Momentum Exchange Devices

ASEN 5010

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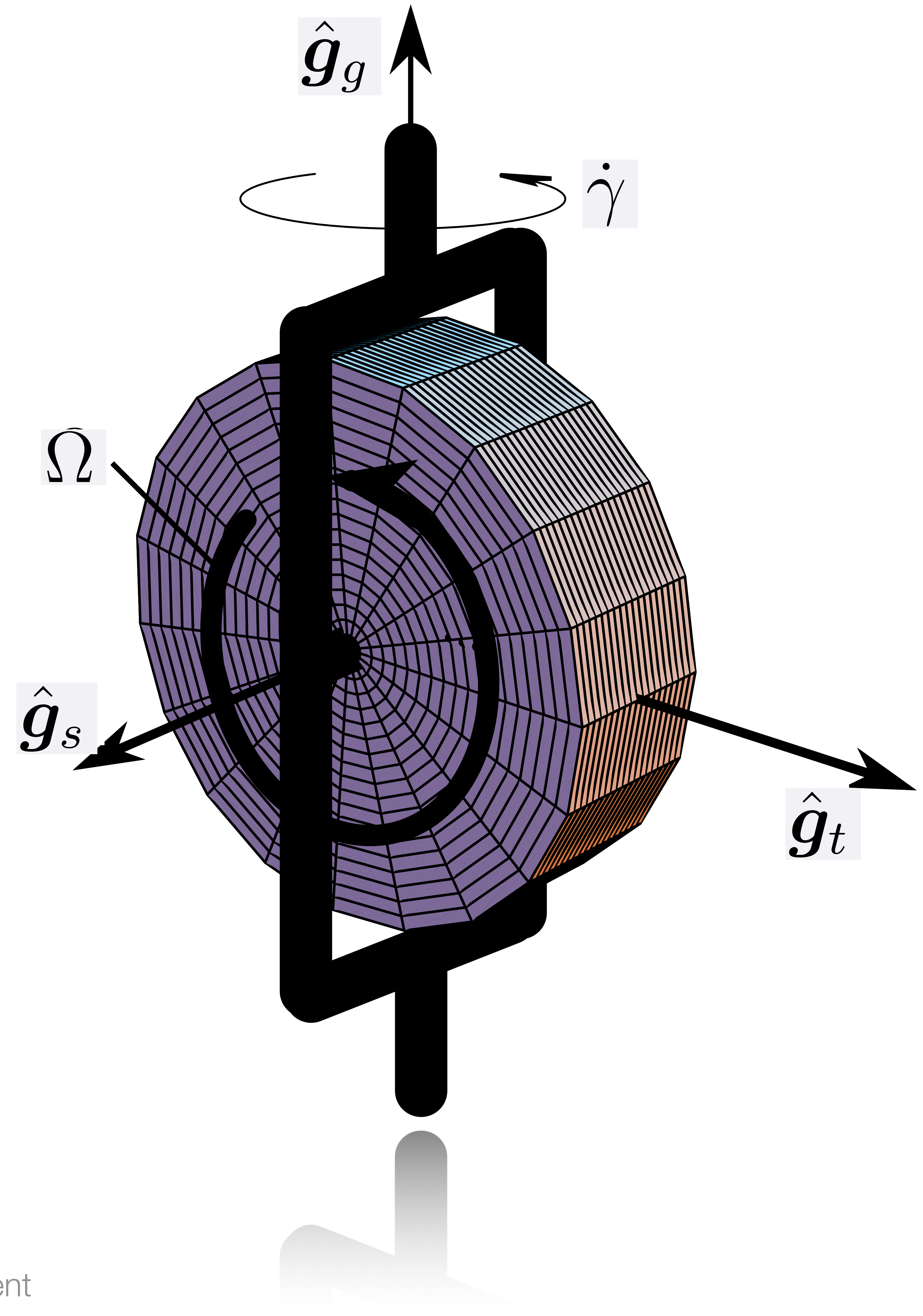
Outline

- Momentum Control Devices
- Equations of motion of VSCMG
  - single VSCMG
  - motor torque calculation
  - cluster of VSCMG
- Momentum Device Control
  - Overview of RW control solution

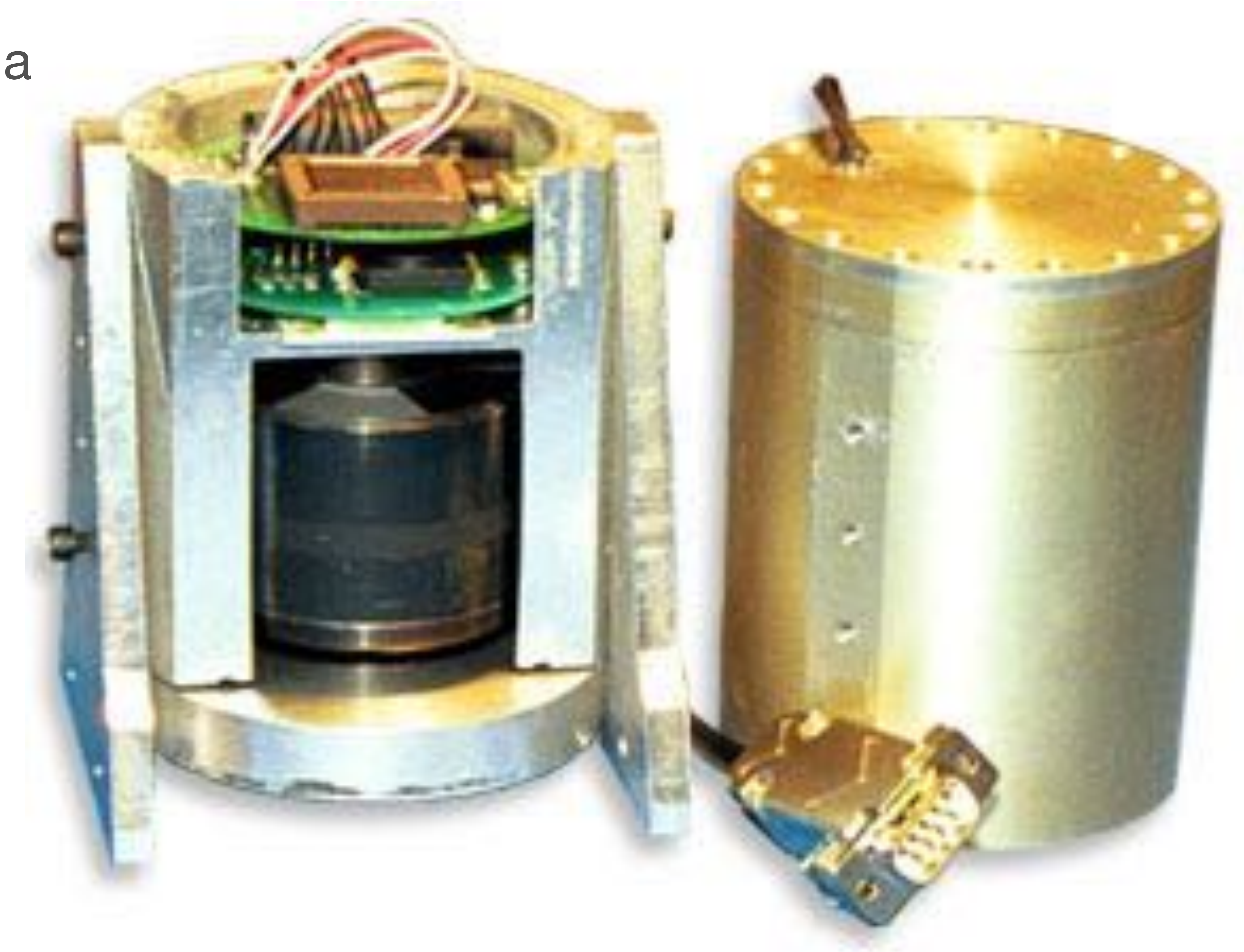


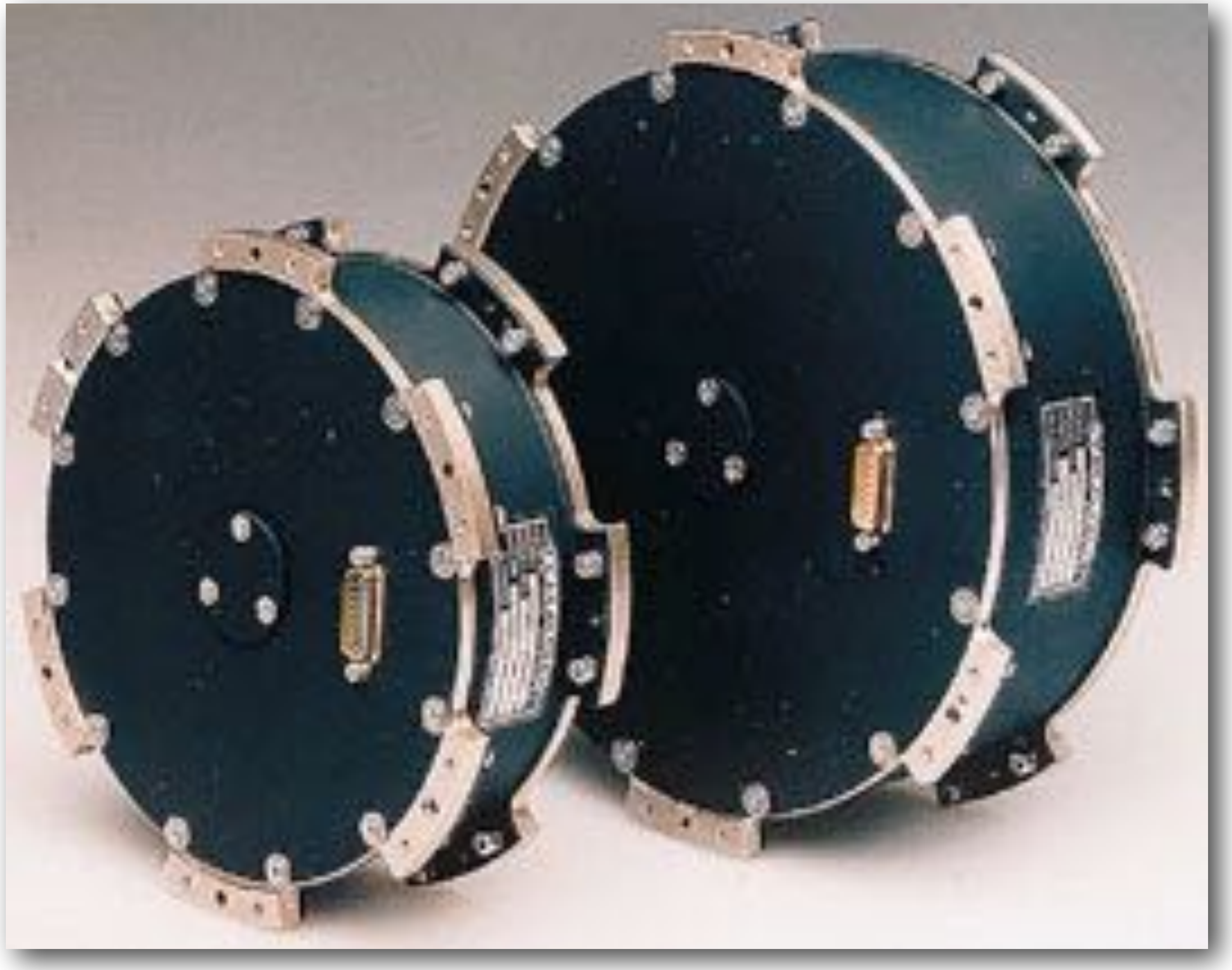
Momentum Control Devices

Spinning hardware “thingies” to rotate the spacecraft...

Reaction Wheels (RW)

- By increasing or decreasing the spin of a disk, a torque is applied onto the spacecraft.
- The torque is parallel to the disk spin axis.
- Simple mechanical device.
- Multiple disks can generate arbitrary torque.
- Wheels can saturate (reach a maximum spin speed.





ITHACO's low-cost but highly reliable reaction wheel designs keep spacecraft correctly oriented as they spin through space. (company description of device)
http://www.sti.nasa.gov/tto/spinoff1997/t2.html

Inside CTA Space System's High Torque Reaction/Momentum Wheel is an innovative flywheel/bearing arrangement that allows the entire rotating system to be balanced after it is assembled.   (company description)
http://www.sti.nasa.gov/tto/spinoff1997/t3.html

Honeywell's reaction wheel assemblies (RWA) and momentum wheel assemblies (MWA) are reliable, lightweight solutions to a variety of momentum control needs, providing stability and attitude-control for small to very large, heavy spacecraft. Earth-pointing satellites and multiple-satellite communication networks are examples of applications that require the fine attitude control that Honeywell RWAs provide. RWAs and MWAs from Honeywell have accumulated more than 9 million hours, or more than a thousand years, in space and have never caused a mission to end prematurely.

Control Moment Gyroscope (CMG)

- Another popular attitude control device.
- A disk is spinning at a constant rate.
- By rotating this disk (called gimbaling), a torque is applied through the gyroscopic effect.
- For a small torque to gimbal the disk, a large torque is produced onto the spacecraft.

Control Moment Gyroscope (CMG)

- Mechanically more complex device than RWs
- Control laws are much more complicated.
- Very large torques can be produced (good for rapid reorientation or large spacecraft such as space station)
- Singular configurations exist where the required torque cannot be produced.

A CMG contains two torque motors. One to keep the disk spinning at a constant rate, the other to gimbal the spinning disk.

A typical CMG setup has 4 devices aligned in a pyramid configuration.

Equations of Motion

Let’s learn to be one with the truth of gyroscopics...

Spacecraft with 1 VSCMG

- A Variable-Speed CMG is a classical CMG device where the disk speed is left to be variable.
- Think of a VSCMG device as a hybrid CMG/RW.
- Convenient when developing the equations of motion, since we get both the CMG and RW equations of motion by doing the work only once!!
- Researchers have started to look into actually building and flying a VSCMG devices.
  - Avoids classical CMG singularities
  - Highly redundant system (more robust to component failure)
  - Can be used as a combined power storage/attitude control device.

Battle Plan...

- To derive the equations of motion of a spacecraft with a single VSCMG, we recall Euler's equation

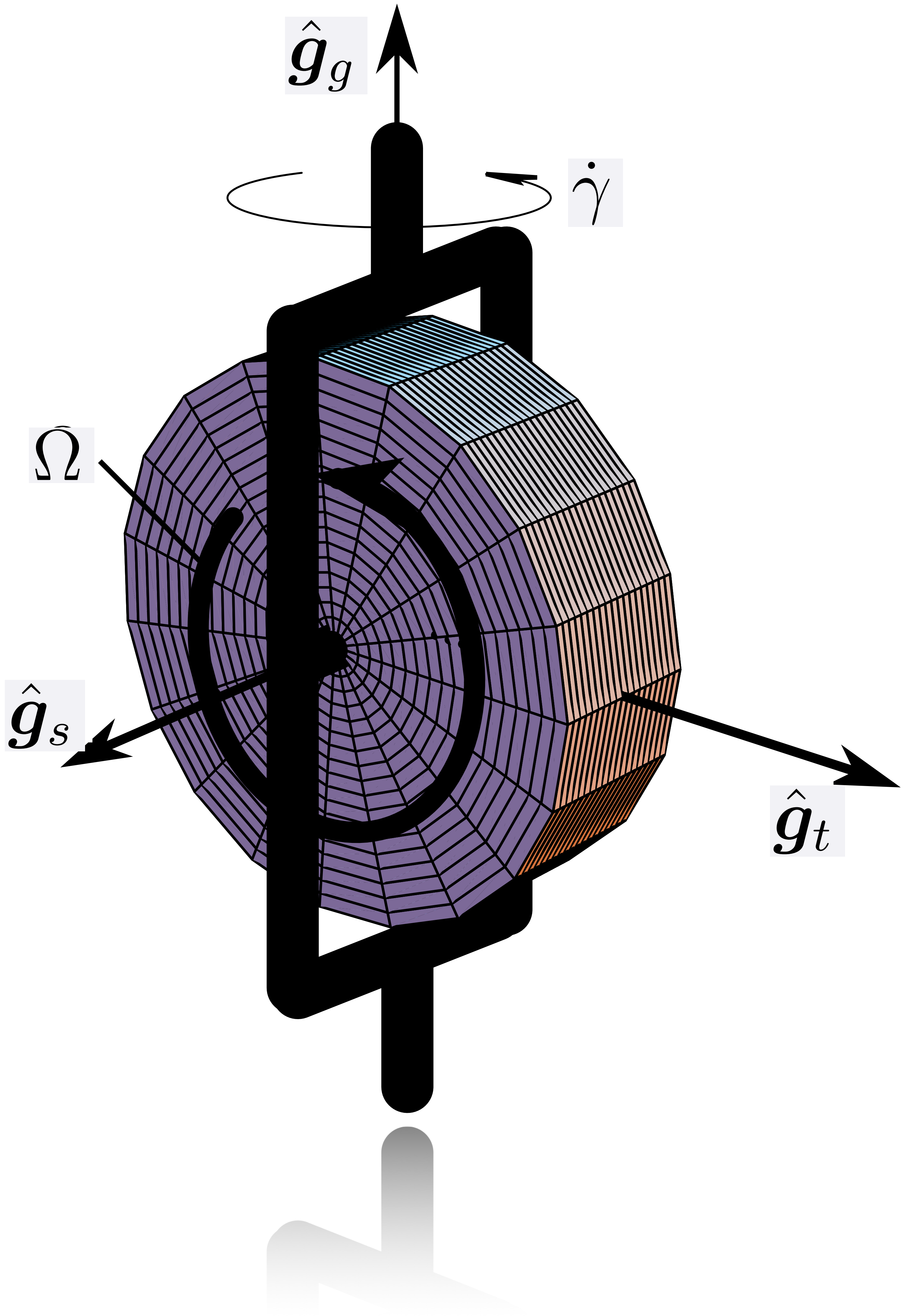
$$\dot{\boldsymbol{H}} = \boldsymbol{L}$$

- We will need to find the total angular momentum vector *H* for the combined spacecraft/VSCMG system. Once we have this expression, we can then differentiate it to get the desired equations of motion.
- To manage all this algebra, we will break up the whole system into the spacecraft part, the CMG momentum and the RW momentum.

VSCMG Frames

- The VSCMG spin axis is $\hat{\boldsymbol{g}}_s$
- The gimbal axis is $\hat{\boldsymbol{g}}_g$
- The disk spin rate is $\Omega(t)$
- The gimbal rate is $\dot{\gamma}(t)$
- The gimbal coordinate frame G is

$$\mathcal{G} : \{\hat{\boldsymbol{g}}_s, \hat{\boldsymbol{g}}_t, \hat{\boldsymbol{g}}_g\}$$



VSCMG Frames

- Note that the gimbal axis is fixed with respect to the spacecraft body frame *B*.
- The gimbal frame *G* angular velocity is

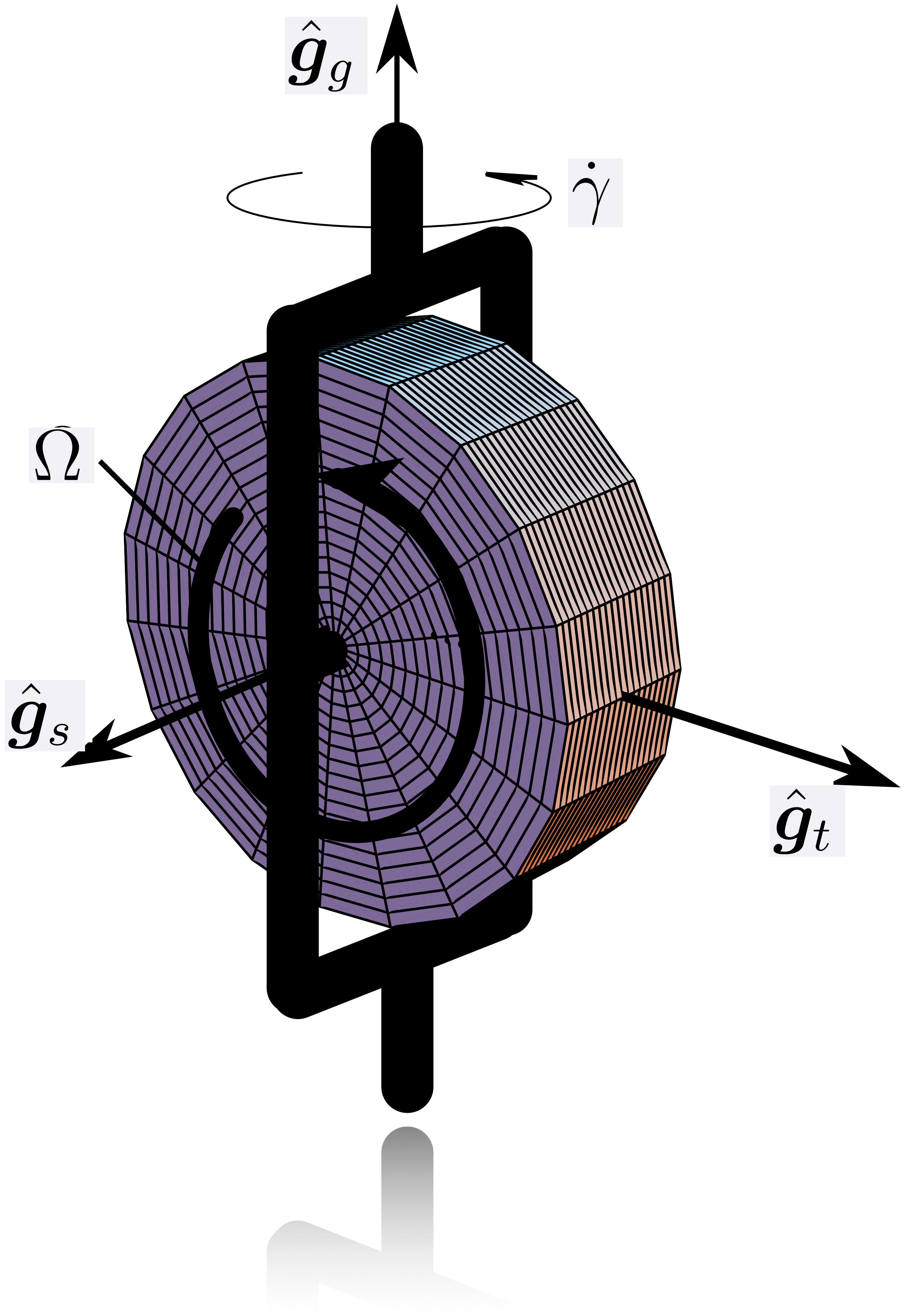
$$\boldsymbol{\omega}_{\mathcal{G}/\mathcal{B}} = \dot{\gamma}\hat{\boldsymbol{g}}_g$$

- Let *W* be a frame that tracks the motion of the reaction wheel.

$$\mathcal{W} : \{\hat{\boldsymbol{g}}_s, \hat{\boldsymbol{w}}_t, \hat{\boldsymbol{w}}_g\}$$

- It's angular velocity is

$$\boldsymbol{\omega}_{\mathcal{W}/\mathcal{G}} = \Omega\hat{\boldsymbol{g}}_s$$



VSCMG Inertias

- Let the gimbal frame inertia be

$$[I_G] = {}^{\mathcal{G}}[I_G] = {}^{\mathcal{G}}\begin{bmatrix} I_{G_s} & 0 & 0 \\ 0 & I_{G_t} & 0 \\ 0 & 0 & I_{G_g} \end{bmatrix}$$

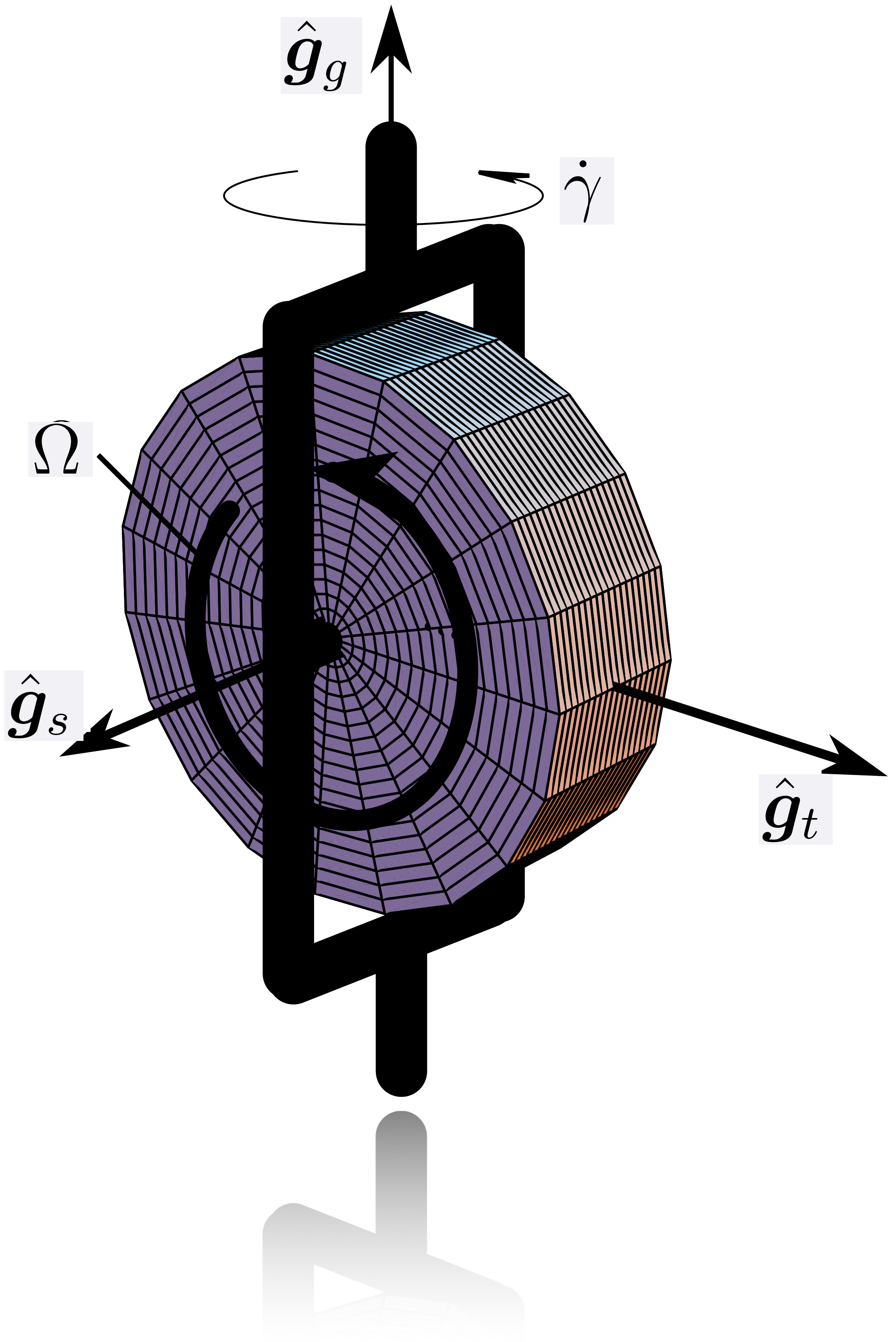
- The wheel (disk) inertia is

Why I_W equal I_G matrix?

$$[I_W] = {}^{\mathcal{W}}[I_W] = {}^{\mathcal{W}}\begin{bmatrix} I_{W_s} & 0 & 0 \\ 0 & I_{W_t} & 0 \\ 0 & 0 & I_{W_t} \end{bmatrix}$$

- Due to symmetry of the disk, we find that

$$ {}^{\mathcal{W}}[I_W] = {}^{\mathcal{G}}[I_W] $$



- Assuming the gimbal frame unit vectors are expressed in body frame vector components, then the rotation matrix [*BG*] can be expressed through

$$[BG] = \left[\hat{\boldsymbol{g}}_s \; \hat{\boldsymbol{g}}_t \; \hat{\boldsymbol{g}}_g\right]$$

- The gimbal frame and disk inertias (which were given in gimbal frame components), can be written in body frame components using

$$^{\mathcal{B}}[I_G] = [BG]^{\mathcal{G}}[I_G][BG]^T = I_{G_s}\hat{\boldsymbol{g}}_s\hat{\boldsymbol{g}}_s^T + I_{G_t}\hat{\boldsymbol{g}}_t\hat{\boldsymbol{g}}_t^T + I_{G_g}\hat{\boldsymbol{g}}_g\hat{\boldsymbol{g}}_g^T$$

$$^{\mathcal{B}}[I_W] = [BG]^{\mathcal{G}}[I_W][BG]^T = I_{W_s}\hat{\boldsymbol{g}}_s\hat{\boldsymbol{g}}_s^T + I_{W_t}\hat{\boldsymbol{g}}_t\hat{\boldsymbol{g}}_t^T + I_{W_g}\hat{\boldsymbol{g}}_g\hat{\boldsymbol{g}}_g^T$$

Angular Momentum...

- We are now ready to express the total angular momentum of the system using

$$\boldsymbol{H} = \boldsymbol{H}_B + \boldsymbol{H}_G + \boldsymbol{H}_W$$

- $\boldsymbol{H}_B$ is the angular momentum of the spacecraft itself, $\boldsymbol{H}_G$ is the angular momentum of the gimbal frame, while $\boldsymbol{H}_W$ is the angular momentum of the spinning disk.

- The spacecraft angular momentum is simply that of a rigid body:

$$\boldsymbol{H}_B = [I_s]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$$

- The inertial angular momentum of the rigid gimbal frame is

$$\boldsymbol{H}_G = [I_G]\boldsymbol{\omega}_{\mathcal{G}/\mathcal{N}}$$

- where $\boldsymbol{\omega}_{\mathcal{G}/\mathcal{N}} = \boldsymbol{\omega}_{\mathcal{G}/\mathcal{B}} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$. This can now be rewritten as

$$\boldsymbol{H}_G = \left(I_{G_s}\hat{\boldsymbol{g}}_s\hat{\boldsymbol{g}}_s^T + I_{G_t}\hat{\boldsymbol{g}}_t\hat{\boldsymbol{g}}_t^T + I_{G_g}\hat{\boldsymbol{g}}_g\hat{\boldsymbol{g}}_g^T\right)\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + I_{G_g}\dot{\gamma}\hat{\boldsymbol{g}}_g$$

- Let use introduce the angular velocity components taken along the gimbal frame axis directions:

$$\omega_s = \hat{\boldsymbol{g}}_s^T\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \qquad \omega_t = \hat{\boldsymbol{g}}_t^T\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \qquad \omega_g = \hat{\boldsymbol{g}}_g^T\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$$

$$^{\mathcal{G}}\boldsymbol{\omega} = \omega_s\hat{\boldsymbol{g}}_s + \omega_t\hat{\boldsymbol{g}}_t + \omega_g\hat{\boldsymbol{g}}_g$$

- This allows us to write the gimbal frame angular momentum expression as

$$\boldsymbol{H}_G = I_{G_s}\omega_s\hat{\boldsymbol{g}}_s + I_{G_t}\omega_t\hat{\boldsymbol{g}}_t + I_{G_g}\left(\omega_g + \dot{\gamma}\right)\hat{\boldsymbol{g}}_g$$

- The inertial angular momentum of the disk is

$$\boldsymbol{H}_W = [I_W]\boldsymbol{\omega}_{\mathcal{W}/\mathcal{N}}$$

- where $\boldsymbol{\omega}_{\mathcal{W}/\mathcal{N}} = \boldsymbol{\omega}_{\mathcal{W}/\mathcal{G}} + \boldsymbol{\omega}_{\mathcal{G}/\mathcal{B}} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$

- The momentum expression can be expanded using

$$\boldsymbol{H}_W = [I_W]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} + [I_W]\boldsymbol{\omega}_{\mathcal{G}/\mathcal{B}} + [I_W]\boldsymbol{\omega}_{\mathcal{W}/\mathcal{G}}$$

It is implied that all vectors are added with components in the same frame.

- The first term can be written as

$$\begin{aligned} [I_W]\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} &= \left(I_{W_s}\hat{\boldsymbol{g}}_s\hat{\boldsymbol{g}}_s^T + I_{W_t}\hat{\boldsymbol{g}}_t\hat{\boldsymbol{g}}_t^T + I_{W_t}\hat{\boldsymbol{g}}_g\hat{\boldsymbol{g}}_g^T\right)\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \\ &= I_{W_s}\omega_s\hat{\boldsymbol{g}}_s + I_{W_t}\omega_t\hat{\boldsymbol{g}}_t + I_{W_g}\omega_g\hat{\boldsymbol{g}}_g \end{aligned}$$

- The second two terms can be written as

$$[I_W]\boldsymbol{\omega}_{\mathcal{G}/\mathcal{B}} = {}^{\mathcal{G}}\begin{bmatrix} I_{W_s} & 0 & 0 \\ 0 & I_{W_t} & 0 \\ 0 & 0 & I_{W_t} \end{bmatrix} {}^{\mathcal{G}}\begin{pmatrix} 0 \\ 0 \\ \dot{\gamma} \end{pmatrix} = I_{W_t}\dot{\gamma}\hat{\boldsymbol{g}}_g$$

$$[I_W]\boldsymbol{\omega}_{\mathcal{W}/\mathcal{G}} = {}^{\mathcal{W}}\begin{bmatrix} I_{W_s} & 0 & 0 \\ 0 & I_{W_t} & 0 \\ 0 & 0 & I_{W_t} \end{bmatrix} {}^{\mathcal{W}}\begin{pmatrix} \Omega \\ 0 \\ 0 \end{pmatrix} = I_{W_s}\Omega\hat{\boldsymbol{g}}_s$$

- Combining all these results, the spinning wheel inertial angular momentum is written as

$$\boldsymbol{H}_W = I_{W_s}\left(\omega_s + \Omega\right)\hat{\boldsymbol{g}}_s + I_{W_t}\omega_t\hat{\boldsymbol{g}}_t + I_{W_t}\left(\omega_g + \dot{\gamma}\right)\hat{\boldsymbol{g}}_g$$

Some final preparation...

- Before we begin to differentiate the system angular momentum vectors, we need to establish some useful relationships.

- The gimbal frame direction vectors can be written in terms of their initial orientations as

$$\hat{\boldsymbol{g}}_s\left(t\right) = \cos\left(\gamma(t) - \gamma_0\right)\hat{\boldsymbol{g}}_s\left(t_0\right) + \sin\left(\gamma(t) - \gamma_0\right)\hat{\boldsymbol{g}}_t\left(t_0\right)$$

$$\hat{\boldsymbol{g}}_t\left(t\right) = -\sin\left(\gamma(t) - \gamma_0\right)\hat{\boldsymbol{g}}_s\left(t_0\right) + \cos\left(\gamma(t) - \gamma_0\right)\hat{\boldsymbol{g}}_t\left(t_0\right)$$

$$\hat{\boldsymbol{g}}_g\left(t\right) = \hat{\boldsymbol{g}}_g\left(t_0\right)$$

- Note that the $\mathcal{B}$ frame derivatives of the gimbal frame unit vectors are

$$\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\left(\hat{\boldsymbol{g}}_s\right) = \dot{\gamma}\hat{\boldsymbol{g}}_t \qquad \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\left(\hat{\boldsymbol{g}}_t\right) = -\dot{\gamma}\hat{\boldsymbol{g}}_s \qquad \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\left(\hat{\boldsymbol{g}}_g\right) = 0$$

- The inertial derivatives of these vectors are

$$\dot{\hat{\boldsymbol{g}}}_s = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\left(\hat{\boldsymbol{g}}_s\right) + \boldsymbol{\omega} \times \hat{\boldsymbol{g}}_s = \left(\dot{\gamma} + \omega_g\right)\hat{\boldsymbol{g}}_t - \omega_t\hat{\boldsymbol{g}}_g$$

$$\dot{\hat{\boldsymbol{g}}}_t = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\left(\hat{\boldsymbol{g}}_t\right) + \boldsymbol{\omega} \times \hat{\boldsymbol{g}}_t = -\left(\dot{\gamma} + \omega_g\right)\hat{\boldsymbol{g}}_s + \omega_s\hat{\boldsymbol{g}}_g$$

$$\dot{\hat{\boldsymbol{g}}}_g = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\left(\hat{\boldsymbol{g}}_g\right) + \boldsymbol{\omega} \times \hat{\boldsymbol{g}}_g = \omega_t\hat{\boldsymbol{g}}_s - \omega_s\hat{\boldsymbol{g}}_t$$

use ${}^{\mathcal{G}}\boldsymbol{\omega} = \omega_s\hat{\boldsymbol{g}}_s + \omega_t\hat{\boldsymbol{g}}_t + \omega_g\hat{\boldsymbol{g}}_g$ to derive this result.

- Finally, the following expressions are derived:

$$\begin{aligned}\dot{\omega}_s &= \dot{\hat{\boldsymbol{g}}}_s^T \boldsymbol{\omega} + \hat{\boldsymbol{g}}_s^T \dot{\boldsymbol{\omega}} = \dot{\gamma}\omega_t + \hat{\boldsymbol{g}}_s^T \dot{\boldsymbol{\omega}} \\ \dot{\omega}_t &= \dot{\hat{\boldsymbol{g}}}_t^T \boldsymbol{\omega} + \hat{\boldsymbol{g}}_t^T \dot{\boldsymbol{\omega}} = -\dot{\gamma}\omega_s + \hat{\boldsymbol{g}}_t^T \dot{\boldsymbol{\omega}} \\ \dot{\omega}_g &= \dot{\hat{\boldsymbol{g}}}_g^T \boldsymbol{\omega} + \hat{\boldsymbol{g}}_g^T \dot{\boldsymbol{\omega}} = \hat{\boldsymbol{g}}_g^T \dot{\boldsymbol{\omega}}\end{aligned}$$

- The following combined gimbal and spinning disk inertia matrix will be useful to simplify some results:

$$[J] = [I_G] + [I_W] = {}^{\mathcal{G}}\begin{bmatrix} J_s & 0 & 0 \\ 0 & J_t & 0 \\ 0 & 0 & J_g \end{bmatrix}$$

And now, the fun...

- At this point we are ready to compute the terms in Euler's equation $\dot{\boldsymbol{H}} = \boldsymbol{L}$. We have all the required expressions and need to simply carry out the required algebra.

- Taking the inertial derivative of the spinning wheel angular momentum expression $\boldsymbol{H}_W$, we find

$$
\begin{aligned}
\dot{\boldsymbol{H}}_W = \hat{\boldsymbol{g}}_s \left[I_{W_s} \left(\dot{\Omega} + \hat{\boldsymbol{g}}_s^T \dot{\boldsymbol{\omega}} + \dot{\gamma}\omega_t \right) \right] + \hat{\boldsymbol{g}}_t \left[I_{W_s} \left(\dot{\gamma}(\omega_s + \Omega) + \Omega\omega_g \right) \right. \\
\left. + I_{W_t} \hat{\boldsymbol{g}}_t^T \dot{\boldsymbol{\omega}} + \left(I_{W_s} - I_{W_t} \right) \omega_s \omega_g - 2 I_{W_t} \omega_s \dot{\gamma} \right] \\
+ \hat{\boldsymbol{g}}_g \left[I_{W_t} \left(\hat{\boldsymbol{g}}_g^T \dot{\boldsymbol{\omega}} + \ddot{\gamma} \right) + \left(I_{W_t} - I_{W_s} \right) \omega_s \omega_t - I_{W_s} \Omega \omega_t \right]
\end{aligned}
$$

- Taking the derivative of the gimbal frame angular momentum expression $\boldsymbol{H}_G$, we find

$$
\begin{aligned}
\dot{\boldsymbol{H}}_G = \hat{\boldsymbol{g}}_s &\left(\left(I_{G_s} - I_{G_t} + I_{G_g}\right)\dot{\gamma}\omega_t + I_{G_s}\hat{\boldsymbol{g}}_s^T\dot{\boldsymbol{\omega}} + \left(I_{G_g} - I_{G_t}\right)\omega_t\omega_g\right) \\
+ \hat{\boldsymbol{g}}_t &\left(\left(I_{G_s} - I_{G_t} - I_{G_g}\right)\dot{\gamma}\omega_s + I_{G_t}\hat{\boldsymbol{g}}_t^T\dot{\boldsymbol{\omega}} + \left(I_{G_s} - I_{G_g}\right)\omega_s\omega_g\right) \\
+ \hat{\boldsymbol{g}}_g &\left(I_{G_g}\left(\hat{\boldsymbol{g}}_g^T\dot{\boldsymbol{\omega}} + \ddot{\gamma}\right) + \left(I_{G_t} - I_{G_s}\right)\omega_s\omega_t\right)
\end{aligned}
$$

- Finally, the spacecraft angular momentum inertial derivative is

$$\dot{\boldsymbol{H}}_B = [I_s]\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times [I_s]\boldsymbol{\omega}$$

- Let us define the time-varying total spacecraft inertia matrix [*I*]:

$$[I] = [I_s] + [J]$$

- Adding up all the terms, and substituting them into Euler's equation $\dot{\boldsymbol{H}} = \boldsymbol{L}$, we finally arrive at the desired equations of motion of a spacecraft with a single VSCMG.

$$\boxed{\begin{aligned}[I]\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega} \times [I]\boldsymbol{\omega} &- \hat{\boldsymbol{g}}_s \left(J_s \dot{\gamma} \omega_t + I_{W_s} \dot{\Omega} - (J_t - J_g)\, \omega_t \dot{\gamma} \right) \\ &- \hat{\boldsymbol{g}}_t \left((J_s \omega_s + I_{W_s} \Omega)\, \dot{\gamma} - (J_t + J_g)\, \omega_s \dot{\gamma} + I_{W_s} \Omega \omega_g \right) \\ &- \hat{\boldsymbol{g}}_g \left(J_g \ddot{\gamma} - I_{W_s} \Omega \omega_t \right) + \boldsymbol{L}\end{aligned}}$$

These equations of motion are valid for both a RW or CMG device!

Comments...

- By changing the wheel speed or by gimbaling the CMG devices, a torque is applied to the spacecraft and the corresponding attitude is changed.
- RW devices are simpler, but have limits on how large the spin speed Ω can grow.
- Adding the gimbaling mode clearly makes the mathematics much more fun and interesting :-)
- To generally control a spacecraft attitude, three of more of these devices would have to be attached to the spacecraft.

RW Motor Torque

- The equations of motion of only the spinning disk could be found by solving Euler's equations for this disk

$$\dot{\boldsymbol{H}}_W = \boldsymbol{L}_w$$

- Note that this is the inertial derivative of the inertial disk angular momentum. We have already found this to be

$$\begin{aligned}\dot{\boldsymbol{H}}_W = \hat{\boldsymbol{g}}_s \left[I_{W_s} \left(\dot{\Omega} + \hat{\boldsymbol{g}}_s^T \dot{\boldsymbol{\omega}} + \dot{\gamma}\omega_t \right) \right] + \hat{\boldsymbol{g}}_t \left[I_{W_s} \left(\dot{\gamma}(\omega_s + \Omega) + \Omega\omega_g \right) \right. \\ \left. + I_{W_t} \hat{\boldsymbol{g}}_t^T \dot{\boldsymbol{\omega}} + (I_{W_s} - I_{W_t})\,\omega_s\omega_g - 2 I_{W_t}\omega_s\dot{\gamma} \right] \\ + \hat{\boldsymbol{g}}_g \left[I_{W_t} \left(\hat{\boldsymbol{g}}_g^T \dot{\boldsymbol{\omega}} + \ddot{\gamma} \right) + (I_{W_t} - I_{W_s})\,\omega_s\omega_t - I_{W_s}\Omega\omega_t \right]\end{aligned}$$

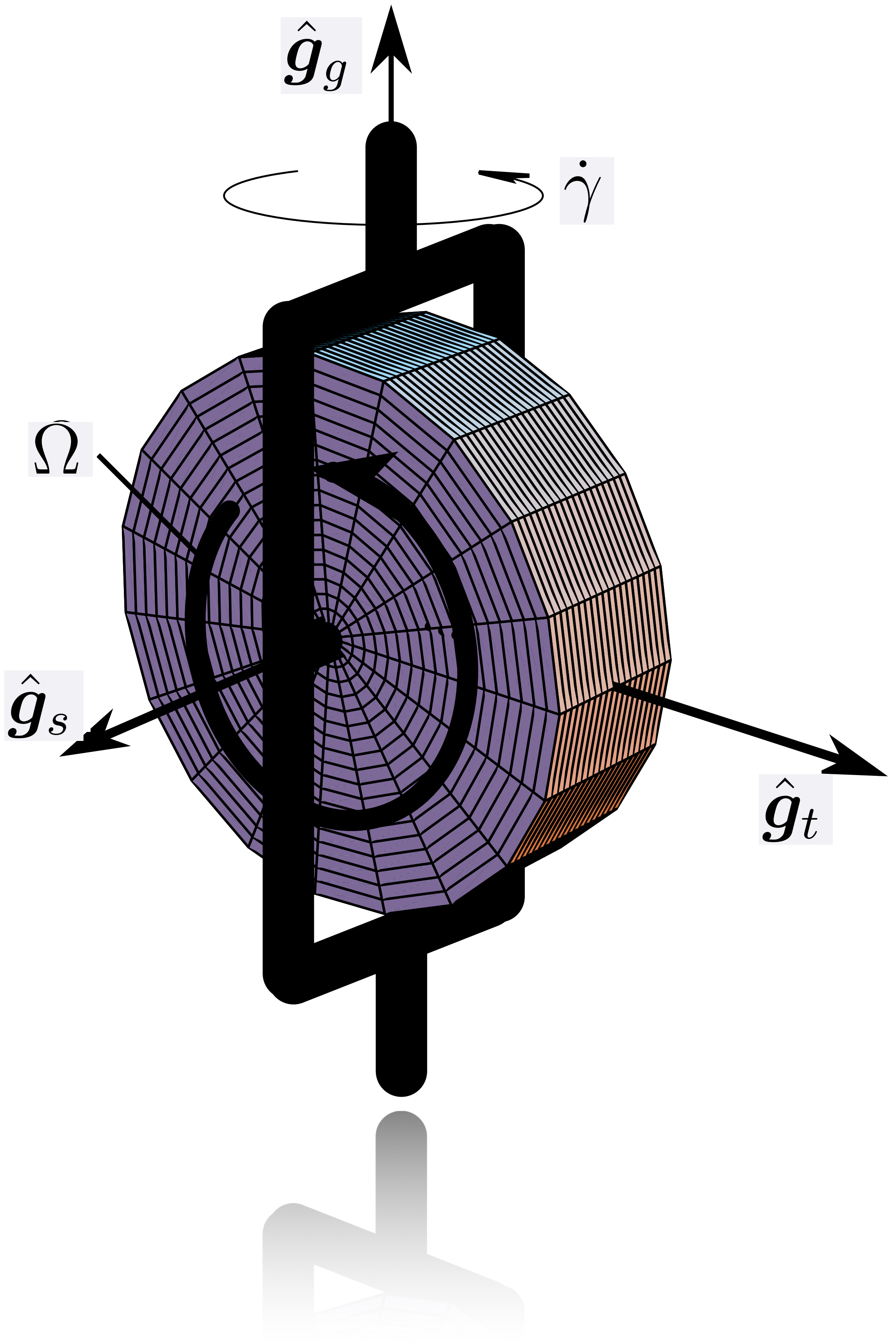
- The only external torque being applied to the spinning disk is through the RW motor.

$$\dot{\boldsymbol{H}}_W = \boldsymbol{L}_W = u_s \hat{\boldsymbol{g}}_s + \tau_{w_t} \hat{\boldsymbol{g}}_t + \tau_{w_g} \hat{\boldsymbol{g}}_g$$

- Thus, equating the $\hat{\boldsymbol{g}}_s$ directions yields:

$$\boxed{u_s = I_{W_s} \left(\dot{\Omega} + \hat{\boldsymbol{g}}_s^T \dot{\boldsymbol{\omega}} + \dot{\gamma} \omega_t \right)}$$

Given the current disk angular acceleration, spacecraft angular acceleration, or the current gimbal rate, this formula shows how hard the RW motor has to work.



CMG Motor Torque

- The compute the motor torque of the CMG gimbal mode, we need to look at both the disk and the gimbal frame as one unit.

$$\dot{\boldsymbol{H}}_G + \dot{\boldsymbol{H}}_W = \boldsymbol{L}_G$$

- Again, we have already computed these inertial angular momentum derivatives. The gimbal momentum rate is:

$$\begin{aligned}
\dot{\boldsymbol{H}}_G = \hat{\boldsymbol{g}}_s &\left(\left(I_{G_s} - I_{G_t} + I_{G_g}\right)\dot{\gamma}\omega_t + I_{G_s}\hat{\boldsymbol{g}}_s^T\dot{\boldsymbol{\omega}} + \left(I_{G_g} - I_{G_t}\right)\omega_t\omega_g\right)\\
+ \hat{\boldsymbol{g}}_t &\left(\left(I_{G_s} - I_{G_t} - I_{G_g}\right)\dot{\gamma}\omega_s + I_{G_t}\hat{\boldsymbol{g}}_t^T\dot{\boldsymbol{\omega}} + \left(I_{G_s} - I_{G_g}\right)\omega_s\omega_g\right)\\
+ \hat{\boldsymbol{g}}_g &\left(I_{G_g}\left(\hat{\boldsymbol{g}}_g^T\dot{\boldsymbol{\omega}} + \ddot{\gamma}\right) + \left(I_{G_t} - I_{G_s}\right)\omega_s\omega_t\right)
\end{aligned}$$

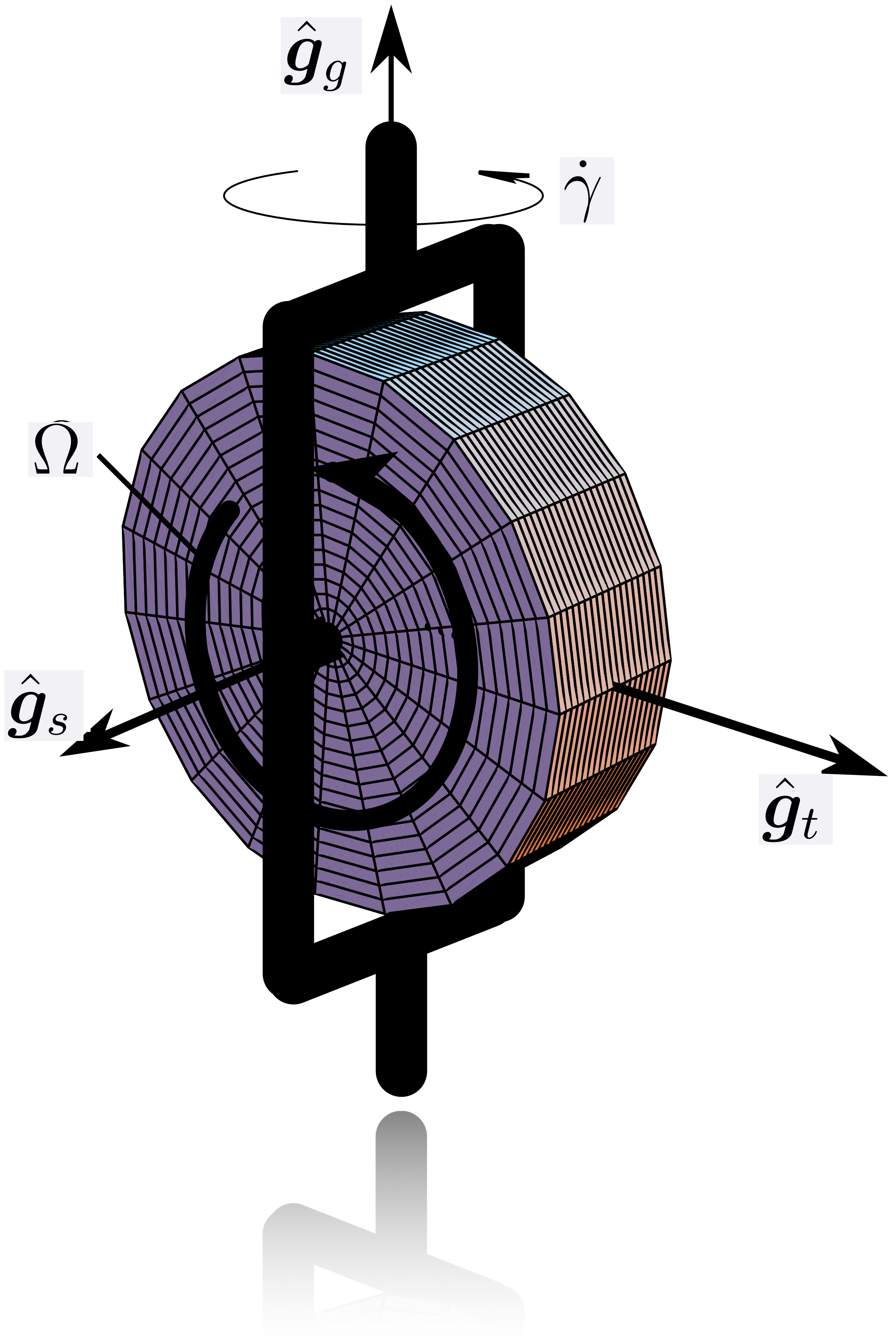
- The only external torque being applied to this two-body system is through the gimbal axis motor.

$$\dot{\boldsymbol{H}}_G + \dot{\boldsymbol{H}}_W = \boldsymbol{L}_G = \tau_{G_s}\hat{\boldsymbol{g}}_s + \tau_{G_t}\hat{\boldsymbol{g}}_t + u_g\hat{\boldsymbol{g}}_g$$

- Thus, equating the $\hat{\boldsymbol{g}}_g$ directions yields:

$$\boxed{\begin{aligned} u_g =& J_g\left(\hat{\boldsymbol{g}}_g^T\dot{\boldsymbol{\omega}} + \ddot{\gamma}\right) \\ & - (J_s - J_t)\,\omega_s\omega_t - I_{W_s}\Omega\omega_t \end{aligned}}$$

Given a commanded gimbal time history $\gamma(t)$, this equation shows us how to compute the actual torque that the gimbal motor must apply.



Example: single RW device

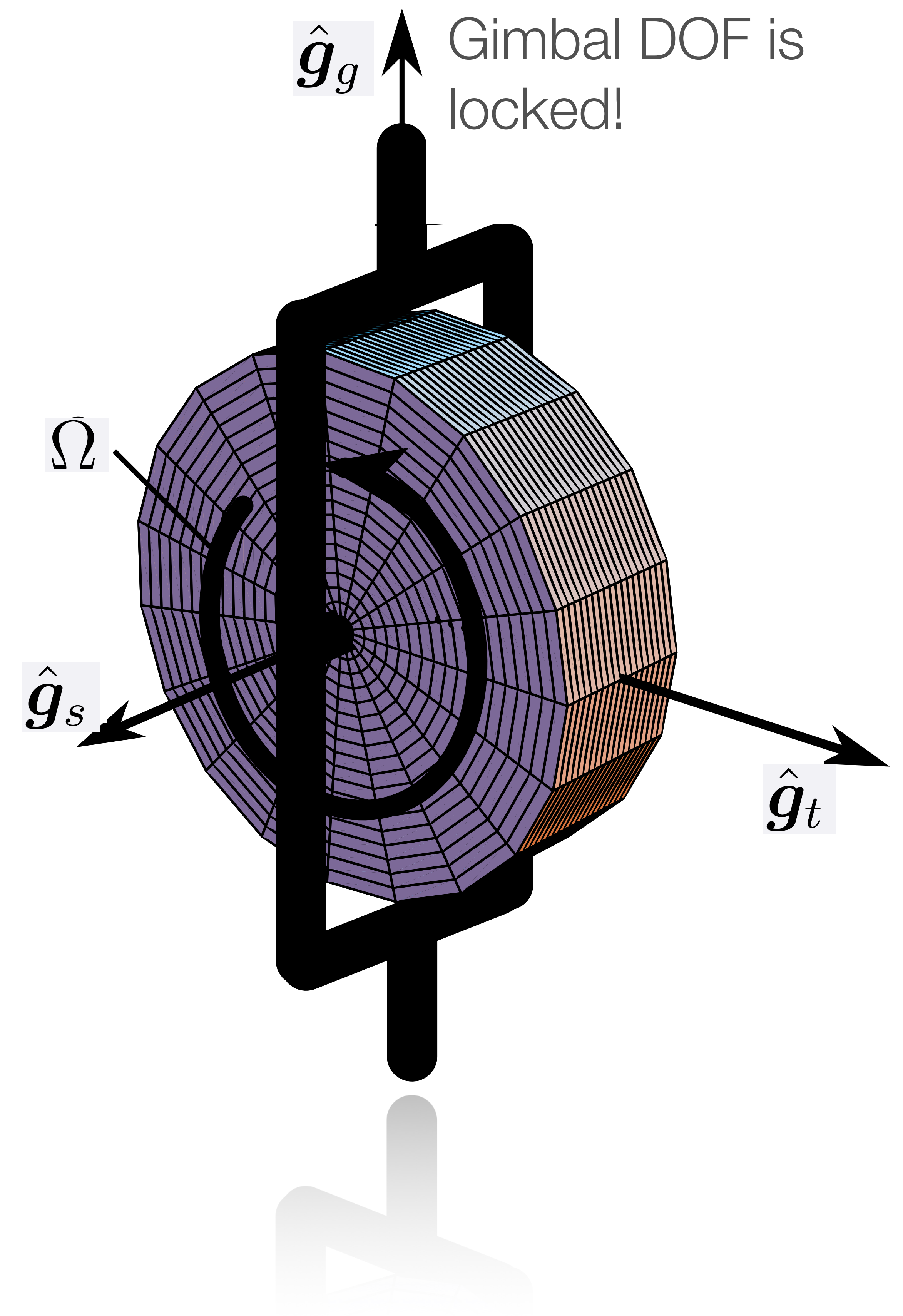
$$\dot{\gamma} = 0 \qquad \ddot{\gamma} = 0$$

$$\boxed{\begin{aligned}[I]\dot{\boldsymbol{\omega}} = &- \boldsymbol{\omega} \times [I]\boldsymbol{\omega} - \hat{\boldsymbol{g}}_s J_s \dot{\Omega} \\ &- J_s \Omega(\omega_g \hat{\boldsymbol{g}}_t - \omega_t \hat{\boldsymbol{g}}_g) + \boldsymbol{L}\end{aligned}}$$

using:

$$\boldsymbol{\omega} \times \hat{\boldsymbol{g}}_s = (\omega_s \hat{\boldsymbol{g}}_s + \omega_t \hat{\boldsymbol{g}}_t + \omega_g \hat{\boldsymbol{g}}_g) \times \hat{\boldsymbol{g}}_s = -\omega_t \hat{\boldsymbol{g}}_g + \omega_g \hat{\boldsymbol{g}}_t$$

$$[I]\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega} \times [I]\boldsymbol{\omega} - \hat{\boldsymbol{g}}_s J_s \dot{\Omega} - \boldsymbol{\omega} \times J_s \Omega \hat{\boldsymbol{g}}_s + \boldsymbol{L}$$



Motor torque: $u_s = J_s \left(\dot{\Omega} + \hat{\boldsymbol{g}}_s^T \dot{\boldsymbol{\omega}} \right)$

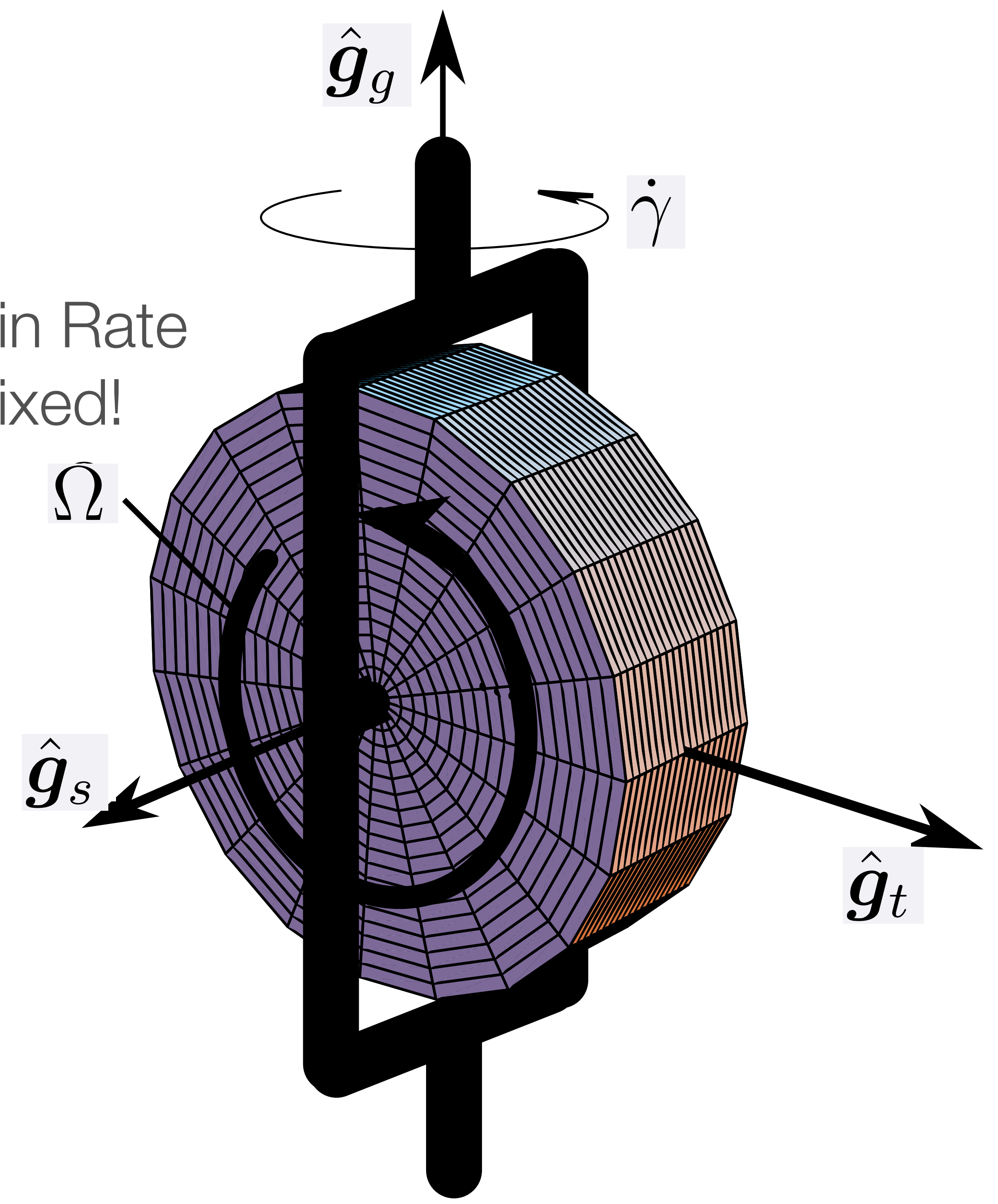
Inertia of spacecraft and non-spin RW axis: $[I_{RW}] = [I_s] + J_t \hat{\boldsymbol{g}}_t \hat{\boldsymbol{g}}_t^T + J_g \hat{\boldsymbol{g}}_g \hat{\boldsymbol{g}}_g^T$

$$\boxed{[I_{RW}]\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega} \times [I_{RW}]\boldsymbol{\omega} - \boldsymbol{\omega} \times J_s \hat{\boldsymbol{g}}_s(\omega_s + \Omega) - u_s \hat{\boldsymbol{g}}_s + \boldsymbol{L}}$$

Example: single CMG device

An inner-servo loop is holding the wheel spin rate fixed:

$$\dot{\Omega} = 0$$



$$\begin{aligned}[I]\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega} \times [I]\boldsymbol{\omega} &- \hat{\boldsymbol{g}}_s \left(J_s \dot{\gamma} \omega_t - (J_t - J_g)\, \omega_t \dot{\gamma}\right) \\ &- \hat{\boldsymbol{g}}_t \left(J_s \left(\omega_s + \Omega\right) \dot{\gamma} - (J_t + J_g)\, \omega_s \dot{\gamma} + J_s \Omega \omega_g\right) \\ &- \hat{\boldsymbol{g}}_g \left(J_g \ddot{\gamma} - J_s \Omega \omega_t\right) + \boldsymbol{L}\end{aligned}$$

CMG controls are discussed shortly...

Multiple VSCMGs

- To accommodate a spacecraft with *N* VSCMG devices, we need to employ a little “book-keeping” to account for the various momentum contributions:

We define the 3x*N* matrices:

$$[G_s] = [\hat{\boldsymbol{g}}_{s_1} \cdots \hat{\boldsymbol{g}}_{s_N}] \qquad [G_t] = [\hat{\boldsymbol{g}}_{t_1} \cdots \hat{\boldsymbol{g}}_{t_N}] \qquad [G_g] = [\hat{\boldsymbol{g}}_{g_1} \cdots \hat{\boldsymbol{g}}_{g_N}]$$

New inertia matrix definition:

$$[I] = [I_s] + \sum_{i=1}^{N} [J_i] = [I_s] + \sum_{i=1}^{N} J_{s_i} \hat{\boldsymbol{g}}_{s_i} \hat{\boldsymbol{g}}_{s_i}^T + J_{t_i} \hat{\boldsymbol{g}}_{t_i} \hat{\boldsymbol{g}}_{t_i}^T + J_{g_i} \hat{\boldsymbol{g}}_{g_i} \hat{\boldsymbol{g}}_{g_i}^T$$

$$\boldsymbol{\tau}_s = \begin{bmatrix} J_{s_1}\left(\dot{\Omega}_1 + \dot{\gamma}_1\omega_{t_1}\right) - \left(J_{t_1} - J_{g_1}\right)\omega_{t_1}\dot{\gamma}_1 \\ \vdots \\ J_{s_N}\left(\dot{\Omega}_N + \dot{\gamma}_N\omega_{t_N}\right) - \left(J_{t_N} - J_{g_N}\right)\omega_{t_N}\dot{\gamma}_N \end{bmatrix}$$

Torque-like vectors:

$$\boldsymbol{\tau}_t = \begin{bmatrix} J_{s_1}\left(\Omega_1 + \omega_{s_1}\right)\dot{\gamma}_1 - \left(J_{t_1} + J_{g_1}\right)\omega_{s_1}\dot{\gamma}_1 + J_{s_1}\Omega_1\omega_{g_1} \\ \vdots \\ J_{s_N}\left(\Omega_N + \omega_{s_N}\right)\dot{\gamma}_N - \left(J_{t_N} + J_{g_N}\right)\omega_{s_N}\dot{\gamma}_N + J_{s_N}\Omega_N\omega_{g_N} \end{bmatrix}$$

$$\boldsymbol{\tau}_g = \begin{bmatrix} J_{g_1}\ddot{\gamma}_1 - J_{s_1}\Omega_1\omega_{t_1} \\ \vdots \\ J_{g_N}\ddot{\gamma}_N - J_{s_N}\Omega_N\omega_{t_N} \end{bmatrix}$$

EOM of spacecraft with *N* VSCMGs:

$$[I]\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega} \times [I]\boldsymbol{\omega} - [G_s]\boldsymbol{\tau}_s - [G_t]\boldsymbol{\tau}_t - [G_g]\boldsymbol{\tau}_g + \boldsymbol{L}$$

Energy expression:

$$T = \frac{1}{2}\boldsymbol{\omega}^T[I_s]\boldsymbol{\omega} + \frac{1}{2}\sum_{i=1}^{N} J_{s_i}\left(\Omega_i + \omega_{s_i}\right)^2 + J_{t_i}\omega_{t_i}^2 + J_{g_i}\left(\omega_{g_i} + \dot{\gamma}_i\right)^2$$

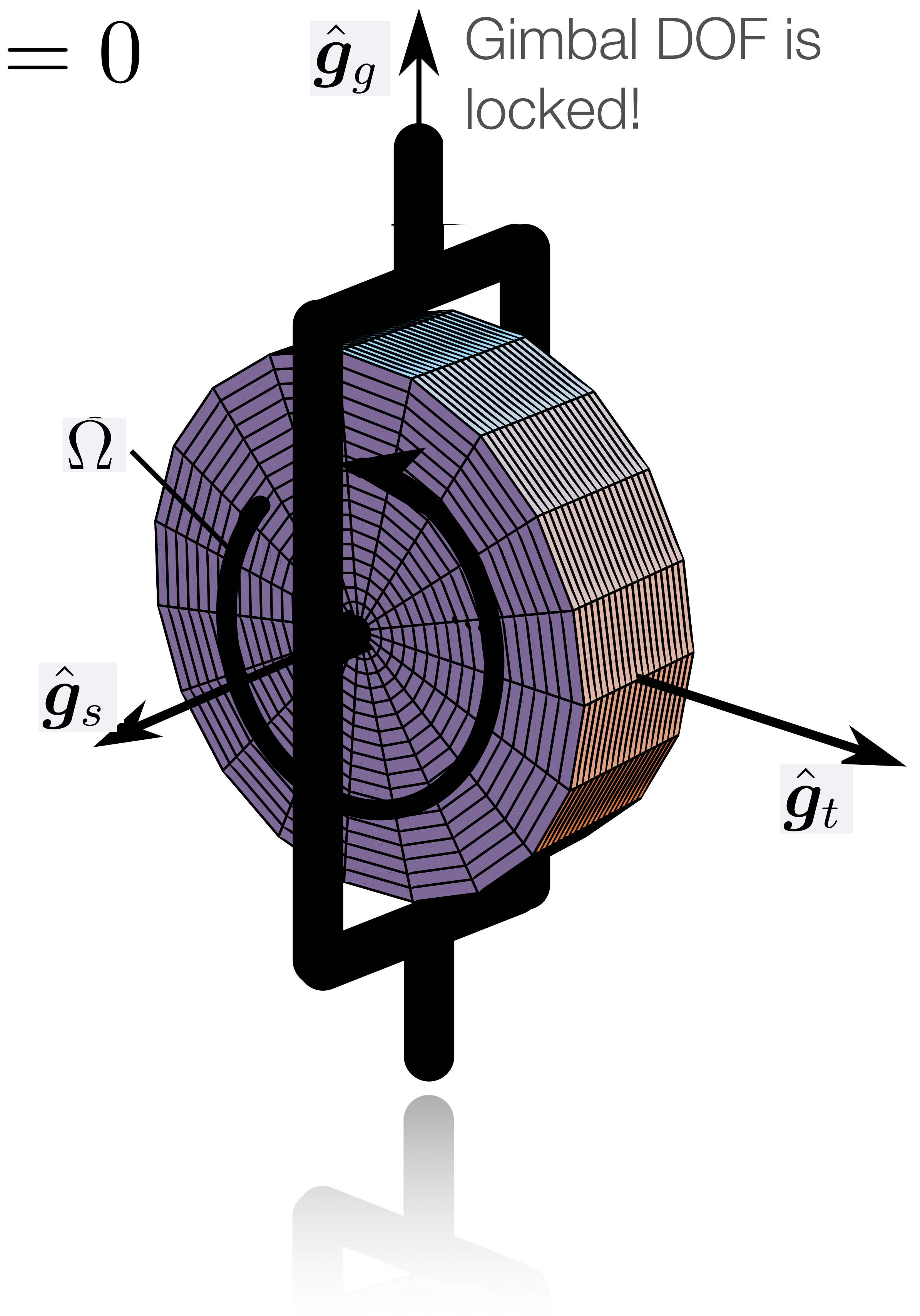
After much algebra, or by using the work-energy-principle...

$$\dot{T} = \boldsymbol{\omega}^T\boldsymbol{L} + \sum_{i=1}^{N} \dot{\gamma}_i u_{g_i} + \Omega_i u_{s_i}$$

Example: multiple RW devices

$$\dot{\gamma} = 0$$

$$\ddot{\gamma} = 0$$



Inertia matrix definition:

$$[I_{RW}] = [I_s] + \sum_{i=1}^{N} \left(J_{t_i} \hat{\boldsymbol{g}}_{t_i} \hat{\boldsymbol{g}}_{t_i}^T + J_{g_i} \hat{\boldsymbol{g}}_{g_i} \hat{\boldsymbol{g}}_{g_i}^T \right)$$

Let us define the momentum vector $\boldsymbol{h}_s$ as:

$$\boldsymbol{h}_s = \begin{pmatrix} \vdots \\ J_{s_i} \left(\omega_{s_i} + \Omega_i \right) \\ \vdots \end{pmatrix}$$

The equations of motion then become:

$$[I_{RW}]\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega} \times [I_{RW}]\boldsymbol{\omega} - \boldsymbol{\omega} \times [G_s]\boldsymbol{h}_s - [G_s]\boldsymbol{u}_s + \boldsymbol{L}$$

For the special case with 3 RWs aligned with the principal axis, [G_s] becomes an identity matrix and the EOM reduce to

$$[I_{RW}]\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega} \times [I_{RW}]\boldsymbol{\omega} - \boldsymbol{\omega} \times \boldsymbol{h}_s - \boldsymbol{u}_s + \boldsymbol{L}$$

Momentum-Device Control Laws

This is where the pudding starts to come together...

RW Control Devices

- First let us develop a feedback control law for a spacecraft with *N* reaction wheels with general orientation.

EOM:

$$[I_{RW}]\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega} \times [I_{RW}]\boldsymbol{\omega} - \boldsymbol{\omega} \times [G_s]\boldsymbol{h}_s - [G_s]\boldsymbol{u}_s + \boldsymbol{L}$$

with

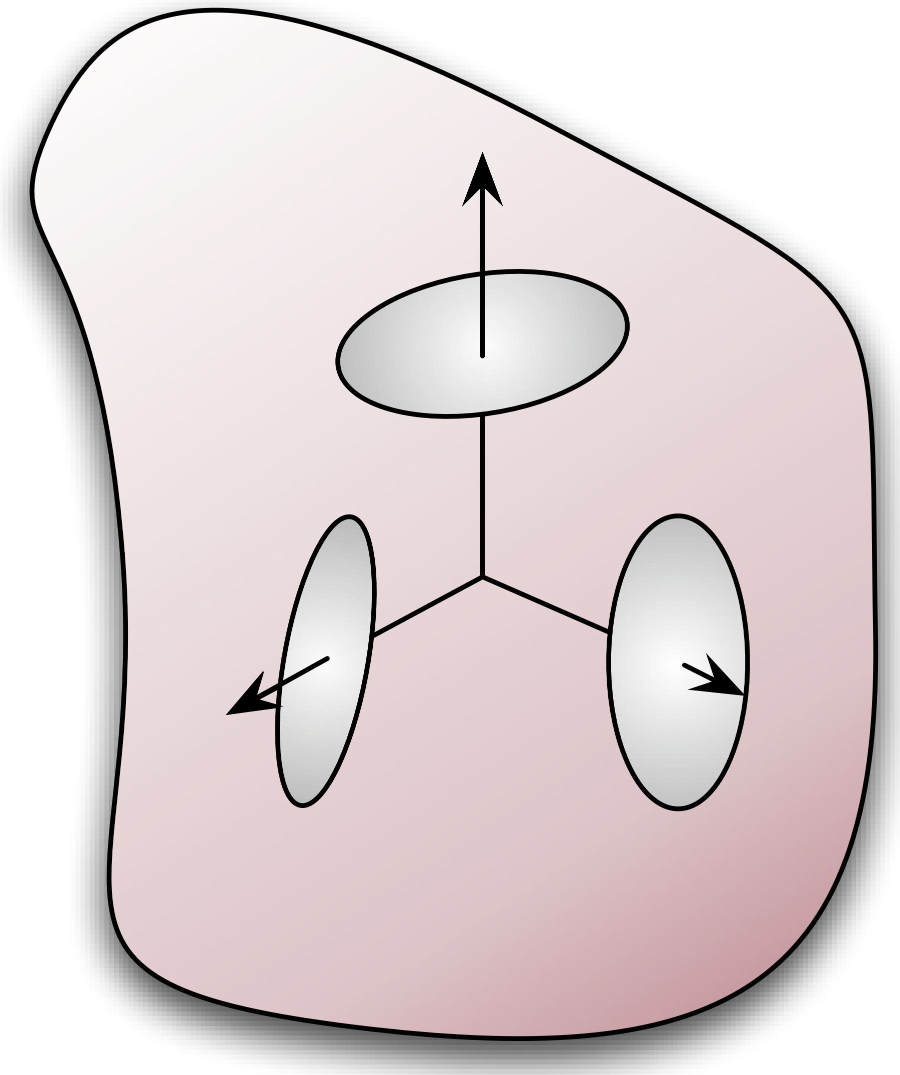
$$\boldsymbol{h}_s = \begin{pmatrix} \vdots \\ J_{s_i}\left(\omega_{s_i} + \Omega_i\right) \\ \vdots \end{pmatrix}$$

Inertia Matrix:

$$[I_{RW}] = [I_s] + \sum_{i=1}^{N}\left(J_{t_i}\hat{\boldsymbol{g}}_{t_i}\hat{\boldsymbol{g}}_{t_i}^T + J_{g_i}\hat{\boldsymbol{g}}_{g_i}\hat{\boldsymbol{g}}_{g_i}^T\right)$$

The RW motor control torque vector is:

$$\boldsymbol{u}_s = \begin{pmatrix} \vdots \\ J_{s_i}\left(\dot{\Omega}_i + \hat{\boldsymbol{g}}_{s_i}^T\dot{\boldsymbol{\omega}}\right) \\ \vdots \end{pmatrix}$$



Spacecraft Tracking Errors:

$\boldsymbol{\sigma}$ - MRP vector of body frame relative to reference frame

$\delta\boldsymbol{\omega} = \boldsymbol{\omega} - \boldsymbol{\omega}_r$ - body angular velocity tracking error vector

Lyapunov Function:

$$V(\boldsymbol{\sigma}, \delta\boldsymbol{\omega}) = \frac{1}{2}\delta\boldsymbol{\omega}^T [I_{RW}]\delta\boldsymbol{\omega} + 2K \ln\left(1 + \boldsymbol{\sigma}^T\boldsymbol{\sigma}\right)$$

components taken in the *B* frame

Let's set the Lyapunov Rate to:

$$\dot{V} = -\delta\boldsymbol{\omega}[P]\delta\boldsymbol{\omega} \leq 0$$

$$[I_{RW}]\frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\delta\boldsymbol{\omega}) + K\boldsymbol{\sigma} + [P]\delta\boldsymbol{\omega} = 0$$

closed-loop dynamics

$$[G_s]\boldsymbol{u}_s = \underbrace{\boxed{K\boldsymbol{\sigma} + [P]\delta\boldsymbol{\omega} - [\tilde{\boldsymbol{\omega}}]([I_{\mathrm{RW}}]\boldsymbol{\omega} + [G_s]\boldsymbol{h}_s) - [I_{\mathrm{RW}}](\dot{\boldsymbol{\omega}}_r - \boldsymbol{\omega}\times\boldsymbol{\omega}_r) + \boldsymbol{L}}}_{\boldsymbol{L}_r}$$

Control condition:

$$[G_s]\boldsymbol{u}_s = \boldsymbol{L}_r$$

Case 1: 3 RWs aligned with principal axes of spacecraft.

$$\boldsymbol{u}_s = \boldsymbol{L}_r$$

Case 2: N RWs aligned generally.

$$\boldsymbol{u}_s = [G_s]^T\left([G_s][G_s]^T\right)^{-1}\boldsymbol{L}_r$$

minimum-norm inverse

Energy rate:

$$\dot{T} = \boldsymbol{\omega}^T\boldsymbol{L} + \sum_{i=1}^{N}\Omega_i u_{s_i}$$

work/energy principle

The End…