Analytical Torque-Free Motion

- Let us assume that there are no external torques acting on the rigid body, and the equations of motion are given by:

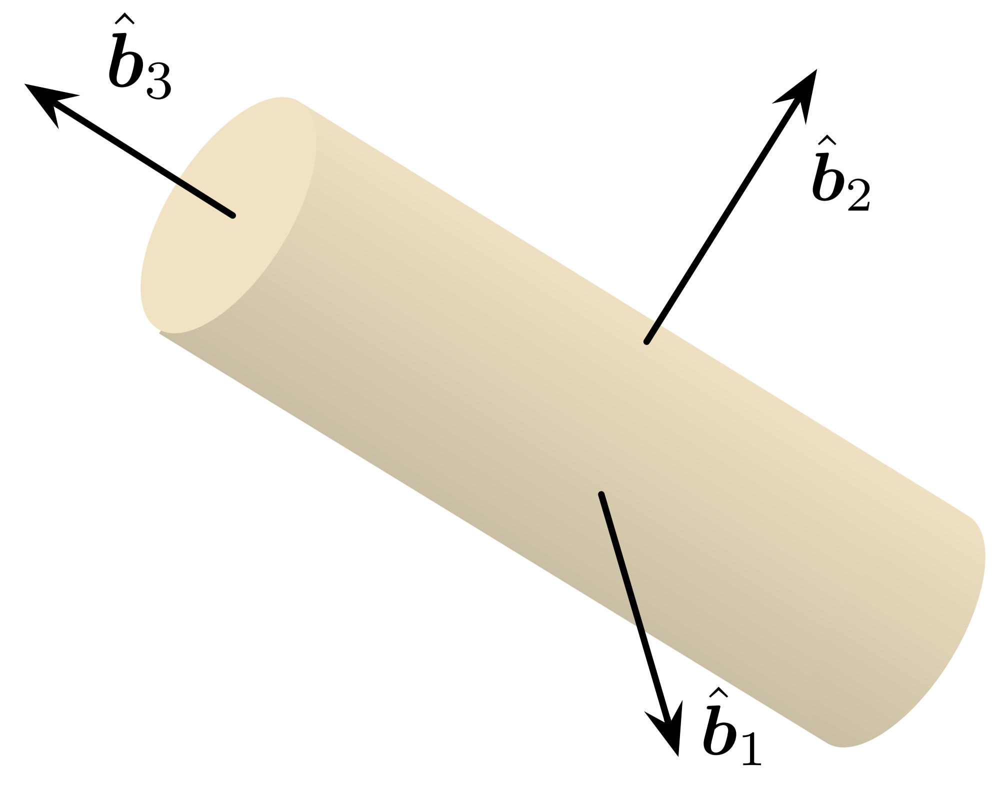
$$
\begin{aligned}
I_{11}\dot{\omega}_1 &= -(I_{33} - I_{22})\omega_2\omega_3 \\
I_{22}\dot{\omega}_2 &= -(I_{11} - I_{33})\omega_3\omega_1 \\
I_{33}\dot{\omega}_3 &= -(I_{22} - I_{11})\omega_1\omega_2
\end{aligned}
$$

- We are looking for analytical solutions to the angular motion.
- Assume that the body coordinate frame is a principal frame, and the inertia matrix is diagonal.

Axi-Symmetric Case

- Let the external torque be zero. Consider the special principal inertia case where

$$I_T = I_{11} = I_{22}$$



- Here the EOM are given by

$$\begin{aligned} I_T\dot{\omega}_1 &= -(I_{33} - I_T)\omega_2\omega_3 \\ I_T\dot{\omega}_2 &= (I_{33} - I_T)\omega_3\omega_1 \\ I_{33}\dot{\omega}_3 &= 0 \end{aligned}$$

- From this equation it is clear that the third angular velocity component will be constant.

$$\omega_3(t) = \omega_3(t_0) = \text{constant}$$

- Let's examine the remaining two differential equations more carefully. Taking the derivative of the first one we find

$$I_t\dot{\omega}_1 = -(I_{33} - I_T)\omega_2\omega_3$$

$$I_T\ddot{\boldsymbol{\omega}}_1 = -(I_{33} - I_T)\dot{\omega}_2\omega_3$$

$$\dot{\omega}_2 = \frac{1}{I_T}((I_{33} - I_T)\omega_3\omega_1)$$

Mathematically equivalent to simple Spring-Mass Systems!

$$\ddot{\omega}_1 + (\frac{I_{33}}{I_T} - 1)^2\omega_3^2\omega_1 = 0$$

Similarly, we can find:

$$\ddot{\omega}_2 + (\frac{I_{33}}{I_T} - 1)^2\omega_3^2\omega_2 = 0$$

- The analytical solution to a spring-mass dynamical system is the simple oscillator equation

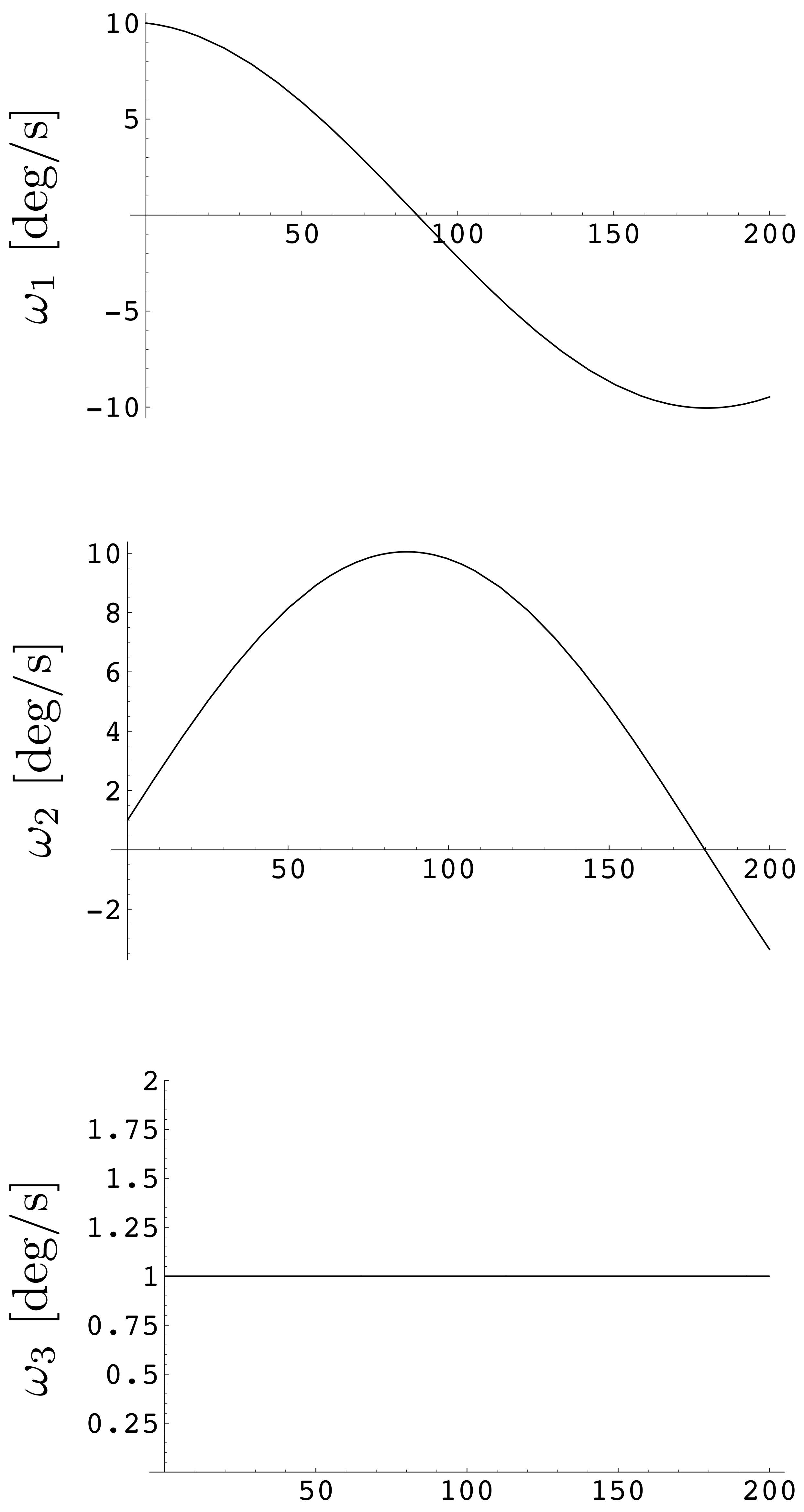
$$\omega_1(t) = A_1 \cos \omega_p t + B_1 \sin \omega_p t$$
$$\omega_2(t) = A_2 \cos \omega_p t + B_2 \sin \omega_p t$$

- Using the initial conditions, we find the analytical solution of the body angular velocity components for the axi-symmetric spacecraft case:

where

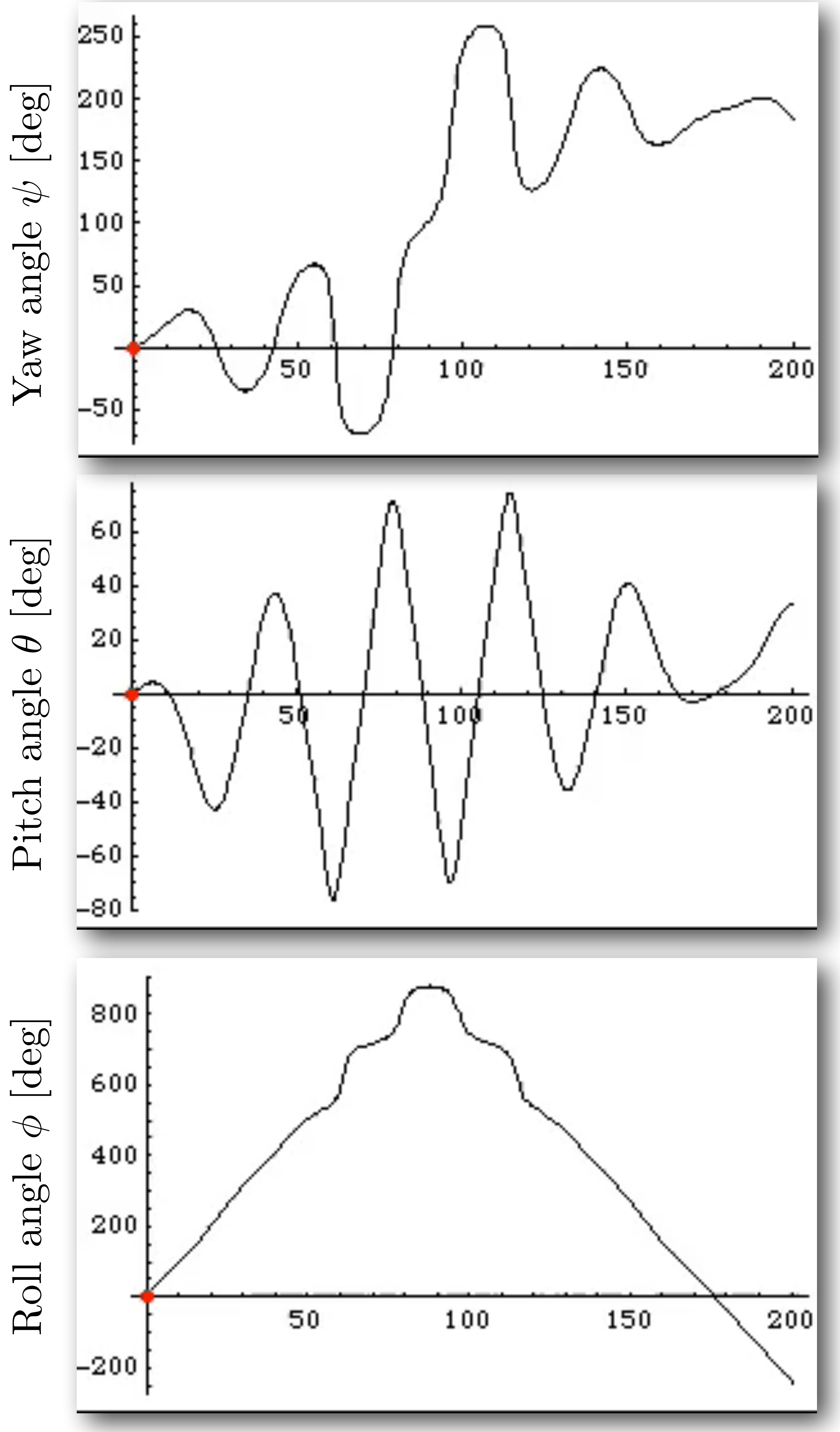
$$\omega_p = \left(\frac{I_{33}}{I_T} - 1 \right) \omega_3$$

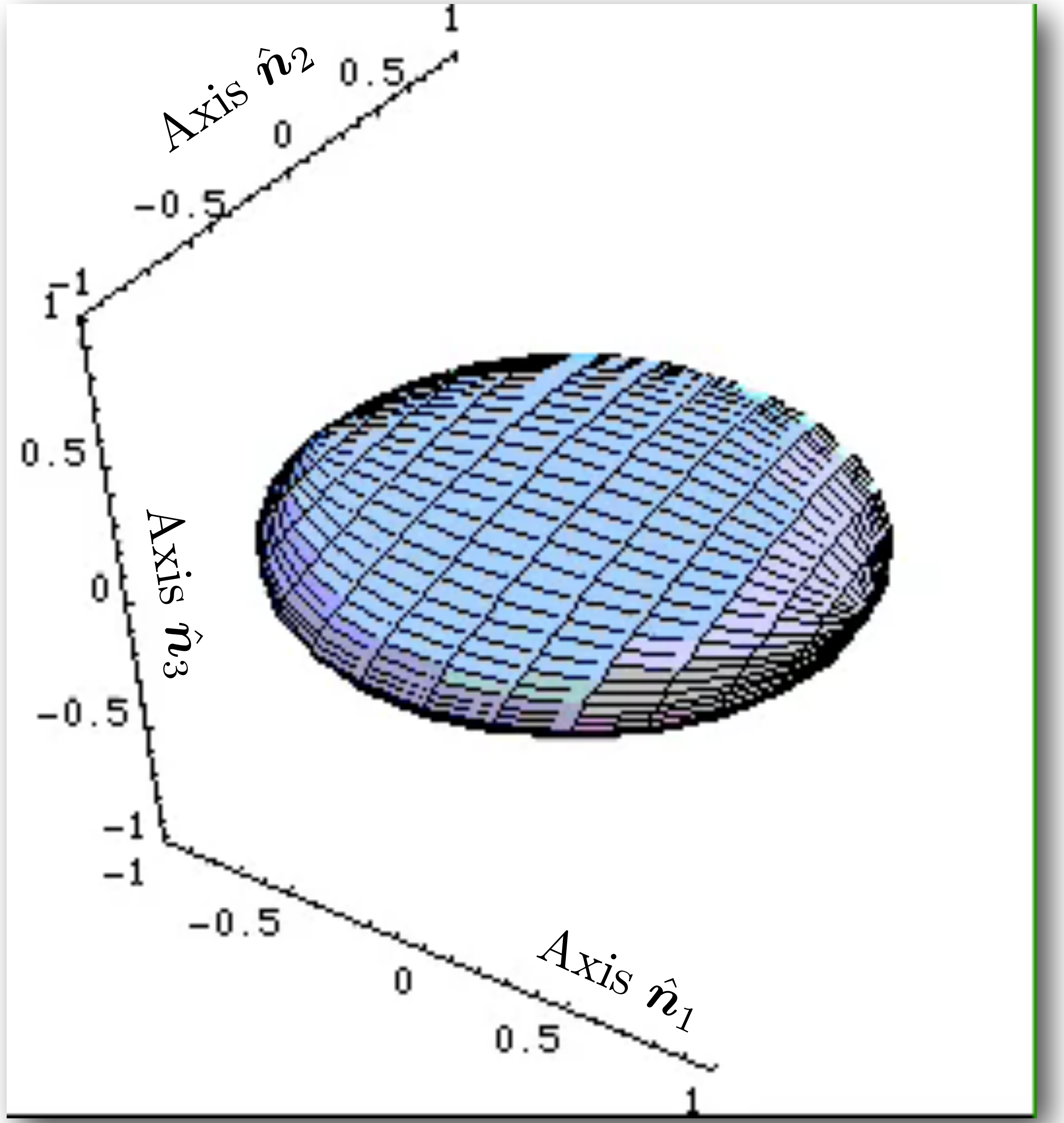
$$\omega_1(t) = \omega_{1_0} \cos \omega_p t - \omega_{2_0} \sin \omega_p t$$
$$\omega_2(t) = \omega_{2_0} \cos \omega_p t + \omega_{1_0} \sin \omega_p t$$
$$\omega_3(t) = \omega_{3_0}$$



The first and second body angular velocity components are sinusoidal in nature.

As predicted, the third body angular velocity component remains constant here.


Yaw angle ψ [deg]
250
200
150
100
50
-50
50
100
150
200
Pitch angle θ [deg]
60
40
20
-20
-40
-60
-80
50
100
150
200
Roll angle ϕ [deg]
800
600
400
200
-200
50
100
150
200


Axis $\hat{\boldsymbol{n}}_2$
Axis $\hat{\boldsymbol{n}}_3$
Axis $\hat{\boldsymbol{n}}_1$
1
0.5
0
-0.5
-1

General Inertia Case*

$$H^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2$$

$$2T = I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2$$

Momentum magnitude and kinetic energy conservation yield two integrals of the torque-free motion.

$$\omega_2^2 = \left(\frac{2I_3T - H^2}{I_2(I_3 - I_2)}\right) - \frac{I_1(I_3 - I_1)}{I_2(I_3 - I_2)}\omega_1^2$$

$$\omega_3^2 = \left(\frac{2I_2T - H^2}{I_3(I_2 - I_3)}\right) - \frac{I_1(I_2 - I_1)}{I_3(I_2 - I_3)}\omega_1^2$$

We can use these two equations to solve for two of the angular rates!

Analogously, we can solve for the two angular velocities in terms of other angular rates.

$$\omega_1^2 = \left(\frac{2I_3T - H^2}{I_1(I_3 - I_1)}\right) - \frac{I_2(I_3 - I_2)}{I_1(I_3 - I_1)}\omega_2^2$$

$$\omega_3^2 = \left(\frac{2I_1T - H^2}{I_3(I_1 - I_3)}\right) - \frac{I_2(I_1 - I_2)}{I_3(I_1 - I_3)}\omega_2^2$$

$$\omega_1^2 = \left(\frac{2I_2T - H^2}{I_1(I_2 - I_1)}\right) - \frac{I_3(I_2 - I_3)}{I_1(I_2 - I_1)}\omega_3^2$$

$$\omega_2^2 = \left(\frac{2I_1T - H^2}{I_2(I_1 - I_2)}\right) - \frac{I_3(I_1 - I_3)}{I_2(I_1 - I_2)}\omega_3^2$$

* Junkins, J. L., Jacobson, I. D., and Blanton, J. N., "A Nonlinear Oscillator Analog of Rigid Body Dynamics," *Celestial Mechanics*, Vol. 7, pp. 398 – 407, 1973.

$$\boxed{\begin{aligned} I_1\dot{\omega}_1 &= -(I_3 - I_2)\omega_2\omega_3 \\ I_2\dot{\omega}_2 &= -(I_1 - I_3)\omega_3\omega_1 \\ I_3\dot{\omega}_3 &= -(I_2 - I_1)\omega_1\omega_2 \end{aligned}}$$

$$\xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}}$$

$$\begin{aligned} \ddot{\omega}_1 &= \frac{I_2 - I_3}{I_1}\left(\boxed{\dot{\omega}_2}\omega_3 + \omega_2\boxed{\dot{\omega}_3}\right) \\ \ddot{\omega}_2 &= \frac{I_3 - I_1}{I_2}\left(\boxed{\dot{\omega}_3}\omega_1 + \omega_3\boxed{\dot{\omega}_1}\right) \\ \ddot{\omega}_3 &= \frac{I_1 - I_2}{I_3}\left(\boxed{\dot{\omega}_1}\omega_2 + \omega_1\boxed{\dot{\omega}_2}\right) \end{aligned}$$

$$\begin{aligned} \ddot{\omega}_1 &= \frac{I_2 - I_3}{I_1}\left(\frac{I_1 - I_2}{I_3}\omega_1\omega_2^2 + \frac{I_3 - I_1}{I_2}\omega_1\omega_3^2\right) \\ \ddot{\omega}_2 &= \frac{I_3 - I_1}{I_2}\left(\frac{I_1 - I_2}{I_3}\omega_2\omega_1^2 + \frac{I_2 - I_3}{I_1}\omega_2\omega_3^2\right) \\ \ddot{\omega}_3 &= \frac{I_1 - I_2}{I_3}\left(\frac{I_3 - I_1}{I_2}\omega_3\omega_1^2 + \frac{I_2 - I_3}{I_1}\omega_3\omega_2^2\right) \end{aligned}$$

$$\ddot{\omega}_1 = \frac{I_2 - I_3}{I_1}\left(\frac{I_1 - I_2}{I_3}\omega_1 \boxed{\omega_2^2} + \frac{I_3 - I_1}{I_2}\omega_1 \boxed{\omega_3^2}\right)$$

$$\ddot{\omega}_2 = \frac{I_3 - I_1}{I_2}\left(\frac{I_1 - I_2}{I_3}\omega_2 \boxed{\omega_1^2} + \frac{I_2 - I_3}{I_1}\omega_2 \boxed{\omega_3^2}\right)$$

$$\ddot{\omega}_3 = \frac{I_1 - I_2}{I_3}\left(\frac{I_3 - I_1}{I_2}\omega_3 \boxed{\omega_1^2} + \frac{I_2 - I_3}{I_1}\omega_3 \boxed{\omega_2^2}\right)$$

$$\omega_2^2 = \left(\frac{2I_3T - H^2}{I_2(I_3 - I_2)}\right) - \frac{I_1(I_3 - I_1)}{I_2(I_3 - I_2)}\omega_1^2$$

$$\omega_3^2 = \left(\frac{2I_2T - H^2}{I_3(I_2 - I_3)}\right) - \frac{I_1(I_2 - I_1)}{I_3(I_2 - I_3)}\omega_1^2$$

$$\omega_1^2 = \left(\frac{2I_3T - H^2}{I_1(I_3 - I_1)}\right) - \frac{I_2(I_3 - I_2)}{I_1(I_3 - I_1)}\omega_2^2$$

$$\omega_3^2 = \left(\frac{2I_1T - H^2}{I_3(I_1 - I_3)}\right) - \frac{I_2(I_1 - I_2)}{I_3(I_1 - I_3)}\omega_2^2$$

$$\omega_1^2 = \left(\frac{2I_2T - H^2}{I_1(I_2 - I_1)}\right) - \frac{I_3(I_2 - I_3)}{I_1(I_2 - I_1)}\omega_3^2$$

$$\omega_2^2 = \left(\frac{2I_1T - H^2}{I_2(I_1 - I_2)}\right) - \frac{I_3(I_1 - I_3)}{I_2(I_1 - I_2)}\omega_3^2$$

$$\ddot{\omega}_i + A_i\omega_i + B_i\omega_i^3 = 0 \qquad \text{for } i = 1, 2, 3$$

homogenous, undamped Duffing equation

Duffing equations are often found studying nonlinear mechanical oscillations, where the cubic “stiffness” term arises to approximately account for nonlinear departure from Hooke’s law. For the torque-free motion, this equation is the *exact differential equation*!

$$\ddot{\omega}_i + A_i\omega_i + B_i\omega_i^3 = 0 \qquad \text{for } i = 1, 2, 3$$

- These equations form three *uncoupled nonlinear oscillators*.
- Notice that while the oscillators are *uncoupled*, they are not *independent*! The six spring constants are all uniquely determined from initially evaluated inertia, energy and momentum constants.

i	A_i	B_i
1	$\frac{(I_1 - I_2)(2I_3T - H^2) + (I_1 - I_3)(2I_2T - H^2)}{I_1 I_2 I_3}$	$\frac{2(I_1 - I_2)(I_1 - I_3)}{I_2 I_3}$
2	$\frac{(I_2 - I_3)(2I_1T - H^2) + (I_2 - I_1)(2I_3T - H^2)}{I_1 I_2 I_3}$	$\frac{2(I_2 - I_1)(I_2 - I_3)}{I_1 I_3}$
3	$\frac{(I_3 - I_1)(2I_2T - H^2) + (I_3 - I_2)(2I_1T - H^2)}{I_1 I_2 I_3}$	$\frac{2(I_3 - I_1)(I_3 - I_2)}{I_1 I_2}$

- The oscillator differential equations have three immediate integrals of the form

$$\dot{\omega}_i^2 + A_i \omega_i^2 + \frac{B_i}{2}\omega_i^4 = K_i \qquad \text{for } i = 1, 2, 3$$

- Here K_1, K_2 and K_3 are the three oscillator "energy-type" integral constants of the motion.

$$K_1 = \frac{(2I_2T - H^2)(H^2 - 2I_3T)}{I_1^2 I_2 I_3}$$

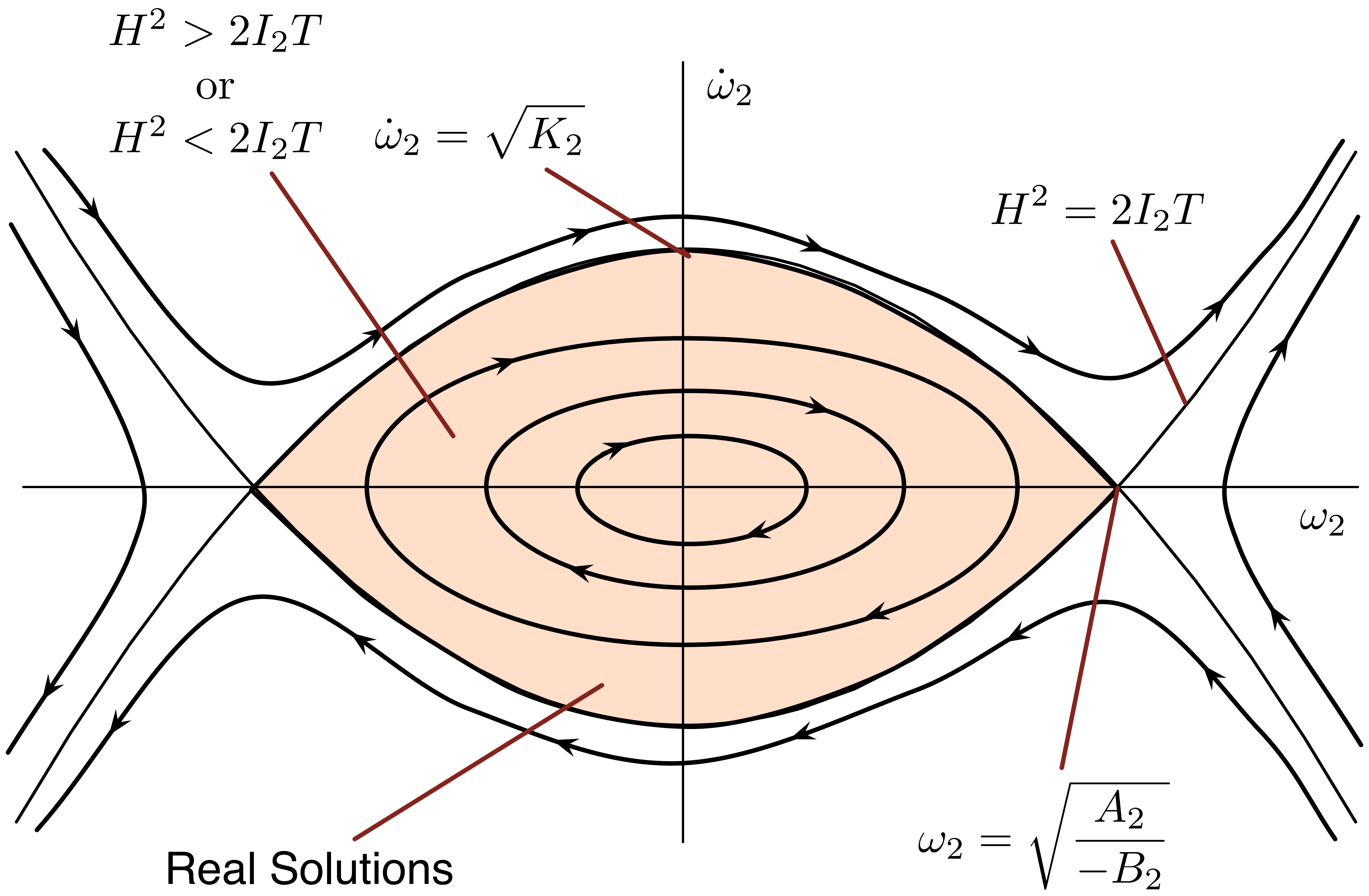
$$K_2 = \frac{(2I_3T - H^2)(H^2 - 2I_1T)}{I_1 I_2^2 I_3}$$

$$K_3 = \frac{(2I_1T - H^2)(H^2 - 2I_2T)}{I_1 I_2 I_3^2}$$

Assume: $I_1 \geq I_2 \geq I_3$

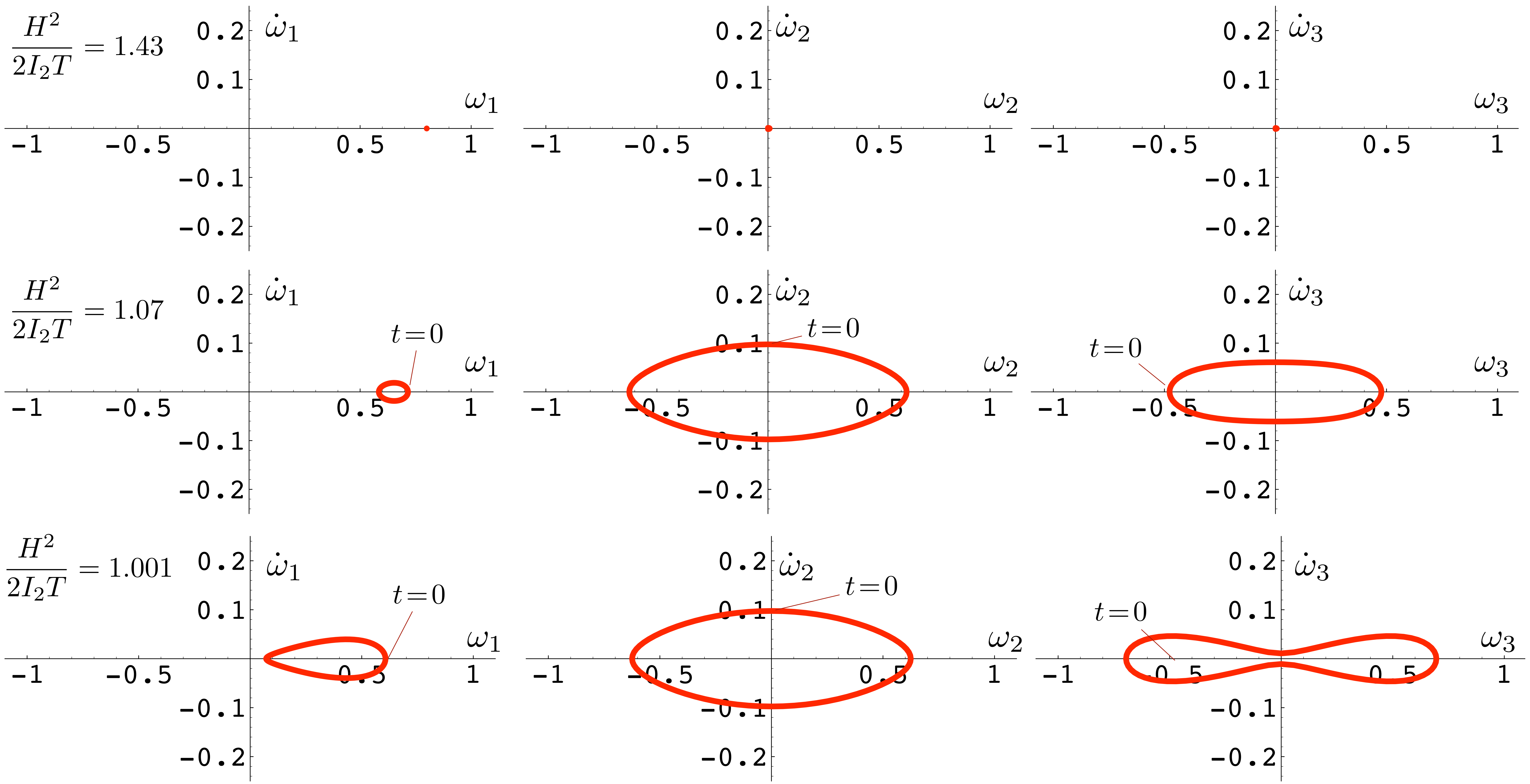
i	A_i	B_i
1	not defined	>0
2	>0	<0
3	not defined	>0

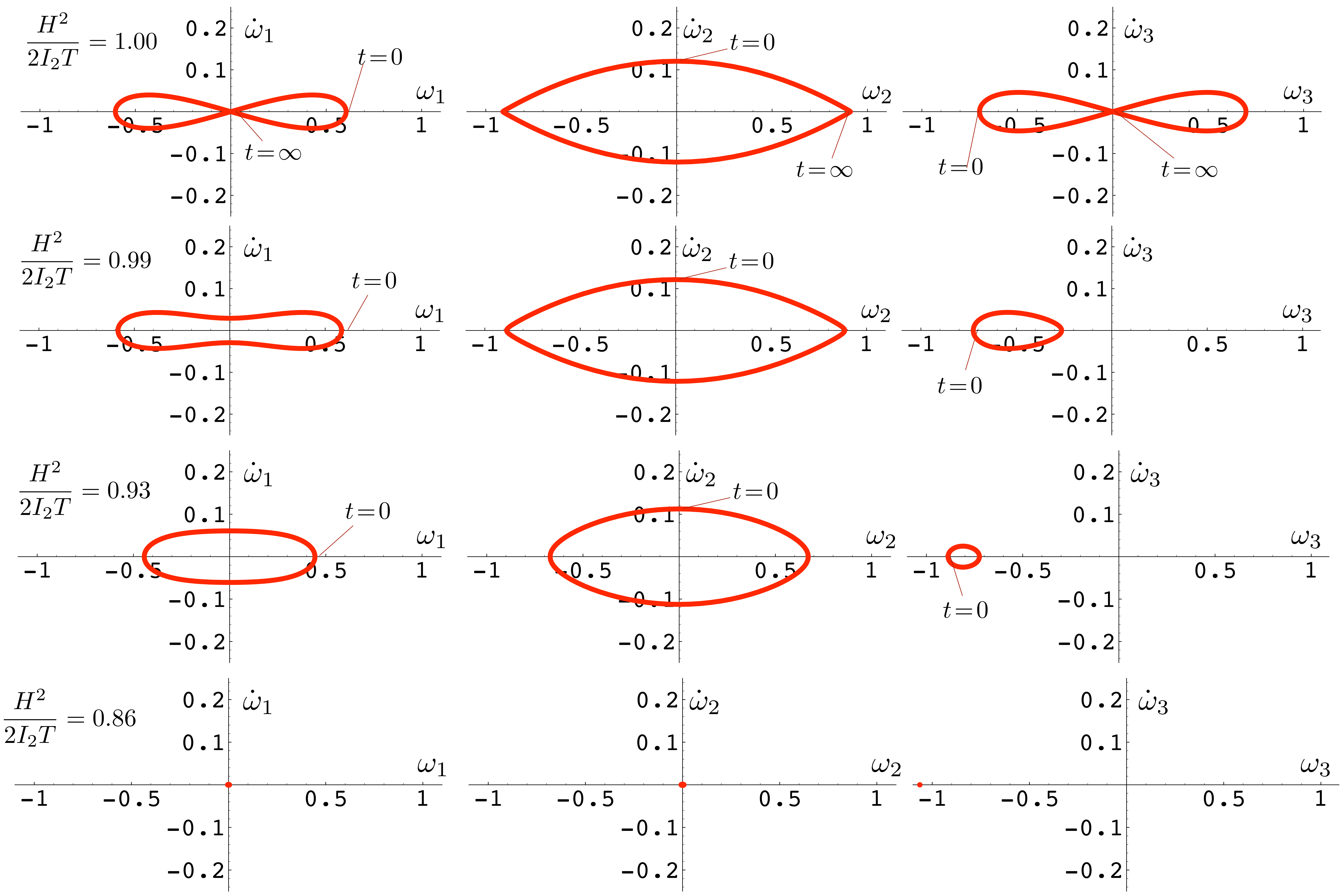
- The linear "spring constants" A_1 and A_3 can produce de-stabilizing spring forces (negative spring effect).
- The positive cubic "spring constants" B_1 and B_3 always produce restoring forces and are therefore hard springs. Because cubic springs will override linear springs for sufficiently large displacements, all trajectories of the 1st and 3rd phase planes must be closed.
- The cubic spring constant B_2 produces a de-stabilizing force (soft spring), and will eventually override the stabilizing linear spring force.



- Only solutions with $K_2 \geq 0$ are physically possible
- The limiting trajectory occurs if
  - $I_1 \rightarrow I_3$
  - $H^2 \rightarrow 2\, I_2 T$ (pure spin about intermediate inertia axis)
  - $I_1 I_2 I_3 \rightarrow \infty$

Let's sweep through cases from a minimum energy case to a maximum energy case. The momentum is held constant here.



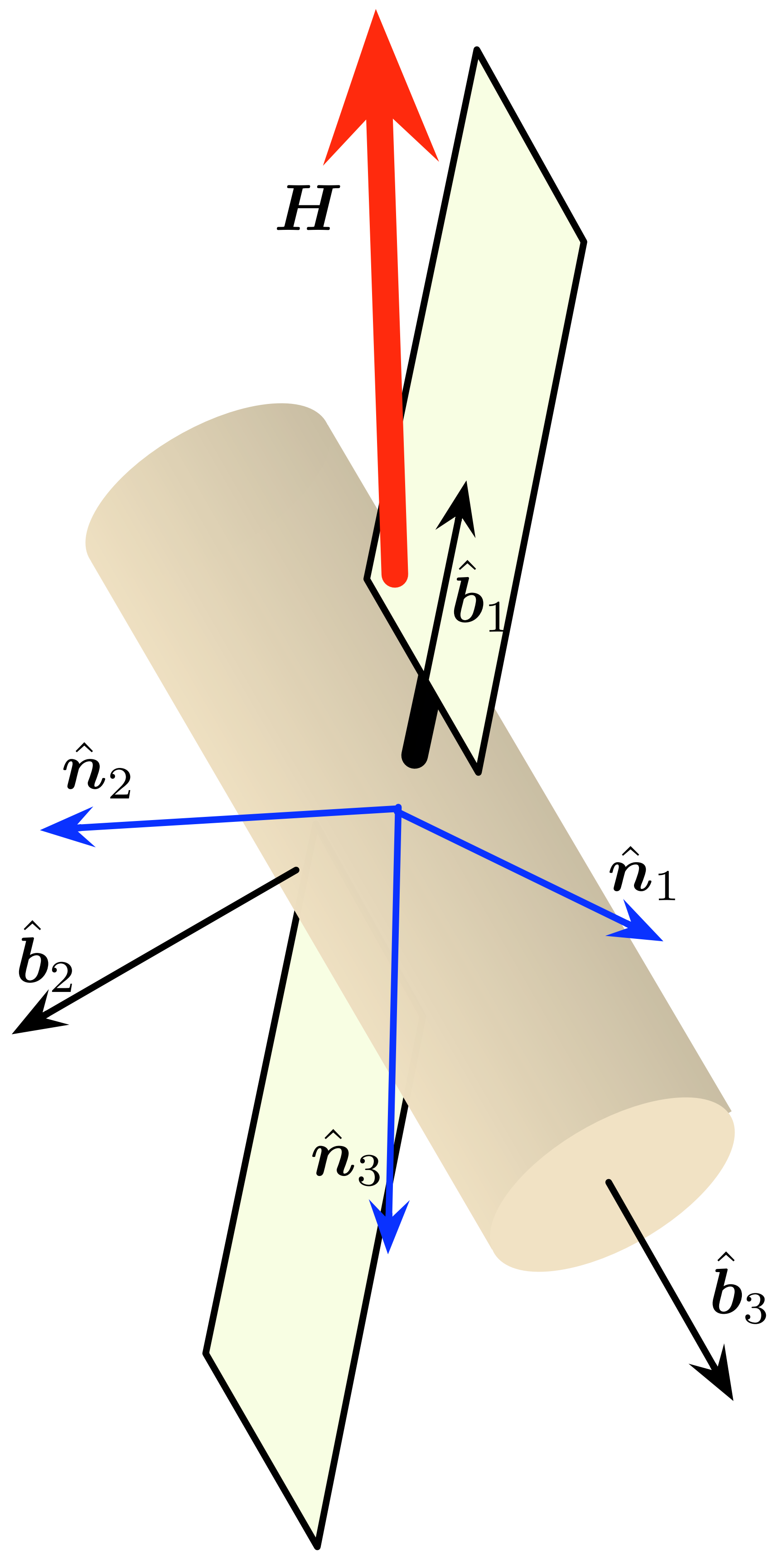


$\frac{H^2}{2I_2T} = 1.00$
$\frac{H^2}{2I_2T} = 0.99$
$\frac{H^2}{2I_2T} = 0.93$
$\frac{H^2}{2I_2T} = 0.86$
$\dot{\omega}_1$
ω_1
$\dot{\omega}_2$
ω_2
$\dot{\omega}_3$
ω_3
$t=0$
$t=\infty$

General Free Rotation

- We would like to study the general free rotation of a rigid body using the 3-2-1 Euler angles.
- Because the inertial angular momentum vector ***H*** is constant as seen by the inertial frame, we can always align our inertial frame such that
- $$\boldsymbol{H} = {}^{\mathcal{N}}\boldsymbol{H} = -H\hat{\boldsymbol{n}}_3 = {}^{\mathcal{N}}\begin{pmatrix} 0 \\ 0 \\ -H \end{pmatrix}$$
- Using the rotation matrix [*BN*], we find

$$^{\mathcal{B}}\boldsymbol{H} = [BN]\,{}^{\mathcal{N}}\boldsymbol{H}$$



- Recall the mapping between the rotation matrix [BN] and the 3-2-1 Euler angles:

$$[BN] = \begin{bmatrix} c\theta_2 c\theta_1 & c\theta_2 s\theta_1 & -s\theta_2 \\ s\theta_3 s\theta_2 c\theta_1 - c\theta_3 s\theta_1 & s\theta_3 s\theta_2 s\theta_1 + c\theta_3 c\theta_1 & s\theta_3 c\theta_2 \\ c\theta_3 s\theta_2 c\theta_1 + s\theta_3 s\theta_1 & c\theta_3 s\theta_2 s\theta_1 - s\theta_3 c\theta_1 & c\theta_3 c\theta_2 \end{bmatrix}$$

This leads to

$$^{\mathcal{B}}\boldsymbol{H} = \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} = [BN]\,^{\mathcal{N}}\boldsymbol{H} = \begin{pmatrix} H \sin\theta \\ -H \sin\phi \cos\theta \\ -H \cos\phi \cos\theta \end{pmatrix} = \begin{pmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{pmatrix}$$

Which can be solved for the rigid body angular velocity.

$$\begin{pmatrix} \frac{H}{I_1} \sin\theta \\ -\frac{H}{I_2} \sin\phi \cos\theta \\ -\frac{H}{I_3} \cos\phi \cos\theta \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

- Recall the 3-2-1 Euler angle differential kinematic equation:

$$\boldsymbol{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{bmatrix} -\sin\theta & 0 & 1 \\ \sin\phi\cos\theta & \cos\phi & 0 \\ \cos\phi\cos\theta & -\sin\phi & 0 \end{bmatrix} \begin{pmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Solving these equations for the Euler angle rates, we obtain:

$$\dot{\psi} = -H\left(\frac{\sin^2\phi}{I_2} + \frac{\cos^2\phi}{I_3}\right) \Rightarrow \text{cannot be positive}$$

$$\dot{\theta} = \frac{H}{2}\left(\frac{1}{I_3} - \frac{1}{I_2}\right)\sin 2\phi\cos\theta$$

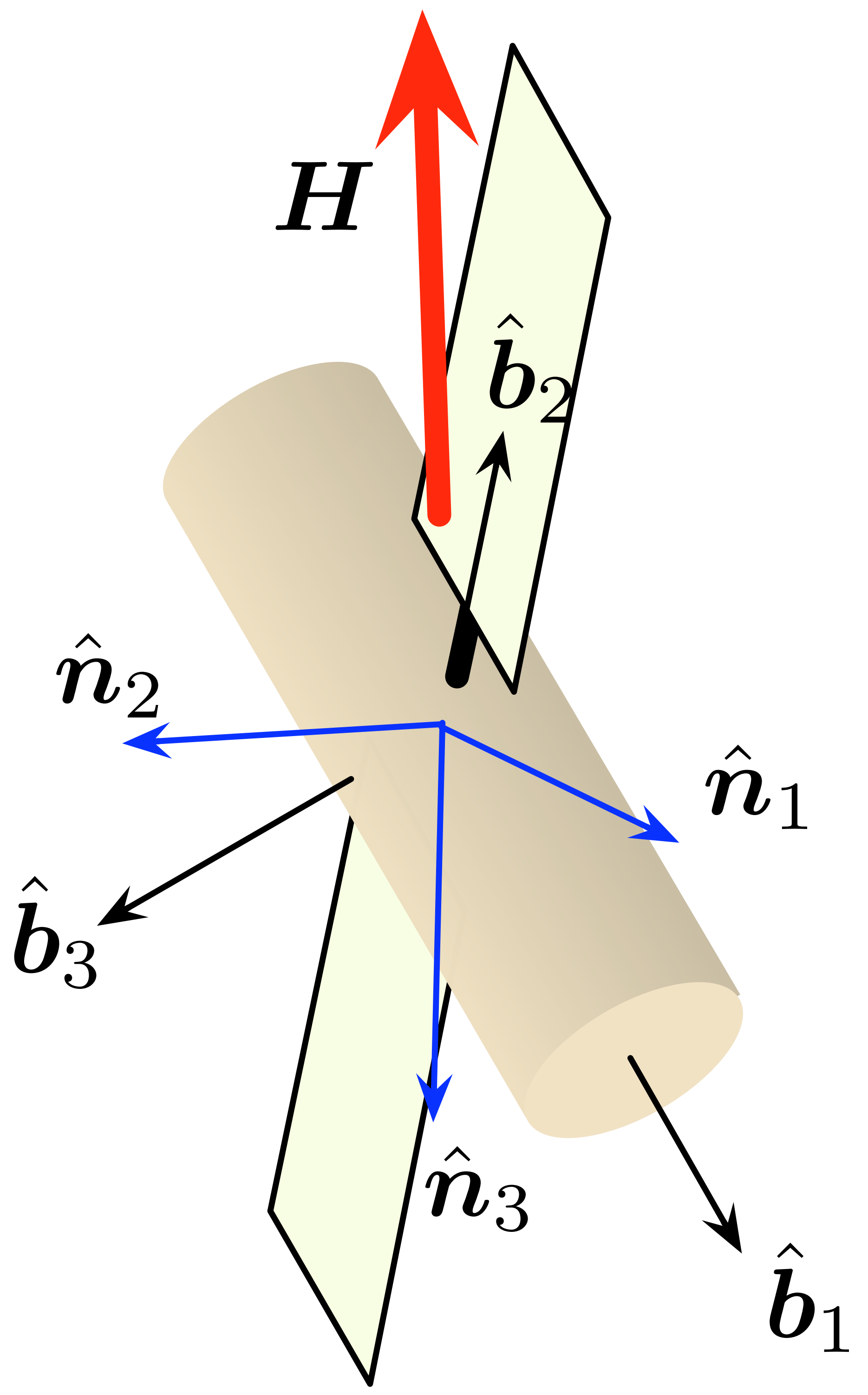
$$\dot{\phi} = H\left(\frac{1}{I_1} - \frac{\sin^2\phi}{I_2} - \frac{\cos^2\phi}{I_3}\right)\sin\theta$$

These are the spinning top equations of motion.

Axi-Symmetric Coning Motion

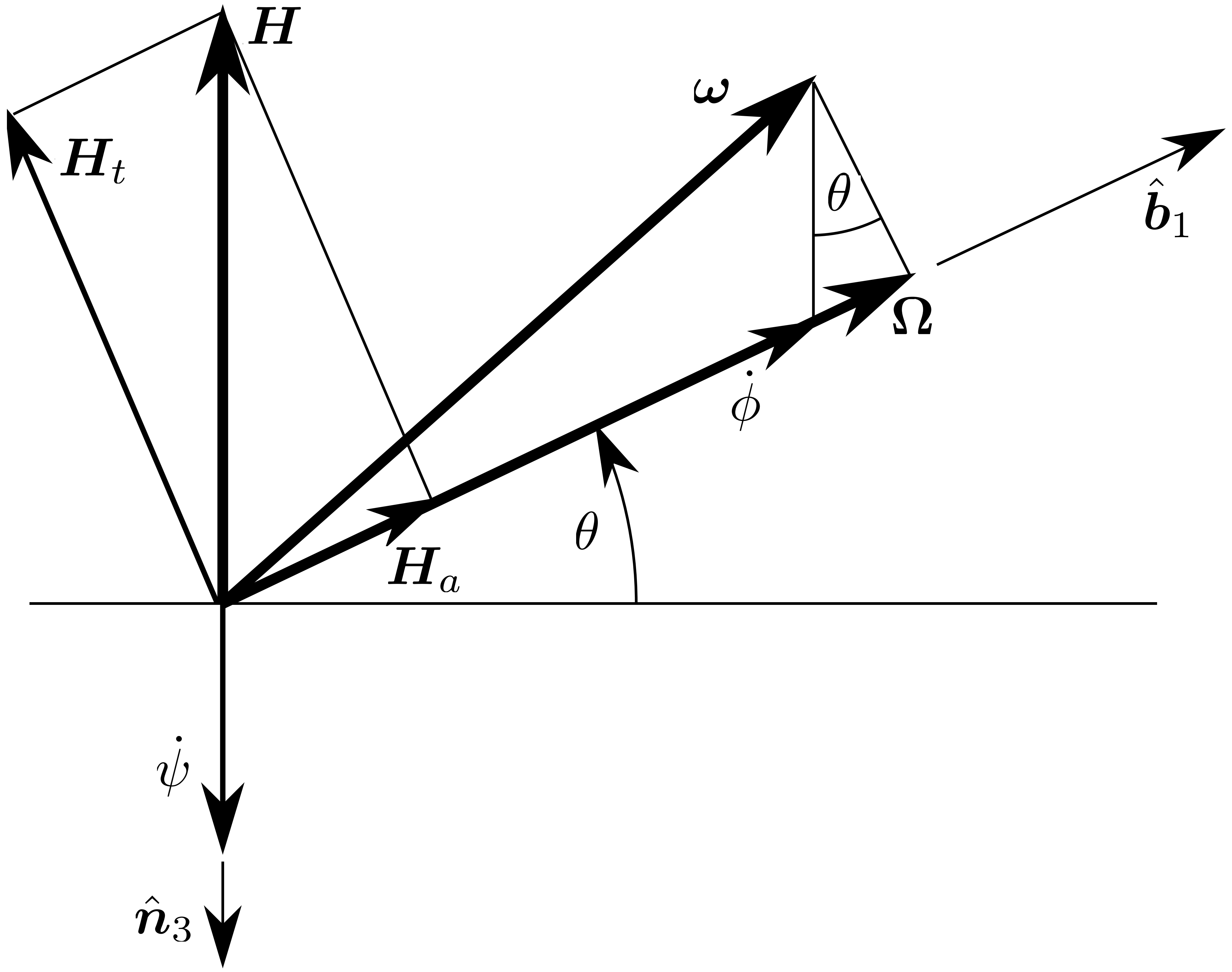
- Assume the spacecraft is axi-symmetric with $I_2 = I_3$, and align the inertial frame such that

$$\boldsymbol{H} = {}^{\mathcal{N}}\boldsymbol{H} = -H\hat{\boldsymbol{n}}_3 = {}^{\mathcal{N}}\begin{pmatrix} 0 \\ 0 \\ -H \end{pmatrix}$$



- The 3-2-1 Euler angle differential equation are then given by:

$$\dot{\psi} = -\frac{H}{I_2}$$
$$\dot{\theta} = 0$$
$$\dot{\phi} = H\left(\frac{I_2 - I_1}{I_1 I_2}\right)\sin\theta$$

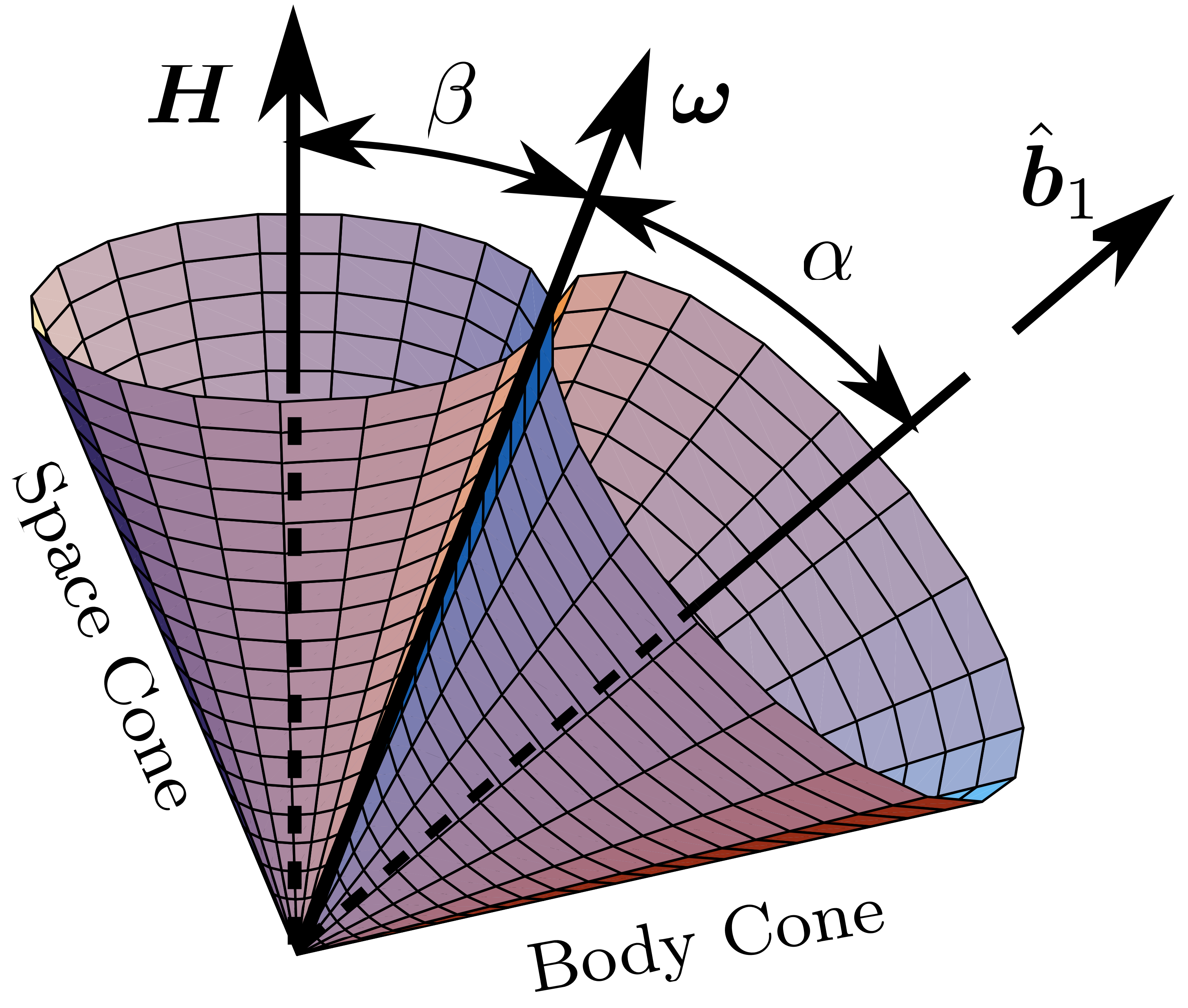


Let $\Omega = \omega_1 \longrightarrow \Omega = \dfrac{H}{I_1}\sin\theta$

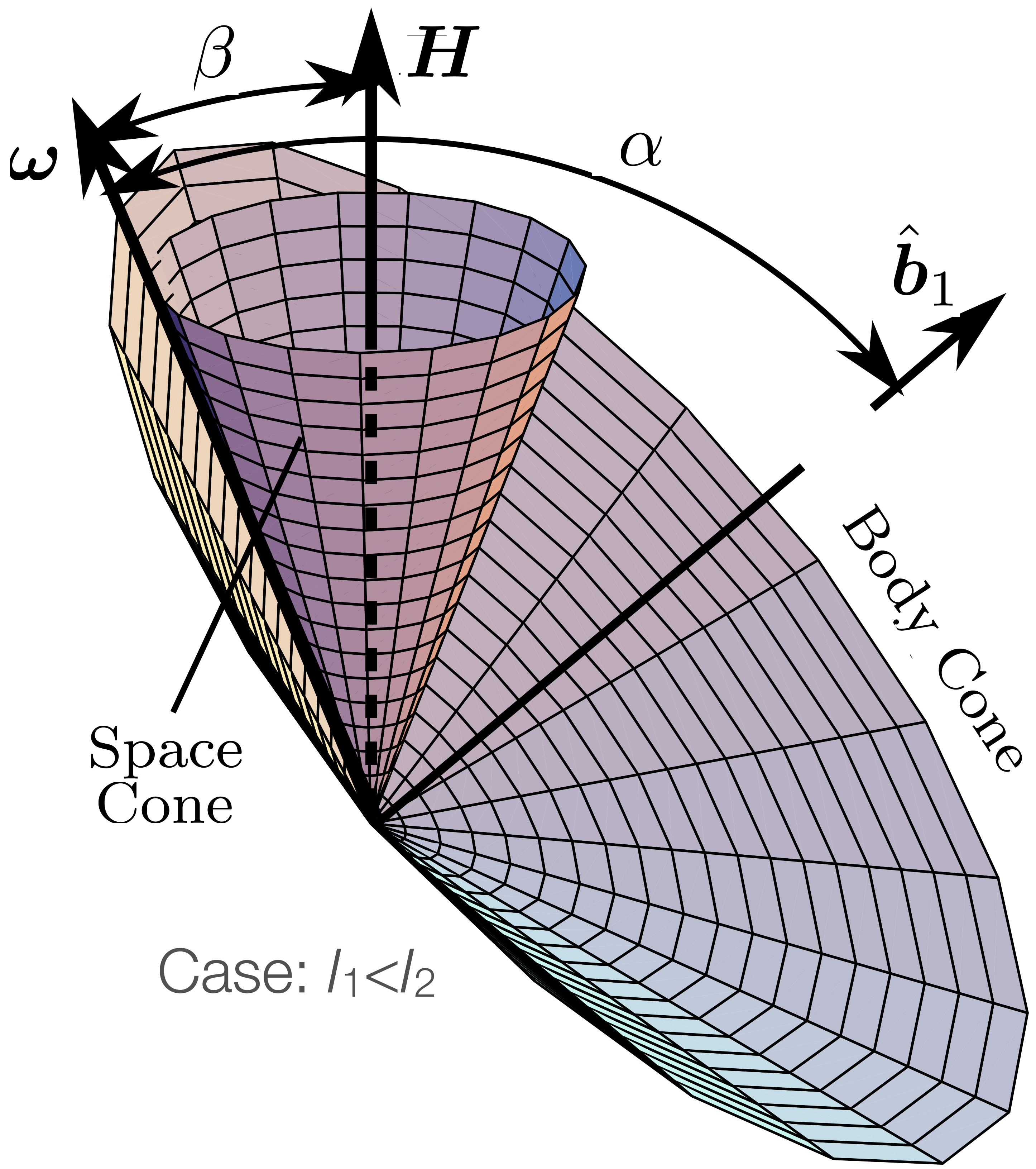
Note that for $0 \leq \theta \leq \pi/2$
we find that $\Omega > 0$

The EOM can be written as

$$\dot{\psi} = -\frac{I_1}{I_2}\frac{\Omega}{\sin\theta} \qquad \dot{\phi} = \frac{I_2 - I_1}{I_2}\Omega$$



Case: $I_1>I_2$



Case: $I_1<I_2$

Since the pitch angle θ is shown to remain constant during this torque-free rotation, the resulting motion can be visualized by two cones rolling on each other. The *space cone* is fixed in space and its cone axis is always aligned with the angular momentum vector ***H***. The cone angle β is defined as the angle between the vectors ***H*** and **ω**. The *body cone* axis is aligned with the first body axis and has the cone angle α which is the angle between **ω** and first body axis.