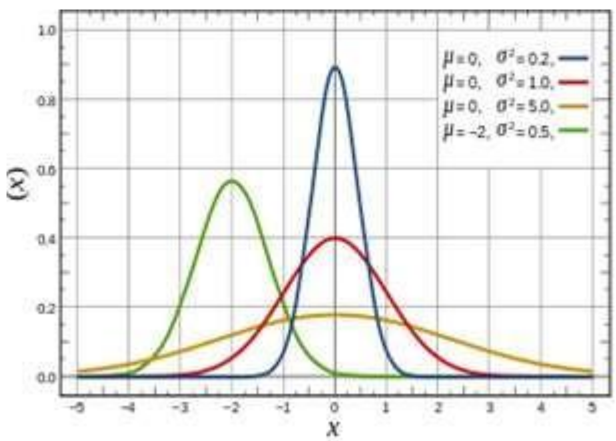
1.1 Continuous Distributions

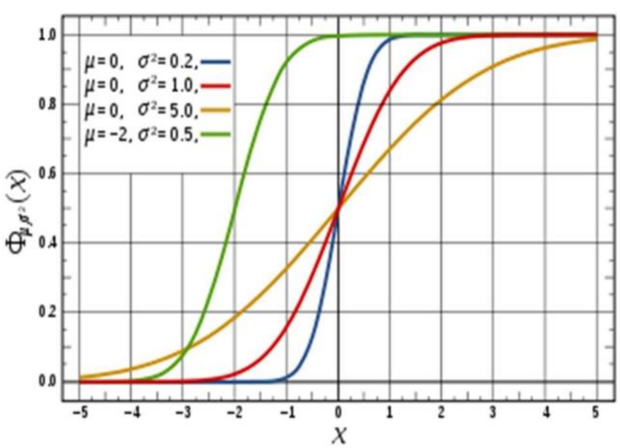
Normal Distribution and Standard Normal Distribution

$$(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)} \quad ; Normal\ pdf$$

$$(x) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \quad ; Standard\ pdf$$



Cummulative density function



Exponential Distribution

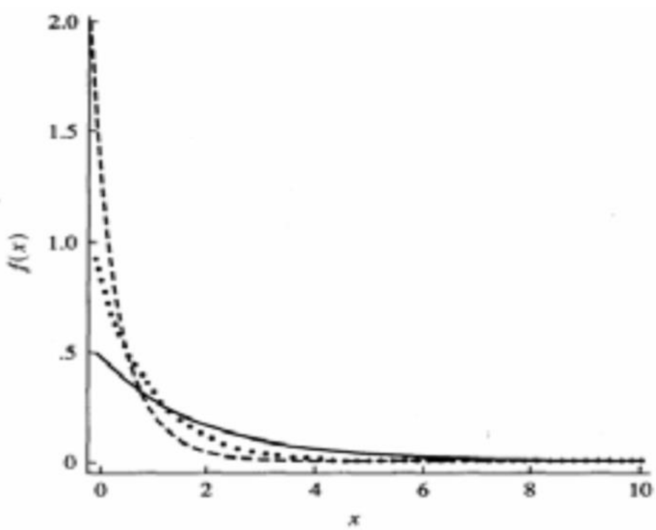
If a random variable X has the *pdf*

$$(x) = \lambda e^{-\lambda x}, \qquad x \geq 0 \; and \; \lambda > 0,$$

then it is said to have the exponential distribution with parameter λ and written as $X \sim Exp(\lambda)$.

The exponential distribution is often used to model the length of time until an event occurs. The exponential distribution can be thought of as the continuous analogue of the geometric distribution.

This parameter $\boldsymbol{\lambda}$ represents the "**mean number of events per unit time**" e.g. the rate of arrivals or the rate of failures as same as in Poisson distribution.



$\lambda = 0.5(solid)$

$\lambda = 1(dotted)$

$\lambda = 2(dashed)$

Applications

- Model inter arrival times (time between arrivals) when arrivals are completely random; $\boldsymbol{\lambda}$ = arrivals / hour
- Model service times; $\boldsymbol{\lambda}$ = services / minute
- Model the lifetime of a component that fails catastrophically (i.e. light bulb); $\boldsymbol{\lambda}$ = failure rate

Properties of the random variable X which has exponential distribution

1. It is closely related to the Poisson distribution – if X describes the time between two failures then the number of failures per unit time has the Poisson distribution with parameter $\boldsymbol{\lambda}$, the same.

2. The *cdf* is $F_X(x) = \lambda \int_0^x e^{-\lambda y}\ dy\ = 1 - e^{-\lambda x}$

3. The 100(1 −∝)% percentile is $x_\propto = -\frac{1}{\lambda}\ln \propto$

4. Mean $\mu_x = 1/\lambda$

5. Variance $V_x = 1/\lambda^2$

6. Moment Generating Function (*mgf)* $M_X(t) = \lambda/(\lambda - t)$

7. "Memoryless" property
   For all s>= 0 and t >=0
   P(X > s + t | X > s) = P(X > t)

   Instance 1: If it is known that a component has survived s hours so far, the remaining amount of time that it survives follows the same distribution as the original distribution. It does not remember that it already has been used for s amount of time.

   Instance 2: This means that the distribution of the waiting time to the next event remains the same regardless of how long we have already been waiting. This only happens when events occur (or not) totally at random, i.e., independent of past history

Exercise: Suppose the life of an industrial lamp is exponentially distributed with failure rate λ=1/3 (one failure every 3000 hours on the avg.) Determine the probability that

a) the lamp will last no longer than its mean life time. (constant for any λ**)**
b) the lamp will last longer than its mean life time
c) the industrial lamp will last between 2000 and 3000 hours.
d) the lamp will last for another 1000 hours given that it is operating after 2500 hours.

Answer:

a) $(X \leq 3)=$
b) $(X > 3)=$
c) $(2 \leq X \leq 3)=$
d) $(X > 3.5|X > 2.5)=P(X > 2.5 + 1|X > 2.5)=P(X > 1)$

Theorem: X has an exponential distribution **iff** X is a positive continuous r.v. and P(X>s+t | X>s) = P(X>t) for all s,t> 0.

Proof: *Omitted*

Gamma distribution

Gamma distribution is more suitable to describe some of the real world applications when they follow exponential patterns. The general command of a such probability density is given by

$$f(x) = \begin{cases} kx^{\alpha-1}e^{-x/\beta}; & for\ x > 0 \\ 0 & ;\ elsewhere \end{cases}$$

where $\alpha > 0$ and $\beta > 0$, and k must be such that the total area under the curve is equal to 1.

In evaluating k, using calculus theory, the ***Gamma function*** which only depends on α is derived:

$$\tau(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y}dy \qquad for\ \alpha > 0$$

The ***Gamma function*** follows the recursion formula

$$\tau\ (\alpha) = (\alpha - 1)\tau\ (\alpha - 1);$$

$$\tau(\alpha) = (\alpha - 1)!$$

where $\tau(1) = \int_0^1 y^0e^{-y}dy = 1 \quad and \quad \tau(1/2) = \sqrt{\pi}$

Thus $\int_0^\infty kx^{\alpha-1}e^{-x/\beta}dx = k\beta^\alpha\tau\ (\alpha) = 1$

A random variable X has a ***Gamma distribution*** has the probability density function

$$g(x;\alpha,\beta) = \begin{cases} \frac{1}{\beta^\alpha\tau(\alpha)}x^{\alpha-1}e^{\frac{-x}{\beta}} & ;\ for\ x > 0 \\ 0 & ;elsewhere \end{cases}$$

where $\alpha > 0$ and $\beta > 0$.

- The mean $\mu = \alpha\beta$ and $V(X) = \alpha\beta^2$
- Observe the graphs of gamma functions for different pairs of values for $\alpha$and β

Exercise: In a certain city, the daily consumption of electric power in millions of kilowatt-hours can be treated as a random variable having a Gamma distribution with α = 3 and β = 2.

(i) What is the average consumption of electric power per day by the city?

(ii) If the power plant of this city has a daily capacity of 12 million kilowatt-hours, what is the probability that this power supply will be inadequate on any given day?

Answer:

(i) Average = $\alpha\beta = 3 * 2 = 6$

(ii) P(daily consumption of electric power ≥ 12) $=\int_{12}^{\infty} \frac{1}{2^3\,\tau(3)} x^{3-1} e^{-\frac{x}{2}}\,dx$

$$=1-\int_{0}^{12} \frac{1}{2^3\,\tau(3)} x^{3-1} e^{-\frac{x}{2}}\,dx$$

<u>Exponential Distribution: Derived from Gamma Distribution</u>

Gamma Distribution: $Gam(x;\alpha,\beta) = \frac{1}{\beta^{\alpha}\tau(\alpha)} x^{\alpha-1} e^{\frac{-x}{\beta}} \quad for\ x > 0$

Exponential Distribution: $Exponential(x;\lambda) = \lambda e^{-\lambda x}$

$$x^0 \equiv x^{\alpha-1} \qquad\qquad e^{-\lambda x} \equiv e^{\frac{-x}{\beta}}$$

$$\alpha - 1 = 0 \qquad\qquad \beta = {}^{1}/_{\lambda}$$

$$\alpha = 1$$

If we let $\alpha = 1$ and $\beta = {}^{1}/_{\lambda}$, we obtain $Gam\left(x; 1, {}^{1}/_{\lambda}\right)$ which is equivalent to the exponential distribution.