Nonlinear Feedback

Finally, we look at the complete 3-axis control of spacecraft attitude...

Unconstrained Control

- First let us assume that the external control (thrusters) is unconstrained in magnitude, and that the thruster can point in any direction.

External torque
(atmospheric torque)

EOM:

$$[I]\dot{\boldsymbol{\omega}} = -[\tilde{\omega}][I]\boldsymbol{\omega} + \boldsymbol{u} + \boldsymbol{L}$$

control vector
(thrusters)

Goal:

$$\delta\boldsymbol{\omega} = \boldsymbol{\omega} - \boldsymbol{\omega}_r \rightarrow 0 \qquad \boldsymbol{\sigma} \rightarrow 0$$

angular velocity
error

body angular
velocity

reference angular
velocity

Attitude error between body
frame *B* and reference frame *R*
using MRPs

Exact attitude tracking error kinematic differential equations:

$$\dot{\boldsymbol{\sigma}} = \frac{1}{4}\left[(1-\sigma^2)I + 2[\tilde{\boldsymbol{\sigma}}] + 2\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right]\delta\boldsymbol{\omega}$$

Lyapunov function definition:

$$V(\delta\boldsymbol{\omega},\boldsymbol{\sigma}) = \underbrace{\frac{1}{2}\delta\boldsymbol{\omega}^T[I]\delta\boldsymbol{\omega}}_{\text{kinetic-energy-like p.d. function}} + \underbrace{2K\ln\left(1+\boldsymbol{\sigma}^T\boldsymbol{\sigma}\right)}_{\text{p.d. MRP attitude error function}}$$

Note that the angular rate and inertia components are taken with respect to the body frame.

$$\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}([I]) = 0 \qquad \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\delta\boldsymbol{\omega})$$

To guarantee stability, we force $\dot{V}$ to be **negative semi-definite** by setting it equal to

$$\dot{V} = -\delta\boldsymbol{\omega}^T[P]\delta\boldsymbol{\omega}$$

$$[P] = [P]^T > 0$$

Differentiating V we find:

$$\dot{V} = \delta\boldsymbol{\omega}^T\left([I]\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\delta\boldsymbol{\omega}) + K\boldsymbol{\sigma}\right) = -\delta\boldsymbol{\omega}^T[P]\delta\boldsymbol{\omega}$$

$$\boxed{[I]\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\delta\boldsymbol{\omega}) + [P]\delta\boldsymbol{\omega} + K\boldsymbol{\sigma} = 0}$$

closed-loop dynamics

Using $\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\delta\boldsymbol{\omega}) = \dot{\boldsymbol{\omega}} - \dot{\boldsymbol{\omega}}_r + \boldsymbol{\omega}\times\boldsymbol{\omega}_r$ yields

$$\boxed{[I](\dot{\boldsymbol{\omega}}} - \dot{\boldsymbol{\omega}}_r + \boldsymbol{\omega}\times\boldsymbol{\omega}_r) + [P](\boldsymbol{\omega}-\boldsymbol{\omega}_r) + K\boldsymbol{\sigma} = 0$$

Substitute EOM: $\boxed{[I]\dot{\boldsymbol{\omega}}} = -[\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} + \boldsymbol{u} + \boldsymbol{L}$

$$\boxed{\begin{aligned}\boldsymbol{u} = -K\boldsymbol{\sigma} - [P]\delta\boldsymbol{\omega} + [I]\left(\dot{\boldsymbol{\omega}}_r - [\tilde{\boldsymbol{\omega}}]\boldsymbol{\omega}_r\right)\\ + [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} - \boldsymbol{L}\end{aligned}}$$

$${}^{\mathcal{B}}\boldsymbol{\omega}_r = [BR]\,{}^{\mathcal{R}}\boldsymbol{\omega}_r \qquad {}^{\mathcal{B}}\dot{\boldsymbol{\omega}}_r = [BR]\,{}^{\mathcal{R}}\dot{\boldsymbol{\omega}}_r$$

Global Stability?

The feedback law found is of the form

$$\boldsymbol{u} = -K\boldsymbol{\sigma} - [P]\delta\boldsymbol{\omega} + [I]\left(\dot{\boldsymbol{\omega}}_r - [\tilde{\boldsymbol{\omega}}]\boldsymbol{\omega}_r\right) + [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} - \boldsymbol{L}$$

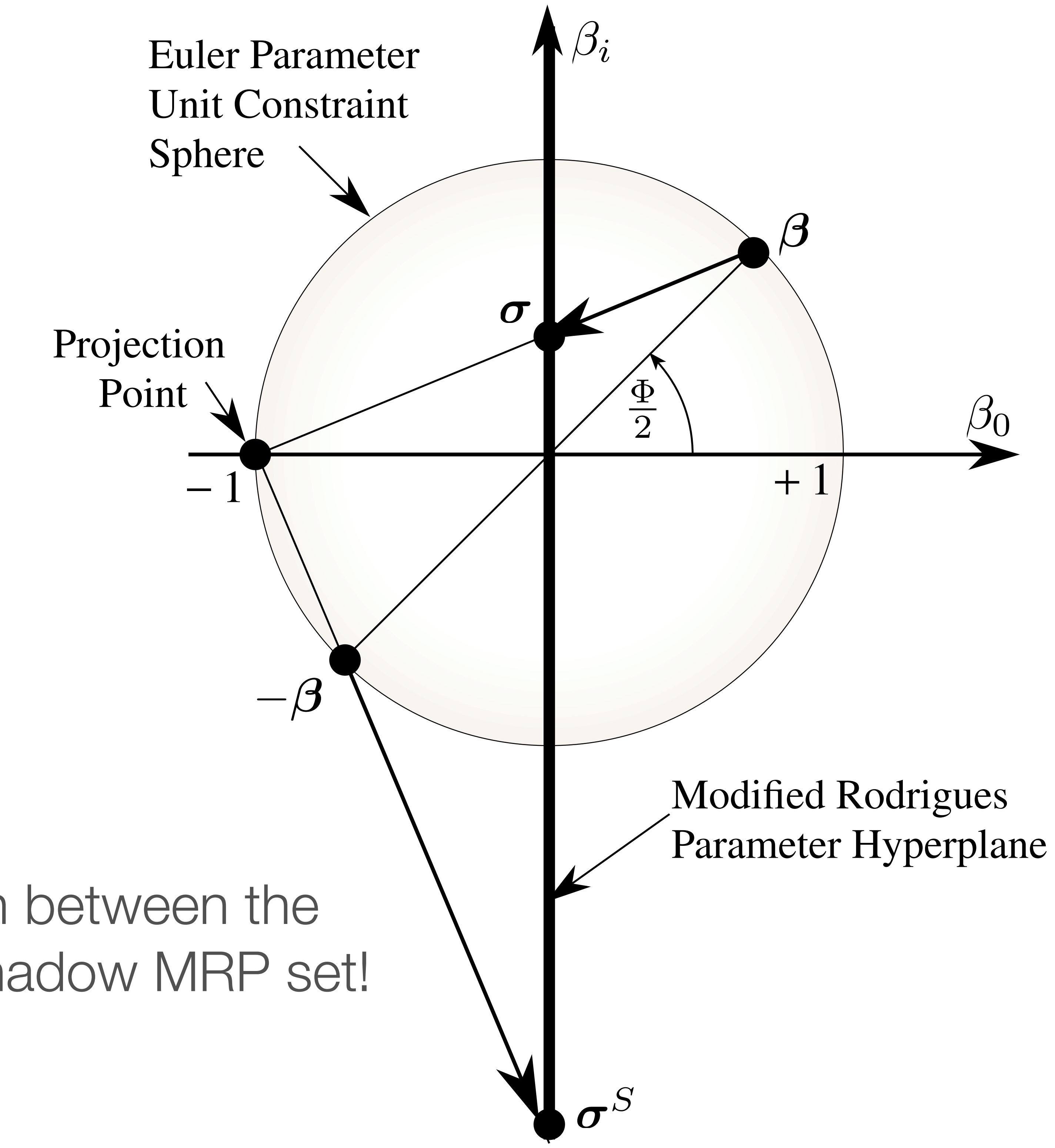
With the associated Lyapunov function being defined as:

$$V(\boldsymbol{\omega}, \boldsymbol{\sigma}) = \frac{1}{2}\delta\boldsymbol{\omega}^T[I]\delta\boldsymbol{\omega} + 2K\ln\left(1 + \boldsymbol{\sigma}^T\boldsymbol{\sigma}\right)$$

$$V \to \infty \quad \text{as} \quad \delta\boldsymbol{\omega}, \boldsymbol{\sigma} \to \infty$$

Globally Stabilizing

However, the MRP attitude can go singular? What if the body is tumbling and we make a 360° revolution?



We can switch between the original and shadow MRP set!

A convenient MRP switching surface is

$$\boldsymbol{\sigma}^T \boldsymbol{\sigma} = \sigma^2 = 1$$

where the body is "upside-down" relative to the reference attitude. The mapping to the shadow set is simply

$$\boldsymbol{\sigma}^S = -\boldsymbol{\sigma}$$

Note that the Lyapunov function *V* is *continuous* during this MRP switching with this switching surface!

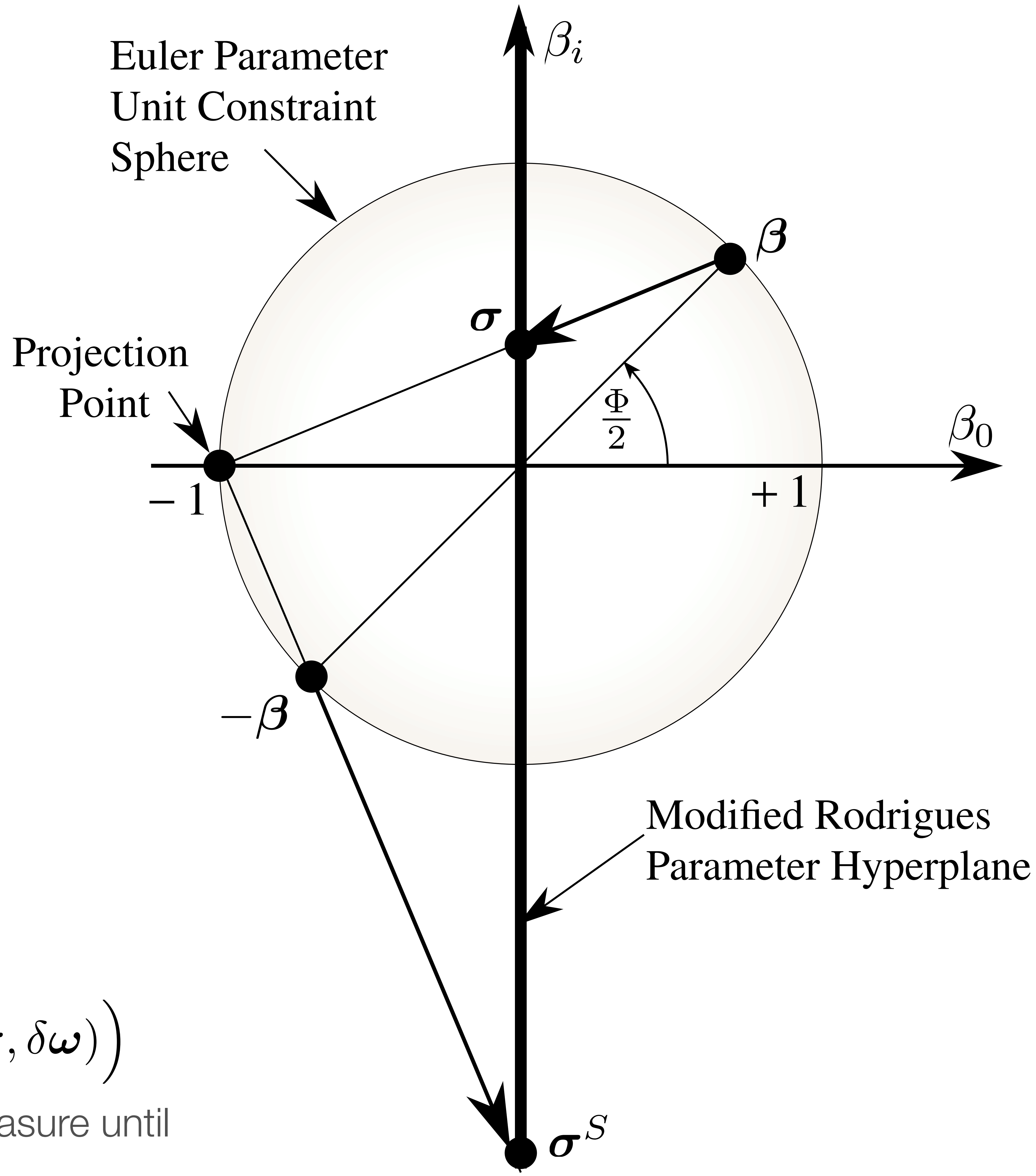
Assume that V_1 is the Lyapunov tracking how the original state errors are being reduced.

$V_1(\boldsymbol{\sigma}, \delta\boldsymbol{\omega})$ is reduced until $|\boldsymbol{\sigma}| = 1$

MRP mapping: $\boldsymbol{\sigma}^S = -\boldsymbol{\sigma}$

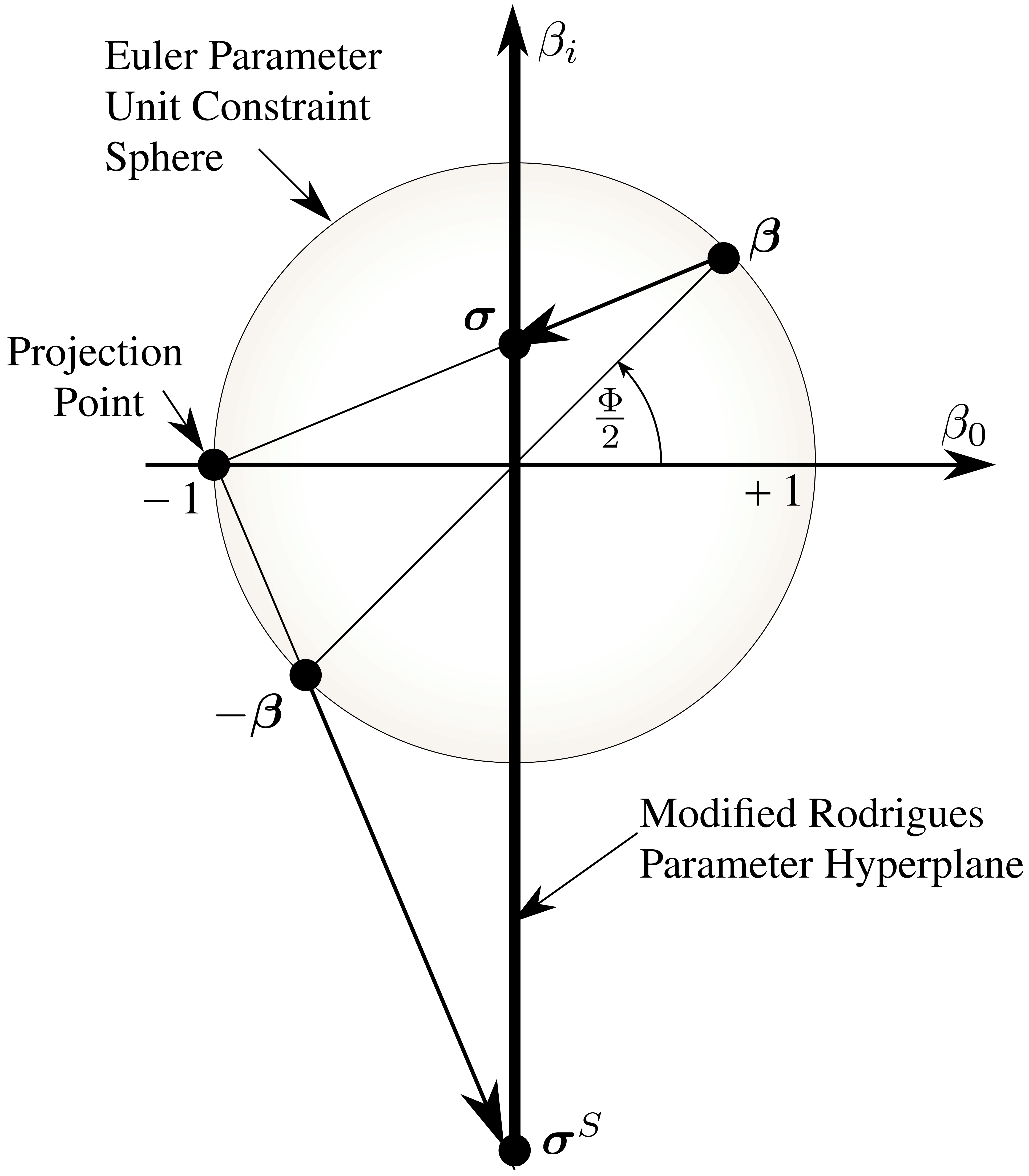
New Lyapunov function: $V_2(\boldsymbol{\sigma}^S, \delta\boldsymbol{\omega}) \quad \left(= V_1(\boldsymbol{\sigma}, \delta\boldsymbol{\omega}) \right)$

We can reset the stability analysis now to track this new error measure until either $|\boldsymbol{\sigma}^S| = 1$ or $|\boldsymbol{\sigma}^S| \rightarrow= 0$.



Comment:
During the switching the V partial derivatives are well defined, but not continuous. Thus, we cannot use standard Lyapunov theory to argue global stability. However, by breaking up the problem into a series of every decreasing Lyapunov functions, we can argue that global stability will be achieved.

Application:
This MRP switching provides very elegant attitude feedback laws which are linear in MRP, and will automatically de-tumble a body by always rotating it back to the reference attitude using the shortest rotation distance. Attitude error wind-up is avoided.



Tumbling Body Example:

Single-axis rotation with

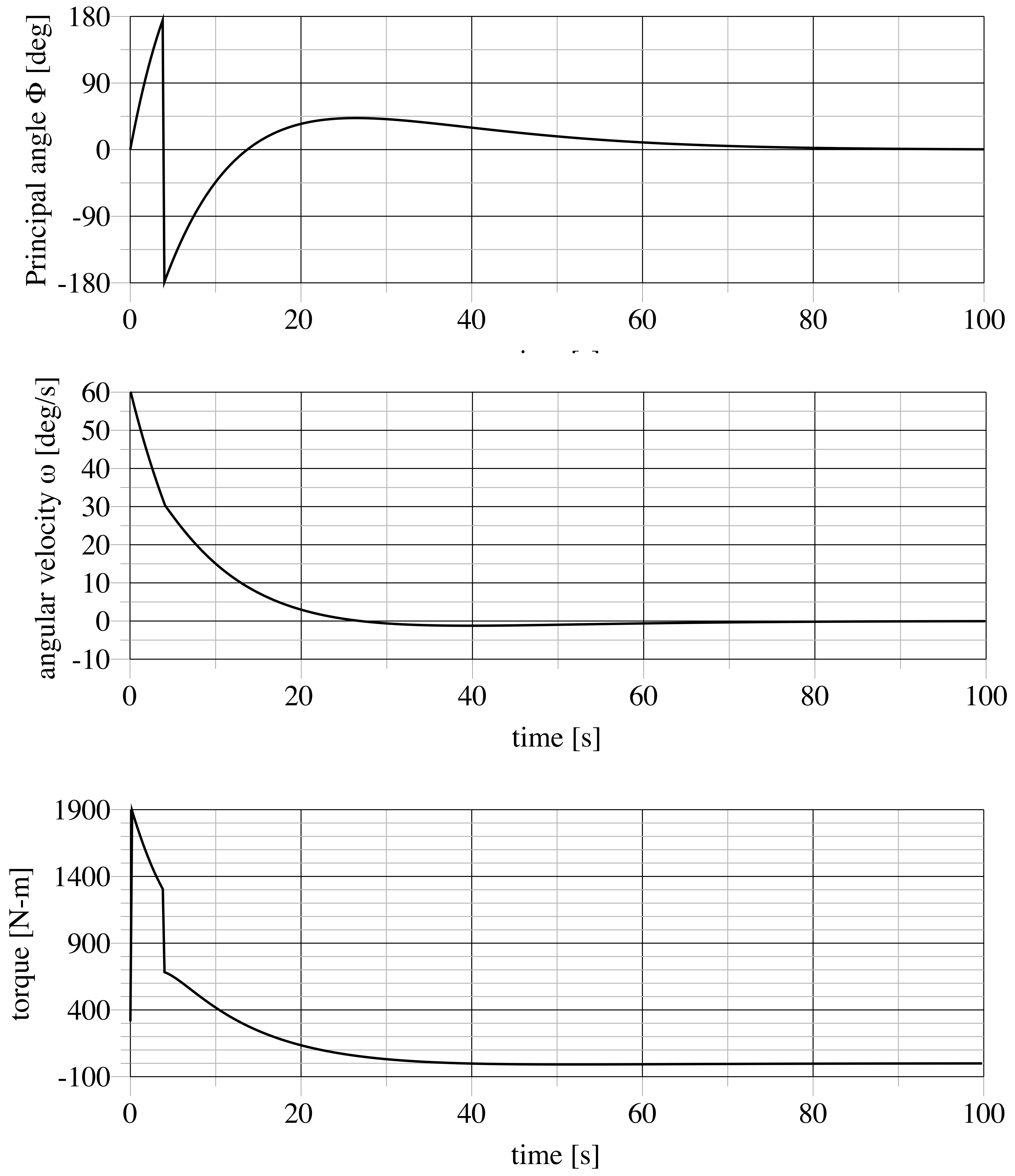
$$J = 12000 \text{ kg m}^2$$

$$K = 300$$

$$P = 1800$$

$$\omega = 60^{\circ}/\text{s}$$

Goal: Bring the body to rest at the zero attitude. (regulator problem)



Asymptotic Convergence

Note that the previous Lyapunov function only had a negative semi-definite derivative.

$$\dot{V}(\boldsymbol{\sigma}, \delta\boldsymbol{\omega}) = -\delta\boldsymbol{\omega}^T [P] \delta\boldsymbol{\omega}$$

$\Rightarrow$ globally stabilizing

Let us analyze this control to see when it is asymptotically stabilizing. We do so by investigating higher derivatives of *V*.

Note $\dot{V} = 0 \quad \Rightarrow \quad \Omega = \{\delta\boldsymbol{\omega} = 0\}$

2nd: $\ddot{V} = -2\delta\boldsymbol{\omega}^T [P] \delta\dot{\boldsymbol{\omega}}$

$$\ddot{V}(\boldsymbol{\sigma}, \delta\boldsymbol{\omega} = 0) = 0$$

3rd: $\dddot{V} = -2\delta\boldsymbol{\omega}^T [P] \delta\ddot{\boldsymbol{\omega}} - 2\delta\dot{\boldsymbol{\omega}}^T [P] \delta\dot{\boldsymbol{\omega}}$

Recall $[I]\delta\dot{\boldsymbol{\omega}} + [P]\delta\boldsymbol{\omega} + K\boldsymbol{\sigma} = 0$

closed-loop dynamics

$$\delta\dot{\boldsymbol{\omega}} = -[I]^{-1}\left([P]\delta\boldsymbol{\omega} + K\boldsymbol{\sigma}\right)$$

$$\delta\dot{\boldsymbol{\omega}} = -[I]^{-1} K\boldsymbol{\sigma} \quad \text{if} \quad \delta\boldsymbol{\omega} = 0$$

$\Downarrow$

$$\dddot{V}(\boldsymbol{\sigma}, \delta\boldsymbol{\omega} = 0) = -K^2 \boldsymbol{\sigma}^T \left([I]^{-1}\right)[P][I]\boldsymbol{\sigma}$$

This 3rd derivative is negative definite in MRPs, and thus the control is asymptotically stabilizing.

External Torque Model Error

If some *un-modeled* external torque $\Delta \boldsymbol{L}$ is present, then the EOM are written as:

$$[I]\dot{\boldsymbol{\omega}} = -[\tilde{\omega}][I]\boldsymbol{\omega} + \boldsymbol{u} + \boldsymbol{L} + \Delta\boldsymbol{L}$$

The Lyapunov rate is now written as

$$\dot{V} = -\delta\boldsymbol{\omega}^T[P]\delta\boldsymbol{\omega} + \delta\boldsymbol{\omega}^T\Delta\boldsymbol{L}$$

This is no longer negative semi-definite! However, assume that the un-modeled torque vector is constant and bounded (as seen by body frame *B*).

However, this $\dot{V}$ does show that the $\delta\boldsymbol{\omega}$ errors will be bounded by the control. For large $\delta\boldsymbol{\omega}$ the quadratic term will dominate.

The new closed-loop EOM are:

$$[I]\delta\dot{\boldsymbol{\omega}} + [P]\delta\boldsymbol{\omega} + K\boldsymbol{\sigma} = \Delta\boldsymbol{L}$$

Differentiating using $\dot{\boldsymbol{\sigma}} = \frac{1}{4}[B(\boldsymbol{\sigma})]\delta\boldsymbol{\omega}$ yields:

$$[I]\delta\ddot{\boldsymbol{\omega}} + [P]\delta\dot{\boldsymbol{\omega}} + \frac{K}{4}[B(\boldsymbol{\sigma})]\delta\boldsymbol{\omega} = \Delta\dot{\boldsymbol{L}} \approx 0$$

This is a spring-mass-damper system with a nonlinear spring. To show that the stiffness matrix here is positive definite note that

$$\boldsymbol{\omega}^T[B(\boldsymbol{\sigma})]\boldsymbol{\omega} = \boldsymbol{\omega}^T\left[(1-\sigma^2)I + 2[\tilde{\boldsymbol{\sigma}}] + 2\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right]\boldsymbol{\omega}$$
$$= (1-\sigma^2)\boldsymbol{\omega}^T\boldsymbol{\omega} + 2\left(\boldsymbol{\omega}^T\boldsymbol{\sigma}\right)^2 > 0$$

Note that we assume that the MRP vector is maintained to have a magnitude less than 1!

$$[I]\delta\ddot{\boldsymbol{\omega}} + [P]\delta\dot{\boldsymbol{\omega}} + \frac{K}{4}[B(\boldsymbol{\sigma})]\delta\boldsymbol{\omega} = 0$$

Because the closed-loop dynamics above are stable, the angular velocity tracking errors will reach a steady state value.

$$K[B(\boldsymbol{\sigma}_{ss})]\delta\boldsymbol{\omega}_{ss} = 0$$

However, the matrix $[B(\sigma_{ss})]$ has been shown to be near-orthogonal and is always full-rank. This leads to

$$\delta\boldsymbol{\omega}_{ss} = 0$$

Using this result in the closed-loop dynamics:

$$[I]\delta\dot{\boldsymbol{\omega}} + [P]\delta\boldsymbol{\omega} + K\boldsymbol{\sigma} = \Delta\boldsymbol{L}$$

the attitude steady-state error is found to be

$$\boldsymbol{\sigma}_{ss} = \lim_{t\to\infty}\boldsymbol{\sigma} = \frac{1}{K}\Delta\boldsymbol{L}$$

The larger the attitude feedback gain *K* is, the smaller the steady-state attitude error will be.

Example:

A symmetric rigid body with all three principal inertias being 10 kg m^2 is studied. An unmodeled external torque is present.

$$\Delta \boldsymbol{L} = (0.05, 0, 10, -0.10)^T \text{ Nm}$$

Goal: $\boldsymbol{\omega} \rightarrow 0 \qquad \boldsymbol{\sigma} \rightarrow 0$

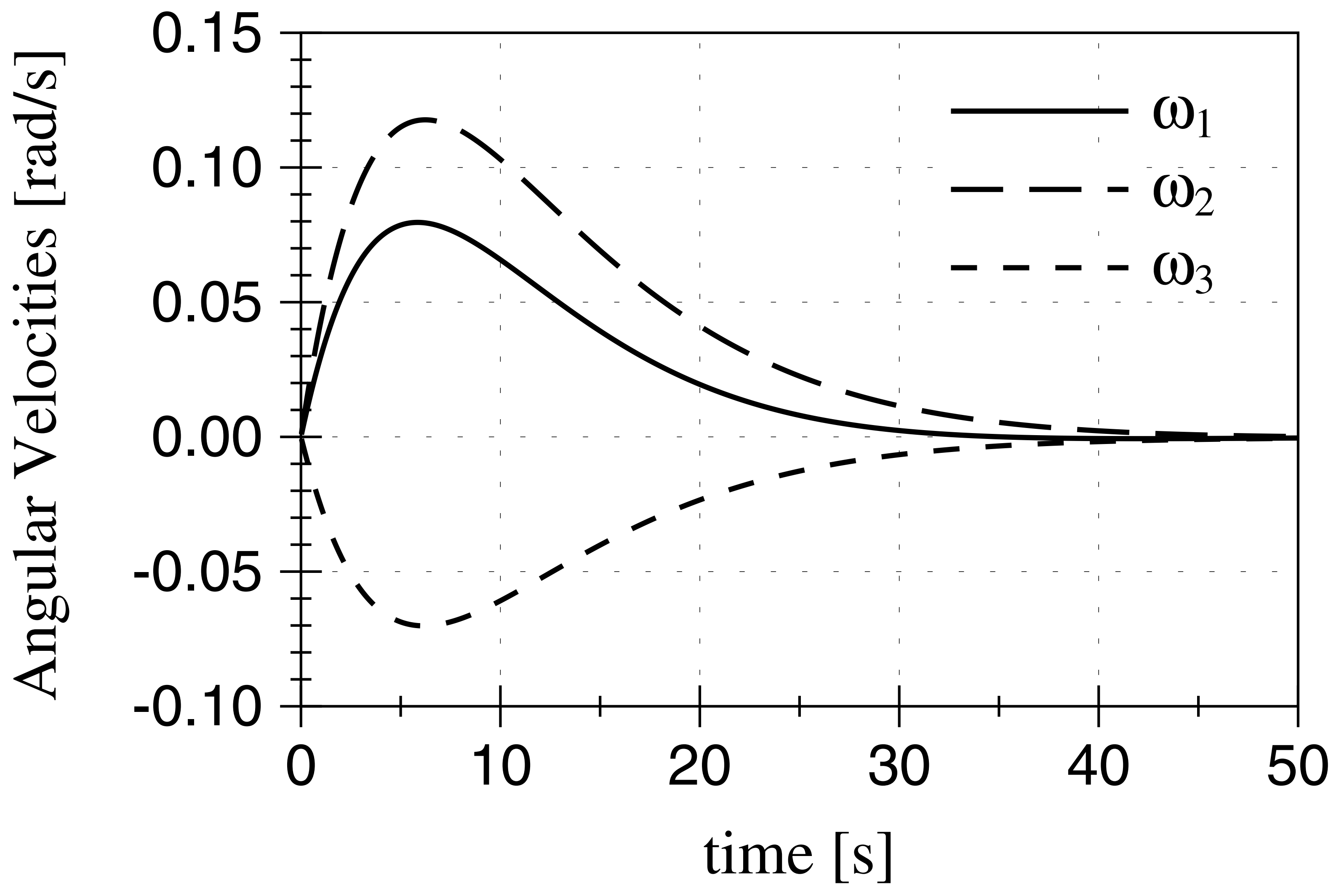
Initial Error:
$$\boldsymbol{\sigma}(t_0) = (-0.3, -0.4, 0.2)^T$$
$$\boldsymbol{\omega}(t_0) = (0.0, 0.0, 0.0)^T$$

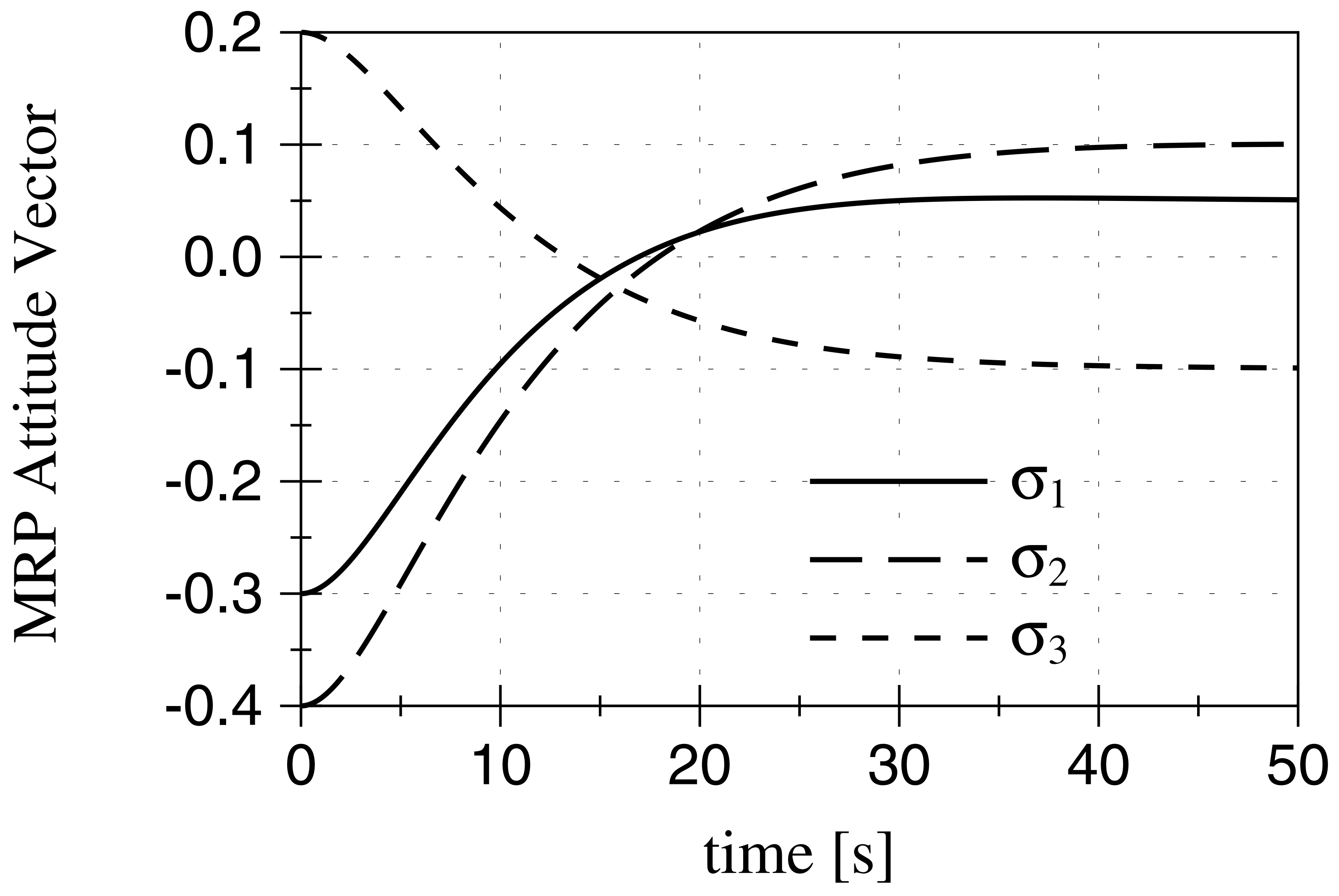
Gains:
$$K = 1 \text{ kg m}^2/\text{s}^2$$
$$P = 3 \text{ kg m}^2/\text{s}$$

Predicted steady state errors:

$$\delta \boldsymbol{\omega}_{ss} = 0$$

$$\boldsymbol{\sigma}_{ss} = \frac{1}{K}\Delta \boldsymbol{L} = \begin{pmatrix} 0.05 \\ 0.10 \\ -0.10 \end{pmatrix}$$





Do Regulation problem by hand

Integral Feedback

- Next, let us investigate adding an integral feedback term to make the attitude control more robust to un-modeled external torques.

Let us introduce the new state vector ***z***:

$$\boldsymbol{z}(t) = \int_0^t (K\boldsymbol{\sigma} + [I]\delta\dot{\boldsymbol{\omega}})\,\mathrm{d}t$$

Note that ***z*** will grow unbounded if there is any finite steady state attitude errors!

Thus, we want a new control law that will force ***z*** to go to zero, and thus drive any steady-state attitude errors to zero as well.

New Lyapunov function:

$$V(\delta\boldsymbol{\omega}, \boldsymbol{\sigma}, \boldsymbol{z}) = \frac{1}{2}\delta\boldsymbol{\omega}^T[I]\delta\boldsymbol{\omega} + 2K\log\left(1 + \boldsymbol{\sigma}^T\boldsymbol{\sigma}\right) + \frac{1}{2}\boldsymbol{z}^T\boxed{[K_I]}\boldsymbol{z}$$

s.p.d.

Assume at first that there is no un-modeled external torque. In this case we set the Lyapunov rate equal to

$$\dot{V} = -\left(\delta\boldsymbol{\omega} + [K_I]\boldsymbol{z}\right)^T[P]\left(\delta\boldsymbol{\omega} + [K_I]\boldsymbol{z}\right)$$

and solve for the control vector ***u***:

$$\boldsymbol{u} = -K\boldsymbol{\sigma} - [P]\delta\boldsymbol{\omega} - [P][K_I]\boldsymbol{z} + [I]\left(\dot{\boldsymbol{\omega}}_r - [\tilde{\boldsymbol{\omega}}]\boldsymbol{\omega}_r\right) + [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} - \boldsymbol{L}$$

Measuring ***z*** direction is not convenient because it required the derivative of $\delta\boldsymbol{\omega}$. Instead, note that we can write

$$\boldsymbol{z}(t) = K\int_0^t \boldsymbol{\sigma}\,\mathrm{d}t + [I]\left(\delta\boldsymbol{\omega} - \delta\boldsymbol{\omega}_0\right)$$

This allows us to re-write the feedback control law in the final form:

$$\boldsymbol{u} = -K\boldsymbol{\sigma} - ([P] + [P][K_I][I])\,\delta\boldsymbol{\omega} - K[P][K_I]\int_0^t \boldsymbol{\sigma}\mathrm{d}t + [P][K_I][I]\delta\boldsymbol{\omega}_0 + [I]\left(\dot{\boldsymbol{\omega}}_r - [\tilde{\omega}]\boldsymbol{\omega}_r\right) + [\tilde{\omega}][I]\boldsymbol{\omega} - \boldsymbol{L}$$

Next, let's analyze the stability of this control law. The Lyapunov rate is semi-negative definite

$$\dot{V} = -\left(\delta\boldsymbol{\omega} + [K_I]\boldsymbol{z}\right)^T [P]\left(\delta\boldsymbol{\omega} + [K_I]\boldsymbol{z}\right)$$

This guarantees that **ω**, **σ**, and ***z*** are stable, and that

$$\delta\boldsymbol{\omega} + [K_I]\boldsymbol{z} \rightarrow 0$$

To study asymptotic convergence, we investigate the higher order derivative of the Lyapunov function *V*.

The first non-zero higher derivative evaluated on the set where $\dot{V}$ is zero is

$$\dddot{V}(\boldsymbol{\sigma}, \delta\boldsymbol{\omega} + [K_I]\boldsymbol{z} = 0) = -K^2\boldsymbol{\sigma}^T\left([I]^{-1}\right)[P][I]\boldsymbol{\sigma}$$

$$\Rightarrow \quad \boldsymbol{\sigma} \rightarrow 0 \quad \Rightarrow \quad \delta\boldsymbol{\omega} \rightarrow 0$$

Kinematic Relationship

$$\delta\boldsymbol{\omega} + [K_I]\boldsymbol{z} \rightarrow 0 \quad \Rightarrow \quad \boldsymbol{z} \rightarrow 0$$

If unmodeled external torques are included, then the Lyapunov rate is expressed as:

$$\dot{V} = -\left(\delta\boldsymbol{\omega} + [K_I]\boldsymbol{z}\right)^T\left([P](\delta\boldsymbol{\omega} + [K_I]\boldsymbol{z}) - \Delta\boldsymbol{L}\right)$$

This is no longer n.s.d. However, we can conclude for bounded Δ***L*** the states δ**ω** and ***z*** must remain bounded.

Recall $\quad \boldsymbol{z}(t) = \int_0^t \left(K\boldsymbol{\sigma} + [I]\delta\dot{\boldsymbol{\omega}}\right)\mathrm{d}t$

$$\Rightarrow \quad \boldsymbol{\sigma} \rightarrow 0 \quad \Rightarrow \quad \delta\boldsymbol{\omega} \rightarrow 0$$

Next, let's study the state $\boldsymbol{z}$ as the Lyapunov rate approaches zero at steady-state. This requires that

$$\lim_{t\to\infty}\left([P]\left(\delta\boldsymbol{\omega}+[K_I]\boldsymbol{z}\right)-\Delta\boldsymbol{L}\right)=0$$

Because $\delta\boldsymbol{\omega}\to 0$, the steady-state value of $\boldsymbol{z}$ is expressed as:

$$\boxed{\lim_{t\to\infty}\boldsymbol{z}=[K_I]^{-1}[P]^{-1}\Delta\boldsymbol{L}}$$

Example:

A symmetric rigid body with all three principal inertias being 10 kg m^2 is studied. An unmodeled external torque is present.

$$\Delta \boldsymbol{L} = (0.05, 0, 10, -0.10)^T \text{ Nm}$$

Goal: $\boldsymbol{\omega} \to 0 \qquad \boldsymbol{\sigma} \to 0$

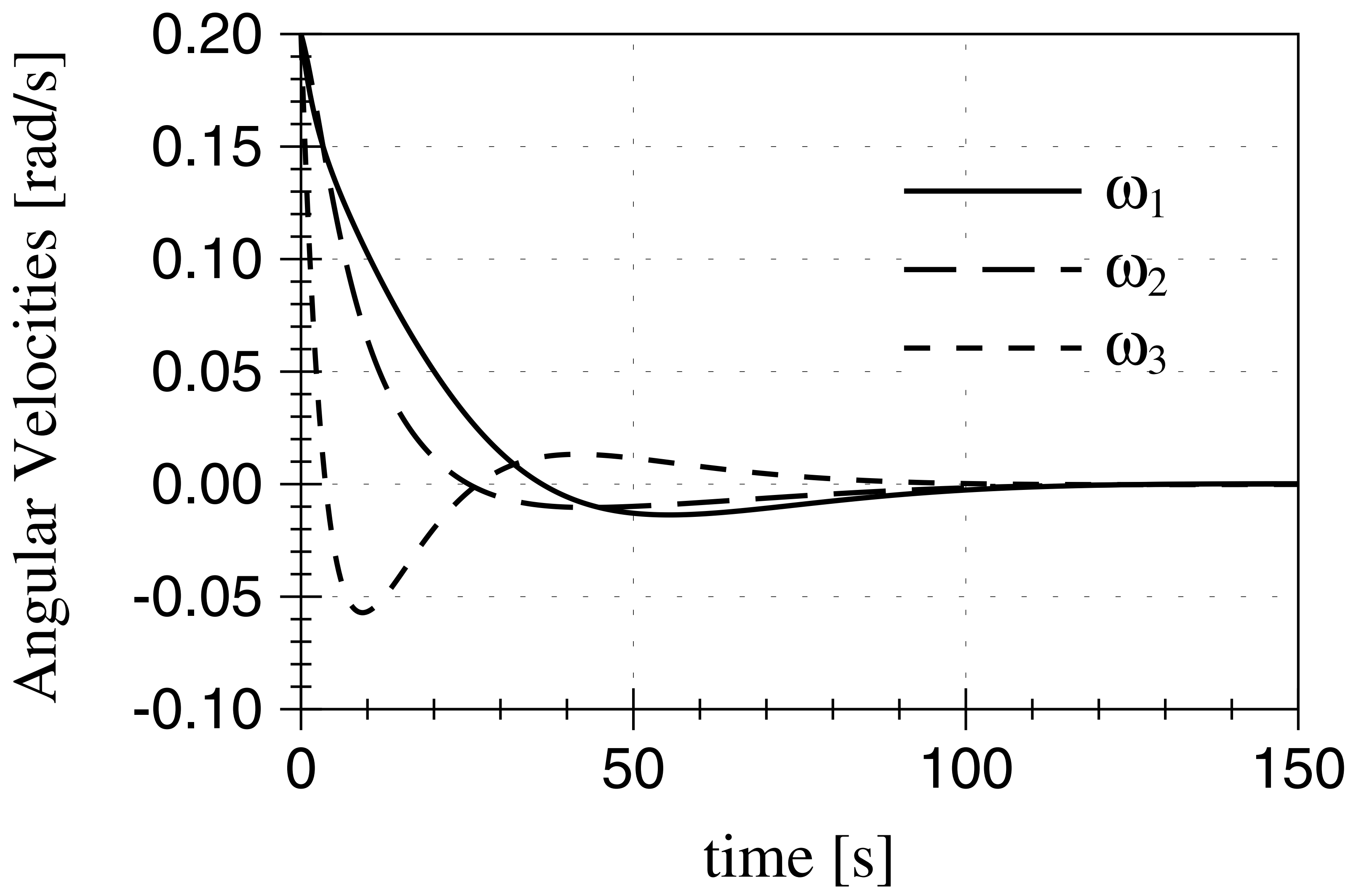
Initial Error:
$$\boldsymbol{\sigma}(t_0) = (-0.3, -0.4, 0.2)^T$$
$$\boldsymbol{\omega}(t_0) = (0.2, 0.2, 0.2)^T \text{ rad/s}$$

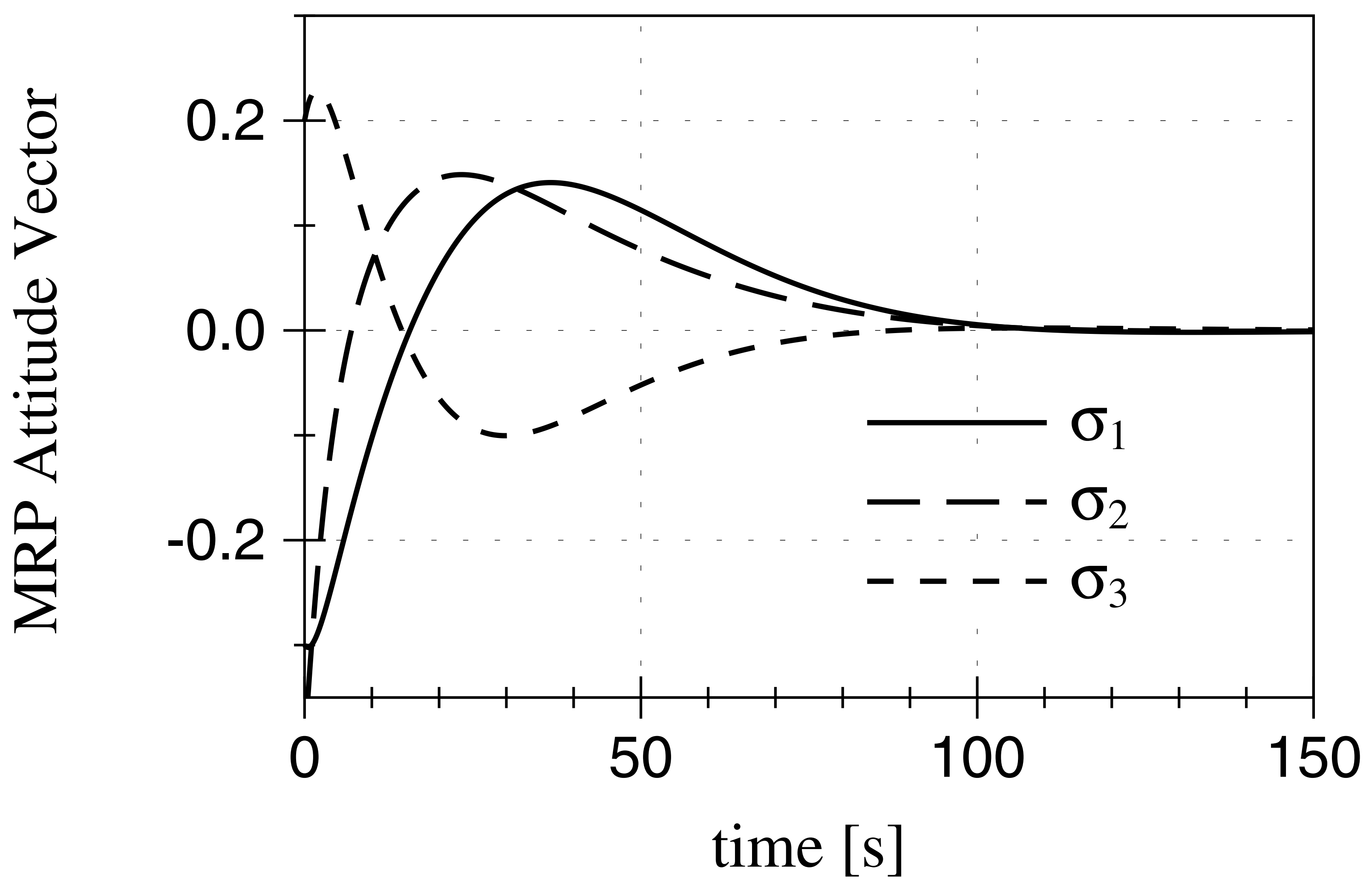
Gains:
$$K = 1 \text{ kg m}^2/\text{s}^2$$
$$P = 3 \text{ kg m}^2/\text{s}$$
$$K_I = 0.01 \text{ s}^{-1}$$

Predicted steady state errors:

$$\delta\boldsymbol{\omega}_{ss} = 0 \qquad \boldsymbol{\sigma}_{ss} = 0$$

$$\lim_{t\to\infty} \boldsymbol{z} = \frac{\Delta \boldsymbol{L}}{K_I P} = \begin{pmatrix} 1.66 \\ 3.33 \\ -3.33 \end{pmatrix} \text{kg-m}^2/\text{sec}$$





Example:

A symmetric rigid body with all three principal inertias being 10 kg m^2 is studied. An unmodeled external torque is present.

$$\Delta \boldsymbol{L} = (0.05, 0, 10, -0.10)^T \text{ Nm}$$

Goal: $\boldsymbol{\omega} \rightarrow 0 \qquad \boldsymbol{\sigma} \rightarrow 0$

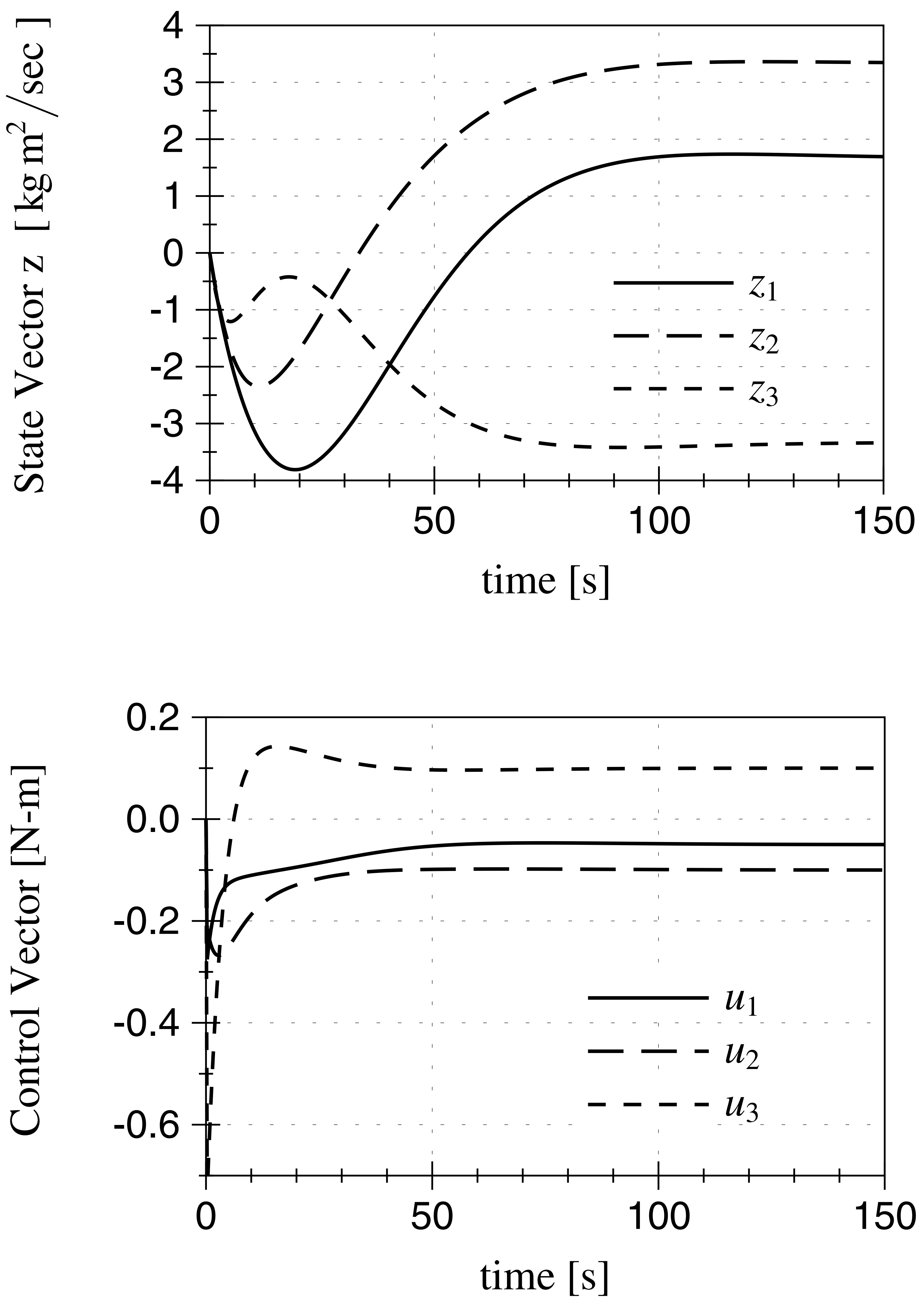
Initial Error:
$$\boldsymbol{\sigma}(t_0) = (-0.3, -0.4, 0.2)^T$$
$$\boldsymbol{\omega}(t_0) = (0.2, 0.2, 0.2)^T \text{ rad/s}$$

Gains:
$$K = 1 \text{ kg m}^2/\text{s}^2$$
$$P = 3 \text{ kg m}^2/\text{s}$$
$$K_I = 0.01 \text{ s}^{-1}$$

Predicted steady state errors:

$$\delta\boldsymbol{\omega}_{ss} = 0 \qquad \boldsymbol{\sigma}_{ss} = 0$$

$$\lim_{t\rightarrow\infty} \boldsymbol{z} = \frac{1}{K_I I}\Delta \boldsymbol{L} = \begin{pmatrix} 1.66 \\ 3.33 \\ -3.33 \end{pmatrix} \text{kg-m}^2/\text{sec}$$



Feedback Gain Selection

- Lyapunov theory is great to develop globally stabilizing nonlinear feedback control law. However, how does one select the feedback gains to get good performance?

Consider the "PD-like" nonlinear feedback control law:

$$\boldsymbol{u} = -K\boldsymbol{\sigma} - [P]\delta\boldsymbol{\omega} + [I]\left(\dot{\boldsymbol{\omega}}_r - [\tilde{\boldsymbol{\omega}}]\boldsymbol{\omega}_r\right) + [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} - \boldsymbol{L}$$

Without external torque modeling errors, the closed-loop dynamics are written as:

$$[I]\delta\dot{\boldsymbol{\omega}} + [P]\delta\boldsymbol{\omega} + K\boldsymbol{\sigma} = 0$$

Linear in $\boldsymbol{\sigma}$ thanks to the use of MRP

Differential kinematic eqn: $$\dot{\boldsymbol{\sigma}} = \frac{1}{4}[B(\boldsymbol{\sigma})]\delta\boldsymbol{\omega}$$

Let's write the tracking error state vector $\boldsymbol{x}$ as:

$$\boldsymbol{x} = \begin{pmatrix} \boldsymbol{\sigma} \\ \delta\boldsymbol{\omega} \end{pmatrix}$$

Nonlinear state-space formulation:

$$\dot{\boldsymbol{x}} = \begin{pmatrix} \dot{\boldsymbol{\sigma}} \\ \delta\dot{\boldsymbol{\omega}} \end{pmatrix} = \begin{bmatrix} 0 & \boxed{\frac{1}{4}B(\boldsymbol{\sigma})} \\ -K[I]^{-1} & -[I]^{-1}[P] \end{bmatrix} \begin{pmatrix} \boldsymbol{\sigma} \\ \delta\boldsymbol{\omega} \end{pmatrix}$$

Approximate differential kinematic equations: $$\dot{\boldsymbol{\sigma}} \simeq \frac{1}{4}\delta\boldsymbol{\omega}$$

$$\begin{pmatrix} \dot{\boldsymbol{\sigma}} \\ \delta\dot{\boldsymbol{\omega}} \end{pmatrix} = \begin{bmatrix} 0 & \frac{1}{4}I \\ -K[I]^{-1} & -[I]^{-1}[P] \end{bmatrix} \begin{pmatrix} \boldsymbol{\sigma} \\ \delta\boldsymbol{\omega} \end{pmatrix}$$

Linear state-space form

If a principal body coordinate frame is chosen, then the inertia matrix is diagonal and the linearized tracking error simplify to:

$$\begin{pmatrix} \dot{\sigma}_i \\ \delta\dot{\omega}_i \end{pmatrix} = \begin{bmatrix} 0 & \frac{1}{4} \\ -\frac{K}{I_i} & -\frac{P_i}{I_i} \end{bmatrix} \begin{pmatrix} \sigma_i \\ \delta\omega_i \end{pmatrix} \qquad i = 1, 2, 3$$

3 uncoupled differential equations

The roots of the corresponding characteristic equation are expressed as

$$\lambda_i = -\frac{1}{2I_i}\left(P_i \pm \sqrt{-KI_i + P_i{}^2}\right) \qquad i = 1, 2, 3$$

The feedback gains P_i and K can now be chosen such that the system is either under-damped (complex roots), critically damped (double real root), or over-damped (two unique real roots).

Let's consider an under-damped response.

$$\omega_{n_i} = \frac{\sqrt{KI_i}}{2I_i}$$ natural frequency

$$\xi_i = \frac{P_i}{\sqrt{KI_i}}$$ damping ratio

$$T_i = \frac{2I_i}{P_i}$$ Time decay constant

$$\omega_{d_i} = \frac{1}{2I_i}\sqrt{KI_i - P_i^2}$$ damped natural frequency

Feedback Gain Selection Example:

Parameter	Value	Units
I_1	140.0	kg-m^2
I_2	100.0	kg-m^2
I_3	80.0	kg-m^2
$\boldsymbol{\sigma}(t_0)$	$[0.60\ -0.40\ 0.20]$	
$\boldsymbol{\omega}(t_0)$	$[0.70\ 0.20\ -0.15]$	rad/sec
$[P]$	$[18.67\ 2.67\ 10.67]$	kg-m^2/sec
K	7.11	kg-m^2/sec^2

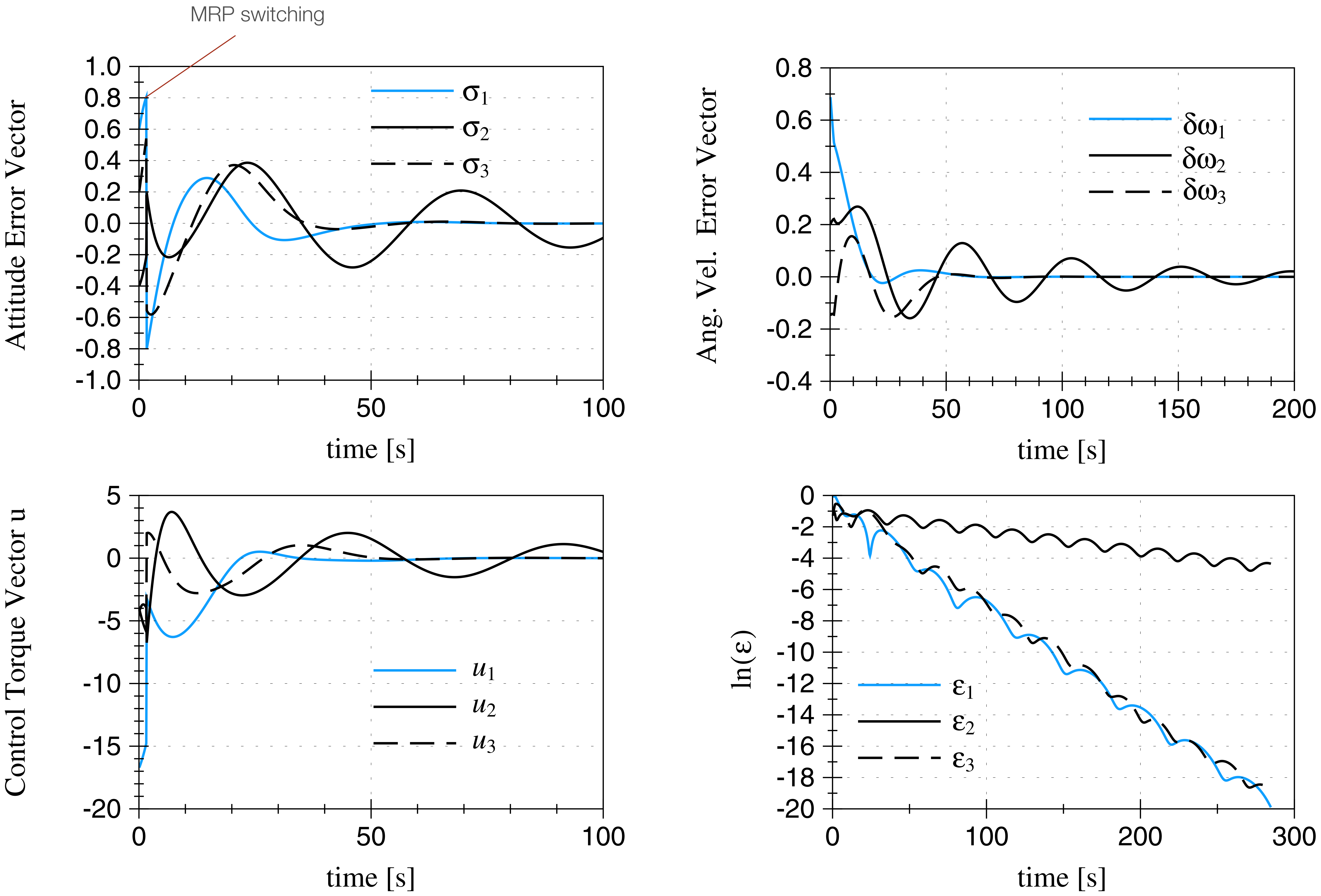
A principal body coordinate frame is chosen which diagonalizes the inertia matrix.

Large initial errors will cause the body to tumble "up-side-down".

Let us define the new state ε_i to track the 3 decoupled linearized tracking error dynamics.

$$\epsilon_i = \sqrt{\sigma_i^2 + \omega_i^2} \qquad i = 1, 2, 3$$

We can then evaluate this state in the nonlinear simulation, and compare to the predicted decay rate of the linearized analysis.


MRP switching
Attitude Error Vector
σ1
σ2
σ3
time [s]
Ang. Vel. Error Vector
δω1
δω2
δω3
time [s]
Control Torque Vector u
u1
u2
u3
time [s]
ln(ε)
ε1
ε2
ε3
time [s]

Parameter	Actual Average	Predicted Value	Percent Difference
T_1	14.71 s	15.00 s	1.97%
T_2	76.92 s	75.00 s	-2.50%
T_3	14.71 s	15.00 s	1.97%
ω_{d_1}	0.0938 rad/s	0.0909 rad/s	-3.12%
ω_{d_2}	0.1326 rad/s	0.1326 rad/s	0.08%
ω_{d_3}	0.1343 rad/s	0.1333 rad/s	-0.74%

This table compares the actual, nonlinear response to that of the linearized prediction of the gain selection method.

Because the MRP behave very linearly, the predicted tracking error dynamics matches the nonlinear motion very well, even though the is tumbling up-side-down!