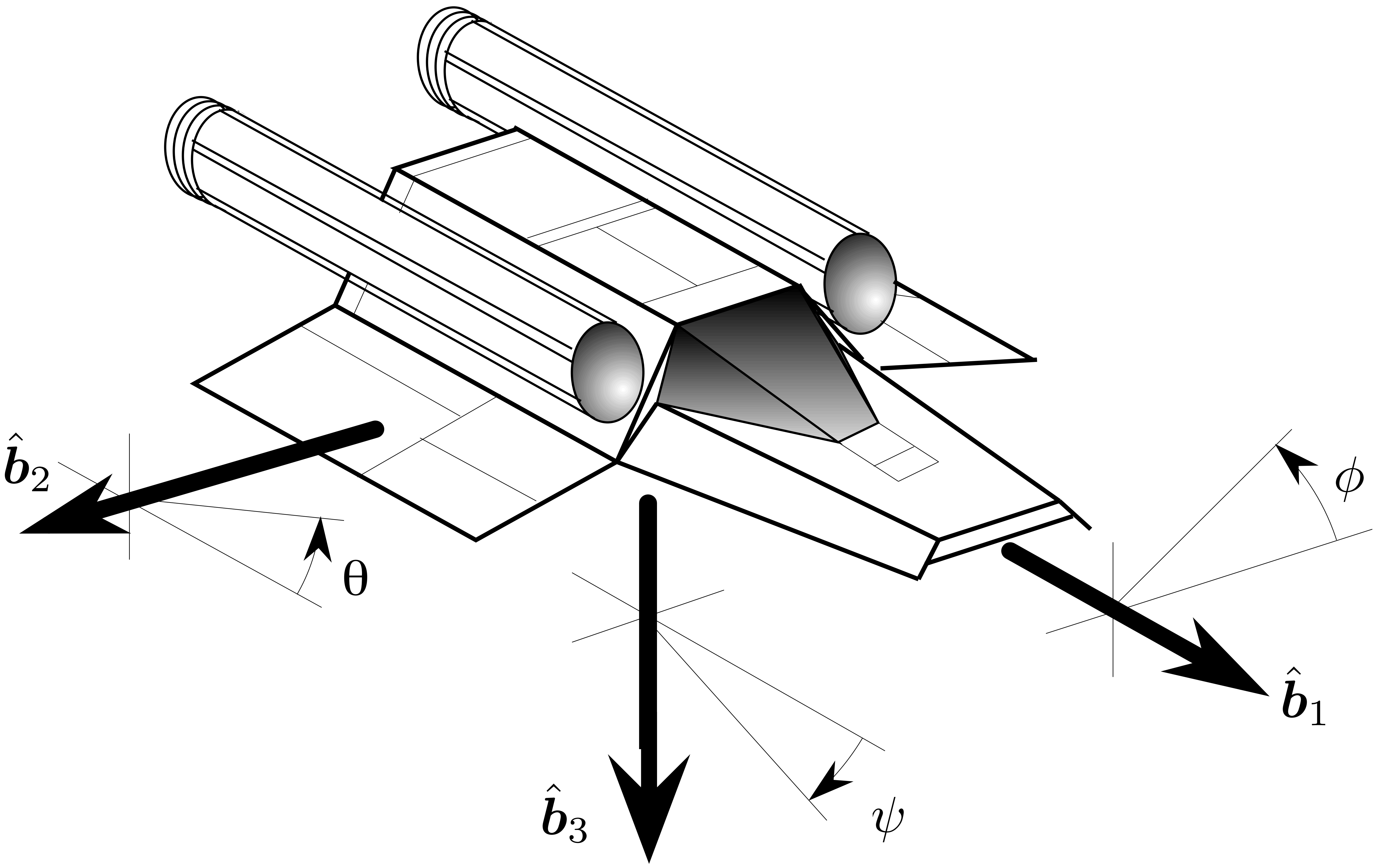
Euler Angles

The 101 of attitude coordinates...

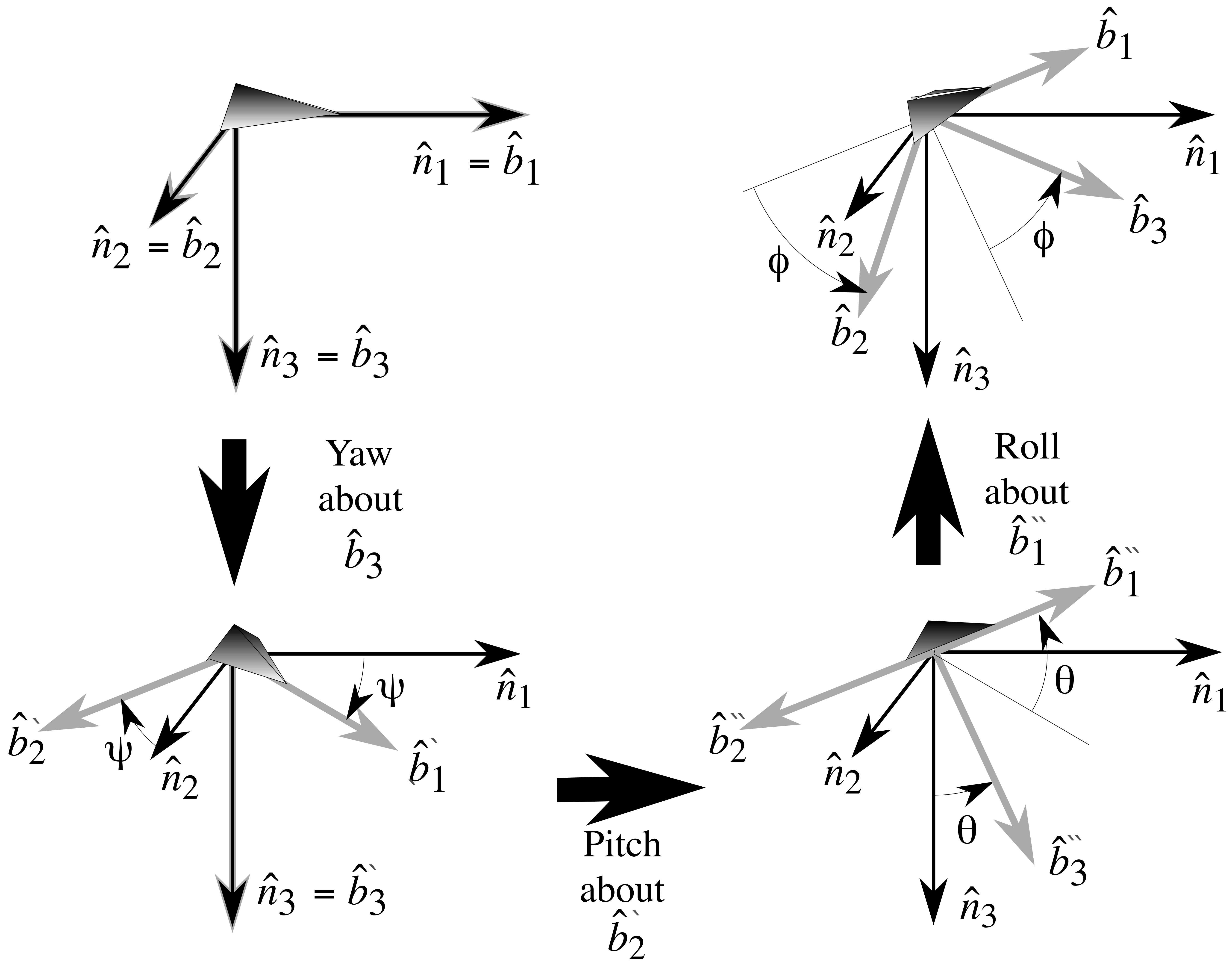
Description

- Most common set of attitude coordinates
- Describe the orientation between two frames using three *sequential* rotations
- Note that the order of rotation is important
- (i-j-k) Euler angles means we rotate first about the i^{th} axis, then about the j^{th} axis, and lastly about the k^{th} axis
- (3-2-1) Euler angles are the typical aircraft and spacecraft attitude angles
- Simple to visualize for small rotations

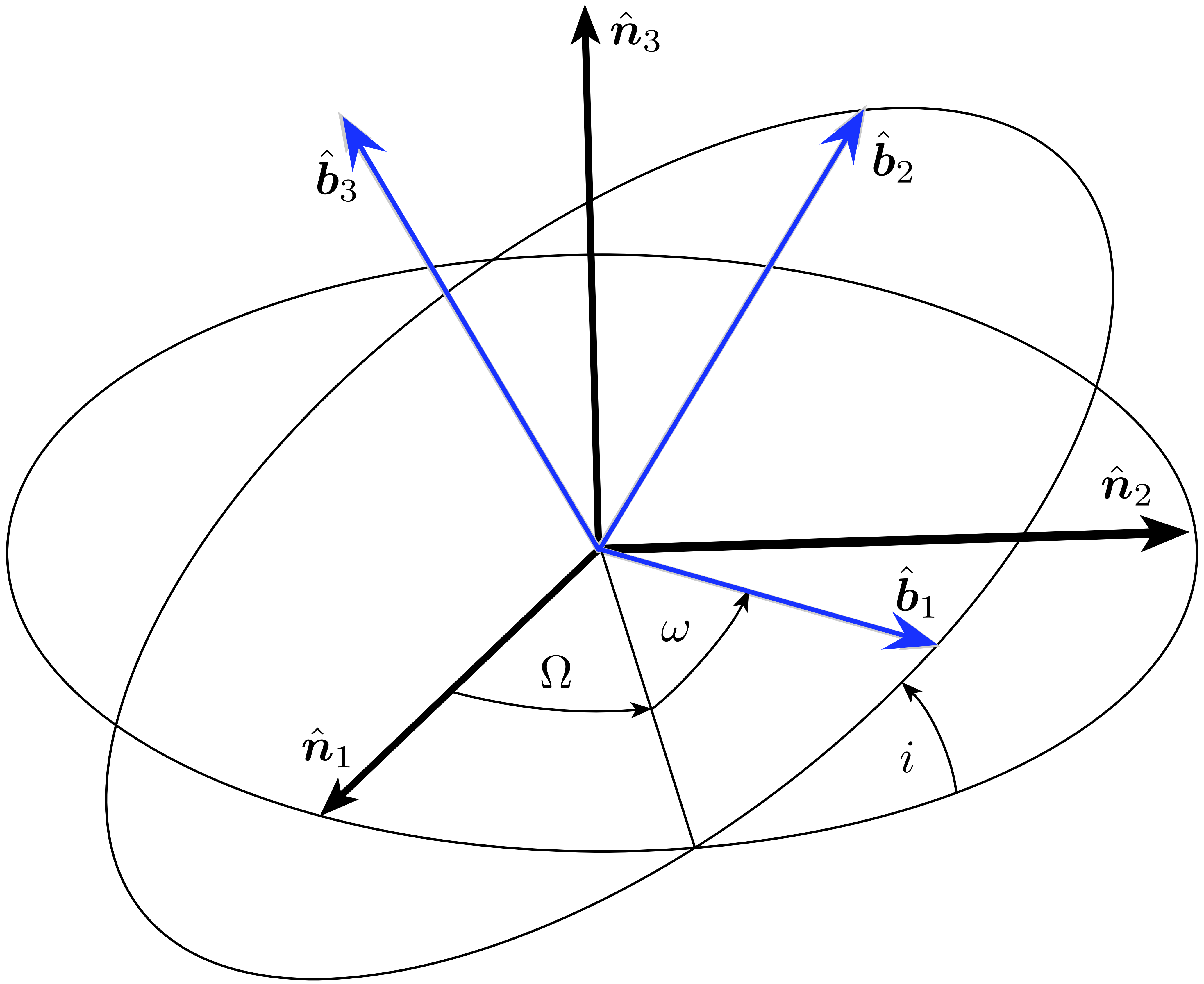
Aircraft/Spacecraft Orientation Angles



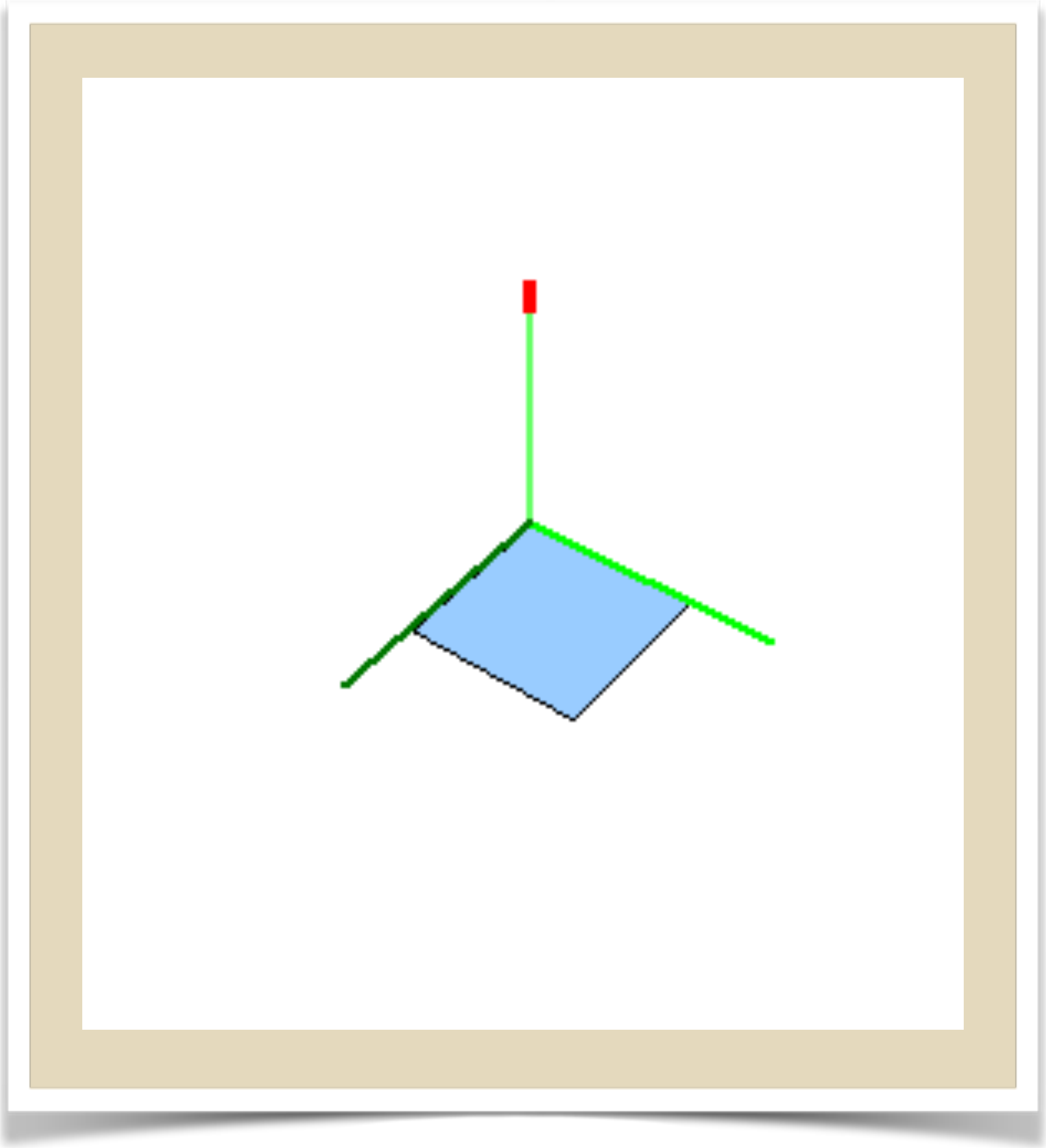
(3-2-1) Euler Angle Illustration



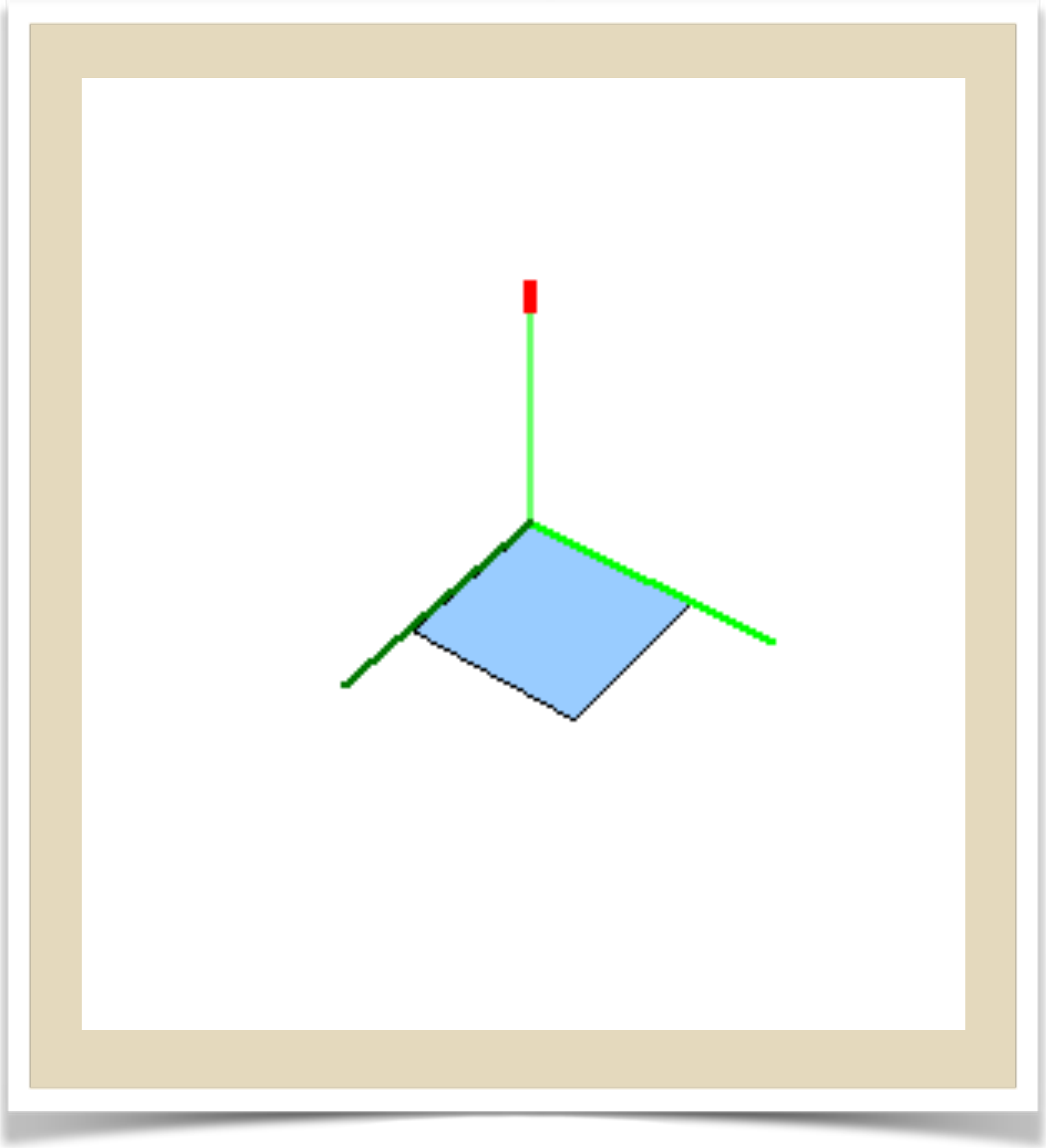
(3-1-3) Euler Angles



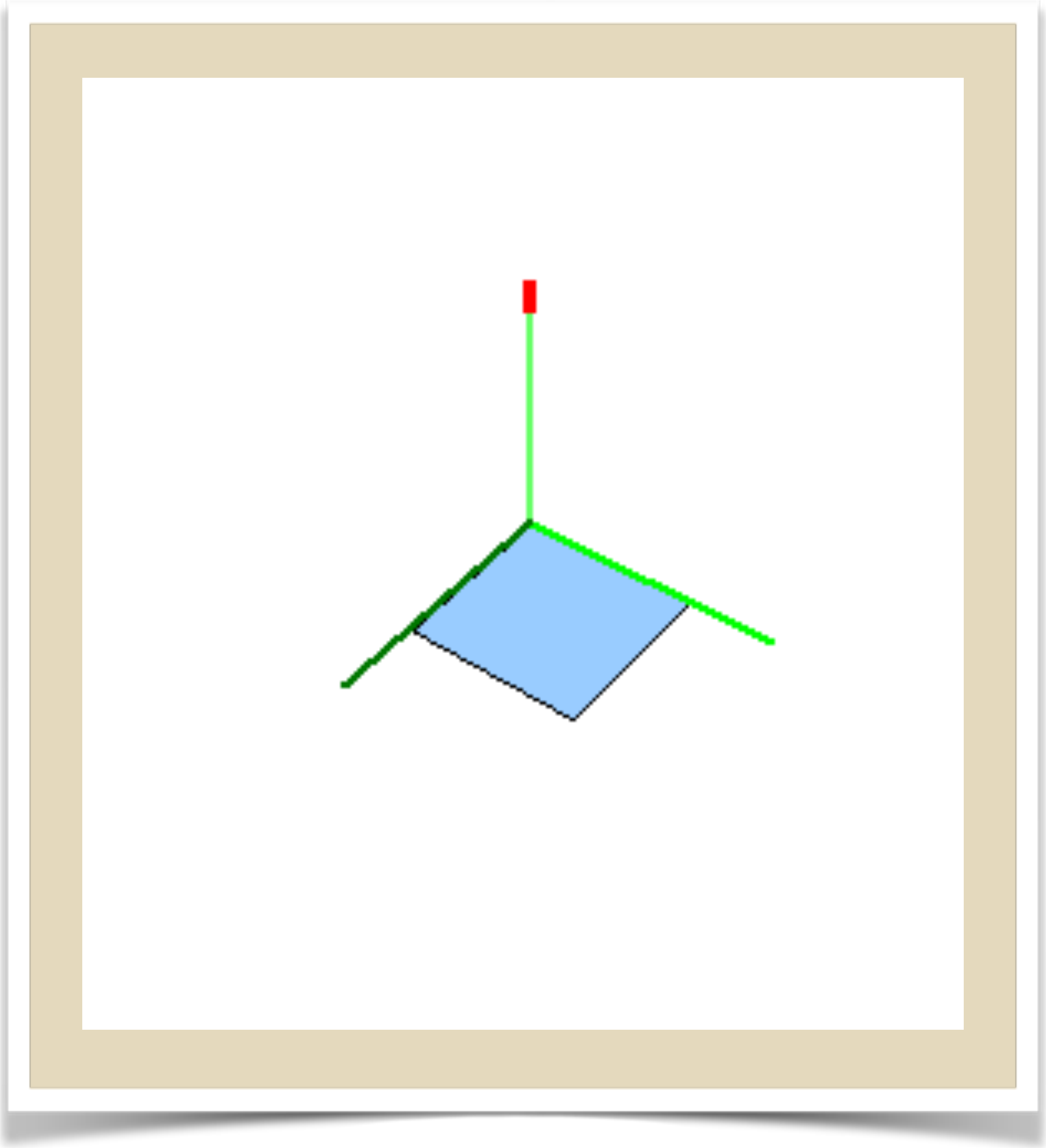
Commonly used to describe the orbit frame orientation relative to the inertial Frame *N*.



(3-2-1) Euler Angles
(60,50,70) Degrees



(3-1-3) Euler Angles
(60,50,70) Degrees



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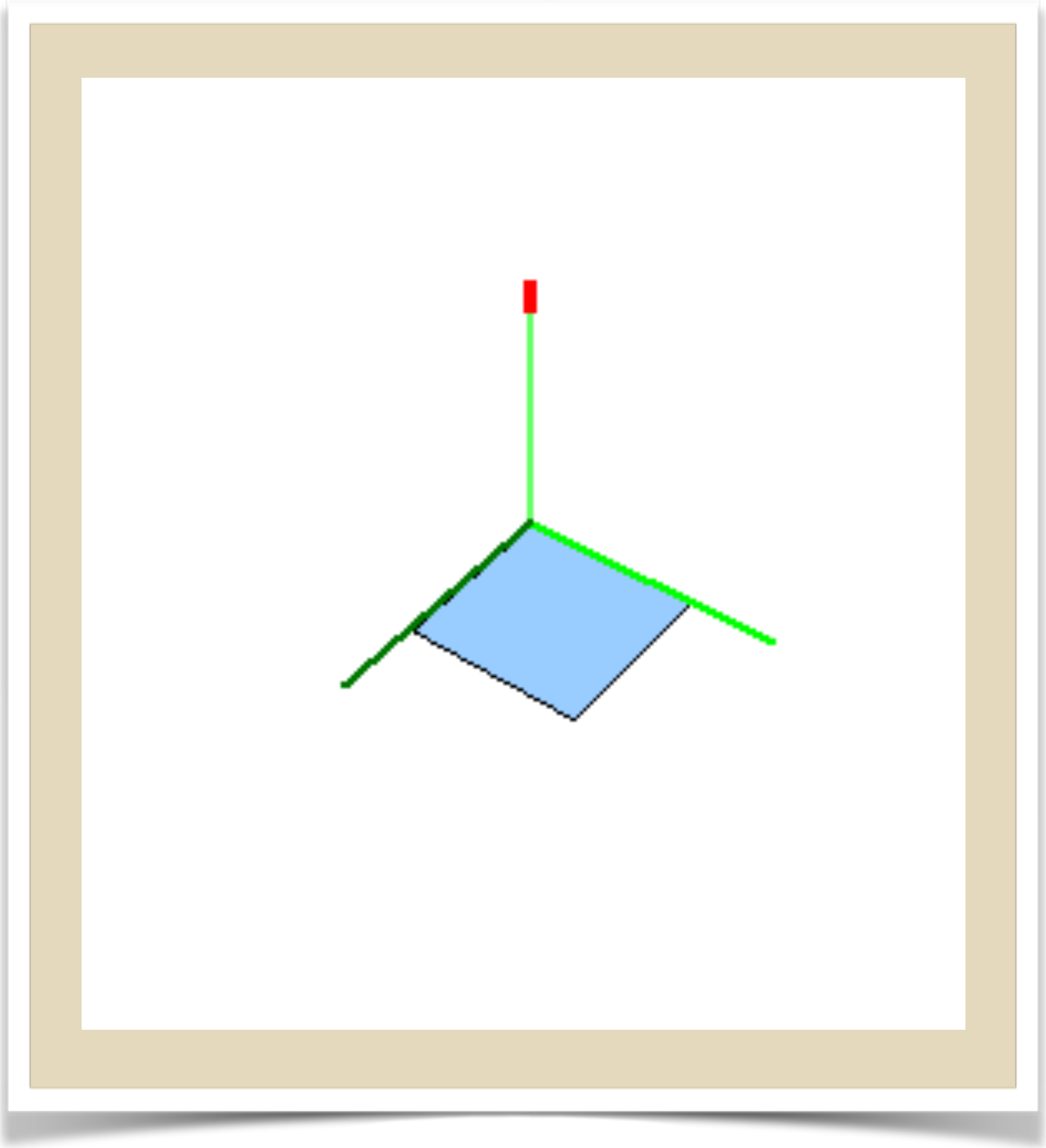
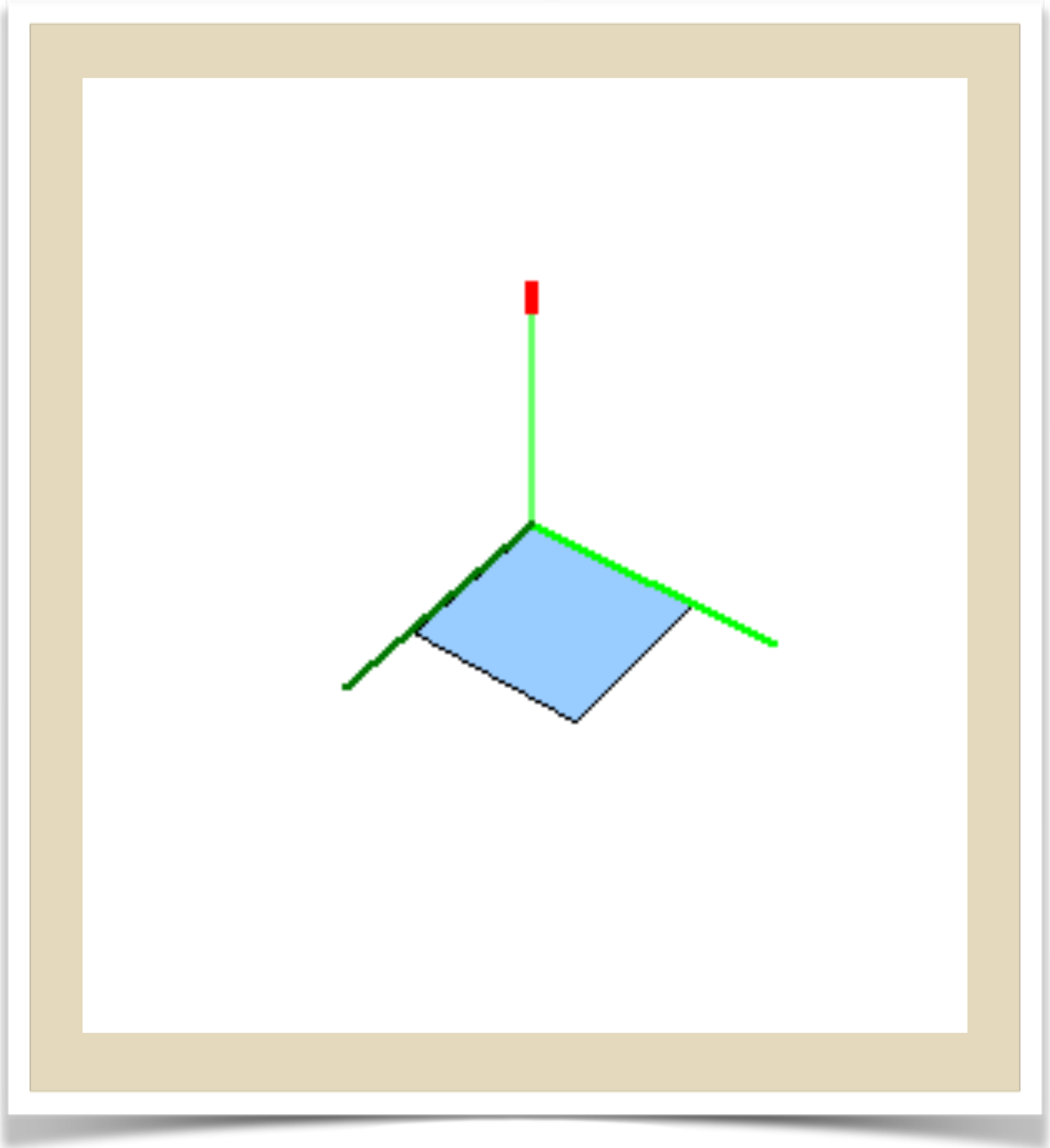
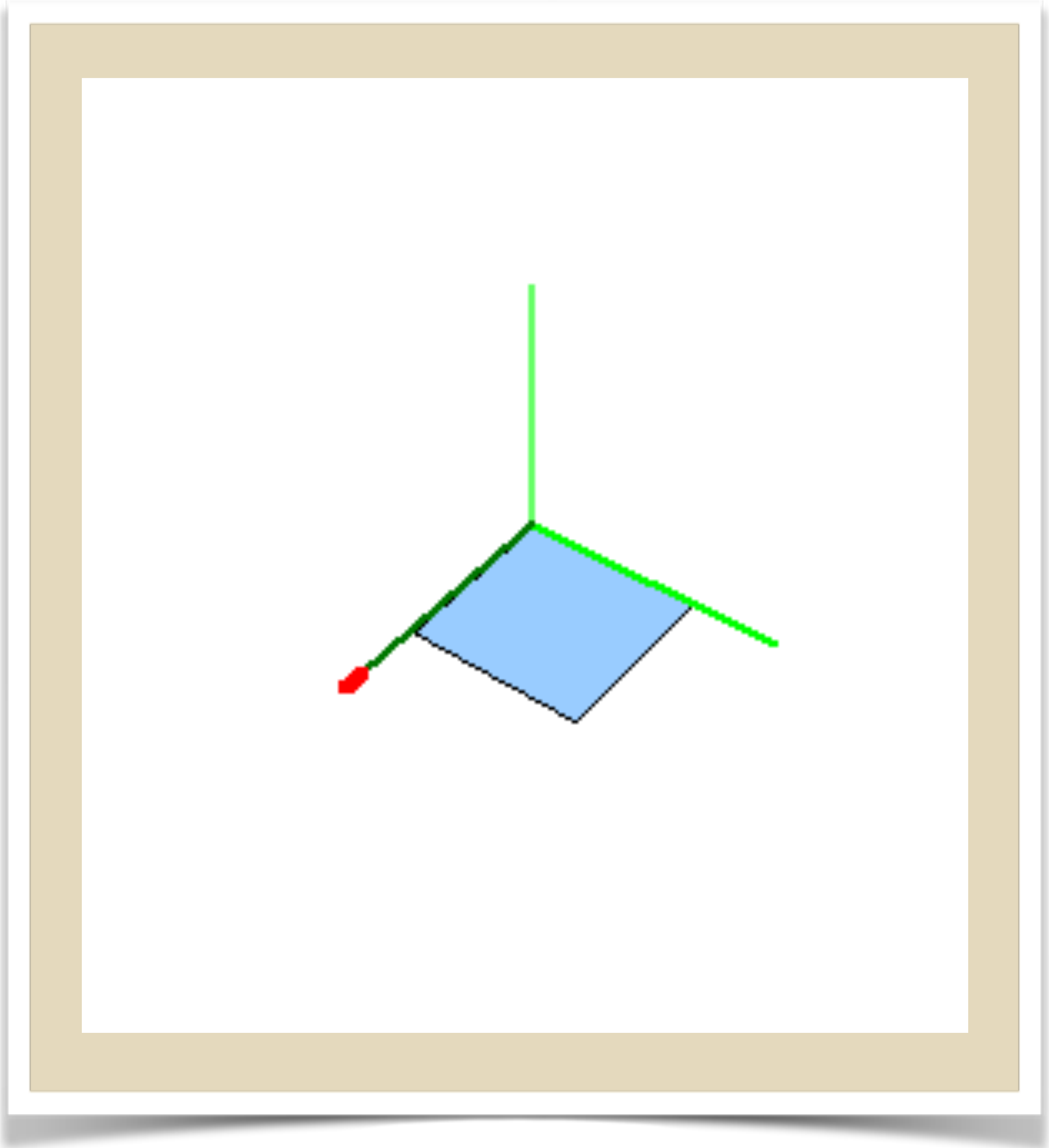


(3-1-3) Euler Angles
(75.6,77.3,-51.7) Degrees



(3-2-1) Euler Angles
(60,50,70) Degrees



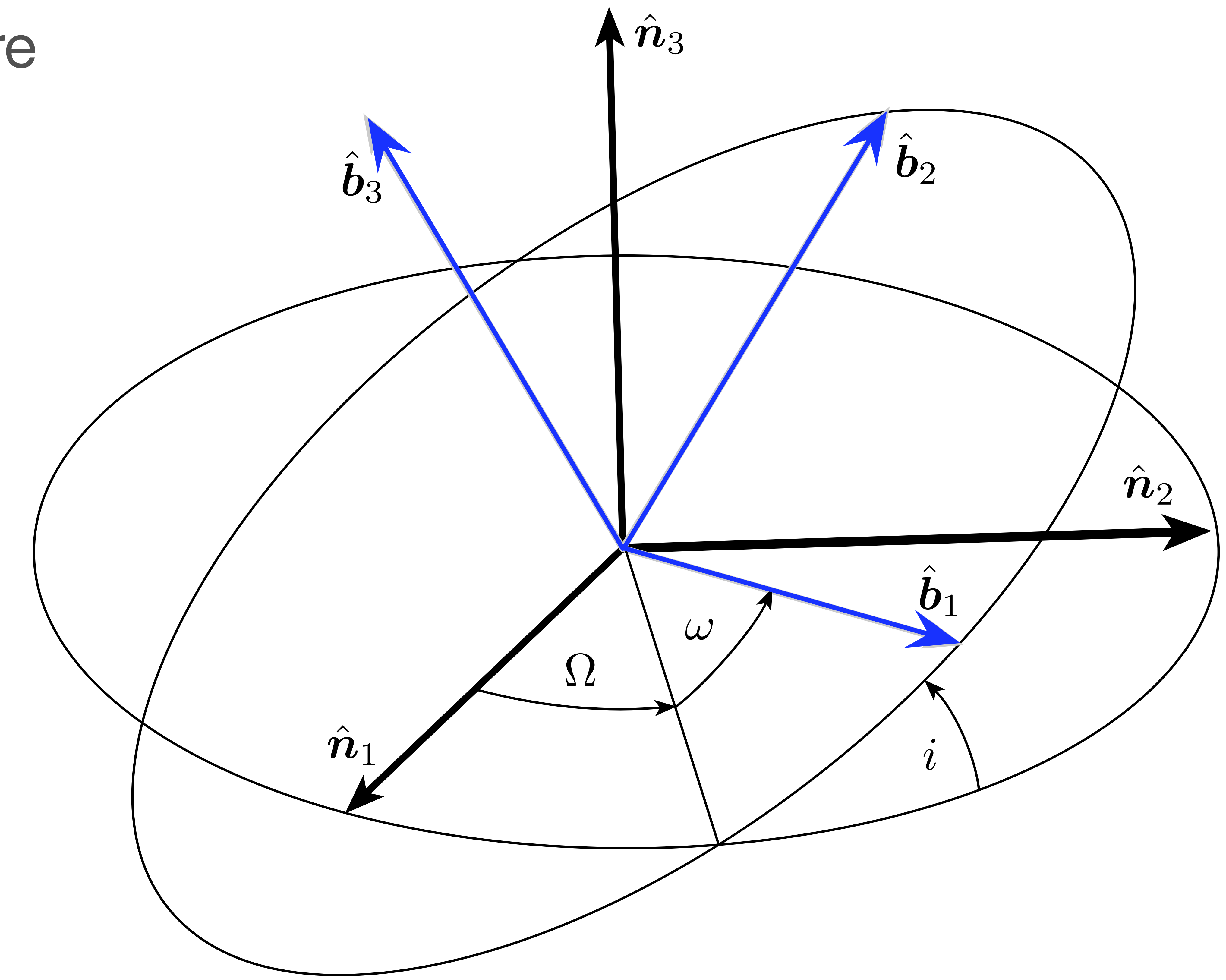
(1-3-2) Euler Angles
(37.2,-3.7,71.2) Degrees

Types of Euler Angles

- There are two types of Euler angles
  - Symmetric Set: Here the first and last rotation axis number is repeated. For example: 3-1-3 set used in astrodynamics to describe the orbit plane
  - Asymmetric Set: Here no axis rotation number is repeated. For example, the 3-2-1 (yaw-pitch-roll) angles used to describe many vehicles.
- Each type of Euler angles will have common mathematical properties and singularities.

Singularities

- Each set of Euler angles has a geometric singularity where two angles are not uniquely defined.
- It is always the second angle which defines this singular orientation.
  - Symmetric Set: 2nd angle is 0 or 180 degrees. For example, the 3-1-3 orbit angles with zero inclination.
  - Asymmetric Set: 2nd angle is +/- 90 degrees. For example, the 3-2-1 angle of an aircraft with 90 degrees pitch.



Single-Axis DCM

- The rotation matrix $[M_i]$ for a single axis rotation about the i^{th} body axis is given by

$$[M_1(\theta)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

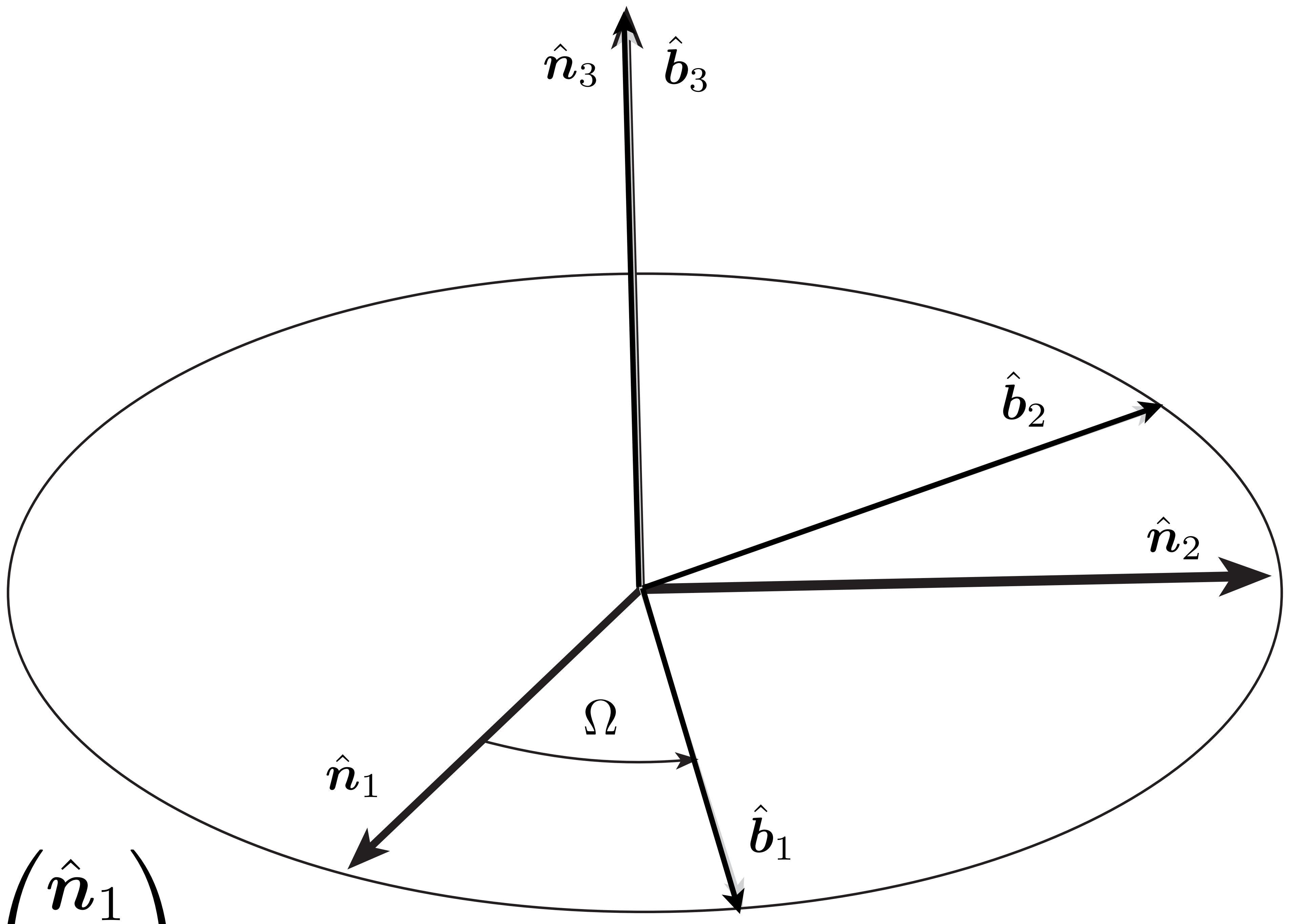
$$[M_2(\theta)] = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$[M_3(\theta)] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example

- Consider the 3-axis rotation using Ω
- The *B* and *N* frame axis are related through

$$\hat{\boldsymbol{b}}_1 = \cos\Omega\hat{\boldsymbol{n}}_1 + \sin\Omega\hat{\boldsymbol{n}}_2$$
$$\hat{\boldsymbol{b}}_2 = -\sin\Omega\hat{\boldsymbol{n}}_1 + \cos\Omega\hat{\boldsymbol{n}}_2$$
$$\hat{\boldsymbol{b}}_3 = \hat{\boldsymbol{n}}_3$$



This allows us to write

$$\begin{pmatrix} \hat{\boldsymbol{b}}_1 \\ \hat{\boldsymbol{b}}_2 \\ \hat{\boldsymbol{b}}_3 \end{pmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \hat{\boldsymbol{n}}_1 \\ \hat{\boldsymbol{n}}_2 \\ \hat{\boldsymbol{n}}_3 \end{pmatrix}$$

Mapping Euler Angles to Rotation Matrix

- Let the (α, β, γ) Euler angle sequence be $(\theta_1, \theta_2, \theta_3)$. To obtain the final rotation matrix [*BN*] which maps inertial frame vector components to body frame vector components, we make use of the composite rotation matrix property [*RN*]=[*RB*][*BN*].

$$\boxed{[C(\theta_1, \theta_2, \theta_3)] = [M_\gamma(\theta_3)][M_\beta(\theta_2)][M_\alpha(\theta_1)]}$$

- Carrying out this matrix algebra, we can find formulas which will map any Euler angle set to the corresponding rotation matrix.

3-2-1 Euler Angles

- Given the yaw, pitch and roll angles, we can compute the DCM using the three elemental rotation matrices:

$$[BN] = [M_1(\theta_3)][M_2(\theta_2)][M_3(\theta_1)] = [M_1(\phi)][M_2(\theta)][M_3(\psi)]$$

$$[BN] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3-2-1 Euler Angles

- Forward mapping is given by:

$$[BN] = \begin{bmatrix} c\theta_2 c\theta_1 & c\theta_2 s\theta_1 & -s\theta_2 \\ s\theta_3 s\theta_2 c\theta_1 - c\theta_3 s\theta_1 & s\theta_3 s\theta_2 s\theta_1 + c\theta_3 c\theta_1 & s\theta_3 c\theta_2 \\ c\theta_3 s\theta_2 c\theta_1 + s\theta_3 s\theta_1 & c\theta_3 s\theta_2 s\theta_1 - s\theta_3 c\theta_1 & c\theta_3 c\theta_2 \end{bmatrix}$$

- Inverse mapping back to Euler angles is found by examining the matrix element entries.

$$\psi = \theta_1 = \tan^{-1}\left(\frac{C_{12}}{C_{11}}\right)$$

$$\theta = \theta_2 = -\sin^{-1}(C_{13})$$

$$\phi = \theta_3 = \tan^{-1}\left(\frac{C_{23}}{C_{33}}\right)$$

Note that the quadrants must be checked with the inverse tangent function!

3-1-3 Euler Angles

- Forward mapping is given by:

$$[BN] = \begin{bmatrix} c\theta_3 c\theta_1 - s\theta_3 c\theta_2 s\theta_1 & c\theta_3 s\theta_1 + s\theta_3 c\theta_2 c\theta_1 & s\theta_3 s\theta_2 \\ -s\theta_3 c\theta_1 - c\theta_3 c\theta_2 s\theta_1 & -s\theta_3 s\theta_1 + c\theta_3 c\theta_2 c\theta_1 & c\theta_3 s\theta_2 \\ s\theta_2 s\theta_1 & -s\theta_2 c\theta_1 & c\theta_2 \end{bmatrix}$$

- Inverse mapping back to Euler angles is found by examining the matrix element entries.

$$\Omega = \theta_1 = \tan^{-1}\left(\frac{C_{31}}{-C_{32}}\right)$$

$$i = \theta_2 = \cos^{-1}\left(C_{33}\right)$$

$$\omega = \theta_3 = \tan^{-1}\left(\frac{C_{13}}{C_{23}}\right)$$

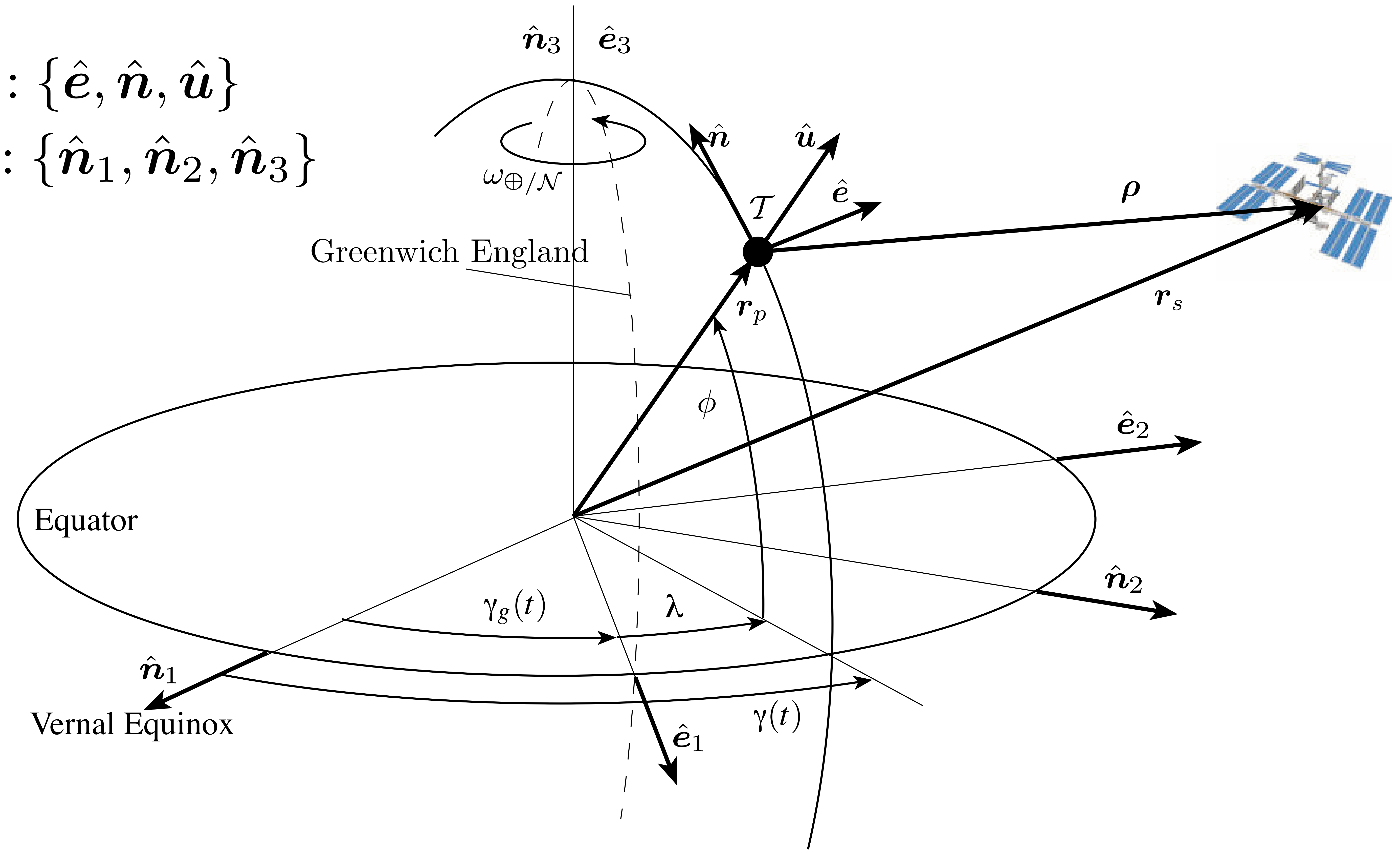
Note that the quadrants must be checked with the inverse tangent function!

Example

- Consider the astrodynamics problem, where the topographic frame (surface frame) T is defined as shown in the figure below.

$$\mathcal{T} : \{\hat{\boldsymbol{e}}, \hat{\boldsymbol{n}}, \hat{\boldsymbol{u}}\}$$

$$\mathcal{N} : \{\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{n}}_3\}$$



- Here the rotation matrix [*TN*] was given as

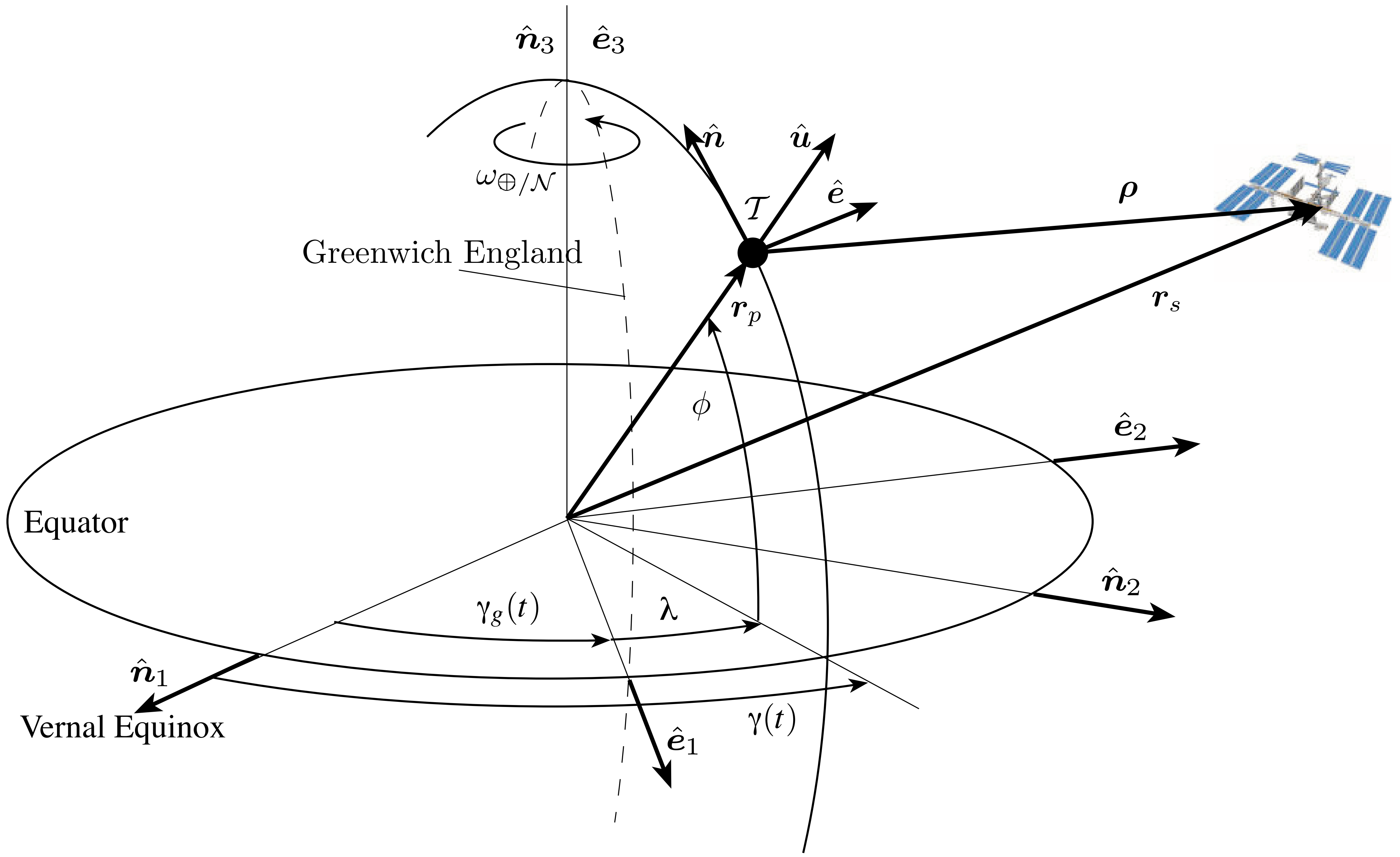
$$[TN] = \begin{bmatrix} -\sin\gamma(t) & \cos\gamma(t) & 0 \\ -\cos\gamma(t)\sin\phi & -\sin\gamma(t)\sin\phi & \cos\phi \\ \cos\gamma(t)\cos\phi & \sin\gamma(t)\cos\phi & \sin\phi \end{bmatrix}$$

- Let's derive this rotation matrix expression. To go from the *N* frame to the *T* frame, the first rotation is a 3-axis rotation by the angle Ω.

$$[M_3(\gamma(t))] = \begin{bmatrix} \cos\gamma(t) & \sin\gamma(t) & 0 \\ -\sin\gamma(t) & \cos\gamma(t) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- The next rotation is about the 2-axis with the angle Φ.

$$[M_2(-\phi)] = \begin{bmatrix} \cos(-\phi) & 0 & -\sin(-\phi) \\ 0 & 1 & 0 \\ \sin(-\phi) & 0 & \cos(-\phi) \end{bmatrix}$$

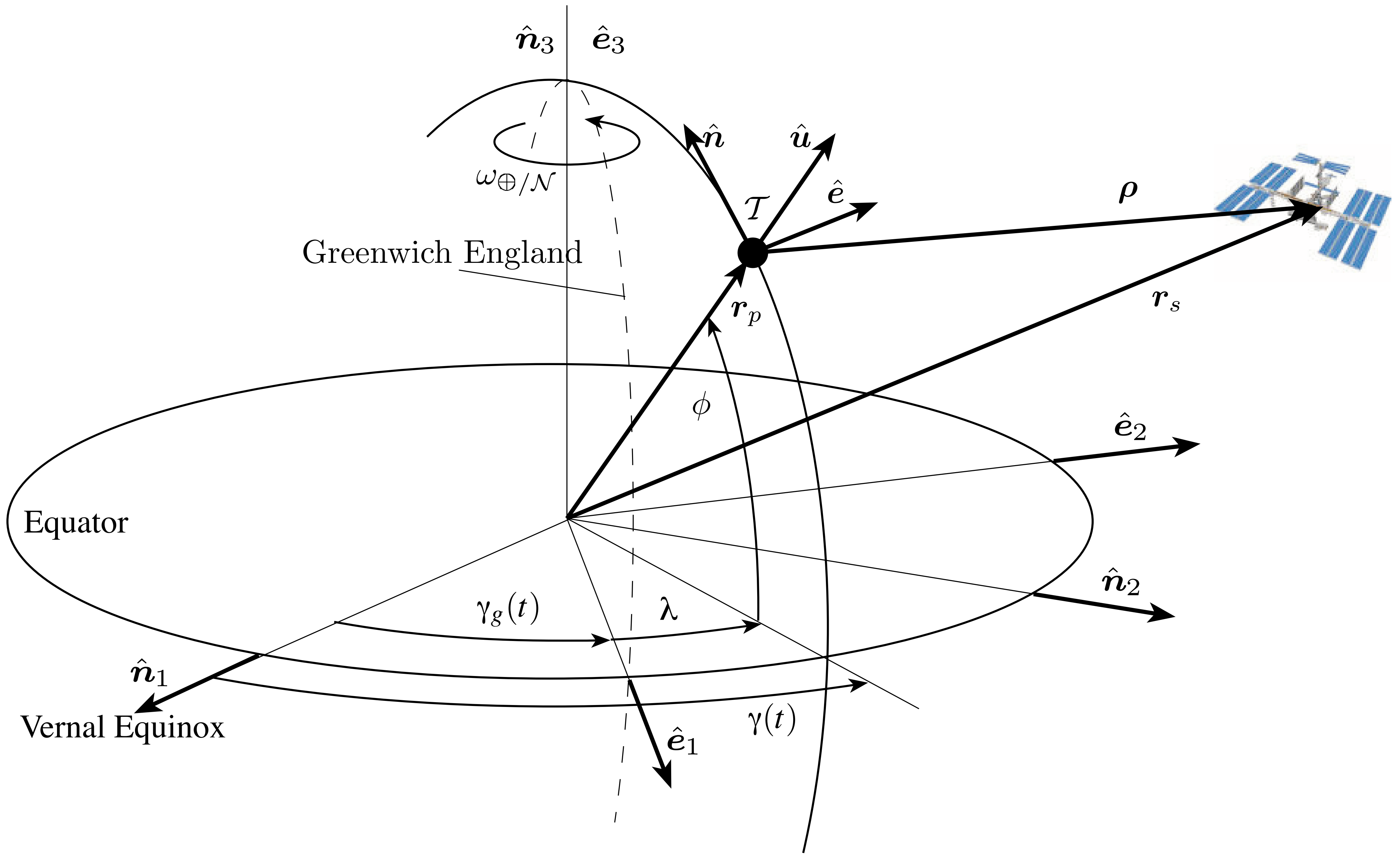


- However, we are not yet done. We still need to align the 1,2 and 3 axis of our current frame to that of the *T* frame. First we correct the 1-axis by doing a 90 degree rotation about our current 3-axis

$$[M_3(90^o)] = \begin{bmatrix} \cos 90^o & \sin 90^o & 0 \\ -\sin 90^o & \cos 90^o & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Next, we fix both the 2 and 3 axis orientation by doing 90 degree rotation about the current 1-axis.

$$[M_1(90^o)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 90^o & \sin 90^o \\ 0 & -\sin 90^o & \cos 90^o \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$



- Finally, we add up all these rotation matrices to find the desired [*TN*] direction cosine matrix:

$$[TN] = [M_1(90^o)][M_3(90^o)][M_2(-\phi)][M_3(\gamma(t))]$$

$$[TN] = \begin{bmatrix} -\sin\gamma(t) & \cos\gamma(t) & 0 \\ -\cos\gamma(t)\sin\phi & -\sin\gamma(t)\sin\phi & \cos\phi \\ \cos\gamma(t)\cos\phi & \sin\gamma(t)\cos\phi & \sin\phi \end{bmatrix}$$

Rotation Addition

- Assume we have a yaw-pitch-roll rotation defined from the inertial frame *N* to the reference frame *R* through

$$\boldsymbol{\theta}_{RN} = \{\psi_{RN}, \theta_{RN}, \phi_{RN}\}$$

- Assume we also know the yaw-pitch-roll rotation defined from the reference frame *R* to the body frame *B* through

$$\boldsymbol{\theta}_{BR} = \{\psi_{BR}, \theta_{BR}, \phi_{BR}\}$$

- The question is, what are the yaw-pitch-roll angles that will take us directly from the inertial frame *N* to the body frame *B*.

$$\boldsymbol{\theta}_{BN} = \{\psi_{BN}, \theta_{BN}, \phi_{BN}\}$$

- Note that $\boldsymbol{\theta}_{BN} \neq \boldsymbol{\theta}_{BR} + \boldsymbol{\theta}_{RN}$

Rotation Addition

- To add two Euler angle rotations, we go back to the rotation matrix addition property. First, we find:

$$\boldsymbol{\theta}_{BR} \quad \Rightarrow \quad [BR(\boldsymbol{\theta}_{BR})] \qquad \boldsymbol{\theta}_{RN} \quad \Rightarrow \quad [RN(\boldsymbol{\theta}_{RN})]$$

- Then, we compute [*BN*] using:

$$[BN(\boldsymbol{\theta}_{BN})] = [BR(\boldsymbol{\theta}_{BR})][RN(\boldsymbol{\theta}_{RN})]$$

- Last, we find the desired 3-2-1 Euler angles using the inverse mapping:

$$[BN(\boldsymbol{\theta}_{BN})] \quad \Rightarrow \quad \boldsymbol{\theta}_{BN} = \{\psi_{BN}, \theta_{BN}, \phi_{BN}\}$$

Rotation Subtraction

- Similarly, assume that we are given:

$$\boldsymbol{\theta}_{BN} = \{\psi_{BN}, \theta_{BN}, \phi_{BN}\}$$
$$\boldsymbol{\theta}_{RN} = \{\psi_{RN}, \theta_{RN}, \phi_{RN}\}$$

- In this case we would like to find the attitude tracking error of body *B* relative to the reference orientation *R*.

$$\boldsymbol{\theta}_{BN} \quad \Rightarrow \quad [BN(\boldsymbol{\theta}_{BN})]$$
$$\boldsymbol{\theta}_{RN} \quad \Rightarrow \quad [RN(\boldsymbol{\theta}_{RN})]$$
$$[BR(\boldsymbol{\theta}_{BR})] = [BN(\boldsymbol{\theta}_{BN})][RN(\boldsymbol{\theta}_{RN})]^T$$
$$[BR(\boldsymbol{\theta}_{BR})] \quad \Rightarrow \quad \boldsymbol{\theta}_{BR} = \{\psi_{BR}, \theta_{BR}, \phi_{BR}\}$$

Example 3.2

- Let the orientation of two spacecraft *B* and *F* relative to an inertial frame *N* be given through the (3-2-1) Euler angles:
- The orientation matrices of these Euler angles are found using Eq. (3.20):

$$\boldsymbol{\theta}_{\mathcal{B}} = (30^o, -45^o, 60^o)^T \qquad \boldsymbol{\theta}_{\mathcal{F}} = (10^o, 25^o, -15^o)^T$$

$$[BN] = \begin{bmatrix} 0.612372 & 0.353553 & 0.707107 \\ -0.78033 & 0.126826 & 0.612372 \\ 0.126826 & -0.926777 & 0.353553 \end{bmatrix}$$

$$[FN] = \begin{bmatrix} 0.892539 & 0.157379 & -0.422618 \\ -0.275451 & 0.932257 & -0.234570 \\ 0.357073 & 0.325773 & 0.875426 \end{bmatrix}$$

- The rotation matrix relating the *B* and *F* frames is found to be

$$[BF] = [BN][FN]^T = \begin{bmatrix} 0.303372 & -0.0049418 & 0.952859 \\ -0.935315 & 0.1895340 & 0.298769 \\ -0.182075 & -0.9818620 & 0.052877 \end{bmatrix}$$

- Using the transformations in Eq. (3.34), the Euler angles are computed using

$$\begin{aligned} \psi &= \tan^{-1}\left(\frac{-0.0049418}{0.303372}\right) = -0.933242 \text{ deg} \\ \theta &= -\sin^{-1}(0.952859) = -72.3373 \text{ deg} \\ \phi &= \tan^{-1}\left(\frac{0.298769}{0.052877}\right) = 79.9636 \text{ deg} \end{aligned}$$

(3-2-1) Kinematic Differential Equation

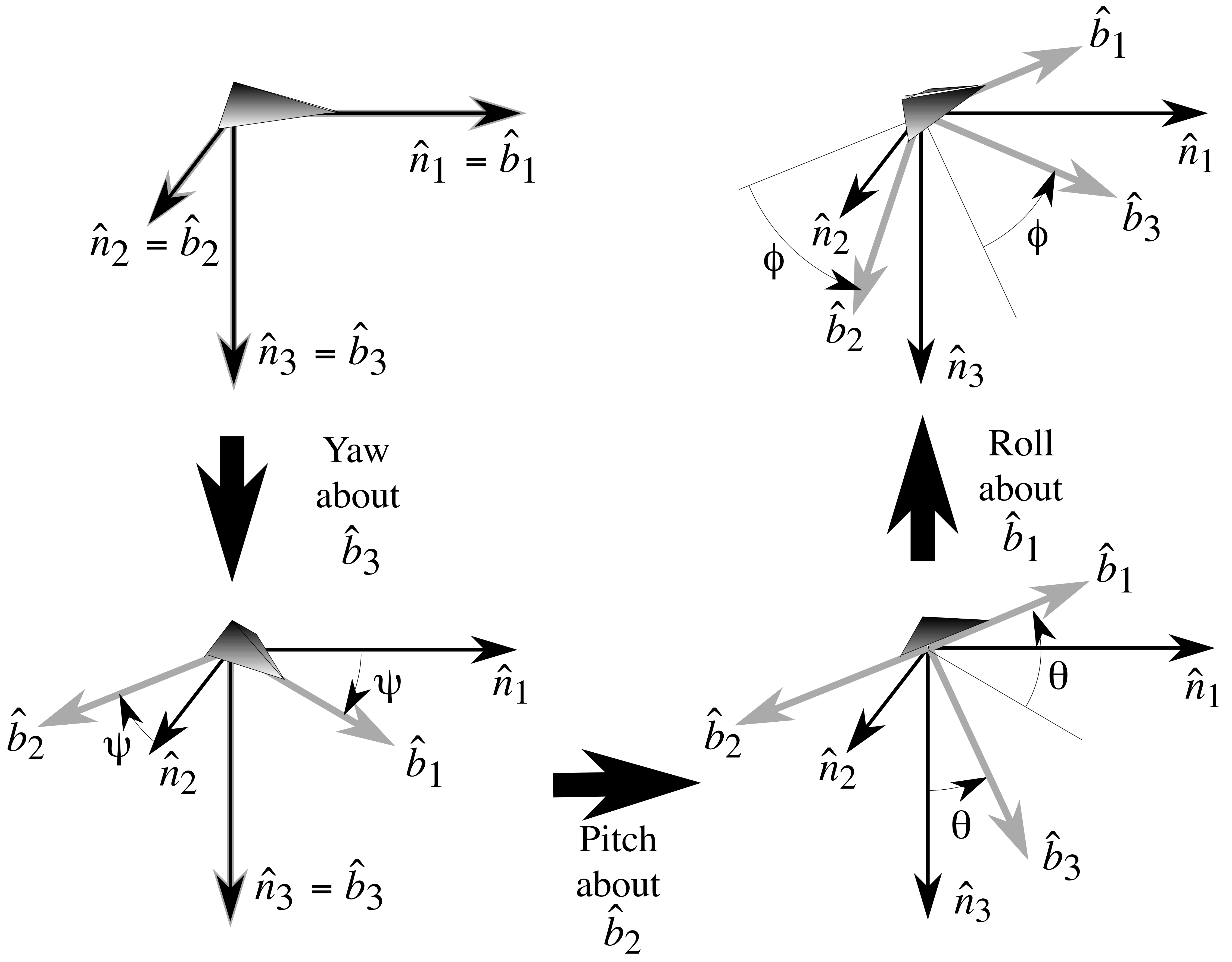
- We would like to find the differential equations of the Euler angles (i.e. yaw, pitch and roll angles).

$$\dot{\psi}(t) \qquad \dot{\theta}(t) \qquad \dot{\phi}(t)$$

- The angular rotation rate is not measured as yaw, pitch and roll rates, but rather through the body angular velocity vector

$$\boldsymbol{\omega} = \omega_1 \hat{\boldsymbol{b}}_1 + \omega_2 \hat{\boldsymbol{b}}_2 + \omega_3 \hat{\boldsymbol{b}}_3$$

- We need to find out how these Euler angle rates and the body angular velocity components are related.



Using the above figure, it is evident that $\boldsymbol{\omega} = \dot{\psi}\hat{\boldsymbol{n}}_3 + \dot{\theta}\hat{\boldsymbol{b}}_2' + \dot{\phi}\hat{\boldsymbol{b}}_1$

Recall that angular velocity vectors are truly vectors and can be simply added up.

- Next, we need to express the $\hat{\boldsymbol{b}}'_2$ in terms of $\{\hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2, \hat{\boldsymbol{b}}_3\}$ vectors:

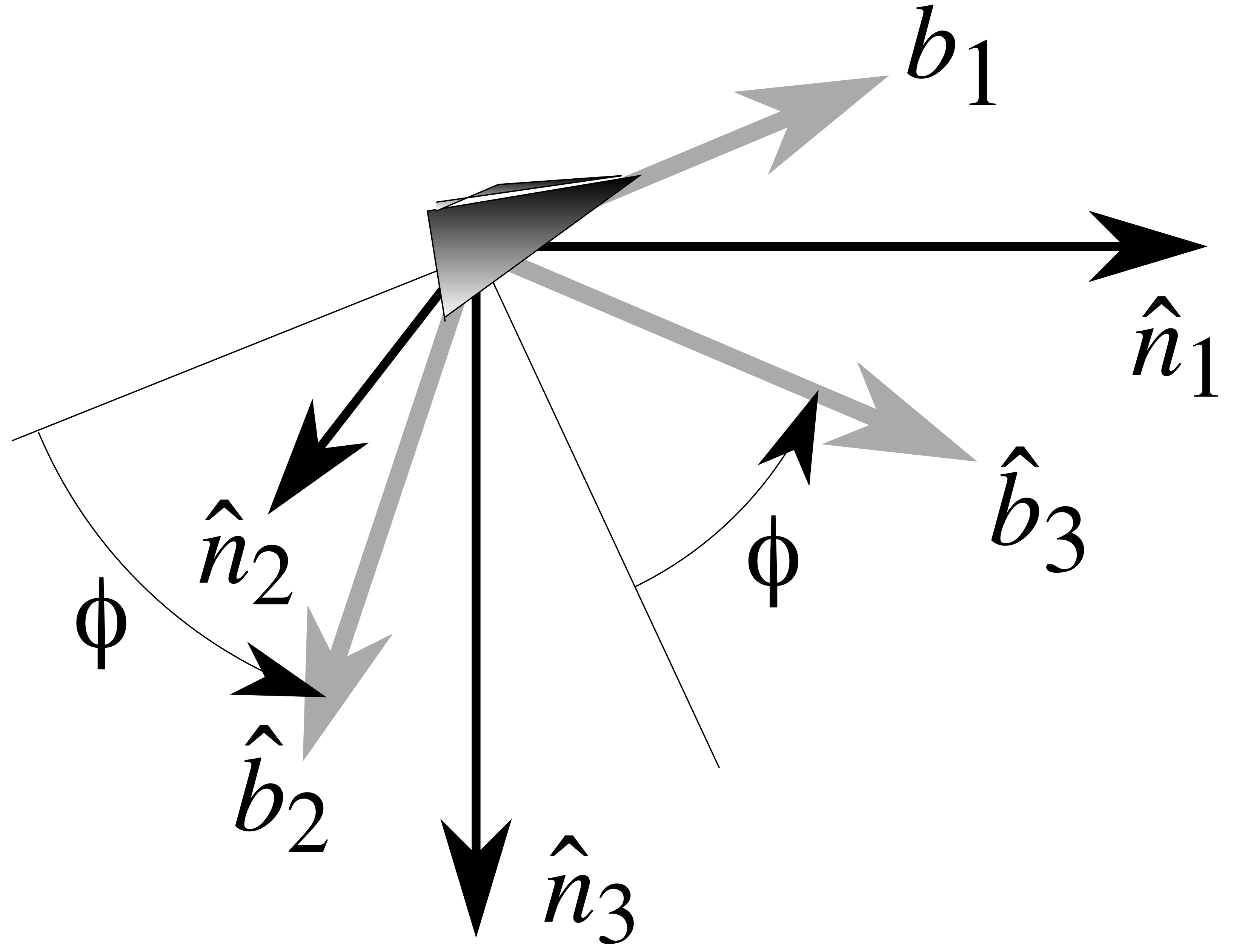
$$\hat{\boldsymbol{b}}'_2 = \cos\phi \hat{\boldsymbol{b}}_2 - \sin\phi \hat{\boldsymbol{b}}_3$$

- To write the $\hat{\boldsymbol{n}}_3$ in terms of $\{\hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2, \hat{\boldsymbol{b}}_3\}$ vector, we use the mapping between the (3-2-1) Euler angles and [*BN*]:

$$\hat{\boldsymbol{n}}_3 = -\sin\theta \hat{\boldsymbol{b}}_1 + \sin\phi\cos\theta \hat{\boldsymbol{b}}_2 + \cos\phi\cos\theta \hat{\boldsymbol{b}}_3$$

- The last step is to equate the vector components by setting

$$\boldsymbol{\omega} = \omega_1 \hat{\boldsymbol{b}}_1 + \omega_2 \hat{\boldsymbol{b}}_2 + \omega_3 \hat{\boldsymbol{b}}_3 = \dot{\psi}\hat{\boldsymbol{n}}_3 + \dot{\theta}\hat{\boldsymbol{b}}'_2 + \dot{\phi}\hat{\boldsymbol{b}}_1$$



- Finally, we can relate the Euler angle rates and the body angular velocity vector components through:

$$ {}^{\mathcal{B}}\boldsymbol{\omega} = {}^{\mathcal{B}}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{bmatrix} -\sin\theta & 0 & 1 \\ \sin\phi\cos\theta & \cos\phi & 0 \\ \cos\phi\cos\theta & -\sin\phi & 0 \end{bmatrix} \begin{pmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} $$

- The inverse relationship (the kinematic differential equation of the (3-2-1) Euler angles) is found to be

$$ \begin{pmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{\cos\theta} \begin{bmatrix} 0 & \sin\phi & \cos\phi \\ 0 & \cos\phi\cos\theta & -\sin\phi\cos\theta \\ \cos\theta & \sin\phi\sin\theta & \cos\phi\sin\theta \end{bmatrix} {}^{\mathcal{B}}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} $$

$$ = [B(\psi, \theta, \phi)]\, {}^{\mathcal{B}}\boldsymbol{\omega} $$

(3-1-3) Kinematic Differential Eqn

- Similarly, the body angular velocity vector is written in terms of the (3-1-3) Euler angles as

$$
{}^{\mathcal{B}}\boldsymbol{\omega} = \begin{bmatrix} \sin\theta_3 \sin\theta_2 & \cos\theta_3 & 0 \\ \cos\theta_3 \sin\theta_2 & -\sin\theta_3 & 0 \\ \cos\theta_2 & 0 & 1 \end{bmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix}
$$

- with the inverse transformation (the kinematic differential equation of the Euler angles) being

$$
\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix} = \frac{1}{\sin\theta_2} \begin{bmatrix} \sin\theta_3 & \cos\theta_3 & 0 \\ \cos\theta_3 \sin\theta_2 & -\sin\theta_3 \sin\theta_2 & 0 \\ -\sin\theta_3 \cos\theta_2 & -\cos\theta_3 \cos\theta_2 & \sin\theta_2 \end{bmatrix} {}^{\mathcal{B}}\boldsymbol{\omega}
$$

$$
= [B(\boldsymbol{\theta})]\,{}^{\mathcal{B}}\boldsymbol{\omega}
$$

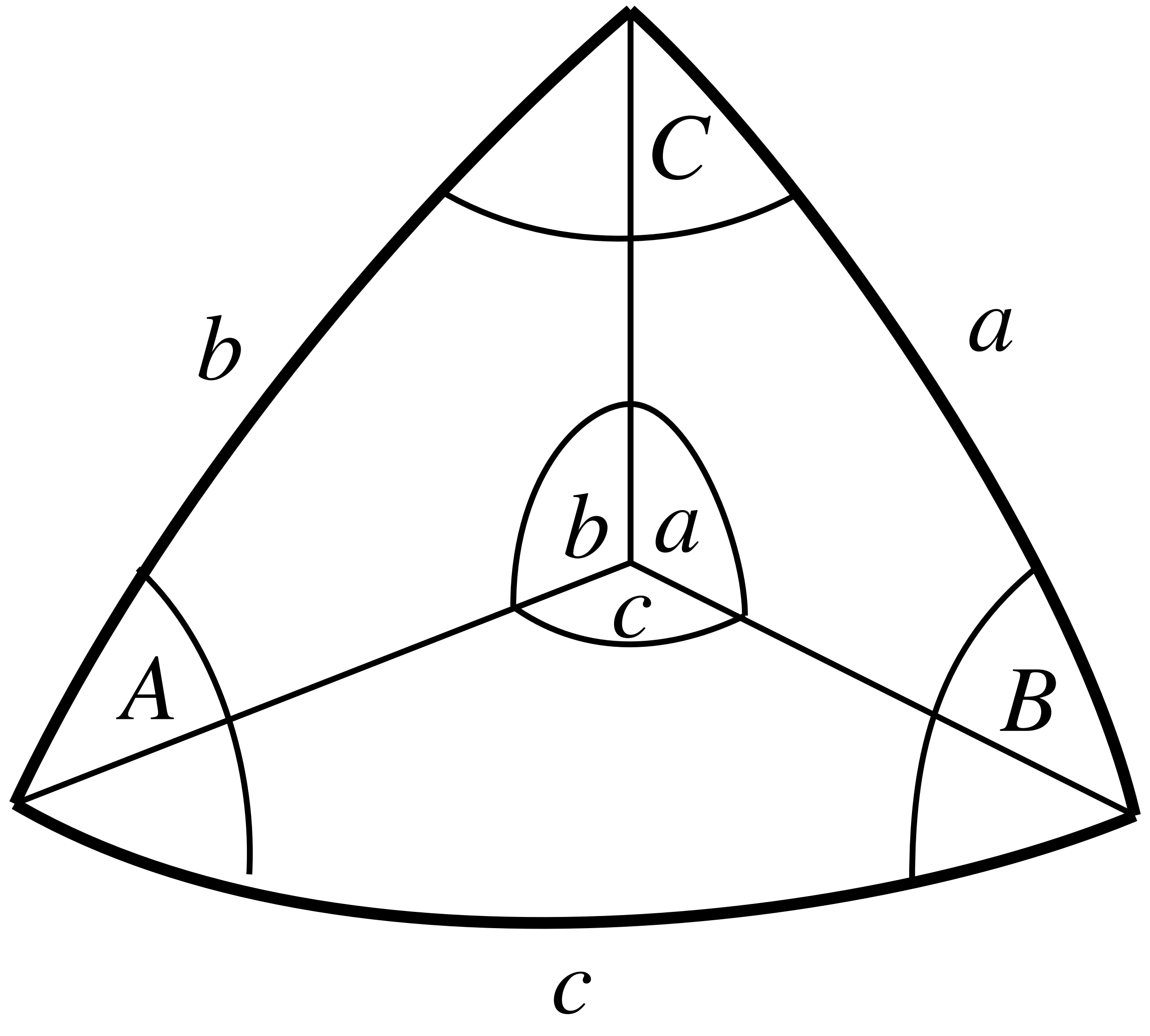
Comments

- Note that it is always the second Euler angle which causes the kinematic differential equations to become singular.
- As with the Euler angle geometric singularities, we find that for
  - Asymmetric Euler angles: differential equations are singular at $\theta_2 = \pm 90^{\text{o}}$
  - Symmetric Euler angles: differential equations are singular at $\theta_2 = 0^{\text{o}} \text{ or } 180^{\text{o}}$
- With Euler angles, one is never more than a 90 degree removed from a singularity. This makes these attitude coordinates less attractive for large reorientations.

Addition of Symmetric Euler Angles

Spherical Law of Sines:

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$



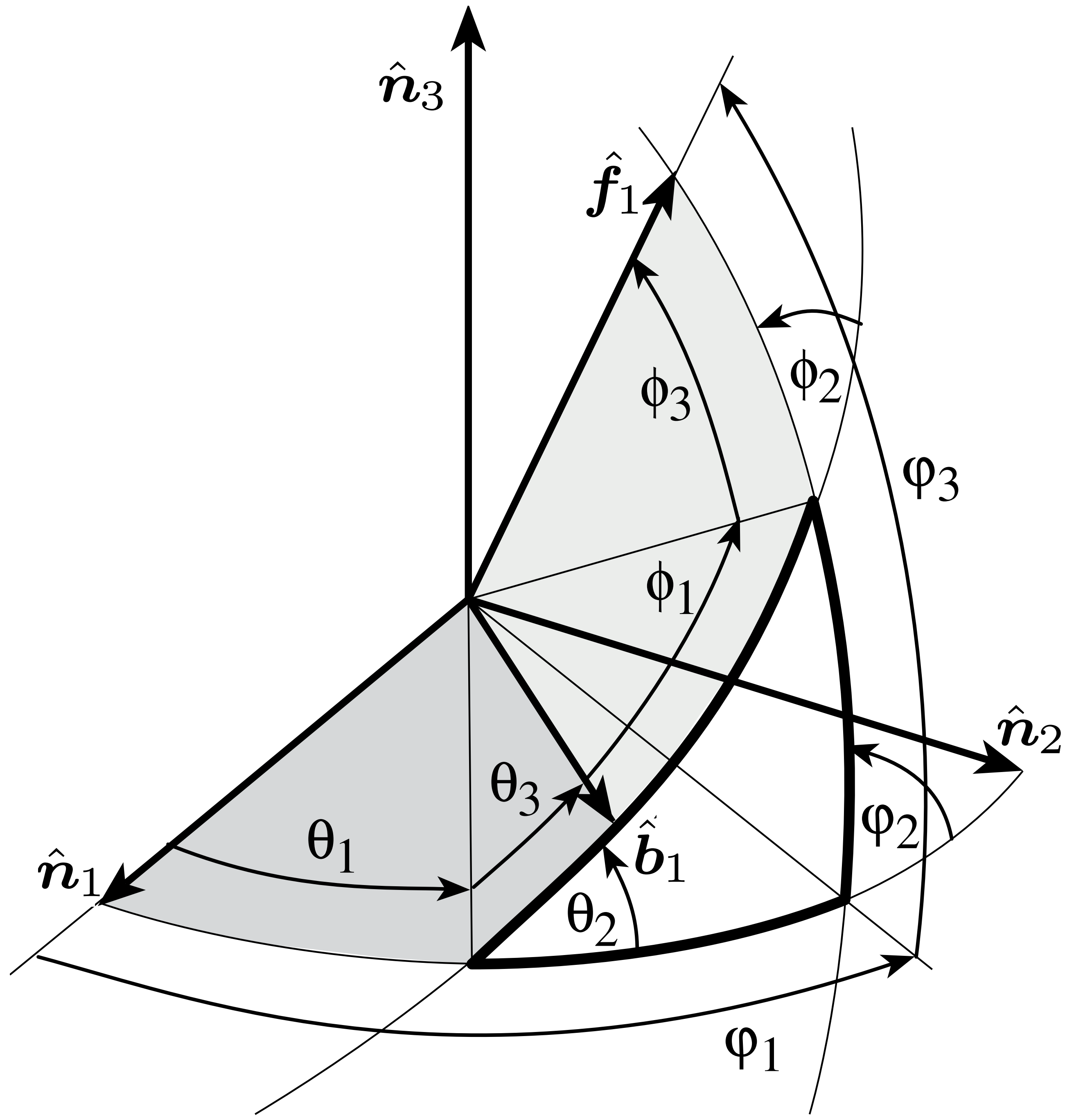
Spherical Law of Cosines:

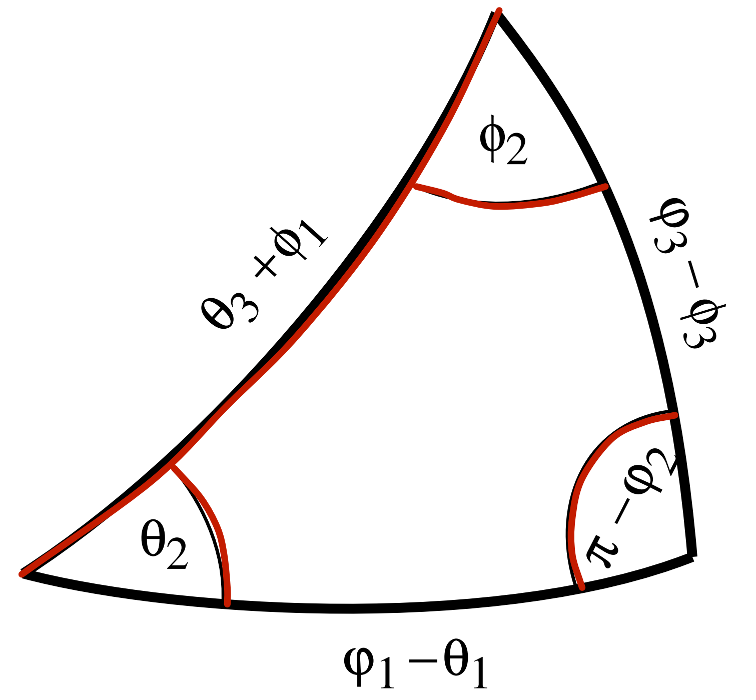
$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$

$$\cos B = -\cos A \cos C + \sin A \sin C \cos b$$

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c$$

3-1-3 Euler Angles:

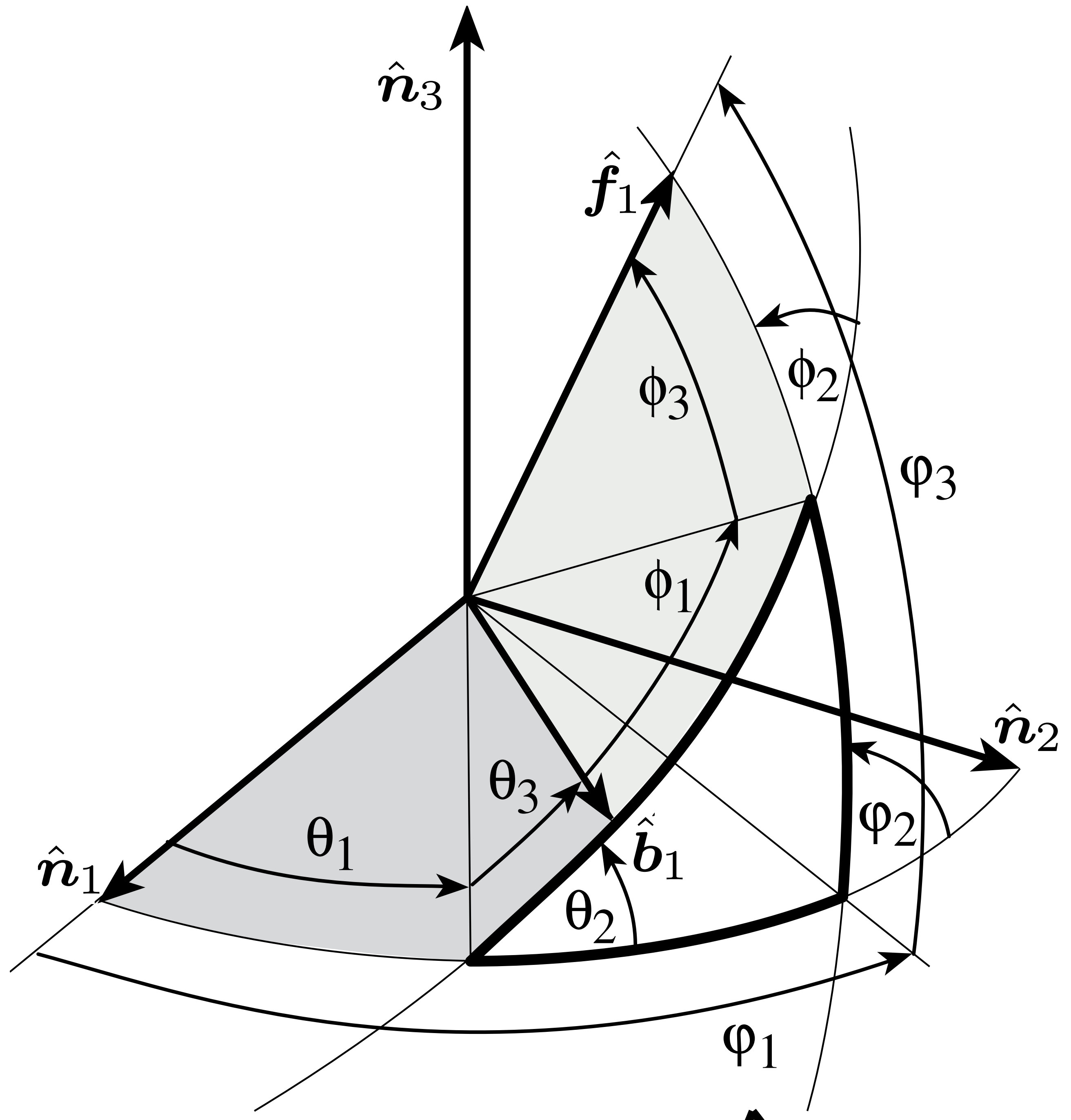


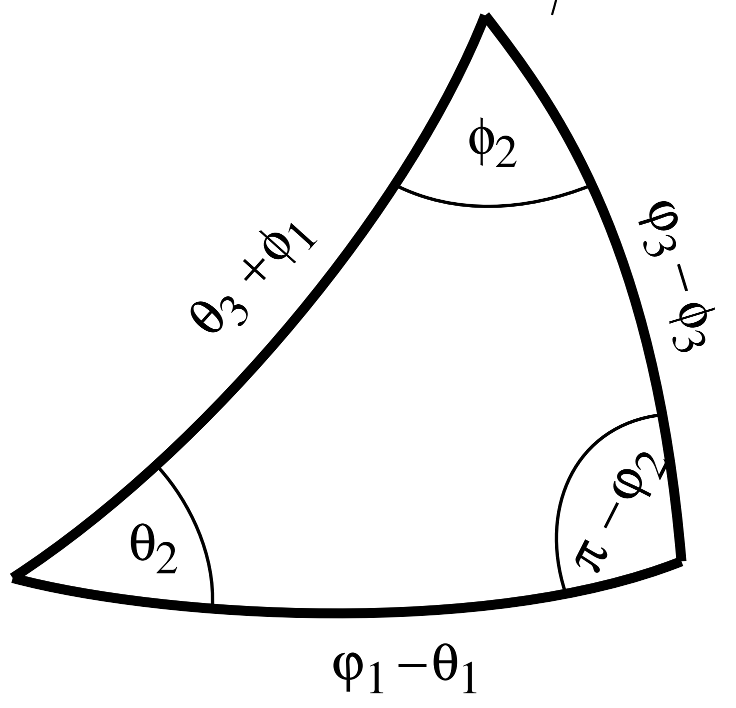


$$\cos(\pi - \varphi_2) = -\cos\theta_2 \cos\phi_2 + \sin\theta_2 \sin\phi_2 \cos(\theta_3 + \phi_1)$$

$$\varphi_2 = \cos^{-1}\left(\cos\theta_2 \cos\phi_2 - \sin\theta_2 \sin\phi_2 \cos(\theta_3 + \phi_1)\right)$$

3-1-3 Euler Angles:





Note: This formula actually holds for *any* symmetric Euler angle set!

Spherical Law of Sines:

$$\sin(\varphi_1 - \theta_1) = \frac{\sin\phi_2}{\sin\varphi_2}\sin(\theta_3 + \phi_1)$$

$$\sin(\varphi_3 - \phi_3) = \frac{\sin\theta_2}{\sin\varphi_2}\sin(\theta_3 + \phi_1)$$

Spherical Law of Cosines:

$$\cos(\varphi_1 - \theta_1) = \frac{\cos\phi_2 - \cos\theta_2\cos\varphi_2}{\sin\theta_2\sin\varphi_2}$$

$$\cos(\varphi_3 - \phi_3) = \frac{\cos\theta_2 - \cos\phi_2\cos\varphi_2}{\sin\phi_2\sin\varphi_2}$$

$$\varphi_1 = \theta_1 + \tan^{-1}\left(\frac{\sin\theta_2\sin\phi_2\sin(\theta_3 + \phi_1)}{\cos\phi_2 - \cos\theta_2\cos\varphi_2}\right)$$

$$\varphi_3 = \phi_3 + \tan^{-1}\left(\frac{\sin\theta_2\sin\phi_2\sin(\theta_3 + \phi_1)}{\cos\theta_2 - \cos\phi_2\cos\varphi_2}\right)$$

Symmetric Euler Angle Subtraction

- Using the equivalent spherical trigonometric formulas as for the EA addition problem, we can find a direct analytical solution to compute the relative symmetric EAs (i.e. EA subtraction).

$$\phi_1 = -\theta_3 + \tan^{-1}\left(\frac{\sin\theta_2 \sin\varphi_2 \sin(\varphi_1 - \theta_1)}{\cos\theta_2 \cos\phi_2 - \cos\varphi_2}\right)$$

$$\phi_2 = \cos^{-1}\left(\cos\theta_2 \cos\varphi_2 + \sin\theta_2 \sin\varphi_2 \cos(\varphi_1 - \theta_1)\right)$$

$$\phi_3 = \varphi_3 - \tan^{-1}\left(\frac{\sin\theta_2 \sin\varphi_2 \sin(\varphi_1 - \theta_1)}{\cos\theta_2 - \cos\phi_2 \cos\varphi_2}\right)$$