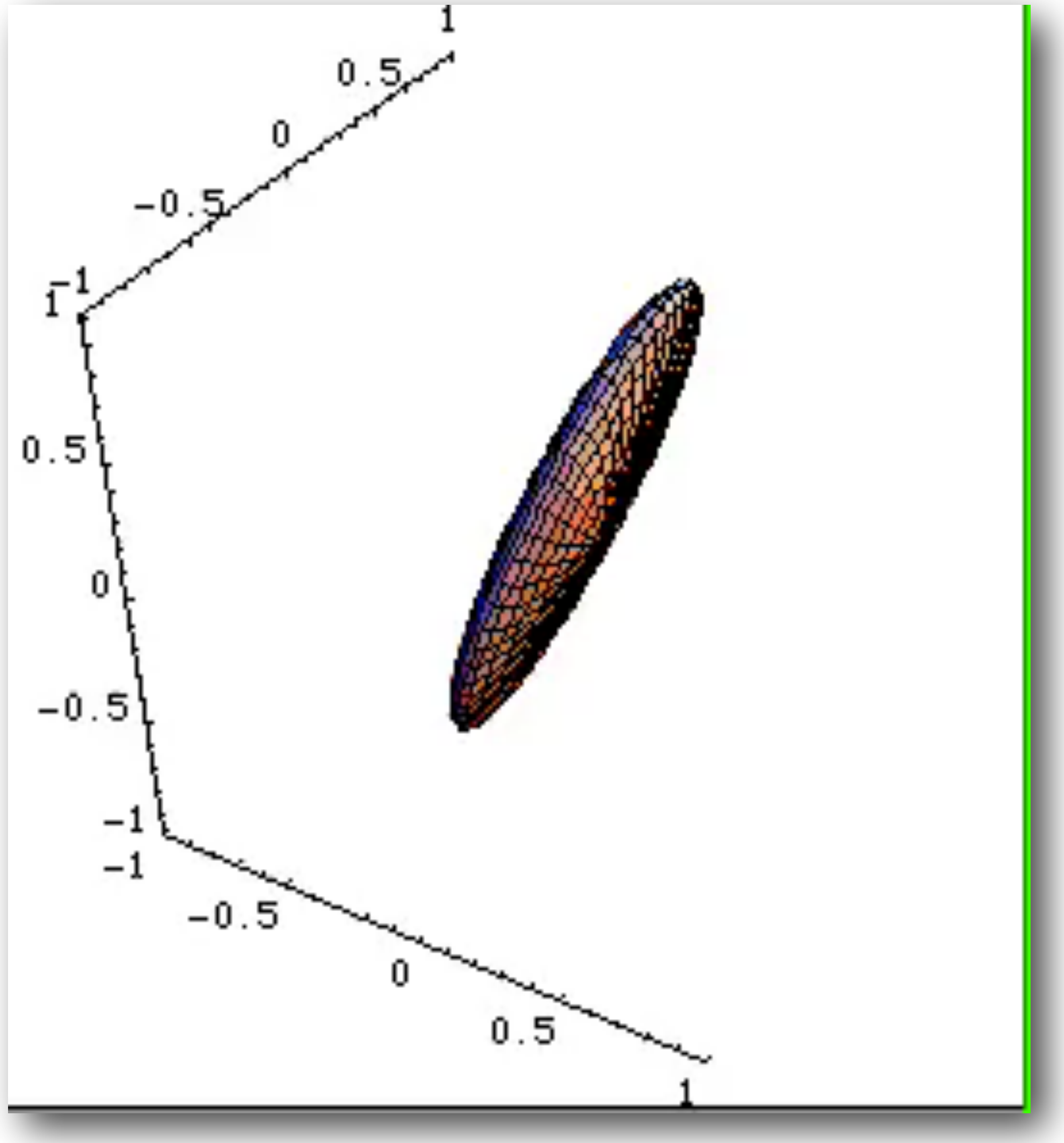
The kinetic energy is:

$$ 1 = \frac{H_1^2}{2I_1T} + \frac{H_2^2}{2I_2T} + \frac{H_3^2}{2I_3T} $$

$$ \boxed{T_{\max} = \frac{H^2}{2I_3}} $$

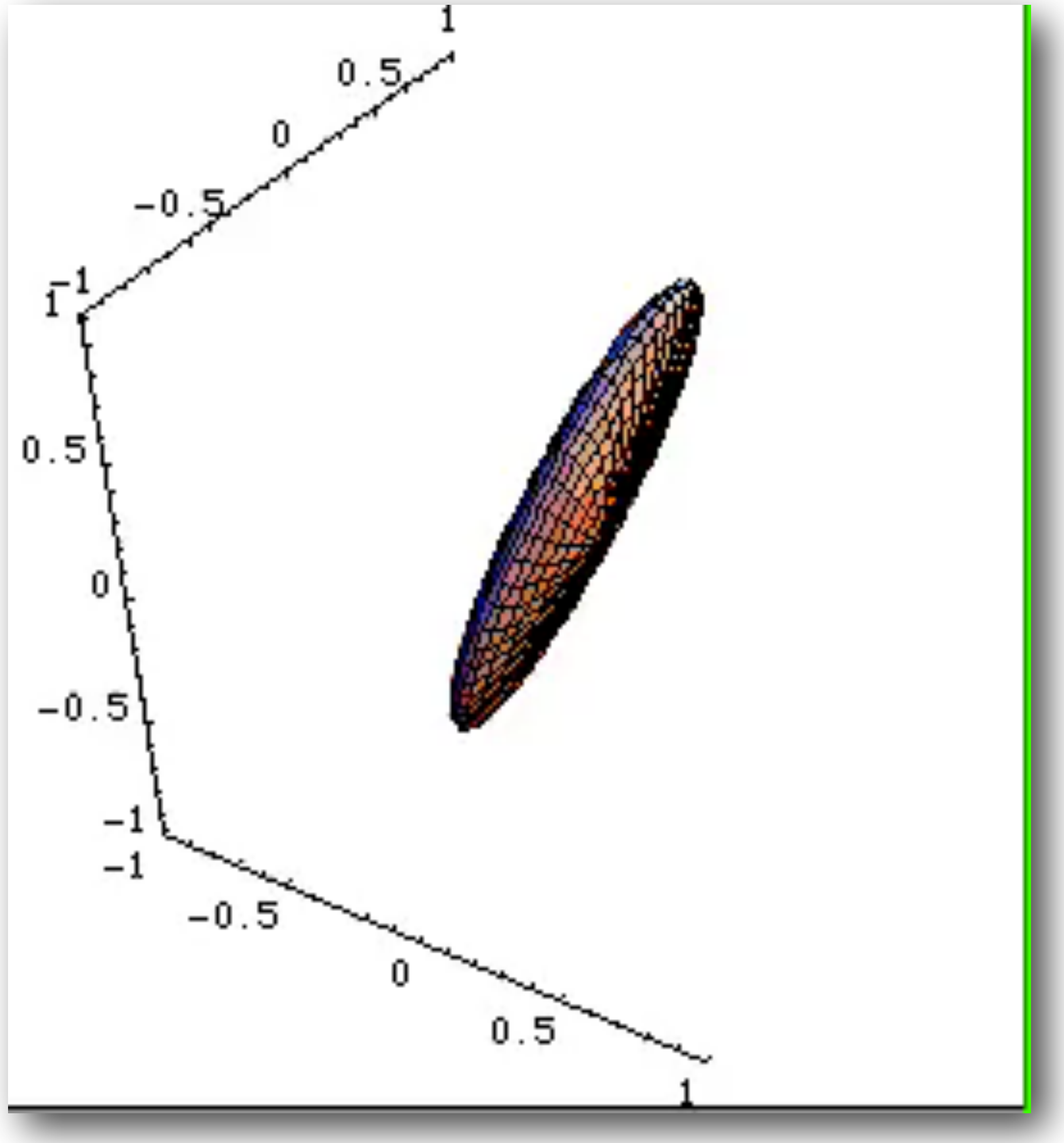


General Rotation of Rigid Body with $I_1 \geq I_2 \geq I_3$



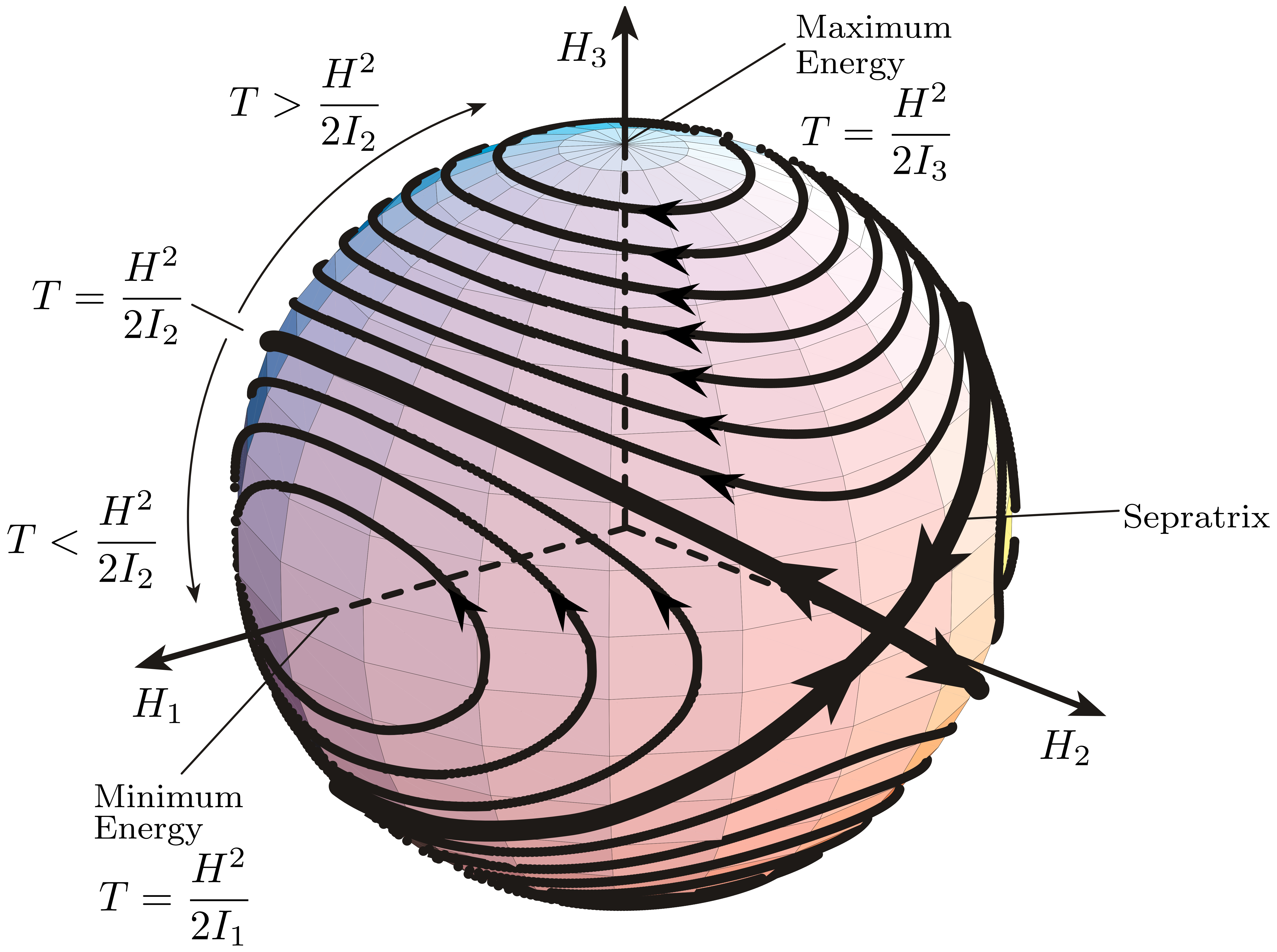
Pure Spin

$$\boldsymbol{\omega}_0 = (0^o, 0^o, 10^o)\ \text{/s}$$

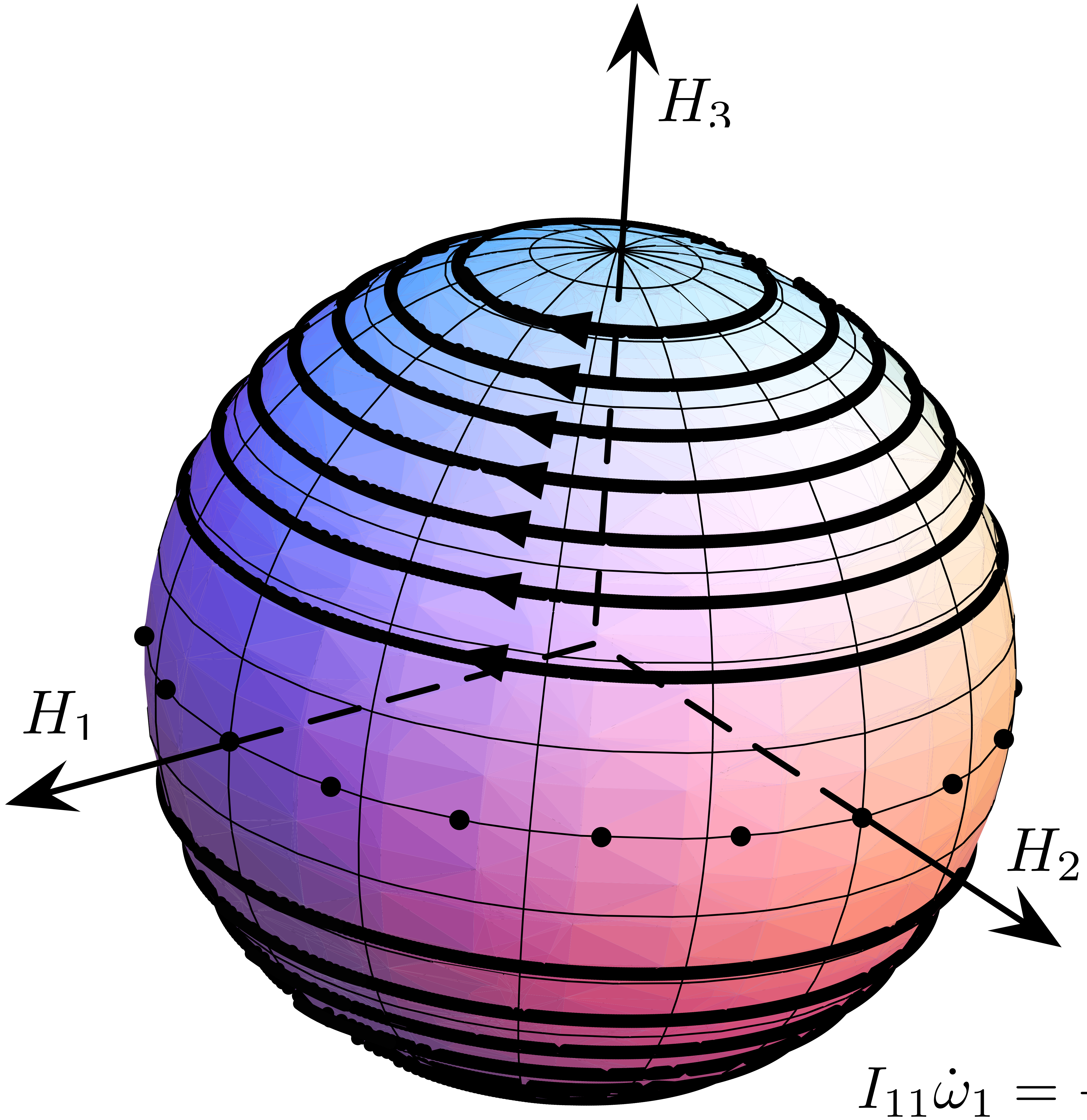


Slightly Off Spin

$$\boldsymbol{\omega}_0 = (0.5^o, 0.5^o, 10^o)\ \text{/s}$$

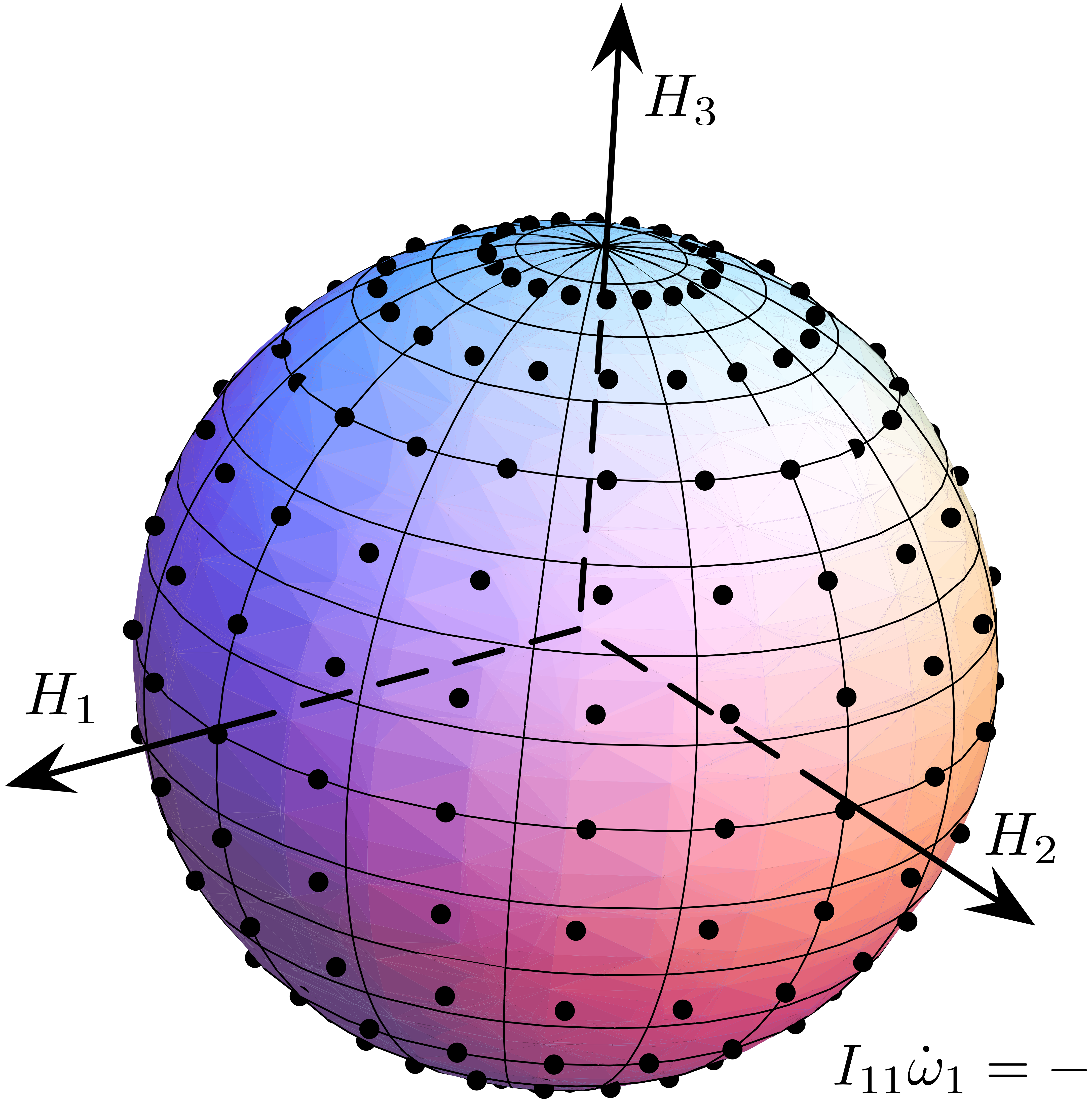


Family of energy ellipsoid and momentum sphere intersections.



What type of spacecraft body would yield this?

$$I_{11}\dot{\omega}_1 = -(I_{33} - I_{22})\omega_2\omega_3$$
$$I_{22}\dot{\omega}_2 = -(I_{11} - I_{33})\omega_3\omega_1$$
$$I_{33}\dot{\omega}_3 = -(I_{22} - I_{11})\omega_1\omega_2$$



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$$I_{11}\dot{\omega}_1 = -(I_{33} - I_{22})\omega_2\omega_3$$
$$I_{22}\dot{\omega}_2 = -(I_{11} - I_{33})\omega_3\omega_1$$
$$I_{33}\dot{\omega}_3 = -(I_{22} - I_{11})\omega_1\omega_2$$

Analytical Torque-Free Motion

- Let us assume that there are no external torques acting on the rigid body, and the equations of motion are given by:

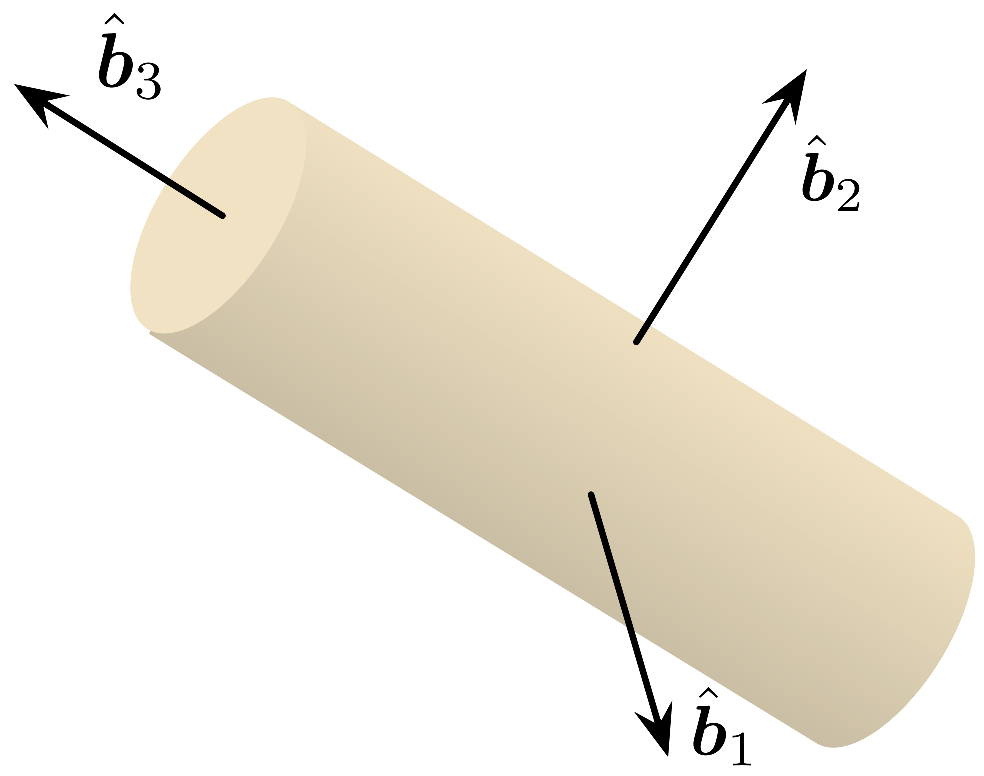
$$
\begin{aligned}
I_{11}\dot{\omega}_1 &= -(I_{33} - I_{22})\omega_2\omega_3 \\
I_{22}\dot{\omega}_2 &= -(I_{11} - I_{33})\omega_3\omega_1 \\
I_{33}\dot{\omega}_3 &= -(I_{22} - I_{11})\omega_1\omega_2
\end{aligned}
$$

- We are looking for analytical solutions to the angular motion.
- Assume that the body coordinate frame is a principal frame, and the inertia matrix is diagonal.

Axi-Symmetric Case

- Let the external torque be zero. Consider the special principal inertia case where

$$I_T = I_{11} = I_{22}$$



- Here the EOM are given by

$$\begin{aligned}
I_T \dot{\omega}_1 &= -(I_{33} - I_T)\omega_2\omega_3 \\
I_T \dot{\omega}_2 &= (I_{33} - I_T)\omega_3\omega_1 \\
I_{33} \dot{\omega}_3 &= 0
\end{aligned}$$

- From this equation it is clear that the third angular velocity component will be constant.

$$\omega_3(t) = \omega_3(t_0) = \text{constant}$$

- Let's examine the remaining two differential equations more carefully. Taking the derivative of the first one we find

$$I_t\dot{\omega}_1 = -(I_{33} - I_T)\omega_2\omega_3$$

$$I_T\ddot{\boldsymbol{\omega}}_1 = -(I_{33} - I_T)\dot{\omega}_2\omega_3$$

$$\dot{\omega}_2 = \frac{1}{I_T}((I_{33} - I_T)\omega_3\omega_1)$$

$$\ddot{\omega}_1 + (\frac{I_{33}}{I_T} - 1)^2\omega_3^2\omega_1 = 0$$

Mathematically equivalent to simple Spring-Mass Systems!

Similarly, we can find:

$$\ddot{\omega}_2 + (\frac{I_{33}}{I_T} - 1)^2\omega_3^2\omega_2 = 0$$

- The analytical solution to a spring-mass dynamical system is the simple oscillator equation

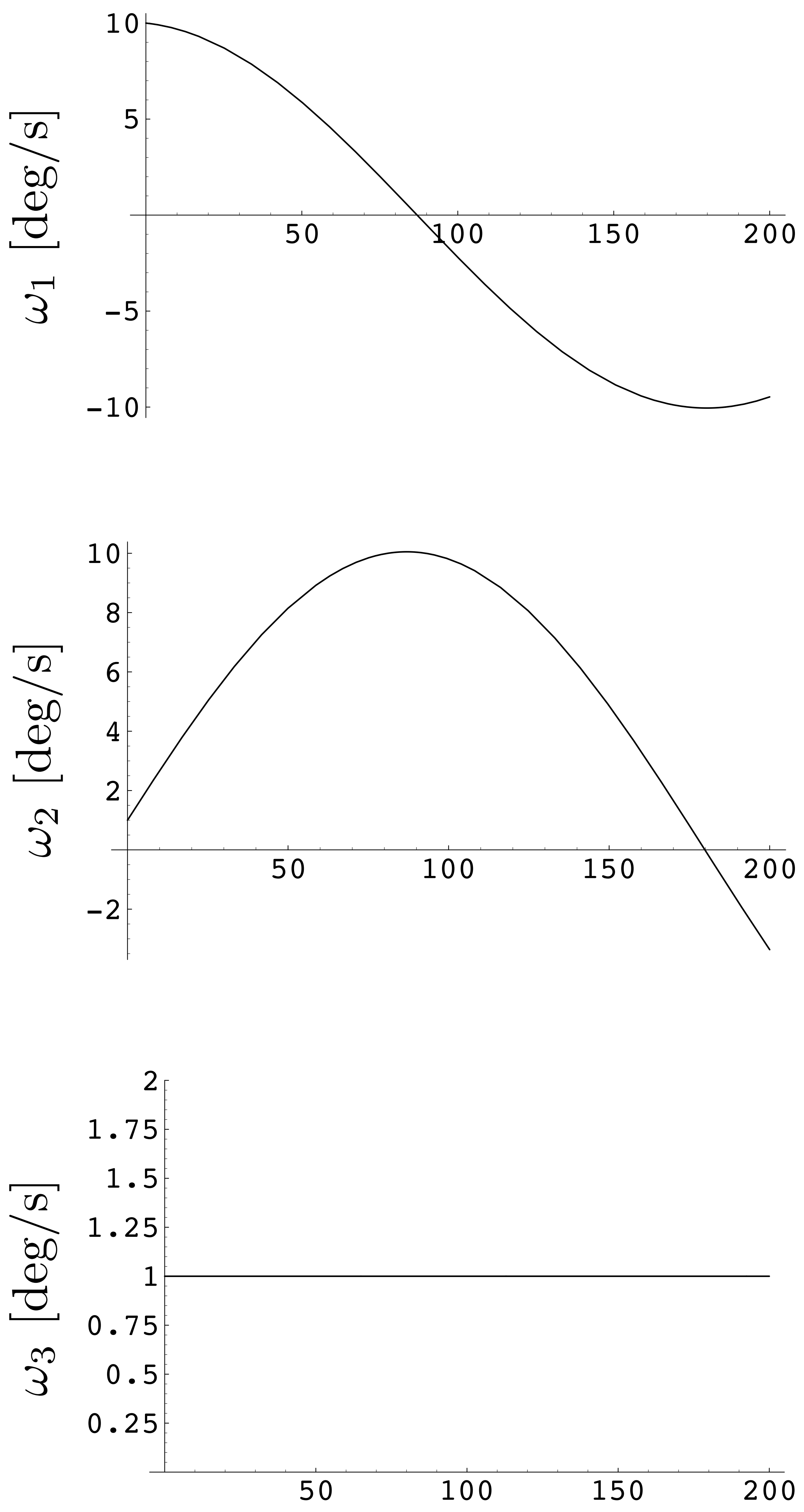
$$\omega_1(t) = A_1 \cos \omega_p t + B_1 \sin \omega_p t$$
$$\omega_2(t) = A_2 \cos \omega_p t + B_2 \sin \omega_p t$$

- Using the initial conditions, we find the analytical solution of the body angular velocity components for the axi-symmetric spacecraft case:

$$\omega_p = \left(\frac{I_{33}}{I_T} - 1 \right) \omega_3$$

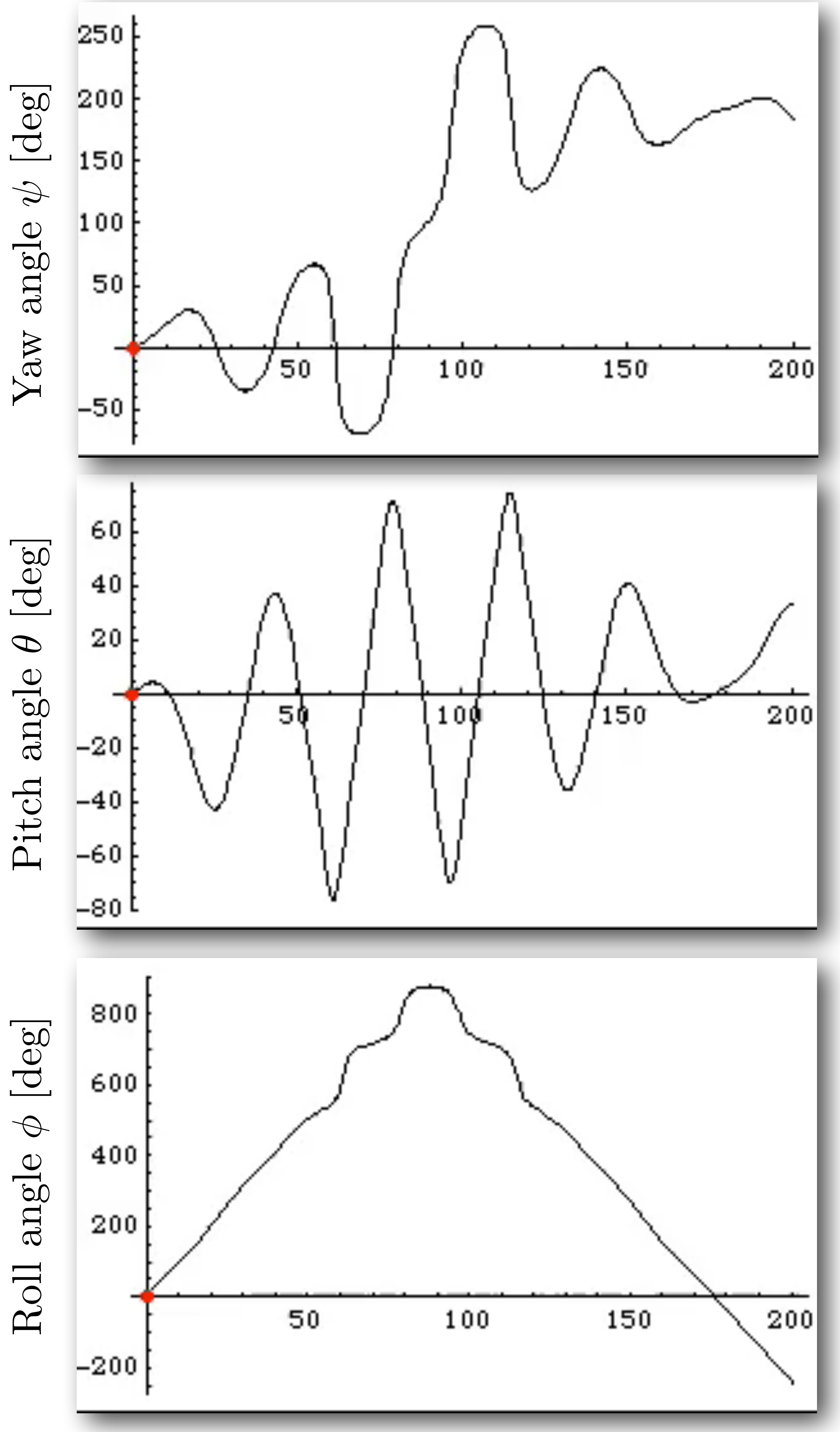
where

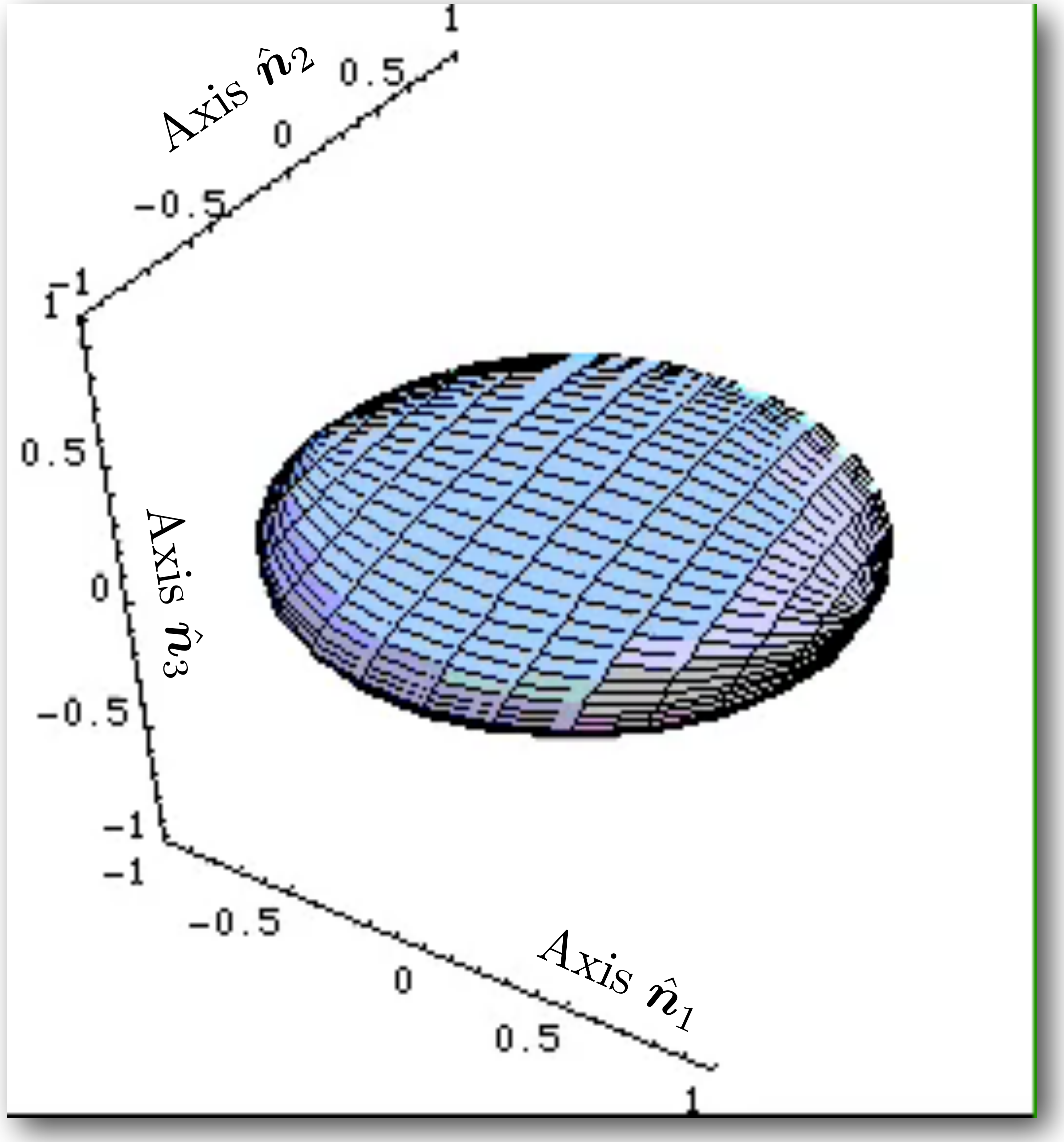
$$\omega_1(t) = \omega_{1_0} \cos \omega_p t - \omega_{2_0} \sin \omega_p t$$
$$\omega_2(t) = \omega_{2_0} \cos \omega_p t + \omega_{1_0} \sin \omega_p t$$
$$\omega_3(t) = \omega_{3_0}$$



The first and second body angular velocity components are sinusoidal in nature.

As predicted, the third body angular velocity component remains constant here.


Yaw angle ψ [deg]
250
200
150
100
50
-50
50
100
150
200
Pitch angle θ [deg]
60
40
20
-20
-40
-60
-80
50
100
150
200
Roll angle ϕ [deg]
800
600
400
200
-200
50
100
150
200


Axis $\hat{n}_2$
Axis $\hat{n}_3$
Axis $\hat{n}_1$
1
0.5
0
-0.5
-1

General Inertia Case*

$$H^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2$$

$$2T = I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2$$

Momentum magnitude and kinetic energy conservation yield two integrals of the torque-free motion.

$$\omega_2^2 = \left(\frac{2I_3T - H^2}{I_2(I_3 - I_2)}\right) - \frac{I_1(I_3 - I_1)}{I_2(I_3 - I_2)}\omega_1^2$$

$$\omega_3^2 = \left(\frac{2I_2T - H^2}{I_3(I_2 - I_3)}\right) - \frac{I_1(I_2 - I_1)}{I_3(I_2 - I_3)}\omega_1^2$$

We can use these two equations to solve for two of the angular rates!

Analogously, we can solve for the two angular velocities in terms of other angular rates.

$$\omega_1^2 = \left(\frac{2I_3T - H^2}{I_1(I_3 - I_1)}\right) - \frac{I_2(I_3 - I_2)}{I_1(I_3 - I_1)}\omega_2^2$$

$$\omega_3^2 = \left(\frac{2I_1T - H^2}{I_3(I_1 - I_3)}\right) - \frac{I_2(I_1 - I_2)}{I_3(I_1 - I_3)}\omega_2^2$$

$$\omega_1^2 = \left(\frac{2I_2T - H^2}{I_1(I_2 - I_1)}\right) - \frac{I_3(I_2 - I_3)}{I_1(I_2 - I_1)}\omega_3^2$$

$$\omega_2^2 = \left(\frac{2I_1T - H^2}{I_2(I_1 - I_2)}\right) - \frac{I_3(I_1 - I_3)}{I_2(I_1 - I_2)}\omega_3^2$$

* Junkins, J. L., Jacobson, I. D., and Blanton, J. N., "A Nonlinear Oscillator Analog of Rigid Body Dynamics," *Celestial Mechanics*, Vol. 7, pp. 398 – 407, 1973.

$$\boxed{\begin{aligned} I_1\dot{\omega}_1 &= -(I_3 - I_2)\omega_2\omega_3 \\ I_2\dot{\omega}_2 &= -(I_1 - I_3)\omega_3\omega_1 \\ I_3\dot{\omega}_3 &= -(I_2 - I_1)\omega_1\omega_2 \end{aligned}}$$

$$\xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}}$$

$$\begin{aligned} \ddot{\omega}_1 &= \frac{I_2 - I_3}{I_1}\left(\boxed{\dot{\omega}_2}\omega_3 + \omega_2\boxed{\dot{\omega}_3}\right) \\ \ddot{\omega}_2 &= \frac{I_3 - I_1}{I_2}\left(\boxed{\dot{\omega}_3}\omega_1 + \omega_3\boxed{\dot{\omega}_1}\right) \\ \ddot{\omega}_3 &= \frac{I_1 - I_2}{I_3}\left(\boxed{\dot{\omega}_1}\omega_2 + \omega_1\boxed{\dot{\omega}_2}\right) \end{aligned}$$

$$\begin{aligned} \ddot{\omega}_1 &= \frac{I_2 - I_3}{I_1}\left(\frac{I_1 - I_2}{I_3}\omega_1\omega_2^2 + \frac{I_3 - I_1}{I_2}\omega_1\omega_3^2\right) \\ \ddot{\omega}_2 &= \frac{I_3 - I_1}{I_2}\left(\frac{I_1 - I_2}{I_3}\omega_2\omega_1^2 + \frac{I_2 - I_3}{I_1}\omega_2\omega_3^2\right) \\ \ddot{\omega}_3 &= \frac{I_1 - I_2}{I_3}\left(\frac{I_3 - I_1}{I_2}\omega_3\omega_1^2 + \frac{I_2 - I_3}{I_1}\omega_3\omega_2^2\right) \end{aligned}$$

$$\ddot{\omega}_1 = \frac{I_2 - I_3}{I_1}\left(\frac{I_1 - I_2}{I_3}\omega_1\boxed{\omega_2^2} + \frac{I_3 - I_1}{I_2}\omega_1\boxed{\omega_3^2}\right)$$

$$\ddot{\omega}_2 = \frac{I_3 - I_1}{I_2}\left(\frac{I_1 - I_2}{I_3}\omega_2\boxed{\omega_1^2} + \frac{I_2 - I_3}{I_1}\omega_2\boxed{\omega_3^2}\right)$$

$$\ddot{\omega}_3 = \frac{I_1 - I_2}{I_3}\left(\frac{I_3 - I_1}{I_2}\omega_3\boxed{\omega_1^2} + \frac{I_2 - I_3}{I_1}\omega_3\boxed{\omega_2^2}\right)$$

$$\omega_2^2 = \left(\frac{2I_3T - H^2}{I_2(I_3 - I_2)}\right) - \frac{I_1(I_3 - I_1)}{I_2(I_3 - I_2)}\omega_1^2$$

$$\omega_3^2 = \left(\frac{2I_2T - H^2}{I_3(I_2 - I_3)}\right) - \frac{I_1(I_2 - I_1)}{I_3(I_2 - I_3)}\omega_1^2$$

$$\omega_1^2 = \left(\frac{2I_3T - H^2}{I_1(I_3 - I_1)}\right) - \frac{I_2(I_3 - I_2)}{I_1(I_3 - I_1)}\omega_2^2$$

$$\omega_3^2 = \left(\frac{2I_1T - H^2}{I_3(I_1 - I_3)}\right) - \frac{I_2(I_1 - I_2)}{I_3(I_1 - I_3)}\omega_2^2$$

$$\omega_1^2 = \left(\frac{2I_2T - H^2}{I_1(I_2 - I_1)}\right) - \frac{I_3(I_2 - I_3)}{I_1(I_2 - I_1)}\omega_3^2$$

$$\omega_2^2 = \left(\frac{2I_1T - H^2}{I_2(I_1 - I_2)}\right) - \frac{I_3(I_1 - I_3)}{I_2(I_1 - I_2)}\omega_3^2$$

$$\ddot{\omega}_i + A_i\omega_i + B_i\omega_i^3 = 0 \qquad \text{for } i = 1, 2, 3$$

homogenous, undamped Duffing equation

Duffing equations are often found studying nonlinear mechanical oscillations, where the cubic “stiffness” term arises to approximately account for nonlinear departure from Hooke’s law. For the torque-free motion, this equation is the *exact differential equation*!

$$\ddot{\omega}_i + A_i\omega_i + B_i\omega_i^3 = 0 \qquad \text{for } i = 1, 2, 3$$

- These equations form three *uncoupled nonlinear oscillators*.
- Notice that while the oscillators are *uncoupled*, they are not *independent*! The six spring constants are all uniquely determined from initially evaluated inertia, energy and momentum constants.

i	A_i	B_i
1	$\dfrac{(I_1 - I_2)(2I_3T - H^2) + (I_1 - I_3)(2I_2T - H^2)}{I_1I_2I_3}$	$\dfrac{2(I_1 - I_2)(I_1 - I_3)}{I_2I_3}$
2	$\dfrac{(I_2 - I_3)(2I_1T - H^2) + (I_2 - I_1)(2I_3T - H^2)}{I_1I_2I_3}$	$\dfrac{2(I_2 - I_1)(I_2 - I_3)}{I_1I_3}$
3	$\dfrac{(I_3 - I_1)(2I_2T - H^2) + (I_3 - I_2)(2I_1T - H^2)}{I_1I_2I_3}$	$\dfrac{2(I_3 - I_1)(I_3 - I_2)}{I_1I_2}$

- The oscillator differential equations have three immediate integrals of the form

$$\dot{\omega}_i^2 + A_i\omega_i^2 + \frac{B_i}{2}\omega_i^4 = K_i \qquad \text{for } i = 1, 2, 3$$

- Here K_1, K_2 and K_3 are the three oscillator "energy-type" integral constants of the motion.

$$K_1 = \frac{(2I_2T - H^2)(H^2 - 2I_3T)}{I_1^2 I_2 I_3}$$

$$K_2 = \frac{(2I_3T - H^2)(H^2 - 2I_1T)}{I_1 I_2^2 I_3}$$

$$K_3 = \frac{(2I_1T - H^2)(H^2 - 2I_2T)}{I_1 I_2 I_3^2}$$

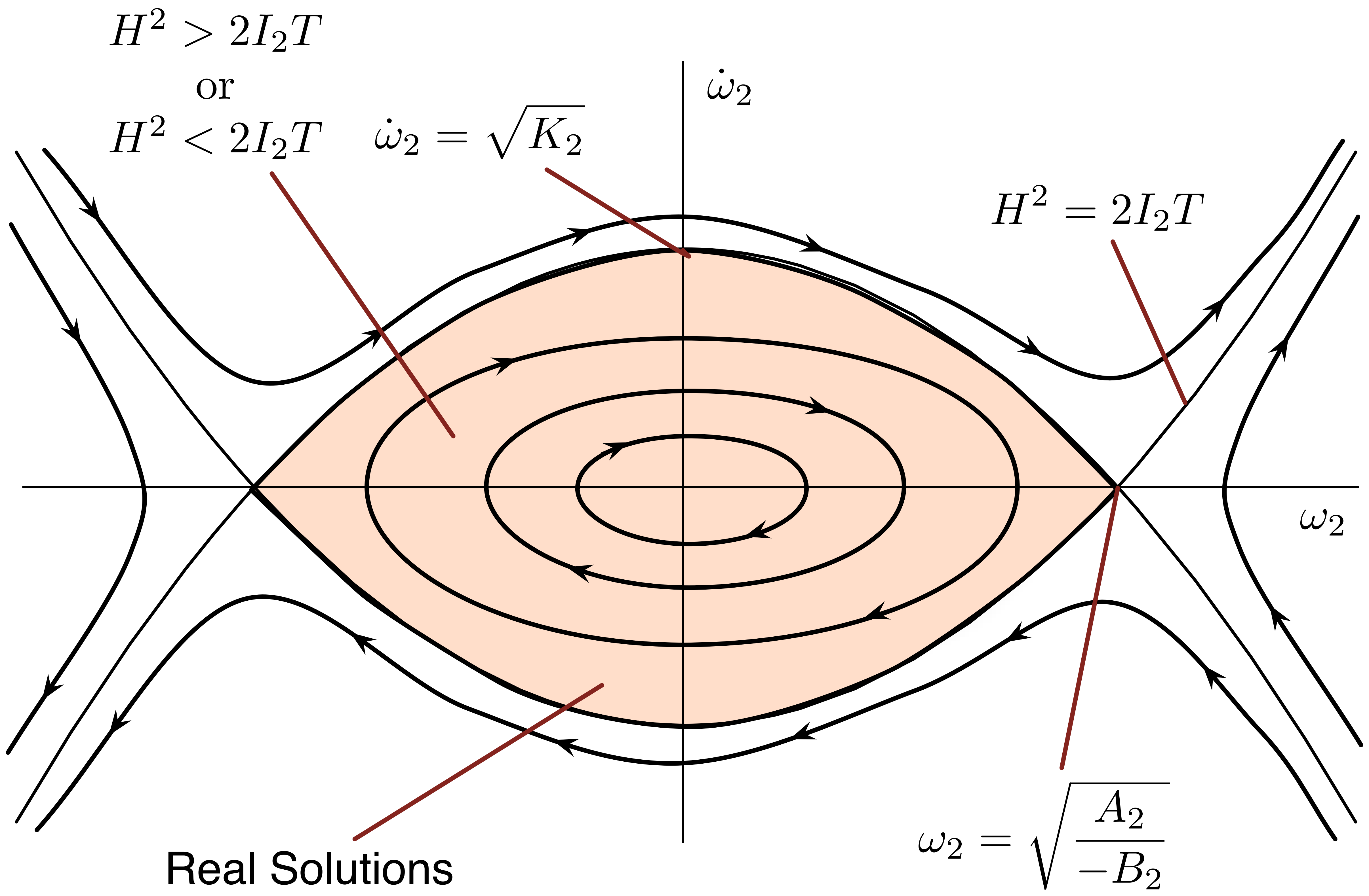
Assume: $I_1 \geq I_2 \geq I_3$

i	A_i	B_i
1	not defined	>0
2	>0	<0
3	not defined	>0

- The linear "spring constants" A_1 and A_3 can produce de-stabilizing spring forces (negative spring effect).

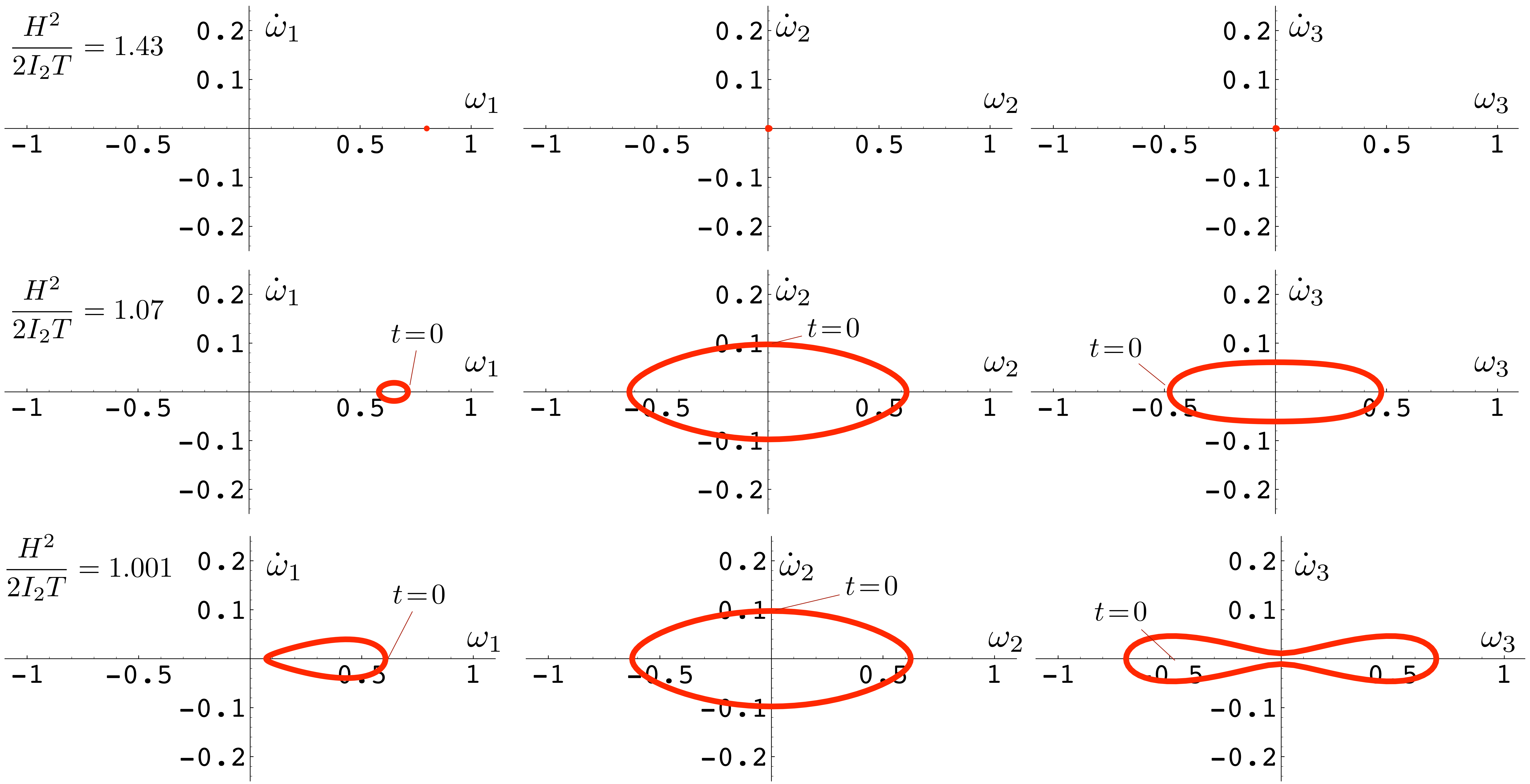
- The positive cubic "spring constants" B_1 and B_3 always produce restoring forces and are therefore hard springs. Because cubic springs will override linear springs for sufficiently large displacements, all trajectories of the 1st and 3rd phase planes must be closed.

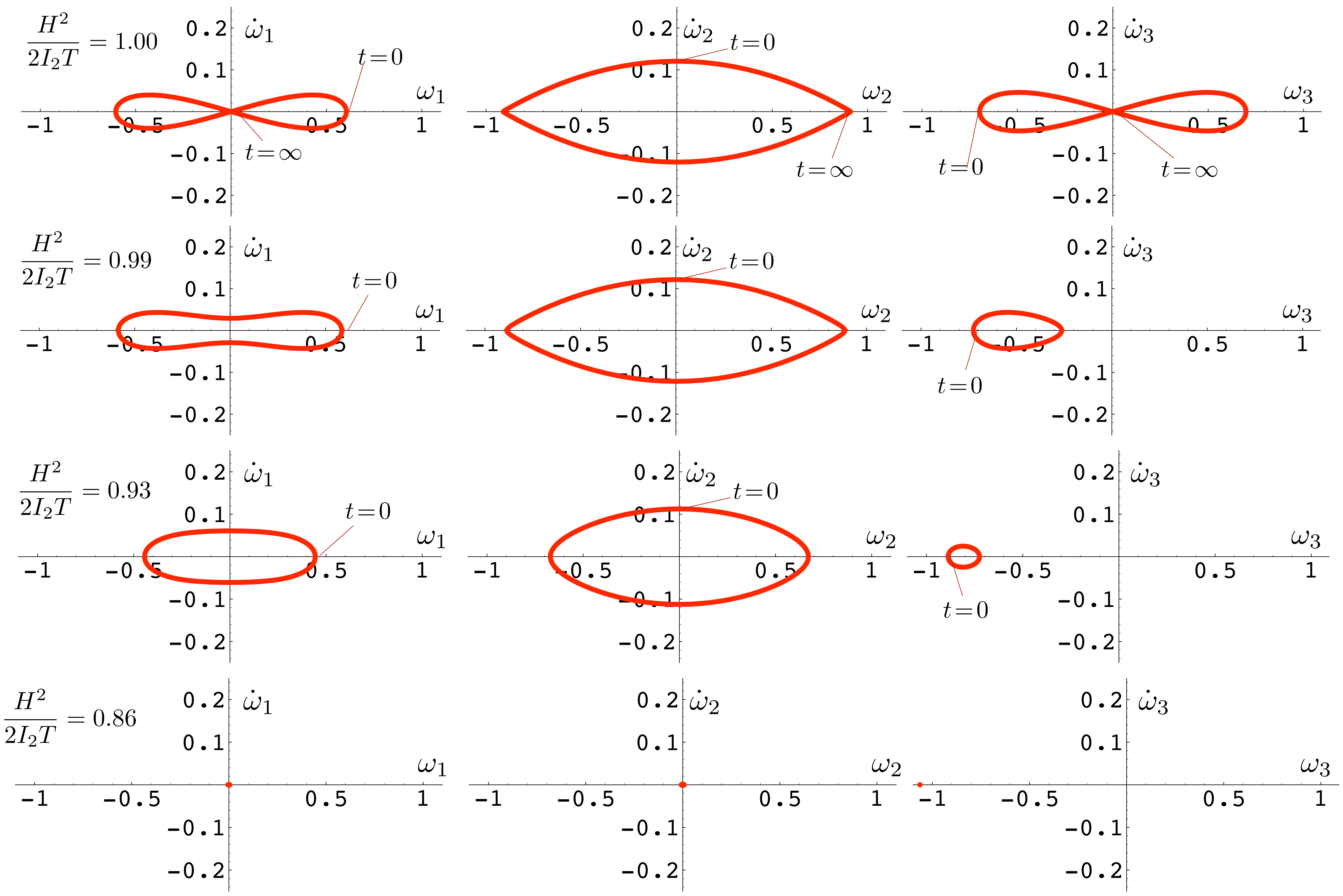
- The cubic spring constant B_2 produces a de-stabilizing force (soft spring), and will eventually override the stabilizing linear spring force.



- Only solutions with $K_2 \geq 0$ are physically possible
- The limiting trajectory occurs if
  - $I_1 \to I_3$
  - $H^2 \to 2\, I_2 T$ (pure spin about intermediate inertia axis)
  - $I_1 I_2 I_3 \to \infty$

Let's sweep through cases from a minimum energy case to a maximum energy case. The momentum is held constant here.



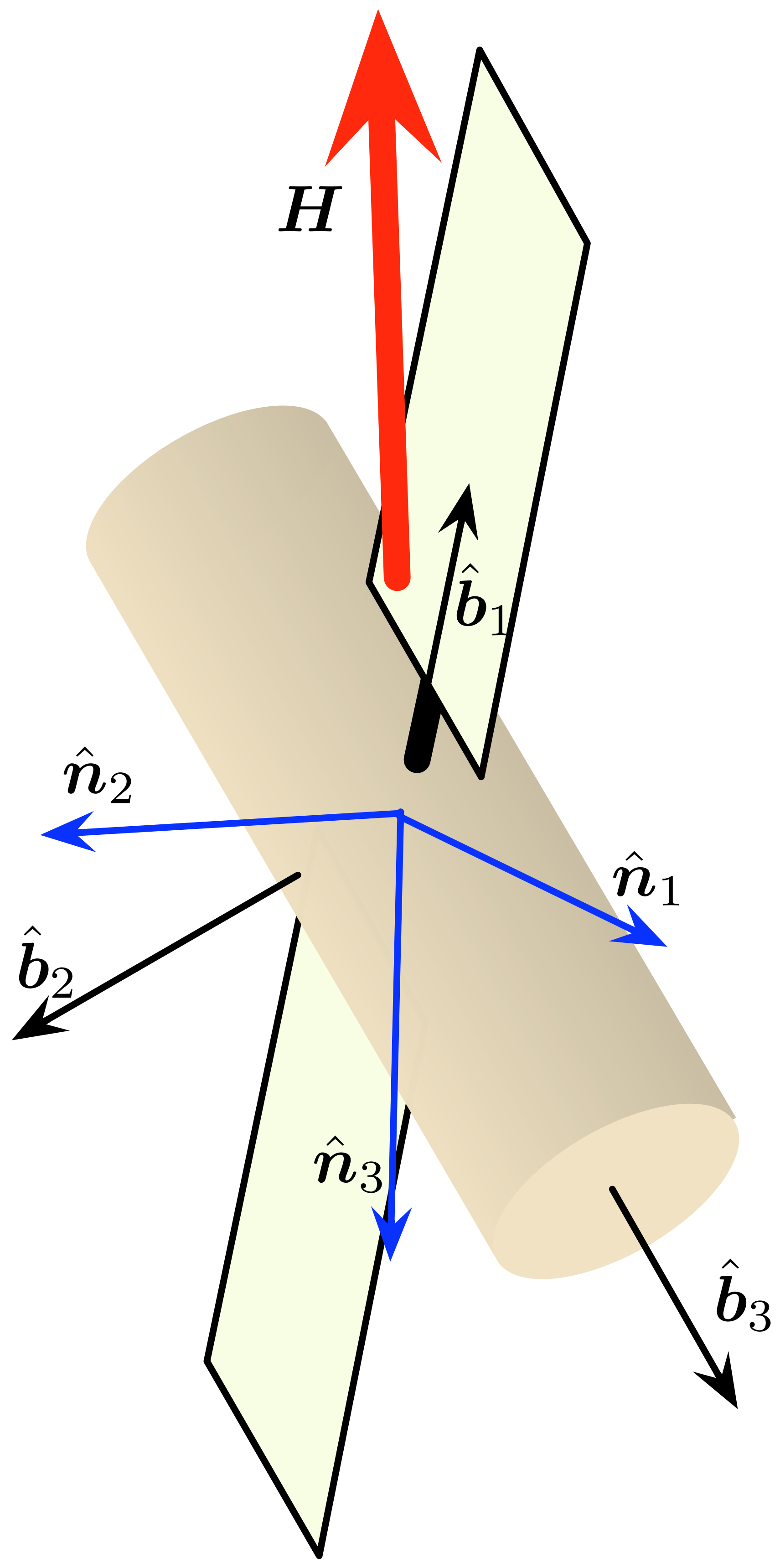


$\frac{H^2}{2I_2T} = 1.00$
$\frac{H^2}{2I_2T} = 0.99$
$\frac{H^2}{2I_2T} = 0.93$
$\frac{H^2}{2I_2T} = 0.86$
$\dot{\omega}_1$
ω_1
$\dot{\omega}_2$
ω_2
$\dot{\omega}_3$
ω_3
$t=0$
$t=\infty$

General Free Rotation

- We would like to study the general free rotation of a rigid body using the 3-2-1 Euler angles.
- Because the inertial angular momentum vector $\boldsymbol{H}$ is constant as seen by the inertial frame, we can always align our inertial frame such that
-
$$\boldsymbol{H} = {}^{\mathcal{N}}\boldsymbol{H} = -H\hat{\boldsymbol{n}}_3 = {}^{\mathcal{N}}\!\begin{pmatrix} 0 \\ 0 \\ -H \end{pmatrix}$$
- Using the rotation matrix [*BN*], we find

$${}^{\mathcal{B}}\boldsymbol{H} = [BN]\,{}^{\mathcal{N}}\boldsymbol{H}$$



- Recall the mapping between the rotation matrix [BN] and the 3-2-1 Euler angles:

$$[BN] = \begin{bmatrix} c\theta_2 c\theta_1 & c\theta_2 s\theta_1 & -s\theta_2 \\ s\theta_3 s\theta_2 c\theta_1 - c\theta_3 s\theta_1 & s\theta_3 s\theta_2 s\theta_1 + c\theta_3 c\theta_1 & s\theta_3 c\theta_2 \\ c\theta_3 s\theta_2 c\theta_1 + s\theta_3 s\theta_1 & c\theta_3 s\theta_2 s\theta_1 - s\theta_3 c\theta_1 & c\theta_3 c\theta_2 \end{bmatrix}$$

This leads to

$$^{\mathcal{B}}\boldsymbol{H} = \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} = [BN]\, ^{\mathcal{N}}\boldsymbol{H} = \begin{pmatrix} H \sin\theta \\ -H \sin\phi \cos\theta \\ -H \cos\phi \cos\theta \end{pmatrix} = \begin{pmatrix} I_1 \omega_1 \\ I_2 \omega_2 \\ I_3 \omega_3 \end{pmatrix}$$

Which can be solved for the rigid body angular velocity.

$$\begin{pmatrix} \frac{H}{I_1} \sin\theta \\ -\frac{H}{I_2} \sin\phi \cos\theta \\ -\frac{H}{I_3} \cos\phi \cos\theta \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

- Recall the 3-2-1 Euler angle differential kinematic equation:

$$\boldsymbol{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{bmatrix} -\sin\theta & 0 & 1 \\ \sin\phi\cos\theta & \cos\phi & 0 \\ \cos\phi\cos\theta & -\sin\phi & 0 \end{bmatrix} \begin{pmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Solving these equations for the Euler angle rates, we obtain:

$$\dot{\psi} = -H\left(\frac{\sin^2\phi}{I_2} + \frac{\cos^2\phi}{I_3}\right) \Rightarrow \text{cannot be positive}$$

$$\dot{\theta} = \frac{H}{2}\left(\frac{1}{I_3} - \frac{1}{I_2}\right)\sin 2\phi\cos\theta$$

$$\dot{\phi} = H\left(\frac{1}{I_1} - \frac{\sin^2\phi}{I_2} - \frac{\cos^2\phi}{I_3}\right)\sin\theta$$

These are the spinning top equations of motion.

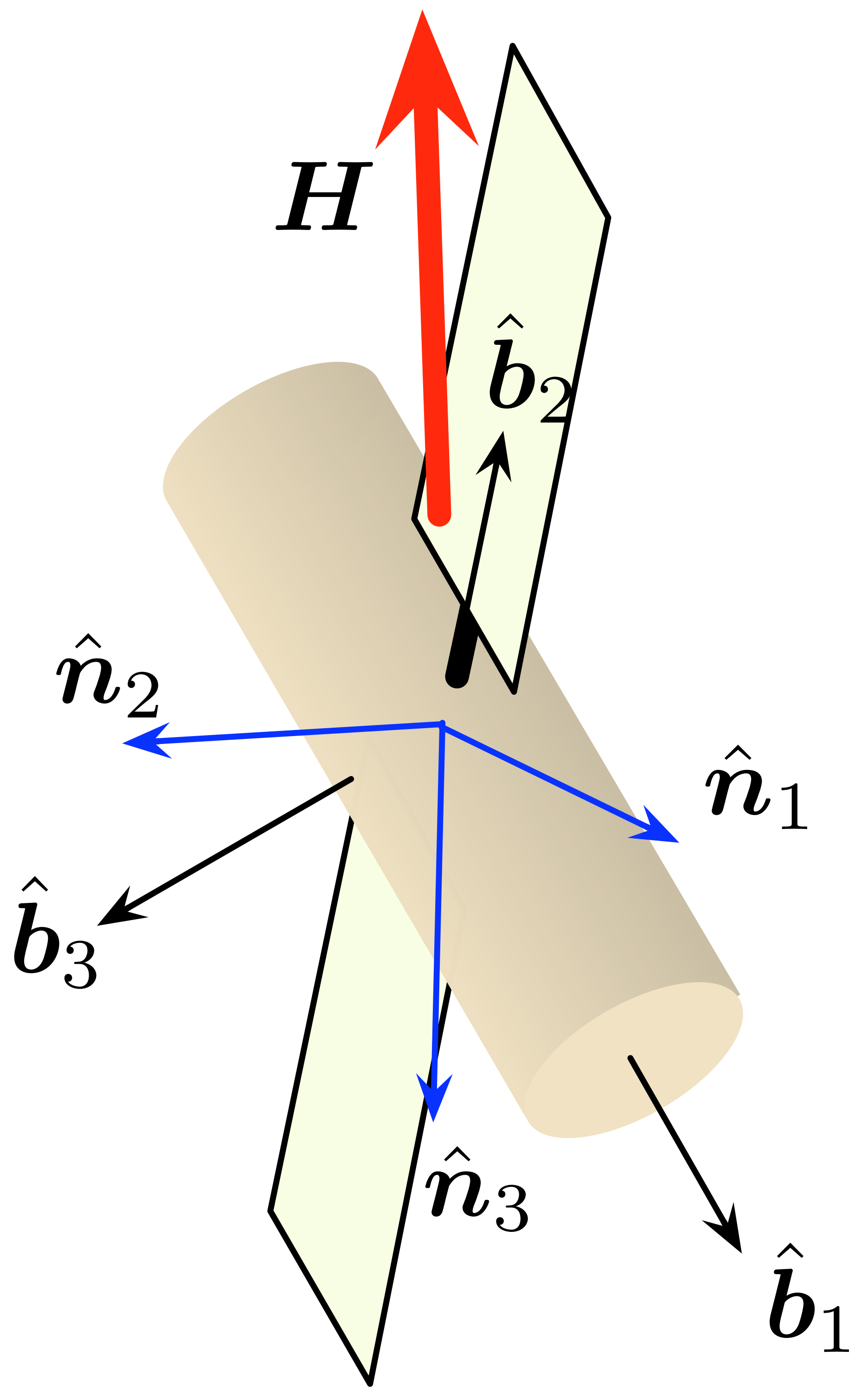
Axi-Symmetric Coning Motion

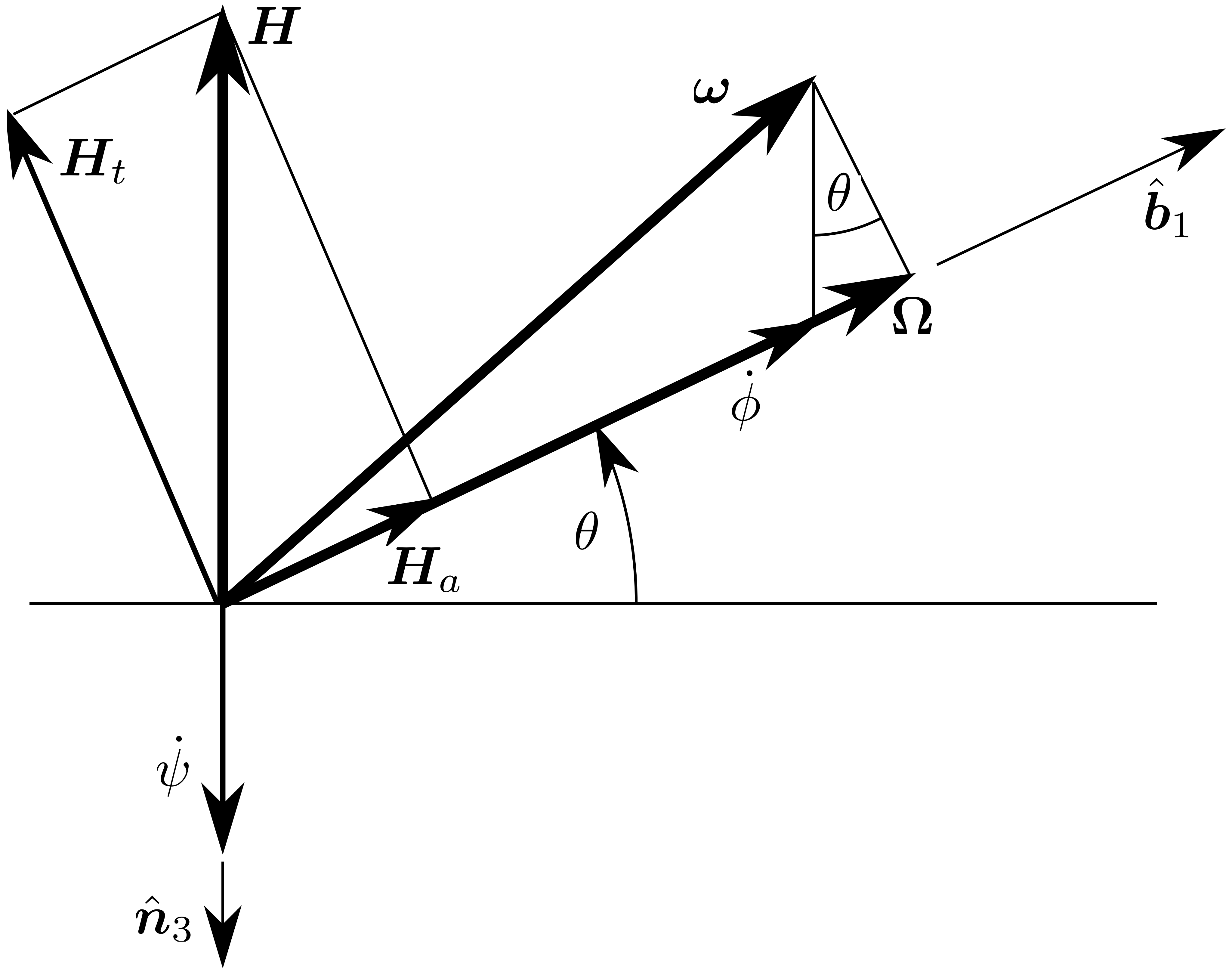
- Assume the spacecraft is axi-symmetric with $I_2 = I_3$, and align the inertial frame such that

$$\boldsymbol{H} = {}^{\mathcal{N}}\boldsymbol{H} = -H\hat{\boldsymbol{n}}_3 = {}^{\mathcal{N}}\begin{pmatrix} 0 \\ 0 \\ -H \end{pmatrix}$$

- The 3-2-1 Euler angle differential equation are then given by:

$$\dot{\psi} = -\frac{H}{I_2}$$
$$\dot{\theta} = 0$$
$$\dot{\phi} = H\left(\frac{I_2 - I_1}{I_1 I_2}\right)\sin\theta$$





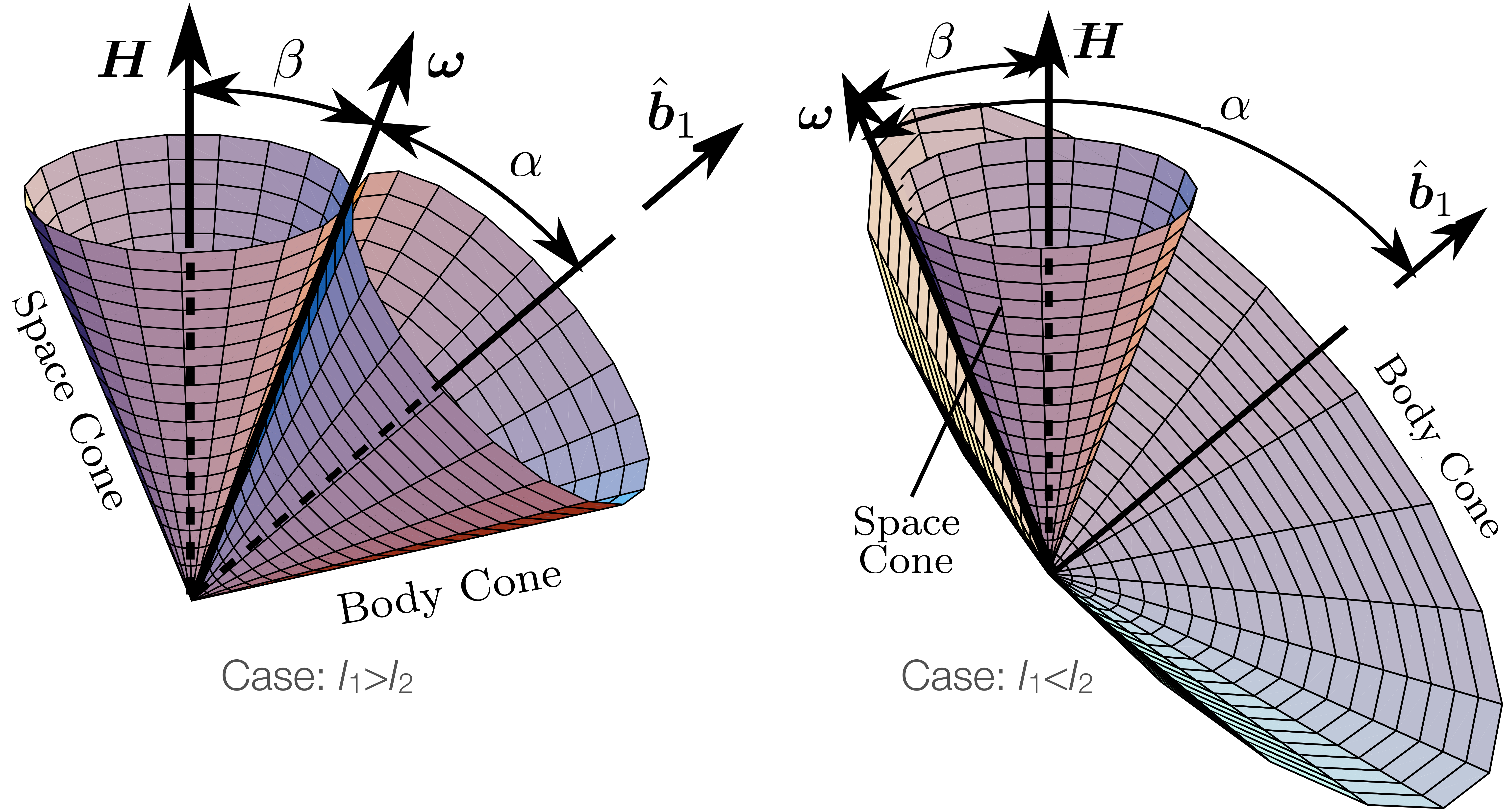
Let $\Omega = \omega_1 \longrightarrow \Omega = \dfrac{H}{I_1}\sin\theta$

Note that for $0 \leq \theta \leq \pi/2$

we find that $\Omega > 0$

The EOM can be written as

$$\dot{\psi} = -\frac{I_1}{I_2}\frac{\Omega}{\sin\theta} \qquad \dot{\phi} = \frac{I_2 - I_1}{I_2}\Omega$$



Case: $I_1 > I_2$ Case: $I_1 < I_2$

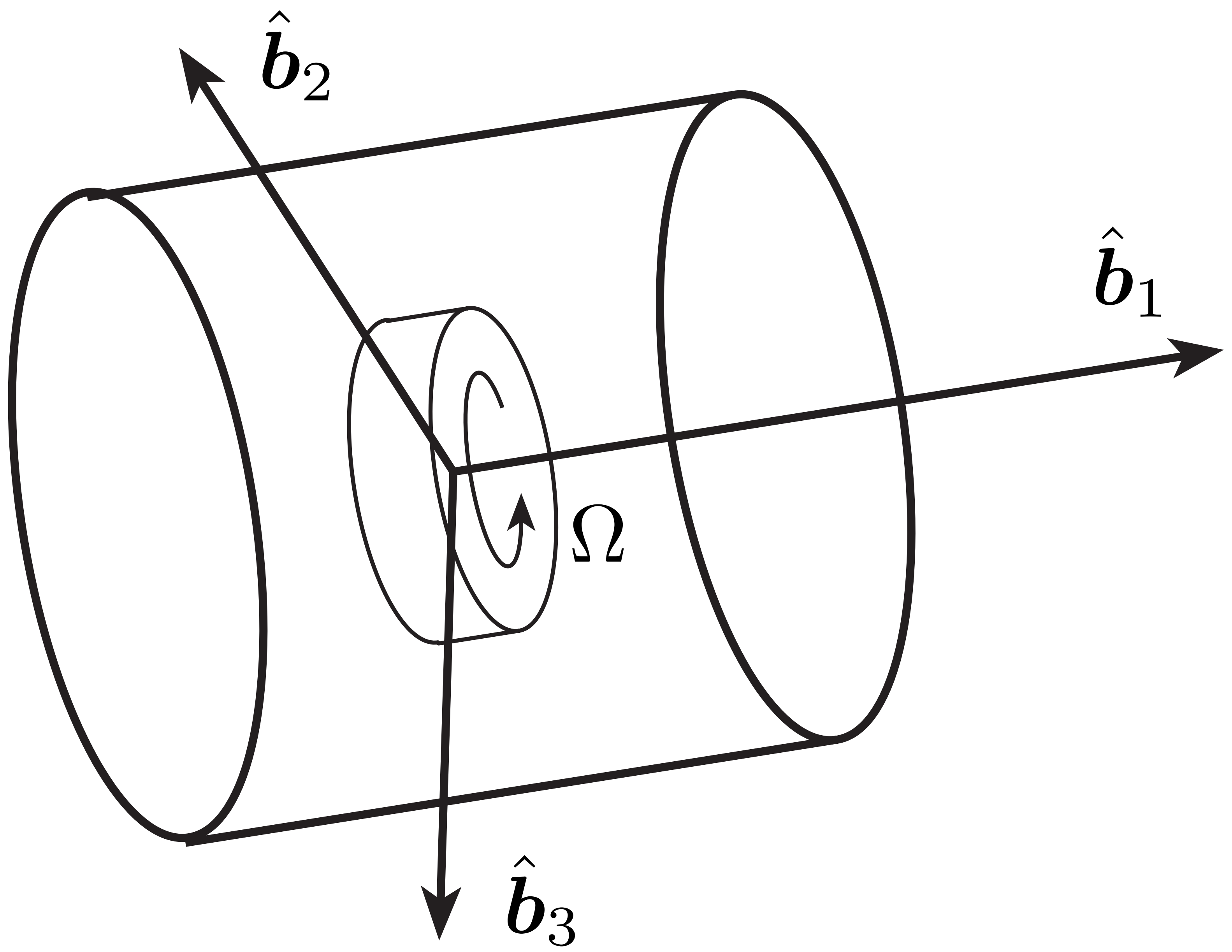
Since the pitch angle θ is shown to remain constant during this torque-free rotation, the resulting motion can be visualized by two cones rolling on each other. The *space cone* is fixed in space and its cone axis is always aligned with the angular momentum vector ***H***. The cone angle β is defined as the angle between the vectors ***H*** and **ω**. The *body cone* axis is aligned with the first body axis and has the cone angle α which is the angle between **ω** and first body axis.

Dual Spin Spacecraft

Elegant attitude stabilization method...

Equations of Motion

- Assume a rigid spacecraft has an internal fly wheel, whose *constant* spin axis is aligned with the first body axis.



Note: The spacecraft inertia magnitude about each body axis is still free to be chosen.

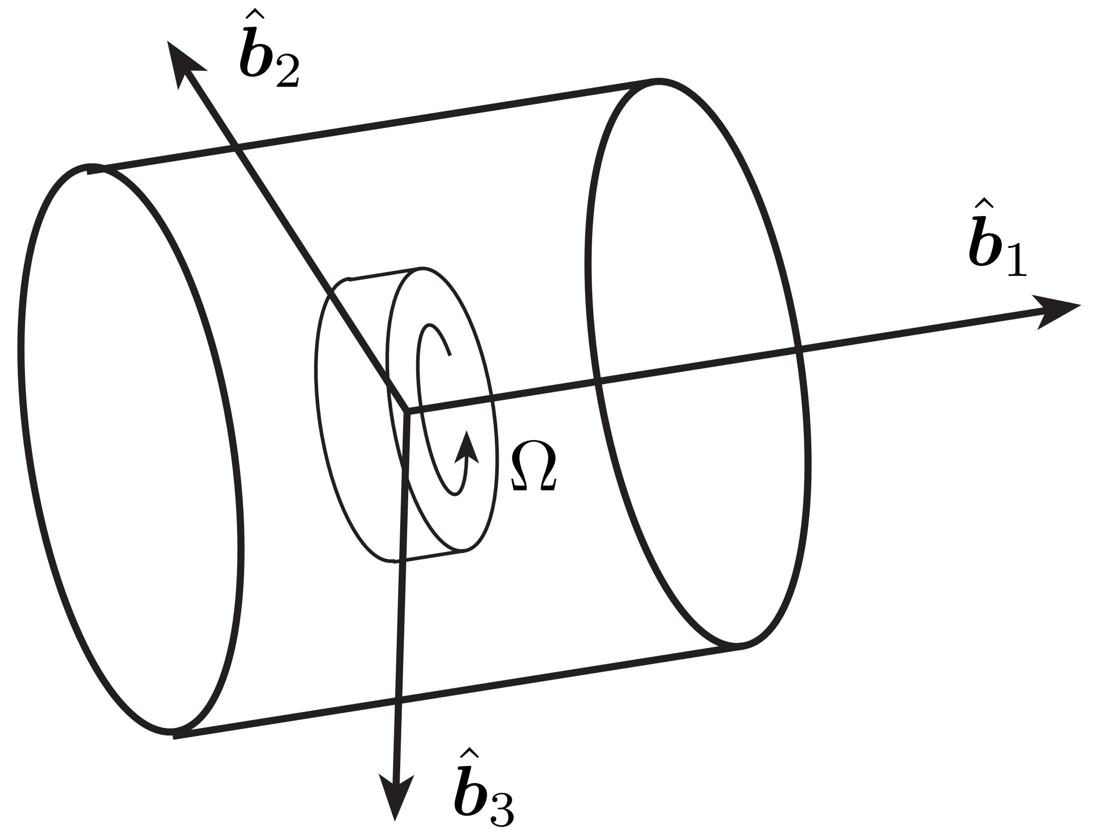
Total inertia matrix:

$$[I] = [I_s] + [I_W] = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

Total ang. momentum:

$$\boldsymbol{H} = [I]\boldsymbol{\omega} + \underbrace{I_W \Omega}_{h} \hat{\boldsymbol{b}}_1$$

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$$



Differentiate to use Euler's equation:

$$\begin{aligned}\dot{\boldsymbol{H}} &= \frac{^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\boldsymbol{H}) + \boldsymbol{\omega} \times \boldsymbol{H} \\ &= [I]\dot{\boldsymbol{\omega}} + \dot{h}\hat{\boldsymbol{b}}_1 + [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} + \boldsymbol{\omega} \times (h\hat{\boldsymbol{b}}_1) = \boldsymbol{L}\end{aligned}$$

Differential equations of motion with no external torque:

$$\boxed{[I]\dot{\boldsymbol{\omega}} = -[\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} - \dot{h}\hat{\boldsymbol{b}}_1 - h\omega_3\hat{\boldsymbol{b}}_2 + h\omega_2\hat{\boldsymbol{b}}_3}$$

with $\dot{h} = I_W \dot{\Omega}$

- Using Euler's equation, we find the spacecraft equations of motion with a constant speed fly wheel to be:

$$\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{pmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{pmatrix} = - \begin{pmatrix} (I_3 - I_2)\omega_2\omega_3 \\ (I_1 - I_3)\omega_1\omega_3 \\ (I_2 - I_1)\omega_1\omega_2 \end{pmatrix} + I_W \begin{pmatrix} -\dot{\Omega} \\ -\Omega\omega_3 \\ \Omega\omega_2 \end{pmatrix}$$

- This vector equation can also be written as three scalar equations:

$$\dot{\omega}_1 = \frac{I_2 - I_3}{I_3}\omega_2\omega_3 - \frac{I_W}{I_1}\dot{\Omega}$$
$$\dot{\omega}_2 = \frac{I_3 - I_1}{I_2}\omega_1\omega_3 - \frac{I_W}{I_2}\omega_3\Omega$$
$$\dot{\omega}_3 = \frac{I_1 - I_2}{I_3}\omega_1\omega_2 + \frac{I_W}{I_3}\omega_2\Omega$$

Linear Stability Analysis

- To determine the stability of this dual-spin spacecraft with constant wheel rate, we assume that the angular rate vector is an equilibrium rotation rate $\boldsymbol{\omega}_e$.
- Next, we study small variations in angular rates about this equilibrium position.
- For the equilibrium motion, note that

$$\boldsymbol{\omega} = \boldsymbol{\omega}_e + \delta\boldsymbol{\omega}$$

$$\begin{aligned}
\dot{\omega}_{e_1} &= \frac{I_2 - I_3}{I_1}\omega_{e_2}\omega_{e_3} = 0 \\
\dot{\omega}_{e_2} &= \frac{I_3 - I_1}{I_2}\omega_{e_1}\omega_{e_3} - \frac{I_{W_s}}{I_2}\omega_{e_3}\Omega = 0 \\
\dot{\omega}_{e_3} &= \frac{I_1 - I_2}{I_3}\omega_{e_1}\omega_{e_2} + \frac{I_{W_s}}{I_3}\omega_{e_2}\Omega = 0
\end{aligned}$$

- Substituting $\boldsymbol{\omega} = \boldsymbol{\omega}_e + \delta\boldsymbol{\omega}$ into the rigid body equations of motion yields:

$$(\dot{\omega}_{e_1} + \delta\dot{\omega}_1) = \frac{I_2 - I_3}{I_1}(\omega_{e_2} + \delta\omega_2)(\omega_{e_3} + \delta\omega_3)$$

$$(\dot{\omega}_{e_2} + \delta\dot{\omega}_2) = \frac{I_3 - I_1}{I_2}(\omega_{e_1} + \delta\omega_1)(\omega_{e_3} + \delta\omega_3) - \frac{I_{W_s}}{I_2}(\omega_{e_3} + \delta\omega_3)\Omega$$

$$(\dot{\omega}_{e_3} + \delta\dot{\omega}_3) = \frac{I_1 - I_2}{I_3}(\omega_{e_1} + \delta\omega_1)(\omega_{e_2} + \delta\omega_2) + \frac{I_{W_s}}{I_3}(\omega_{e_2} + \delta\omega_2)\Omega$$

- Next, assume that the space craft is spinning nominally about its first body axis

$$\boldsymbol{\omega}_e = \omega_{e_1}\hat{\boldsymbol{b}}_1 = \begin{pmatrix} \omega_{e_1} \\ 0 \\ 0 \end{pmatrix}$$

$$\omega_{e_2} = \omega_{e_3} = 0$$

- Dropping the higher order terms, and assuming that the equilibrium spin condition of interest is $\boldsymbol{\omega}_e = \omega_{e_1}\hat{\boldsymbol{b}}_1$, we find the following departure motion differential equations of motion.
-

$$\begin{aligned}
\delta\dot{\omega}_1 &= 0 \\
\delta\dot{\omega}_2 &= \frac{I_3 - I_1}{I_2}\omega_{e_1}\delta\omega_3 - \frac{I_{W_s}}{I_2}\delta\omega_3\Omega \\
\delta\dot{\omega}_3 &= \frac{I_1 - I_2}{I_3}\omega_{e_1}\delta\omega_2 + \frac{I_{W_s}}{I_3}\delta\omega_2\Omega
\end{aligned}$$

- Note that $\delta\omega_1$ is constant and does not appear in the other two equations (decoupled from them).

$$\begin{aligned}
\delta\dot{\omega}_2 &= \left(\frac{I_3 - I_1}{I_2}\omega_{e_1} - \frac{I_{W_s}}{I_2}\Omega\right)\delta\omega_3 \\
\delta\dot{\omega}_3 &= \left(\frac{I_1 - I_2}{I_3}\omega_{e_1} + \frac{I_{W_s}}{I_3}\Omega\right)\delta\omega_2
\end{aligned}$$

- Next, we take the derivative of $\delta\dot{\omega}_2$:

$$\delta\ddot{\omega}_2 = \left(\frac{I_3 - I_1}{I_2}\omega_{e_1} - \frac{I_{W_s}}{I_2}\Omega\right)\delta\dot{\omega}_3$$

- Substituting in the $\delta\dot{\omega}_3$ result from the previous page, we find the following decoupled body rate departure dynamics about the 2nd body axis:

$$\delta\ddot{\omega}_2 + \underbrace{\left(\frac{I_1 - I_3}{I_2}\omega_{e_1} + \frac{I_{W_s}}{I_2}\Omega\right)\left(\frac{I_1 - I_2}{I_3}\omega_{e_1} + \frac{I_{W_s}}{I_3}\Omega\right)}_{k}\delta\omega_2 = 0$$

Compare this to: $\delta\ddot{\omega}_2 + k\delta\omega_2 = 0$

Stability requires that $k > 0$

- The parameter k can be written as:

$$k = \frac{\omega_{e_1}^2}{I_2 I_3}\left(I_1 - I_3 + I_{W_s}\hat{\Omega}\right)\left(I_1 - I_2 + I_{W_s}\hat{\Omega}\right)$$

where $\hat{\Omega} = \dfrac{\Omega}{\omega_{e_1}}$

Zero RW Spin Rate

- First, let's verify the classical rigid body spin stability analysis if the RW spin rate is zero.
- For stability, we require *k*>0:

$$k = \frac{\omega_{e_1}^2}{I_2 I_3}\left(I_1 - I_3\right)\left(I_1 - I_2\right) > 0$$

True if: $I_1 > I_3$ $I_1 > I_2$ $\Rightarrow$ Max. Inertia Case

$I_1 < I_3$ $I_1 < I_2$ $\Rightarrow$ Min. Inertia Case

Non-Zero RW Spin

- Next, let's look at the stability requirement if the RW spin rate is nonzero:

$$k = \frac{\omega_{e_1}^2}{I_2 I_3}\left(I_1 - I_3 + I_{W_s}\hat{\Omega}\right)\left(I_1 - I_2 + I_{W_s}\hat{\Omega}\right) > 0$$

True if:

$I_1 > I_3 - I_{W_s}\hat{\Omega}$	$I_1 > I_2 - I_{W_s}\hat{\Omega}$
$I_1 < I_3 - I_{W_s}\hat{\Omega}$	$I_1 < I_2 - I_{W_s}\hat{\Omega}$

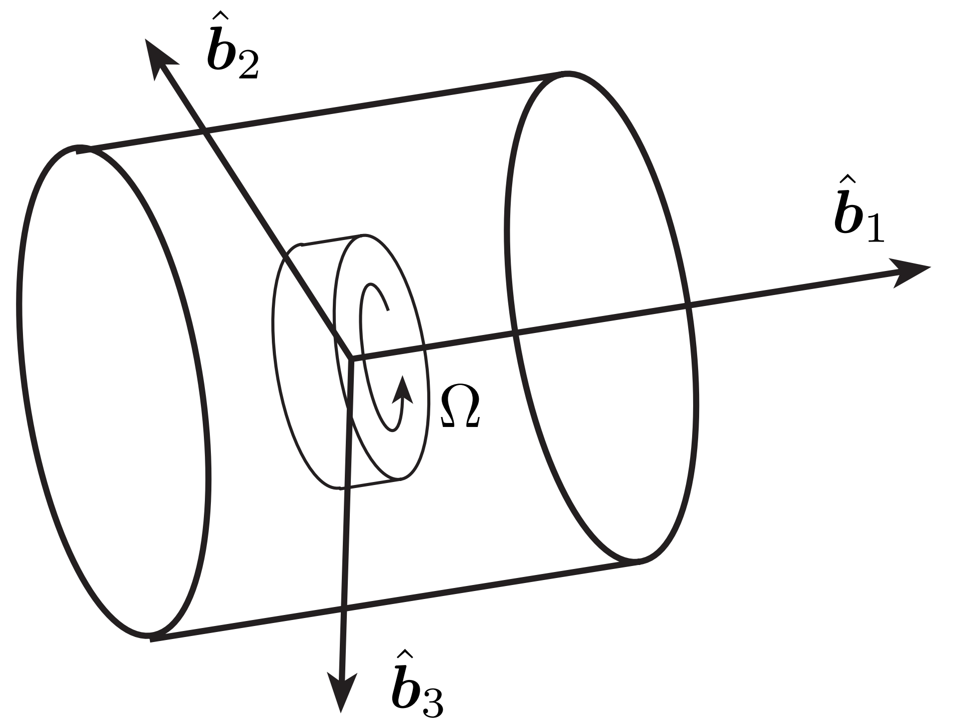
Note: The spacecraft spin can be made stable, regardless if I_1 is a major, intermediate or minor inertia!

Note: Careless use of the RW mode can also cause the spacecraft spin too become unstable.

Example

- Consider a spacecraft to have the following inertias:

$$
\begin{aligned}
I_1 &= 350 \text{ kgm}^2 \\
I_2 &= 300 \text{ kgm}^2 \\
I_3 &= 400 \text{ kgm}^2 \\
I_{W_s} &= 10 \text{ kgm}^2
\end{aligned}
$$



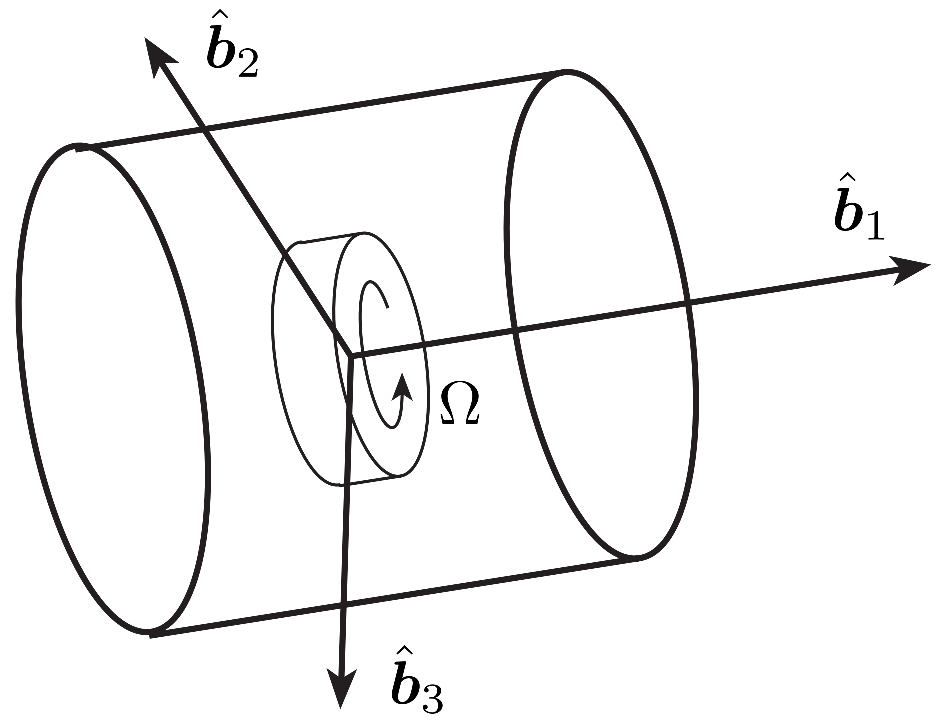
- Without the fly-wheel, note that spinning abut the first body axis would be unstable.
- The spacecraft spin about $\hat{\boldsymbol{b}}_1$ is 60 RPM.
- How fast does the wheel have to spin to make this spacecraft a stable dual-spin system?

- The stability conditions are:

Set 1: $I_1 > I_3 - I_{W_s}\hat{\Omega}$ $\quad$ $I_1 > I_2 - I_{W_s}\hat{\Omega}$

Set 2: $I_1 < I_3 - I_{W_s}\hat{\Omega}$ $\quad$ $I_1 < I_2 - I_{W_s}\hat{\Omega}$

Since $I_1 > I_2$, the second condition of set 1 is satisfied if $\hat{\Omega} > -5$

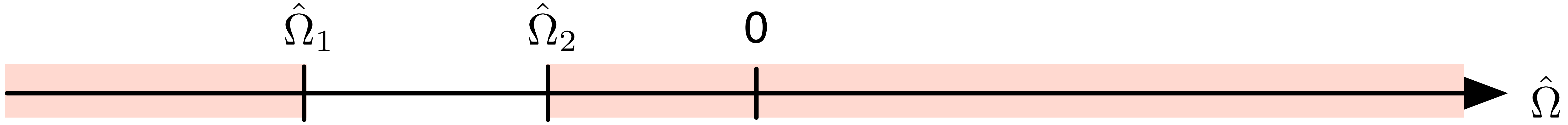


The first condition of set 1 then requires that:

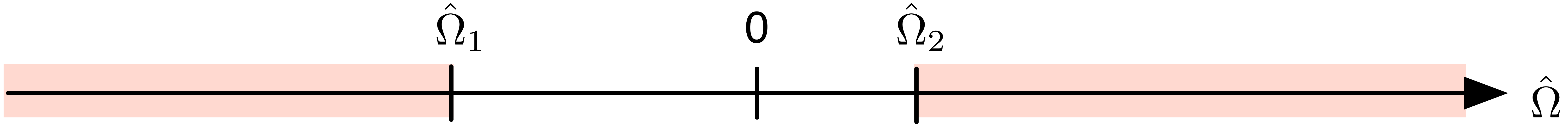
$$
\begin{aligned}
I_1 &> I_3 - I_{W_s}\hat{\Omega} \\
I_{W_s}\hat{\Omega} &> I_3 - I_1 \\
\hat{\Omega} &> \frac{I_3 - I_1}{I_{W_s}} \\
\hat{\Omega} &> 5
\end{aligned}
$$

$$
\begin{aligned}
\Omega &= \hat{\Omega}\omega_{e_1} \\
&= 300 \text{ RPM}
\end{aligned}
$$

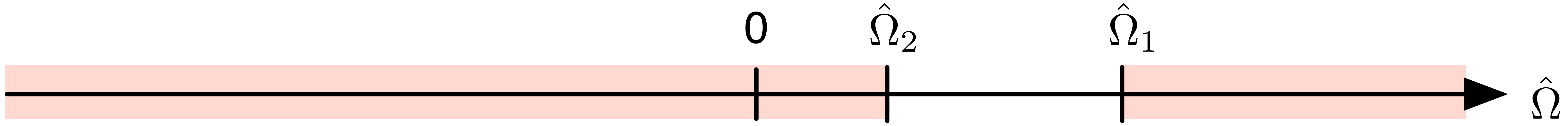
Rotor spinning about major axis:



Rotor spinning about intermediate axis:

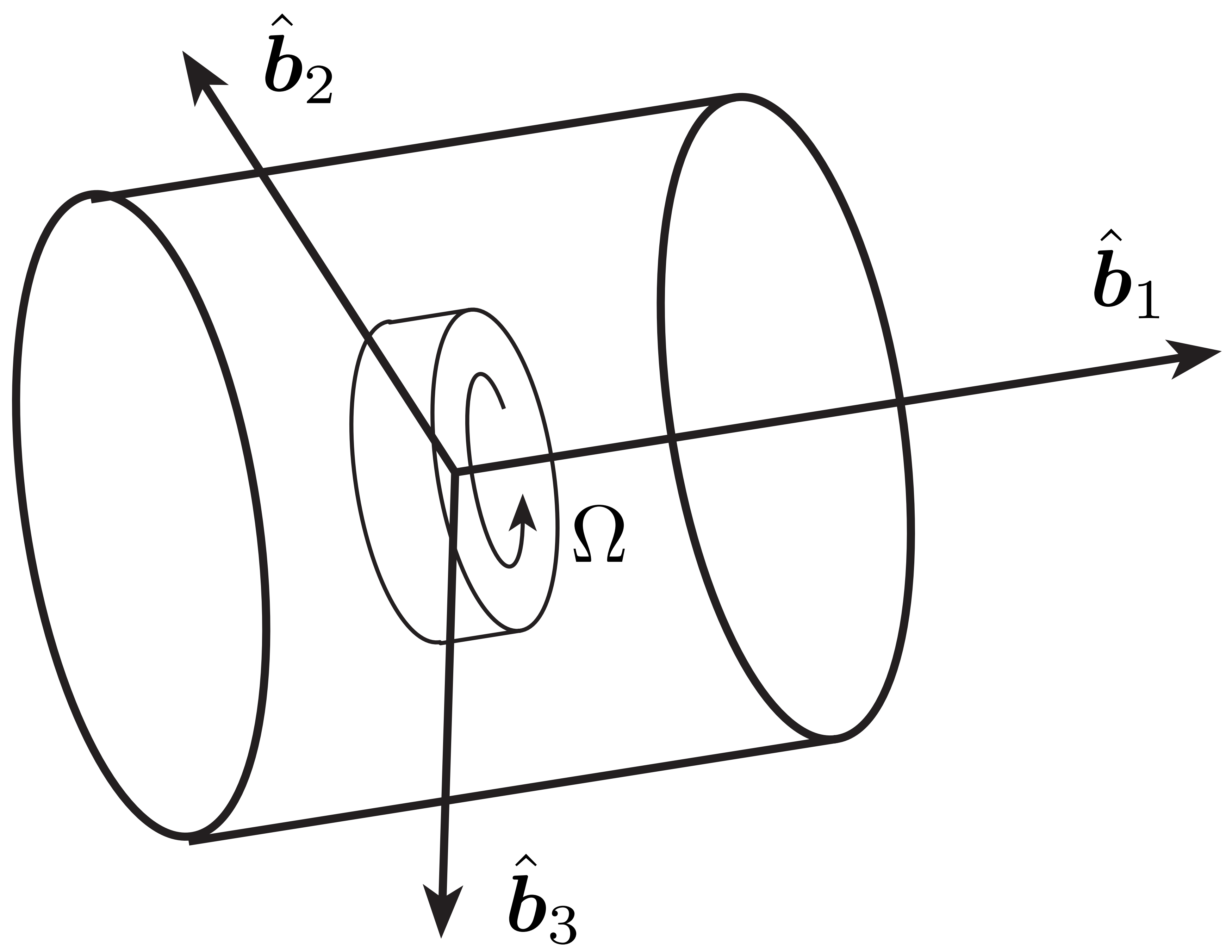


Rotor spinning about minor axis:



Spin-Up Study

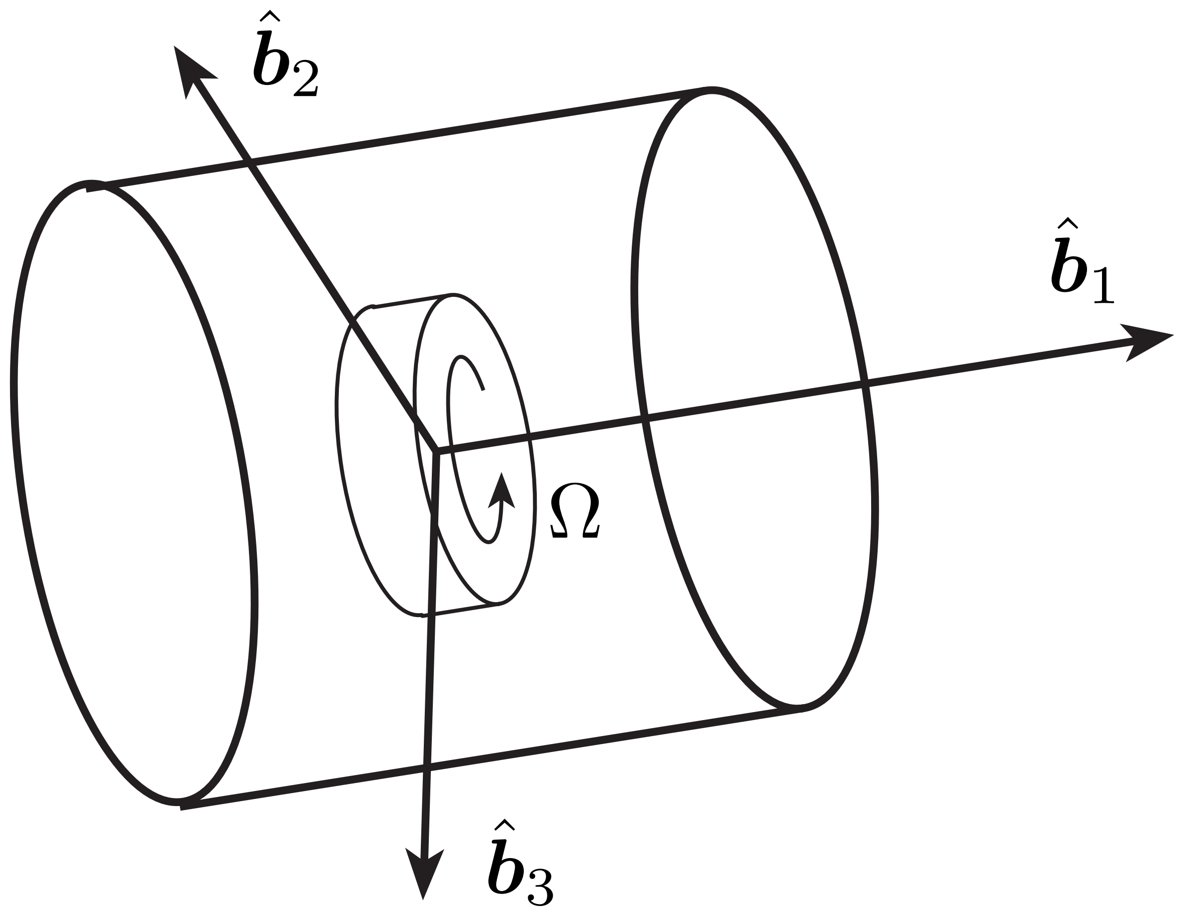
- Next we investigate a classical spin-up maneuver with a dual-spin spacecraft. Assume the wheel is initial at rest relative to the spacecraft.
- The spacecraft is assumed to have a pure spin about a principal axis which his not aligned with the reaction wheel spin axis.
- Then the RW is spun up until it has the same amount of angular momentum as the spacecraft had initially.
- What will happen to the spacecraft attitude during this spin-up maneuver?



- Because no external torques are present, the angular momentum magnitude is constant and given by:

$$H = |\boldsymbol{H}(t_0)|$$

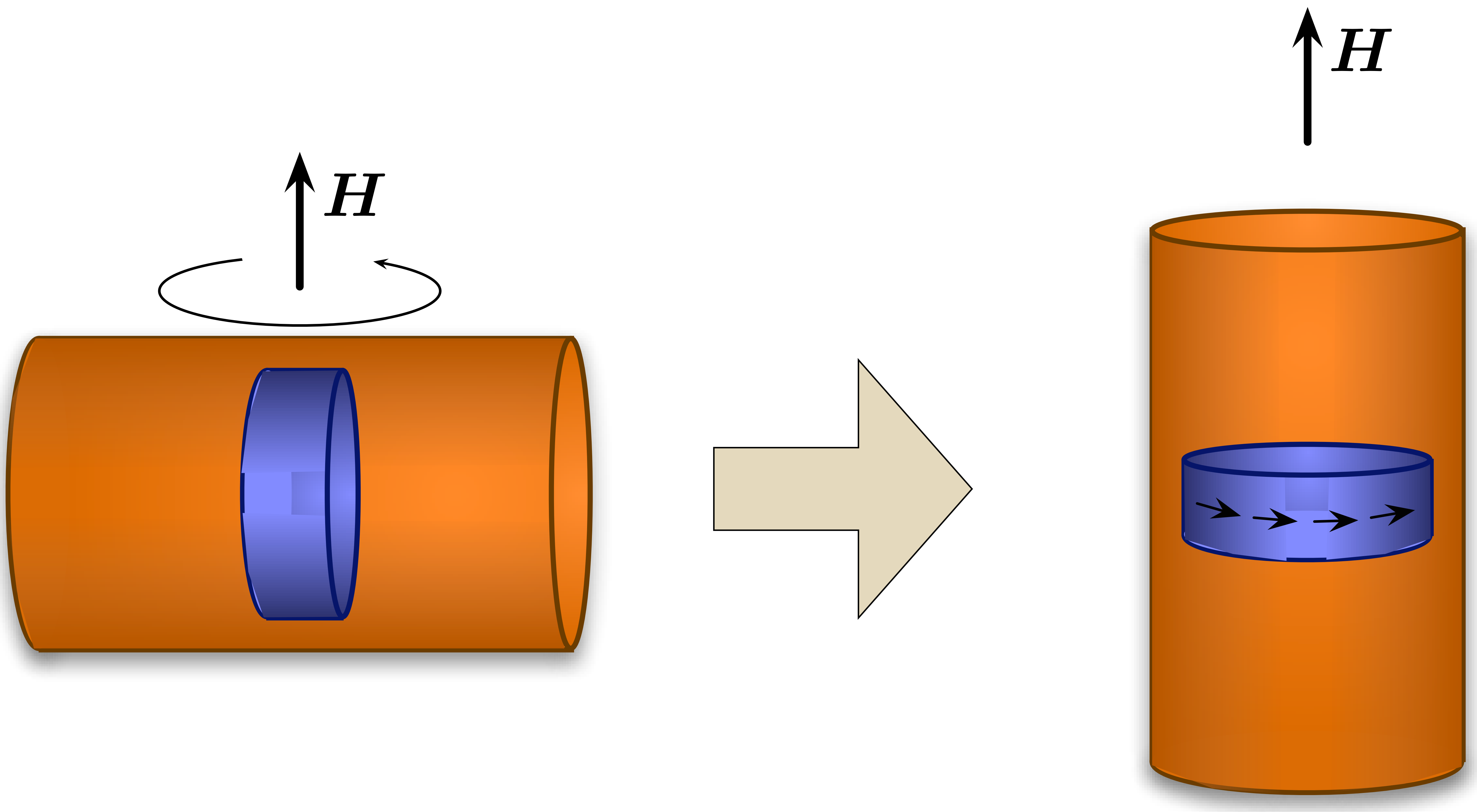
- The wheel angular momentum is increased at a constant rate through:



$$\dot{h} = I_W \dot{\Omega} = C = \text{constant}$$
$$h(t) = Ct$$

- The total maneuver time is

$$T_{\max} = \frac{H}{C} = \frac{H}{I_W \dot{\Omega}}$$



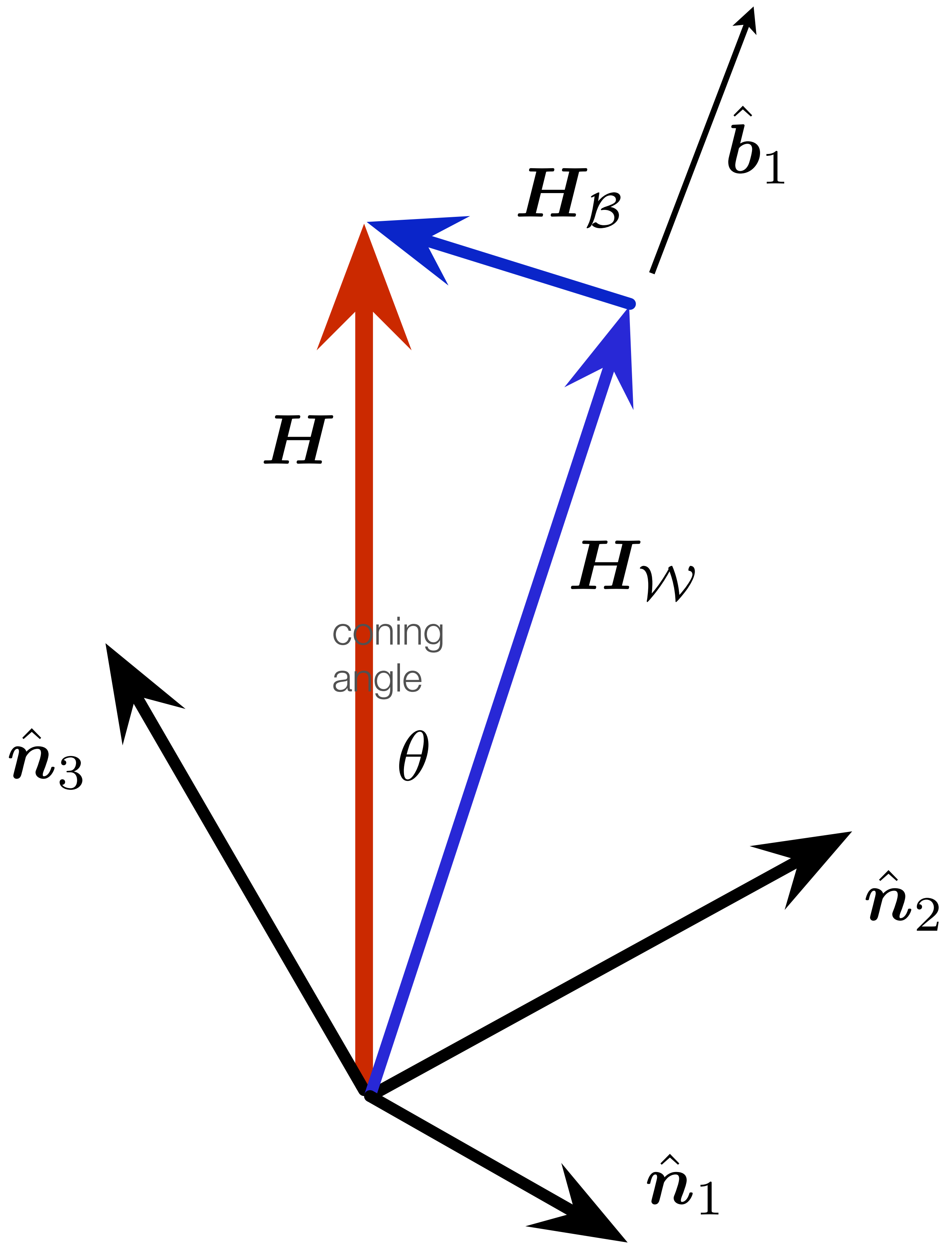
Question: As the RW assumes the same amount of angular momentum as the initially spinning spacecraft possessed, won’t the angular momentum conservation cause the craft to realign RW spin axis along the momentum vector with the spacecraft at rest?

The answer is, not necessarily...

We are only controlling a single-degree of freedom, which is influencing the three-dimension motion of the spacecraft.

If the wheel angular momentum has the same magnitude as the initial system angular momentum ***H***, this does not mean that the wheel angular momentum $\boldsymbol{H}_W$ is aligned with the ***H*** vector.

Let's study this through an numerical example...



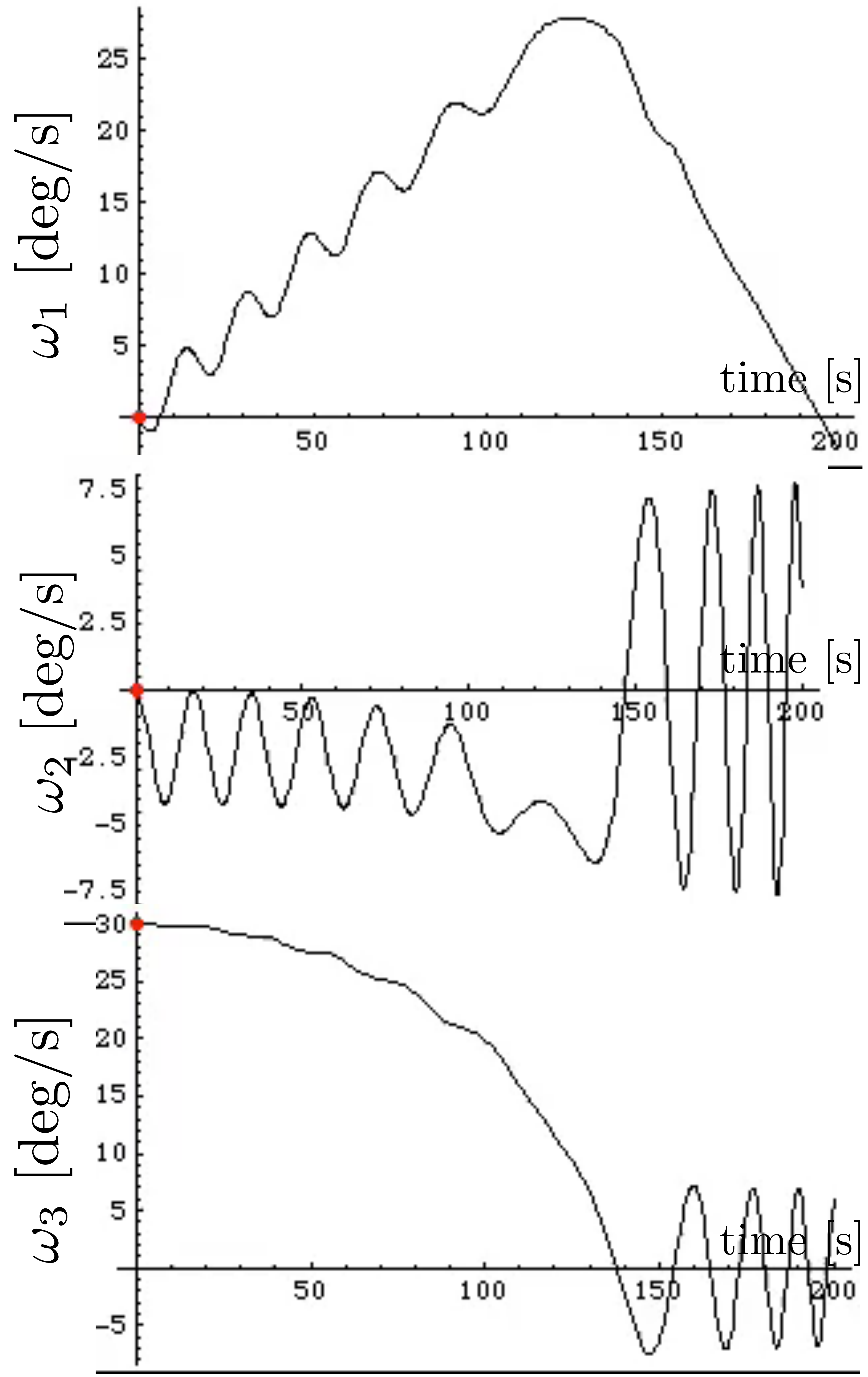
Example: Numerical Simulations of Spin up Maneuvers

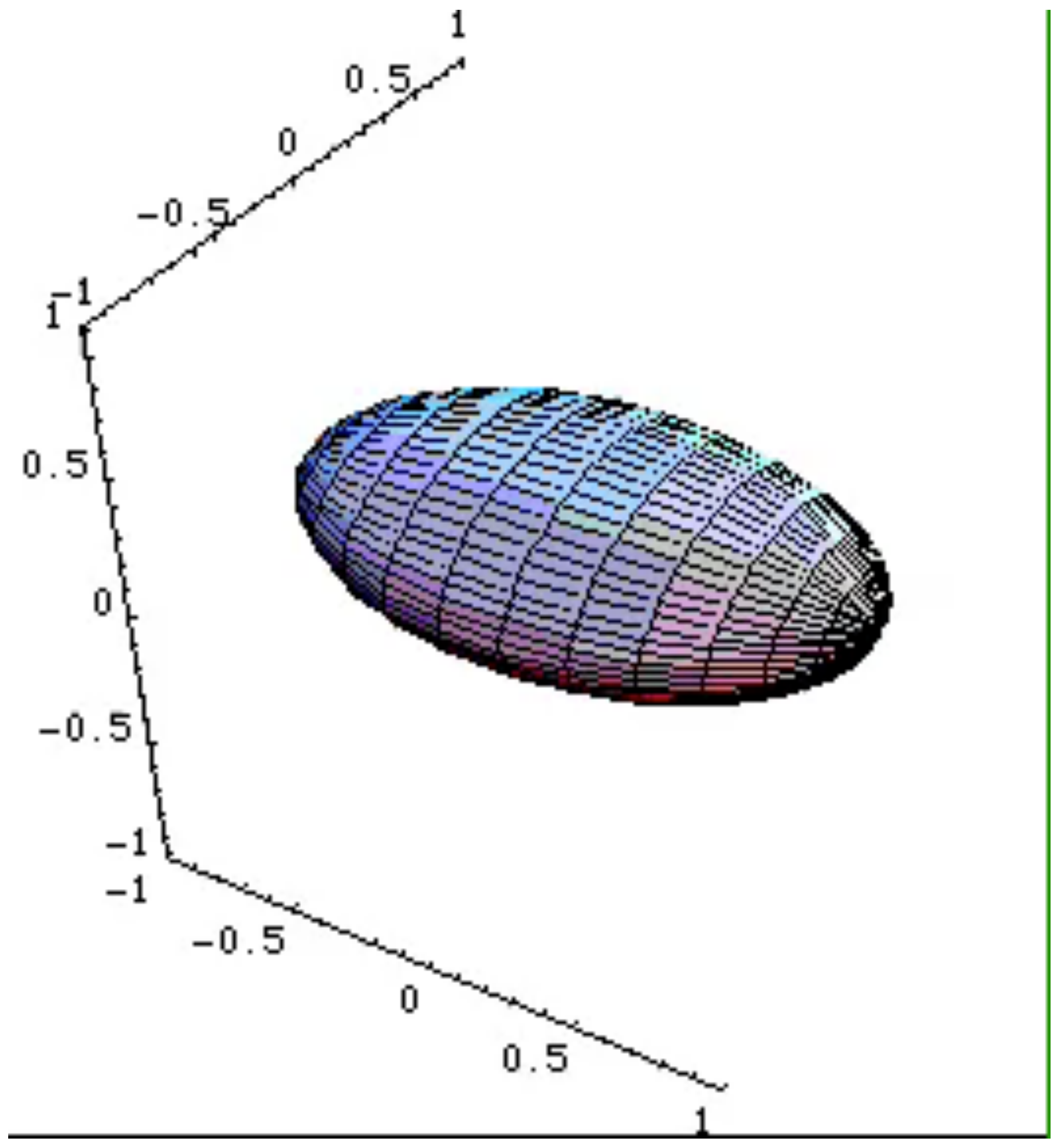
Spacecraft is initially in a flat spin about the major inertia principal axis.

The RW is spun up at different rates and the final coning angle θ is investigated.

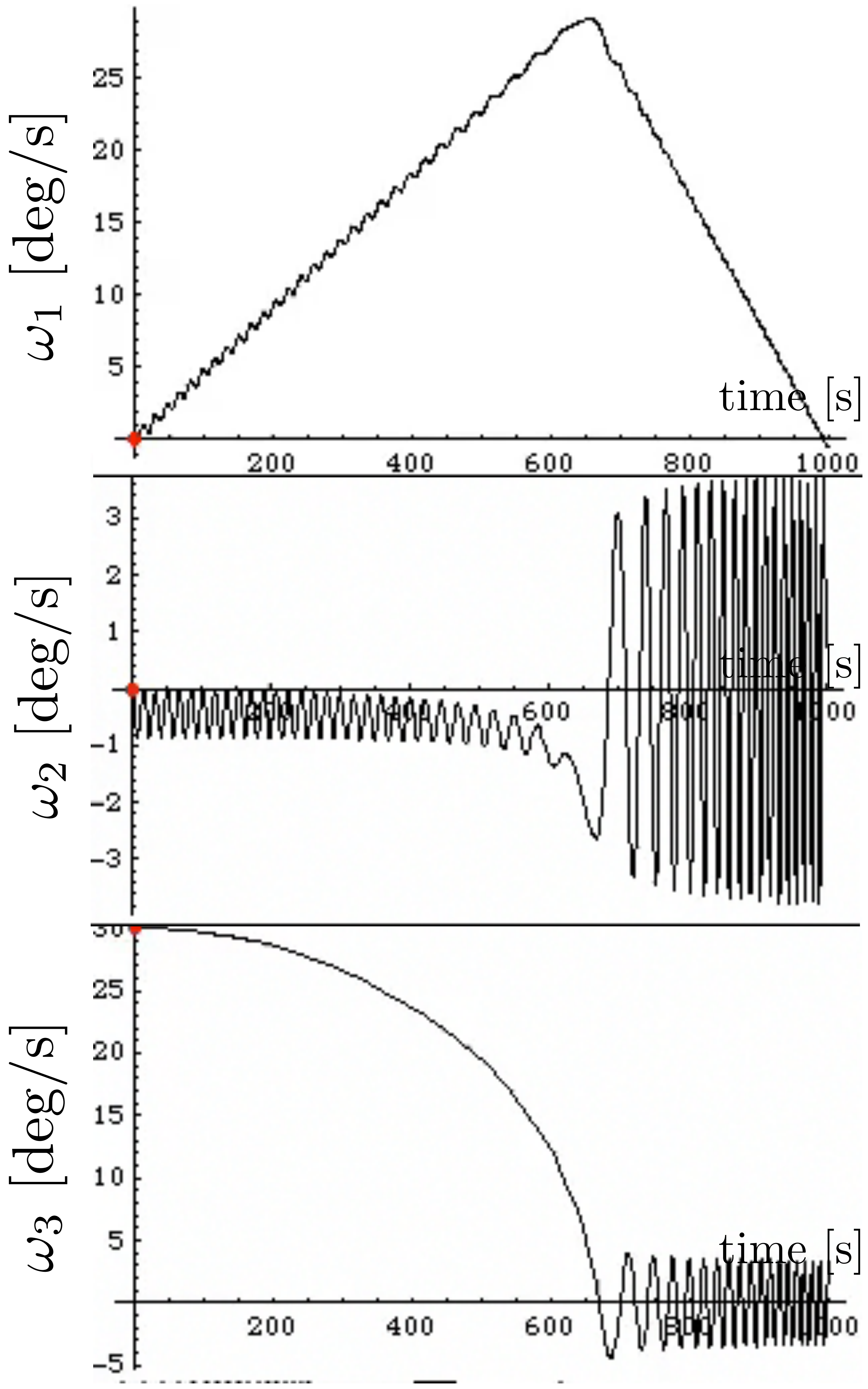
Simulation Parameters

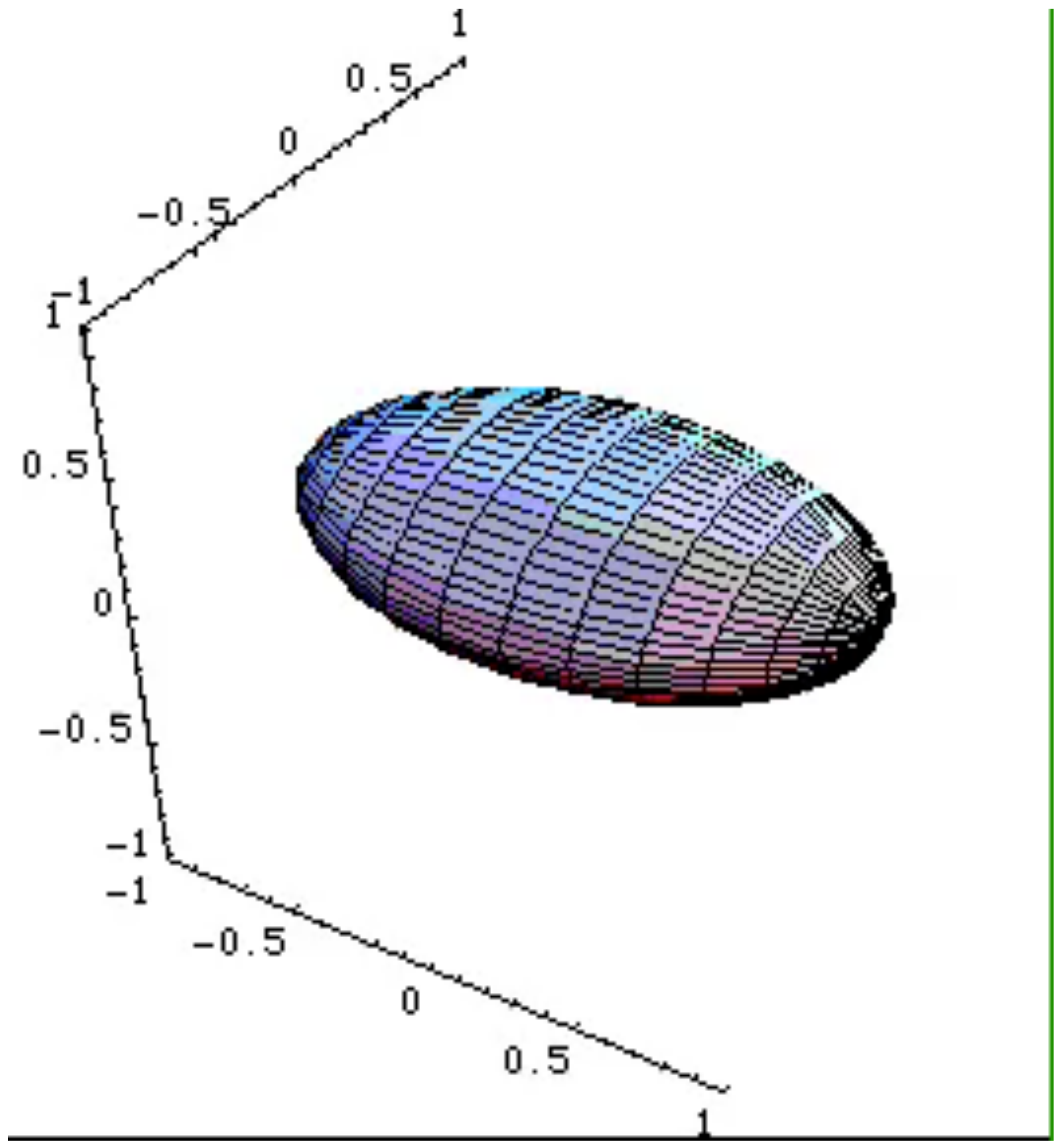
I_1	9.47 kgm^2
I_2	21.90 kgm^2
I_3	27.57 kgm^2
$\omega_3(t_0)$	30 °/s
$\omega_1(t_0)$, $\omega_2(t_0)$	0 °/s
$\Omega(t_0)$	0 °/s
I_W	1.89 kgm^2



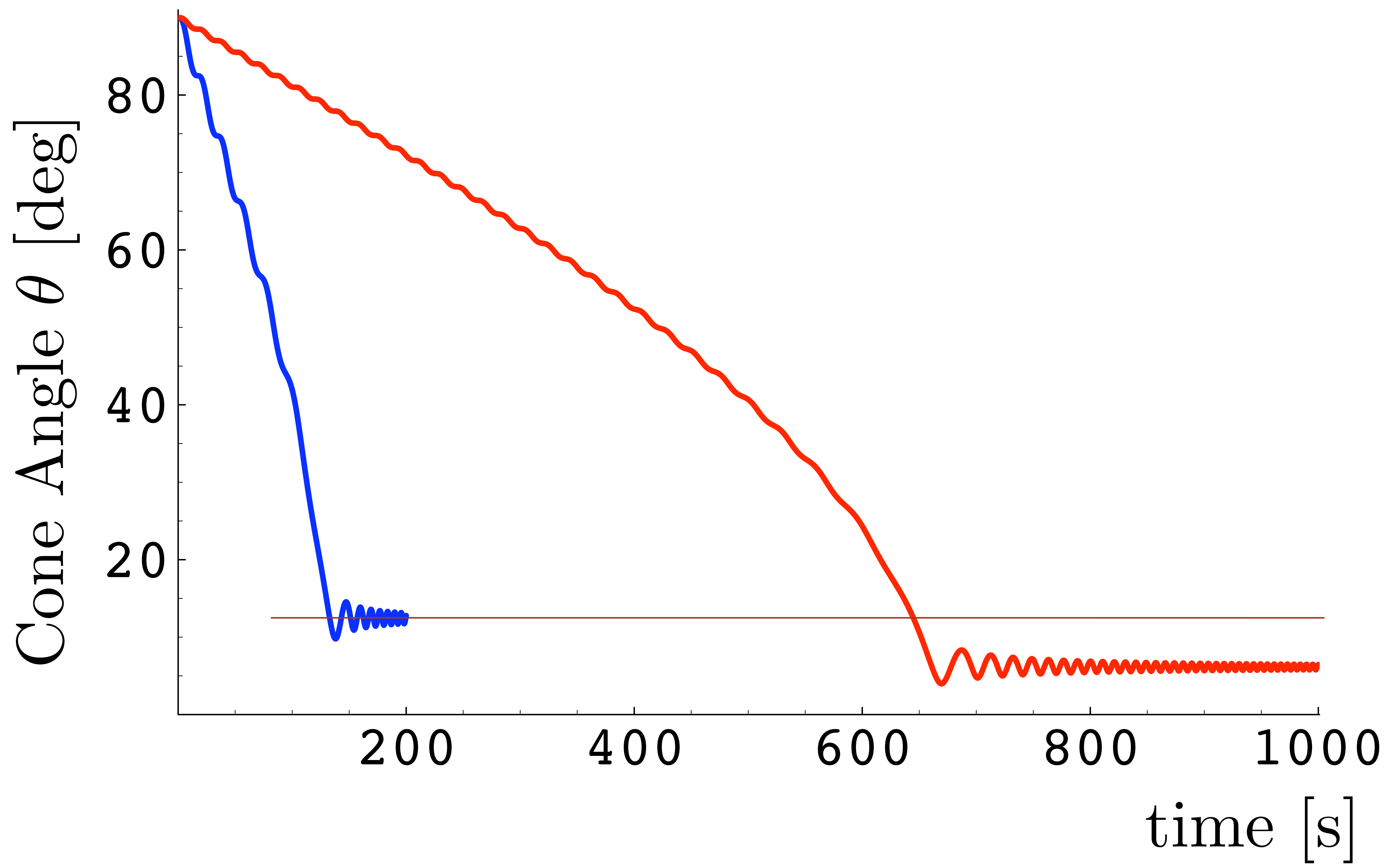


Maneuver Time: 200 seconds





Maneuver Time: 1000 seconds


80
60
40
20
200
400
600
800
1000
Cone Angle θ [deg]
time [s]

- To study the spin-up dynamics, we can use the momentum sphere –energy ellipsoid method.*

Spacecraft Kinetic Energy:

$$E^* = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$$

The control objective is to drive this positive definite measure of the spacecraft motion to zero!

Momentum Sphere:

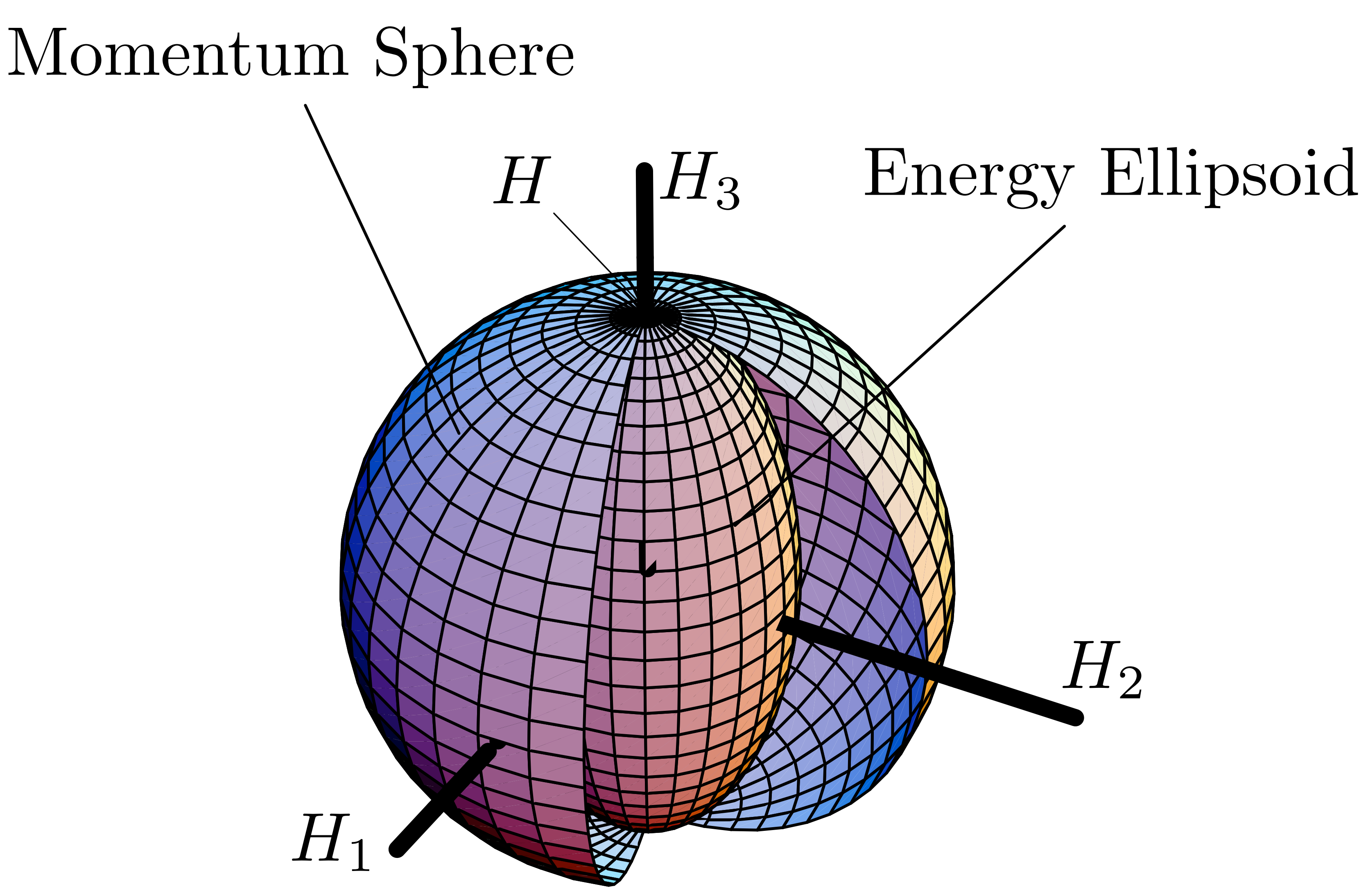
$$H^2 = H_1^2 + H_2^2 + H_3^2$$

Energy Ellipsoid:

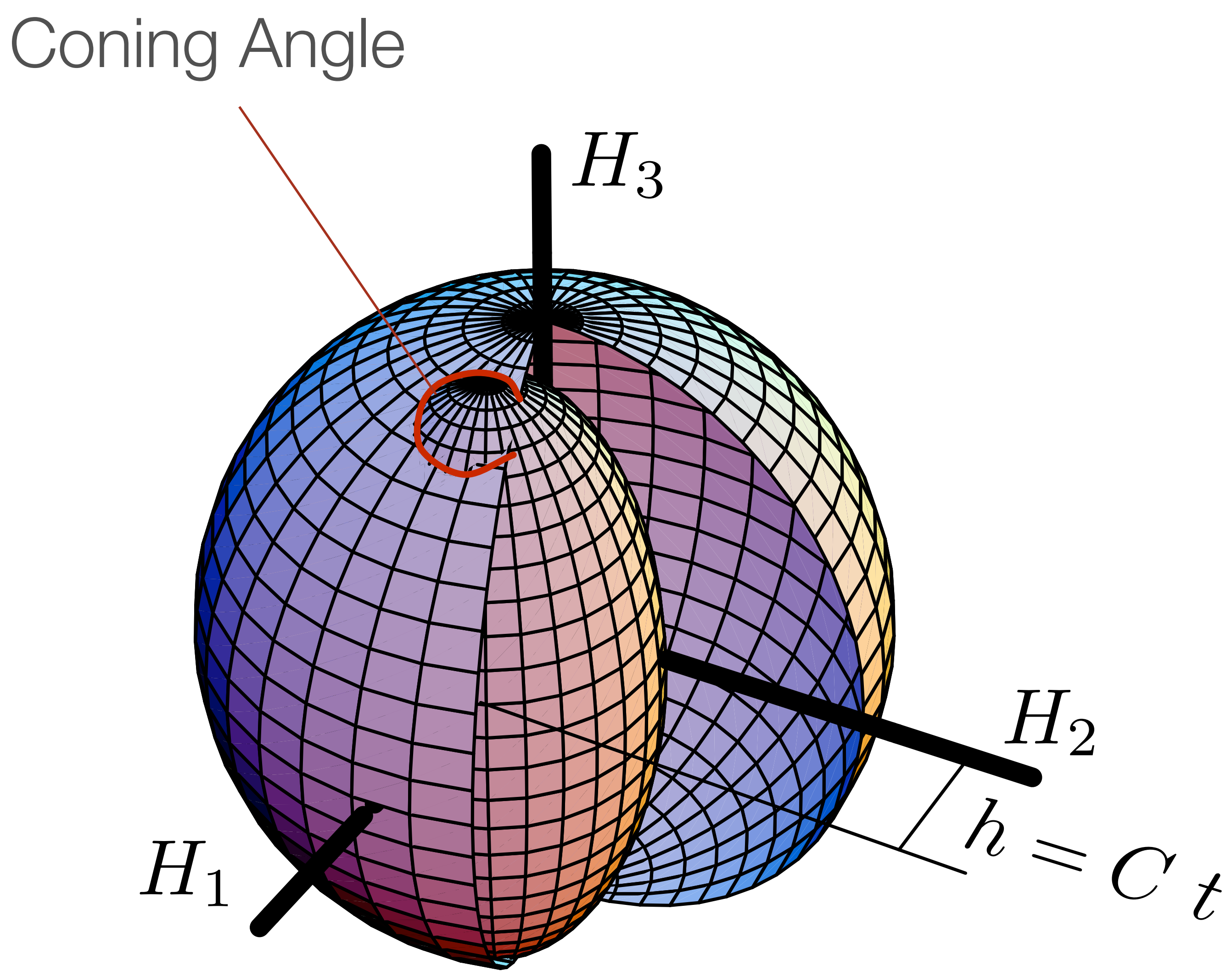
$$1 = \frac{(H_1 - h)^2}{2I_1E^*} + \frac{H_2^2}{2I_21E^*} + \frac{H_3^2}{2I_3E^*}$$

Note how the energy ellipsoid size will vary as the spacecraft kinetic energy is reduced, and how the ellipsoid center will shift along the first body axis.

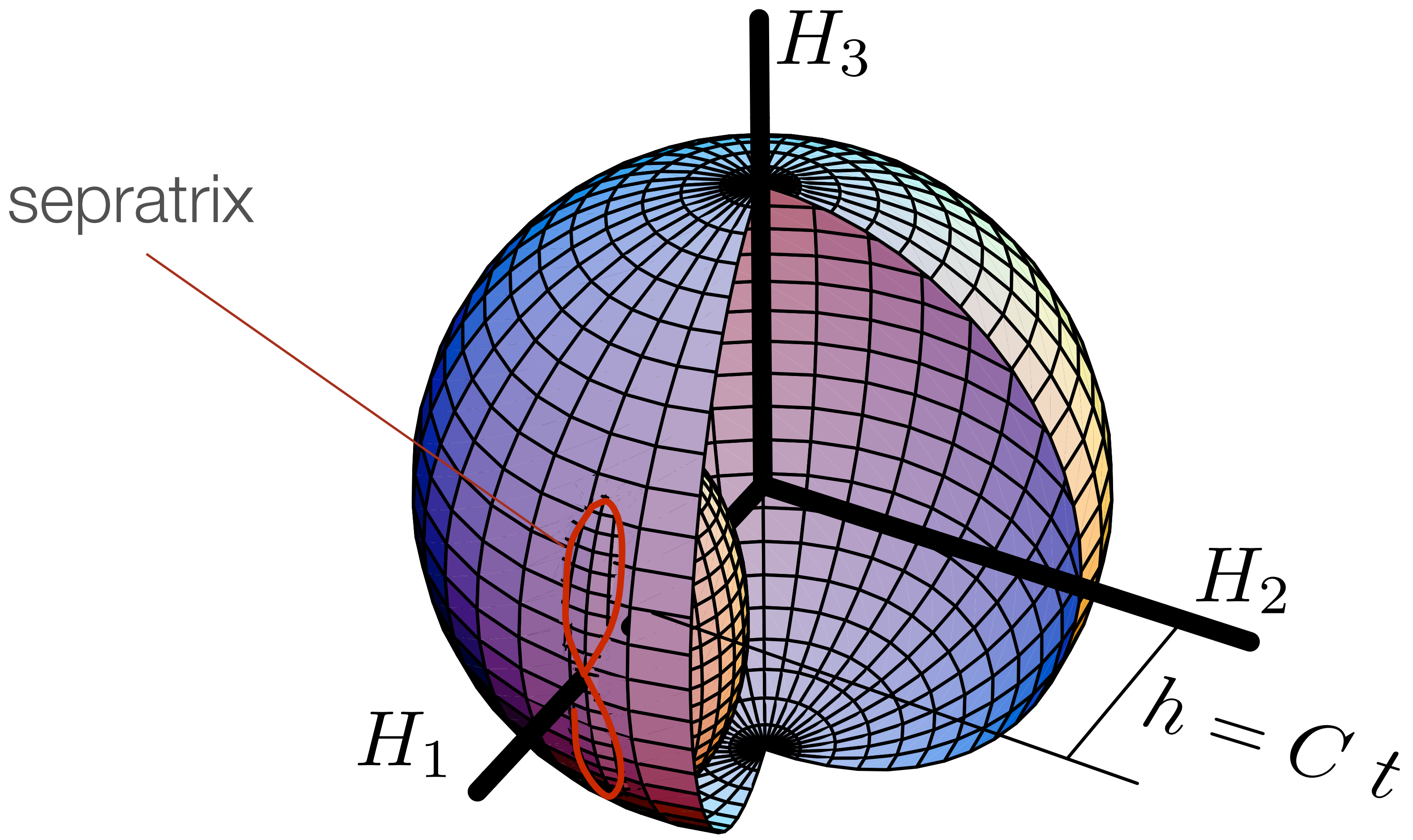
*Barba, P., and Auburn, J., "Satellite Attitude Acquisition by Momentum Transfer," Paper #AAS-75-053, Presented at the AAS/AIAA Astrodynamics Conference, Nasau, Bahamas, July 1975.



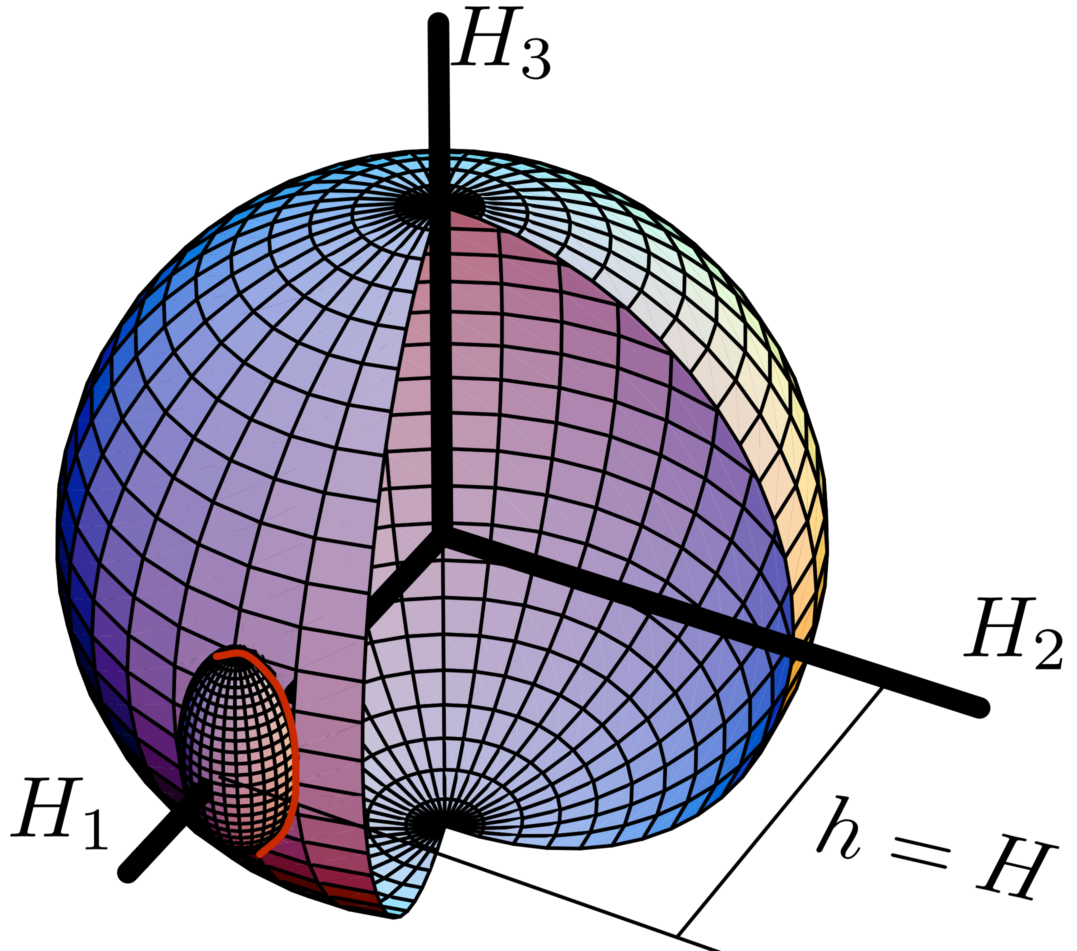
Spacecraft with pure spin about 3rd axis, RW at rest



The RW is spun up and reduces the spacecraft kinetic energy.



A critical energy condition is reached with the sepratrix.



The final stage has the spacecraft with a non-zero energy state, and some coning.

Gravity Gradients

Nature's free stabilization method...

Gravity Gradient Satellite

- For a rigid body in space, the “lower” parts of the body will be heavier than the “upper” parts!
- Techsat 21 satellites were originally planned to be G^2 satellites
- This tidal force will produce a torque onto the body, and cause the CM to move.

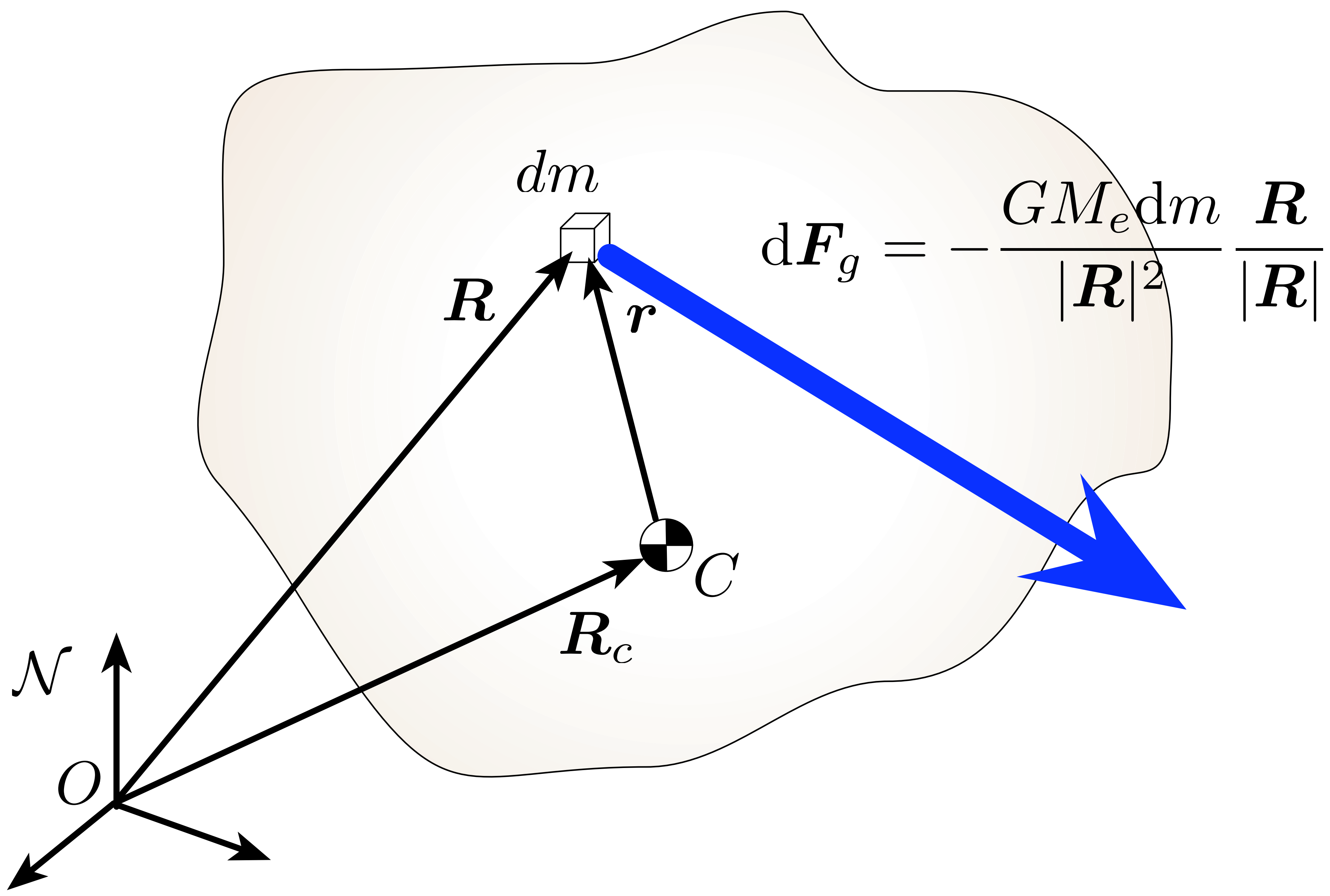
Gravity Gradient Torque

Inertial Frame: $\mathcal{N}:\{O,\hat{\boldsymbol{n}}_1,\hat{\boldsymbol{n}}_2,\hat{\boldsymbol{n}}_3\}$

Inertial Vectors: $\boldsymbol{R},\ \boldsymbol{R}_c$

Relative Vectors: $\boldsymbol{r}$

Note: $\boldsymbol{R}=\boldsymbol{R}_c+\boldsymbol{r}$



The Gravity gradient torque acting on the spacecraft is:

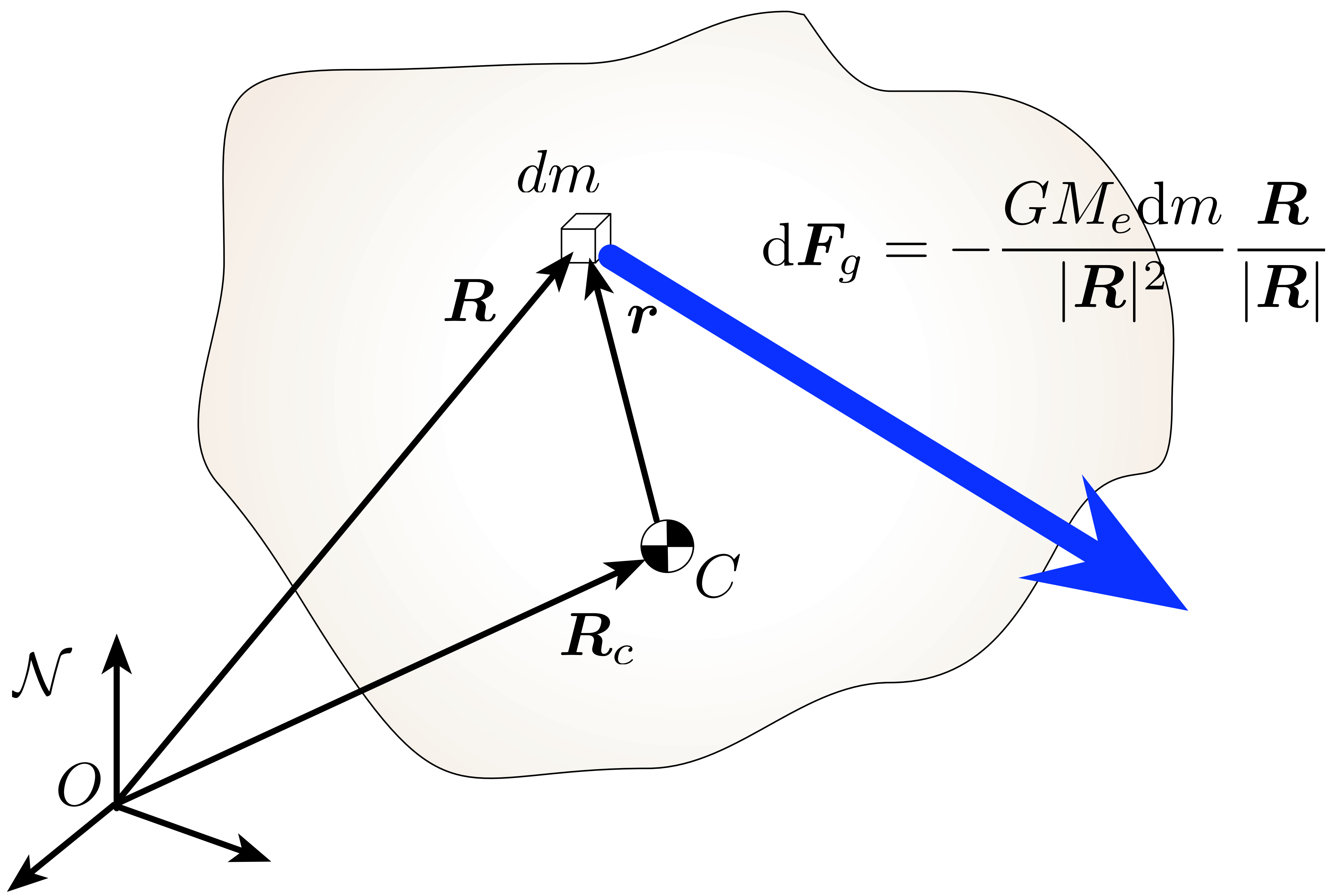
$$\boldsymbol{L}_G = \int_{\mathcal{B}} \boldsymbol{r} \times \mathrm{d}\boldsymbol{F}_G$$

The gravity force acting on the mass element is:

$$\mathrm{d}\boldsymbol{F}_G = -\frac{GM_e}{|\boldsymbol{R}|^3}\boldsymbol{R}\mathrm{d}m$$

The gravity gradient torque is then written as:

$$\boldsymbol{L}_G = -\int_{\mathcal{B}} \boldsymbol{r} \times \frac{GM_e}{|\boldsymbol{R}|^3}\left(\boldsymbol{R}_c + \boldsymbol{r}\right)\mathrm{d}m$$



- Taking all the constants outside of the integral term, we find:

$$\boldsymbol{L}_G = GM_e\boldsymbol{R}_c \times \int_{\mathcal{B}} \frac{\boldsymbol{r}}{|\boldsymbol{R}|^3}\mathrm{d}m$$

- Note that the inertial position vector ***R*** contains both ***R$_c$*** and ***r***. We can simplify the denominator using:

$$\begin{aligned}
|\boldsymbol{R}|^{-3} = |\boldsymbol{R}_c + \boldsymbol{r}|^{-3} &= \left(R_c^2 + 2\boldsymbol{R}_c\cdot\boldsymbol{r} + r^2\right)^{-3/2} \\
&= \frac{1}{R_c^3}\left(1 + \frac{2\boldsymbol{R}_c\cdot\boldsymbol{r}}{R_c^2} + \left(\frac{r}{R_c}\right)^2\right)^{-3/2} \\
&\approx \frac{1}{R_c^3}\left(1 - \frac{3\boldsymbol{R}_c\cdot\boldsymbol{r}}{R_c^2} + \cdots\right)
\end{aligned}$$

- Using this approximation for the $1/R^3$ term, the gravity gradient torque can be approximated as

$$\boldsymbol{L}_G = \frac{GM_e}{R_c^3}\boldsymbol{R}_c \times \int_{\mathcal{B}} \boldsymbol{r}\left(1 - \frac{3\boldsymbol{R}_c \cdot \boldsymbol{r}}{R_c^2}\right)\mathrm{d}m$$

- Using the center of mass definition

$$\int_{\mathcal{B}} \boldsymbol{r}\mathrm{d}m = 0$$

the gravity gradient torque is reduced to

$$\boldsymbol{L}_G = \frac{3GM_e}{R_c^5}\boldsymbol{R}_c \times \int_{\mathcal{B}} -\boldsymbol{r}\left(\boldsymbol{r} \cdot \boldsymbol{R}_c\right)\mathrm{d}m$$

- Using the vector identity

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\, \boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\, \boldsymbol{c}$$

  we can rewrite the following term

$$-\,(\boldsymbol{a} \cdot \boldsymbol{c})\, \boldsymbol{b} = -\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) - (\boldsymbol{a} \cdot \boldsymbol{b})\, \boldsymbol{c}$$

$$-(\boldsymbol{r} \cdot \boldsymbol{R}_c)\boldsymbol{r} = -\boldsymbol{r} \times (\boldsymbol{r} \times \boldsymbol{R}_c) - (\boldsymbol{r} \cdot \boldsymbol{r})\boldsymbol{R}_c$$

- The torque vector is now written as

$$\boldsymbol{L}_G = \frac{3GM_e}{R_c^5}\boldsymbol{R}_c \times \int_{\mathcal{B}} -\left(\boldsymbol{r} \times (\boldsymbol{r} \times \boldsymbol{R}_c) + (\boldsymbol{r} \cdot \boldsymbol{r})\, \boldsymbol{R}_c\right) \mathrm{d}m$$

- Using the tilde matrix definition, we can reduce this expression to

$$\boldsymbol{L}_G = \frac{3GM_e}{R_c^5}\boldsymbol{R}_c \times \left(\int_{\mathcal{B}} -[\tilde{\boldsymbol{r}}][\tilde{\boldsymbol{r}}]\mathrm{d}m\right)\boldsymbol{R}_c - \frac{3GM_e}{R_c^5}\left(\int_{\mathcal{B}} r^2\mathrm{d}m\right)\boldsymbol{R}_c \times \boldsymbol{R}_c$$

- Using the matrix definition

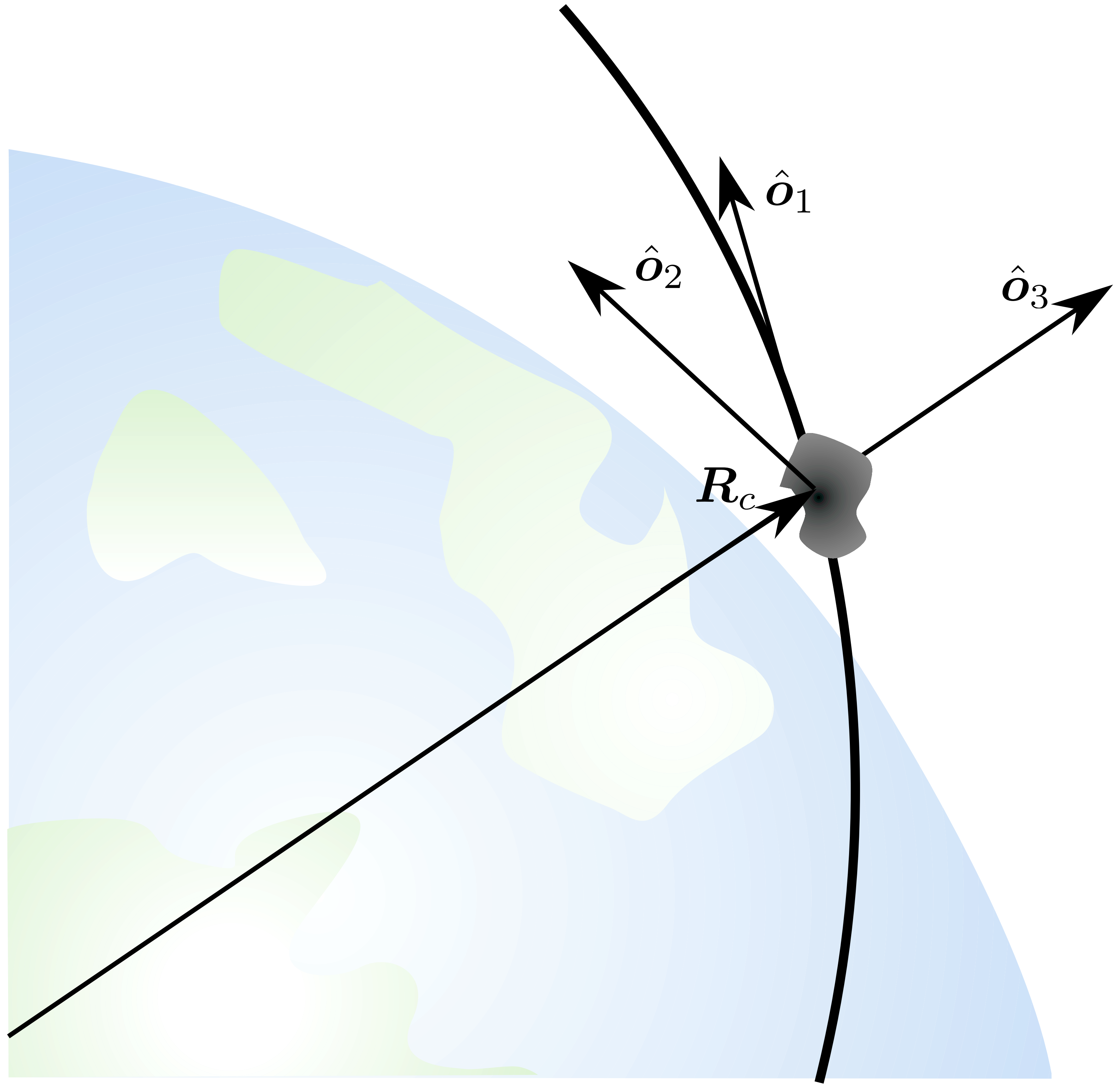
$$^{\mathcal{B}}[I_c] = \int_B -[\tilde{\boldsymbol{r}}][\tilde{\boldsymbol{r}}]\mathrm{d}m = \int_B {}^{\mathcal{B}}\begin{bmatrix} r_2^2 + r_3^2 & -r_1 r_2 & -r_1 r_3 \\ -r_1 r_2 & r_1^2 + r_3^2 & -r_2 r_3 \\ -r_1 r_3 & -r_2 r_3 & r_1^2 + r_2^2 \end{bmatrix} \mathrm{d}m$$

the gravity torque on a rigid body is **finally** written as

$$\boxed{\boldsymbol{L}_G = \frac{3GM_e}{R_c^5}\boldsymbol{R}_c \times [I]\boldsymbol{R}_c}$$

- The previous expression was still a general vector/matrix expression where the specific vector coordinate frame was not specified.

- Let us introduce the orbit frame *O* which tracks the center of mass of the rigid body as it rotates about the Earth.

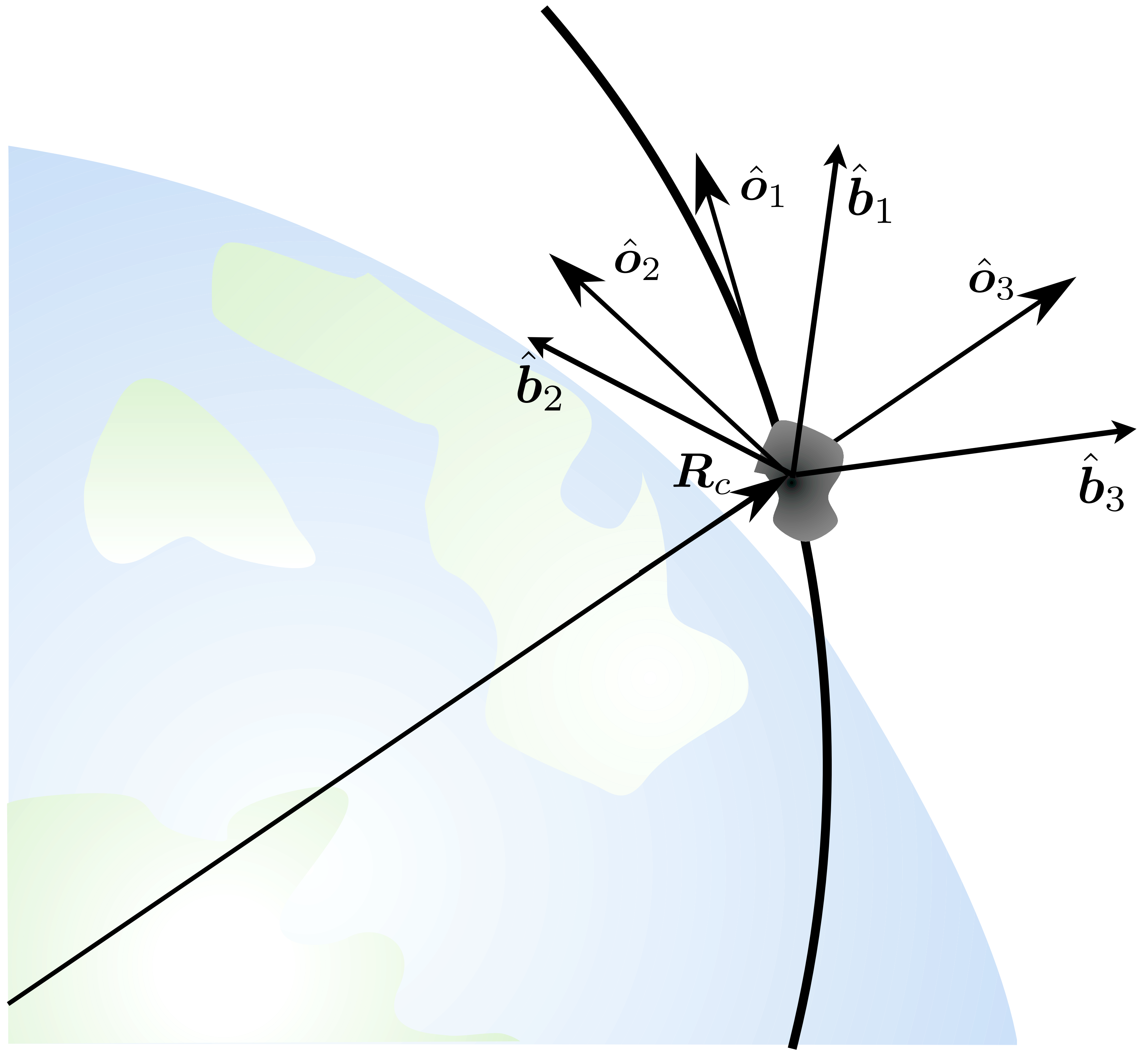


$$O : \{\hat{\boldsymbol{o}}_1, \hat{\boldsymbol{o}}_2, \hat{\boldsymbol{o}}_3\}$$

$$\boldsymbol{R}_c = R_c \hat{\boldsymbol{o}}_3 = {}^{\mathcal{O}}\begin{pmatrix} 0 \\ 0 \\ R_c \end{pmatrix}$$

- Since the rigid body dynamics are written in the body frame $\mathcal{B}$, we can write the center of mass position vector in the body frame using:

$$\boldsymbol{R}_c = {}^{\mathcal{B}}\begin{pmatrix} R_{c_1} \\ R_{c_2} \\ R_{c_3} \end{pmatrix} = [BO]\ {}^{\mathcal{O}}\begin{pmatrix} 0 \\ 0 \\ R_c \end{pmatrix}$$



- Assuming that the inertia matrix [*I*] is taken with respect to a principal coordinate system (i.e. [*I*] is diagonal), the gravity torque vector can now be written as

$$\boldsymbol{L}_G = \frac{3GM_e}{R_c^5}\begin{pmatrix} R_{c_1} \\ R_{c_2} \\ R_{c_3} \end{pmatrix} \times \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}\begin{pmatrix} R_{c_1} \\ R_{c_2} \\ R_{c_3} \end{pmatrix}$$

$$\boxed{\boldsymbol{L}_G = \begin{pmatrix} L_{G_1} \\ L_{G_2} \\ L_{G_3} \end{pmatrix} = \frac{3GM_e}{R_c^5}\begin{pmatrix} R_{c_2}R_{c_3}\left(I_{33} - I_{22}\right) \\ R_{c_1}R_{c_3}\left(I_{11} - I_{33}\right) \\ R_{c_1}R_{c_2}\left(I_{22} - I_{11}\right) \end{pmatrix}}$$

This torque (or components thereof) can be zero if:

$$I_{ii} = I_{jj} \qquad\qquad \boldsymbol{R}_c = R_c \hat{\boldsymbol{b}}_i$$

Center of Mass Motion

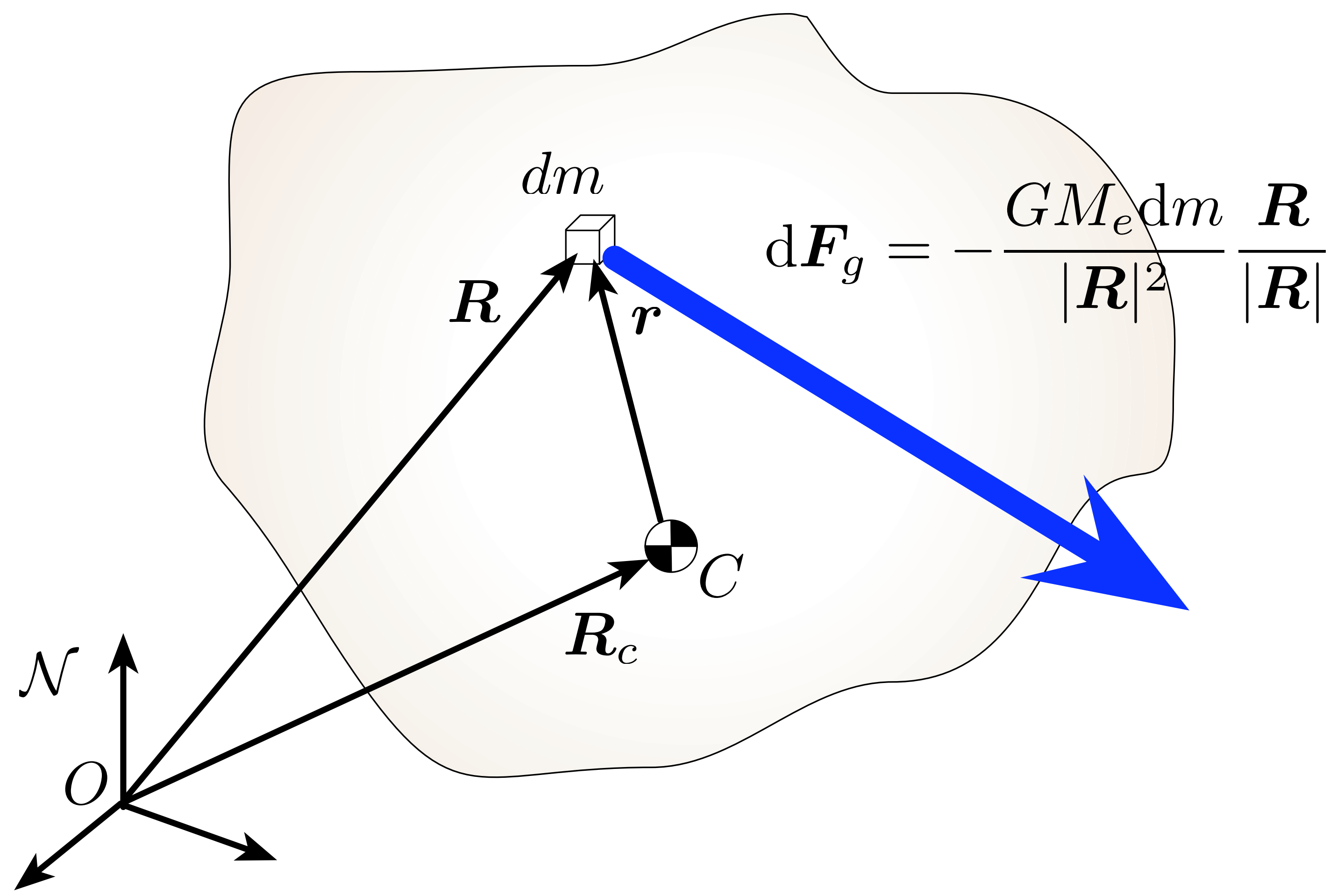
- Next, let's study how the center of mass of a rigid body will move while in orbit.
- From astrodynamics, we have seen that the orbit of a point mass *m* has the differential equations of motion:

$$\ddot{\boldsymbol{R}}_c = -\frac{GM_e}{R_c^3}\boldsymbol{R}_c \qquad \text{or} \qquad m\ddot{\boldsymbol{R}}_c = -\frac{GM_e}{R_c^3}m\boldsymbol{R}_c$$

Question: *How will the center of mass of a rigid body move? The gravitation force on the various mass elements will contribute to accelerate the CM.*

- The total gravity force is computed by integrating all gravity forces over the entire body:

$$\boldsymbol{F}_G = \int_{\mathcal{B}} \mathrm{d}\boldsymbol{F}_G = -GM_e \int_{\mathcal{B}} \frac{\boldsymbol{R}}{|\boldsymbol{R}|^3}\mathrm{d}m$$



Using the simplifying assumption:

$$\frac{1}{|\boldsymbol{R}|^3} \approx \frac{1}{R_c^3}\left[1 - \frac{3}{2}\left(2\frac{\boldsymbol{R}_c \cdot \boldsymbol{r}}{R_c^2} + \frac{\boldsymbol{r}\cdot\boldsymbol{r}}{R_c^2}\right) + \frac{15}{2}\frac{(\boldsymbol{R}_c \cdot \boldsymbol{r})^2}{R_c^4}\right]$$

we find the gravity force expression:

$$\boldsymbol{F}_G = -\frac{GM_e}{R_c^3}\left[\cancel{\int_{\mathcal{B}} \boldsymbol{r}\mathrm{d}m} - \frac{3}{R_c^2}\int_{\mathcal{B}}(\boldsymbol{r}\cdot\boldsymbol{R}_c)\boldsymbol{r}\mathrm{d}m - \cancel{\frac{3}{R_c^2}\int_{\mathcal{B}}(\boldsymbol{R}_c\cdot\boldsymbol{r})\boldsymbol{R}_c\mathrm{d}m}\right.$$
$$\left. + \boldsymbol{R}_c\int_{\mathcal{B}}\mathrm{d}m - \frac{3}{2R_c^2}\int_{\mathcal{B}}\boldsymbol{R}_c(\boldsymbol{r}\cdot\boldsymbol{r})\mathrm{d}m + \frac{15}{2R_c^4}\int_{\mathcal{B}}(\boldsymbol{R}_c\cdot\boldsymbol{r})^2\boldsymbol{R}_c\mathrm{d}m\right]$$

- Using the center of mass definition and the vector identity

$$-(\boldsymbol{r}\cdot\boldsymbol{R}_c)\boldsymbol{r} = -\boldsymbol{r}\times(\boldsymbol{r}\times\boldsymbol{R}_c) - (\boldsymbol{r}\cdot\boldsymbol{r})\boldsymbol{R}_c$$

leads to:

$$\boldsymbol{F}_G = -\frac{GM_e}{R_c^3}\left[m\boldsymbol{R}_c - \frac{3}{R_c^2}\int_{\mathcal{B}}\Big(\boldsymbol{r}\times(\boldsymbol{r}\times\boldsymbol{R}_c) + r^2\boldsymbol{R}_c\Big)\mathrm{d}m\right.$$
$$\left. - \frac{3}{2R_c^2}\int_{\mathcal{B}} r^2\boldsymbol{R}_c\mathrm{d}m + \frac{15}{2R_c^4}\int_{\mathcal{B}}\boldsymbol{R}_c\cdot\Big(\boldsymbol{r}\times(\boldsymbol{r}\times\boldsymbol{R}_c) + r^2\boldsymbol{R}_c\Big)\boldsymbol{R}_c\mathrm{d}m\right]$$

Next, note the identities:

$$\int_{\mathcal{B}} r^2\mathrm{d}m = \frac{1}{2}\mathrm{tr}([I]) \qquad\qquad \hat{\boldsymbol{\imath}}_r = \boldsymbol{R}_c/R_c$$

- Using the inertia matrix definition

$$[I_c] = \int_B -[\tilde{\boldsymbol{r}}][\tilde{\boldsymbol{r}}]\mathrm{d}m$$

The gravity force is finally written as

$$\boxed{\boldsymbol{F}_G = -\frac{\mu m}{R_c^3}\left[1 + \frac{3}{mR_c^2}\left([I] + \frac{1}{2}\Big(\mathrm{tr}([I]) - 5(\hat{\boldsymbol{i}}_r^T[I]\hat{\boldsymbol{i}}_r)\Big)[I_{3\times 3}]\right)\right]\boldsymbol{R}_c}$$

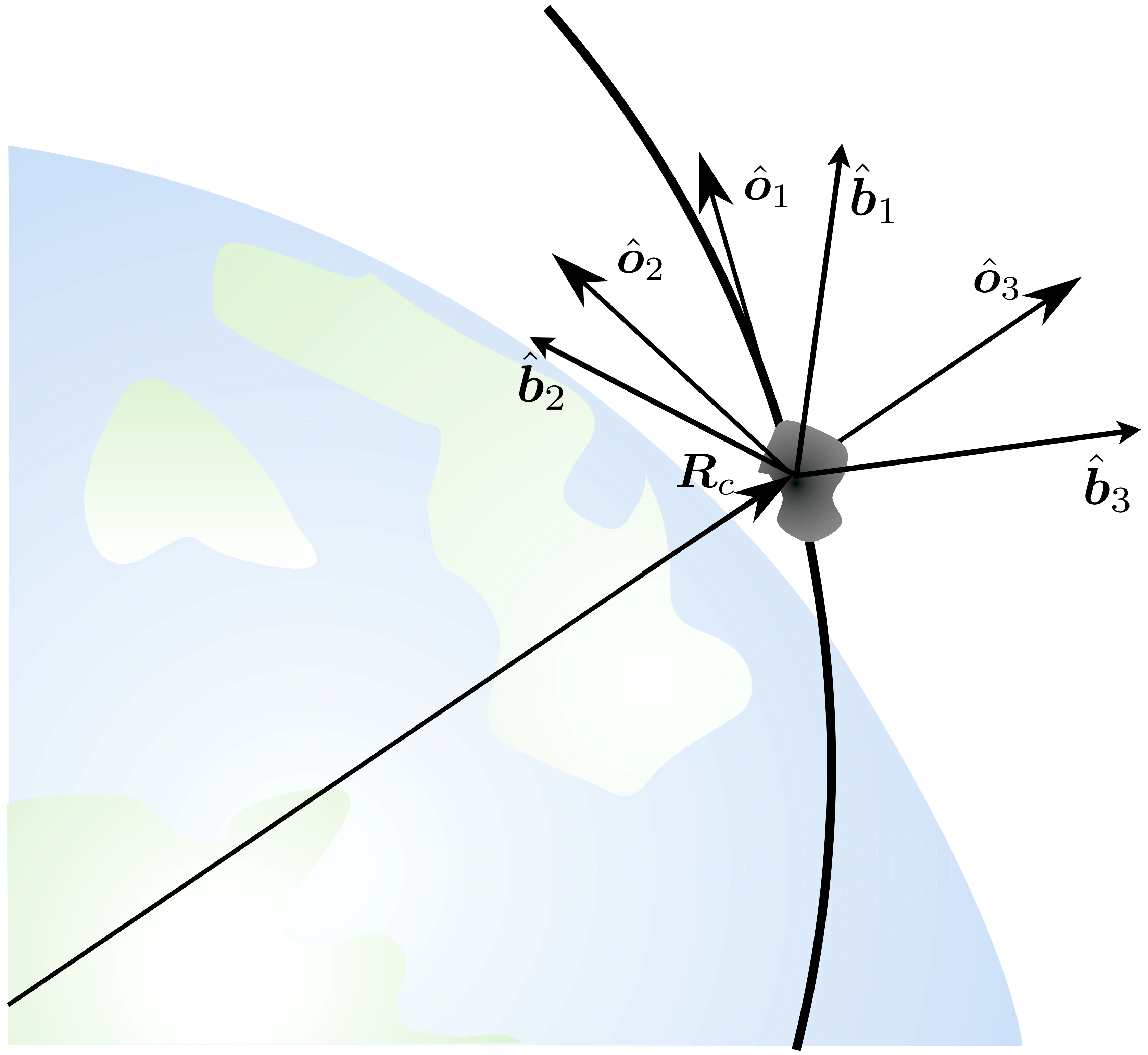
Compare to typical orbit force expression of point mass:

$$m\ddot{\boldsymbol{R}}_c = -\frac{GM_e}{R_c^3}m\boldsymbol{R}_c$$

The rigid body coupling to the center of mass motion is often ignored, because it is many, many orders of magnitude smaller than the orbital acceleration.

Relative Equilibrium State

We seek a rigid body attitude/orientation where the craft will remain stationary as seen by the rotating orbit frame.



Equations of motion:

$$[I]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = \boldsymbol{L}_G$$

Angular velocities:

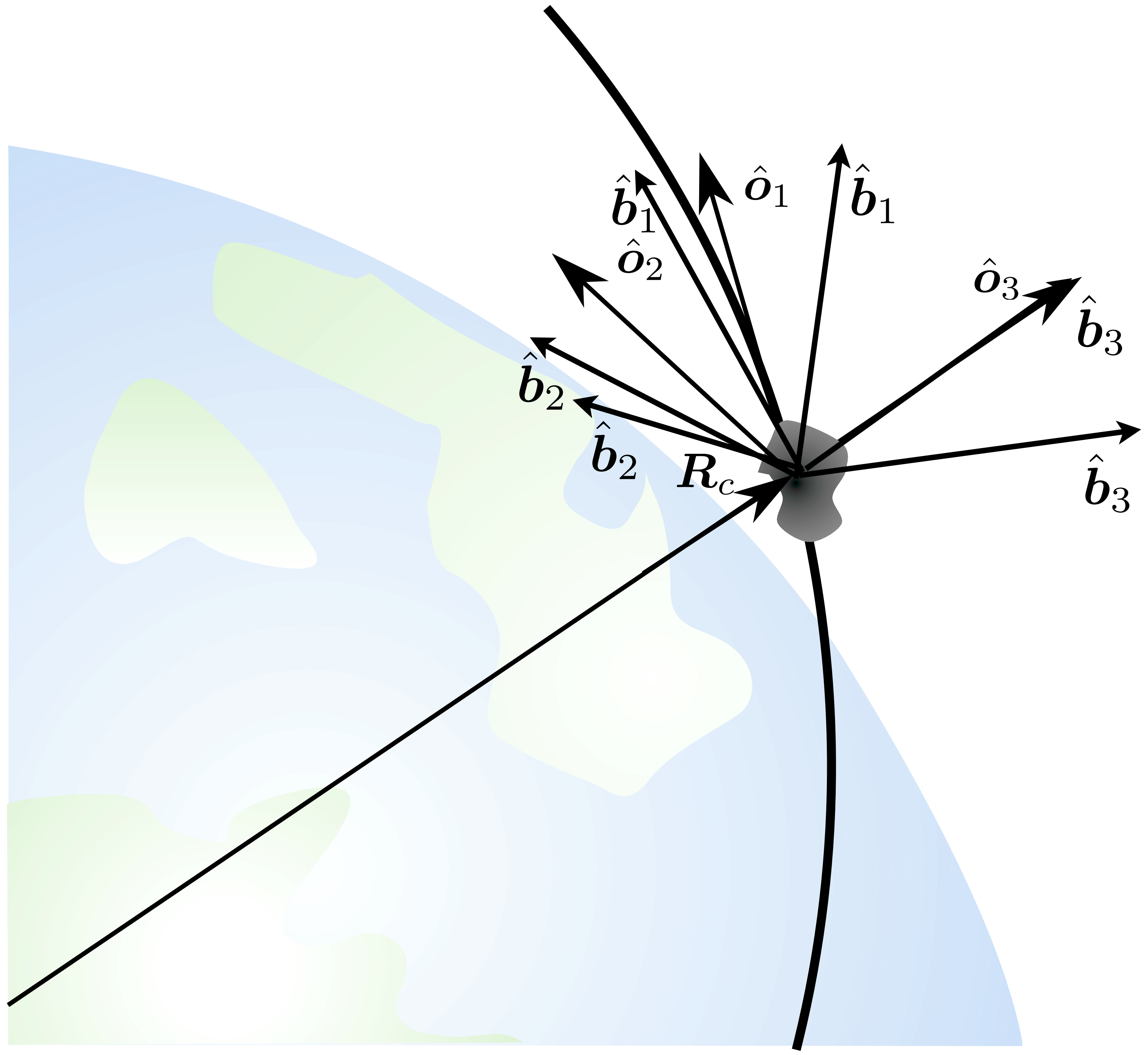
$$\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = \boldsymbol{\omega}_{\mathcal{B}/\mathcal{O}} + \boldsymbol{\omega}_{\mathcal{O}/\mathcal{N}}$$

$$\boldsymbol{\omega}_{\mathcal{O}/\mathcal{N}} = n\hat{\boldsymbol{o}}_2$$

$$\boldsymbol{\omega}_{\mathcal{B}/\mathcal{O}} = \boldsymbol{0} \longleftarrow \boxed{\text{Relative Equilibria Condition}}$$

Equations of motion:

$$[I]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = \boldsymbol{L}_G$$



Gravity Gradient Torque:

$$\boldsymbol{L}_G = \boldsymbol{0}$$

$$\hat{\boldsymbol{b}}_3 = \hat{\boldsymbol{o}}_3$$

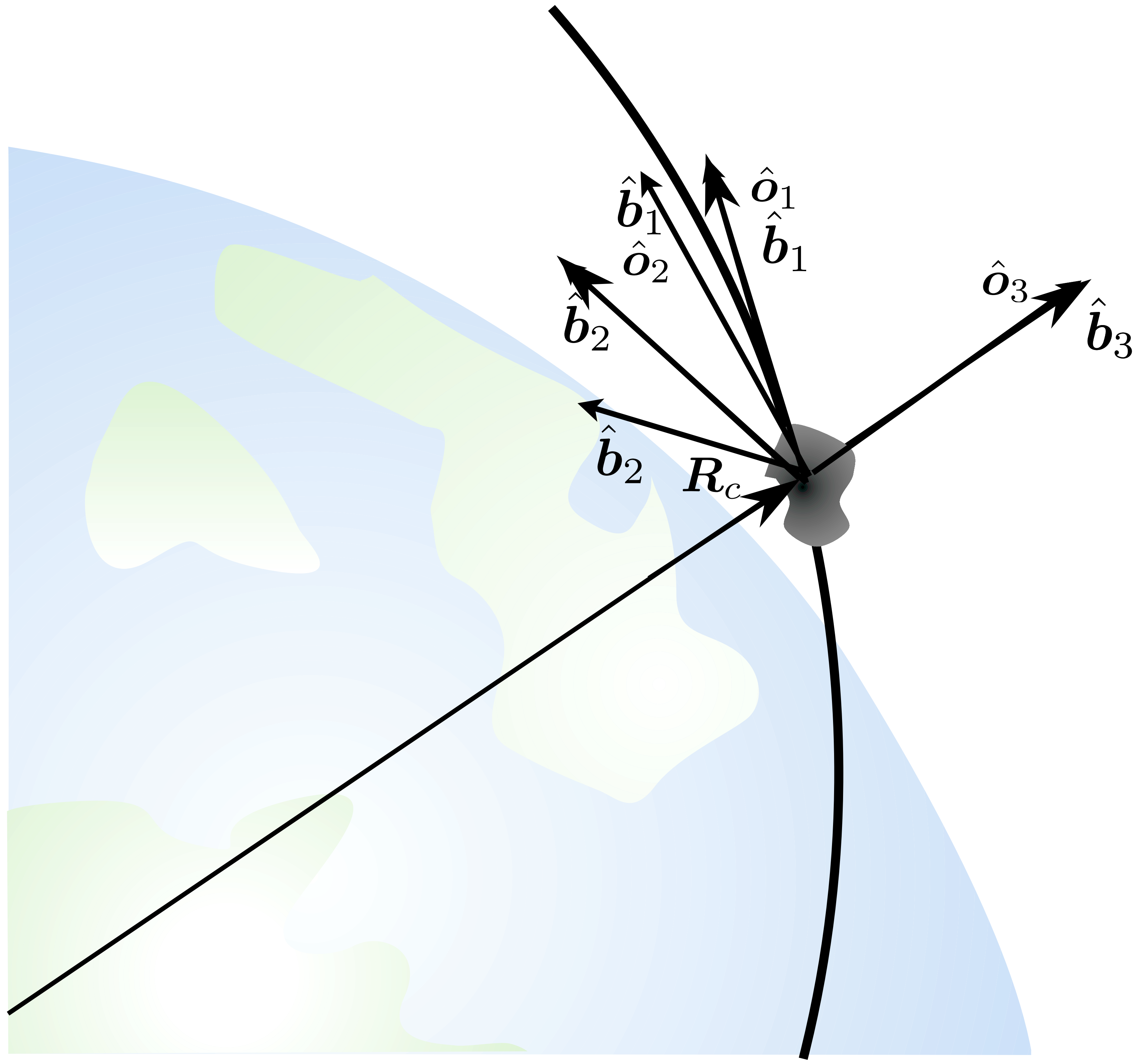
must be principal axis of body

$$[I] = {}^{\mathcal{O}}\begin{bmatrix} I_{11} & I_{12} & 0 \\ I_{12} & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix}$$

This leads to this block diagonal form

Equations of motion:

$$[I]\dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} + [\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = \boldsymbol{L}_G$$



Angular velocity condition:

$$\boldsymbol{\omega}_{\mathcal{B}/\mathcal{O}} = \boldsymbol{0} \qquad \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{O}} = \boldsymbol{0}$$

$$\boldsymbol{\omega}_{\mathcal{O}/\mathcal{N}} = n\hat{\boldsymbol{o}}_2 \qquad \dot{\boldsymbol{\omega}}_{\mathcal{O}/\mathcal{N}} = \boldsymbol{0}$$

$$[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}][I]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = \boldsymbol{0}$$

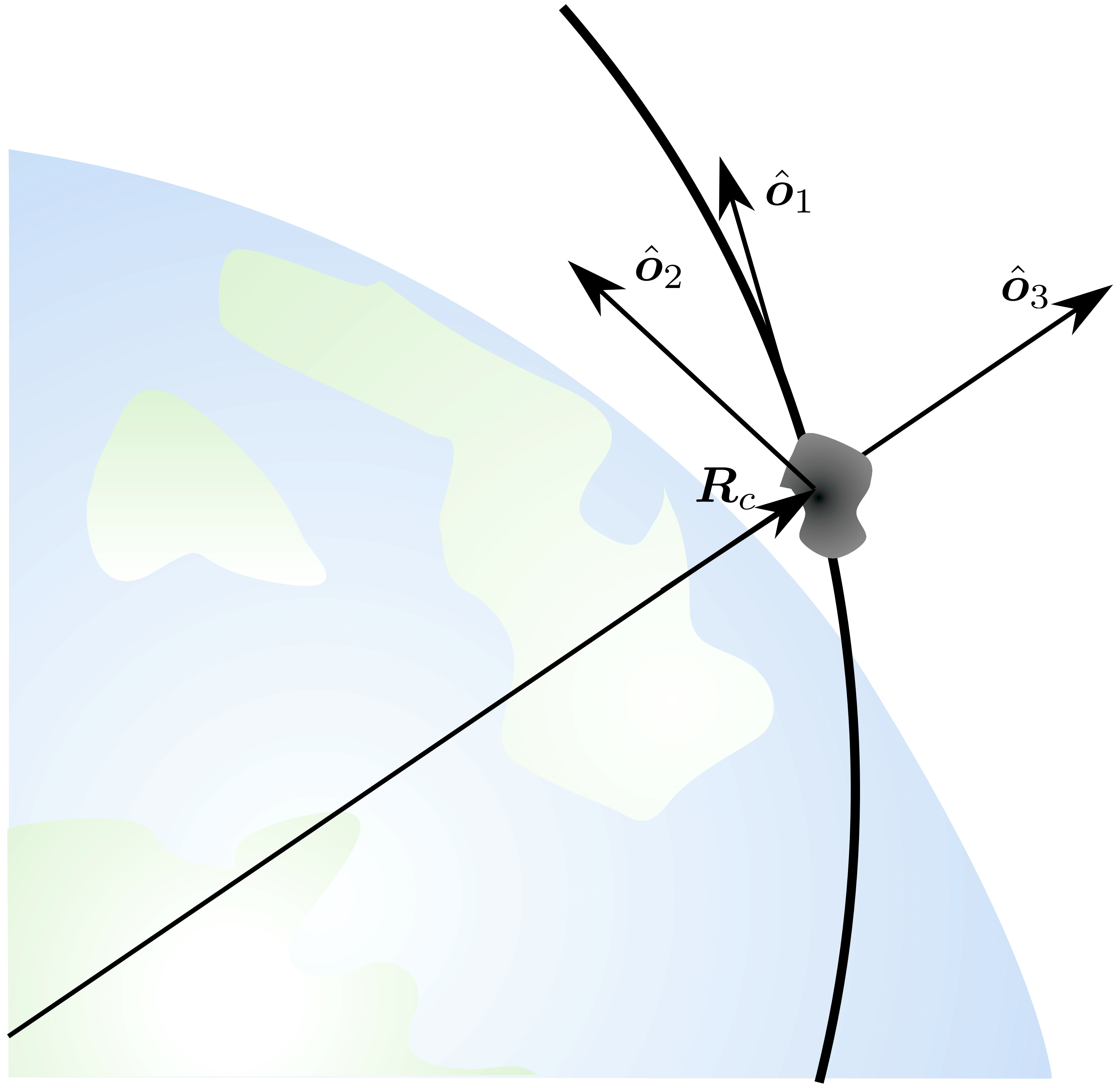
$$^{\mathcal{O}}\begin{pmatrix} 0 \\ 0 \\ -I_{12}n^2 \end{pmatrix} = \boldsymbol{0}$$

$$I_{12} = 0$$

As a result, we find that relative equilibria must have all principal axes aligned with the orbit frame.

Gravity Gradient Motion

- Next, we study how the gravity torque vector will rotate the craft.
- The gravity torque is the only external force acting on the single-rigid body spacecraft.
- We use the “airplane” and “ship” like orbit frame O.
- Note, roll is about $\hat{o}_1$, pitch is about $\hat{o}_2$, yaw is about $\hat{o}_3$.



- The inertial angular velocity of the orbital frame *O* is:

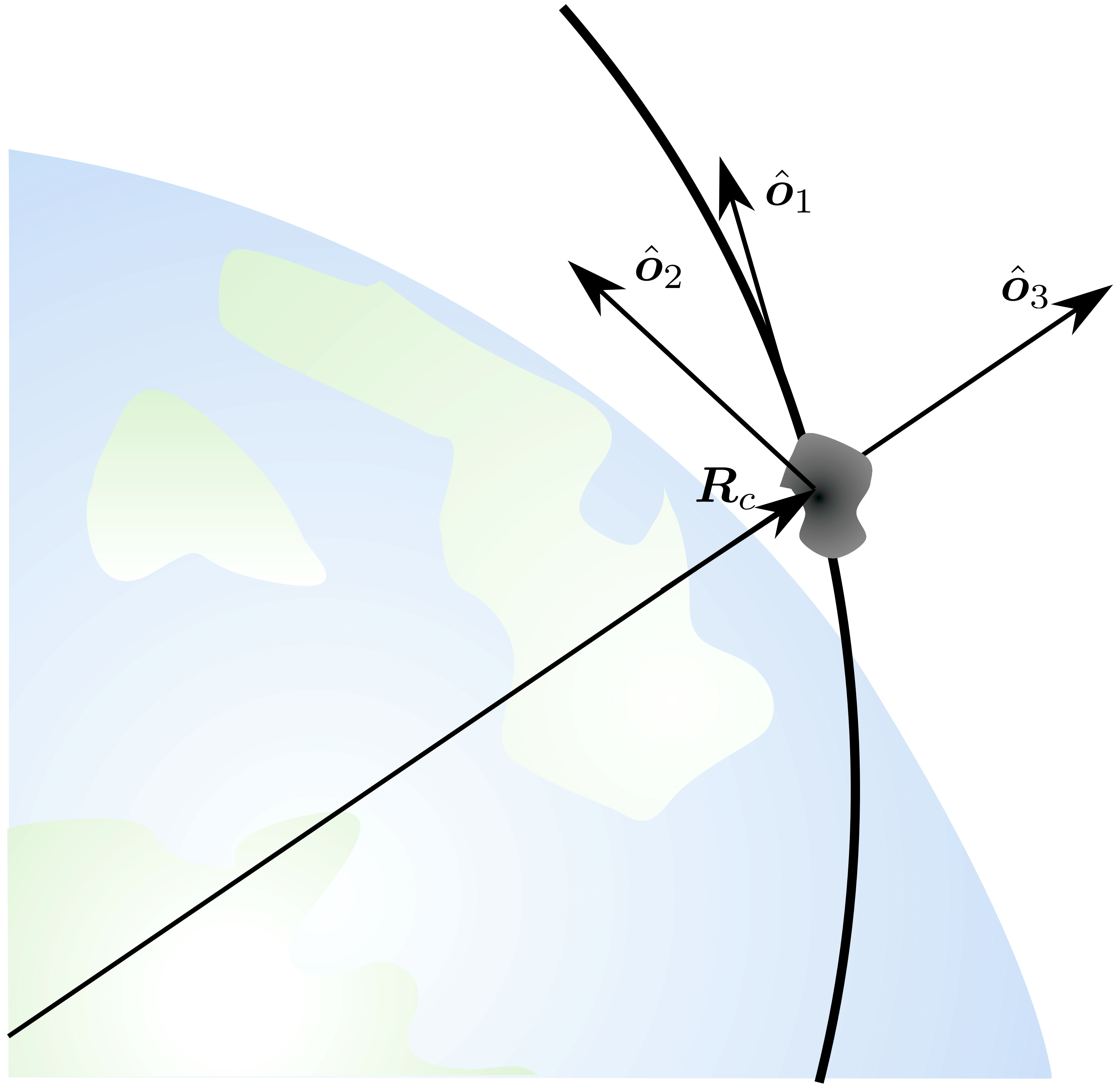
$$\boldsymbol{\omega}_{\mathcal{O}/\mathcal{N}} = \Omega \hat{\boldsymbol{o}}_2$$

- For a circular orbit, the constant orbit rate magnitude is given through Kepler's equation:

$$\Omega^2 = \frac{GM_e}{R_c^3}$$

- The spacecraft frame *B* is assumed to be a principal coordinate system. It's angular rate relative to the orbit frame *O* is:

$$\boldsymbol{\omega}_{\mathcal{B}/\mathcal{O}}$$



- We would like to study the yaw, pitch and roll motion (3-2-1 Euler angles) of the rigid body, if the gravitational torque is acting on it.

- From rigid body kinematics, we can relative the yaw, pitch and roll rates to body angular velocities through:

$$^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{O}} = \begin{bmatrix} -\sin\theta & 0 & 1 \\ \sin\phi\cos\theta & \cos\phi & 0 \\ \cos\phi\cos\theta & -\sin\phi & 0 \end{bmatrix} \begin{pmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

- The spacecraft angular velocity relative to the *inertial* frame *N* is:

$$\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = \boldsymbol{\omega}_{\mathcal{B}/\mathcal{O}} + \boldsymbol{\omega}_{\mathcal{O}/\mathcal{N}}$$

- Further, also from rigid body kinematics, we can express the [*BO*] rotation matrix using the yaw, pitch and roll angles of the spacecraft with respect to the orbit frame *O* as:

$$[BO] = \begin{bmatrix} c\theta c\psi & c\theta s\psi & -s\theta \\ s\phi s\theta c\psi - c\phi s\psi & s\phi s\theta s\psi + c\phi c\psi & s\phi c\theta \\ c\phi s\theta c\psi + s\phi s\psi & c\phi s\theta s\psi - s\phi c\psi & c\phi c\theta \end{bmatrix}$$

- To be able to add up the angular rate vectors in *B* frame components, we find

$$\begin{aligned} {}^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{O}/\mathcal{N}} &= [BO]\,{}^{\mathcal{O}}\boldsymbol{\omega}_{\mathcal{O}/\mathcal{N}} = [BO](\Omega\hat{\boldsymbol{o}}_2) \\ &= \Omega \begin{pmatrix} c\theta s\psi \\ s\phi s\theta s\psi + c\phi c\psi \\ c\phi s\theta s\psi - s\phi c\psi \end{pmatrix} \end{aligned}$$

- Now we are able to compute the inertial body angular velocity vector:

$$\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = \boldsymbol{\omega}_{\mathcal{B}/\mathcal{O}} + \boldsymbol{\omega}_{\mathcal{O}/\mathcal{N}}$$

$$^{\mathcal{B}}\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = {}^{\mathcal{B}}\begin{pmatrix} \dot{\phi} - s\theta\dot{\psi} + \Omega c\theta s\psi \\ s\phi c\theta\dot{\psi} + c\phi\dot{\theta} + \Omega(s\phi s\theta s\psi + c\phi c\psi) \\ c\phi c\theta\dot{\psi} - s\phi\dot{\theta} + \Omega(c\phi s\theta s\psi - s\phi c\psi) \end{pmatrix}$$

- This expression is valid for any large rotation of the spacecraft with respect to the orbit frame.

- Next, we would like to look at small rotation about the *O* frame. Here the yaw, pitch and roll angles are all treated as small angles.

$$\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = \boldsymbol{\omega} = {}^{\mathcal{B}}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \approx {}^{\mathcal{B}}\begin{pmatrix} \dot{\phi} + \Omega\psi \\ \dot{\theta} + \Omega \\ \dot{\psi} - \Omega\phi \end{pmatrix}$$

- The inertial angular acceleration is approximated as:

$$\dot{\boldsymbol{\omega}} = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\omega}) + \boldsymbol{\omega} \times \boldsymbol{\omega} \approx {}^{\mathcal{B}}\begin{pmatrix} \ddot{\phi} + \Omega\dot{\psi} \\ \ddot{\theta} \\ \ddot{\psi} - \Omega\dot{\phi} \end{pmatrix}$$

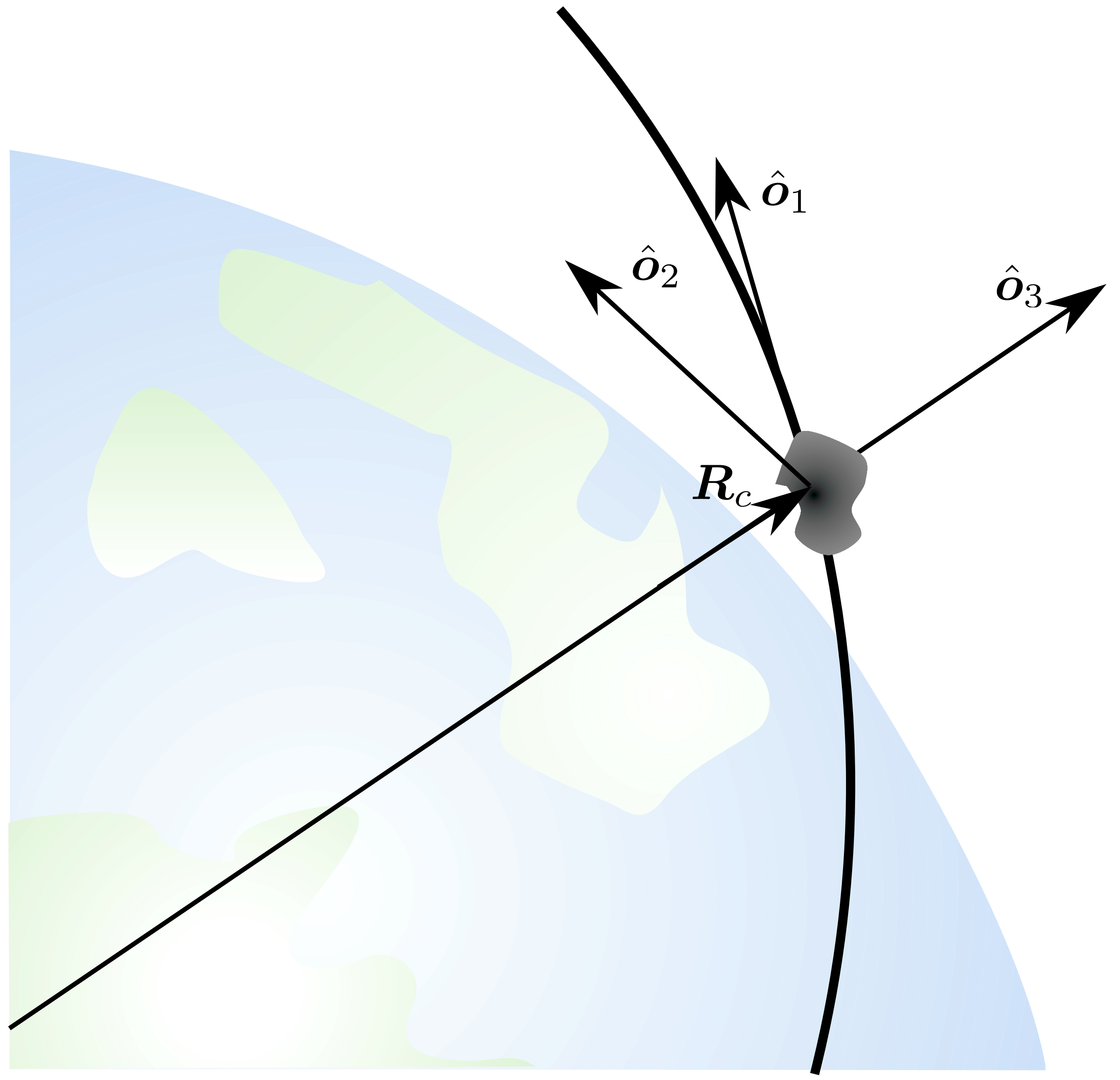
- These two equations can later be used in the rigid body equations of motion:

$$[I]\dot{\boldsymbol{\omega}} = -[\tilde{\omega}][I]\boldsymbol{\omega} + \boldsymbol{L}_c$$

- We still need to simplify the external gravity gradient torque expression for the case where yaw, pitch and roll are small angles

$$\boldsymbol{R}_c = {}^{\mathcal{B}}\begin{pmatrix} R_{c_1} \\ R_{c_2} \\ R_{c_3} \end{pmatrix} = [BO] \, {}^{\mathcal{O}}\begin{pmatrix} 0 \\ 0 \\ R_c \end{pmatrix}$$

$${}^{\mathcal{B}}\begin{pmatrix} R_{c_1} \\ R_{c_2} \\ R_{c_3} \end{pmatrix} = {}^{\mathcal{B}}\begin{pmatrix} -\sin\theta \\ \sin\phi\cos\theta \\ \cos\phi\cos\theta \end{pmatrix} R_c$$



Substituting this $\boldsymbol{R}_c$ expression into the gravity gradient torque definition, we find:

$$\boldsymbol{L}_G = \begin{pmatrix} L_{G_1} \\ L_{G_2} \\ L_{G_3} \end{pmatrix} = \frac{3GM_e}{R_c^5} \begin{pmatrix} R_{c_2}R_{c_3}\left(I_{33}-I_{22}\right) \\ R_{c_1}R_{c_3}\left(I_{11}-I_{33}\right) \\ R_{c_1}R_{c_2}\left(I_{22}-I_{11}\right) \end{pmatrix} \Rightarrow {}^{\mathcal{B}}\boldsymbol{L}_G = \frac{3}{2}\Omega^2 \, {}^{\mathcal{B}}\begin{pmatrix} \left(I_{33}-I_{22}\right)\cos^2\theta\sin 2\phi \\ -\left(I_{11}-I_{33}\right)\cos\phi\sin 2\theta \\ -\left(I_{22}-I_{11}\right)\sin\phi\sin 2\theta \end{pmatrix}$$

- The nonlinear gravity torque vector is repeated here as:

$$
{}^{\mathcal{B}}\boldsymbol{L}_G = \frac{3}{2}\Omega^2 \, {}^{\mathcal{B}}\!\begin{pmatrix} (I_{33} - I_{22}) \cos^2\theta \sin 2\phi \\ -(I_{11} - I_{33}) \cos\phi \sin 2\theta \\ -(I_{22} - I_{11}) \sin\phi \sin 2\theta \end{pmatrix}
$$

- Note that the body frame torque components **do not depend on the yaw angle**.

- Linearizing this torque for small attitude angles, we find:

$$
{}^{\mathcal{B}}\boldsymbol{L}_G \approx 3\Omega^2 \, {}^{\mathcal{B}}\!\begin{pmatrix} (I_{33} - I_{22}) \phi \\ -(I_{11} - I_{33}) \theta \\ 0 \end{pmatrix}
$$

Note: the linearized torque will never have a yaw component.

- Now we are able to write the equations of motion of a rigid spacecraft in a circular orbit subject to an inverse square gravity field. We use

$$[I]\dot{\boldsymbol{\omega}} = -[\tilde{\omega}][I]\boldsymbol{\omega} + \boldsymbol{L}_c$$

and substitute in the previous linearized results to find

$$\begin{aligned} I_{11}\left(\ddot{\phi} + \Omega\dot{\psi}\right) &= -\left(I_{33} - I_{22}\right)\left(\dot{\theta} + \Omega\right)\left(\dot{\psi} - \Omega\phi\right) + 3\Omega^2\left(I_{33} - I_{22}\right)\phi \\ I_{22}\ddot{\theta} &= -\left(I_{11} - I_{33}\right)\left(\dot{\psi} - \Omega\phi\right)\left(\dot{\phi} + \Omega\psi\right) - 3\Omega^2\left(I_{11} - I_{33}\right)\theta \\ I_{33}\left(\ddot{\psi} - \Omega\dot{\phi}\right) &= -\left(I_{22} - I_{11}\right)\left(\dot{\phi} + \Omega\psi\right)\left(\dot{\theta} + \Omega\right) \end{aligned}$$

Note: These expression contain products of angles! Since we are assuming small angles here, these equations can be further simplified.

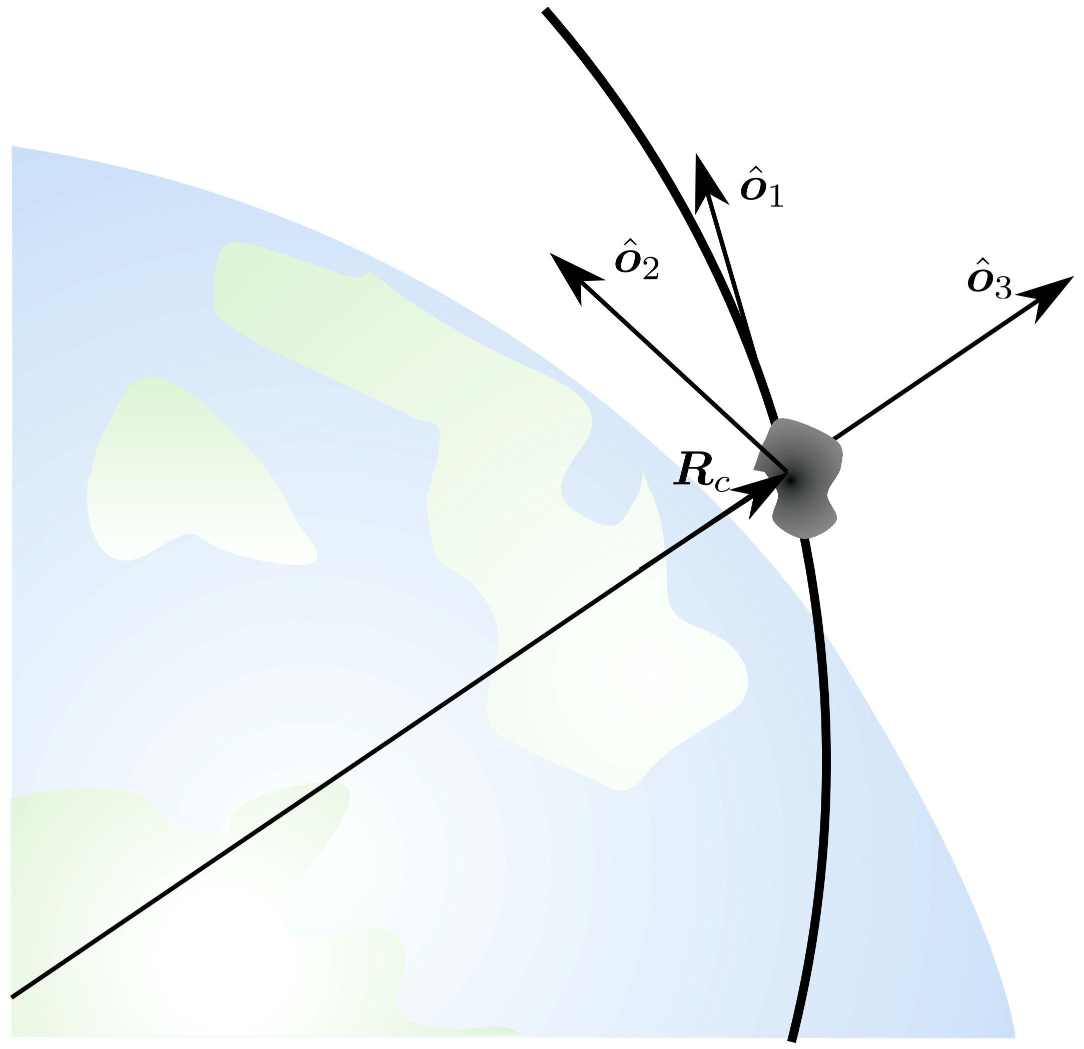
- The pitch equations can be decoupled from the yaw and roll equations!

$$\ddot{\theta} + 3\Omega^2 \left(\frac{I_{11} - I_{33}}{I_{22}} \right) \theta = 0$$

- Compare these equations to the spring-mass system

$$\ddot{x} + \frac{k}{m} x = 0$$

- This spring system is stable if the spring stiffness $k \geq 0$. Thus, the decoupled pitch motion is stable if



$$3\Omega^2 \left(\frac{I_{11} - I_{33}}{I_{22}} \right) \geq 0 \quad \Rightarrow \quad \boxed{I_{11} \geq I_{33}}$$

- The yaw and roll motion of the spacecraft are coupled through the gravity gradient torque:

$$\begin{pmatrix} \ddot{\phi} \\ \ddot{\psi} \end{pmatrix} + \begin{bmatrix} 0 & \Omega\left(1 - k_Y\right) \\ \Omega\left(k_R - 1\right) & 0 \end{bmatrix} \begin{pmatrix} \dot{\phi} \\ \dot{\psi} \end{pmatrix} + \begin{bmatrix} 4\Omega^2 k_Y & 0 \\ 0 & \Omega^2 k_R \end{bmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = 0$$

where we introduce the inertia ratios:

$$k_R = \frac{I_{22} - I_{11}}{I_{33}} \qquad k_Y = \frac{I_{22} - I_{33}}{I_{11}}$$

- To prove stability of this coupled linear time-invariant system, we need to examine the characteristic equation.

$$\lambda^4 + \lambda^2 \Omega^2 \left(1 + 3k_Y + k_Y k_R\right) + 4\Omega^4 k_Y k_R = 0$$

The system is stable if **NO** roots have positive real components!

- Let's rewrite the characteristic equation into the convenient form:

$$\lambda^4 + b_1\lambda^2 + b_0 = 0$$

- We can solve this as a quadratic equations for λ^2.

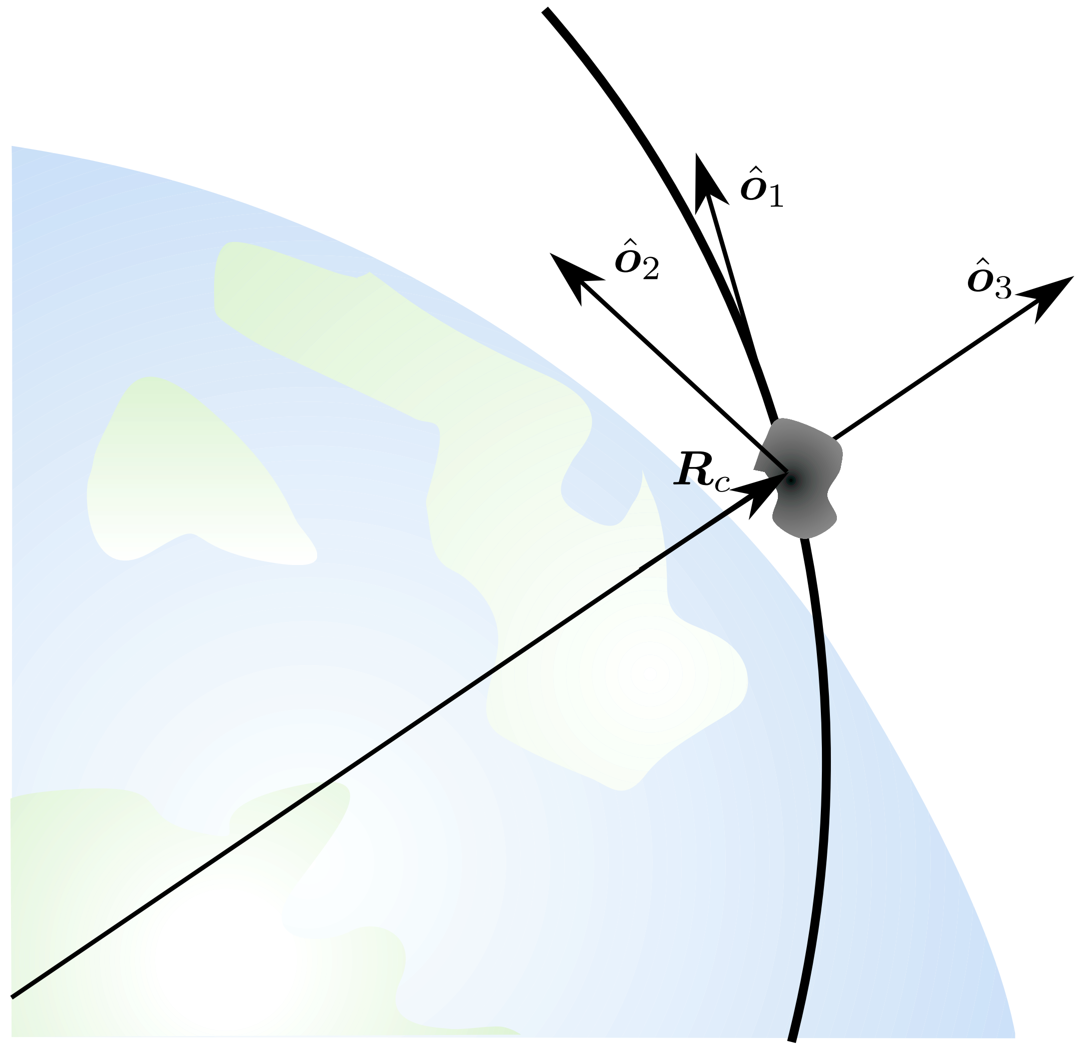
$$\lambda^2 = \frac{-b_1 \pm \sqrt{b_1^2 - 4b_0}}{2}$$

- Next, we need to check what conditions apply to guarantee that no root of this characteristic equation has positive real components.

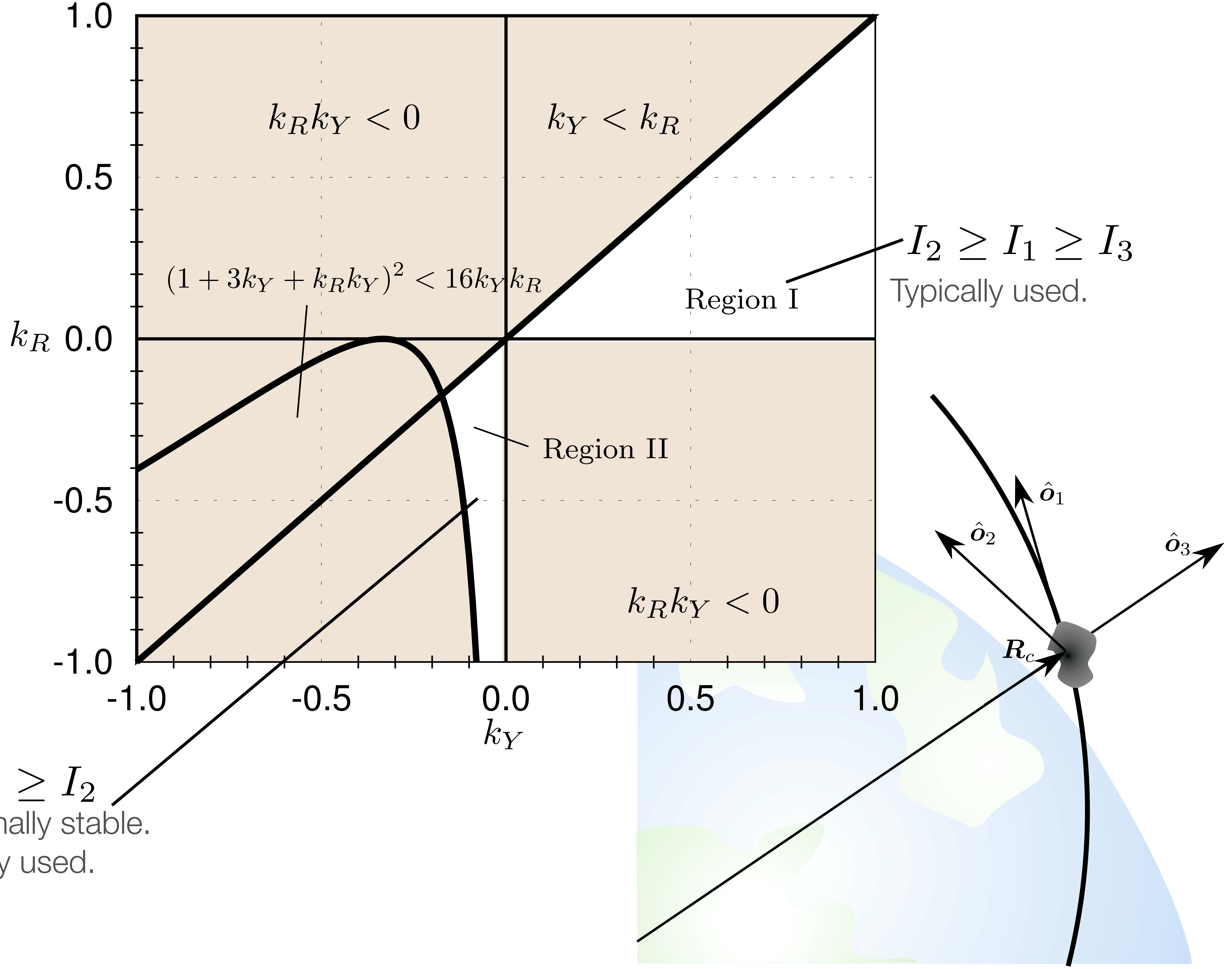
- The yaw-roll motion stability conditions can be summarized as:

$$\begin{aligned} k_R k_Y &> 0 \\ 1 + 3k_Y + k_Y k_R &> 0 \\ (1 + 3k_Y + k_Y k_R)^2 &> 16 k_Y k_R \\ k_Y &> k_R \end{aligned} \iff \begin{aligned} b_0 &> 0 \\ b_1 &> 0 \\ b_1^2 - 4b_0 &> 0 \\ I_{11} &> I_{33} \end{aligned}$$

Note: the first condition states that the only solutions will be in the first and third quadrant. This is equivalent to



$$\begin{aligned} I_{22} &> I_{11}, I_{33} \\ I_{22} &< I_{11}, I_{33} \end{aligned}$$



$I_2 \geq I_1 \geq I_3$

Typically used.

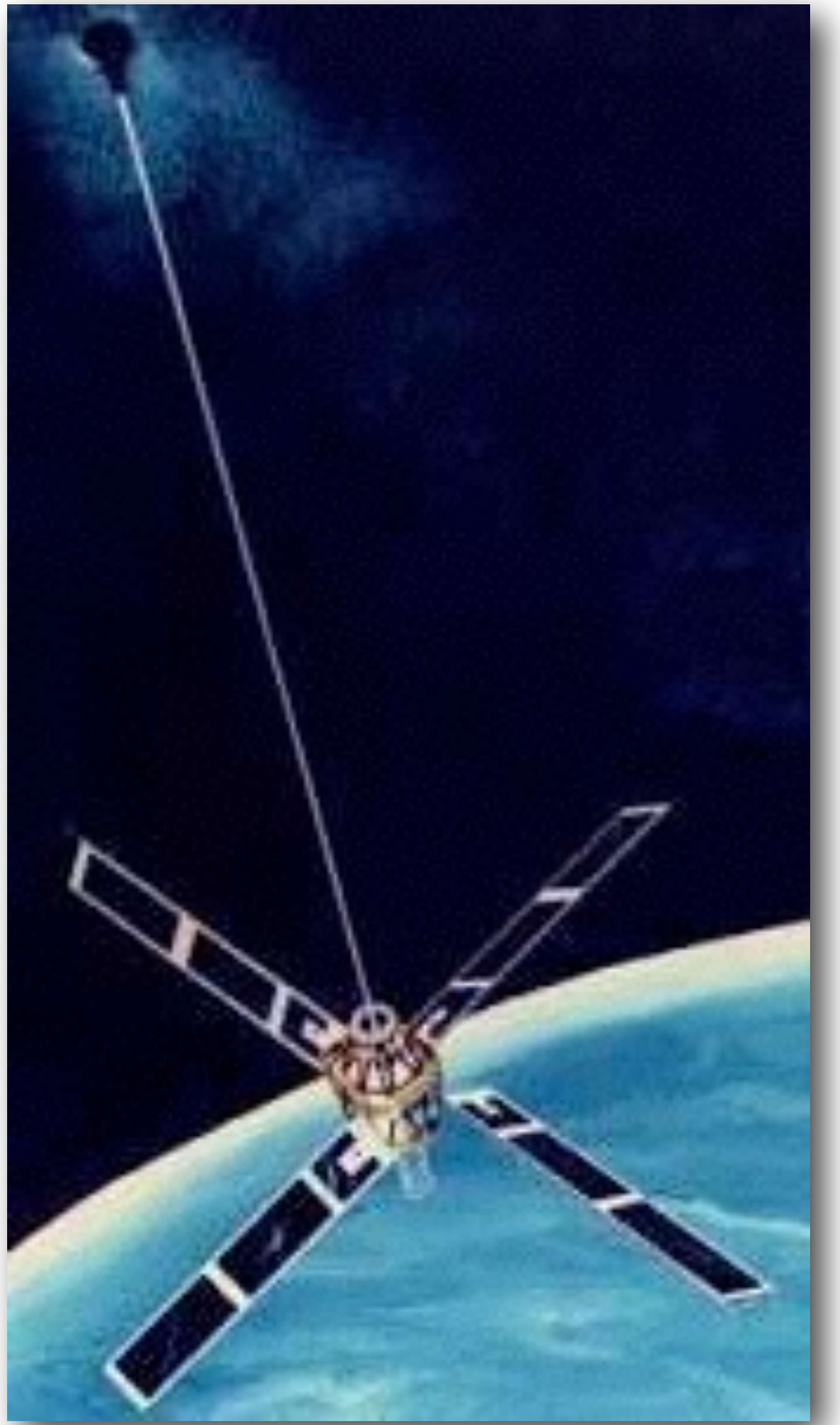
$I_1 \geq I_3 \geq I_2$

Only marginally stable.

Not typically used.

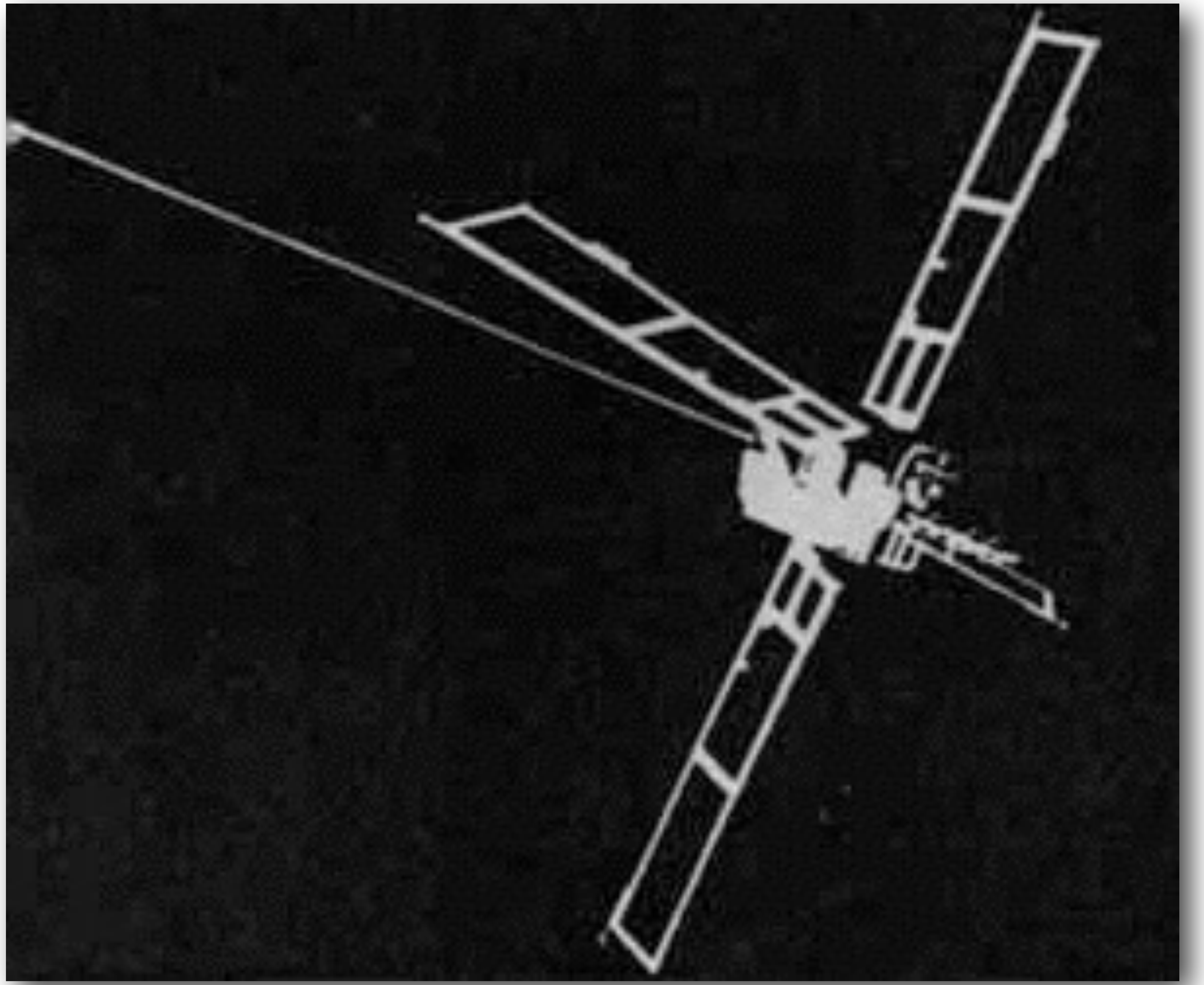
Polar Bear G^2 Mission

- Mission: Polar Bear (P87-1)
- Launched: Nov. 1986
- Goal: measure near-Earth plasma properties
- Attitude: gravity stabilized spacecraft
- Mass: 125 kg



Polar Bear G^2 Mission

- February 1987: After completing its first period of fully sunlit orbit, the attitude degraded significantly!
- May 1987: spacecraft inverted its attitude.
- Several attempts were undertaken to re-invert.
- Third attempt proved successful when the momentum wheel was allowed to despin for an orbit, before returning to max spin rate.



Momentum Exchange Devices

ASEN 5010

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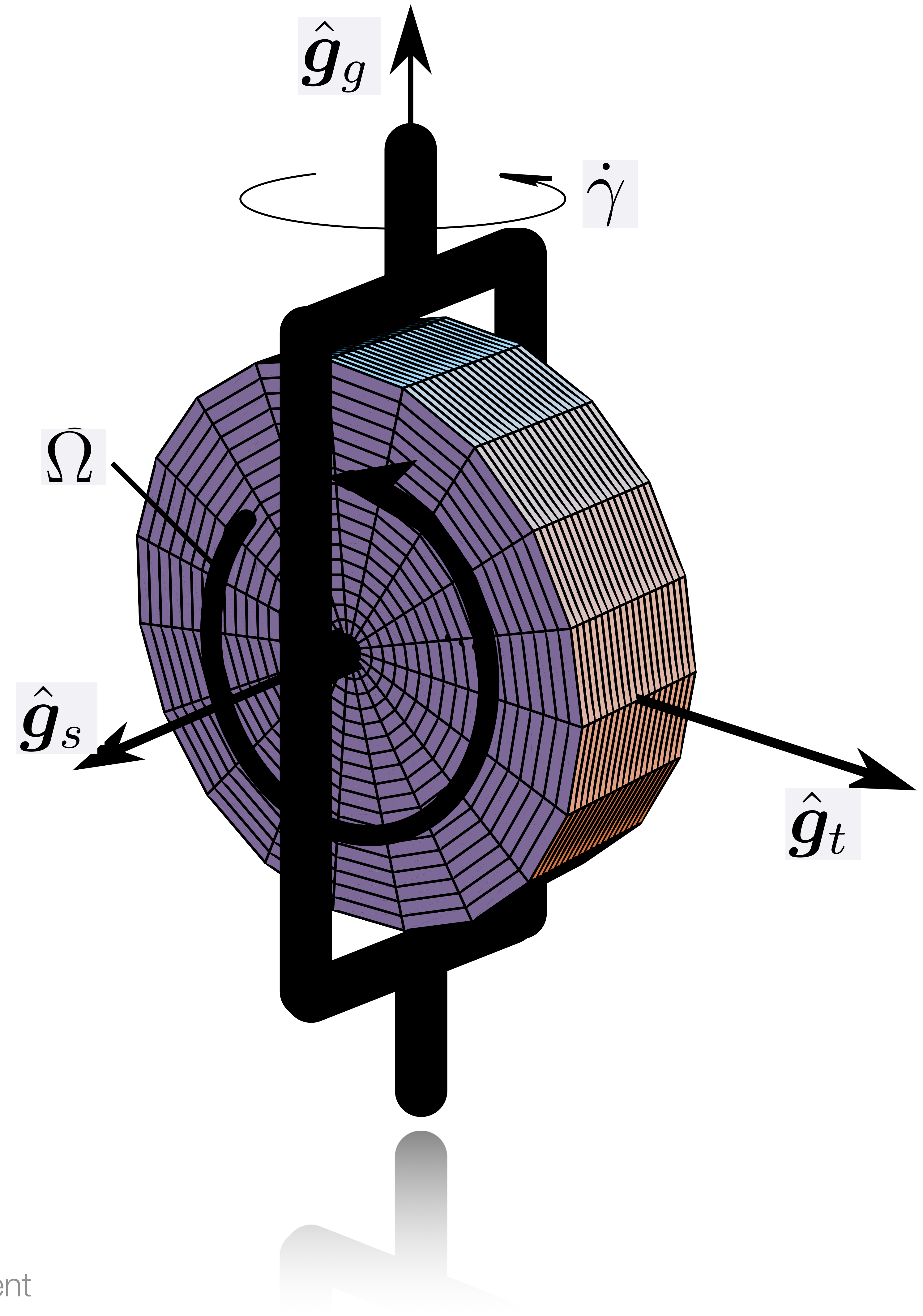
Outline

- Momentum Control Devices
- Equations of motion of VSCMG
  - single VSCMG
  - motor torque calculation
  - cluster of VSCMG
- Momentum Device Control
  - Overview of RW control solution

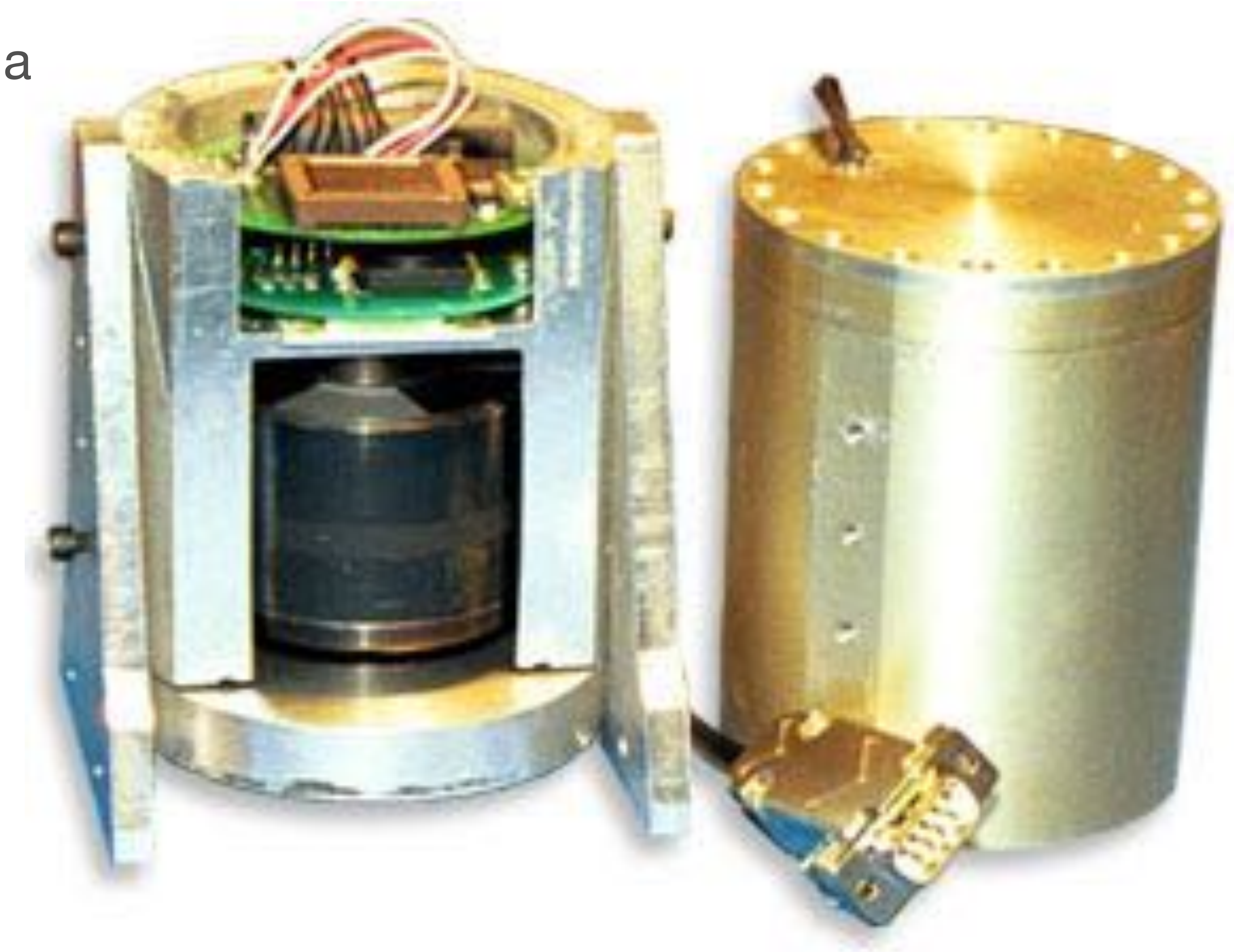


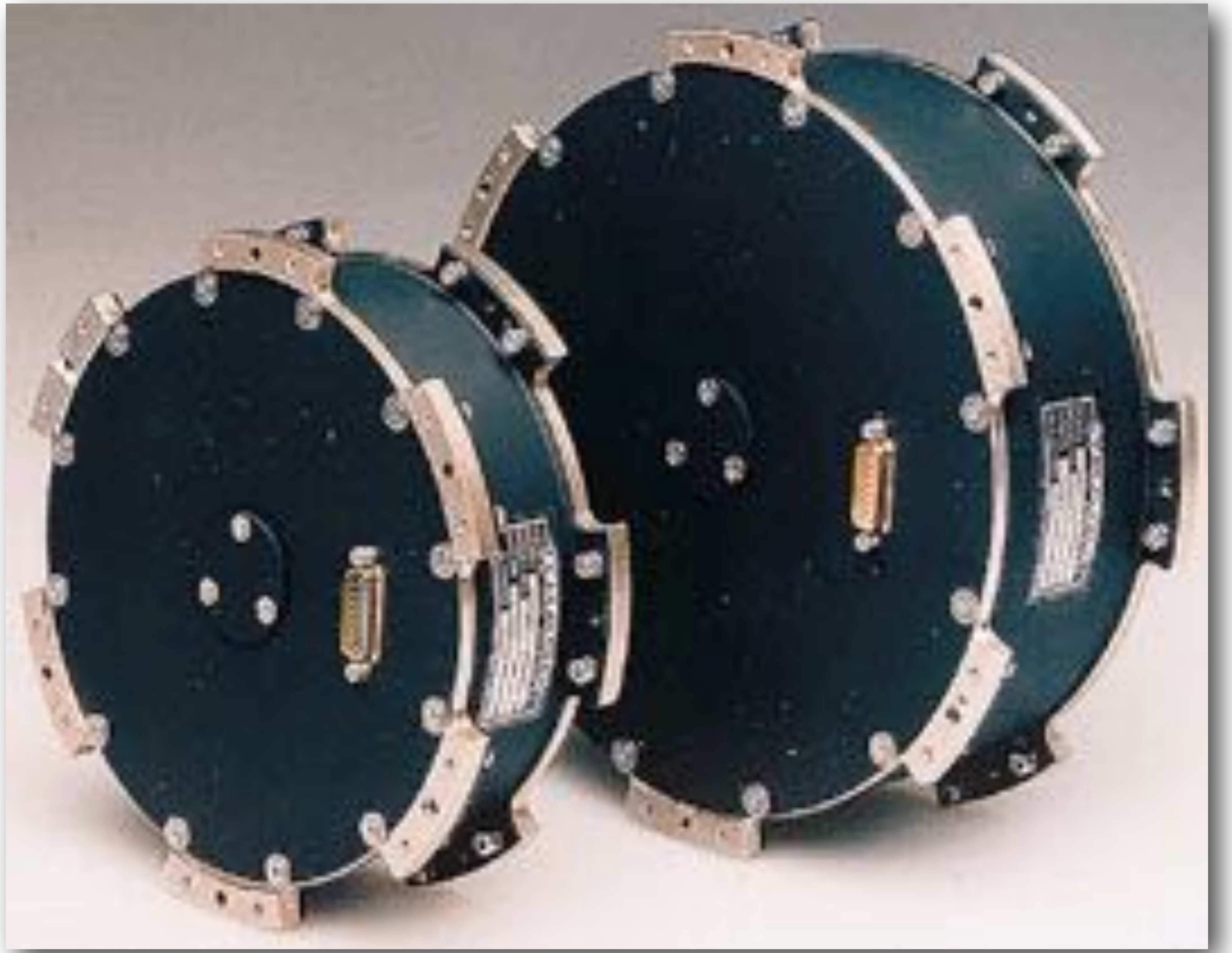
Momentum Control Devices

Spinning hardware “thingies” to rotate the spacecraft...

Reaction Wheels (RW)

- By increasing or decreasing the spin of a disk, a torque is applied onto the spacecraft.
- The torque is parallel to the disk spin axis.
- Simple mechanical device.
- Multiple disks can generate arbitrary torque.
- Wheels can saturate (reach a maximum spin speed.





ITHACO's low-cost but highly reliable reaction wheel designs keep spacecraft correctly oriented as they spin through space. (company description of device)
http://www.sti.nasa.gov/tto/spinoff1997/t2.html

Inside CTA Space System's High Torque Reaction/Momentum Wheel is an innovative flywheel/bearing arrangement that allows the entire rotating system to be balanced after it is assembled.   (company description)
http://www.sti.nasa.gov/tto/spinoff1997/t3.html

Honeywell’s reaction wheel assemblies (RWA) and momentum wheel assemblies (MWA) are reliable, lightweight solutions to a variety of momentum control needs, providing stability and attitude-control for small to very large, heavy spacecraft. Earth-pointing satellites and multiple-satellite communication networks are examples of applications that require the fine attitude control that Honeywell RWAs provide. RWAs and MWAs from Honeywell have accumulated more than 9 million hours, or more than a thousand years, in space and have never caused a mission to end prematurely.

Control Moment Gyroscope (CMG)

- Another popular attitude control device.
- A disk is spinning at a constant rate.
- By rotating this disk (called gimbaling), a torque is applied through the gyroscopic effect.
- For a small torque to gimbal the disk, a large torque is produced onto the spacecraft.

Control Moment Gyroscope (CMG)

- Mechanically more complex device than RWs
- Control laws are much more complicated.
- Very large torques can be produced (good for rapid reorientation or large spacecraft such as space station)
- Singular configurations exist where the required torque cannot be produced.

A CMG contains two torque motors. One to keep the disk spinning at a constant rate, the other to gimbal the spinning disk.

A typical CMG setup has 4 devices aligned in a pyramid configuration.

Equations of Motion

Let’s learn to be one with the truth of gyroscopics...

Spacecraft with 1 VSCMG

- A Variable-Speed CMG is a classical CMG device where the disk speed is left to be variable.
- Think of a VSCMG device as a hybrid CMG/RW.
- Convenient when developing the equations of motion, since we get both the CMG and RW equations of motion by doing the work only once!!
- Researchers have started to look into actually building and flying a VSCMG devices.
    - Avoids classical CMG singularities
    - Highly redundant system (more robust to component failure)
    - Can be used as a combined power storage/attitude control device.

Battle Plan...

- To derive the equations of motion of a spacecraft with a single VSCMG, we recall Euler's equation

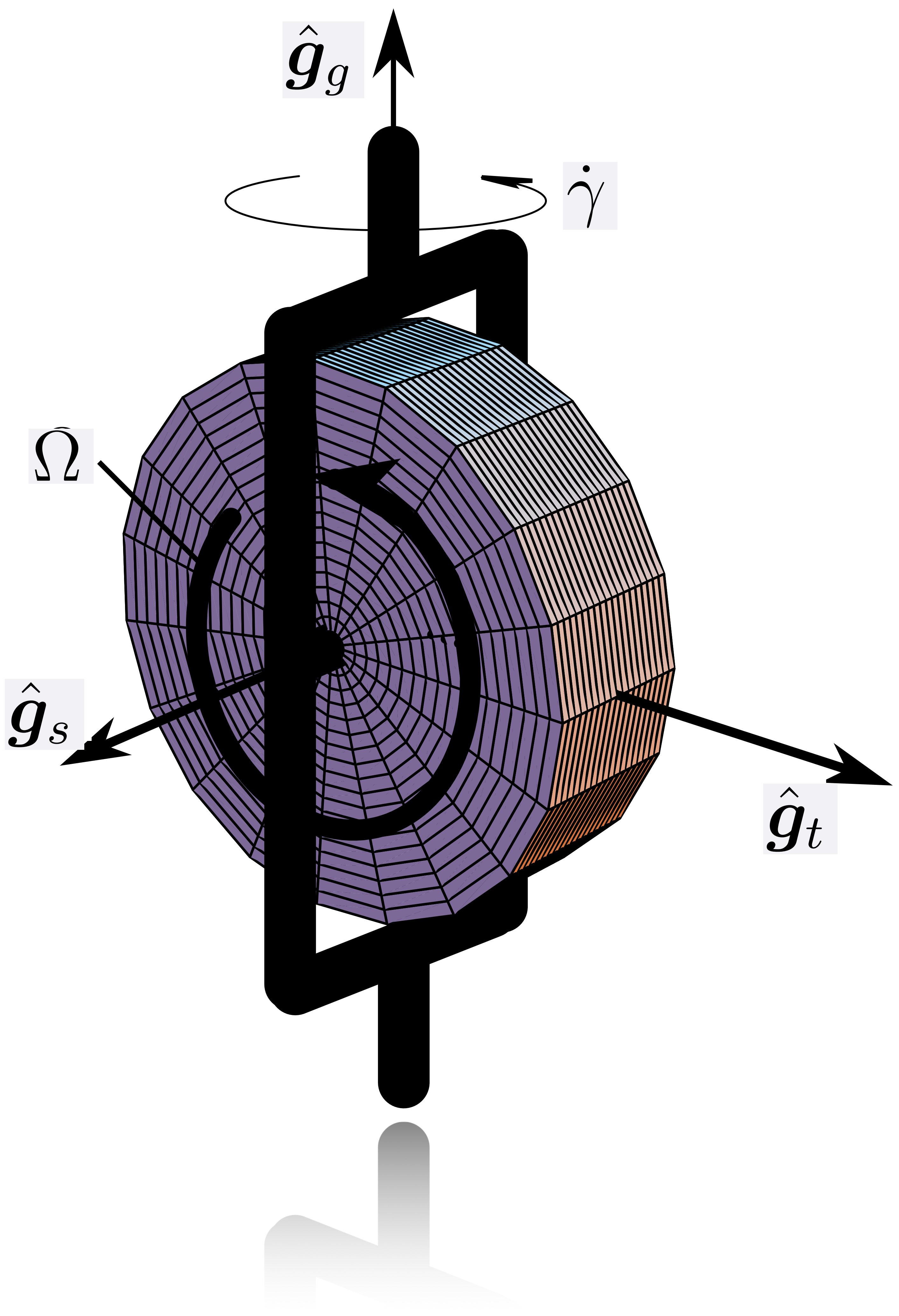
$$\dot{\boldsymbol{H}} = \boldsymbol{L}$$

- We will need to find the total angular momentum vector *H* for the combined spacecraft/VSCMG system. Once we have this expression, we can then differentiate it to get the desired equations of motion.
- To manage all this algebra, we will break up the whole system into the spacecraft part, the CMG momentum and the RW momentum.

VSCMG Frames

- The VSCMG spin axis is $\hat{\boldsymbol{g}}_s$
- The gimbal axis is $\hat{\boldsymbol{g}}_g$
- The disk spin rate is $\Omega(t)$
- The gimbal rate is $\dot{\gamma}(t)$
- The gimbal coordinate frame G is

$$\mathcal{G} : \{\hat{\boldsymbol{g}}_s, \hat{\boldsymbol{g}}_t, \hat{\boldsymbol{g}}_g\}$$



VSCMG Frames

- Note that the gimbal axis is fixed with respect to the spacecraft body frame *B*.
- The gimbal frame *G* angular velocity is

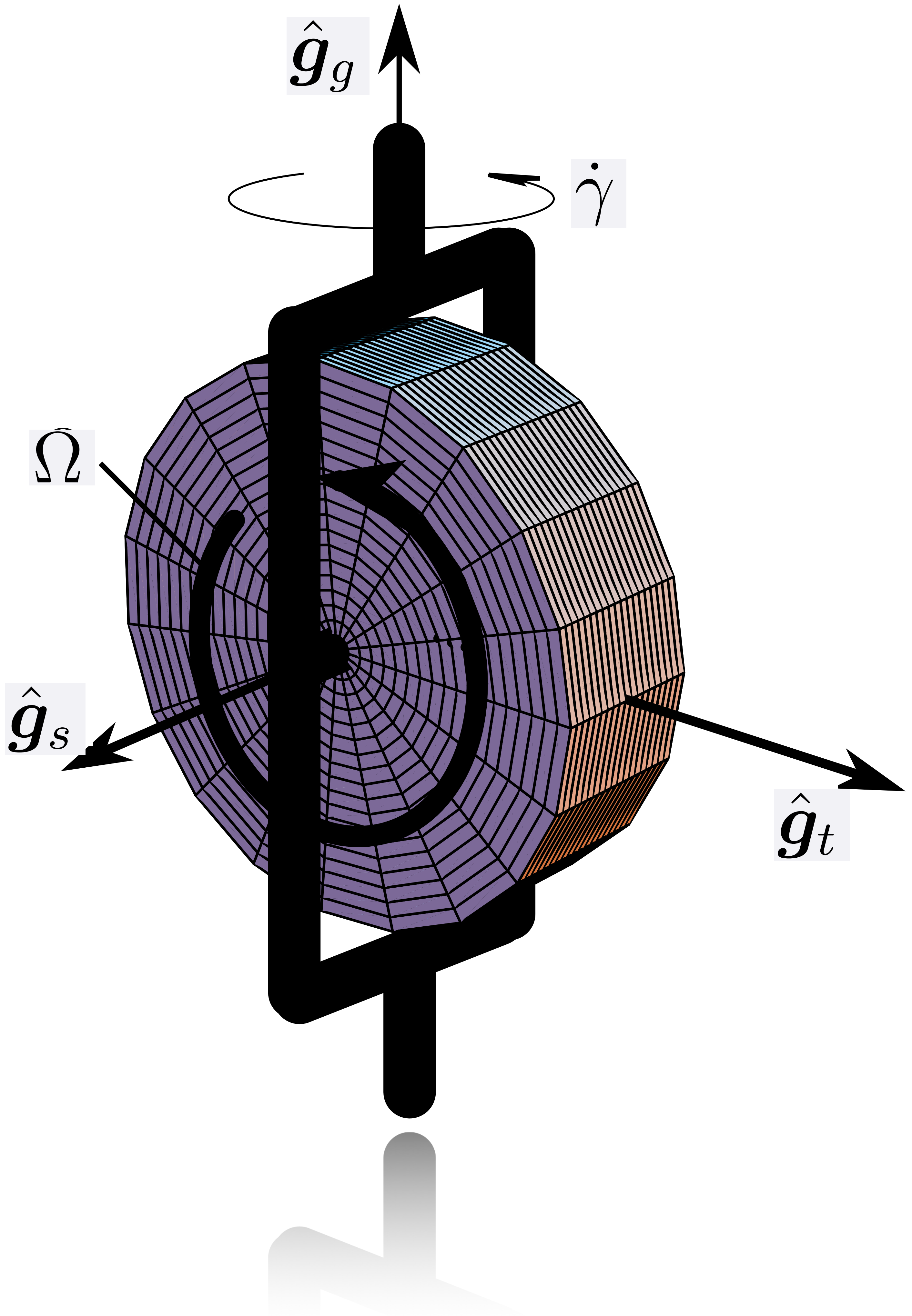
$$\boldsymbol{\omega}_{\mathcal{G}/\mathcal{B}} = \dot{\gamma}\hat{\boldsymbol{g}}_g$$

- Let *W* be a frame that tracks the motion of the reaction wheel.

$$\mathcal{W} : \{\hat{\boldsymbol{g}}_s, \hat{\boldsymbol{w}}_t, \hat{\boldsymbol{w}}_g\}$$

- It's angular velocity is

$$\boldsymbol{\omega}_{\mathcal{W}/\mathcal{G}} = \Omega\hat{\boldsymbol{g}}_s$$



VSCMG Inertias

- Let the gimbal frame inertia be

$$[I_G] = {}^{\mathcal{G}}[I_G] = {}^{\mathcal{G}}\begin{bmatrix} I_{G_s} & 0 & 0 \\ 0 & I_{G_t} & 0 \\ 0 & 0 & I_{G_g} \end{bmatrix}$$

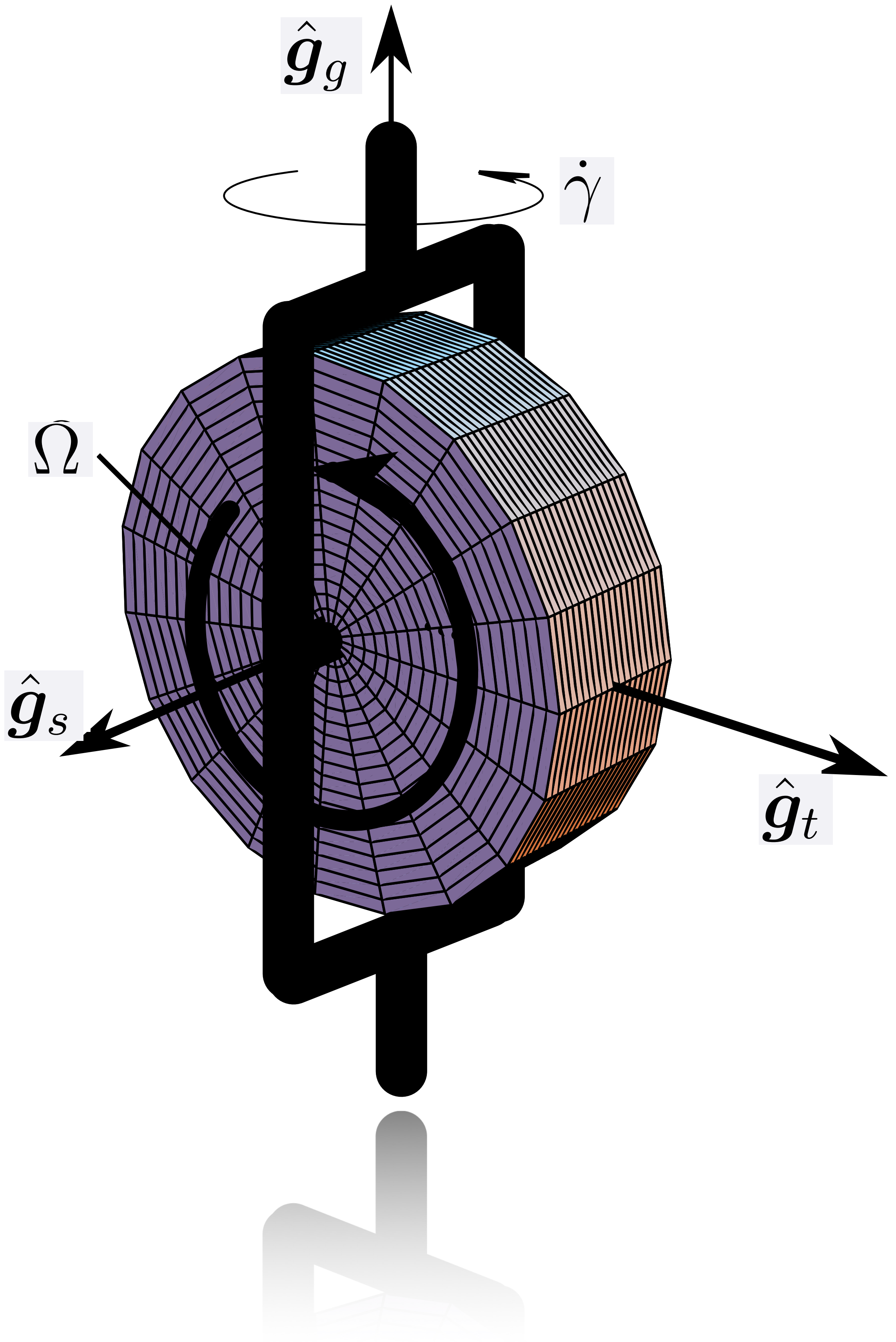
- The wheel (disk) inertia is

Why I_W equal I_G matrix?

$$[I_W] = {}^{\mathcal{W}}[I_W] = {}^{\mathcal{W}}\begin{bmatrix} I_{W_s} & 0 & 0 \\ 0 & I_{W_t} & 0 \\ 0 & 0 & I_{W_t} \end{bmatrix}$$

- Due to symmetry of the disk, we find that

$$^{\mathcal{W}}[I_W] = {}^{\mathcal{G}}[I_W]$$



- Assuming the gimbal frame unit vectors are expressed in body frame vector components, then the rotation matrix [*BG*] can be expressed through

$$[BG] = \left[\hat{\boldsymbol{g}}_s \ \hat{\boldsymbol{g}}_t \ \hat{\boldsymbol{g}}_g\right]$$

- The gimbal frame and disk inertias (which were given in gimbal frame components), can be written in body frame components using

$$^{\mathcal{B}}[I_G] = [BG]^{\mathcal{G}}[I_G][BG]^T = I_{G_s}\hat{\boldsymbol{g}}_s\hat{\boldsymbol{g}}_s^T + I_{G_t}\hat{\boldsymbol{g}}_t\hat{\boldsymbol{g}}_t^T + I_{G_g}\hat{\boldsymbol{g}}_g\hat{\boldsymbol{g}}_g^T$$

$$^{\mathcal{B}}[I_W] = [BG]^{\mathcal{G}}[I_W][BG]^T = I_{W_s}\hat{\boldsymbol{g}}_s\hat{\boldsymbol{g}}_s^T + I_{W_t}\hat{\boldsymbol{g}}_t\hat{\boldsymbol{g}}_t^T + I_{W_g}\hat{\boldsymbol{g}}_g\hat{\boldsymbol{g}}_g^T$$

Angular Momentum...

- We are now ready to express the total angular momentum of the system using

$$\boldsymbol{H} = \boldsymbol{H}_B + \boldsymbol{H}_G + \boldsymbol{H}_W$$

- $\boldsymbol{H}_B$ is the angular momentum of the spacecraft itself, $\boldsymbol{H}_G$ is the angular momentum of the gimbal frame, while $\boldsymbol{H}_W$ is the angular momentum of the spinning disk.

- The spacecraft angular momentum is simply that of a rigid body:

$$\boldsymbol{H}_B = [I_s]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$$

- The inertial angular momentum of the rigid gimbal frame is

$$\boldsymbol{H}_G = [I_G]\boldsymbol{\omega}_{\mathcal{G}/\mathcal{N}}$$

- where $\boldsymbol{\omega}_{\mathcal{G}/\mathcal{N}} = \boldsymbol{\omega}_{\mathcal{G}/\mathcal{B}} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$. This can now be rewritten as

$$\boldsymbol{H}_G = \left(I_{G_s}\hat{\boldsymbol{g}}_s\hat{\boldsymbol{g}}_s^T + I_{G_t}\hat{\boldsymbol{g}}_t\hat{\boldsymbol{g}}_t^T + I_{G_g}\hat{\boldsymbol{g}}_g\hat{\boldsymbol{g}}_g^T\right)\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + I_{G_g}\dot{\gamma}\hat{\boldsymbol{g}}_g$$

- Let use introduce the angular velocity components taken along the gimbal frame axis directions:

$$\omega_s = \hat{\boldsymbol{g}}_s^T\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \qquad \omega_t = \hat{\boldsymbol{g}}_t^T\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \qquad \omega_g = \hat{\boldsymbol{g}}_g^T\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$$

$$^{\mathcal{G}}\boldsymbol{\omega} = \omega_s\hat{\boldsymbol{g}}_s + \omega_t\hat{\boldsymbol{g}}_t + \omega_g\hat{\boldsymbol{g}}_g$$

- This allows us to write the gimbal frame angular momentum expression as

$$\boldsymbol{H}_G = I_{G_s}\omega_s\hat{\boldsymbol{g}}_s + I_{G_t}\omega_t\hat{\boldsymbol{g}}_t + I_{G_g}\left(\omega_g + \dot{\gamma}\right)\hat{\boldsymbol{g}}_g$$

- The inertial angular momentum of the disk is

$$\boldsymbol{H}_W = [I_W]\boldsymbol{\omega}_{\mathcal{W}/\mathcal{N}}$$

- where $\boldsymbol{\omega}_{\mathcal{W}/\mathcal{N}} = \boldsymbol{\omega}_{\mathcal{W}/\mathcal{G}} + \boldsymbol{\omega}_{\mathcal{G}/\mathcal{B}} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$

- The momentum expression can be expanded using

$$\boldsymbol{H}_W = [I_W]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + [I_W]\boldsymbol{\omega}_{\mathcal{G}/\mathcal{B}} + [I_W]\boldsymbol{\omega}_{\mathcal{W}/\mathcal{G}}$$

It is implied that all vectors are added with components in the same frame.

- The first term can be written as

$$\begin{aligned} [I_W]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} &= \left(I_{W_s}\hat{\boldsymbol{g}}_s\hat{\boldsymbol{g}}_s^T + I_{W_t}\hat{\boldsymbol{g}}_t\hat{\boldsymbol{g}}_t^T + I_{W_t}\hat{\boldsymbol{g}}_g\hat{\boldsymbol{g}}_g^T\right)\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\ &= I_{W_s}\omega_s\hat{\boldsymbol{g}}_s + I_{W_t}\omega_t\hat{\boldsymbol{g}}_t + I_{W_g}\omega_g\hat{\boldsymbol{g}}_g \end{aligned}$$

- The second two terms can be written as

$$[I_W]\boldsymbol{\omega}_{\mathcal{G}/\mathcal{B}} = {}^{\mathcal{G}}\!\begin{bmatrix} I_{W_s} & 0 & 0 \\ 0 & I_{W_t} & 0 \\ 0 & 0 & I_{W_t} \end{bmatrix} {}^{\mathcal{G}}\!\begin{pmatrix} 0 \\ 0 \\ \dot{\gamma} \end{pmatrix} = I_{W_t}\dot{\gamma}\hat{\boldsymbol{g}}_g$$

$$[I_W]\boldsymbol{\omega}_{\mathcal{W}/\mathcal{G}} = {}^{\mathcal{W}}\!\begin{bmatrix} I_{W_s} & 0 & 0 \\ 0 & I_{W_t} & 0 \\ 0 & 0 & I_{W_t} \end{bmatrix} {}^{\mathcal{W}}\!\begin{pmatrix} \Omega \\ 0 \\ 0 \end{pmatrix} = I_{W_s}\Omega\hat{\boldsymbol{g}}_s$$

- Combining all these results, the spinning wheel inertial angular momentum is written as

$$\boldsymbol{H}_W = I_{W_s}\left(\omega_s + \Omega\right)\hat{\boldsymbol{g}}_s + I_{W_t}\omega_t\hat{\boldsymbol{g}}_t + I_{W_t}\left(\omega_g + \dot{\gamma}\right)\hat{\boldsymbol{g}}_g$$

Some final preparation...

- Before we begin to differentiate the system angular momentum vectors, we need to establish some useful relationships.

- The gimbal frame direction vectors can be written in terms of their initial orientations as

$$\hat{\boldsymbol{g}}_s\left(t\right) = \cos\left(\gamma(t) - \gamma_0\right)\hat{\boldsymbol{g}}_s\left(t_0\right) + \sin\left(\gamma(t) - \gamma_0\right)\hat{\boldsymbol{g}}_t\left(t_0\right)$$

$$\hat{\boldsymbol{g}}_t\left(t\right) = -\sin\left(\gamma(t) - \gamma_0\right)\hat{\boldsymbol{g}}_s\left(t_0\right) + \cos\left(\gamma(t) - \gamma_0\right)\hat{\boldsymbol{g}}_t\left(t_0\right)$$

$$\hat{\boldsymbol{g}}_g\left(t\right) = \hat{\boldsymbol{g}}_g\left(t_0\right)$$

- Note that the $\mathcal{B}$ frame derivatives of the gimbal frame unit vectors are

$$\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\left(\hat{\boldsymbol{g}}_s\right) = \dot{\gamma}\hat{\boldsymbol{g}}_t \qquad \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\left(\hat{\boldsymbol{g}}_t\right) = -\dot{\gamma}\hat{\boldsymbol{g}}_s \qquad \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\left(\hat{\boldsymbol{g}}_g\right) = 0$$

- The inertial derivatives of these vectors are

$$\dot{\hat{\boldsymbol{g}}}_s = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\left(\hat{\boldsymbol{g}}_s\right) + \boldsymbol{\omega} \times \hat{\boldsymbol{g}}_s = \left(\dot{\gamma} + \omega_g\right)\hat{\boldsymbol{g}}_t - \omega_t \hat{\boldsymbol{g}}_g$$

$$\dot{\hat{\boldsymbol{g}}}_t = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\left(\hat{\boldsymbol{g}}_t\right) + \boldsymbol{\omega} \times \hat{\boldsymbol{g}}_t = -\left(\dot{\gamma} + \omega_g\right)\hat{\boldsymbol{g}}_s + \omega_s \hat{\boldsymbol{g}}_g$$

$$\dot{\hat{\boldsymbol{g}}}_g = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\left(\hat{\boldsymbol{g}}_g\right) + \boldsymbol{\omega} \times \hat{\boldsymbol{g}}_g = \omega_t \hat{\boldsymbol{g}}_s - \omega_s \hat{\boldsymbol{g}}_t$$

use ${}^{\mathcal{G}}\boldsymbol{\omega} = \omega_s \hat{\boldsymbol{g}}_s + \omega_t \hat{\boldsymbol{g}}_t + \omega_g \hat{\boldsymbol{g}}_g$ to derive this result.

- Finally, the following expressions are derived:

$$
\begin{aligned}
\dot{\omega}_s &= \dot{\hat{\boldsymbol{g}}}_s^T \boldsymbol{\omega} + \hat{\boldsymbol{g}}_s^T \dot{\boldsymbol{\omega}} = \dot{\gamma}\omega_t + \hat{\boldsymbol{g}}_s^T \dot{\boldsymbol{\omega}} \\
\dot{\omega}_t &= \dot{\hat{\boldsymbol{g}}}_t^T \boldsymbol{\omega} + \hat{\boldsymbol{g}}_t^T \dot{\boldsymbol{\omega}} = -\dot{\gamma}\omega_s + \hat{\boldsymbol{g}}_t^T \dot{\boldsymbol{\omega}} \\
\dot{\omega}_g &= \dot{\hat{\boldsymbol{g}}}_g^T \boldsymbol{\omega} + \hat{\boldsymbol{g}}_g^T \dot{\boldsymbol{\omega}} = \hat{\boldsymbol{g}}_g^T \dot{\boldsymbol{\omega}}
\end{aligned}
$$

- The following combined gimbal and spinning disk inertia matrix will be useful to simplify some results:

$$
[J] = [I_G] + [I_W] = {}^{\mathcal{G}}\begin{bmatrix} J_s & 0 & 0 \\ 0 & J_t & 0 \\ 0 & 0 & J_g \end{bmatrix}
$$

And now, the fun...

- At this point we are ready to compute the terms in Euler's equation $\dot{\boldsymbol{H}} = \boldsymbol{L}$. We have all the required expressions and need to simply carry out the required algebra.

- Taking the inertial derivative of the spinning wheel angular momentum expression $\boldsymbol{H}_W$, we find

$$
\begin{aligned}
\dot{\boldsymbol{H}}_W = {} & \hat{\boldsymbol{g}}_s \left[I_{W_s} \left(\dot{\Omega} + \hat{\boldsymbol{g}}_s^T \dot{\boldsymbol{\omega}} + \dot{\gamma}\omega_t \right) \right] + \hat{\boldsymbol{g}}_t \left[I_{W_s} \left(\dot{\gamma}(\omega_s + \Omega) + \Omega\omega_g \right) \right. \\
& \left. + I_{W_t} \hat{\boldsymbol{g}}_t^T \dot{\boldsymbol{\omega}} + \left(I_{W_s} - I_{W_t} \right) \omega_s \omega_g - 2 I_{W_t} \omega_s \dot{\gamma} \right] \\
& + \hat{\boldsymbol{g}}_g \left[I_{W_t} \left(\hat{\boldsymbol{g}}_g^T \dot{\boldsymbol{\omega}} + \ddot{\gamma} \right) + \left(I_{W_t} - I_{W_s} \right) \omega_s \omega_t - I_{W_s} \Omega \omega_t \right]
\end{aligned}
$$

- Taking the derivative of the gimbal frame angular momentum expression $\boldsymbol{H}_G$, we find

$$
\begin{aligned}
\dot{\boldsymbol{H}}_G = {} & \hat{\boldsymbol{g}}_s \left(\left(I_{G_s} - I_{G_t} + I_{G_g} \right) \dot{\gamma} \omega_t + I_{G_s} \hat{\boldsymbol{g}}_s^T \dot{\boldsymbol{\omega}} + \left(I_{G_g} - I_{G_t} \right) \omega_t \omega_g \right) \\
& + \hat{\boldsymbol{g}}_t \left(\left(I_{G_s} - I_{G_t} - I_{G_g} \right) \dot{\gamma} \omega_s + I_{G_t} \hat{\boldsymbol{g}}_t^T \dot{\boldsymbol{\omega}} + \left(I_{G_s} - I_{G_g} \right) \omega_s \omega_g \right) \\
& + \hat{\boldsymbol{g}}_g \left(I_{G_g} \left(\hat{\boldsymbol{g}}_g^T \dot{\boldsymbol{\omega}} + \ddot{\gamma} \right) + \left(I_{G_t} - I_{G_s} \right) \omega_s \omega_t \right)
\end{aligned}
$$

- Finally, the spacecraft angular momentum inertial derivative is

$$
\dot{\boldsymbol{H}}_B = [I_s] \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times [I_s] \boldsymbol{\omega}
$$

- Let us define the time-varying total spacecraft inertia matrix [*I*]:

$$[I] = [I_s] + [J]$$

- Adding up all the terms, and substituting them into Euler's equation $\dot{\boldsymbol{H}} = \boldsymbol{L}$, we finally arrive at the desired equations of motion of a spacecraft with a single VSCMG.

$$\boxed{\begin{aligned}[I]\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega} \times [I]\boldsymbol{\omega} &- \hat{\boldsymbol{g}}_s \left(J_s \dot{\gamma} \omega_t + I_{W_s} \dot{\Omega} - (J_t - J_g)\, \omega_t \dot{\gamma} \right) \\ &- \hat{\boldsymbol{g}}_t \left((J_s \omega_s + I_{W_s} \Omega)\, \dot{\gamma} - (J_t + J_g)\, \omega_s \dot{\gamma} + I_{W_s} \Omega \omega_g \right) \\ &- \hat{\boldsymbol{g}}_g \left(J_g \ddot{\gamma} - I_{W_s} \Omega \omega_t \right) + \boldsymbol{L}\end{aligned}}$$

These equations of motion are valid for both a RW or CMG device!

Comments...

- By changing the wheel speed or by gimbaling the CMG devices, a torque is applied to the spacecraft and the corresponding attitude is changed.
- RW devices are simpler, but have limits on how large the spin speed Ω can grow.
- Adding the gimbaling mode clearly makes the mathematics much more fun and interesting :-)
- To generally control a spacecraft attitude, three of more of these devices would have to be attached to the spacecraft.

RW Motor Torque

- The equations of motion of only the spinning disk could be found by solving Euler's equations for this disk

$$\dot{\boldsymbol{H}}_W = \boldsymbol{L}_w$$

- Note that this is the inertial derivative of the inertial disk angular momentum. We have already found this to be

$$\begin{aligned}\dot{\boldsymbol{H}}_W = \hat{\boldsymbol{g}}_s \left[I_{W_s} \left(\dot{\Omega} + \hat{\boldsymbol{g}}_s^T \dot{\boldsymbol{\omega}} + \dot{\gamma}\omega_t \right) \right] + \hat{\boldsymbol{g}}_t \left[I_{W_s} \left(\dot{\gamma}(\omega_s + \Omega) + \Omega\omega_g \right) \right. \\ \left. + I_{W_t} \hat{\boldsymbol{g}}_t^T \dot{\boldsymbol{\omega}} + \left(I_{W_s} - I_{W_t} \right) \omega_s \omega_g - 2 I_{W_t} \omega_s \dot{\gamma} \right] \\ + \hat{\boldsymbol{g}}_g \left[I_{W_t} \left(\hat{\boldsymbol{g}}_g^T \dot{\boldsymbol{\omega}} + \ddot{\gamma} \right) + \left(I_{W_t} - I_{W_s} \right) \omega_s \omega_t - I_{W_s} \Omega \omega_t \right]\end{aligned}$$

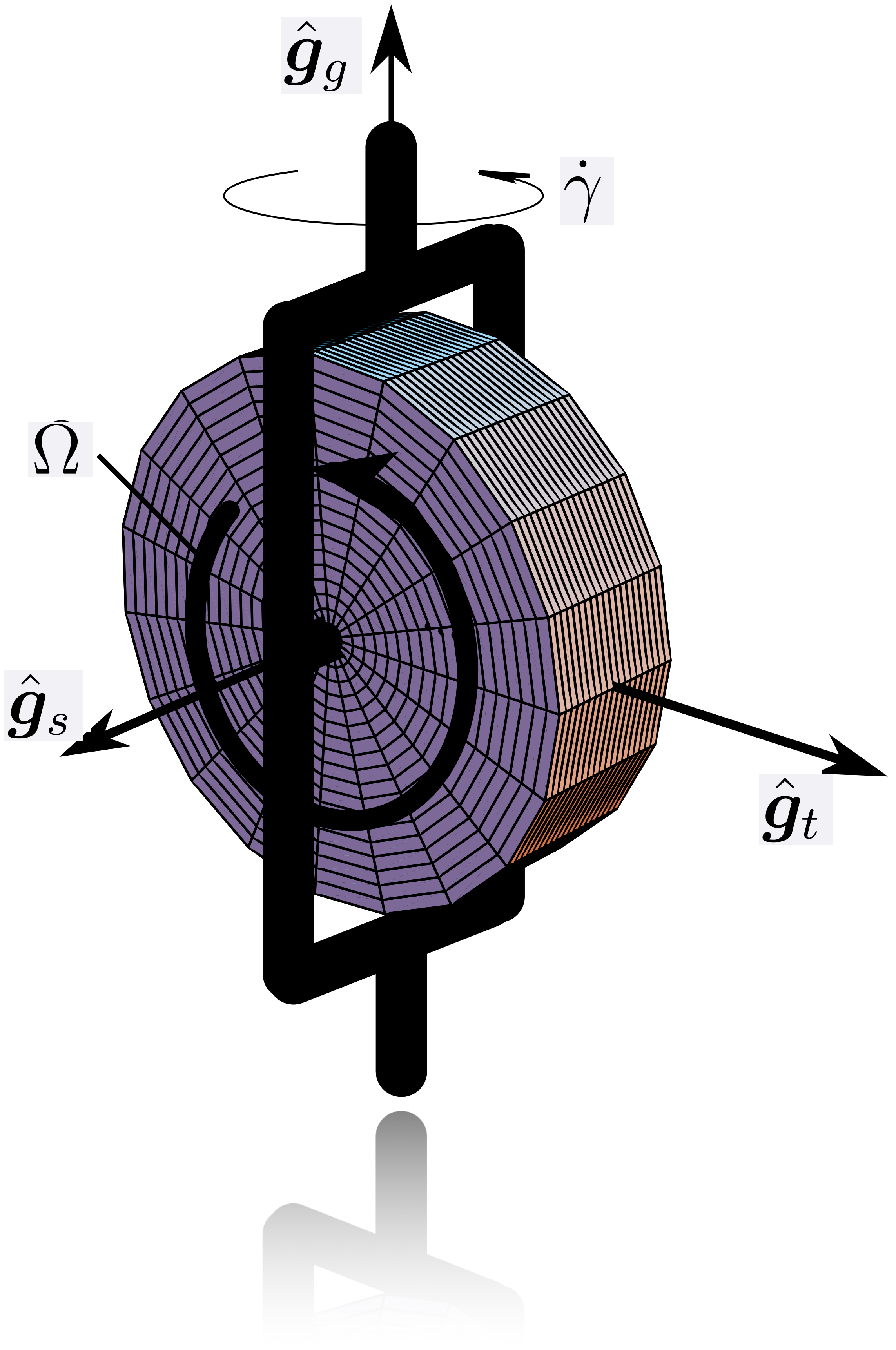
- The only external torque being applied to the spinning disk is through the RW motor.

$$\dot{\boldsymbol{H}}_W = \boldsymbol{L}_W = u_s \hat{\boldsymbol{g}}_s + \tau_{w_t} \hat{\boldsymbol{g}}_t + \tau_{w_g} \hat{\boldsymbol{g}}_g$$

- Thus, equating the $\hat{\boldsymbol{g}}_s$ directions yields:

$$\boxed{u_s = I_{W_s} \left(\dot{\Omega} + \hat{\boldsymbol{g}}_s^T \dot{\boldsymbol{\omega}} + \dot{\gamma} \omega_t \right)}$$

Given the current disk angular acceleration, spacecraft angular acceleration, or the current gimbal rate, this formula shows how hard the RW motor has to work.



CMG Motor Torque

- The compute the motor torque of the CMG gimbal mode, we need to look at both the disk and the gimbal frame as one unit.

$$\dot{\boldsymbol{H}}_G + \dot{\boldsymbol{H}}_W = \boldsymbol{L}_G$$

- Again, we have already computed these inertial angular momentum derivatives. The gimbal momentum rate is:

$$\begin{aligned}\dot{\boldsymbol{H}}_G = \hat{\boldsymbol{g}}_s & \left(\left(I_{G_s} - I_{G_t} + I_{G_g}\right)\dot{\gamma}\omega_t + I_{G_s}\hat{\boldsymbol{g}}_s^T\dot{\boldsymbol{\omega}} + \left(I_{G_g} - I_{G_t}\right)\omega_t\omega_g\right) \\ + \hat{\boldsymbol{g}}_t & \left(\left(I_{G_s} - I_{G_t} - I_{G_g}\right)\dot{\gamma}\omega_s + I_{G_t}\hat{\boldsymbol{g}}_t^T\dot{\boldsymbol{\omega}} + \left(I_{G_s} - I_{G_g}\right)\omega_s\omega_g\right) \\ + \hat{\boldsymbol{g}}_g & \left(I_{G_g}\left(\hat{\boldsymbol{g}}_g^T\dot{\boldsymbol{\omega}} + \ddot{\gamma}\right) + \left(I_{G_t} - I_{G_s}\right)\omega_s\omega_t\right)\end{aligned}$$

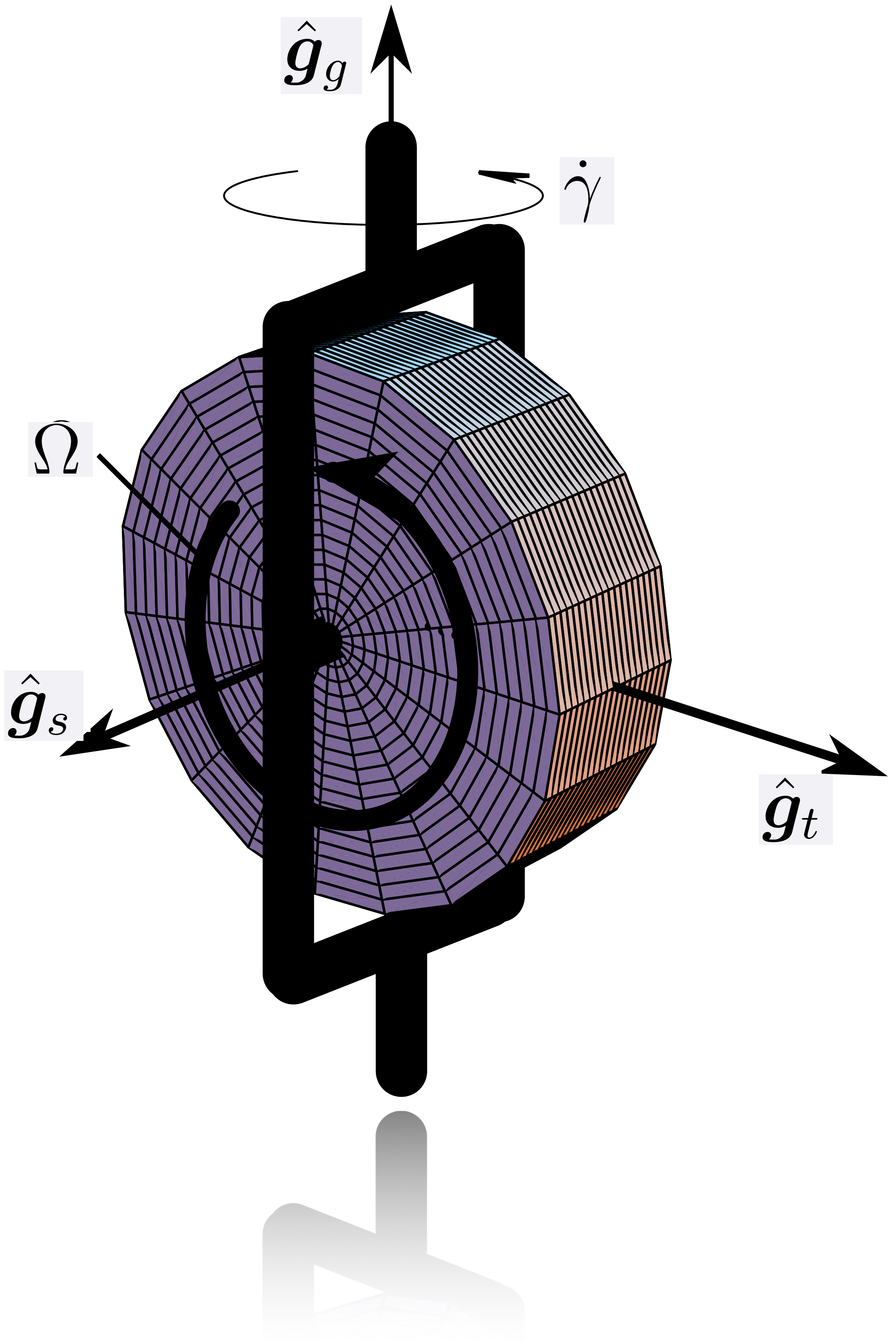
- The only external torque being applied to this two-body system is through the gimbal axis motor.

$$\dot{\boldsymbol{H}}_G + \dot{\boldsymbol{H}}_W = \boldsymbol{L}_G = \tau_{G_s}\hat{\boldsymbol{g}}_s + \tau_{G_t}\hat{\boldsymbol{g}}_t + u_g\hat{\boldsymbol{g}}_g$$

- Thus, equating the $\hat{\boldsymbol{g}}_g$ directions yields:

$$\boxed{\begin{aligned} u_g =& J_g\left(\hat{\boldsymbol{g}}_g^T\dot{\boldsymbol{\omega}} + \ddot{\gamma}\right) \\ & - \left(J_s - J_t\right)\omega_s\omega_t - I_{W_s}\Omega\omega_t \end{aligned}}$$

Given a commanded gimbal time history $\gamma(t)$, this equation shows us how to compute the actual torque that the gimbal motor must apply.



Example: single RW device

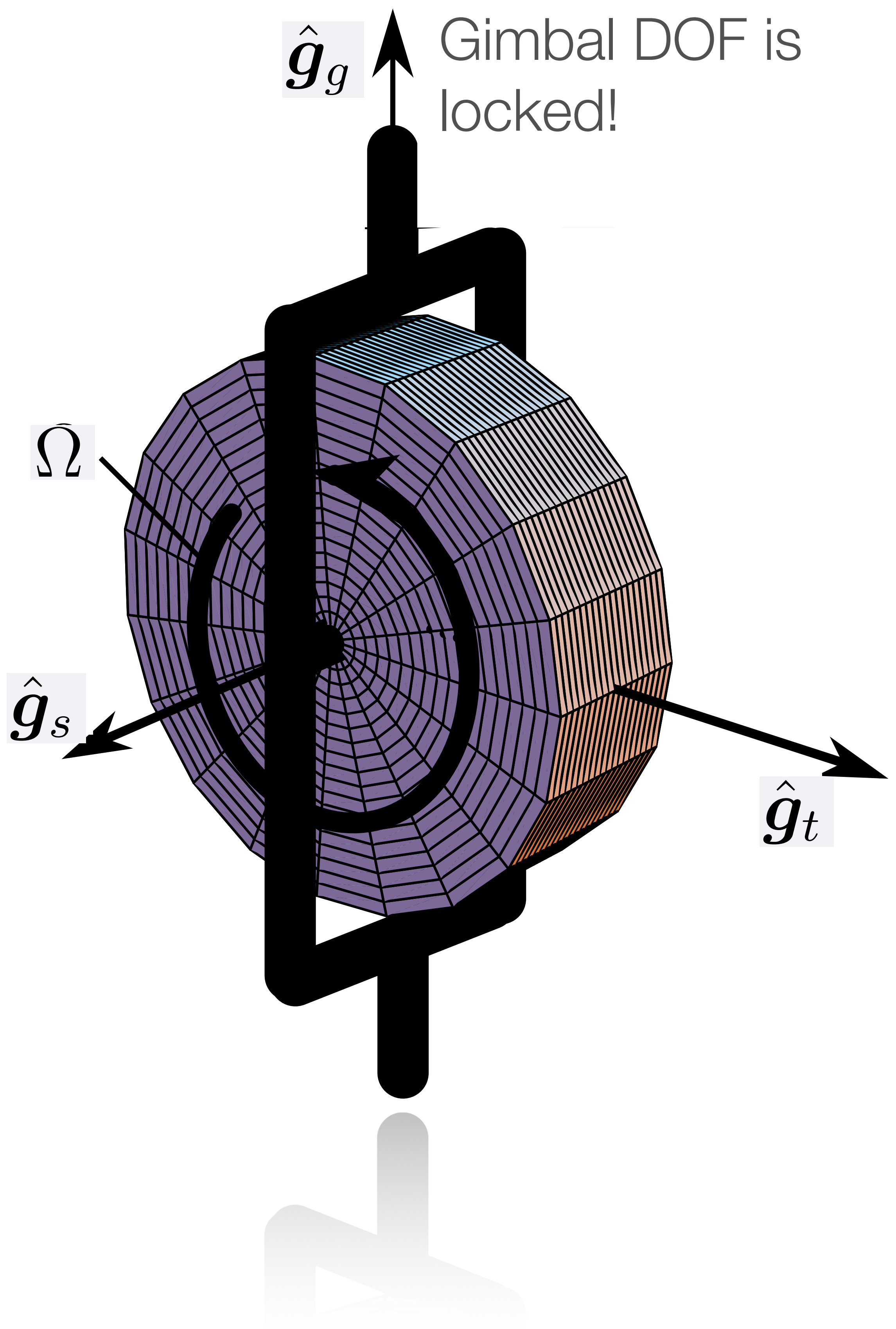
$$\dot{\gamma} = 0 \qquad \ddot{\gamma} = 0$$

$$[I]\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega} \times [I]\boldsymbol{\omega} - \hat{\boldsymbol{g}}_s J_s \dot{\Omega} - J_s \Omega(\omega_g \hat{\boldsymbol{g}}_t - \omega_t \hat{\boldsymbol{g}}_g) + \boldsymbol{L}$$

using:

$$\boldsymbol{\omega} \times \hat{\boldsymbol{g}}_s = (\omega_s \hat{\boldsymbol{g}}_s + \omega_t \hat{\boldsymbol{g}}_t + \omega_g \hat{\boldsymbol{g}}_g) \times \hat{\boldsymbol{g}}_s = -\omega_t \hat{\boldsymbol{g}}_g + \omega_g \hat{\boldsymbol{g}}_t$$

$$[I]\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega} \times [I]\boldsymbol{\omega} - \hat{\boldsymbol{g}}_s J_s \dot{\Omega} - \boldsymbol{\omega} \times J_s \Omega \hat{\boldsymbol{g}}_s + \boldsymbol{L}$$



Motor torque: $u_s = J_s \left(\dot{\Omega} + \hat{\boldsymbol{g}}_s^T \dot{\boldsymbol{\omega}}\right)$

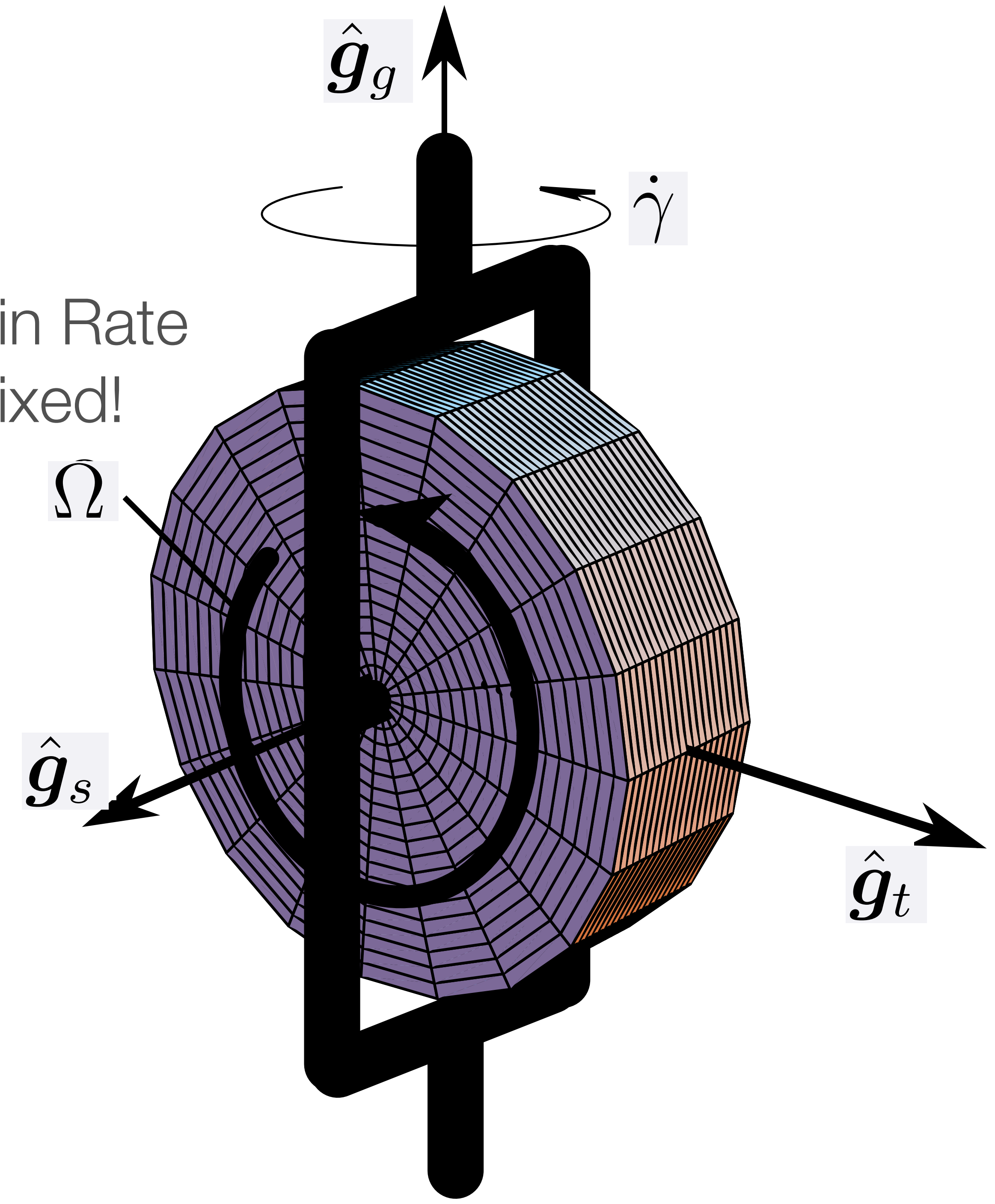
Inertia of spacecraft and non-spin RW axis: $[I_{RW}] = [I_s] + J_t \hat{\boldsymbol{g}}_t \hat{\boldsymbol{g}}_t^T + J_g \hat{\boldsymbol{g}}_g \hat{\boldsymbol{g}}_g^T$

$$[I_{RW}]\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega} \times [I_{RW}]\boldsymbol{\omega} - \boldsymbol{\omega} \times J_s \hat{\boldsymbol{g}}_s(\omega_s + \Omega) - u_s \hat{\boldsymbol{g}}_s + \boldsymbol{L}$$

Example: single CMG device

An inner-servo loop is holding the wheel spin rate fixed:

$$\dot{\Omega} = 0$$



$$\begin{aligned}[I]\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega} \times [I]\boldsymbol{\omega} &- \hat{\boldsymbol{g}}_s \left(J_s \dot{\gamma} \omega_t - (J_t - J_g)\, \omega_t \dot{\gamma}\right) \\ &- \hat{\boldsymbol{g}}_t \left(J_s \left(\omega_s + \Omega\right) \dot{\gamma} - (J_t + J_g)\, \omega_s \dot{\gamma} + J_s \Omega \omega_g\right) \\ &- \hat{\boldsymbol{g}}_g \left(J_g \ddot{\gamma} - J_s \Omega \omega_t\right) + \boldsymbol{L}\end{aligned}$$

CMG controls are discussed shortly...

Multiple VSCMGs

- To accommodate a spacecraft with *N* VSCMG devices, we need to employ a little “book-keeping” to account for the various momentum contributions:

We define the 3x*N* matrices:

$$[G_s] = [\hat{\boldsymbol{g}}_{s_1} \cdots \hat{\boldsymbol{g}}_{s_N}] \qquad [G_t] = [\hat{\boldsymbol{g}}_{t_1} \cdots \hat{\boldsymbol{g}}_{t_N}] \qquad [G_g] = [\hat{\boldsymbol{g}}_{g_1} \cdots \hat{\boldsymbol{g}}_{g_N}]$$

New inertia matrix definition:

$$[I] = [I_s] + \sum_{i=1}^{N} [J_i] = [I_s] + \sum_{i=1}^{N} J_{s_i} \hat{\boldsymbol{g}}_{s_i} \hat{\boldsymbol{g}}_{s_i}^T + J_{t_i} \hat{\boldsymbol{g}}_{t_i} \hat{\boldsymbol{g}}_{t_i}^T + J_{g_i} \hat{\boldsymbol{g}}_{g_i} \hat{\boldsymbol{g}}_{g_i}^T$$

Torque-like vectors:

$$\boldsymbol{\tau}_s = \begin{bmatrix} J_{s_1}\left(\dot{\Omega}_1 + \dot{\gamma}_1\omega_{t_1}\right) - \left(J_{t_1} - J_{g_1}\right)\omega_{t_1}\dot{\gamma}_1 \\ \vdots \\ J_{s_N}\left(\dot{\Omega}_N + \dot{\gamma}_N\omega_{t_N}\right) - \left(J_{t_N} - J_{g_N}\right)\omega_{t_N}\dot{\gamma}_N \end{bmatrix}$$

$$\boldsymbol{\tau}_t = \begin{bmatrix} J_{s_1}\left(\Omega_1 + \omega_{s_1}\right)\dot{\gamma}_1 - \left(J_{t_1} + J_{g_1}\right)\omega_{s_1}\dot{\gamma}_1 + J_{s_1}\Omega_1\omega_{g_1} \\ \vdots \\ J_{s_N}\left(\Omega_N + \omega_{s_N}\right)\dot{\gamma}_N - \left(J_{t_N} + J_{g_N}\right)\omega_{s_N}\dot{\gamma}_N + J_{s_N}\Omega_N\omega_{g_N} \end{bmatrix}$$

$$\boldsymbol{\tau}_g = \begin{bmatrix} J_{g_1}\ddot{\gamma}_1 - J_{s_1}\Omega_1\omega_{t_1} \\ \vdots \\ J_{g_N}\ddot{\gamma}_N - J_{s_N}\Omega_N\omega_{t_N} \end{bmatrix}$$

EOM of spacecraft with N VSCMGs:

$$[I]\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega} \times [I]\boldsymbol{\omega} - [G_s]\boldsymbol{\tau}_s - [G_t]\boldsymbol{\tau}_t - [G_g]\boldsymbol{\tau}_g + \boldsymbol{L}$$

Energy expression:

$$T = \frac{1}{2}\boldsymbol{\omega}^T[I_s]\boldsymbol{\omega} + \frac{1}{2}\sum_{i=1}^{N} J_{s_i}\left(\Omega_i + \omega_{s_i}\right)^2 + J_{t_i}\omega_{t_i}^2 + J_{g_i}\left(\omega_{g_i} + \dot{\gamma}_i\right)^2$$

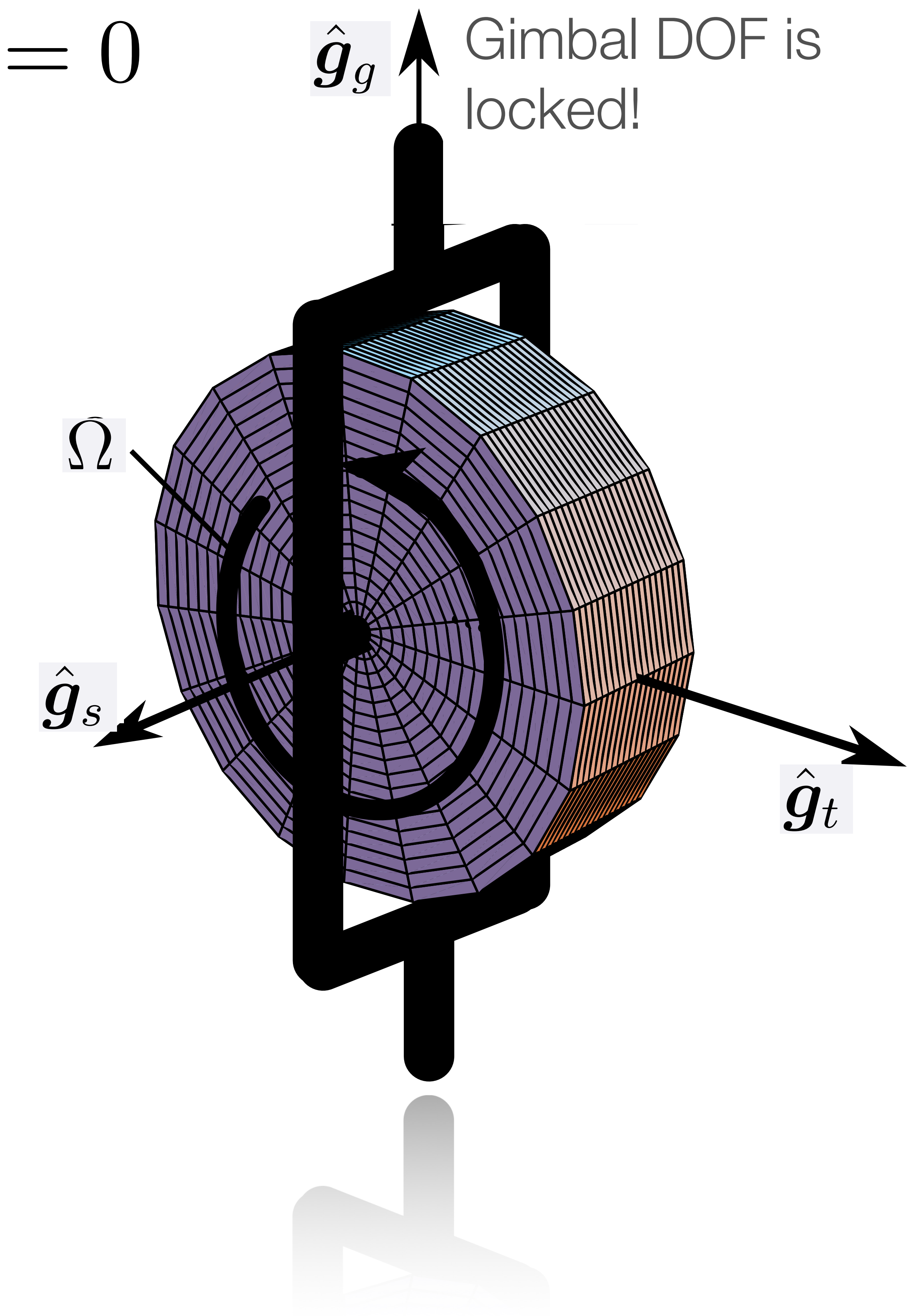
After much algebra, or by using the work-energy-principle...

$$\dot{T} = \boldsymbol{\omega}^T\boldsymbol{L} + \sum_{i=1}^{N} \dot{\gamma}_i u_{g_i} + \Omega_i u_{s_i}$$

Example: multiple RW devices

$$\dot{\gamma} = 0$$

$$\ddot{\gamma} = 0$$



Inertia matrix definition:

$$[I_{RW}] = [I_s] + \sum_{i=1}^{N} \left(J_{t_i} \hat{\boldsymbol{g}}_{t_i} \hat{\boldsymbol{g}}_{t_i}^T + J_{g_i} \hat{\boldsymbol{g}}_{g_i} \hat{\boldsymbol{g}}_{g_i}^T \right)$$

Let us define the momentum vector $\boldsymbol{h}_s$ as:

$$\boldsymbol{h}_s = \begin{pmatrix} \vdots \\ J_{s_i} \left(\omega_{s_i} + \Omega_i \right) \\ \vdots \end{pmatrix}$$

The equations of motion then become:

$$[I_{RW}]\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega} \times [I_{RW}]\boldsymbol{\omega} - \boldsymbol{\omega} \times [G_s]\boldsymbol{h}_s - [G_s]\boldsymbol{u}_s + \boldsymbol{L}$$

For the special case with 3 RWs aligned with the principal axis, $[G_s]$ becomes an identity matrix and the EOM reduce to

$$[I_{RW}]\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega} \times [I_{RW}]\boldsymbol{\omega} - \boldsymbol{\omega} \times \boldsymbol{h}_s - \boldsymbol{u}_s + \boldsymbol{L}$$

Momentum-Device Control Laws

This is where the pudding starts to come together...

RW Control Devices

- First let us develop a feedback control law for a spacecraft with *N* reaction wheels with general orientation.

EOM:

$$[I_{RW}]\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega} \times [I_{RW}]\boldsymbol{\omega} - \boldsymbol{\omega} \times [G_s]\boldsymbol{h}_s - [G_s]\boldsymbol{u}_s + \boldsymbol{L}$$

with

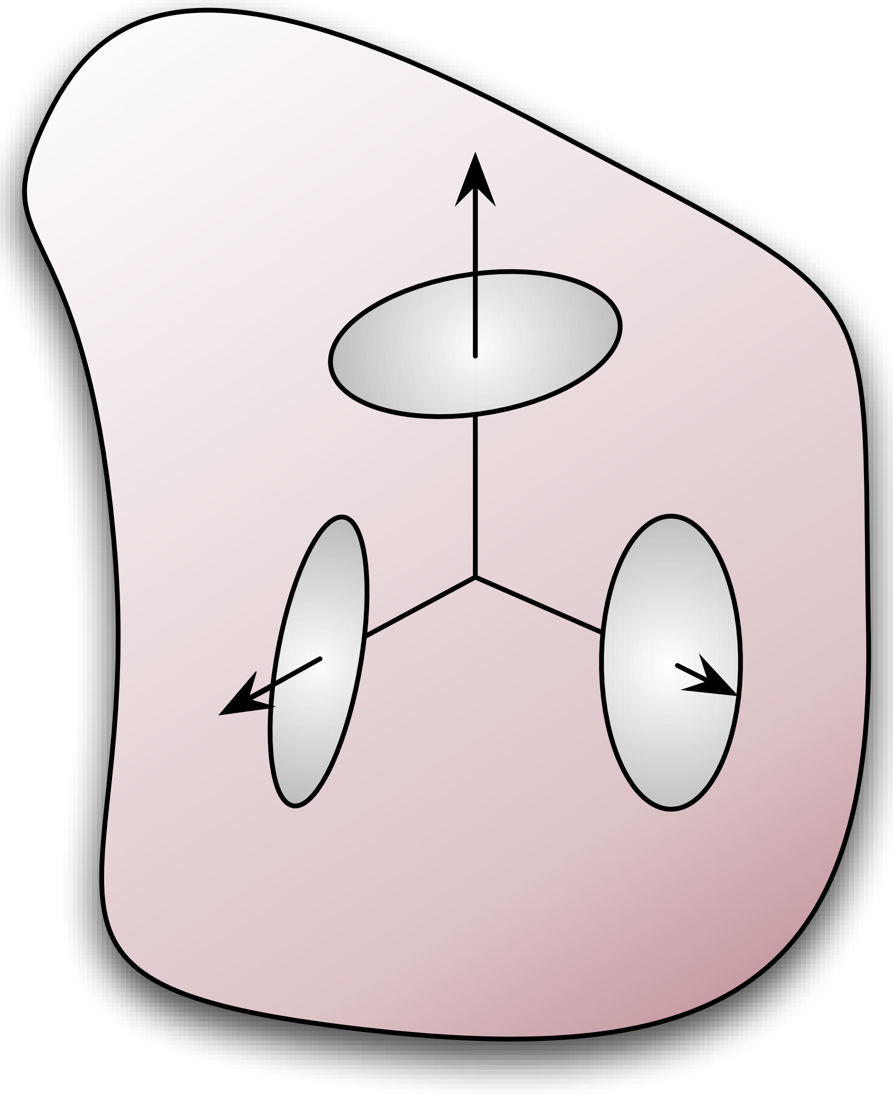
$$\boldsymbol{h}_s = \begin{pmatrix} \vdots \\ J_{s_i}\left(\omega_{s_i} + \Omega_i\right) \\ \vdots \end{pmatrix}$$

Inertia Matrix:

$$[I_{RW}] = [I_s] + \sum_{i=1}^{N} \left(J_{t_i}\hat{\boldsymbol{g}}_{t_i}\hat{\boldsymbol{g}}_{t_i}^T + J_{g_i}\hat{\boldsymbol{g}}_{g_i}\hat{\boldsymbol{g}}_{g_i}^T\right)$$

The RW motor control torque vector is:

$$\boldsymbol{u}_s = \begin{pmatrix} \vdots \\ J_{s_i}\left(\dot{\Omega}_i + \hat{\boldsymbol{g}}_{s_i}^T\dot{\boldsymbol{\omega}}\right) \\ \vdots \end{pmatrix}$$



Spacecraft Tracking Errors:

$\boldsymbol{\sigma}$ - MRP vector of body frame relative to reference frame

$\delta\boldsymbol{\omega} = \boldsymbol{\omega} - \boldsymbol{\omega}_r$ - body angular velocity tracking error vector

Lyapunov Function:

$$V(\boldsymbol{\sigma}, \delta\boldsymbol{\omega}) = \frac{1}{2}\delta\boldsymbol{\omega}^T[I_{RW}]\delta\boldsymbol{\omega} + 2K\ln\left(1 + \boldsymbol{\sigma}^T\boldsymbol{\sigma}\right)$$

components taken in the *B* frame

Let's set the Lyapunov Rate to:

$$\dot{V} = -\delta\boldsymbol{\omega}[P]\delta\boldsymbol{\omega} \leq 0$$

$$[I_{RW}]\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\delta\boldsymbol{\omega}) + K\boldsymbol{\sigma} + [P]\delta\boldsymbol{\omega} = 0$$

closed-loop dynamics

$$[G_s]\boldsymbol{u}_s = \underbrace{\boxed{\begin{aligned}&= K\boldsymbol{\sigma} + [P]\delta\boldsymbol{\omega} - [\tilde{\boldsymbol{\omega}}]([I_{\mathrm{RW}}]\boldsymbol{\omega} + [G_s]\boldsymbol{h}_s)\\ &\quad - [I_{\mathrm{RW}}](\dot{\boldsymbol{\omega}}_r - \boldsymbol{\omega}\times\boldsymbol{\omega}_r) + \boldsymbol{L}\end{aligned}}}_{\boldsymbol{L}_r}$$

Control condition:

$$[G_s]\boldsymbol{u}_s = \boldsymbol{L}_r$$

Case 1: 3 RWs aligned with principal axes of spacecraft.

$$\boldsymbol{u}_s = \boldsymbol{L}_r$$

Case 2: N RWs aligned generally.

$$\boldsymbol{u}_s = [G_s]^T\left([G_s][G_s]^T\right)^{-1}\boldsymbol{L}_r$$

minimum-norm inverse

Energy rate:

$$\dot{T} = \boldsymbol{\omega}^T\boldsymbol{L} + \sum_{i=1}^{N}\Omega_i u_{s_i}$$

work/energy principle

The End…

Nonlinear Spacecraft Control

ASEN 5010

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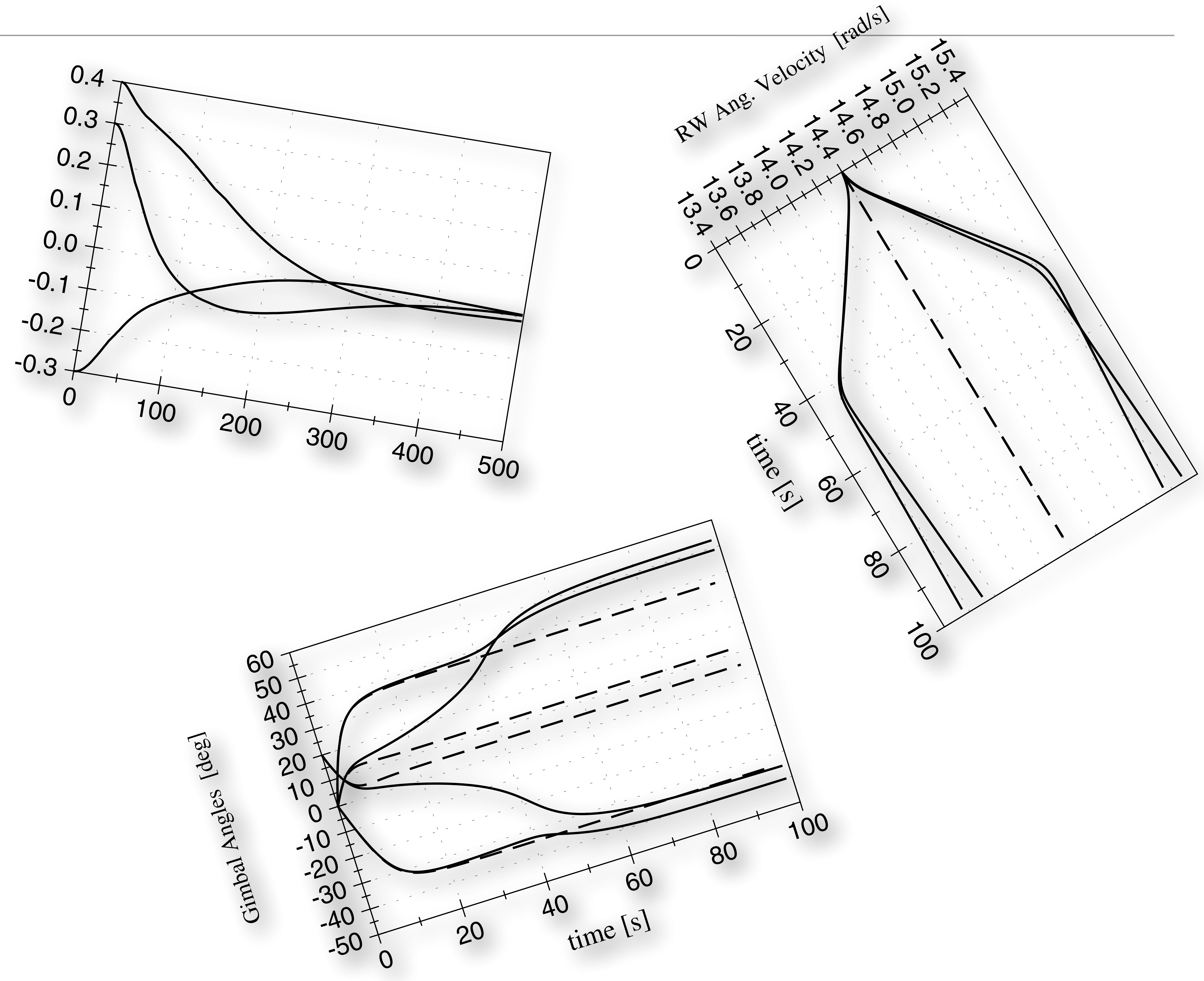
Outline

- Stability Definitions
- Lyapunov Functions
  - Velocity-based feedback
  - Position-based feedback
  - Lyapunov's Direct Method
- Nonlinear Feedback of Spacecraft Attitude
  - Full-state feedback for regulator and tracking problems
  - Feedback Gain Selection
- Lyapunov Optimal Feedback
- Linear Closed-Loop Dynamics



Stability Definitions

Why isn’t stable just stable?

Definitions

State Vector: $\boldsymbol{x} = (x_1 \cdots\ x_N)^T$

EOM: $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t)$ —— Non-Autonomous System

$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ —— Autonomous System

Control Vector: $\boldsymbol{u} = \boldsymbol{g}(\boldsymbol{x})$

Closed-Loop System: $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t)$

Equilibrium State: A state vector point $\boldsymbol{x}_e$ is said to be an equilibrium state (or equilibrium point) of a dynamical system described by $\dot{\boldsymbol{x}}=\boldsymbol{f}(\boldsymbol{x},t)$ at time t_0 if

$$\boldsymbol{f}(\boldsymbol{x}_e, t) = 0 \qquad \forall\ t > t_0$$

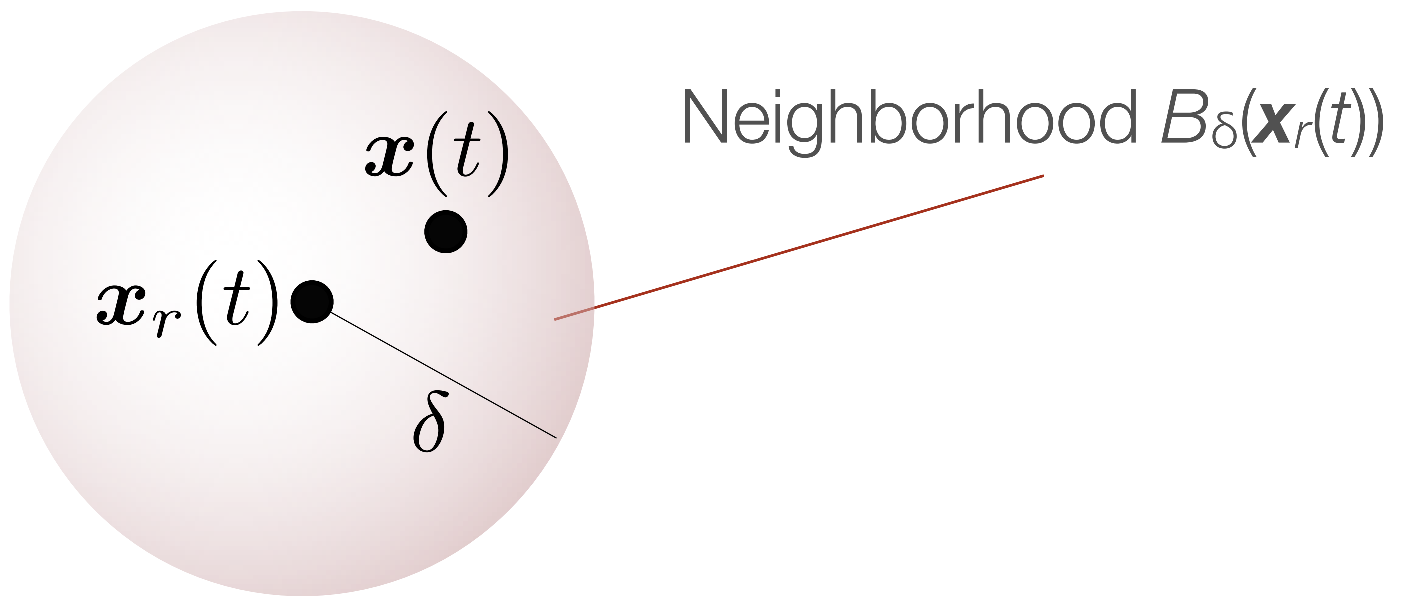
$$\dot{\boldsymbol{x}}_e = 0 \qquad \boldsymbol{x}_e = \text{constant}$$

Neighborhood: Given δ>0, a state vector ***x***(*t*) is said to be in the neighborhood $B_\delta(\boldsymbol{x}_r(t))$ of the state $\boldsymbol{x}_r(t)$ if

$$||\boldsymbol{x}(t) - \boldsymbol{x}_r(t)|| \ < \ \delta$$

then

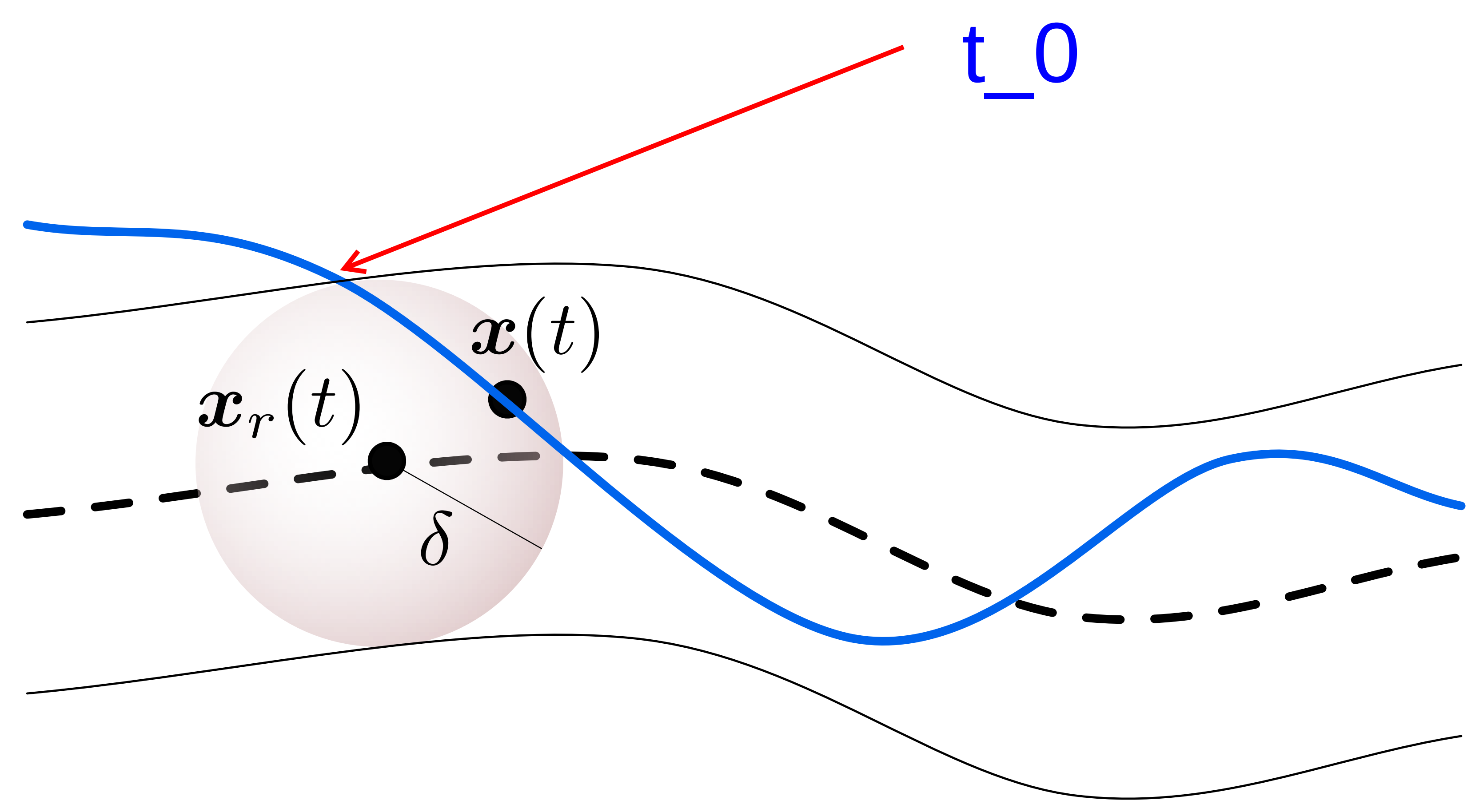
$$\boldsymbol{x}(t) \in B_\delta(\boldsymbol{x}_r(t))$$



Doesn't depend on initial condition

Lagrange Stability: The motion ***x***(*t*) is said to be Lagrange stable (or bounded) relative to $\boldsymbol{x}_r(t)$ if there exists a δ>0 such that

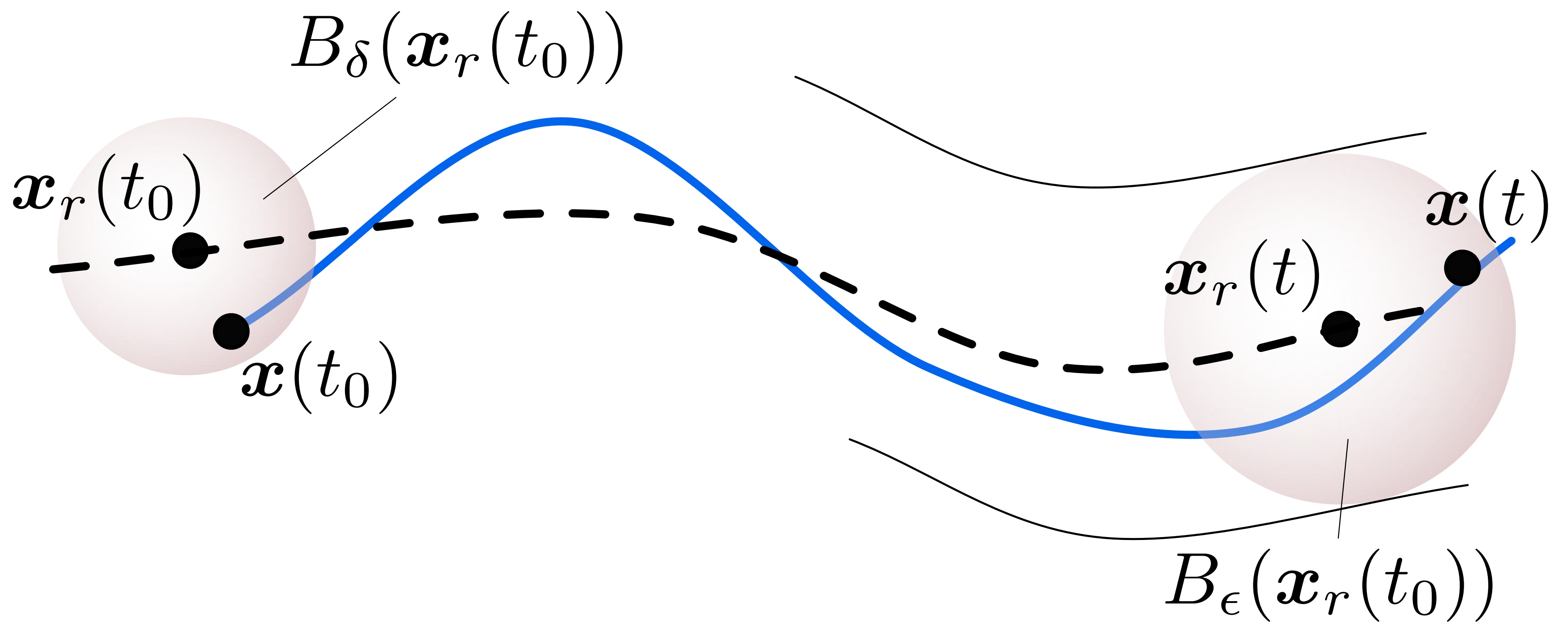
$$\boldsymbol{x}(t) \in B_\delta(\boldsymbol{x}_r(t)) \qquad \forall\ t > t_0$$



Depend on initial condition

Lyapunov Stability: The motion ***x***(*t*) is said to be Lyapunov stable (or stable) relative to $\boldsymbol{x}_r$(t) if for each ε>0 there exists a δ(ε)>0 such that

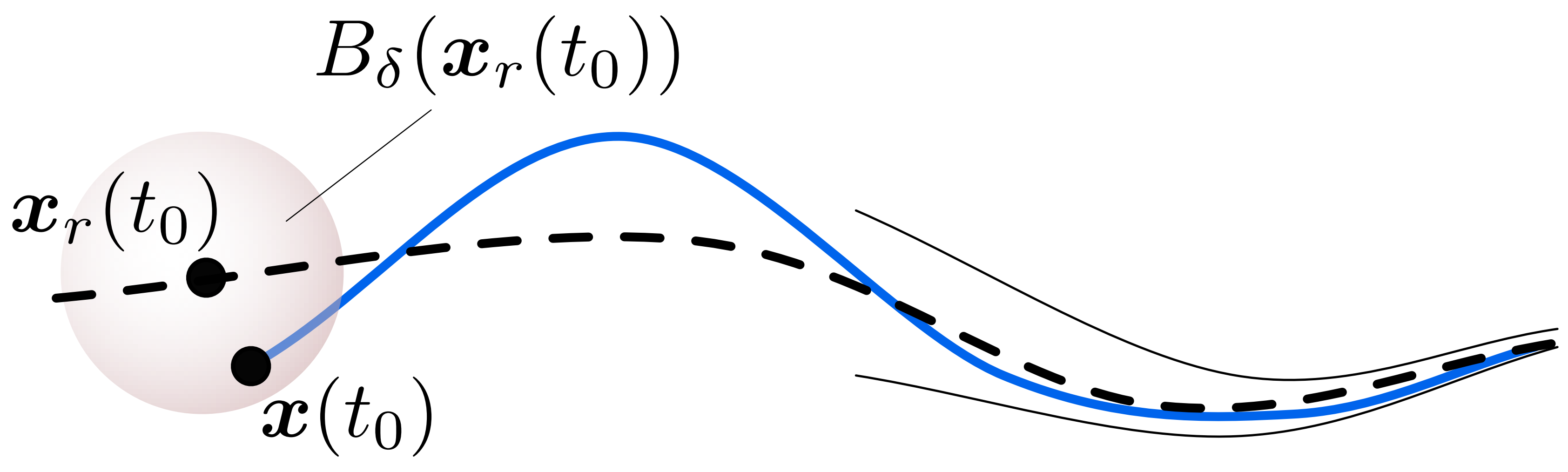
$$\boldsymbol{x}(t_0) \in B_\delta(\boldsymbol{x}_r(t_0)) \Longrightarrow \boldsymbol{x}(t) \in B_\epsilon(\boldsymbol{x}_r(t)) \quad \forall\ t > t_0$$

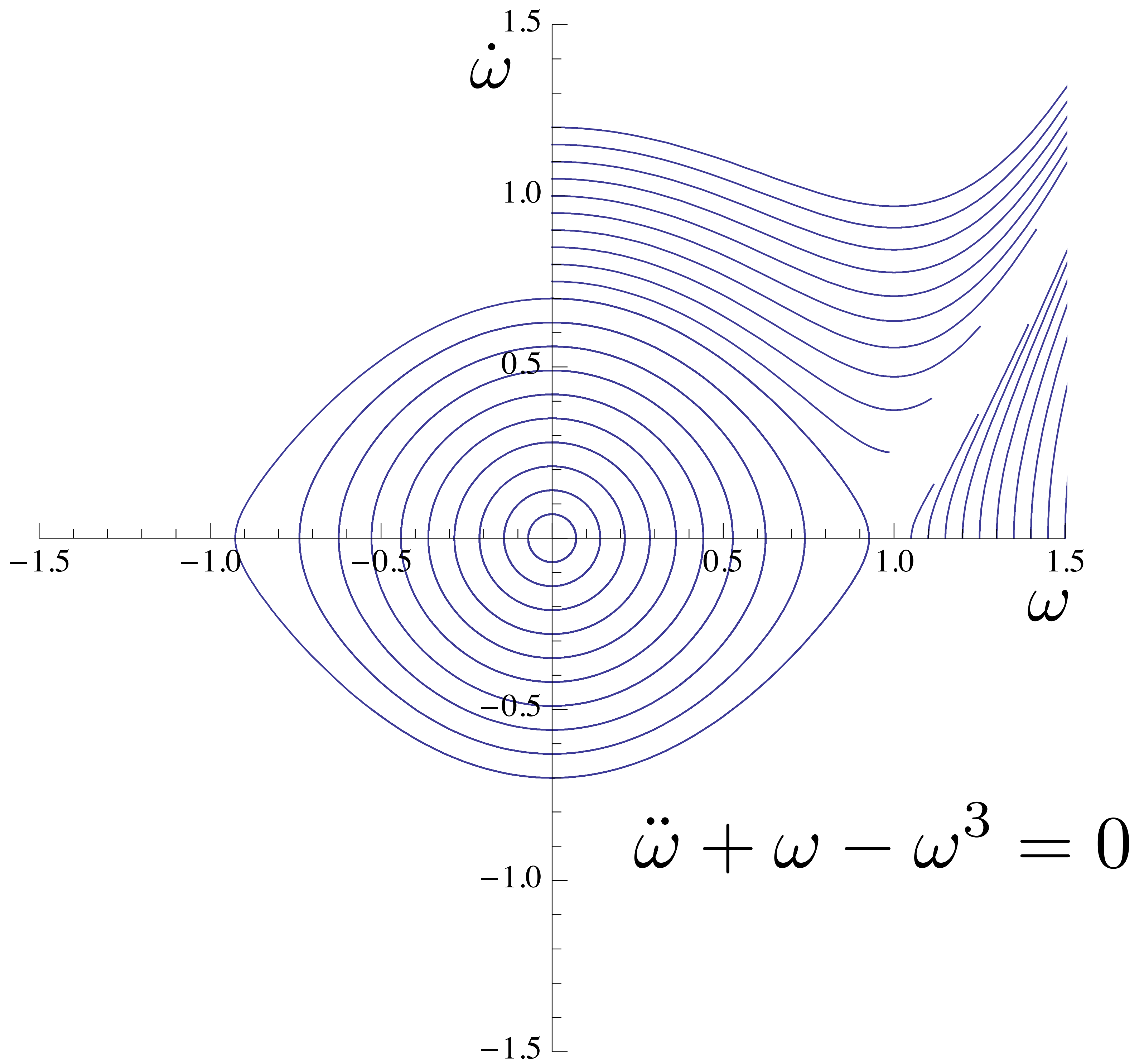


Asymptotic Stability: The motion $\boldsymbol{x}(t)$ is asymptotically stable relative to $\boldsymbol{x}_r(t)$ if $\boldsymbol{x}(t)$ is Lyapunov stable and there exists a $\delta>0$ such that

$$\boldsymbol{x}(t_0) \in B_\delta(\boldsymbol{x}_r(t_0)) \Longrightarrow \lim_{t\to\infty} \boldsymbol{x}(t) = \boldsymbol{x}_r(t)$$

Global Stability: The motion $\boldsymbol{x}(t)$ is globally stable relative to $\boldsymbol{x}_r(t)$ if $\boldsymbol{x}(t)$ is stable for any initial state vector $\boldsymbol{x}(t_0)$.





(Show Mathematica Example)

Linearization of Dynamical System

Feedforward control

Reference motion given by: $\dot{\boldsymbol{x}}_r = \boldsymbol{f}(\boldsymbol{x}_r, \boldsymbol{u}_r)$

Nonlinear EOM: $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$

Feedback control: $\delta\boldsymbol{u} = \boldsymbol{u} - \boldsymbol{u}_r$

Departure motion: $\delta\boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}_r$

Performing a Taylor Series expansion of $\boldsymbol{x}$ about ($\boldsymbol{x}_r$,$\boldsymbol{u}_r$) we obtain

$$\delta\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}_r, \boldsymbol{u}_r) + \frac{\partial \boldsymbol{f}(\boldsymbol{x}_r, \boldsymbol{u}_r)}{\partial \boldsymbol{x}}\delta\boldsymbol{x} + \frac{\partial \boldsymbol{f}(\boldsymbol{x}_r, \boldsymbol{u}_r)}{\partial \boldsymbol{u}}\delta\boldsymbol{u} + H.O.T - \boldsymbol{f}(\boldsymbol{x}_r, \boldsymbol{u}_r)$$

$$\delta\dot{\boldsymbol{x}} \simeq \frac{\partial \boldsymbol{f}(\boldsymbol{x}_r, \boldsymbol{u}_r)}{\partial \boldsymbol{x}}\delta\boldsymbol{x} + \frac{\partial \boldsymbol{f}(\boldsymbol{x}_r, \boldsymbol{u}_r)}{\partial \boldsymbol{u}}\delta\boldsymbol{u}$$

Let us define:

$$[A] = \frac{\partial \boldsymbol{f}(\boldsymbol{x}_r, \boldsymbol{u}_r)}{\partial \boldsymbol{x}}$$

$$[B] = \frac{\partial \boldsymbol{f}(\boldsymbol{x}_r, \boldsymbol{u}_r)}{\partial \boldsymbol{u}}$$

The linearized system is then written in standard form as

$$\boxed{\delta\dot{\boldsymbol{x}} \simeq [A]\delta\boldsymbol{x} + [B]\delta\boldsymbol{u}}$$

If the nominal reference motion is an equilibrium state $\boldsymbol{x}_e$, then the linearized EOM simplify to:

$$\dot{\boldsymbol{x}} \simeq [A]\boldsymbol{x} + [B]\boldsymbol{u}$$

Lyapunov’s Direct Method

Powerful method to prove nonlinear stability using energy methods...

More Definitions...

Positive (Negative) Definite Function: A scalar continuous function $V(\boldsymbol{x})$ is said to be locally positive (negative) definite about $\boldsymbol{x}_r$ if

$$\boldsymbol{x} = \boldsymbol{x}_r \implies V(\boldsymbol{x}) = 0$$

and there exists a δ>0 such that

$$\forall\ \boldsymbol{x} \in B_\delta(\boldsymbol{x}_r) \implies V(\boldsymbol{x}) > 0 \quad (V(\boldsymbol{x}) < 0)$$

Positive (Negative) Semi-Definite Function: A scalar continuous function $V(\boldsymbol{x})$ is said to be locally positive (negative) semi-definite about $\boldsymbol{x}_r$ if

$$\boldsymbol{x} = \boldsymbol{x}_r \implies V(\boldsymbol{x}) = 0$$

and there exists a δ>0 such that

$$\forall\ \boldsymbol{x} \in B_\delta(\boldsymbol{x}_r) \implies V(\boldsymbol{x}) \geq 0 \quad (V(\boldsymbol{x}) \leq 0)$$

Examples:

$$V(x, \dot{x}) = \frac{1}{2}x^2 + \frac{1}{2}\dot{x}^2$$

A matrix $[K]$ is said to be positive or negative (semi-) definite if for every state vector $\boldsymbol{x}$:

$$\boldsymbol{x}^T[K]\boldsymbol{x} \begin{cases} > 0 & \Rightarrow \text{positive definite} \\ \geq 0 & \Rightarrow \text{positive semi-definite} \\ < 0 & \Rightarrow \text{negative definite} \\ \leq 0 & \Rightarrow \text{negative semi-definite} \end{cases}$$

Lyapunov Function

Lyapunov Function: The scalar function $V(\boldsymbol{x})$ is a Lyapunov function for the dynamical system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ if it is continuous and there exists a $\delta>0$ such that for any $\boldsymbol{x} \in B_\delta(\boldsymbol{x}_r)$

1) $V(\boldsymbol{x})$ is a positive definite function about $\boldsymbol{x}_r$
2) $V(\boldsymbol{x})$ has continuous partial derivatives
3) $\dot{V}(\boldsymbol{x})$ is negative semi-definite

Example: Consider the spring-mass system

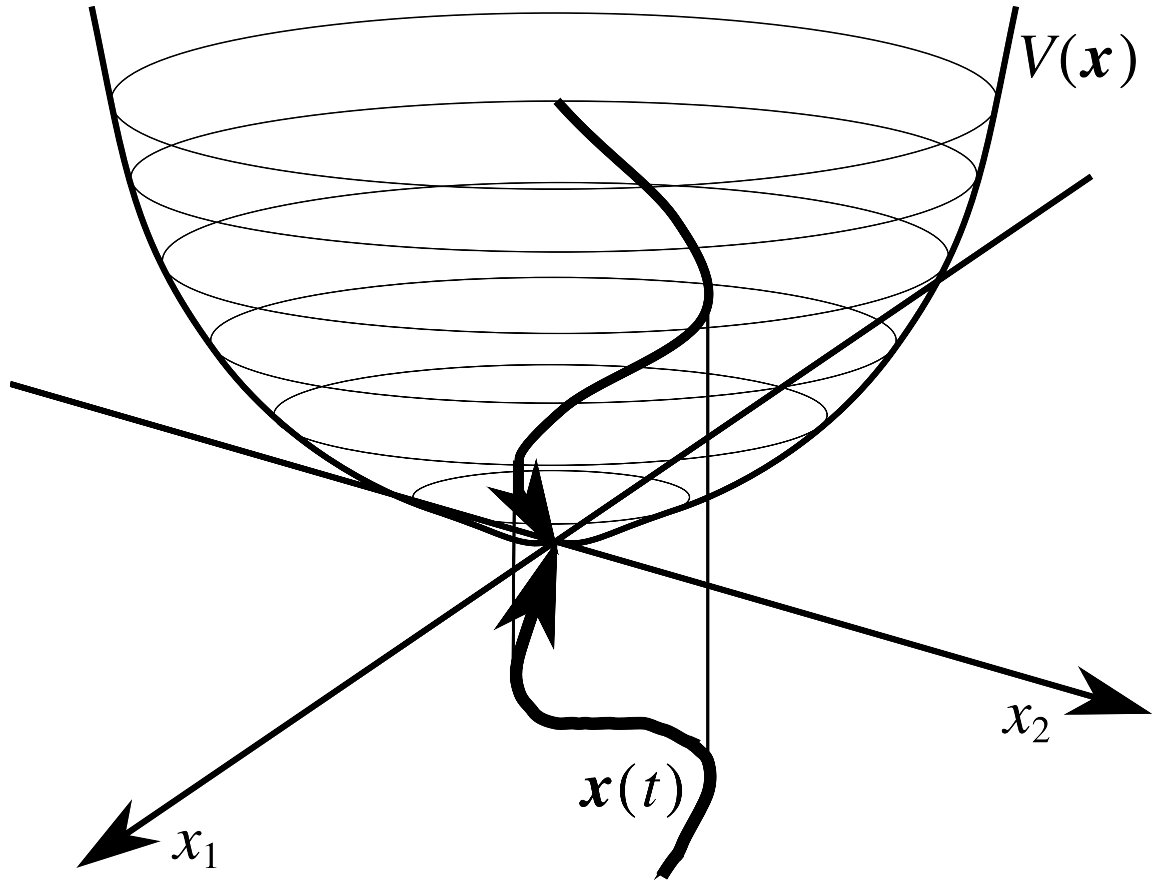
$$m\ddot{x} + kx = 0$$

Let us use the total system energy as a candidate Lyapunov function.

$$V(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

The Lyapunov rate is then expressed as

$$\dot{V}(x, \dot{x}) = (m\ddot{x} + kx)\,\dot{x} = 0 \leq 0$$



$$\dot{V} = \frac{\partial V}{\partial \boldsymbol{x}}^T \dot{\boldsymbol{x}} = \frac{\partial V}{\partial \boldsymbol{x}}^T \boldsymbol{f}(\boldsymbol{x}) \leq 0$$

All projections of the dynamical motion on to the Lyapunov function surface must point toward the reference state $\boldsymbol{x}_r$.

Lyapunov Stability: If a Lyapunov function $V(\boldsymbol{x})$ exists for the dynamical system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, then this system is stable about the origin.

Asymptotic Stability: Assume $V(\boldsymbol{x})$ is a Lyapunov function about $\boldsymbol{x}_r(t)$ for the dynamical system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$; then the system is asymptotically stable if

1) the system is stable about $\boldsymbol{x}_r$
2) $\dot{V}(\boldsymbol{x})$ is negative definite about $\boldsymbol{x}_r$

Example: Consider the spring-mass-damper system:

$$m\ddot{x} + c\dot{x} + kx = 0$$

with the Lyapunov function

$$V(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

Taking the derivative we only determine stability.

$$\dot{V}(x, \dot{x}) = (m\ddot{x} + kx)\,\dot{x} = -c\dot{x}^2 \leq 0$$

Evaluating the higher derivatives on the set $\dot{x} = 0$ yields:

$$\ddot{V}(\dot{x} = 0) = -2c\ddot{x}\dot{x} = 2\frac{c}{m}(c\dot{x} + kx)\,\dot{x} = 0$$

$$\dddot{V} = -2\frac{c}{m^2}\left((c\dot{x} + kx)^2 + c^2\dot{x}^2 + ckx\dot{x} - k\dot{x}^2\right) \quad \Rightarrow \quad \dddot{V}(\dot{x} = 0) = -2\frac{ck^2}{m^2}x^2 < 0$$

Theorem:* Assume there exists a Lyapunov function $V(\boldsymbol{x})$ of the dynamical system $\dot{\boldsymbol{x}}=\boldsymbol{f}(\boldsymbol{x})$. Let Ω be the non-empty set of state vectors such that

$$\boldsymbol{x} \in \Omega \implies \dot{V}(\boldsymbol{x}) = 0$$

If the first k-1 derivatives of $V(\boldsymbol{x})$, evaluated on the set Ω, are zero

$$\frac{\mathrm{d}^i V(\boldsymbol{x})}{\mathrm{d}t^i} = 0 \quad \forall\ \boldsymbol{x} \in \Omega \qquad i = 1, 2, \ldots, k-1$$

and the k^{th} derivative is negative definite on the set Ω

$$\frac{\mathrm{d}^k V(\boldsymbol{x})}{\mathrm{d}t^k} < 0 \quad \forall\ \boldsymbol{x} \in \Omega$$

then the system $\boldsymbol{x}(t)$ is asymptotically stable if k is an odd number.

*R. Mukherjee and D. Chen, "Asymptotic Stability Theorem for Autonomous Systems," *Journal of Guidance, Control and Dynamics*, Vol. 16, Sept. –Oct. 1993, pp. 961–963.

Lyapunov Stability of Linear System

- Assume that the dynamical system is of the linear form:

$$\dot{\boldsymbol{x}} = [A]\boldsymbol{x}$$

- Let [P]>0 be a symmetric, p.d. matrix, then we define

$$V(\boldsymbol{x}) = \boldsymbol{x}^T[P]\boldsymbol{x}$$

$$\Downarrow$$

$$\dot{V} = \dot{\boldsymbol{x}}^T[P]\boldsymbol{x} + \boldsymbol{x}^T[P]\dot{\boldsymbol{x}}$$

$$\Downarrow$$

$$\dot{V} = \boldsymbol{x}^T\left([A]^T[P] + [P][A]\right)\boldsymbol{x} < 0 \quad \checkmark$$

is this negative definite?

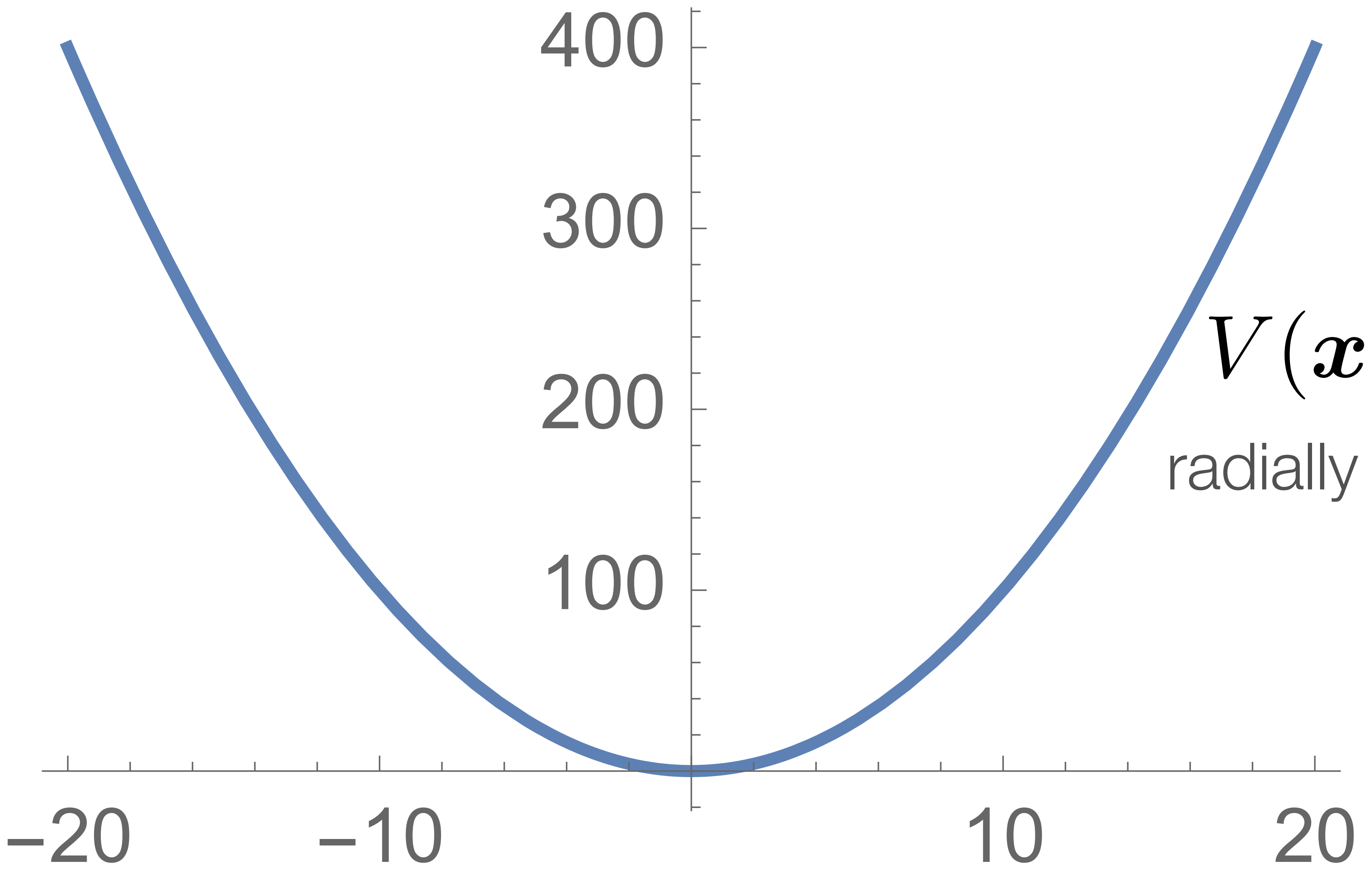
Theorem: An autonomous linear system $\dot{\boldsymbol{x}} = [A]\boldsymbol{x}$ is stable if and only if for any symmetric, positive definite [R] there exists a corresponding symmetric, positive definite [P] such that

$$[A]^T[P] + [P][A] = -[R]$$

algebraic Lyapunov equation

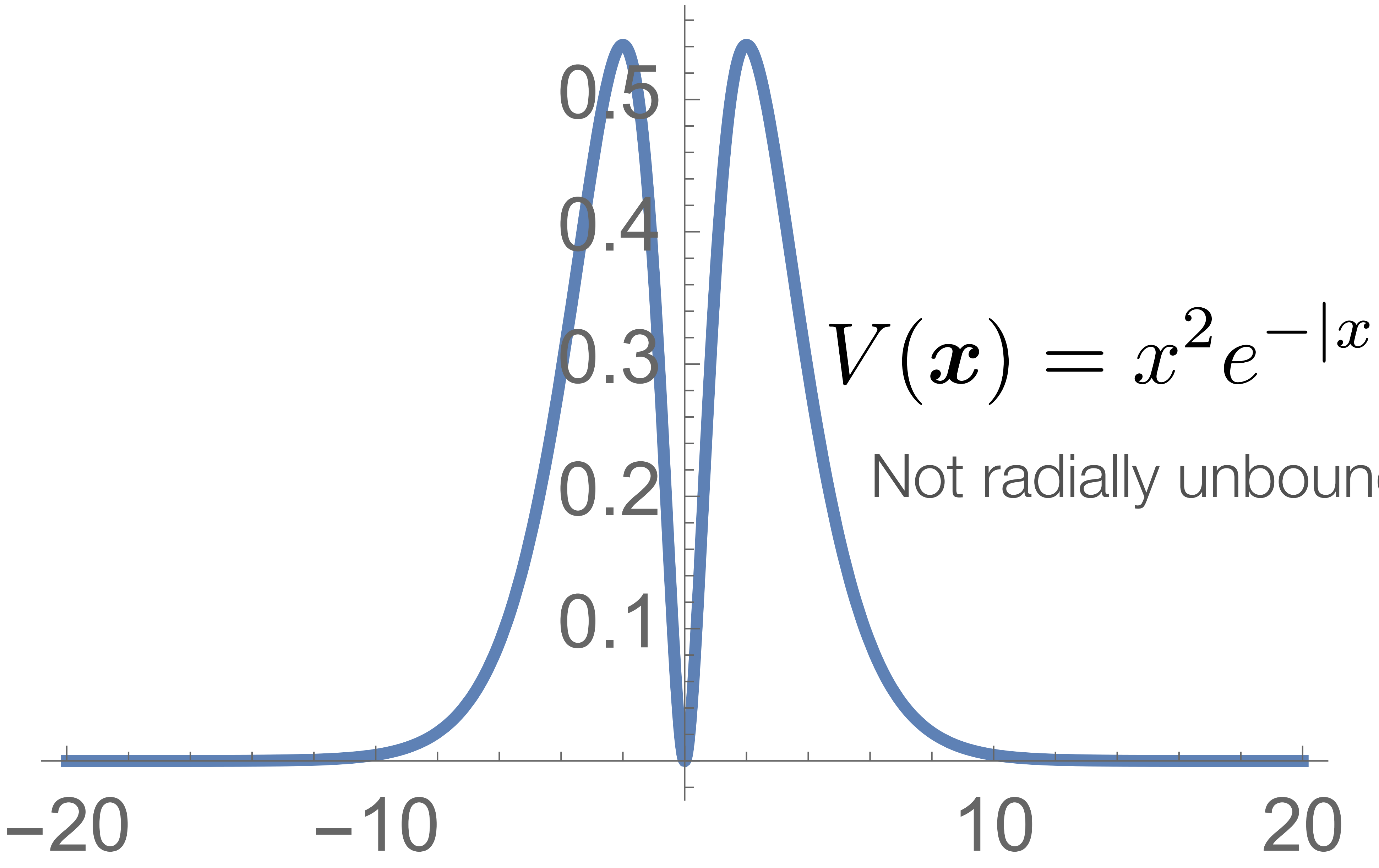
Global Stability

- The stability argument holds for any initial conditions
- The V(**x**) function is radially unbounded $\lim_{|\boldsymbol{x}|\to\infty} V(\boldsymbol{x}) = \infty$



$V(\boldsymbol{x}) = x^2$

radially unbounded



$V(\boldsymbol{x}) = x^2 e^{-|x|}$

Not radially unbounded

Lyapunov Functions

Elegant energy functions to make the control design/analysis simpler...

Elemental Velocity-Based Lyapunov Functions

Finding proper Lyapunov functions can be a difficult task for many systems.

We will break up this search into rate and position based Lyapunov functions.

Goal: drive only the state rates to zero

$$\dot{\boldsymbol{q}} \to 0$$

General Mechanical System

State Vector: $(\boldsymbol{q}, \dot{\boldsymbol{q}})$

Goal: $\dot{\boldsymbol{q}} \rightarrow 0$

Kinetic Energy: $\boxed{T = \frac{1}{2}\dot{\boldsymbol{q}}^T[M]\dot{\boldsymbol{q}}}$ True for any natural mechanical system

with $[M] = [M]^T > 0$

EOM: $\boxed{[M(\boldsymbol{q})]\ddot{\boldsymbol{q}}} = -[\dot{M}(\boldsymbol{q}, \dot{\boldsymbol{q}})]\dot{\boldsymbol{q}} + \frac{1}{2}\dot{\boldsymbol{q}}^T[M_q(\boldsymbol{q})]\dot{\boldsymbol{q}} + \boxed{\boldsymbol{Q}}$ with $\dot{\boldsymbol{q}}^T[M_{\boldsymbol{q}}(\boldsymbol{q})]\dot{\boldsymbol{q}} \equiv \begin{pmatrix} \dot{\boldsymbol{q}}^T \left[\frac{\partial M}{\partial q_1}\right] \dot{\boldsymbol{q}} \\ \vdots \\ \dot{\boldsymbol{q}}^T \left[\frac{\partial M}{\partial q_N}\right] \dot{\boldsymbol{q}} \end{pmatrix}$

Generalized Force Vector

Lyapunov Function: $\boxed{V(\dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}}^T[M(\boldsymbol{q})]\dot{\boldsymbol{q}}}$

Let's chose to use the kinetic energy function as our Lyapunov function!

Lyapunov Rate: $\dot{V} = \dot{\boldsymbol{q}}^T\boxed{[M]\ddot{\boldsymbol{q}}} + \frac{1}{2}\dot{\boldsymbol{q}}^T[\dot{M}]\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}^T\left(-\frac{1}{2}[\dot{M}]\dot{\boldsymbol{q}} + \frac{1}{2}\dot{\boldsymbol{q}}^T[M_{\boldsymbol{q}}]\dot{\boldsymbol{q}} + \boldsymbol{Q}\right)$

Note that $\dot{\boldsymbol{q}}^T\left(\dot{\boldsymbol{q}}^T[M_{\boldsymbol{q}}]\dot{\boldsymbol{q}}\right) = \sum_{i=1}^{N} \dot{q}_i \left(\dot{\boldsymbol{q}}^T[M_{q_i}]\dot{\boldsymbol{q}}\right) = \dot{\boldsymbol{q}}^T[\dot{M}]\dot{\boldsymbol{q}}$

$\boxed{\dot{V} = \dot{\boldsymbol{q}}^T\boldsymbol{Q}}$ This is the generalized work/energy equation! $\Rightarrow$ $\boldsymbol{Q} = -[P]\dot{\boldsymbol{q}}$ $\Rightarrow$ $\dot{V} = -\dot{\boldsymbol{q}}^T[P]\dot{\boldsymbol{q}} < 0$ Globally asymptotically stabilizing

- Next, let us consider the tracking problem of a generalized mechanical system:

Reference States: $(\boldsymbol{q}_r, \dot{\boldsymbol{q}}_r)$

State Vector: $(\boldsymbol{q}, \dot{\boldsymbol{q}})$

Tracking Error: $\delta\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}_r$

Goal: $\delta\dot{\boldsymbol{q}} \to 0$

Lyapunov Function:
$$V(\dot{\boldsymbol{q}}) = \frac{1}{2}\delta\dot{\boldsymbol{q}}^T[M(\boldsymbol{q})]\delta\dot{\boldsymbol{q}}$$
Energy-function-like positive definite measure of tracking error.

Lyapunov Rate:
$$\dot{V} = \delta\dot{\boldsymbol{q}}^T\left(-\frac{1}{2}[\dot{M}]\left(\dot{\boldsymbol{q}} + \dot{\boldsymbol{q}}_r\right) + \frac{1}{2}\dot{\boldsymbol{q}}^T[M_q]\dot{\boldsymbol{q}} - [M]\ddot{\boldsymbol{q}}_r + \boldsymbol{Q}\right)$$

Note that the work/energy principle doesn't hold with these non-mechanical energy function, and the Lyapunov rate is no longer the simply the power equation.

Proposed Control:
$$\boldsymbol{Q} = \frac{1}{2}[\dot{M}]\left(\dot{\boldsymbol{q}} + \dot{\boldsymbol{q}}_r\right) - \frac{1}{2}\dot{\boldsymbol{q}}^T[M_q]\dot{\boldsymbol{q}} + [M]\ddot{\boldsymbol{q}}_r - [P]\delta\dot{\boldsymbol{q}}$$

Feedback linearization

Feedforward compensation

Proportional Feedback

$$\Rightarrow \quad \dot{V}(\delta\dot{\boldsymbol{q}}) = -\delta\dot{\boldsymbol{q}}^T[P]\delta\dot{\boldsymbol{q}} < 0$$

Globally asymptotically stabilizing

- Example of Mechanical System Stabilization: (Ex: 8.8 in S&J)

State vector: $\boldsymbol{q} = (\theta_1, \theta_2, \theta_3)^T$

Goal: $\dot{\boldsymbol{q}} \to 0$

EOM:

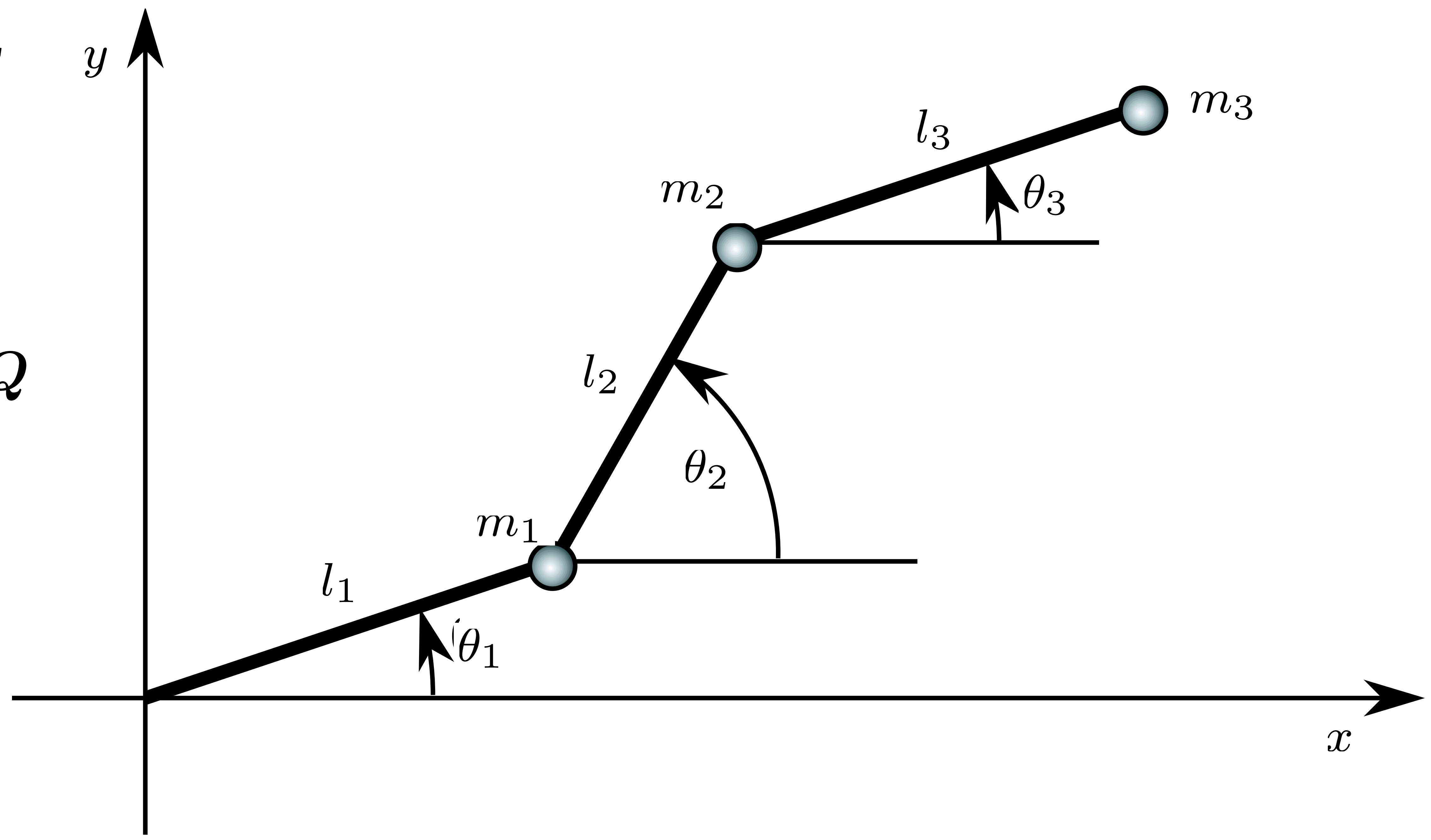
$$[M]\ddot{\boldsymbol{q}} + [\dot{M}]\dot{\boldsymbol{q}} - \frac{1}{2}\dot{\boldsymbol{q}}^T[M_{\boldsymbol{q}}]\dot{\boldsymbol{q}} = \boldsymbol{Q}$$

Lyapunov Function:

$$V(\dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}}^T[M(\boldsymbol{q})]\dot{\boldsymbol{q}}$$

Lyapunov Rate:

$$\dot{V} = \dot{\boldsymbol{q}}^T\boldsymbol{Q}$$



Control:

$$\boxed{\boldsymbol{Q}_1 = -P_1\dot{\boldsymbol{q}}}$$ Rate Feedback

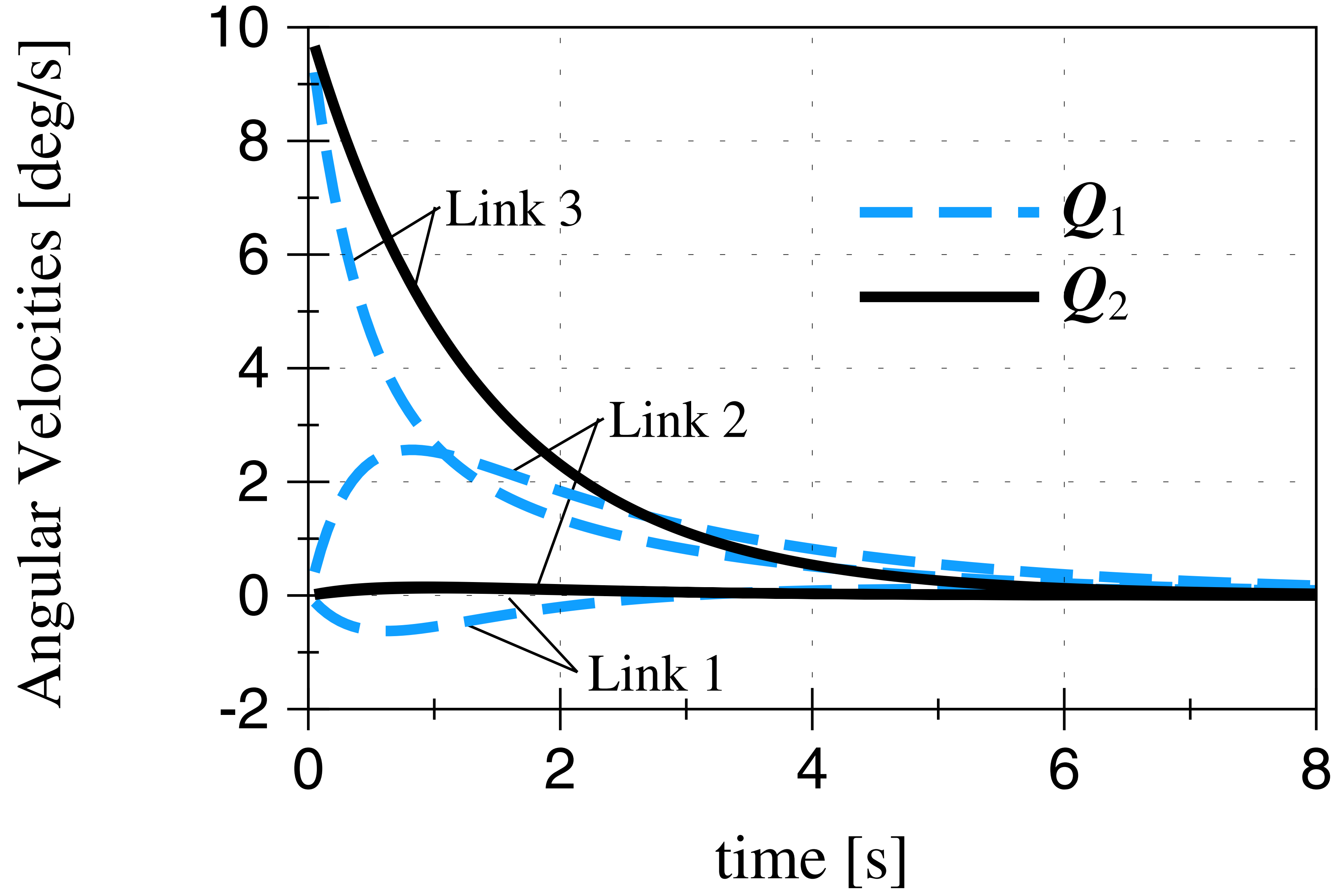
$$\boxed{\boldsymbol{Q}_2 = -P_2[M(\boldsymbol{q})]\dot{\boldsymbol{q}}}$$ "Momentum" Feedback

Symmetric, positive definite Mass Matrix:

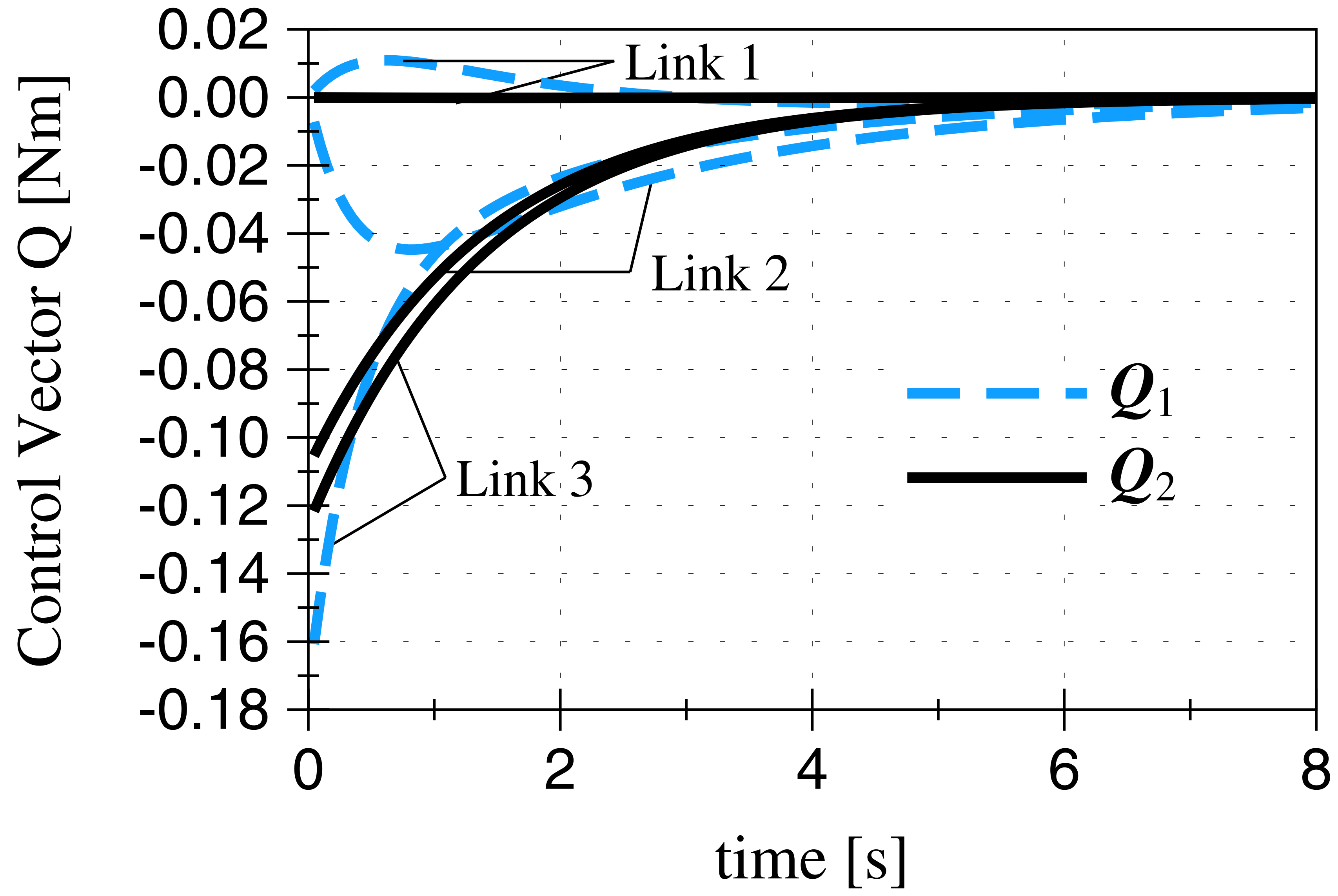
$$M(\boldsymbol{q}) = \begin{bmatrix} (m_1+m_2+m_3)l_1^2 & (m_2+m_3)l_1l_2\cos(\theta_2-\theta_1) & m_3l_1l_3\cos(\theta_3-\theta_1) \\ (m_2+m_3)l_1l_2\cos(\theta_2-\theta_1) & (m_2+m_3)l_2^2 & m_3l_1l_3\cos(\theta_3-\theta_2) \\ m_3l_1l_3\cos(\theta_3-\theta_1) & m_3l_2l_3\cos(\theta_3-\theta_2) & m_3l_3^2 \end{bmatrix}$$

Simulation Parameters

Parameter	Value	Units
l_i	1	m
m_i	1.0	kg
P_1	1.0	kg-m^2/sec
P_2	0.72	kg-m^2/sec
$\boldsymbol{x}(t_0)$	$[-90\ 30\ 0]$	deg
$\dot{\boldsymbol{x}}(t_0)$	$[0.0\ 0.0\ 10]$	deg/sec



While both controls are asymptotically stabilizing, the 2nd control solution is actually exponentially stabilizing, and successfully isolates the motion of the third link from the first link.



Rigid Body Detumbling

State Vector: $\boldsymbol{\omega}$ Goal: $\boldsymbol{\omega} \to 0$

EOM: $[I]\dot{\boldsymbol{\omega}} = -[\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} + \boldsymbol{Q}$

External control torque

Lyapunov Function:

$$V(\boldsymbol{\omega}) = T = \frac{1}{2}\boldsymbol{\omega}^T[I]\boldsymbol{\omega}$$

Constant in Body frame components

Lyapunov Rate:

$$\dot{V} = \boldsymbol{\omega}^T\left([I]\dot{\boldsymbol{\omega}}\right) = \boldsymbol{\omega}^T\left(-[\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} + \boldsymbol{Q}\right)$$

$$= \boldsymbol{\omega}^T\boldsymbol{Q}$$

Power form of work/ energy equation

Control: $\boxed{\boldsymbol{Q} = -[P]\boldsymbol{\omega}}$ with $[P] = [P]^T > 0$

$$\dot{V}(\boldsymbol{\omega}) = -\boldsymbol{\omega}^T[P]\boldsymbol{\omega} < 0$$

Globally asymptotically stabilizing

Note: This control result does not require any knowledge of the inertia matrix! It is very robust to inertia modeling errors.

Reference: $\boldsymbol{\omega}_r$ Goal: $\delta\boldsymbol{\omega} = \boldsymbol{\omega} - \boldsymbol{\omega}_r \to 0$

Note: ${}^{\mathcal{B}}\delta\boldsymbol{\omega} = {}^{\mathcal{B}}\boldsymbol{\omega} - [BR]\,{}^{\mathcal{R}}\boldsymbol{\omega}_r$

Lyapunov Function: $V(\delta\boldsymbol{\omega}) = \frac{1}{2}\delta\boldsymbol{\omega}^T[I]\delta\boldsymbol{\omega}$

Lyapunov Rate: $\dot{V} = \delta\boldsymbol{\omega}^T[I]\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\delta\boldsymbol{\omega})$

Note: $\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\delta\boldsymbol{\omega}) = \dot{\boldsymbol{\omega}} - \dot{\boldsymbol{\omega}}_r + \boldsymbol{\omega} \times \boldsymbol{\omega}_r$

$$\dot{V} = \delta\boldsymbol{\omega}^T\left(-[\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} + [I]\boldsymbol{\omega} \times \boldsymbol{\omega}_r - [I]\dot{\boldsymbol{\omega}}_r + \boldsymbol{Q}\right)$$

Control:

$$\boxed{\boldsymbol{Q} = [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} - [I][\tilde{\boldsymbol{\omega}}]\boldsymbol{\omega}_r + [I]\dot{\boldsymbol{\omega}}_r - [P]\delta\boldsymbol{\omega}}$$

$$\dot{V}(\delta\boldsymbol{\omega}) = -\delta\boldsymbol{\omega}^T[P]\delta\boldsymbol{\omega} < 0$$

Globally asymptotically stabilizing

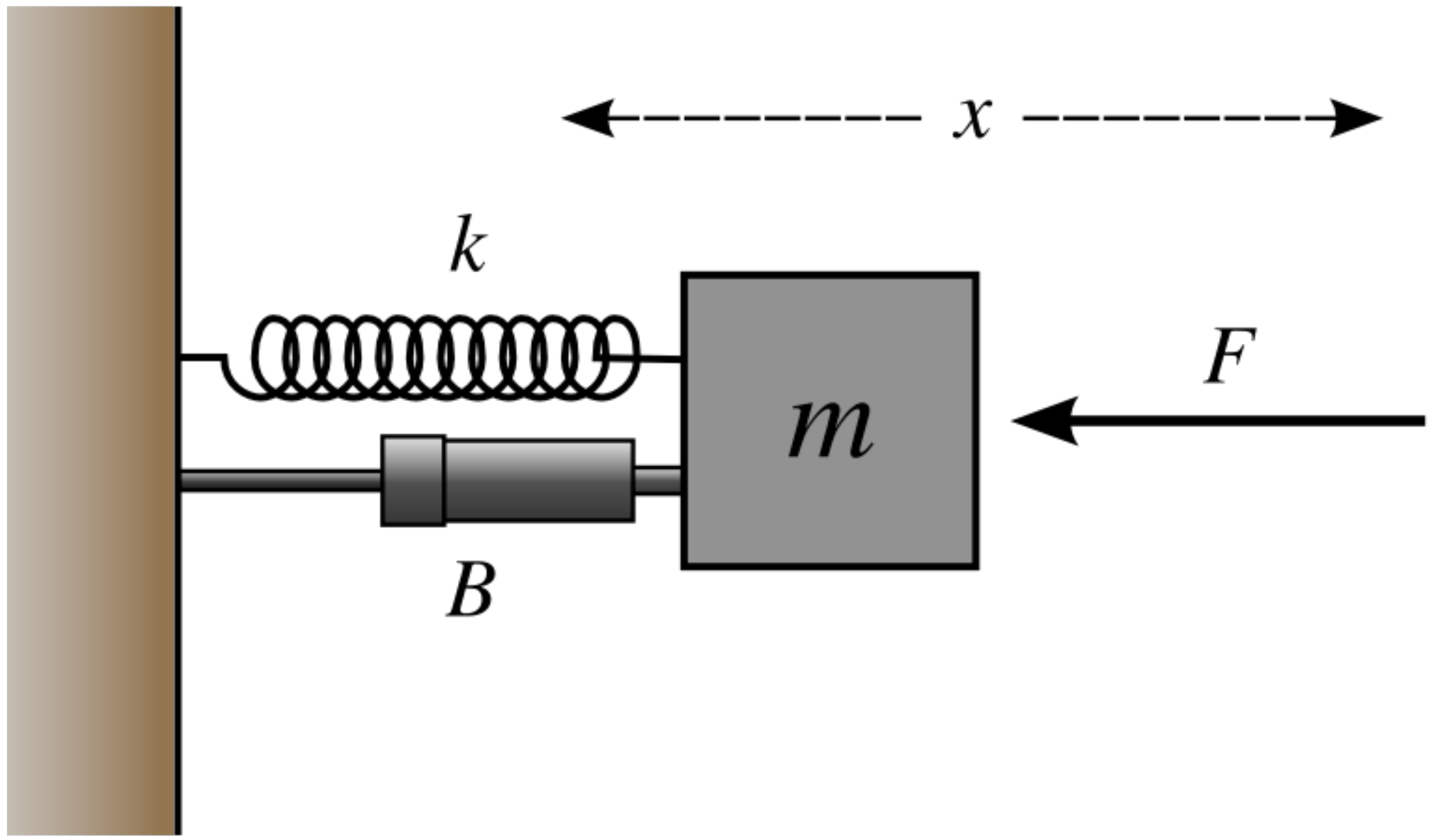
Elemental Position-Based Lyapunov Functions

Physical Motivation is the linear spring energy with stiffness k:

$$T = \frac{1}{2} k x^2$$

We will seek similar energy-like functions which provide positive definite error measures of the position-errors.

The state rate $\dot{x}$ is treated as a control variable in this discussion. This is typically the case in robotic control where a lower level servo loop implements the required x.



Euler Angle Potential Functions

State Vector: $$\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)^T$$

Lyapunov Function: $$V(\boldsymbol{\theta}) = \frac{1}{2}\boldsymbol{\theta}^T[K]\boldsymbol{\theta}$$ with $$[K] = [K]^T > 0$$

Kinematic Differential Equations: $$\dot{\boldsymbol{\theta}} = [B(\boldsymbol{\theta})]\boldsymbol{\omega}$$

Lyapunov Rate: $$\dot{V} = \boldsymbol{\omega}^T\left([B(\boldsymbol{\theta})]^T[K]\boldsymbol{\theta}\right)$$

This function can later on be used in Lyapunov position and rate feedback law developments.

Attitude Tracking Error between *B* and reference frame *R*: $$\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)^T$$

Kinematic Differential Equations: $$\dot{\boldsymbol{\theta}} = [B(\boldsymbol{\theta})]\delta\boldsymbol{\omega}$$ with $$\delta\boldsymbol{\omega} = \boldsymbol{\omega} - \boldsymbol{\omega}_r$$

Lyapunov Function: $$V(\boldsymbol{\theta}) = \frac{1}{2}\boldsymbol{\theta}^T[K]\boldsymbol{\theta}$$

There is no algebraic distinction between position based regulator and tracking Lyapunov functions, and we won't distinguish the two from remainder of the position-based Lyapunov function discussion.

Classical Rodrigues Parameters

A brute force approach would define a candidate Lyapunov function as spring-mass energy-like form:

$$V(\boldsymbol{q}) = \boldsymbol{q}^T [K] \boldsymbol{q}$$

Taking the derivative we find

$$\dot{V} = \boldsymbol{\omega}^T \left(\left(I - [\tilde{\boldsymbol{q}}] + \boldsymbol{q}\boldsymbol{q}^T \right) [K] \boldsymbol{q} \right)$$

which gets reduced to

$$\dot{V} = \boldsymbol{\omega}^T \left(K \left(1 + q^2 \right) \boldsymbol{q} \right)$$

if the gain matrix [*K*] is a scalar *K* and $q^2 = \boldsymbol{q}^T \boldsymbol{q}$.

This term may lead to nonlinear feedback laws.

A more elegant Gibbs-vector Lyapunov function is given by:

$$V(\boldsymbol{q}) = K \ln \left(1 + \boldsymbol{q}^T \boldsymbol{q} \right)$$

Taking the derivative, and substituting the differential kinematic equations, a surprisingly simple form is found:

$$\boxed{\dot{V} = \boldsymbol{\omega}^T \left(K \boldsymbol{q} \right)}$$

leads to linear attitude feedback!

Let's develop an attitude servo law, we define

$$V(\boldsymbol{q}) = \ln \left(1 + \boldsymbol{q}^T \boldsymbol{q} \right) \quad \text{with} \quad \dot{V} = \boldsymbol{\omega}^T \boldsymbol{q}$$

The body rate control vector is then defined as

$$\boldsymbol{\omega} = -[K] \boldsymbol{q} \quad \Rightarrow \quad \dot{V}(\boldsymbol{q}) = -\boldsymbol{q}^T [K] \boldsymbol{q} < 0$$

Other Parameters

Modified Rodrigues Parameters:

Lyapunov Function:

$$V(\boldsymbol{\sigma}) = 2K \ln\left(1 + \boldsymbol{\sigma}^T \boldsymbol{\sigma}\right)$$

If you switch to the shadow MRP set on the $\sigma^2=1$ surface, then this Lyapunov function is continuous.

Lyapunov Rate:

$$\dot{V} = \boldsymbol{\omega}^T \left(K\boldsymbol{\sigma}\right)$$

This leads to elegant linear attitude feedback laws which are globally stabilizing by switching between the original and shadow MRP set.

Euler Parameters:

Ideal Attitude: $\hat{\boldsymbol{\beta}} = \begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}^T$

Lyapunov Function:

$$V(\boldsymbol{\beta}) = K\left(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}\right)^T\left(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}\right)$$

Lyapunov Rate:

$$\dot{V} = K\boldsymbol{\omega}^T[B(\boldsymbol{\beta})]^T\left(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}\right)$$

recall that $[B(\boldsymbol{\beta})]^T\boldsymbol{\beta} = 0$

This leads to

$$\dot{V} = K\boldsymbol{\omega}^T\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix} = \boldsymbol{\omega}^T\left(K\boldsymbol{\epsilon}\right)$$

Note that will will stabilize the attitude to $\beta_0 = \pm 1$, which is the same attitude. However, no guarantee is made if the long or short rotational path is used.

Nonlinear Feedback

Finally, we look at the complete 3-axis control of spacecraft attitude...

Unconstrained Control

- First let us assume that the external control (thrusters) is unconstrained in magnitude, and that the thruster can point in any direction.

External torque
(atmospheric torque)

EOM: $$[I]\dot{\boldsymbol{\omega}} = -[\tilde{\omega}][I]\boldsymbol{\omega} + \boldsymbol{u} + \boldsymbol{L}$$

control vector
(thrusters)

Goal: $$\delta\boldsymbol{\omega} = \boldsymbol{\omega} - \boldsymbol{\omega}_r \to 0 \qquad \boldsymbol{\sigma} \to 0$$

angular velocity
error

body angular
velocity

reference angular
velocity

Attitude error between body
frame *B* and reference frame *R*
using MRPs

Exact attitude tracking error kinematic differential equations:

$$\dot{\boldsymbol{\sigma}} = \frac{1}{4}\left[(1-\sigma^2)I + 2[\tilde{\boldsymbol{\sigma}}] + 2\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right]\delta\boldsymbol{\omega}$$

Lyapunov function definition:

$$V(\delta\boldsymbol{\omega}, \boldsymbol{\sigma}) = \frac{1}{2}\delta\boldsymbol{\omega}^T[I]\delta\boldsymbol{\omega} + 2K\ln\left(1 + \boldsymbol{\sigma}^T\boldsymbol{\sigma}\right)$$

kinetic-energy-like p.d. function

p.d. MRP attitude error function

Note that the angular rate and inertia components are taken with respect to the body frame.

$$\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}([I]) = 0 \qquad \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\delta\boldsymbol{\omega})$$

To guarantee stability, we force $\dot{V}$ to be **negative semi-definite** by setting it equal to

$$\dot{V} = -\delta\boldsymbol{\omega}^T[P]\delta\boldsymbol{\omega}$$

$$[P] = [P]^T > 0$$

Differentiating V we find:

$$\dot{V} = \delta\boldsymbol{\omega}^T\left([I]\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\delta\boldsymbol{\omega}) + K\boldsymbol{\sigma}\right) = -\delta\boldsymbol{\omega}^T[P]\delta\boldsymbol{\omega}$$

$$\boxed{[I]\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\delta\boldsymbol{\omega}) + [P]\delta\boldsymbol{\omega} + K\boldsymbol{\sigma} = 0}$$

closed-loop dynamics

Using $\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\delta\boldsymbol{\omega}) = \dot{\boldsymbol{\omega}} - \dot{\boldsymbol{\omega}}_r + \boldsymbol{\omega}\times\boldsymbol{\omega}_r$ yields

$$\boxed{[I](\dot{\boldsymbol{\omega}}} - \dot{\boldsymbol{\omega}}_r + \boldsymbol{\omega}\times\boldsymbol{\omega}_r) + [P](\boldsymbol{\omega} - \boldsymbol{\omega}_r) + K\boldsymbol{\sigma} = 0$$

Substitute EOM: $\boxed{[I]\dot{\boldsymbol{\omega}}} = -[\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} + \boldsymbol{u} + \boldsymbol{L}$

$$\boldsymbol{u} = -K\boldsymbol{\sigma} - [P]\delta\boldsymbol{\omega} + [I]\left(\dot{\boldsymbol{\omega}}_r - [\tilde{\boldsymbol{\omega}}]\boldsymbol{\omega}_r\right) + [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} - \boldsymbol{L}$$

$${}^{\mathcal{B}}\boldsymbol{\omega}_r = [BR]\,{}^{\mathcal{R}}\boldsymbol{\omega}_r \qquad {}^{\mathcal{B}}\dot{\boldsymbol{\omega}}_r = [BR]\,{}^{\mathcal{R}}\dot{\boldsymbol{\omega}}_r$$

Global Stability?

The feedback law found is of the form

$$\boldsymbol{u} = -K\boldsymbol{\sigma} - [P]\delta\boldsymbol{\omega} + [I]\left(\dot{\boldsymbol{\omega}}_r - [\tilde{\boldsymbol{\omega}}]\boldsymbol{\omega}_r\right) + [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} - \boldsymbol{L}$$

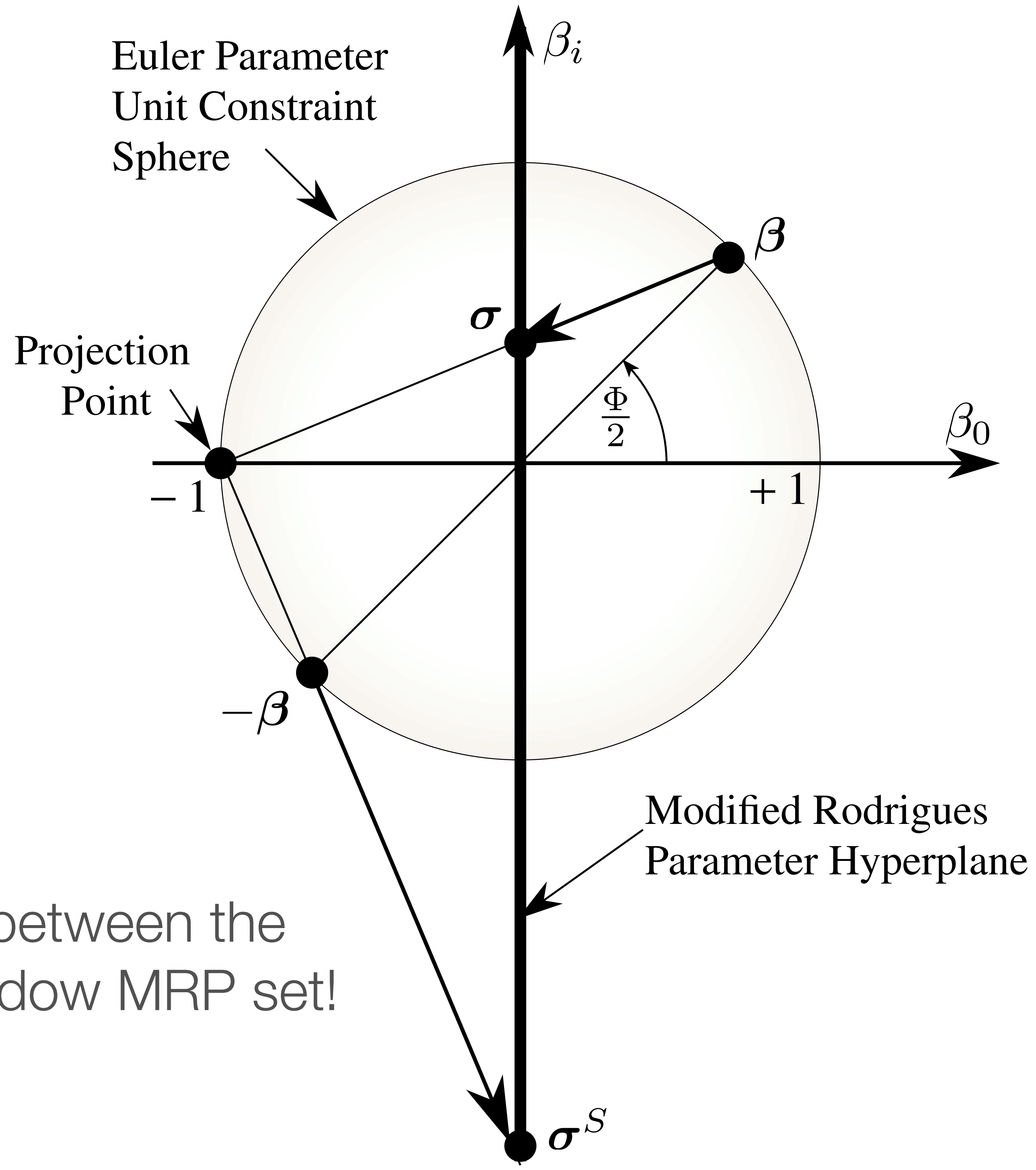
With the associated Lyapunov function being defined as:

$$V(\boldsymbol{\omega},\boldsymbol{\sigma}) = \frac{1}{2}\delta\boldsymbol{\omega}^T[I]\delta\boldsymbol{\omega} + 2K\ln\left(1+\boldsymbol{\sigma}^T\boldsymbol{\sigma}\right)$$

$$V \to \infty \quad \text{as} \quad \delta\boldsymbol{\omega},\boldsymbol{\sigma} \to \infty$$

Globally Stabilizing

However, the MRP attitude can go singular? What if the body is tumbling and we make a 360° revolution?



We can switch between the original and shadow MRP set!

A convenient MRP switching surface is

$$\boldsymbol{\sigma}^T\boldsymbol{\sigma} = \sigma^2 = 1$$

where the body is "upside-down" relative to the reference attitude. The mapping to the shadow set is simply

$$\boldsymbol{\sigma}^S = -\boldsymbol{\sigma}$$

Note that the Lyapunov function *V* is *continuous* during this MRP switching with this switching surface!

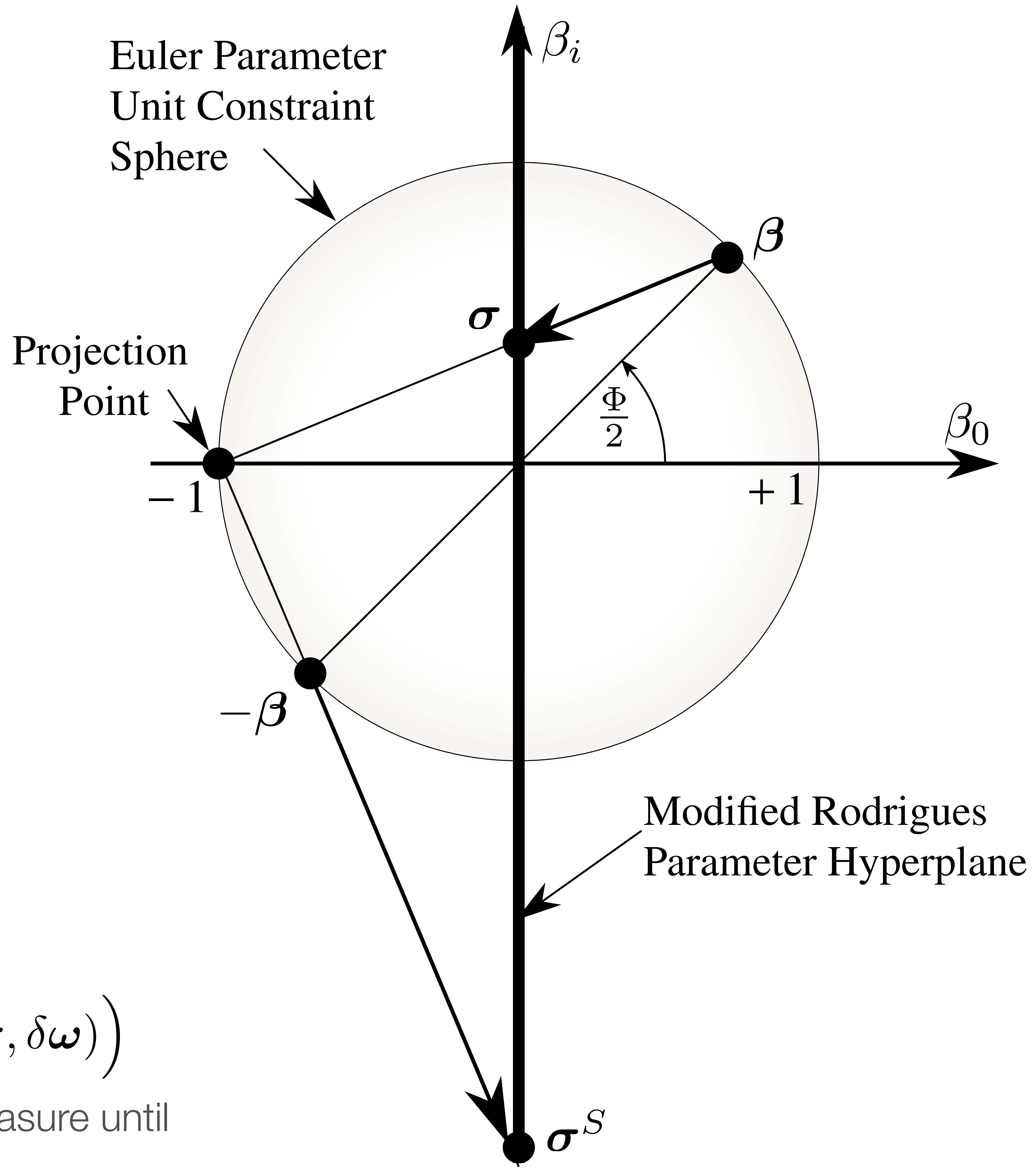
Assume that V_1 is the Lyapunov tracking how the original state errors are being reduced.

$V_1(\boldsymbol{\sigma}, \delta\boldsymbol{\omega})$ is reduced until $|\boldsymbol{\sigma}| = 1$

MRP mapping: $\boldsymbol{\sigma}^S = -\boldsymbol{\sigma}$

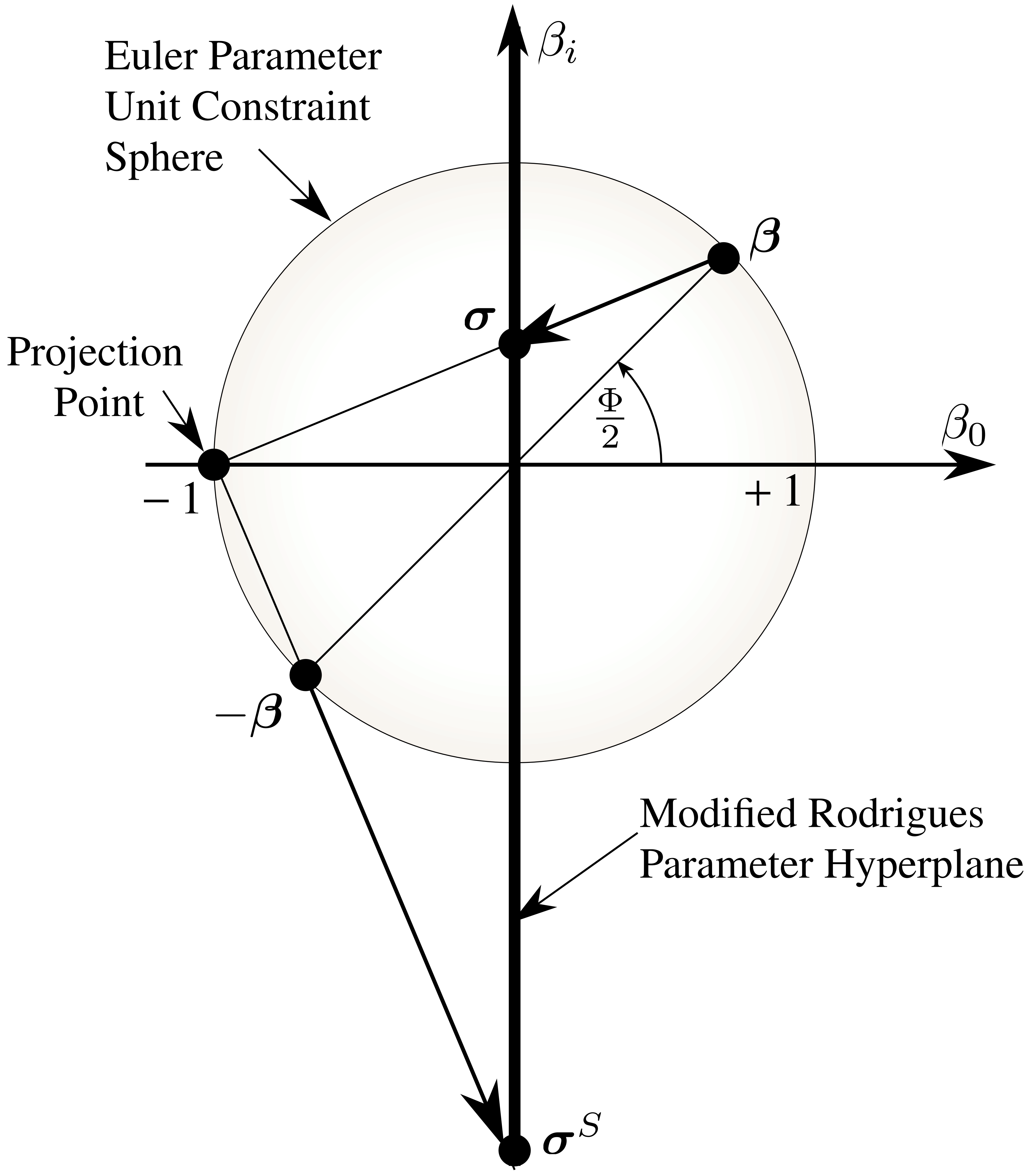
New Lyapunov function: $V_2(\boldsymbol{\sigma}^S, \delta\boldsymbol{\omega}) \quad \Big(= V_1(\boldsymbol{\sigma}, \delta\boldsymbol{\omega})\Big)$

We can reset the stability analysis now to track this new error measure until either $|\boldsymbol{\sigma}^S| = 1$ or $|\boldsymbol{\sigma}^S| \rightarrow= 0$.



Comment:
During the switching the V partial derivatives are well defined, but not continuous. Thus, we cannot use standard Lyapunov theory to argue global stability. However, by breaking up the problem into a series of every decreasing Lyapunov functions, we can argue that global stability will be achieved.

Application:
This MRP switching provides very elegant attitude feedback laws which are linear in MRP, and will automatically de-tumble a body by always rotating it back to the reference attitude using the shortest rotation distance. Attitude error wind-up is avoided.



Tumbling Body Example:

Single-axis rotation with

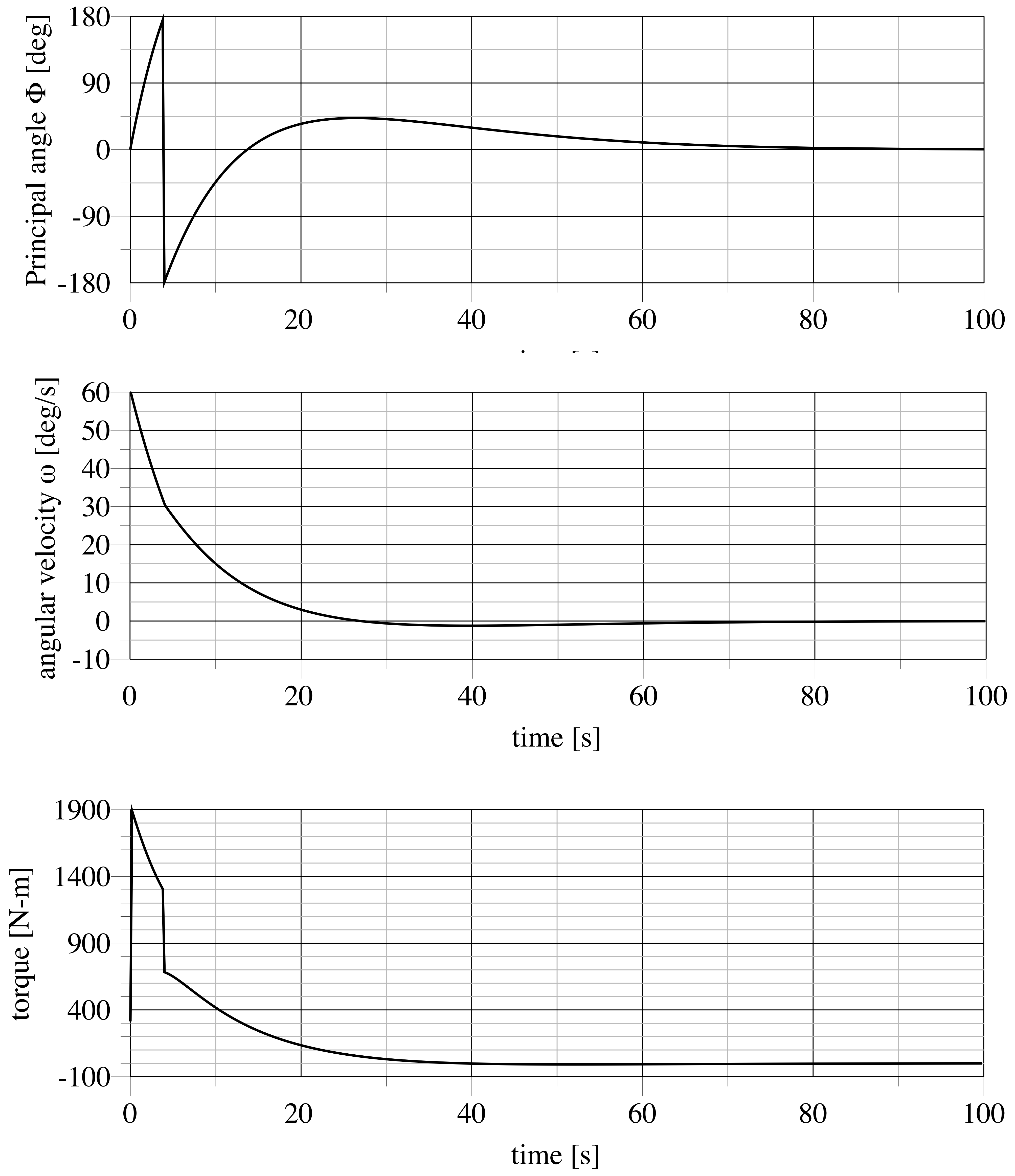
$$J = 12000 \text{ kg m}^2$$

$$K = 300$$

$$P = 1800$$

$$\omega = 60^{\text{o}}/\text{s}$$

Goal: Bring the body to rest at the zero attitude. (regulator problem)



Asymptotic Convergence

Note that the previous Lyapunov function only had a negative semi-definite derivative.

$$\dot{V}(\boldsymbol{\sigma}, \delta\boldsymbol{\omega}) = -\delta\boldsymbol{\omega}^T[P]\delta\boldsymbol{\omega}$$

$\Rightarrow$ globally stabilizing

Let us analyze this control to see when it is asymptotically stabilizing. We do so by investigating higher derivatives of *V*.

Note $\dot{V} = 0 \quad \Rightarrow \quad \Omega = \{\delta\boldsymbol{\omega} = 0\}$

2nd: $$\ddot{V} = -2\delta\boldsymbol{\omega}^T[P]\delta\dot{\boldsymbol{\omega}}$$

$$\ddot{V}(\boldsymbol{\sigma}, \delta\boldsymbol{\omega} = 0) = 0$$

3rd: $$\dddot{V} = -2\delta\boldsymbol{\omega}^T[P]\delta\ddot{\boldsymbol{\omega}} - 2\delta\dot{\boldsymbol{\omega}}^T[P]\delta\dot{\boldsymbol{\omega}}$$

Recall $$[I]\delta\dot{\boldsymbol{\omega}} + [P]\delta\boldsymbol{\omega} + K\boldsymbol{\sigma} = 0$$

closed-loop dynamics

$$\delta\dot{\boldsymbol{\omega}} = -[I]^{-1}\left([P]\delta\boldsymbol{\omega} + K\boldsymbol{\sigma}\right)$$

$$\delta\dot{\boldsymbol{\omega}} = -[I]^{-1}K\boldsymbol{\sigma} \qquad \text{if} \qquad \delta\boldsymbol{\omega} = 0$$

$\Downarrow$

$$\dddot{V}(\boldsymbol{\sigma}, \delta\boldsymbol{\omega} = 0) = -K^2\boldsymbol{\sigma}^T\left([I]^{-1}\right)[P][I]\boldsymbol{\sigma}$$

This 3rd derivative is negative definite in MRPs, and thus the control is asymptotically stabilizing.

External Torque Model Error

If some *un-modeled* external torque $\Delta \boldsymbol{L}$ is present, then the EOM are written as:

$$[I]\dot{\boldsymbol{\omega}} = -[\tilde{\omega}][I]\boldsymbol{\omega} + \boldsymbol{u} + \boldsymbol{L} + \Delta \boldsymbol{L}$$

The Lyapunov rate is now written as

$$\dot{V} = -\delta\boldsymbol{\omega}^T [P]\delta\boldsymbol{\omega} + \delta\boldsymbol{\omega}^T \Delta \boldsymbol{L}$$

This is no longer negative semi-definite! However, assume that the un-modeled torque vector is constant and bounded (as seen by body frame *B*).

However, this $\dot{V}$ does show that the $\delta\boldsymbol{\omega}$ errors will be bounded by the control. For large $\delta\boldsymbol{\omega}$ the quadratic term will dominate.

The new closed-loop EOM are:

$$[I]\delta\dot{\boldsymbol{\omega}} + [P]\delta\boldsymbol{\omega} + K\boldsymbol{\sigma} = \Delta \boldsymbol{L}$$

Differentiating using $\dot{\boldsymbol{\sigma}} = \frac{1}{4}[B(\boldsymbol{\sigma})]\delta\boldsymbol{\omega}$ yields:

$$[I]\delta\ddot{\boldsymbol{\omega}} + [P]\delta\dot{\boldsymbol{\omega}} + \frac{K}{4}[B(\boldsymbol{\sigma})]\delta\boldsymbol{\omega} = \Delta \dot{\boldsymbol{L}} \approx 0$$

This is a spring-mass-damper system with a nonlinear spring. To show that the stiffness matrix here is positive definite note that

$$\begin{aligned}\boldsymbol{\omega}^T [B(\boldsymbol{\sigma})]\boldsymbol{\omega} &= \boldsymbol{\omega}^T \left[(1-\sigma^2)I + 2[\tilde{\boldsymbol{\sigma}}] + 2\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right]\boldsymbol{\omega} \\ &= (1-\sigma^2)\boldsymbol{\omega}^T\boldsymbol{\omega} + 2\left(\boldsymbol{\omega}^T\boldsymbol{\sigma}\right)^2 > 0\end{aligned}$$

Note that we assume that the MRP vector is maintained to have a magnitude less than 1!

$$[I]\delta\ddot{\boldsymbol{\omega}} + [P]\delta\dot{\boldsymbol{\omega}} + \frac{K}{4}[B(\boldsymbol{\sigma})]\delta\boldsymbol{\omega} = 0$$

Because the closed-loop dynamics above are stable, the angular velocity tracking errors will reach a steady state value.

$$K[B(\boldsymbol{\sigma}_{ss})]\delta\boldsymbol{\omega}_{ss} = 0$$

However, the matrix $[B(\sigma_{ss})]$ has been shown to be near-orthogonal and is always full-rank. This leads to

$$\delta\boldsymbol{\omega}_{ss} = 0$$

Using this result in the closed-loop dynamics:

$$[I]\delta\dot{\boldsymbol{\omega}} + [P]\delta\boldsymbol{\omega} + K\boldsymbol{\sigma} = \Delta\boldsymbol{L}$$

the attitude steady-state error is found to be

$$\boldsymbol{\sigma}_{ss} = \lim_{t\to\infty}\boldsymbol{\sigma} = \frac{1}{K}\Delta\boldsymbol{L}$$

The larger the attitude feedback gain *K* is, the smaller the steady-state attitude error will be.

Example:

A symmetric rigid body with all three principal inertias being 10 kg m^2 is studied. An unmodeled external torque is present.

$$\Delta \boldsymbol{L} = (0.05, 0, 10, -0.10)^T \text{ Nm}$$

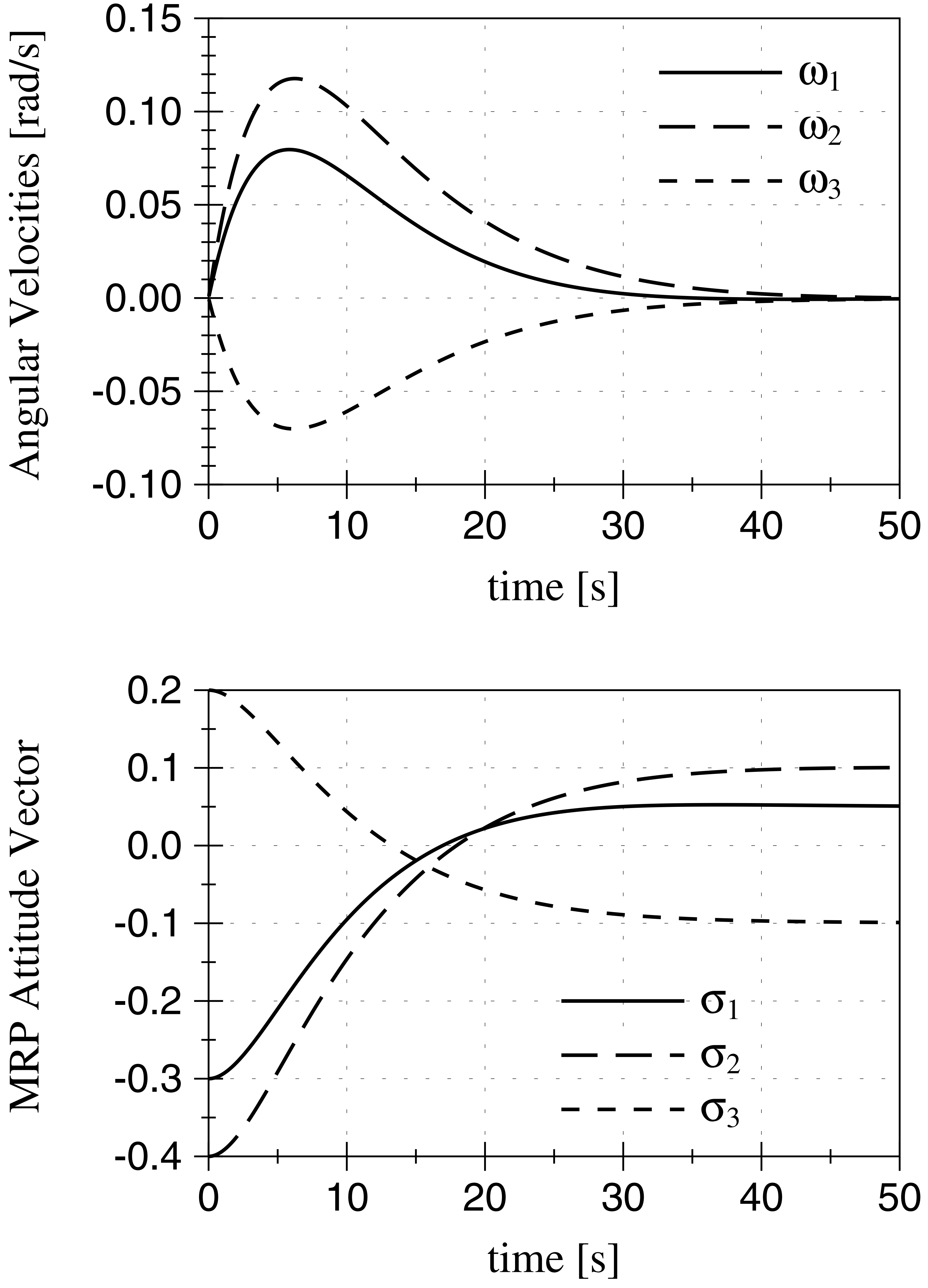
Goal: $\boldsymbol{\omega} \to 0 \qquad \boldsymbol{\sigma} \to 0$

Initial Error:
$$\boldsymbol{\sigma}(t_0) = (-0.3, -0.4, 0.2)^T$$
$$\boldsymbol{\omega}(t_0) = (0.0, 0.0, 0.0)^T$$

Gains:
$$K = 1 \text{ kg m}^2/\text{s}^2$$
$$P = 3 \text{ kg m}^2/\text{s}$$

Predicted steady state errors:

$$\delta\boldsymbol{\omega}_{ss} = 0$$
$$\boldsymbol{\sigma}_{ss} = \frac{1}{K}\Delta\boldsymbol{L} = \begin{pmatrix} 0.05 \\ 0.10 \\ -0.10 \end{pmatrix}$$



Do Regulation problem by hand

Integral Feedback

- Next, let us investigate adding an integral feedback term to make the attitude control more robust to un-modeled external torques.

Let us introduce the new state vector ***z***:

$$\boldsymbol{z}(t) = \int_0^t (K\boldsymbol{\sigma} + [I]\delta\dot{\boldsymbol{\omega}})\,\mathrm{d}t$$

Note that ***z*** will grow unbounded if there is any finite steady state attitude errors!

Thus, we want a new control law that will force ***z*** to go to zero, and thus drive any steady-state attitude errors to zero as well.

New Lyapunov function:

$$V(\delta\boldsymbol{\omega}, \boldsymbol{\sigma}, \boldsymbol{z}) = \frac{1}{2}\delta\boldsymbol{\omega}^T[I]\delta\boldsymbol{\omega} + 2K\log\left(1 + \boldsymbol{\sigma}^T\boldsymbol{\sigma}\right) + \frac{1}{2}\boldsymbol{z}^T\boxed{[K_I]}\boldsymbol{z}$$

s.p.d.

Assume at first that there is no un-modeled external torque. In this case we set the Lyapunov rate equal to

$$\dot{V} = -\left(\delta\boldsymbol{\omega} + [K_I]\boldsymbol{z}\right)^T[P]\left(\delta\boldsymbol{\omega} + [K_I]\boldsymbol{z}\right)$$

and solve for the control vector ***u***:

$$\boldsymbol{u} = -K\boldsymbol{\sigma} - [P]\delta\boldsymbol{\omega} - [P][K_I]\boldsymbol{z} + [I]\left(\dot{\boldsymbol{\omega}}_r - [\tilde{\boldsymbol{\omega}}]\boldsymbol{\omega}_r\right) + [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} - \boldsymbol{L}$$

Measuring ***z*** direction is not convenient because it required the derivative of $\delta\boldsymbol{\omega}$. Instead, note that we can write

$$\boldsymbol{z}(t) = K\int_0^t \boldsymbol{\sigma}\,\mathrm{d}t + [I]\left(\delta\boldsymbol{\omega} - \delta\boldsymbol{\omega}_0\right)$$

This allows us to re-write the feedback control law in the final form:

$$\boxed{\begin{aligned}\boldsymbol{u} = &- K\boldsymbol{\sigma} - ([P] + [P][K_I][I])\,\delta\boldsymbol{\omega} \\ &- K[P][K_I]\int_0^t \boldsymbol{\sigma}\mathrm{d}t + [P][K_I][I]\delta\boldsymbol{\omega}_0 \\ &+ [I]\left(\dot{\boldsymbol{\omega}}_r - [\tilde{\omega}]\boldsymbol{\omega}_r\right) + [\tilde{\omega}][I]\boldsymbol{\omega} - \boldsymbol{L}\end{aligned}}$$

Next, let's analyze the stability of this control law. The Lyapunov rate is semi-negative definite

$$\dot{V} = -\left(\delta\boldsymbol{\omega} + [K_I]\boldsymbol{z}\right)^T [P]\left(\delta\boldsymbol{\omega} + [K_I]\boldsymbol{z}\right)$$

This guarantees that **ω**, **σ**, and ***z*** are stable, and that

$$\delta\boldsymbol{\omega} + [K_I]\boldsymbol{z} \rightarrow 0$$

To study asymptotic convergence, we investigate the higher order derivative of the Lyapunov function *V*.

The first non-zero higher derivative evaluated on the set where $\dot{V}$ is zero is

$$\dddot{V}(\boldsymbol{\sigma}, \delta\boldsymbol{\omega}+[K_I]\boldsymbol{z}=0) = -K^2\boldsymbol{\sigma}^T\left([I]^{-1}\right)[P][I]\boldsymbol{\sigma}$$

$$\Rightarrow \quad \boldsymbol{\sigma} \rightarrow 0 \quad \Rightarrow \quad \delta\boldsymbol{\omega} \rightarrow 0$$

Kinematic Relationship

$$\delta\boldsymbol{\omega} + [K_I]\boldsymbol{z} \rightarrow 0 \quad \Rightarrow \quad \boldsymbol{z} \rightarrow 0$$

If unmodeled external torques are included, then the Lyapunov rate is expressed as:

$$\dot{V} = -\left(\delta\boldsymbol{\omega}+[K_I]\boldsymbol{z}\right)^T\left([P](\delta\boldsymbol{\omega}+[K_I]\boldsymbol{z}) - \Delta\boldsymbol{L}\right)$$

This is no longer n.s.d. However, we can conclude for bounded Δ***L*** the states δ**ω** and ***z*** must remain bounded.

Recall $$\boldsymbol{z}(t) = \int_0^t \left(K\boldsymbol{\sigma} + [I]\delta\dot{\boldsymbol{\omega}}\right)\mathrm{d}t$$

$$\Rightarrow \quad \boldsymbol{\sigma} \rightarrow 0 \quad \Rightarrow \quad \delta\boldsymbol{\omega} \rightarrow 0$$

Next, let's study the state $\boldsymbol{z}$ as the Lyapunov rate approaches zero at steady-state. This requires that

$$\lim_{t\to\infty} \left([P]\left(\delta\boldsymbol{\omega} + [K_I]\boldsymbol{z}\right) - \Delta\boldsymbol{L}\right) = 0$$

Because $\delta\boldsymbol{\omega} \to 0$, the steady-state value of $\boldsymbol{z}$ is expressed as:

$$\boxed{\lim_{t\to\infty} \boldsymbol{z} = [K_I]^{-1}[P]^{-1}\Delta\boldsymbol{L}}$$

Example:

A symmetric rigid body with all three principal inertias being 10 kg m^2 is studied. An unmodeled external torque is present.

$$\Delta \boldsymbol{L} = (0.05, 0, 10, -0.10)^T \text{ Nm}$$

Goal: $\boldsymbol{\omega} \rightarrow 0 \qquad \boldsymbol{\sigma} \rightarrow 0$

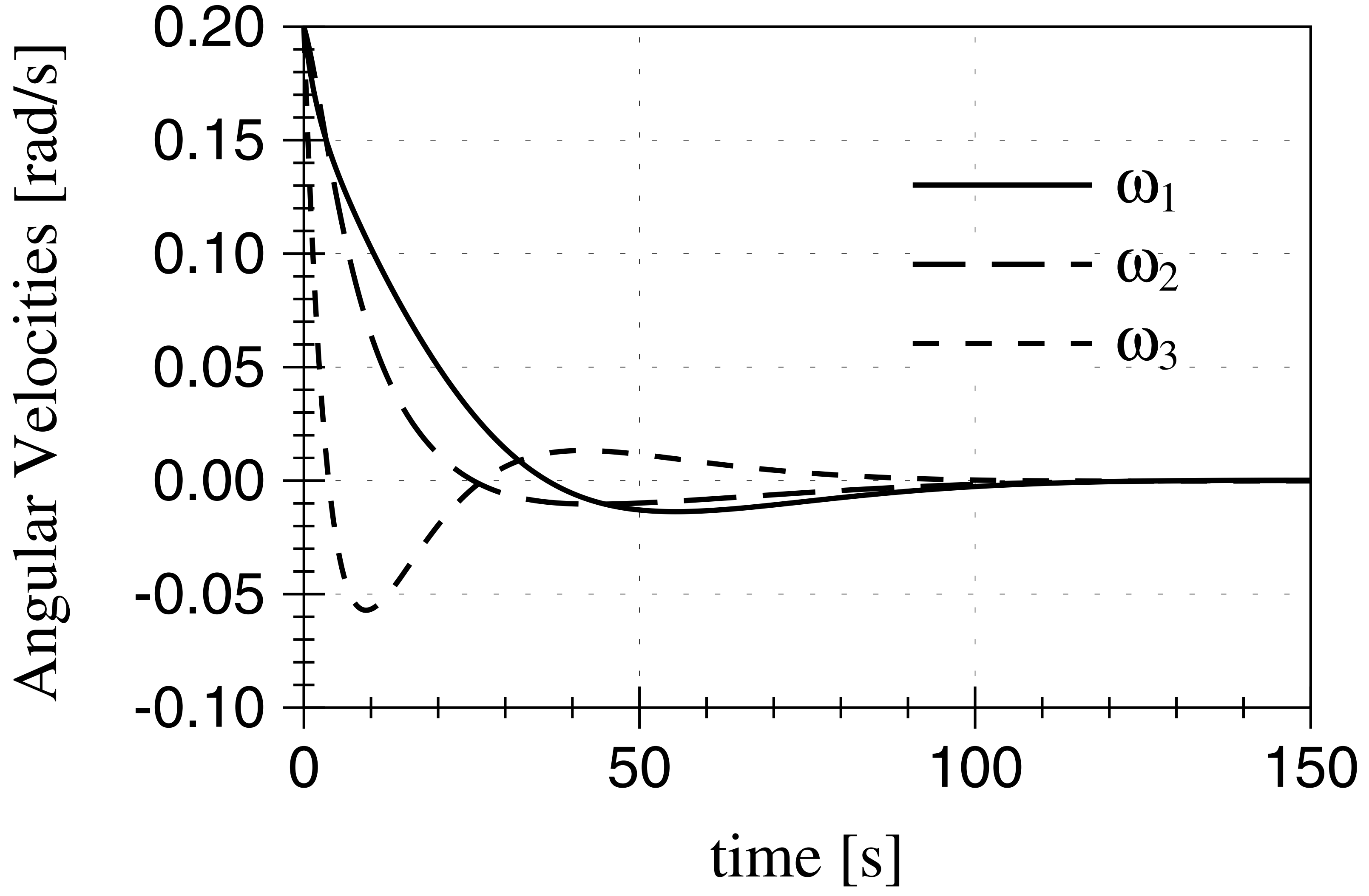
Initial Error:
$$\boldsymbol{\sigma}(t_0) = (-0.3, -0.4, 0.2)^T$$
$$\boldsymbol{\omega}(t_0) = (0.2, 0.2, 0.2)^T \text{ rad/s}$$

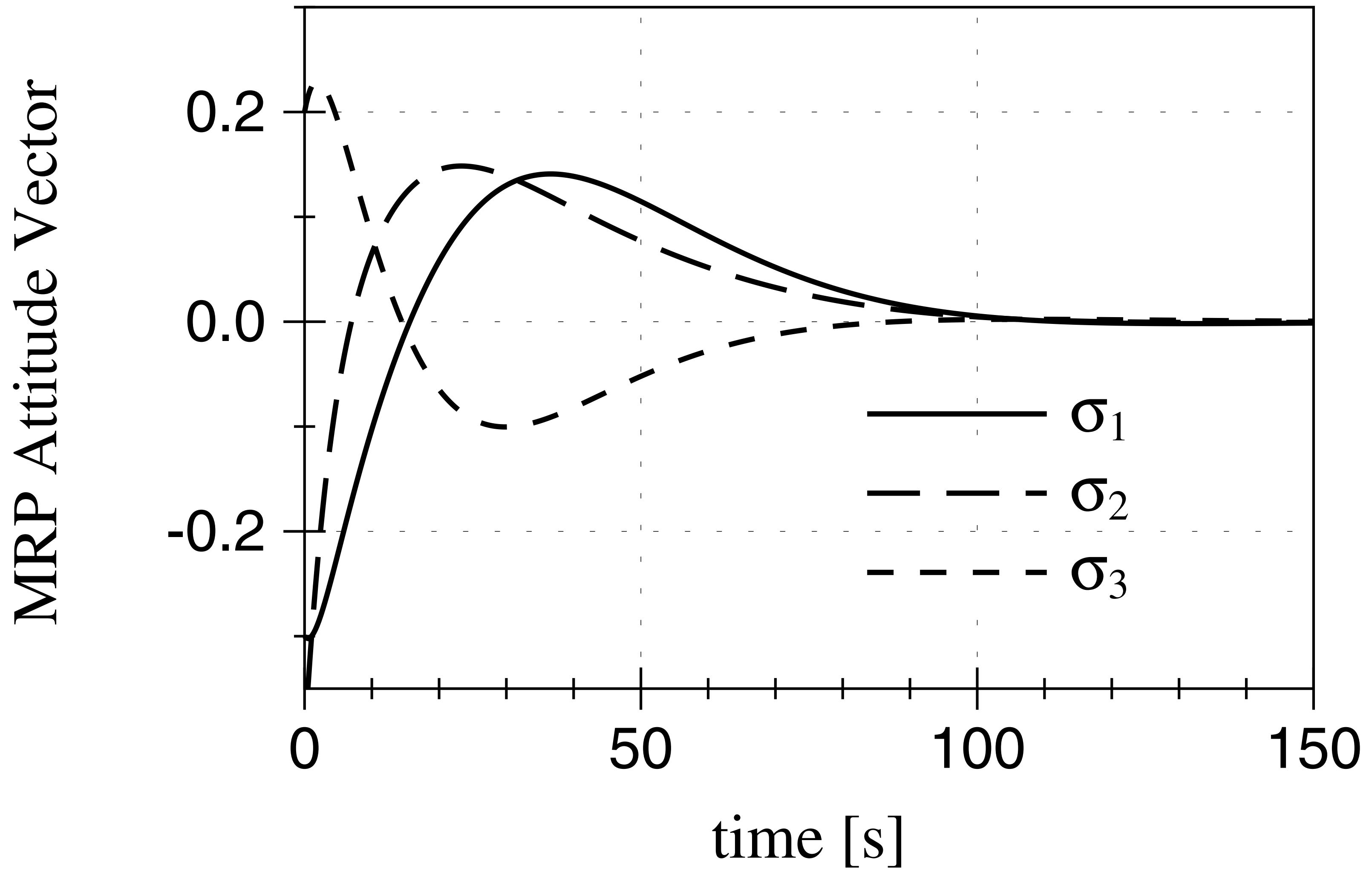
Gains:
$$K = 1 \text{ kg m}^2/\text{s}^2$$
$$P = 3 \text{ kg m}^2/\text{s}$$
$$K_I = 0.01 \text{ s}^{-1}$$

Predicted steady state errors:

$$\delta\boldsymbol{\omega}_{ss} = 0 \qquad \boldsymbol{\sigma}_{ss} = 0$$

$$\lim_{t\rightarrow\infty} \boldsymbol{z} = \frac{\Delta \boldsymbol{L}}{K_I P} = \begin{pmatrix} 1.66 \\ 3.33 \\ -3.33 \end{pmatrix} \text{kg-m}^2/\text{sec}$$





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Predicted steady state errors:

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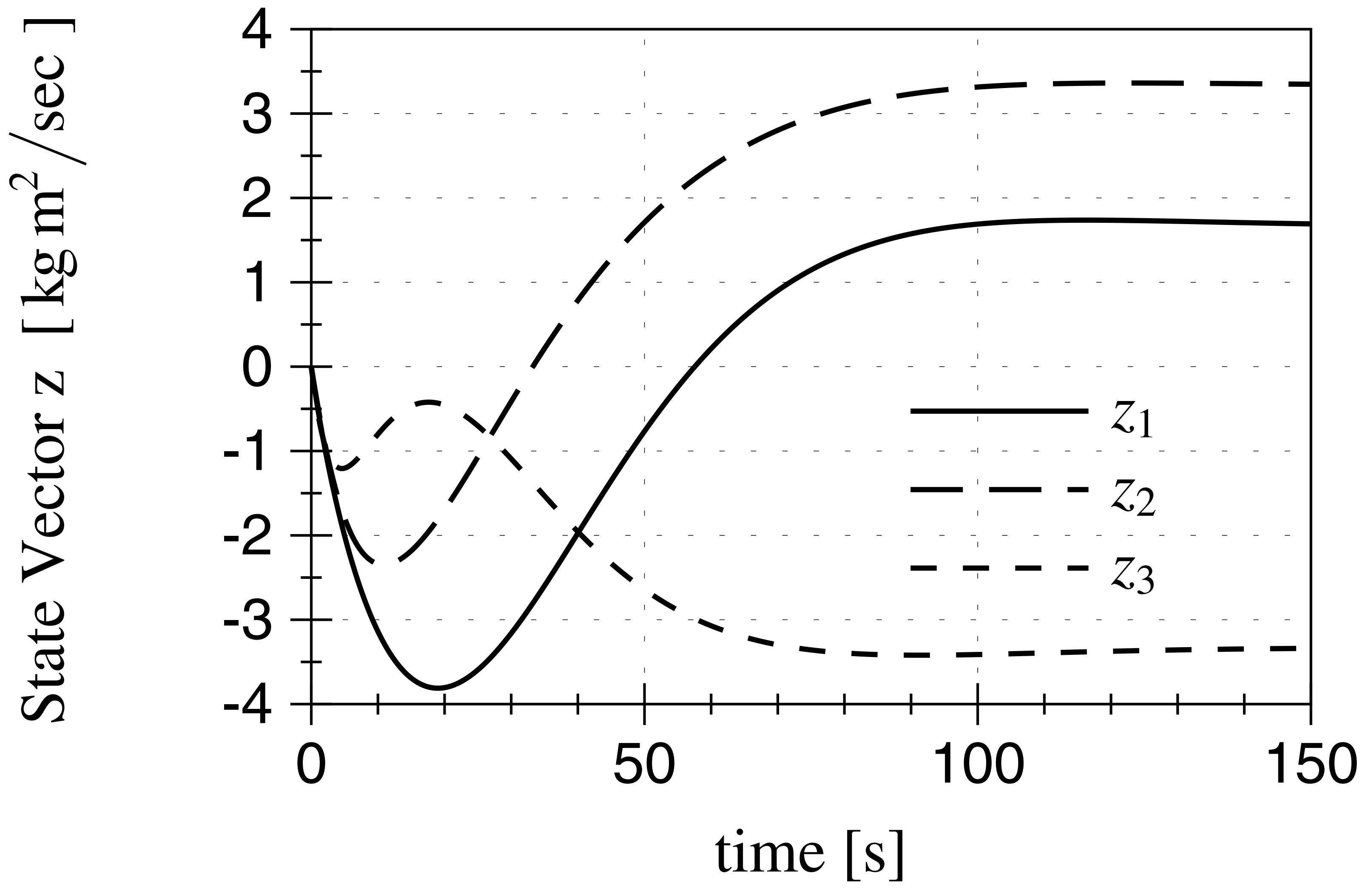
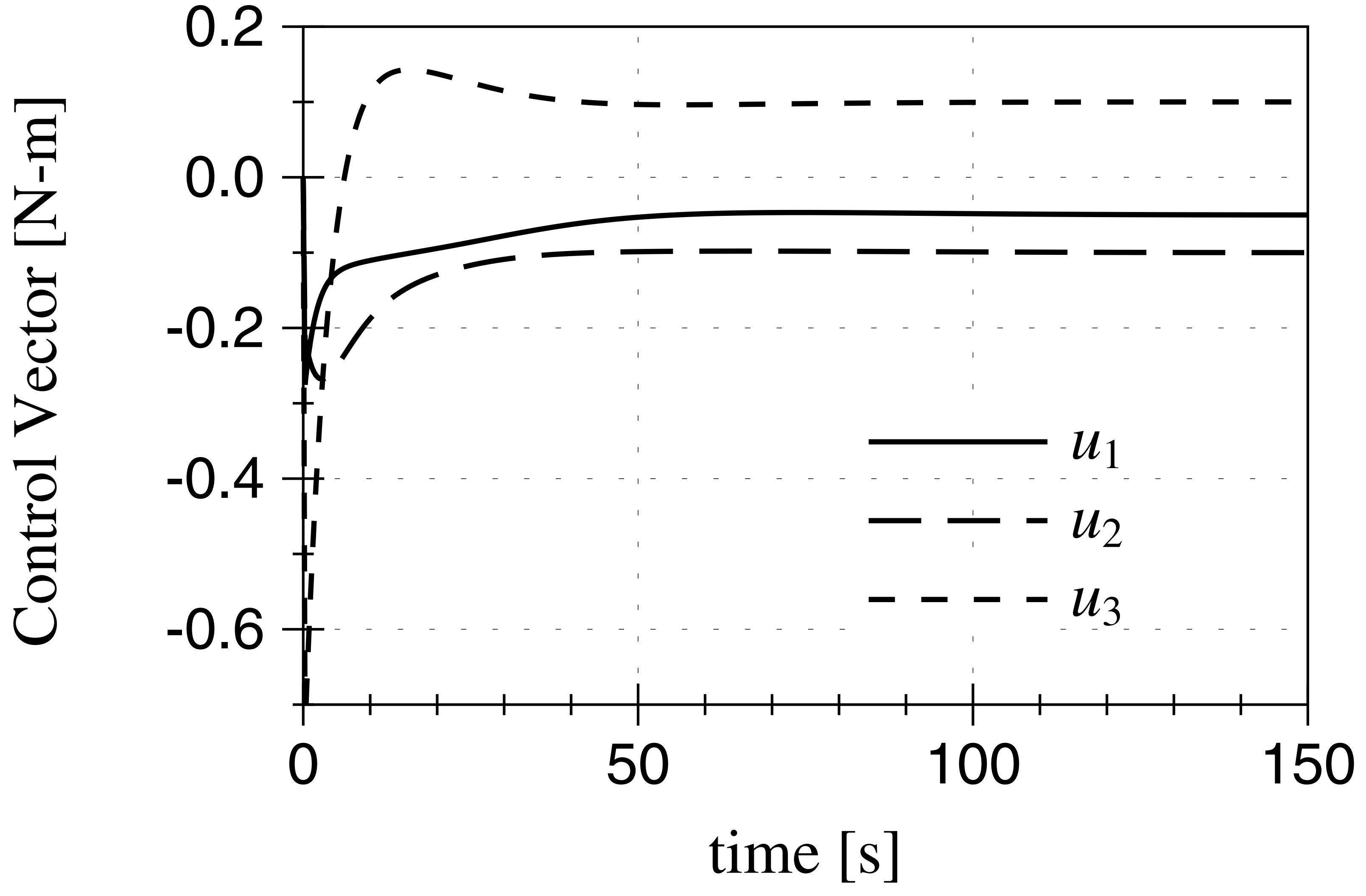
$$\lim_{t \rightarrow \infty} \boldsymbol{z} = \frac{1}{K_I I} \Delta \boldsymbol{L} = \begin{pmatrix} 1.66 \\ 3.33 \\ -3.33 \end{pmatrix} \text{kg-m}^2/\text{sec}$$





Feedback Gain Selection

- Lyapunov theory is great to develop globally stabilizing nonlinear feedback control law. However, how does one select the feedback gains to get good performance?

Consider the "PD-like" nonlinear feedback control law:

$$\boldsymbol{u} = -K\boldsymbol{\sigma} - [P]\delta\boldsymbol{\omega} + [I]\left(\dot{\boldsymbol{\omega}}_r - [\tilde{\boldsymbol{\omega}}]\boldsymbol{\omega}_r\right) + [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} - \boldsymbol{L}$$

Without external torque modeling errors, the closed-loop dynamics are written as:

$$[I]\delta\dot{\boldsymbol{\omega}} + [P]\delta\boldsymbol{\omega} + K\boldsymbol{\sigma} = 0$$

Linear in $\boldsymbol{\sigma}$ thanks to the use of MRP

Differential kinematic eqn: $$\dot{\boldsymbol{\sigma}} = \frac{1}{4}[B(\boldsymbol{\sigma})]\delta\boldsymbol{\omega}$$

Let's write the tracking error state vector $\boldsymbol{x}$ as:

$$\boldsymbol{x} = \begin{pmatrix} \boldsymbol{\sigma} \\ \delta\boldsymbol{\omega} \end{pmatrix}$$

Nonlinear state-space formulation:

$$\dot{\boldsymbol{x}} = \begin{pmatrix} \dot{\boldsymbol{\sigma}} \\ \delta\dot{\boldsymbol{\omega}} \end{pmatrix} = \begin{bmatrix} 0 & \boxed{\frac{1}{4}B(\boldsymbol{\sigma})} \\ -K[I]^{-1} & -[I]^{-1}[P] \end{bmatrix} \begin{pmatrix} \boldsymbol{\sigma} \\ \delta\boldsymbol{\omega} \end{pmatrix}$$

Approximate differential kinematic equations: $$\dot{\boldsymbol{\sigma}} \simeq \frac{1}{4}\delta\boldsymbol{\omega}$$

$$\begin{pmatrix} \dot{\boldsymbol{\sigma}} \\ \delta\dot{\boldsymbol{\omega}} \end{pmatrix} = \begin{bmatrix} 0 & \frac{1}{4}I \\ -K[I]^{-1} & -[I]^{-1}[P] \end{bmatrix} \begin{pmatrix} \boldsymbol{\sigma} \\ \delta\boldsymbol{\omega} \end{pmatrix}$$

Linear state-space form

If a principal body coordinate frame is chosen, then the inertia matrix is diagonal and the linearized tracking error simplify to:

$$\begin{pmatrix} \dot{\sigma}_i \\ \delta\dot{\omega}_i \end{pmatrix} = \begin{bmatrix} 0 & \frac{1}{4} \\ -\frac{K}{I_i} & -\frac{P_i}{I_i} \end{bmatrix} \begin{pmatrix} \sigma_i \\ \delta\omega_i \end{pmatrix} \qquad i = 1, 2, 3$$

3 uncoupled differential equations

The roots of the corresponding characteristic equation are expressed as

$$\lambda_i = -\frac{1}{2I_i}\left(P_i \pm \sqrt{-KI_i + {P_i}^2}\right) \qquad i = 1, 2, 3$$

The feedback gains P_i and K can now be chosen such that the system is either under-damped (complex roots), critically damped (double real root), or over-damped (two unique real roots).

Let's consider an under-damped response.

$$\omega_{n_i} = \frac{\sqrt{KI_i}}{2I_i}$$ natural frequency

$$\xi_i = \frac{P_i}{\sqrt{KI_i}}$$ damping ratio

$$T_i = \frac{2I_i}{P_i}$$ Time decay constant

$$\omega_{d_i} = \frac{1}{2I_i}\sqrt{KI_i - P_i^2}$$ damped natural frequency

Feedback Gain Selection Example:

Parameter	Value	Units
I_1	140.0	kg-m^2
I_2	100.0	kg-m^2
I_3	80.0	kg-m^2
$\boldsymbol{\sigma}(t_0)$	[0.60 – 0.40 0.20]	
$\boldsymbol{\omega}(t_0)$	[0.70 0.20 – 0.15]	rad/sec
$[P]$	[18.67 2.67 10.67]	kg-m^2/sec
K	7.11	kg-m^2/sec^2

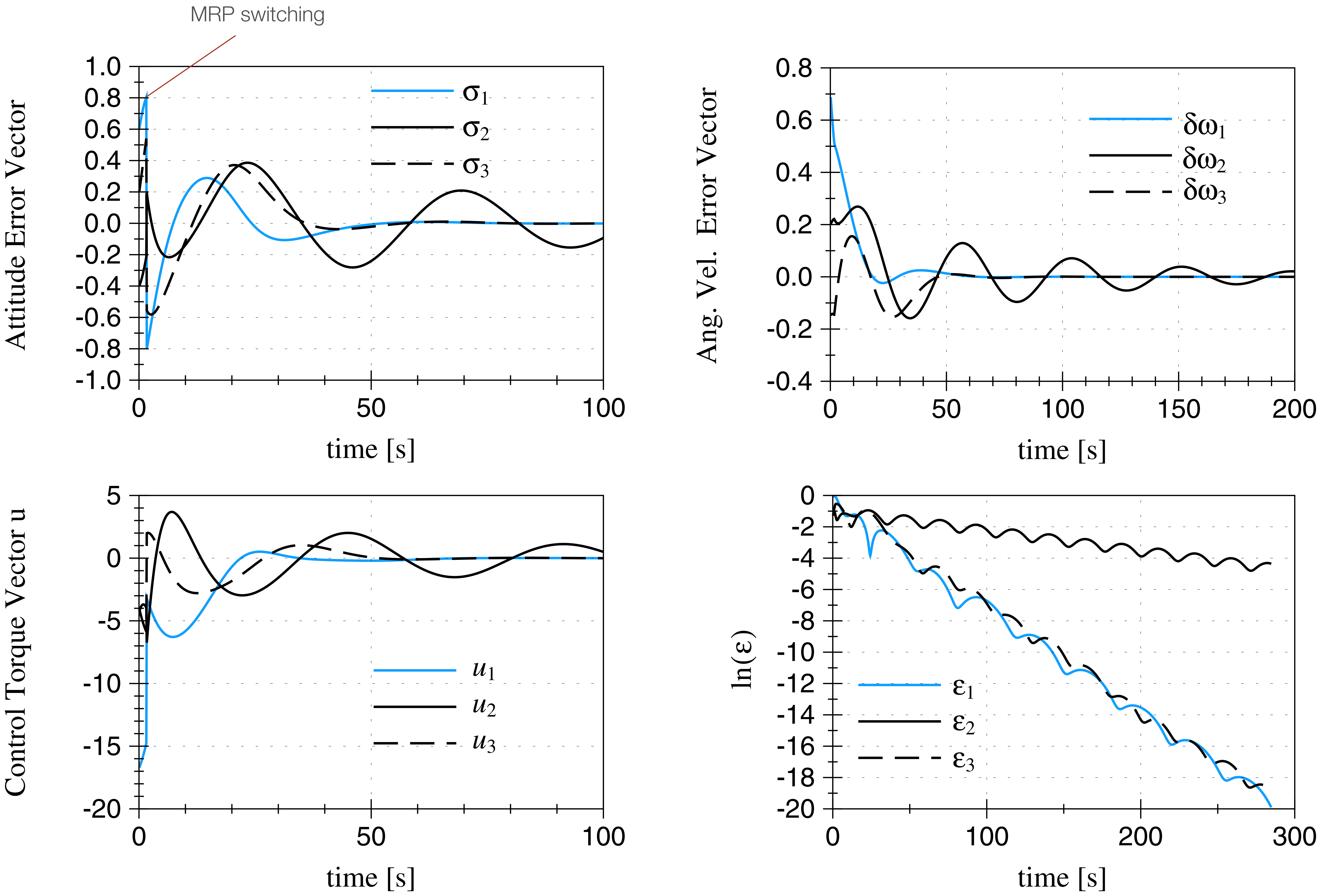
A principal body coordinate frame is chosen which diagonalizes the inertia matrix.

Large initial errors will cause the body to tumble "up-side-down".

Let us define the new state ε_i to track the 3 decoupled linearized tracking error dynamics.

$$\epsilon_i = \sqrt{\sigma_i^2 + \omega_i^2} \qquad i = 1, 2, 3$$

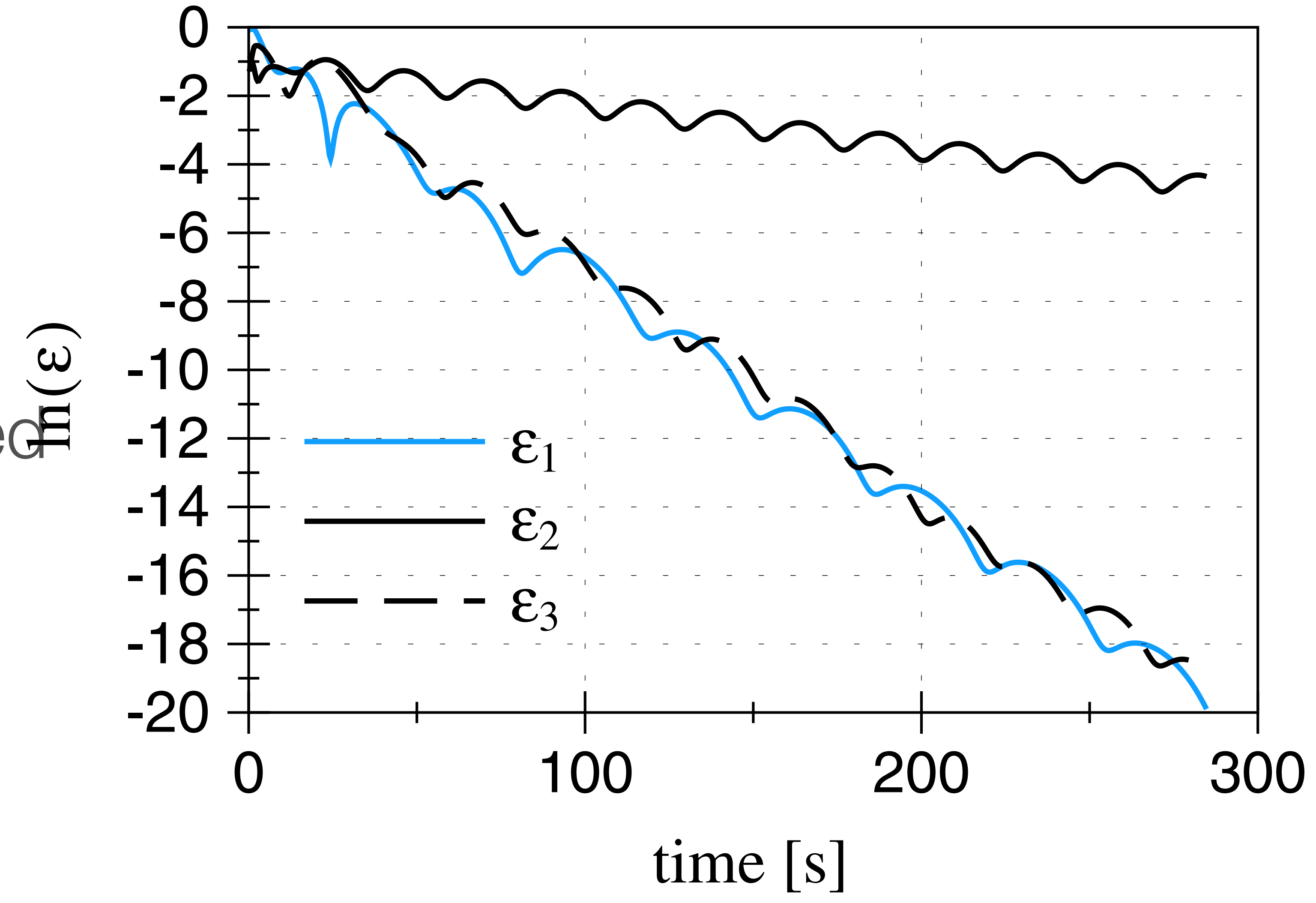
We can then evaluate this state in the nonlinear simulation, and compare to the predicted decay rate of the linearized analysis.


MRP switching
Attitude Error Vector
time [s]
σ_1
σ_2
σ_3
Ang. Vel. Error Vector
$\delta\omega_1$
$\delta\omega_2$
$\delta\omega_3$
Control Torque Vector u
u_1
u_2
u_3
ln(ε)
ε_1
ε_2
ε_3

Parameter	Actual Average	Predicted Value	Percent Difference
T_1	14.71 s	15.00 s	1.97%
T_2	76.92 s	75.00 s	-2.50%
T_3	14.71 s	15.00 s	1.97%
ω_{d_1}	0.0938 rad/s	0.0909 rad/s	-3.12%
ω_{d_2}	0.1326 rad/s	0.1326 rad/s	0.08%
ω_{d_3}	0.1343 rad/s	0.1333 rad/s	-0.74%

This table compares the actual, nonlinear response to that of the linearized prediction of the gain selection method.

Because the MRP behave very linearly, the predicted tracking error dynamics matches the nonlinear motion very well, even though the is tumbling up-side-down!



Lyapunov Optimal Feedback

What if your thrusters are just too wimpy?

Stabilization of General System

Generalized Coordinates: $(q_i, \dot{q}_i)$

Goal: $\dot{q}_i \to 0$

EOM:

$$[M(\boldsymbol{q})]\ddot{\boldsymbol{q}} = -[\dot{M}(\boldsymbol{q}, \dot{\boldsymbol{q}})]\dot{\boldsymbol{q}} + \frac{1}{2}\dot{\boldsymbol{q}}^T[M_{\boldsymbol{q}}(\boldsymbol{q})]\dot{\boldsymbol{q}} + \boldsymbol{Q}$$

If we use the kinetic energy T as the Lyapunov function of this system, then we find:

$$V(\dot{\boldsymbol{q}}) = T = \frac{1}{2}\dot{\boldsymbol{q}}^T[M(\boldsymbol{q})]\dot{\boldsymbol{q}}$$

Using the work/energy relationship, we can write

$$\dot{V} = \sum_{i=1}^{N} \dot{q}_i Q_i$$

Setting the control equal to $Q_i = -K_i \dot{q}_i$

yields $$\dot{V} = \sum_{i=1}^{N} -K_i \dot{q}_i^2 < 0$$

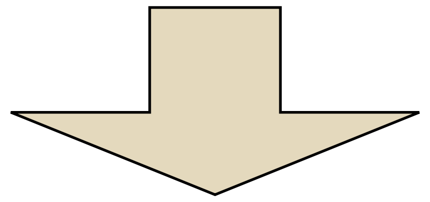
Next, what if the control authority magnitude is limited?

1st approach:

We could reduce our feedback gains K_i such that

$$|Q_i| = |K_i \dot{q}_i| \leq Q_{i_{\max}}$$

is true for all state errors $\dot{q}_i$ considered.



reduced control performance!

2nd approach:

We would like to be able to handle control saturation without sacrificing performance (as much) throughout the maneuver!

Let us treat the saturated control problem as an optimization problem. For stability, we require the Lyapunov rate to be negative semi-definite:

$$\dot{V} \leq 0$$

Thus, we define a cost function which is equal to this Lyapunov rate, and aim to minimize it!

$$\boxed{J = \dot{V} = \sum_{i=1}^{n} \dot{q}_i Q_i}$$

We define a ***Lyapunov optimal control law*** to be one that minimizes the Lyapunov rate function.

Given a limited amount of control authority, the Lyapunov optimal rate control is simply

$$Q_i = -Q_{i_{\max}} \operatorname{sgn}(\dot{q}_i)$$

which yields

$$J = \dot{V} = \sum_{i=1}^{N} -Q_{i_{\max}} \dot{q}_i \operatorname{sgn}(\dot{q}_i)$$

Note that this direct implementation will have chatter issues around the target state values.

The following control is only Lyapunov optimal during saturation periods, but avoids the zero crossing chatter.

$$Q_i = \begin{cases} -K_i \dot{q}_i & \text{for } |K_i \dot{q}_i| \leq Q_{i_{\max}} \\ -Q_{i_{\max}} \operatorname{sgn}(\dot{q}_i) & \text{for } |K_i \dot{q}_i| > Q_{i_{\max}} \end{cases}$$

Note that individual control components can be saturated, while others are not!

Saturated Attitude Control

- Next we study attitude control laws when the external control torque is saturated in one or more of its components.

Case 1: Attitude Tracking Problem

Un-saturated control found previously:

$$\boldsymbol{u}_{\text{us}} = -K\boldsymbol{\sigma} - [P]\delta\boldsymbol{\omega} + [I]\left(\dot{\boldsymbol{\omega}}_r - [\tilde{\omega}]\boldsymbol{\omega}_r\right) + [\tilde{\omega}][I]\boldsymbol{\omega} - \boldsymbol{L}$$

Corresponding Lyapunov rate:

$$\dot{V} = \delta\boldsymbol{\omega}^T\left(-[\tilde{\omega}][I]\boldsymbol{\omega} + \boldsymbol{u} - [I]\left(\dot{\boldsymbol{\omega}}_r - [\tilde{\omega}]\boldsymbol{\omega}_r\right) + K\boldsymbol{\sigma} + \boldsymbol{L}\right)$$

Lyapunov optimal saturated control strategy:

$$u_i = \begin{cases} u_{\text{us}_i} & \text{for } |u_{\text{us}_i}| \leq u_{\text{max}_i} \\ u_{\text{us}_i} \cdot \text{sgn}(u_{\text{us}_i}) & \text{for } |u_{\text{us}_i}| > u_{\text{max}_i} \end{cases}$$

Conservative stability boundary (sufficient condition):

$$\left|\left([I]\left(\dot{\boldsymbol{\omega}}_r - [\tilde{\omega}]\boldsymbol{\omega}_r\right) + [\tilde{\omega}][I]\boldsymbol{\omega} - K\boldsymbol{\sigma} - \boldsymbol{L}\right)_i\right| \leq u_{\text{max}_i}$$

If this is violated, we don't necessarily have instability!

Case 2: Attitude Regulator Problem

In this case the unsaturated control torque on the previous slide simplifies to:

$$\boldsymbol{u}_{\text{us}} = -K\boldsymbol{\sigma} - [P]\boldsymbol{\omega}$$

while the Lyapunov rate expression reduces to:

$$\dot{V} = \boldsymbol{\omega}^T \left(\boldsymbol{u} + K\boldsymbol{\sigma}\right)$$

A conservative stability boundary which guarantees that $\dot{V} \leq 0$ is

$$K|\sigma_i| \leq u_{\text{max}_i}$$

However, note that the MRP attitude error are typically bounded by switching between the original and shadow sets!

$$|\boldsymbol{\sigma}| \leq 1$$

$$K \leq u_{\text{max}_i}$$

Case 3: Rate Regulator Problem

A common situation just requires the current spacecraft spin to be arrested. Essentially, the final attitude is irrelevant and we set $K = 0$.

The Lyapunov optimal saturated control strategy in this case reduces to:

$$u_i = \begin{cases} -P_{ii}\omega_i & \text{for } |P_{ii}\omega_i| \leq u_{\text{max}_i} \\ -u_{\text{max}_i} \cdot \text{sgn}(\omega_i) & \text{for } |P_{ii}\omega_i| > u_{\text{max}_i} \end{cases}$$

The corresponding Lyapunov rate function is

$$\dot{V}(\boldsymbol{\omega}) = -\sum_{i=1}^{M} P_{ii}\omega_i^2 - \sum_{i=M+1}^{N} \omega_i u_{\text{maxi}} \cdot \text{sgn}(\omega_i) < 0$$

Note, this function is *negative definite*, and thus globally asymptotically stabilizing!

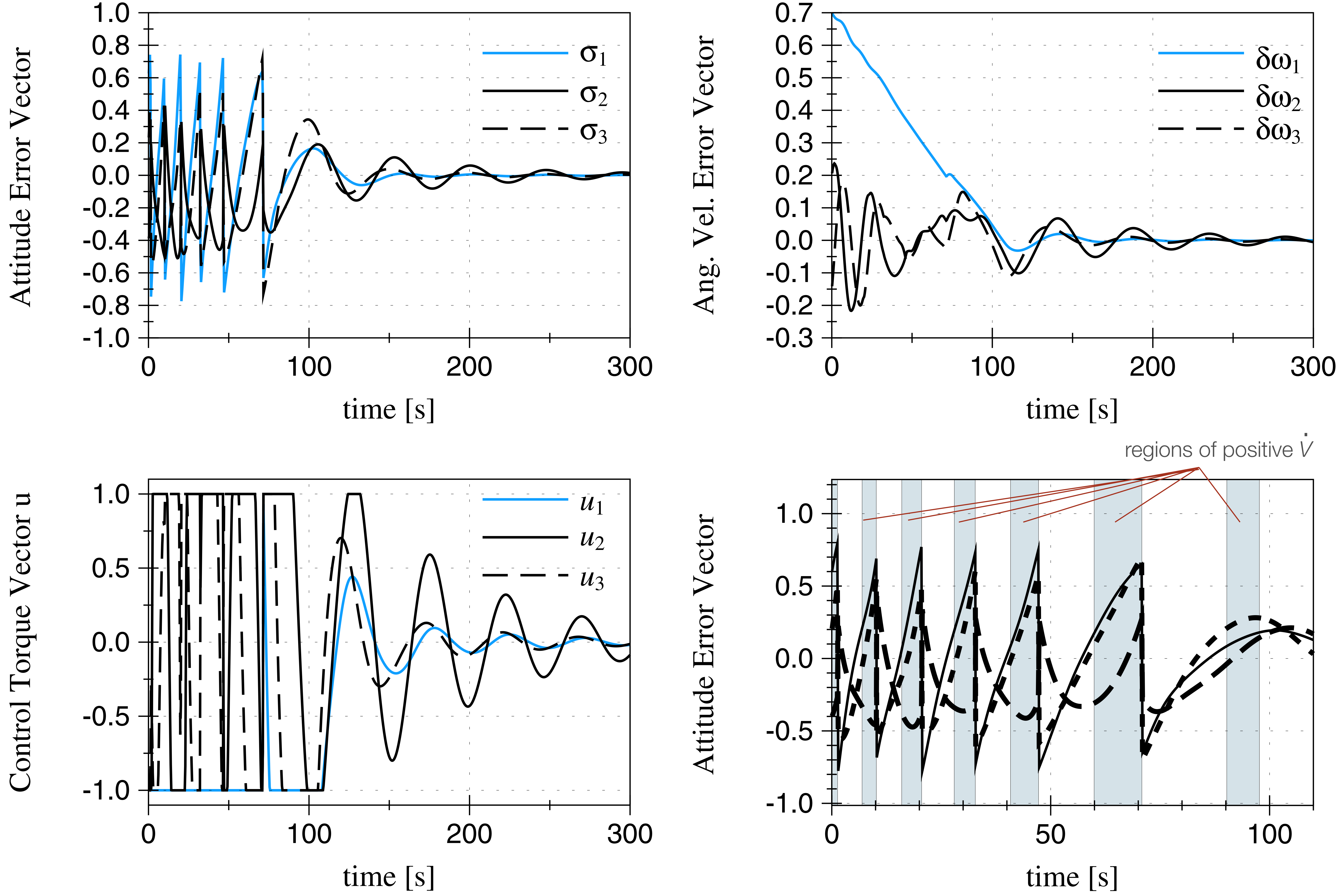
Further, because no inertia terms are used in this saturated control, it is very robust to modeling errors.

Saturated Attitude Control Example:

Parameter	Value	Units
I_1	140.0	kg-m^2
I_2	100.0	kg-m^2
I_3	80.0	kg-m^2
$\boldsymbol{\sigma}(t_0)$	$[0.60\ -0.40\ 0.20]$	
$\boldsymbol{\omega}(t_0)$	$[0.70\ 0.20\ -0.15]$	rad/sec
$[P]$	$[18.67\ 2.67\ 10.67]$	kg-m^2/sec
K	7.11	kg-m^2/sec^2

Torque saturation level: $u_{\max_i} = 1 \text{ N m}$

Unsaturated control law: $\boldsymbol{u} = -K\boldsymbol{\sigma} - [P]\boldsymbol{\omega}$


Attitude Error Vector
σ_1
σ_2
σ_3
time [s]
Ang. Vel. Error Vector
$\delta\omega_1$
$\delta\omega_2$
$\delta\omega_3$
time [s]
Control Torque Vector u
u_1
u_2
u_3
time [s]
regions of positive $\dot{V}$
Attitude Error Vector
time [s]

Linear Closed-Loop Dynamics

Surprisingly elegant inverse-kinematic solutions...

The idea...

- An extensive body of literature exists on the behavior of ***linear*** closed-loop dynamics (CLD) of the form:

$$\boxed{}\,\ddot{\boldsymbol{\epsilon}} + P\dot{\boldsymbol{\epsilon}} + K\boldsymbol{\epsilon} = 0$$

PD-feedback control

$$\boxed{}\,\ddot{\boldsymbol{\epsilon}} + P\dot{\boldsymbol{\epsilon}} + K\boldsymbol{\epsilon} + K_i \int_0^t \boldsymbol{\epsilon}\mathrm{d}t = 0$$

PID-feedback control

Note: the proposed CLD doesn't require any knowledge of the system mass or inertia properties compared to

$$[I]\delta\dot{\boldsymbol{\omega}} + [P]\delta\boldsymbol{\omega} + K\boldsymbol{\sigma} = 0$$

$$[\dot{B}]^{-1}\dot{\boldsymbol{\sigma}} + [B]^{-1}\ddot{\boldsymbol{\sigma}} \qquad [B]^{-1}\dot{\boldsymbol{\sigma}}$$

With the complicated differential kinematic equations, what a mess this will be!

...or will it?

Setup

- Let us solve for the linear CLD using Euler parameters (β_1,β_2,β_3).*

Desired CLD:

$$\boxed{\ddot{\boldsymbol{\epsilon}} + P\dot{\boldsymbol{\epsilon}} + K\boldsymbol{\epsilon} = 0}$$

Find $\boldsymbol{u}$ to achieve CLD

EOM:

$$[I]\dot{\boldsymbol{\omega}} + [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} = \boxed{\boldsymbol{u}}$$

Attitude State Vector:

$$\boldsymbol{\epsilon} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \sin\left(\frac{\Phi}{2}\right)\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

Differential Kinematic Equations:

$$\dot{\boldsymbol{\epsilon}} = \frac{1}{2}[T]\boldsymbol{\omega}$$

$$[T(\beta_0, \boldsymbol{\epsilon})] = \begin{bmatrix} \beta_0 & -\beta_3 & \beta_2 \\ \beta_3 & \beta_0 & -\beta_1 \\ -\beta_2 & \beta_1 & \beta_0 \end{bmatrix} = \beta_0[I_{3\times3}] + [\tilde{\boldsymbol{\epsilon}}]$$

*Paielli, R. A. and Bach, R. E., “Attitude Control with Realization of Linear Error Dynamics,” Journal of Guidance, Control and Dynamics, Vol. 16, No. 1, Jan.–Feb. 1993, pp. 182–189.

$$\boxed{\dot{\boldsymbol{\epsilon}} = \frac{1}{2}[T]\boldsymbol{\omega}}$$

$$\Big\downarrow \frac{\mathrm{d}}{\mathrm{d}t}$$

$$\ddot{\boldsymbol{\epsilon}} = \frac{1}{2}[T]\dot{\boldsymbol{\omega}} + \frac{1}{2}\boxed{[\dot{T}]\boldsymbol{\omega}}$$

$$[T]\boldsymbol{\omega} = \beta_0\boldsymbol{\omega} + [\tilde{\boldsymbol{\epsilon}}]\boldsymbol{\omega}$$

$$\Big\downarrow \frac{\mathrm{d}}{\mathrm{d}t}$$

$$\boxed{[\dot{T}]\boldsymbol{\omega}} = \boxed{\dot{\beta}_0}\boldsymbol{\omega} - [\tilde{\boldsymbol{\omega}}]\boxed{\dot{\boldsymbol{\epsilon}}}$$

$$\boxed{\dot{\beta}_0 = -\frac{1}{2}\boldsymbol{\epsilon}^T\boldsymbol{\omega}}$$

Note:

$$[\tilde{\boldsymbol{\omega}}][T]\boldsymbol{\omega} = [\tilde{\boldsymbol{\omega}}]\left(\beta_0[I_{3\times 3}] + [\tilde{\boldsymbol{\epsilon}}]\right)\boldsymbol{\omega}$$

$$= \cancel{[\tilde{\boldsymbol{\omega}}]\boldsymbol{\omega}}\beta_0 + [\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{\epsilon}}]\boldsymbol{\omega}$$

$$= -[\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{\omega}}]\boldsymbol{\epsilon}$$

$$[\tilde{\boldsymbol{a}}][\tilde{\boldsymbol{a}}] = \boldsymbol{a}\boldsymbol{a}^T - \boldsymbol{a}^T\boldsymbol{a}I_{3\times 3}$$

$$[\tilde{\boldsymbol{\omega}}][T]\boldsymbol{\omega} = -\left(\boldsymbol{\omega}\boldsymbol{\omega}^T - \omega^2[I_{3\times 3}]\right)\boldsymbol{\epsilon}$$

$$= \boxed{-\boldsymbol{\epsilon}^T\boldsymbol{\omega}\boldsymbol{\omega} + \omega^2\boldsymbol{\epsilon}}$$

$$\ddot{\boldsymbol{\epsilon}} = \frac{1}{2}[T]\dot{\boldsymbol{\omega}} - \frac{1}{4}\left(\boldsymbol{\epsilon}^T\boldsymbol{\omega}\boldsymbol{\omega} + \boxed{[\tilde{\boldsymbol{\omega}}][T]\boldsymbol{\omega}}\right) \Rightarrow \ddot{\boldsymbol{\epsilon}} = \frac{1}{2}[T]\dot{\boldsymbol{\omega}} - \frac{1}{4}\omega^2\boldsymbol{\epsilon}$$

$$\ddot{\boldsymbol{\epsilon}} = \frac{1}{2}[T]\dot{\boldsymbol{\omega}} - \frac{1}{4}\omega^2\boldsymbol{\epsilon} \qquad \dot{\boldsymbol{\epsilon}} = \frac{1}{2}[T]\boldsymbol{\omega}$$

$$\ddot{\boldsymbol{\epsilon}} + P\dot{\boldsymbol{\epsilon}} + K\boldsymbol{\epsilon} = 0$$

$$[T]\left(\boxed{\dot{\boldsymbol{\omega}} + P\boldsymbol{\omega} + [T]^{-1}\left(-\frac{1}{2}\omega^2\boldsymbol{\epsilon} + 2K\boldsymbol{\epsilon}\right)}\right) = 0$$

must be zero

Can [*T*] be inverted?

$$[T]^{-1} = [T]^T + \frac{1}{\beta_0}\boldsymbol{\epsilon}\boldsymbol{\epsilon}^T$$

Always possible
except 180º case.

$$[T] = \beta_0 [I_{3\times3}] + [\tilde{\boldsymbol{\epsilon}}]$$

$$[T]^{-1} = [T]^T + \frac{1}{\beta_0}\boldsymbol{\epsilon}\boldsymbol{\epsilon}^T$$

$$\dot{\boldsymbol{\omega}} + P\boldsymbol{\omega} + [T]^{-1}\left(-\tfrac{1}{2}\omega^2\boldsymbol{\epsilon} + 2K\boldsymbol{\epsilon}\right) = 0$$

Bunch of algebra...

$$\boxed{\dot{\boldsymbol{\omega}} = -P\boldsymbol{\omega} - 2\left(K - \frac{\omega^2}{4}\right)\frac{\boldsymbol{\epsilon}}{\beta_0}}$$

Remarkably simply required angular acceleration expression!

$$[I]\dot{\boldsymbol{\omega}} + [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} = \boldsymbol{u}$$

$$\boxed{\boldsymbol{u} = [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} + [I]\left(-P\boldsymbol{\omega} - 2\left(K - \frac{\omega^2}{4}\right)\frac{\boldsymbol{\epsilon}}{\beta_0}\right)}$$

Not that even though we started out with the non-singular Euler parameters, the attitude feedback terms is the Gibbs vector which is singular for 180º rotations!

Comparison to Gibbs Feedback

Paielli/Back feedback control law:

$$\boldsymbol{u} = [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} + [I]\left(-P\boldsymbol{\omega} - 2\left(K - \frac{\omega^2}{4}\right)\frac{\boldsymbol{\epsilon}}{\beta_0}\right)$$

Gibbs-vector Feedback

$$\boldsymbol{u} = [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} - P\boldsymbol{\omega} - K\boldsymbol{q}$$

Integral Feedback

- By starting out with a different desired linear CLD, we can also include an integral feedback term:

$$\ddot{\boldsymbol{\epsilon}} + P\dot{\boldsymbol{\epsilon}} + K\boldsymbol{\epsilon} + K_i \int_0^t \boldsymbol{\epsilon} \mathrm{d}t = 0$$

Bunch of more algebra following earlier steps...

$$\dot{\boldsymbol{\omega}} = -P\boldsymbol{\omega} - 2\left(K - \frac{\omega^2}{4}\right)\frac{\boldsymbol{\epsilon}}{\beta_0} - 2K_i\left([T]^T + \frac{1}{\beta_0}\boldsymbol{\epsilon}\boldsymbol{\epsilon}^T\right)\int_0^t \boldsymbol{\epsilon}\mathrm{d}t$$

$$[I]\dot{\boldsymbol{\omega}} + [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} = \boldsymbol{u}$$

Tracking Problem

- This linear CLD behavior can also be achieved for an attitude tracking problem with a time-varying reference attitude *R*.

Kinematic expressions:

$$\dot{\boldsymbol{\epsilon}} = \frac{1}{2}[T]\delta\boldsymbol{\omega}$$

$$\delta\boldsymbol{\omega} = \boldsymbol{\omega} - \boldsymbol{\omega}_r$$

Even more algebra of the same sort...

$$\dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_r - P\delta\boldsymbol{\omega} - 2\left(K - \frac{\delta\omega^2}{4}\right)\frac{\boldsymbol{\epsilon}}{\beta_0}$$

$$[I]\dot{\boldsymbol{\omega}} + [\tilde{\omega}][I]\boldsymbol{\omega} = \boldsymbol{u}$$

Linear MRP CLD

- The previous linear CLD development only yielded elegant results because of some special properties of Euler parameter kinematic equations. However, the resulting feedback was singular at 180º.

$$[T]^{-1} = [T]^T + \frac{1}{\beta_0}\boldsymbol{\epsilon}\boldsymbol{\epsilon}^T$$

Important property for simple linear EP CLD.

MRP Kinematic Equations:

$$\dot{\boldsymbol{\sigma}} = \frac{1}{4}[B(\boldsymbol{\sigma})]\boldsymbol{\omega}$$

$$[B]^{-1} = \frac{1}{(1+\sigma^2)^2}[B]^T$$

elegant near-orthogonal inverse property.

Desired Linear MRP CLD:

$$\ddot{\boldsymbol{\sigma}} + P\dot{\boldsymbol{\sigma}} + K\boldsymbol{\sigma} = 0$$

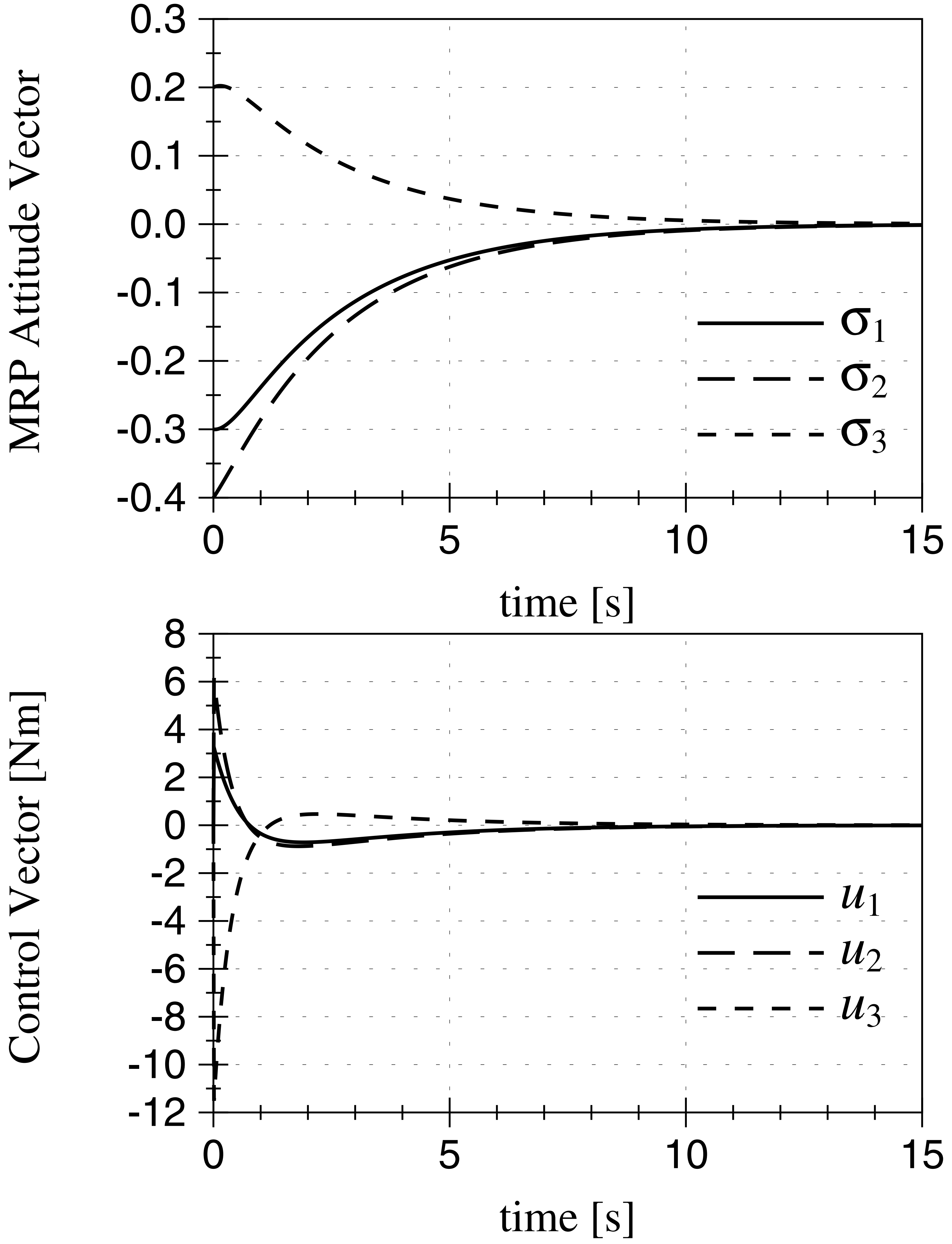
MRP Feedback Control

$$\dot{\boldsymbol{\omega}} = -P\boldsymbol{\omega} - \left(\boldsymbol{\omega}\boldsymbol{\omega}^T + \left(\frac{4K}{1+\sigma^2} - \frac{\omega^2}{2}\right)[I_{3\times 3}]\right)\boldsymbol{\sigma}$$

This feedback is non-singular at 180º and globally valid by switching to the shadow set!

Linear MRP CLD Example:

Parameter	Value	Units
I_1	30.0	kg-m^2
I_2	20.0	kg-m^2
I_3	10.0	kg-m^2
$\boldsymbol{\sigma}(t_0)$	$[-0.30\ -0.40\ 0.20]$	
$\boldsymbol{\omega}(t_0)$	$[0.20\ 0.20\ 0.20]$	rad/sec
$[P]$	3.0	kg-m^2/sec
K	1.0	kg-m^2/sec^2



Goal: Regulator problem which arrests satellite motion at the zero attitude.