Euler Parameters (Quaternions)

Voted most popular attitude coordinates in the non-singular category...

Introduction

- Very popular redundant set of attitude coordinates
- Are called either Euler Parameters (EPs) or quaternions
- Major benefits:
  - Non-singular attitude description
  - Linear differential kinematic equation
  - Works well for small and large rotations
- Drawbacks:
  - Constraint equation must be identified as all times
  - Not as simple to visualize

Definition of EP

- The redundant Euler Parameters are defined using the principal rotation components as

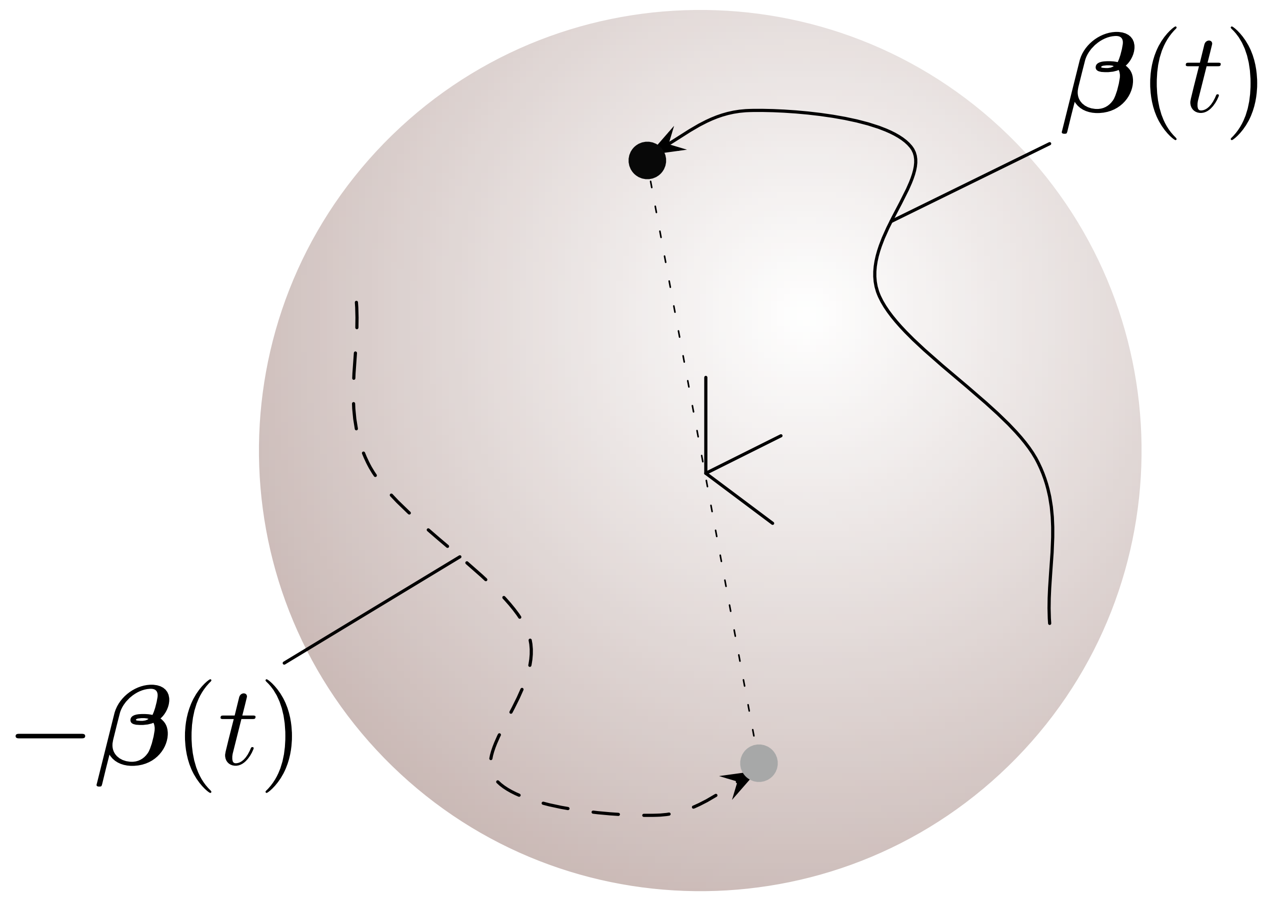
$$
\begin{aligned}
\beta_0 &= \cos(\Phi/2) \\
\beta_1 &= e_1 \sin(\Phi/2) \\
\beta_2 &= e_2 \sin(\Phi/2) \\
\beta_3 &= e_3 \sin(\Phi/2)
\end{aligned}
$$

Constraints:

$$e_1^2 + e_2^2 + e_3^2 = 1$$

$$\beta_0^2 + \beta_1^2 + \beta_2^2 + \beta_3^2 = 1$$

- Note that the 4-coordinate set has a single constraint equation! All EPs must lie on the three-dimensional surface of a 4-dimensional hypersphere.



Unit Hypersphere

- Since the PRV components are not unique, we find that the EP also isn't unique:

$$(-\hat{\boldsymbol{e}}, -\Phi) \qquad \beta_0' = \cos\left(-\frac{\Phi}{2}\right) = \cos\left(\frac{\Phi}{2}\right) = \beta_0$$

$$\beta_i' = -e_i \sin\left(-\frac{\Phi}{2}\right) = e_i \sin\left(\frac{\Phi}{2}\right) = \beta_i$$

$$(\hat{\boldsymbol{e}}, \Phi') \qquad \beta_0' = \cos\left(\frac{\Phi'}{2}\right) = \cos\left(\frac{\Phi}{2} - \pi\right) = -\cos\left(\frac{\Phi}{2}\right) = -\beta_0$$

$$\beta_i' = e_i \sin\left(\frac{\Phi'}{2}\right) = e_i \sin\left(\frac{\Phi'}{2} - \pi\right) = -e_i \sin\left(\frac{\Phi}{2}\right) = -\beta_i$$

- Note that the alternate EP set corresponds to performing the larger principle rotation angle (i.e., rotating the long way round)

Euler Parameter to DCM Relationship

- The rotation matrix can be expressed in terms or EPs as:

$$[C] = \begin{bmatrix} \beta_0^2+\beta_1^2-\beta_2^2-\beta_3^2 & 2\left(\beta_1\beta_2+\beta_0\beta_3\right) & 2\left(\beta_1\beta_3-\beta_0\beta_2\right) \\ 2\left(\beta_1\beta_2-\beta_0\beta_3\right) & \beta_0^2-\beta_1^2+\beta_2^2-\beta_3^2 & 2\left(\beta_2\beta_3+\beta_0\beta_1\right) \\ 2\left(\beta_1\beta_3+\beta_0\beta_2\right) & 2\left(\beta_2\beta_3-\beta_0\beta_1\right) & \beta_0^2-\beta_1^2-\beta_2^2+\beta_3^2 \end{bmatrix}$$

- The inverse relationship is found by inspection to be

$$\beta_0 = \pm\frac{1}{2}\sqrt{C_{11}+C_{22}+C_{33}+1}$$

$$\beta_1 = \frac{C_{23}-C_{32}}{4\beta_0}$$

$$\beta_2 = \frac{C_{31}-C_{13}}{4\beta_0}$$

$$\beta_3 = \frac{C_{12}-C_{21}}{4\beta_0}$$

Singular if: $\beta_0 \rightarrow 0$

- Sheppard's method is a robust method to compute the EP from a rotation matrix:

1st step: Find largest value of

$$\beta_0^2 = \frac{1}{4}\left(1 + \operatorname{trace}\left([C]\right)\right) \qquad \beta_2^2 = \frac{1}{4}\left(1 + 2C_{22} - \operatorname{trace}\left([C]\right)\right)$$

$$\beta_1^2 = \frac{1}{4}\left(1 + 2C_{11} - \operatorname{trace}\left([C]\right)\right) \qquad \beta_3^2 = \frac{1}{4}\left(1 + 2C_{33} - \operatorname{trace}\left([C]\right)\right)$$

2nd step: Compute the remaining EPs using

$$\beta_0\beta_1 = (C_{23} - C_{32})/4 \qquad \beta_1\beta_2 = (C_{12} + C_{21})/4$$

$$\beta_0\beta_2 = (C_{31} - C_{13})/4 \qquad \beta_3\beta_1 = (C_{31} + C_{13})/4$$

$$\beta_0\beta_3 = (C_{12} - C_{21})/4 \qquad \beta_2\beta_3 = (C_{23} + C_{32})/4$$

Adding Euler Parameters

- A very useful advantage of EPs is how you can add or subtract two orientations using them. Using DCMs, we can add two rotations using:

$$[FN(\boldsymbol{\beta})] = [FB(\boldsymbol{\beta}'')][BN(\boldsymbol{\beta}')]$$

- However, using EPs directly, we find the elegant result:

$$\begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{bmatrix} \beta_0'' & -\beta_1'' & -\beta_2'' & -\beta_3'' \\ \beta_1'' & \beta_0'' & \beta_3'' & -\beta_2'' \\ \beta_2'' & -\beta_3'' & \beta_0'' & \beta_1'' \\ \beta_3'' & \beta_2'' & -\beta_1'' & \beta_0'' \end{bmatrix} \begin{pmatrix} \beta_0' \\ \beta_1' \\ \beta_2' \\ \beta_3' \end{pmatrix}$$

- Note that this matrix is orthogonal!

- By reshuffling the terms (i.e. permutation) in the last EP addition equation, we can also write this as

$$\begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{bmatrix} \beta_0' & -\beta_1' & -\beta_2' & -\beta_3' \\ \beta_1' & \beta_0' & -\beta_3' & \beta_2' \\ \beta_2' & \beta_3' & \beta_0' & -\beta_1' \\ \beta_3' & -\beta_2' & \beta_1' & \beta_0' \end{bmatrix} \begin{pmatrix} \beta_0'' \\ \beta_1'' \\ \beta_2'' \\ \beta_3'' \end{pmatrix}$$

- To subtract two orientations described through EPs, we can use the last two equations and exploit the orthogonality property of the 4x4 matrix to invert it and solve for either $\boldsymbol{\beta}'$ or $\boldsymbol{\beta}''$.

Euler Parameter Differential Equation

- Using the differential equation of DCMs, and the relationship between EPs and DCMs, we can derive the differential kinematic equations of Euler parameters.
- However, this is a rather lengthy and algebraically complex task. The end result is the amazingly simple bi-linear result:

$$\begin{pmatrix} \dot{\beta}_0 \\ \dot{\beta}_1 \\ \dot{\beta}_2 \\ \dot{\beta}_3 \end{pmatrix} = \frac{1}{2} \begin{bmatrix} \beta_0 & -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_1 & \beta_0 & -\beta_3 & \beta_2 \\ \beta_2 & \beta_3 & \beta_0 & -\beta_1 \\ \beta_3 & -\beta_2 & \beta_1 & \beta_0 \end{bmatrix} \begin{pmatrix} 0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

- Rearranging the terms on the right hand side of this differential equation, we can also write this as

$$\begin{pmatrix} \dot{\beta}_0 \\ \dot{\beta}_1 \\ \dot{\beta}_2 \\ \dot{\beta}_3 \end{pmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{bmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$$

- Having the differential equation depend linearly on the EPs is important in estimation theory. This makes the EPs ideal coordinate candidates to be used in a Kalman filter (Spacecraft orientation estimator).

2^{nd} Euler Parameter Differential Kinematic Eqs.

- The EP differential equations can also be written in the following convenient form for numerical integration:

$$\dot{\boldsymbol{\beta}} = \frac{1}{2}[B(\boldsymbol{\beta})]\boldsymbol{\omega} \qquad [B(\boldsymbol{\beta})] = \begin{bmatrix} -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_0 & -\beta_3 & \beta_2 \\ \beta_3 & \beta_0 & -\beta_1 \\ -\beta_2 & \beta_1 & \beta_0 \end{bmatrix}$$

- The [*B*] matrix satisfies the following useful identities:

$$[B(\boldsymbol{\beta})]^T \boldsymbol{\beta} = \mathbf{0}$$

$$[B(\boldsymbol{\beta})]^T \boldsymbol{\beta}' = -[B(\boldsymbol{\beta}')]^T \boldsymbol{\beta}$$

3rd Euler Parameter Differential Kinematic Eqs.

- In control applications, the scalar and vector components of the Euler parameters are sometimes treated separately.

Define: $$\boldsymbol{\epsilon} \equiv (\beta_1, \beta_2, \beta_3)^T$$

Define: $$[T(\beta_0, \boldsymbol{\epsilon})] = \beta_0[I_{3\times3}] + [\tilde{\boldsymbol{\epsilon}}]$$

Differential Equation:

$$\dot{\beta}_0 = -\frac{1}{2}\boldsymbol{\epsilon}^T\boldsymbol{\omega} = -\frac{1}{2}\boldsymbol{\omega}^T\boldsymbol{\epsilon}$$

$$\dot{\boldsymbol{\epsilon}} = \frac{1}{2}[T]\boldsymbol{\omega}$$

Conclusion

- Non-singular, redundant set of attitude coordinates
- Euler parameter vector must abide by the unit length constraint
- There are two sets of EPs that describe a particular orientation (short and long way round)
- Convenient method to add two EP vectors
- Linear differential kinematic equations