Classical Rodrigues Parameters (Gibbs Vector or CRPs)

Popular coordinates for large rotations and robotics....

CRP Definitions

Euler parameter relationship:

$$q_i = \frac{\beta_i}{\beta_0} \qquad i = 1, 2, 3$$

Singular if 0
(±180° case)

$$\beta_0 = \frac{1}{\sqrt{1 + \boldsymbol{q}^T \boldsymbol{q}}}$$

$$\beta_i = \frac{q_i}{\sqrt{1 + \boldsymbol{q}^T \boldsymbol{q}}} \qquad i = 1, 2, 3$$

Singular if ∞
(±180° case)

Principal rotation parameter relationship:

$$\boldsymbol{q} = \tan \frac{\Phi}{2} \hat{\boldsymbol{e}}$$

Singular for ±180°

$$\boldsymbol{q} \approx \frac{\Phi}{2} \hat{\boldsymbol{e}}$$

Linearizes to angles over 2.

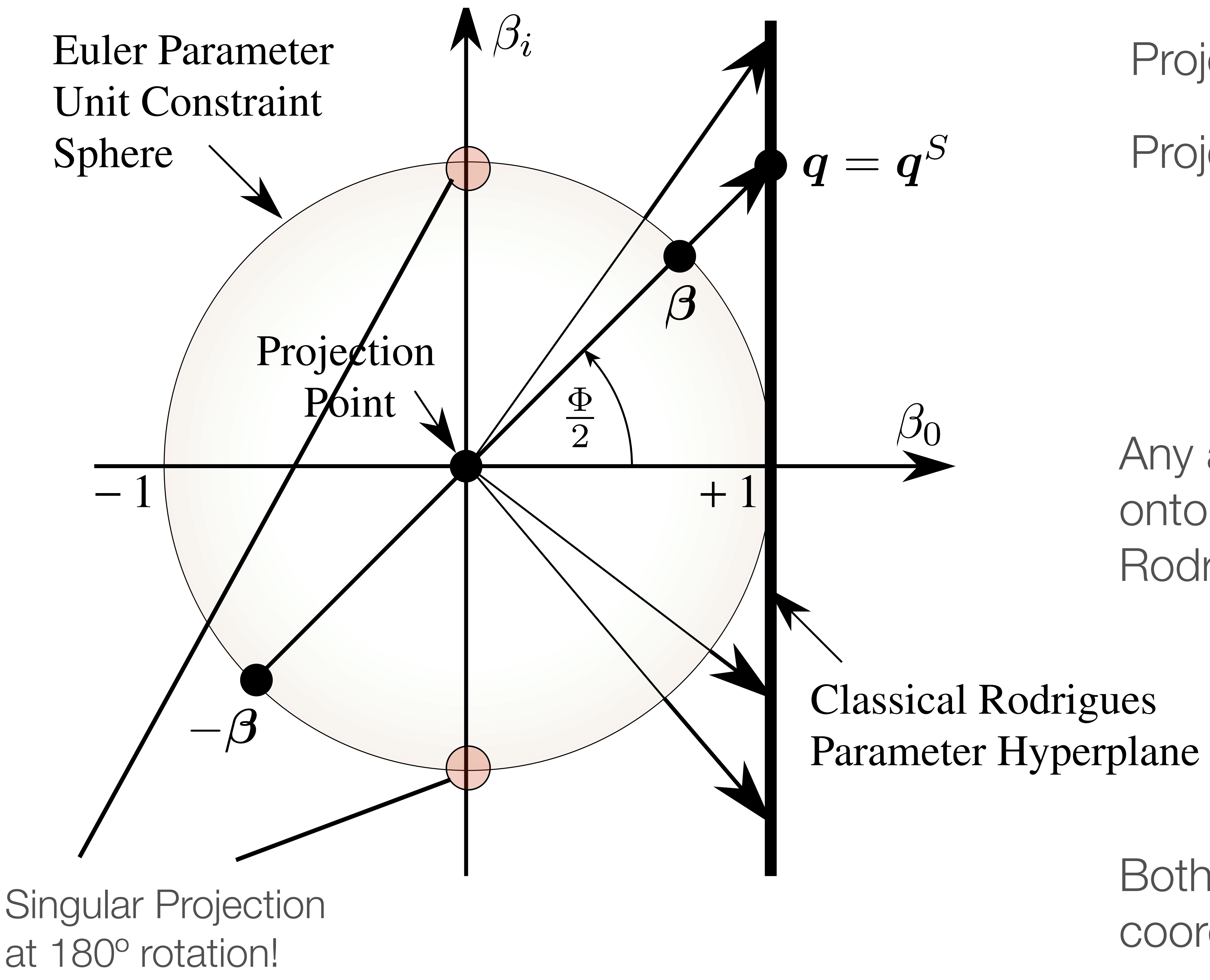
These parameters are much better suited for large spacecraft rotations than Euler angles, while remaining a minimal coordinate set.
Only the upside down description is singular.

CRP Definitions

Relationship to DCM: $$[\tilde{\boldsymbol{q}}] = \frac{[C]^T - [C]}{\zeta^2}$$ $$\zeta = \sqrt{\text{trace}([C]) + 1} = 2\beta_0$$

$$\boldsymbol{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \frac{1}{\zeta^2} \begin{pmatrix} C_{23} - C_{32} \\ C_{31} - C_{13} \\ C_{12} - C_{21} \end{pmatrix}$$

Stereographic Projection

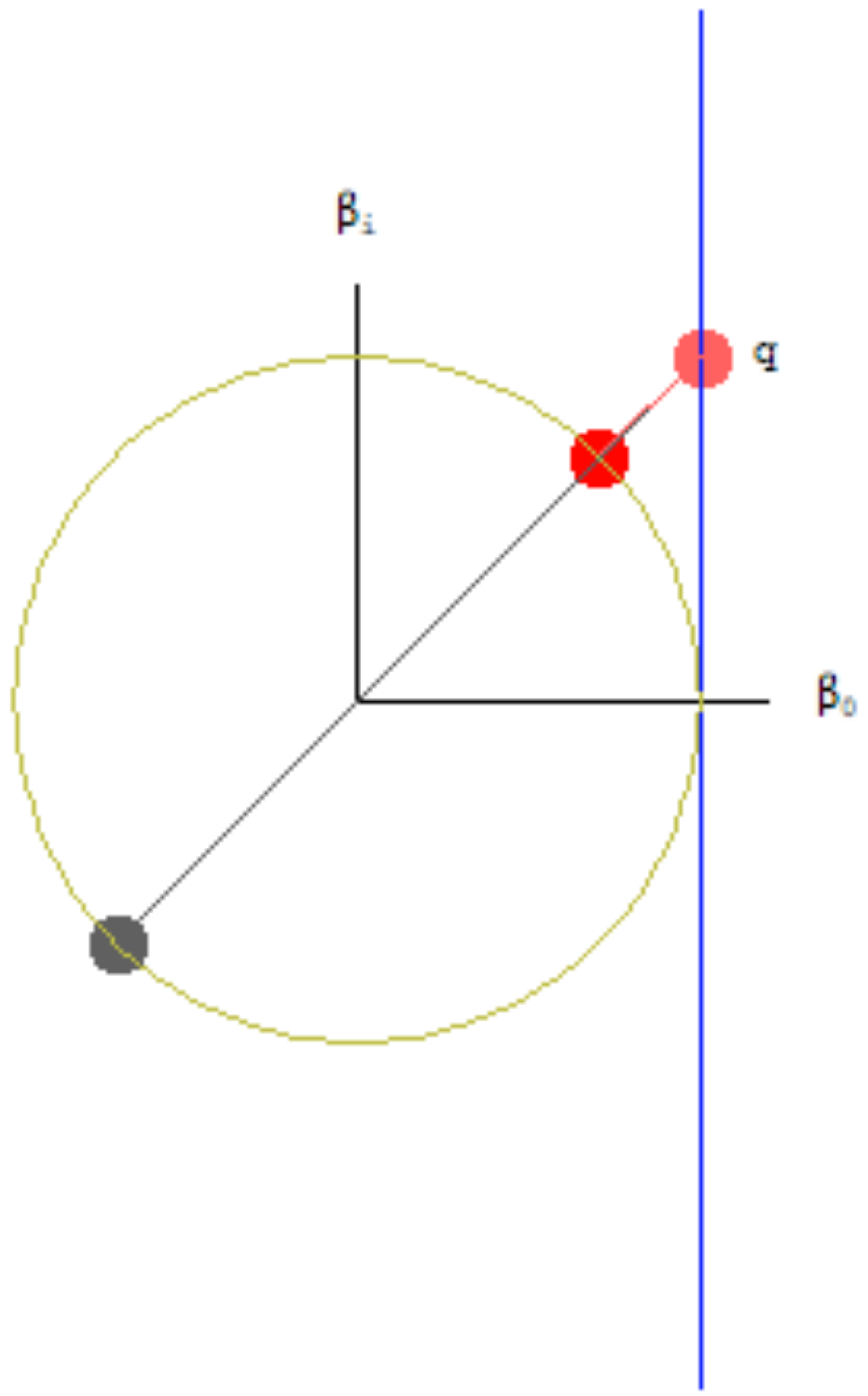


Projection Point: $(0, 0, 0, 0)$

Projection Plane: $\beta_0 = +1$

Any attitude (surface point) is projected onto the hyper-plane to form the classical Rodrigues parameters.

Both EP sets yield the identical CRP coordinates.



http://hanspeterschaub.info/crp.html

Shadow CRP Set

- Using the alternate set of Euler parameters, we can find the “shadow” set of CRP parameters:

$$q_i^S = \frac{-\beta_i}{-\beta_0} = q_i$$

- For the case of CRPs, the shadow set and the original set of attitude parameters are identical. Thus, the shadow set cannot be used to avoid the 180º singularity.

Direction Cosine Matrix

Matrix components:

$$[C] = \frac{1}{1+\boldsymbol{q}^T\boldsymbol{q}} \begin{bmatrix} 1+q_1^2-q_2^2-q_3^2 & 2(q_1q_2+q_3) & 2(q_1q_3-q_2) \\ 2(q_2q_1-q_3) & 1-q_1^2+q_2^2-q_3^2 & 2(q_2q_3+q_1) \\ 2(q_3q_1+q_2) & 2(q_3q_2-q_1) & 1-q_1^2-q_2^2+q_3^2 \end{bmatrix}$$

$$[C] = \frac{1}{1+\boldsymbol{q}^T\boldsymbol{q}} \left(\left(1-\boldsymbol{q}^T\boldsymbol{q}\right)[I_{3\times3}] + 2\boldsymbol{q}\boldsymbol{q}^T - 2[\tilde{\boldsymbol{q}}] \right)$$

$$[C(\boldsymbol{q})]^{-1} = [C(\boldsymbol{q})]^T = [C(-\boldsymbol{q})]$$

Attitude Addition/Subtraction

- DCM method:

$$[FN(\boldsymbol{q})] = [FB(\boldsymbol{q}'')][BN(\boldsymbol{q}')]$$

- Direct method:

$$\boldsymbol{q} = \frac{\boldsymbol{q}'' + \boldsymbol{q}' - \boldsymbol{q}'' \times \boldsymbol{q}'}{1 - \boldsymbol{q}'' \cdot \boldsymbol{q}'}$$

Attitude Addition

$$\boldsymbol{q}'' = \frac{\boldsymbol{q} - \boldsymbol{q}' + \boldsymbol{q} \times \boldsymbol{q}'}{1 + \boldsymbol{q} \cdot \boldsymbol{q}'}$$

Relative Attitude (Subtraction)

Note: Using $\delta\boldsymbol{q} = \boldsymbol{q} - \boldsymbol{q}'$ to compute the relative attitude, or attitude error, still yields a result that is a proper CRP attitude measure. However, also note that the approximation $\delta\boldsymbol{q} \approx \boldsymbol{q}''$ only holds for small attitude differences.

Differential Kinematic Equations

Matrix components:

$$\dot{\boldsymbol{q}} = \frac{1}{2} \begin{bmatrix} 1+q_1^2 & q_1q_2 - q_3 & q_1q_3 + q_2 \\ q_2q_1 + q_3 & 1+q_2^2 & q_2q_3 - q_1 \\ q_3q_1 - q_2 & q_3q_2 + q_1 & 1+q_3^2 \end{bmatrix} {}^{\mathcal{B}}\!\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

Vector computation:

$$\dot{\boldsymbol{q}} = \frac{1}{2}\left[[I_{3\times3}] + [\tilde{\boldsymbol{q}}] + \boldsymbol{q}\boldsymbol{q}^T\right] {}^{\mathcal{B}}\boldsymbol{\omega}$$

$${}^{\mathcal{B}}\boldsymbol{\omega} = \frac{2}{1+\boldsymbol{q}^T\boldsymbol{q}}\left([I_{3\times3}] - [\tilde{\boldsymbol{q}}]\right)\dot{\boldsymbol{q}}$$

Note: Only contains quadratic nonlinearities, but is singular for $\Phi = 180^\circ$.

Cayley Transform

- Amazingly elegant matrix transformation, that allows us to use attitude parameters in higher dimensional spaces. ☺️

- Let [*Q*] be a skew-symmetric matrix, [*C*] be a proper orthogonal matrix, and [*I*] be a identity matrix. These matrices can be of any dimension *N*. The Cayley Transform is then defined as:

$$\boxed{[C] = ([I] - [Q])\left([I] + [Q]\right)^{-1} = \left([I] + [Q]\right)^{-1}([I] - [Q])}$$

$$[Q] = ([I] - [C])\left([I] + [C]\right)^{-1} = \left([I] + [C]\right)^{-1}([I] - [C])$$

Note: Both the forward and backwards mapping between [*Q*] and [*C*] has the same algebraic form!

Example:

- For 3D space, the proper orthogonal [*C*] matrix is also a rotation or direction cosine matrix. In this case we find that

$$[Q] = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix}$$

where the unique matrix elements are the CRP!

$$[C] = \begin{bmatrix} 0.813797 & 0.296198 & -0.5 \\ 0.235888 & 0.617945 & 0.75 \\ 0.531121 & -0.728292 & 0.433012 \end{bmatrix}$$

$$\Downarrow$$

$$[Q] = \begin{bmatrix} 0 & -0.021052 & 0.359933 \\ 0.021052 & 0 & -0.516027 \\ -0.359933 & 0.516027 & 0 \end{bmatrix} \Rightarrow \boldsymbol{q} = \begin{pmatrix} 0.516027 \\ 0.359933 \\ 0.021052 \end{pmatrix}$$

CRP vector

- Higher Dimensional Example:

$$[C] = \begin{bmatrix} 0.505111 & -0.503201 & -0.215658 & 0.667191 \\ 0.563106 & -0.034033 & -0.538395 & -0.626006 \\ 0.560111 & 0.748062 & 0.272979 & 0.228387 \\ -0.337714 & 0.431315 & -0.767532 & 0.332884 \end{bmatrix}$$

$$[Q] = \begin{bmatrix} 0 & 0.5 & 0.2 & -0.3 \\ -0.5 & 0 & 0.7 & -0.6 \\ -0.2 & -0.7 & 0 & -0.4 \\ 0.3 & -0.6 & 0.4 & 0 \end{bmatrix}$$

4D space CRP

Note: The *N*-dimensional proper orthogonal matrices can be parameterized with higher dimensional attitude coordinates.

That's nice, but is there also a higher dimensional equivalent to the differential kinematic equations to solve [*Q(t)*]?

- Recall that regardless of the dimensionality of the orthogonal matrix [*C*(*t*)], it must evolve according to:

$$[\dot{C}] = -[\tilde{\boldsymbol{\omega}}][C]$$

- These higher-dimensional “body angular velocities” can be related to the higher dimensional CRPs using:

$$[\dot{Q}] = \frac{1}{2}\left([I] + [Q]\right)[\tilde{\boldsymbol{\omega}}]\left([I] - [Q]\right)$$
$$[\tilde{\boldsymbol{\omega}}] = 2\left([I] + [Q]\right)^{-1}[\dot{Q}]\left([I] - [Q]\right)^{-1}$$

- Thus, can solve for the [*C*(*t*)] using a reduced coordinate set.

- This parameterization is singular whenever a principal rotation of 180º is performed.

- **Physical Example:**

Consider a typical mechanical system. The EOM can be written in the form

$$[M(\boldsymbol{x}, t)]\ddot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)$$

To solve this system for the state accelerations, the system positive definite system mass matrix must be inverted, a numerically expensive operations for large dimensions.

This inverse could be avoided by using the spectral decomposition:

$$[M] = [V][D][V]^T \qquad\qquad [M]^{-1} = [V]^T[D]^{-1}[V]$$

where [*V*] is a proper orthogonal eigenvector matrix and [*D*] is a diagonal eigenvalue matrix. To determine [*V*(*t*)] the Cayley transform could be used to track a reduced parameter set:

$$[Q] = ([I] - [V])([I] + [V])^{-1}$$