Lyapunov Optimal Feedback

What if your thrusters are just too wimpy?

Stabilization of General System

Generalized Coordinates: $(q_i, \dot{q}_i)$

Goal: $\dot{q}_i \to 0$

EOM:

$$[M(\boldsymbol{q})]\ddot{\boldsymbol{q}} = -[\dot{M}(\boldsymbol{q}, \dot{\boldsymbol{q}})]\dot{\boldsymbol{q}} + \frac{1}{2}\dot{\boldsymbol{q}}^T[M_{\boldsymbol{q}}(\boldsymbol{q})]\dot{\boldsymbol{q}} + \boldsymbol{Q}$$

If we use the kinetic energy T as the Lyapunov function of this system, then we find:

$$V(\dot{\boldsymbol{q}}) = T = \frac{1}{2}\dot{\boldsymbol{q}}^T[M(\boldsymbol{q})]\dot{\boldsymbol{q}}$$

Using the work/energy relationship, we can write

$$\dot{V} = \sum_{i=1}^{N} \dot{q}_i Q_i$$

Setting the control equal to $Q_i = -K_i \dot{q}_i$

yields $$\dot{V} = \sum_{i=1}^{N} -K_i \dot{q}_i^2 < 0$$

Next, what if the control authority magnitude is limited?

1st approach:

We could reduce our feedback gains K_i such that

$$|Q_i| = |K_i \dot{q}_i| \leq Q_{i_{\max}}$$

is true for all state errors $\dot{q}_i$ considered.

reduced control performance!

2nd approach:

We would like to be able to handle control saturation without sacrificing performance (as much) throughout the maneuver!

Let us treat the saturated control problem as an optimization problem. For stability, we require the Lyapunov rate to be negative semi-definite:

$$\dot{V} \leq 0$$

Thus, we define a cost function which is equal to this Lyapunov rate, and aim to minimize it!

$$\boxed{J = \dot{V} = \sum_{i=1}^{n} \dot{q}_i Q_i}$$

We define a ***Lyapunov optimal control law*** to be one that minimizes the Lyapunov rate function.

Given a limited amount of control authority, the Lyapunov optimal rate control is simply

$$Q_i = -Q_{i_{\max}} \operatorname{sgn}(\dot{q}_i)$$

which yields

$$J = \dot{V} = \sum_{i=1}^{N} -Q_{i_{\max}} \dot{q}_i \operatorname{sgn}(\dot{q}_i)$$

Note that this direct implementation will have chatter issues around the target state values.

The following control is only Lyapunov optimal during saturation periods, but avoids the zero crossing chatter.

$$Q_i = \begin{cases} -K_i \dot{q}_i & \text{for } |K_i \dot{q}_i| \leq Q_{i_{\max}} \\ -Q_{i_{\max}} \operatorname{sgn}(\dot{q}_i) & \text{for } |K_i \dot{q}_i| > Q_{i_{\max}} \end{cases}$$

Note that individual control components can be saturated, while others are not!

Saturated Attitude Control

- Next we study attitude control laws when the external control torque is saturated in one or more of its components.

Case 1: Attitude Tracking Problem

Un-saturated control found previously:

$$\boldsymbol{u}_{\text{us}} = -K\boldsymbol{\sigma} - [P]\delta\boldsymbol{\omega} + [I]\left(\dot{\boldsymbol{\omega}}_r - [\tilde{\omega}]\boldsymbol{\omega}_r\right) + [\tilde{\omega}][I]\boldsymbol{\omega} - \boldsymbol{L}$$

Corresponding Lyapunov rate:

$$\dot{V} = \delta\boldsymbol{\omega}^T\left(-[\tilde{\omega}][I]\boldsymbol{\omega} + \boldsymbol{u} - [I]\left(\dot{\boldsymbol{\omega}}_r - [\tilde{\omega}]\boldsymbol{\omega}_r\right) + K\boldsymbol{\sigma} + \boldsymbol{L}\right)$$

Lyapunov optimal saturated control strategy:

$$u_i = \begin{cases} u_{\text{us}_i} & \text{for } |u_{\text{us}_i}| \le u_{\max_i} \\ u_{\text{us}_i} \cdot \operatorname{sgn}(u_{\text{us}_i}) & \text{for } |u_{\text{us}_i}| > u_{\max_i} \end{cases}$$

Conservative stability boundary (sufficient condition):

$$\left|\left([I]\left(\dot{\boldsymbol{\omega}}_r - [\tilde{\omega}]\boldsymbol{\omega}_r\right) + [\tilde{\omega}][I]\boldsymbol{\omega} - K\boldsymbol{\sigma} - \boldsymbol{L}\right)_i\right| \le u_{\max_i}$$

If this is violated, we don't necessarily have instability!

Case 2: Attitude Regulator Problem

In this case the unsaturated control torque on the previous slide simplifies to:

$$\boldsymbol{u}_{\text{us}} = -K\boldsymbol{\sigma} - [P]\boldsymbol{\omega}$$

while the Lyapunov rate expression reduces to:

$$\dot{V} = \boldsymbol{\omega}^T \left(\boldsymbol{u} + K\boldsymbol{\sigma}\right)$$

A conservative stability boundary which guarantees that $\dot{V} \leq 0$ is

$$K|\sigma_i| \leq u_{\max_i}$$

However, note that the MRP attitude error are typically bounded by switching between the original and shadow sets!

$$|\boldsymbol{\sigma}| \leq 1$$

$$K \leq u_{\max_i}$$

Case 3: Rate Regulator Problem

A common situation just requires the current spacecraft spin to be arrested. Essentially, the final attitude is irrelevant and we set $K = 0$.

The Lyapunov optimal saturated control strategy in this case reduces to:

$$u_i = \begin{cases} -P_{ii}\omega_i & \text{for } |P_{ii}\omega_i| \leq u_{\max_i} \\ -u_{\max_i} \cdot \operatorname{sgn}(\omega_i) & \text{for } |P_{ii}\omega_i| > u_{\max_i} \end{cases}$$

The corresponding Lyapunov rate function is

$$\dot{V}(\boldsymbol{\omega}) = -\sum_{i=1}^{M} P_{ii}\omega_i^2 - \sum_{i=M+1}^{N} \omega_i u_{\text{maxi}} \cdot \operatorname{sgn}(\omega_i) < 0$$

Note, this function is *negative definite*, and thus globally asymptotically stabilizing!

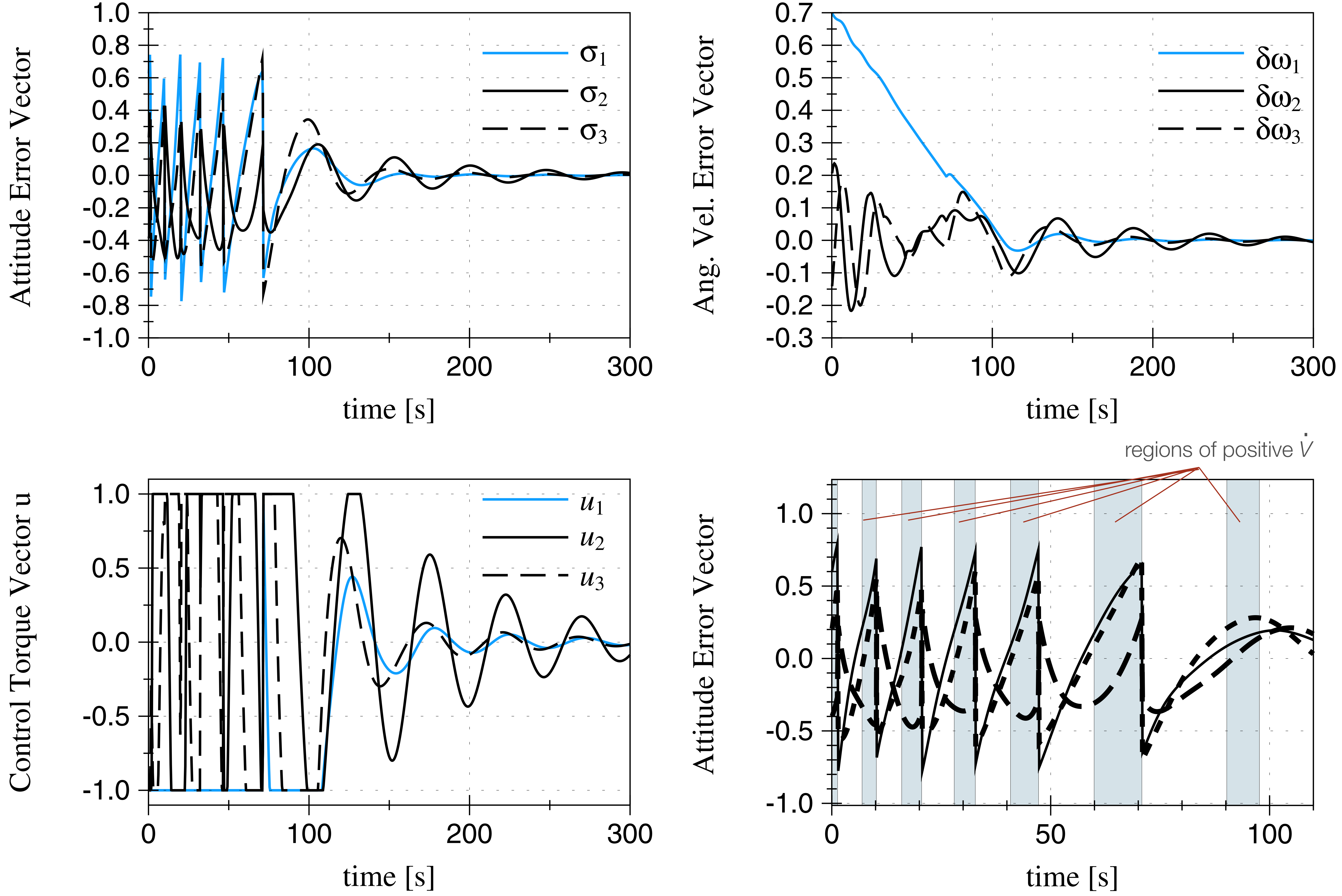
Further, because no inertia terms are used in this saturated control, it is very robust to modeling errors.

Saturated Attitude Control Example:

Parameter	Value	Units
I_1	140.0	kg-m^2
I_2	100.0	kg-m^2
I_3	80.0	kg-m^2
$\boldsymbol{\sigma}(t_0)$	$[0.60\ -0.40\ 0.20]$	
$\boldsymbol{\omega}(t_0)$	$[0.70\ 0.20\ -0.15]$	rad/sec
$[P]$	$[18.67\ 2.67\ 10.67]$	kg-m^2/sec
K	7.11	kg-m^2/sec^2

Torque saturation level: $u_{\text{max}_i} = 1 \text{ N m}$

Unsaturated control law: $\boldsymbol{u} = -K\boldsymbol{\sigma} - [P]\boldsymbol{\omega}$



Attitude Error Vector
time [s]
σ1
σ2
σ3
Ang. Vel. Error Vector
δω1
δω2
δω3
Control Torque Vector u
u1
u2
u3
regions of positive $\dot{V}$
Attitude Error Vector

Linear Closed-Loop Dynamics

Surprisingly elegant inverse-kinematic solutions...

The idea...

- An extensive body of literature exists on the behavior of ***linear*** closed-loop dynamics (CLD) of the form:

$$\boxed{}\,\ddot{\boldsymbol{\epsilon}} + P\dot{\boldsymbol{\epsilon}} + K\boldsymbol{\epsilon} = 0$$

PD-feedback control

$$\boxed{}\,\ddot{\boldsymbol{\epsilon}} + P\dot{\boldsymbol{\epsilon}} + K\boldsymbol{\epsilon} + K_i \int_0^t \boldsymbol{\epsilon}\mathrm{d}t = 0$$

PID-feedback control

Note: the proposed CLD doesn't require any knowledge of the system mass or inertia properties compared to

$$[I]\underbrace{\delta\dot{\boldsymbol{\omega}}}_{[\dot{B}]^{-1}\dot{\boldsymbol{\sigma}} + [B]^{-1}\ddot{\boldsymbol{\sigma}}} + [P]\underbrace{\delta\boldsymbol{\omega}}_{[B]^{-1}\dot{\boldsymbol{\sigma}}} + K\boldsymbol{\sigma} = 0$$

With the complicated differential kinematic equations, what a mess this will be!

...or will it?

Setup

- Let us solve for the linear CLD using Euler parameters (β_1,β_2,β_3).*

Desired CLD:

$$\boxed{\ddot{\boldsymbol{\epsilon}} + P\dot{\boldsymbol{\epsilon}} + K\boldsymbol{\epsilon} = 0}$$

Find $\boldsymbol{u}$ to achieve CLD

EOM:

$$[I]\dot{\boldsymbol{\omega}} + [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} = \boxed{\boldsymbol{u}}$$

Attitude State Vector:

$$\boldsymbol{\epsilon} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \sin\left(\frac{\Phi}{2}\right) \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

Differential Kinematic Equations:

$$\dot{\boldsymbol{\epsilon}} = \frac{1}{2}[T]\boldsymbol{\omega}$$

$$[T(\beta_0, \boldsymbol{\epsilon})] = \begin{bmatrix} \beta_0 & -\beta_3 & \beta_2 \\ \beta_3 & \beta_0 & -\beta_1 \\ -\beta_2 & \beta_1 & \beta_0 \end{bmatrix} = \beta_0[I_{3\times 3}] + [\tilde{\boldsymbol{\epsilon}}]$$

*Paielli, R. A. and Bach, R. E., “Attitude Control with Realization of Linear Error Dynamics,” Journal of Guidance, Control and Dynamics, Vol. 16, No. 1, Jan.–Feb. 1993, pp. 182–189.

$$\boxed{\dot{\boldsymbol{\epsilon}} = \frac{1}{2}[T]\boldsymbol{\omega}}$$

$$\Big\Downarrow \frac{\mathrm{d}}{\mathrm{d}t}$$

$$\ddot{\boldsymbol{\epsilon}} = \frac{1}{2}[T]\dot{\boldsymbol{\omega}} + \frac{1}{2}\boxed{[\dot{T}]\boldsymbol{\omega}}$$

$$[T]\boldsymbol{\omega} = \beta_0\boldsymbol{\omega} + [\tilde{\boldsymbol{\epsilon}}]\boldsymbol{\omega}$$

$$\Big\Downarrow \frac{\mathrm{d}}{\mathrm{d}t}$$

$$\boxed{[\dot{T}]\boldsymbol{\omega}} = \boxed{\dot{\beta}_0}\boldsymbol{\omega} - [\tilde{\boldsymbol{\omega}}]\boxed{\dot{\boldsymbol{\epsilon}}}$$

$$\boxed{\dot{\beta}_0 = -\frac{1}{2}\boldsymbol{\epsilon}^T\boldsymbol{\omega}}$$

Note:

$$[\tilde{\boldsymbol{\omega}}][T]\boldsymbol{\omega} = [\tilde{\boldsymbol{\omega}}]\left(\beta_0[I_{3\times3}] + [\tilde{\boldsymbol{\epsilon}}]\right)\boldsymbol{\omega}$$

$$= \cancel{[\tilde{\boldsymbol{\omega}}]\boldsymbol{\omega}}\beta_0 + [\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{\epsilon}}]\boldsymbol{\omega}$$

$$= -[\tilde{\boldsymbol{\omega}}][\tilde{\boldsymbol{\omega}}]\boldsymbol{\epsilon}$$

$$[\tilde{\boldsymbol{a}}][\tilde{\boldsymbol{a}}] = \boldsymbol{a}\boldsymbol{a}^T - \boldsymbol{a}^T\boldsymbol{a}I_{3\times3}$$

$$[\tilde{\boldsymbol{\omega}}][T]\boldsymbol{\omega} = -\left(\boldsymbol{\omega}\boldsymbol{\omega}^T - \omega^2[I_{3\times3}]\right)\boldsymbol{\epsilon}$$

$$= \boxed{-\boldsymbol{\epsilon}^T\boldsymbol{\omega}\boldsymbol{\omega} + \omega^2\boldsymbol{\epsilon}}$$

$$\ddot{\boldsymbol{\epsilon}} = \frac{1}{2}[T]\dot{\boldsymbol{\omega}} - \frac{1}{4}\left(\boldsymbol{\epsilon}^T\boldsymbol{\omega}\boldsymbol{\omega} + \boxed{[\tilde{\boldsymbol{\omega}}][T]\boldsymbol{\omega}}\right) \quad \Rightarrow \quad \ddot{\boldsymbol{\epsilon}} = \frac{1}{2}[T]\dot{\boldsymbol{\omega}} - \frac{1}{4}\omega^2\boldsymbol{\epsilon}$$

$$\ddot{\boldsymbol{\epsilon}} = \tfrac{1}{2}[T]\dot{\boldsymbol{\omega}} - \tfrac{1}{4}\omega^2\boldsymbol{\epsilon} \qquad \dot{\boldsymbol{\epsilon}} = \tfrac{1}{2}[T]\boldsymbol{\omega}$$

$$\ddot{\boldsymbol{\epsilon}} + P\dot{\boldsymbol{\epsilon}} + K\boldsymbol{\epsilon} = 0$$

$$[T]\left(\underbrace{\dot{\boldsymbol{\omega}} + P\boldsymbol{\omega} + [T]^{-1}\left(-\tfrac{1}{2}\omega^2\boldsymbol{\epsilon} + 2K\boldsymbol{\epsilon}\right)}_{\text{must be zero}}\right) = 0$$

Can [*T*] be inverted?

$$[T]^{-1} = [T]^T + \tfrac{1}{\beta_0}\boldsymbol{\epsilon}\boldsymbol{\epsilon}^T$$ Always possible except 180º case.

$$[T] = \beta_0 [I_{3\times 3}] + [\tilde{\boldsymbol{\epsilon}}]$$

$$[T]^{-1} = [T]^T + \frac{1}{\beta_0}\boldsymbol{\epsilon}\boldsymbol{\epsilon}^T$$

$$\dot{\boldsymbol{\omega}} + P\boldsymbol{\omega} + [T]^{-1}\left(-\frac{1}{2}\omega^2\boldsymbol{\epsilon} + 2K\boldsymbol{\epsilon}\right) = 0$$

Bunch of algebra...

$$\boxed{\dot{\boldsymbol{\omega}} = -P\boldsymbol{\omega} - 2\left(K - \frac{\omega^2}{4}\right)\frac{\boldsymbol{\epsilon}}{\beta_0}}$$

Remarkably simply required angular acceleration expression!

$$[I]\dot{\boldsymbol{\omega}} + [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} = \boldsymbol{u}$$

$$\boxed{\boldsymbol{u} = [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} + [I]\left(-P\boldsymbol{\omega} - 2\left(K - \frac{\omega^2}{4}\right)\frac{\boldsymbol{\epsilon}}{\beta_0}\right)}$$

Not that even though we started out with the non-singular Euler parameters, the attitude feedback terms is the Gibbs vector which is singular for 180º rotations!

Comparison to Gibbs Feedback

Paielli/Back feedback control law:

$$\boldsymbol{u} = [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} + [I]\left(-P\boldsymbol{\omega} - 2\left(K - \frac{\omega^2}{4}\right)\frac{\boldsymbol{\epsilon}}{\beta_0}\right)$$

Gibbs-vector Feedback

$$\boldsymbol{u} = [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} - P\boldsymbol{\omega} - K\boldsymbol{q}$$

Integral Feedback

- By starting out with a different desired linear CLD, we can also include an integral feedback term:

$$\ddot{\boldsymbol{\epsilon}} + P\dot{\boldsymbol{\epsilon}} + K\boldsymbol{\epsilon} + K_i \int_0^t \boldsymbol{\epsilon}\mathrm{d}t = 0$$

Bunch of more algebra following earlier steps...

$$\dot{\boldsymbol{\omega}} = -P\boldsymbol{\omega} - 2\left(K - \frac{\omega^2}{4}\right)\frac{\boldsymbol{\epsilon}}{\beta_0} - 2K_i\left([T]^T + \frac{1}{\beta_0}\boldsymbol{\epsilon}\boldsymbol{\epsilon}^T\right)\int_0^t \boldsymbol{\epsilon}\mathrm{d}t$$

$$[I]\dot{\boldsymbol{\omega}} + [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} = \boldsymbol{u}$$

Tracking Problem

- This linear CLD behavior can also be achieved for an attitude tracking problem with a time-varying reference attitude *R*.

Kinematic expressions:

$$\dot{\boldsymbol{\epsilon}} = \frac{1}{2}[T]\delta\boldsymbol{\omega}$$

$$\delta\boldsymbol{\omega} = \boldsymbol{\omega} - \boldsymbol{\omega}_r$$

Even more algebra of the same sort...

$$\dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_r - P\delta\boldsymbol{\omega} - 2\left(K - \frac{\delta\omega^2}{4}\right)\frac{\boldsymbol{\epsilon}}{\beta_0}$$

$$[I]\dot{\boldsymbol{\omega}} + [\tilde{\omega}][I]\boldsymbol{\omega} = \boldsymbol{u}$$

Linear MRP CLD

- The previous linear CLD development only yielded elegant results because of some special properties of Euler parameter kinematic equations. However, the resulting feedback was singular at 180°.

$$[T]^{-1} = [T]^T + \frac{1}{\beta_0}\boldsymbol{\epsilon}\boldsymbol{\epsilon}^T$$

Important property for simple linear EP CLD.

MRP Kinematic Equations:

$$\dot{\boldsymbol{\sigma}} = \frac{1}{4}[B(\boldsymbol{\sigma})]\boldsymbol{\omega}$$

$$[B]^{-1} = \frac{1}{(1+\sigma^2)^2}[B]^T$$

elegant near-orthogonal inverse property.

Desired Linear MRP CLD:

$$\ddot{\boldsymbol{\sigma}} + P\dot{\boldsymbol{\sigma}} + K\boldsymbol{\sigma} = 0$$

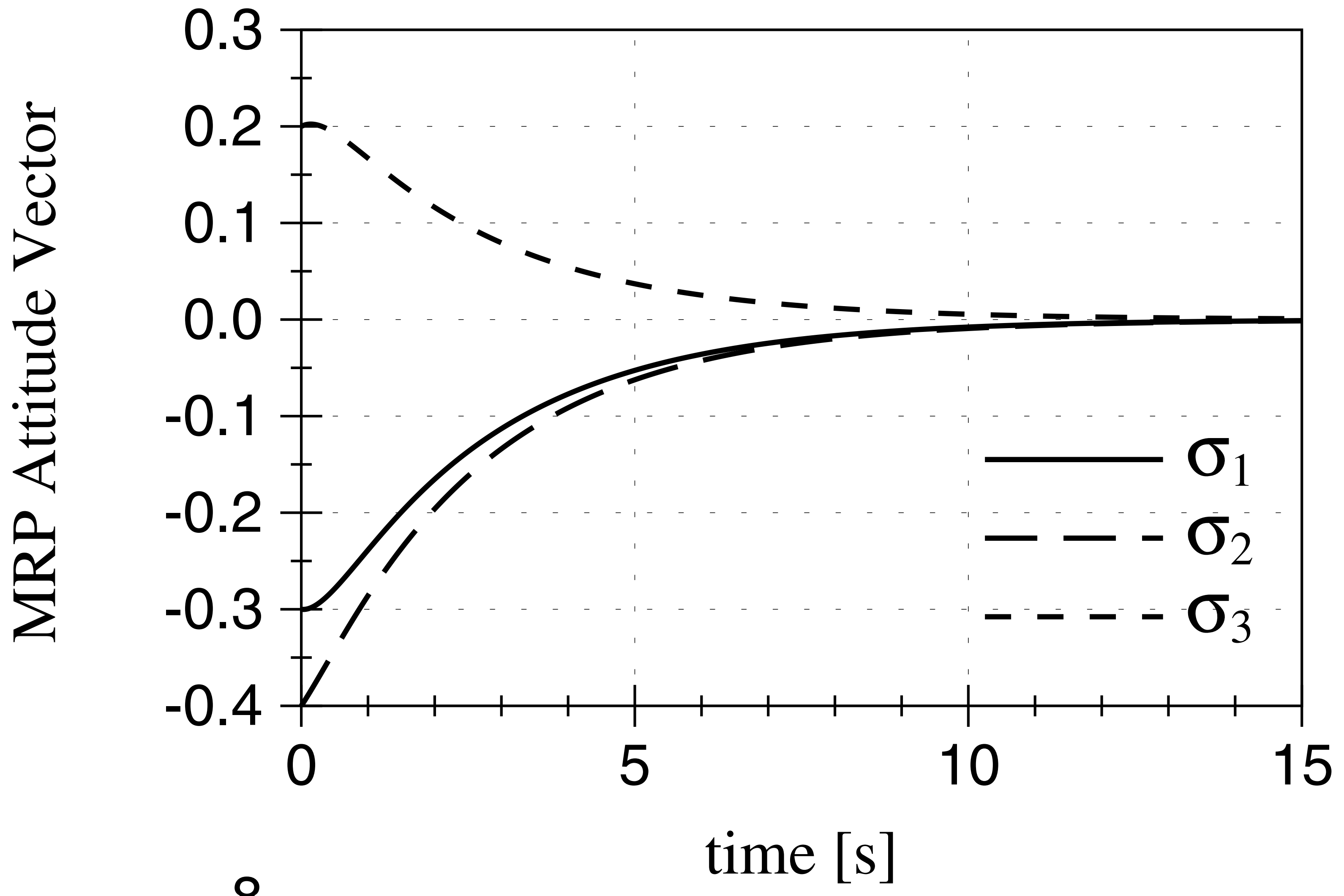
MRP Feedback Control

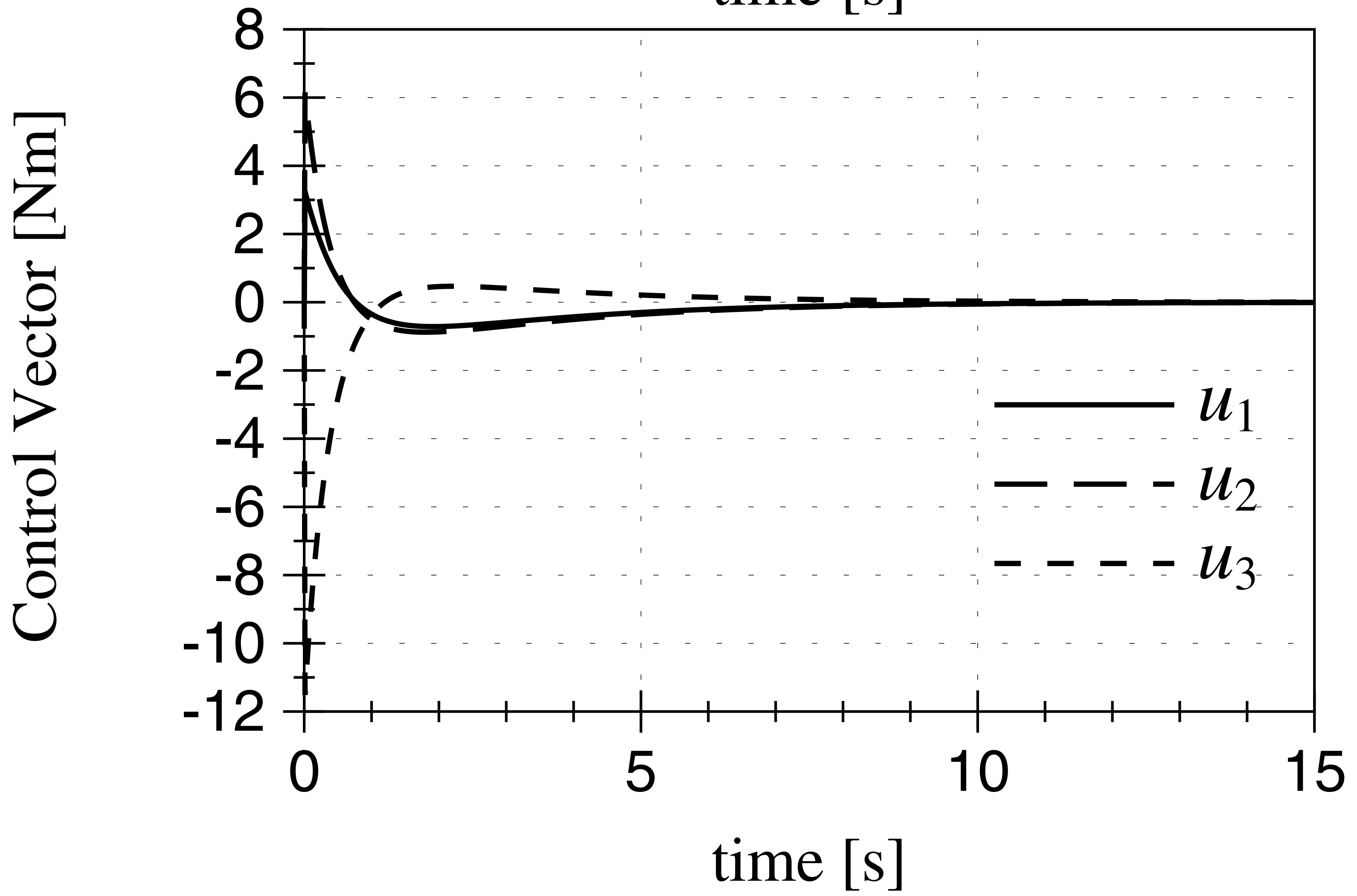
$$\dot{\boldsymbol{\omega}} = -P\boldsymbol{\omega} - \left(\boldsymbol{\omega}\boldsymbol{\omega}^T + \left(\frac{4K}{1+\sigma^2} - \frac{\omega^2}{2}\right)[I_{3\times3}]\right)\boldsymbol{\sigma}$$

This feedback is non-singular at 180° and globally valid by switching to the shadow set!

Linear MRP CLD Example:

Parameter	Value	Units
I_1	30.0	kg-m^2
I_2	20.0	kg-m^2
I_3	10.0	kg-m^2
$\boldsymbol{\sigma}(t_0)$	$[-0.30\ -0.40\ 0.20]$	
$\boldsymbol{\omega}(t_0)$	$[0.20\ 0.20\ 0.20]$	rad/sec
$[P]$	3.0	kg-m^2/sec
K	1.0	kg-m^2/sec^2





Goal: Regulator problem which arrests satellite motion at the zero attitude.