Week 2 - Choice under Uncertainty

1. Proof the following proposition:

Let $\succsim$ be a preference relation satisfying the Monotonicity Axiom and the Archimedean Axiom. Then it has the continuity property.

Proof: Given $p \succ p' \succ p''$. Define $\alpha \equiv \sup\{k \in [0,1] \mid p' \succsim kp + (1-k)p''\}$. By monotonicity, we have that $p' \succ kp + (1-k)p''$ for all $k < \alpha$ and that $kp + (1-k)p'' \succsim p'$ for all $k > \alpha$. We want to show that $p' \sim \alpha p + (1-\alpha)p''$.

By the Archimedean axiom, there is some $k > 0$ such that $p' \succ kp + (1-k)p''$ and some $k < 1$ such that $kp + (1-k)p'' \succ p'$. Hence, $\alpha \in (0,1)$.

Suppose that $p' \succ \alpha p + (1-\alpha)p''$. Then according to the Archimedean Axiom, we can find $b \in (0,1)$ such that $p' \succ bp + (1-b)(\alpha p + (1-\alpha)p'')$.

The compound lottery on the right-side has total weight $k = b + (1-b)\alpha = \alpha + (1-\alpha)b > \alpha$ on p. Thus we have found some $k > \alpha$ for which $p' \succ kp + (1-k)p''$. This contradicts the definition of α. Therefore, we must have $p' \precsim \alpha p + (1-\alpha)p''$.

Now suppose that $\alpha p + (1-\alpha)p'' \succ p'$. From the Archimedean Axiom we know that there exists $a \in (0,1)$ such that $a(\alpha p + (1-\alpha)p'') + (1-a)p'' \succ p'$. The compound lottery on the left-hand side has total weight $a\alpha$ on p. Hence we have found some $k < \alpha$ for which $kp + (1-k)p'' \succ p'$ contradicting the definition of α.

Therefore we must have $\alpha p + (1-\alpha)p'' \precsim p'$.

Taken together, it must be that $p' \sim \alpha p + (1-\alpha)p''$.

2. Proof the following proposition:

If a preference relation $\succsim$ satisfies the Independence Axiom, it will also satisfy the Monotonicity Axiom.

Proof: Given $p, p' \in \mathcal{P}(Z)$, with $p \succ p'$ and $a, b \in (0,1)$, we need to show that

$$a > b \quad \Leftrightarrow \quad ap + (1-a)p' \succ bp + (1-b)p'$$

Wlog we can assume that $a > 0$. Assume first that $a > b$. It follows from independence that

$$ap + (1-a)p' \succ ap' + (1-a)p'$$

so that $ap + (1-a)p' \succ p'$. Similarly, for any $p'' \succ p'$ it follows from independence that

$$p'' \sim \left(1 - \frac{b}{a}\right)p'' + \frac{b}{a}p'' \succ \left(1 - \frac{b}{a}\right)p' + \frac{b}{a}p''$$

Consider now the case of $p'' = ap + (1-a)p'$. Then we get

$$
\begin{aligned}
ap + (1-a)p' &\succ \frac{b}{a}(ap + (1-a)p') + \left(1 - \frac{b}{a}\right)p' \\
&\sim bp + (1-b)p'
\end{aligned}
$$

as desired. Conversely, assuming that

$$ap + (1-a)p' \succ bp + (1-b)p'$$

we must show that $a > b$. If $b > a$ the above arguments (interchanged) would lead to a contradiction.

3. Proof the von Neumann-Morgenstern Theorem:

Assume that Z is finite and that $\succsim$ is a preference relation on $\mathcal{P}(Z)$. Then $\succsim$ can be represented by a linear utility index if and only if $\succsim$ satisfies the Archimedean and the Independence Axiom.

Proof: First, assume that the preference relation $\succ$ satisfies the Archimedean and the Independence Axiom. Define a utility index $\mathcal{U}: \mathcal{P}(Z) \to \mathbb{R}$ by $\mathcal{U}(p) = \alpha_p$, where α_p is the unique number that satisfies

$$p \sim \alpha_p \mathbb{1}_{z_1} + (1-\alpha_p)\mathbb{1}_{z_0}$$

We want to show that as a consequence of the Independence Axiom, $\mathcal{U}$ is indeed linear. To start, pick $p, p' \in \mathcal{P}(Z)$ and any number $a \in (0,1)$. We want to show that $\mathcal{U}(ap + (1-a)p') = a\mathcal{U}(p) + (1-a)\mathcal{U}(p')$. This follows if we show that

$$ap + (1-a)p' \sim (a\mathcal{U}(p) + (1-a)\mathcal{U}(p'))\mathbb{1}_{z_1} + (1 - (a\mathcal{U}(p) + (1-a)\mathcal{U}(p')))\mathbb{1}_{z_0}$$

The latter, however, follows from the Independence Axiom:

$$
\begin{aligned}
ap + (1-a)p' &\sim a(\mathcal{U}(p)\mathbb{1}_{z_1} + (1-\mathcal{U}(p))\mathbb{1}_{z_0}) + (1-a)(\mathcal{U}(p')\mathbb{1}_{z_1} + (1-\mathcal{U}(p'))\mathbb{1}_{z_0}) \\
&\sim (a\mathcal{U}(p) + (1-a)\mathcal{U}(p'))\mathbb{1}_{z_1} + (1 - (a\mathcal{U}(p) + (1-a)\mathcal{U}(p')))\mathbb{1}_{z_0}
\end{aligned}
$$

The converse is straightforward to check.

Week 3 - Risk Attitudes & Risk Aversion

1. (a) Determine the class of utility functions that exhibit constant absolute risk aversion (assume that the coefficient of risk aversion is strictly positive).

The utility functions $u(W)$ that display CARA solve the differential equation

$$-\frac{u''(W)}{u'(W)} = a$$

where a is a positive constant. Therefore

$$\frac{d \log u'(W)}{dW} = -a$$

Integrating,

$$\log u'(W) = -aW + b$$

where b is a constant of integration. This implies that

$$u'(W) = \exp(-aW + b)$$

Integrating again:

$$
\begin{aligned}
u(W) &= -\frac{1}{a}\exp(-aW + b) + c \\
&= -ke^{-aW} + c
\end{aligned}
$$

where $k := \frac{e^b}{a} > 0$

Thus we have established that the CARA utility functions are those that lie in the negative exponential class, i.e. $u(W) = -e^{-aW}$ (with $a > 0$) or an increasing affine transformation thereof.

(b) Determine the class of utility functions that exhibit constant relative risk aversion (assume that the coefficient of risk aversion is strictly positive).

The utility functions $u(W)$ that display CRRA solve the differential equation

$$-\frac{u''(W)}{u'(W)} \cdot W = a$$

where a is a positive constant. Therefore

$$\frac{d \log u'(W)}{dW} = -\frac{a}{W}$$

Integrating

$$
\begin{aligned}
\log u'(W) &= -a\log(W) + b \\
&= \log\left(e^b W^{-a}\right)
\end{aligned}
$$

Therefore,

$$u'(W) = e^b W^{-a}$$

If $a \neq 1$, we obtain:

$$u(W) = k \cdot \frac{W^{1-a}}{1-a} + c$$

where $k := e^b > 0$. For $a = 1$

$$u'(W) = \frac{k}{W}$$

so that

$$u(W) = k\log(W) + c$$

We have shown that the CRRA utility functions are the power utility functions given by $u(w) = \frac{W^{1-a}}{1-a}$, when the CRRA coefficient a is not equal to one, or the logarithmic utility function $u(W) = \log(W)$, when $a = 1$. This representation is unique up to an increasing affine transformation.

2. Consider the HARA (hyperbolic absolute risk aversion) class of von Neumann-Morgenstern utility functions:

$$U(W) = \frac{1-c}{c}\left(\frac{aW}{1-c} + b\right)^c \tag{1}$$

defined over the domain $b + aW/(1-c) > 0$, with $b \geq 0$.

(a) The coefficient of absolute risk tolerance is the reciprocal, $\frac{1}{\mathcal{A}(W)}$, of the coefficient of absolute risk aversion $\mathcal{A}(W)$. Show that for utility function above absolute risk tolerance is affine in wealth.

Direct computation shows that the coefficient of absolute risk tolerance is given by

$$\frac{W}{1-c} + \frac{b}{a}$$

(b) Suppose there is a safe and a risky asset available. Take the (gross) riskless rate of return to be one, and initial wealth to be W_0. There are no constraints on short sales. Derive an implicit expression for the demand for the risky asset (i.e. the quantity of the asset demanded as a function of its price) implied by (1).

Let $\tilde{x}$, p, and θ denote respectively the payoff, price, and quantity demanded of the risky asset. Since the riskless rate of return is one, terminal wealth is given by

$$
\begin{aligned}
\tilde{W} &= \theta\tilde{x} + (W_0 - p\theta) \\
&= W_0 + (\tilde{x} - p)\theta
\end{aligned}
$$

We solve

$$\max_{\theta} EU\left[W_0 + (\tilde{x} - p)\theta\right]$$

The first order condition is

$$E\left[U'(W_0 + (\tilde{x} - p)\theta) \cdot (\tilde{x} - p)\right] = 0$$

Now using the functional form

$$U(W) = \frac{1-c}{c}\left(\frac{aW}{1-c} + b\right)^c$$

we get

$$E\left[\left(\frac{a\left[W_0 + (\tilde{x} - p)\theta\right]}{1-c} + b\right)^{c-1} \cdot (\tilde{x} - p)\right] = 0$$

which gives θ implicitly as a function of p.

Week 5 - Capital Asset Pricing Model

1. (a) Consider an economy with two dates, $t = 0, 1$. Let the random time 1 payoff of a risky security be $\tilde{y}$, and let p_y be its time 0 equilibrium price. Suppose that the Sharpe-Lintner CAPM holds and let the security's beta be denoted by β_{ym}. R_0 is the gross riskfree rate. Show that

$$p_y = \frac{E(\tilde{y})}{R_0 + \beta_{ym}\left[E\left(\tilde{R}_m\right) - R_0\right]} \tag{1}$$

$$p_y = \frac{E(\tilde{y}) - \phi\rho_{ym}\sigma(\tilde{y})}{R_0} \tag{2}$$

where ϕ is the "market price of risk" defined by

$$\phi := \frac{E\left(\tilde{R}_m\right) - R_0}{\sigma\left(\tilde{R}_m\right)}$$

and

$$\rho_{ym} = \frac{\operatorname{cov}\left(\tilde{y}, \tilde{R}_m\right)}{\sigma(\tilde{y})\sigma\left(\tilde{R}_m\right)}$$

Interpret the pricing formulas (1) and (2).

The CAPM relation is

$$E(\tilde{R}) = R_0 + \beta_{ym}\left(E\left(\tilde{R}_m\right) - R_0\right)$$

By definition,

$$\tilde{R} = \frac{\tilde{y}}{p_y}$$

Taking expectations and substituting in the CAPM equation, we get

$$
\begin{aligned}
p_y &= \frac{E(\tilde{y})}{R_0 + \beta_{ym}\left[E\left(\tilde{R}_{\mathbf{m}}\right) - R_0\right]} \\
&= \frac{E(\tilde{y})}{R_0 + \frac{\operatorname{cov}\left(\tilde{R}, \tilde{R}_{\mathbf{m}}\right)}{\sigma^2\left(\tilde{R}_{\mathbf{m}}\right)}\left[E\left(\tilde{R}_{\mathbf{m}}\right) - R_0\right]} \\
&= \frac{E(\tilde{y})}{R_0 + \frac{\operatorname{cov}\left(\tilde{y}, \tilde{R}_{\mathbf{m}}\right)}{p_y \sigma^2\left(\tilde{R}_{\mathbf{m}}\right)}\left[E\left(\tilde{R}_{\mathbf{m}}\right) - R_0\right]} \\
&= \frac{E(\tilde{y})}{R_0 + \frac{\rho_{ym}}{p_y} \cdot \sigma(\tilde{y})\phi}
\end{aligned}
$$

Now we can solve for p_y to get the desired result.

Interpretation: In the first pricing formula, the price is obtained as the expected payoff discounted at a risk-adjusted rate. In the second formula, the price is equal to the expected payoff, minus a risk adjustment, discounted at the risk-free rate.

(b) Suppose the Sharpe-Lintner CAPM holds. Let $\tilde{y}$ and $\tilde{z}$ be the random time 1 payoffs of two securities with time zero prices p_y and p_z, respectively. Using the pricing formula (2), show that the equilibrium price of a security with time 1 payoff $\tilde{y}+\tilde{z}$ is equal to p_y+p_z.

We wish to show that $p_{y+z} = p_y + p_z$. From the previous problem,

$$
\begin{aligned}
p_{y+z} &= \frac{E(\tilde{y}+\tilde{z}) - \phi\rho_{(y+z)m}\sigma(\tilde{y}+\tilde{z})}{R_0} \\
&= \frac{E(\tilde{y}) + E(\tilde{z}) - \phi \cdot \frac{\operatorname{cov}\left(\tilde{y}+\tilde{z}, \tilde{R}_m\right)}{\sigma\left(\tilde{R}_m\right)}}{R_0} \\
&= \frac{E(\tilde{y}) + E(\tilde{z}) - \phi\left[\rho_{ym}\sigma(\tilde{y}) + \rho_{zm}\sigma(\tilde{z})\right]}{R_0} \\
&= \frac{E(\tilde{y}) - \phi\rho_{ym}\sigma(\tilde{y})}{R_0} + \frac{E(\tilde{z}) - \phi\rho_{zm}\sigma(\tilde{z})}{R_0} \\
&= p_y + p_z
\end{aligned}
$$

2. Consider an economy with a riskfree asset and two risky assets. Equilibrium rates of return are as follows. The riskfree asset has a net rate of return of 6%. The risky assets, labelled A and B, have normally distributed rates of return:

$$
\begin{pmatrix} \tilde{r}_A \\ \tilde{r}_B \end{pmatrix} \sim \mathcal{N}\left[\begin{pmatrix} 0.12 \\ 0.18 \end{pmatrix}, \begin{pmatrix} 0.003 & 0.001 \\ 0.001 & 0.005 \end{pmatrix}\right]
$$

Investors maximize expected utility, prefer more wealth to less, and are risk averse. They behave competitively and there are no market frictions (in particular, there are no restrictions on short sales). One of the investors, Mr. Cara, has initial wealth equal to 1 and his coefficient of absolute risk aversion is 20 at all levels of wealth.

(a) Calculate Mr. Cara's optimal portfolio in terms of the amounts invested in each asset.

Mr. Cara has negative exponential utility:

$$U(W) = -\exp(-20W)$$

Let the amounts invested in assets A and B be denoted by a and b, respectively. Terminal wealth is given by

$$\tilde{W} = a\left(1+\tilde{r}_A\right) + b\left(1+\tilde{r}_B\right) + (1-a-b)\left(1+r_0\right)$$

where r_0 is the riskfree rate. Since wealth is normally distributed, the program

$$\max E[-\exp(-20\tilde{W})]$$

has the same solution as the program

$$\max[E(\tilde{W}) - 10\operatorname{var}(\tilde{W})]$$

We have

$$
\begin{aligned}
E(\tilde{W}) &= 1.06 + 0.06a + 0.12b \\
\operatorname{var}(\tilde{W}) &= 0.003a^2 + 0.005b^2 + 0.002ab
\end{aligned}
$$

Therefore, the utility-maximization problem is

$$\max_{a,b}\left[1.06 + 0.06a + 0.12b - 0.03a^2 - 0.05b^2 - 0.02ab\right]$$

Solving the first order conditions, we get $a = \frac{9}{14}$ and $b = \frac{15}{14}$, implying an investment of $-\frac{5}{7}$ in the riskless asset.

(b) Argue that the CAPM must hold for this economy. (You may allude to known results without proving them.) Without calculating the risky asset portfolio frontier, derive the equations of the capital market line and the security market line.

Returns are normally distributed, and we have expected-utility maximizing investors with increasing, concave VNM utility. Therefore, all investors have meanvariance preferences. Moreover, all investors hold the same portfolio m of the two risky assets. Hence the CAPM holds provided the market rate of return r_m exceeds the riskfree rate.

Since Mr. Cara, in particular, holds the market portfolio, we see from his optimal portfolio in part (a) that

$$m = \left(\frac{a}{a+b}, \frac{b}{a+b}\right) = \left(\frac{9}{24}, \frac{15}{24}\right) = (0.375, 0.625)$$

Therefore, $r_m = 15.75\%$, which is higher than the riskfree rate of 6%. Hence the CAPM holds. The security market line is given by

$$
\begin{aligned}
r &= r_0 + \left(r_m - r_0\right)\beta \\
&= 0.06 + 0.0975\beta
\end{aligned}
$$

The market portfolio is on the capital market line. Hence, the equation of the capital market line is

$$r = r_0 + \frac{r_m - r_0}{\sigma_m} \cdot \sigma$$

Now

$$\sigma_m^2 = (0.375)^2(0.003) + (0.625)^2(0.005) + 2(0.375)(0.625)(0.001) = 0.00284375$$

which gives us the capital market line:

$$r = 0.06 + 1.83\sigma$$

Week 7 - Preparation Midterm Exam

1. Consider an investor with logarithmic VNM utility and initial wealth W_0. The investor chooses a portfolio at date 0 to maximize the expected utility of wealth at date 1. There are two assets: a riskless asset with (gross) rate of return R_f and a risky asset with rate of return $\tilde{R}$ that takes values R_1 and R_2 $(0 < R_1 < R_2)$ with probability π and $(1-\pi)$ respectively. Denote $\mathbb{E}(\tilde{R})$ by $\bar{R}$, and the amount invested in the risky asset by $a \in \mathbb{R}$. Note that there are no restrictions on short sales.

(a) Determine a condition on rates of return such that the investor's portfolio problem has a solution. Interpret this condition. Assume that it holds for the remainder of this question.

NOTE: Terminal wealth is $\tilde{W} = W_0 R_f + a\left(\tilde{R} - R_f\right)$. The agent chooses a to maximize $\mathbb{E}U(\tilde{W})$. The first order condition is

$$\mathbb{E}\left[U'(\tilde{W})\left(\tilde{R} - R_f\right)\right] = 0$$

A necessary condition for the FOC to have a solution is that $\operatorname{Prob}\left(\tilde{R} > R_f\right) \in (0,1)$ so that neither asset dominates the other. In this example the condition is simply

$$R_1 < R_f < R_2 \tag{1}$$

This condition is also sufficient for a solution to the agent's problem as we shall see below.

(b) Solve explicitly for the optimal amount a^* invested in the risky asset. You should be able to answer the remaining questions, at least partially, even if you are not able to obtain an explicit expression for a^*.

With log utility, the FOC is

$$\mathbb{E}\left[\frac{\tilde{R} - R_f}{\tilde{W}}\right] = 0$$

or

$$\pi \cdot \frac{R_1 - R_f}{W_0 R_f + a\left(R_1 - R_f\right)} + (1-\pi) \cdot \frac{R_2 - R_f}{W_0 R_f + a\left(R_2 - R_f\right)} = 0$$

Multiplying across by the product of the terms in the denominators, we get an equation that is linear in a. This gives us:

$$a^* = W_0 R_f \cdot \frac{\bar{R} - R_f}{\left(R_f - R_1\right)\left(R_2 - R_f\right)} \tag{2}$$

2. Suppose there are two assets with distinct expected rates of return. Show that the set of minimum-variance portfolios is the set of all portfolios.

Consider the portfolio frontier generated by portfolios $\mathbf{p}$ and $\mathbf{q}$, with $R_{\mathbf{p}} \neq R_{\mathbf{q}}$. For a given R, there is one and only one portfolio of $\mathbf{p}$ and $\mathbf{q}$ that yields expected return R. Its weights $(w, 1-w)$ solve

$$w R_{\mathrm{p}} + (1-w) R_{\mathrm{q}} = R$$

i.e

$$w = \frac{R - R_{\mathrm{q}}}{R_{\mathrm{p}} - R_{\mathrm{q}}}$$

This portfolio is clearly the minimum-variance portfolio for expected return R. As we vary R we generate the set of feasible portfolios, all of which are on the portfolio frontier.

Week 10 - State Prices & Arrow Debreu

1. Consider a two-date economy with two agents who trade a risky asset at price p (which you will be asked to determine below). The asset is in zero net supply. Each agent has to decide on what her position in the asset is at date 0, but pays the price (p, per unit of the asset held) at the same time the payoffs are realized, at date 1. There are no restrictions on short sales. The risky asset has payoff $\tilde{v}$. Agent $i (i = 1, 2)$ has constant absolute risk aversion r_i, and initial endowment $\tilde{e}_i$ available at date 1 (there are no endowments at date 0, nor do the agents have any initial holdings of the asset). The random variables $\tilde{v}, \tilde{e}_1$, and $\tilde{e}_2$ are joint normally distributed. Traders behave competitively, taking prices as given, and maximize expected utility of date 1 consumption.

(a) Derive trader i's demand function for the risky asset (i.e. the number of units of the risky asset bought as a function of the price). Provide an interpretation of this demand function in terms of speculation and hedging.

Let p be the price of the risky asset. If trader i takes the position θ_i in the risky asset, his wealth (or consumption) at date 1 is

$$\tilde{W}_i = \tilde{e}_i + \theta_i(\tilde{v} - p)$$

We see that $\tilde{W}_i$ is normally distributed. Since the agent has CARA utility, maximizing expected utility is equivalent to maximizing the mean-variance criterion:

$$
\begin{aligned}
& E\left(\tilde{W}_i\right) - \frac{r_i}{2}\operatorname{var}\left(\tilde{W}_i\right) \\
=& E\left(\tilde{e}_i\right) + \theta_i[E(\tilde{v}) - p] - \frac{r_i}{2}\left[\operatorname{var}\left(\tilde{e}_i\right) + \theta_i^2 \operatorname{var}(\tilde{v}) + 2\theta_i \operatorname{cov}\left(\tilde{e}_i, \tilde{v}\right)\right]
\end{aligned}
$$

The first order condition gives us

$$\theta_i = \frac{E(\tilde{v}) - p - r_i \operatorname{cov}\left(\tilde{e}_i, \tilde{v}\right)}{r_i \operatorname{var}(\tilde{v})}$$

The optimal portfolio is the sum of a speculative component

$$\frac{E(\tilde{v}) - p}{r_i \operatorname{var}(\tilde{v})}$$

and a hedging term

$$-\frac{\operatorname{cov}\left(\tilde{e}_i, \tilde{v}\right)}{\operatorname{var}(\tilde{v})}$$

In the absence of a hedging motive (when $\tilde{e}_i = 0$, or $\operatorname{cov}\left(\tilde{e}_i, \tilde{v}\right) = 0$) , the investor goes long or short in the asset depending on whether the expected asset payoff is higher or lower than the price. The hedging term is just the negative of the "beta" of the investor's endowment with respect to the asset payoff, i.e. it is the regression coefficient from the OLS regression of $\tilde{e}_i$ on $\tilde{v}$. The greater is the correlation of the asset payoff with the investor's endowment, the more he will wish to hedge, selling in the case of a positive correlation, and buying otherwise.

(b) Calculate the equilibrium price of the risky asset. Interpret.

In equilibrium, it must be the case that long and short positions cancel out:

$$\theta_1 + \theta_2 = 0$$

implying that

$$\frac{E(\tilde{v}) - p}{\operatorname{var}(\tilde{v})} \cdot \frac{2}{r} - \frac{\operatorname{cov}\left(\tilde{e}_1 + \tilde{e}_2, \tilde{v}\right)}{\operatorname{var}(\tilde{v})} = 0$$

where r is defined by

$$\frac{2}{r} = \frac{1}{r_1} + \frac{1}{r_2}$$

The coefficient r can be interpreted as the aggregate risk aversion coefficient (it is the harmonic mean of the individual risk aversion coefficients r_1 and r_2). We can now solve for the equilibrium price:

$$p = E(\tilde{v}) - \frac{r}{2}\operatorname{cov}\left(\tilde{e}_1 + \tilde{e}_2, \tilde{v}\right)$$

The second term is the risk premium, which is higher the greater is the aggregate risk aversion coefficient r and/or the greater is the covariance of the aggregate endowment (which is the aggregate risk in the economy) with the asset payoff.

(c) What are the equilibrium holdings of the risky asset? Interpret your results.

By substituting the equilibrium price into the optimal portfolios of agents, we can calculate their equilibrium holdings of the risky asset:

$$
\begin{aligned}
\theta_1 &= \frac{\operatorname{cov}\left(r_2\tilde{e}_2 - r_1\tilde{e}_1, \tilde{v}\right)}{\left(r_1+r_2\right)\operatorname{var}(\tilde{v})} \\
\theta_2 &= \frac{\operatorname{cov}\left(r_1\tilde{e}_1 - r_2\tilde{e}_2, \tilde{v}\right)}{\left(r_1+r_2\right)\operatorname{var}(\tilde{v})}
\end{aligned}
$$

Suppose, for example, that both $\tilde{e}_1$ and $\tilde{e}_2$ are positively correlated with $\tilde{v}$. Then, all else equal, an agent buys more of the asset in equilibrium, the greater is the risk aversion weighted endowment of the other agent. Why? Because the other agent will have a greater desire to hedge, in other words the other agent will have a greater desire to go short.

Week 11 - Complete Markets & CCAPM

1. Consider a two-period Arrow-Debreu economy with a single consumption good, two states, and two consumers. There is no consumption or endowments at the initial date (date 0). Uncertainty is resolved at date 1. The consumers have utility functions:

$$
\begin{aligned}
U^A\left(c_1^A, c_2^A\right) &= \pi^A u^A\left(c_1^A\right) + \left(1-\pi^A\right) u^A\left(c_2^A\right) \\
U^B\left(c_1^B, c_2^B\right) &= \pi^B u^B\left(c_1^B\right) + \left(1-\pi^B\right) u^B\left(c_2^B\right)
\end{aligned}
$$

where c_s^i is the amount of the s -contingent good consumed by consumer i, and π^i is the subjective probability of consumer i for state 1. The functions u^i are concave (though not necessarily strictly concave). The total initial endowments of the two contingent commodities are

$$\overline{\mathrm{e}} := \left(\bar{e}_1, \bar{e}_2\right)$$

Each consumer has half of the random variable $\overline{\mathrm{e}}$, that is,

$$\mathrm{e}^A = \mathrm{e}^B = \frac{1}{2}\overline{\mathrm{e}}$$

For the following problems, show your algebraic calculations, and illustrate using Edgeworth boxes. Assume interior solutions.

(a) Suppose that consumer A is risk-neutral and consumer B is risk averse. Both consumers have the same subjective probabilities $\left(\pi^A = \pi^B\right)$. Compute the ArrowDebreu equilibrium prices and show that consumer B insures completely.

The marginal rate of substitution of agent $i (i = A, B)$ is given by

$$\mathrm{MRS}^i = \frac{\pi^i}{1-\pi^i} \cdot \frac{u^{i\prime}\left(c_1^i\right)}{u^{i\prime}\left(c_2^i\right)}$$

At an interior equilibrium

$$\mathrm{MRS}^A = \mathrm{MRS}^B = \frac{p_1}{p_2} \tag{1}$$

Let $\pi = \pi^A = \pi^B$. Since agent A is risk neutral,

$$\mathrm{MRS}^A = \frac{\pi}{1-\pi}$$

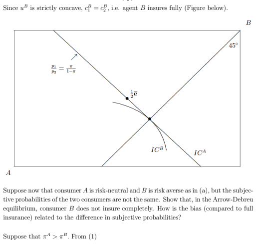
Therefore, from (1)

$$\frac{p_1}{p_2} = \frac{\pi}{1-\pi}$$

and

$$\frac{u^{B\prime}\left(c_1^B\right)}{u^{B\prime}\left(c_2^B\right)} = 1$$

Since u^B is strictly concave, $c_1^B = c_2^B$, i.e. agent B insures fully (Figure below).



(b) Suppose now that consumer A is risk-neutral and B is risk averse as in (a), but the subjective probabilities of the two consumers are not the same. Show that, in the Arrow-Debreu equilibrium, consumer B does not insure completely. How is the bias (compared to full insurance) related to the difference in subjective probabilities?

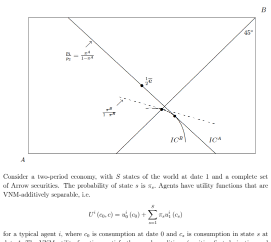
Suppose that $\pi^A > \pi^B$. From (1)

$$\frac{\pi^A}{1-\pi^A} = \frac{\pi^B}{1-\pi^B} \cdot \frac{u^{B\prime}\left(c_1^B\right)}{u^{B\prime}\left(c_2^B\right)}$$

implying that

$$\frac{u^{B\prime}\left(c_1^B\right)}{u^{B\prime}\left(c_2^B\right)} > 1$$

By the strict concavity of u^B, $c_1^B < c_2^B$. Agent B consumes more in the state that he considers more likely (compared to agent A). See Figure below.



2. Consider a two-period economy, with S states of the world at date 1 and a complete set of Arrow securities. The probability of state s is π_s. Agents have utility functions that are VNM-additively separable, i.e.

$$U^i\left(c_0, c\right) = u_0^i\left(c_0\right) + \sum_{s=1}^{S} \pi_s u_1^i\left(c_s\right)$$

for a typical agent i, where c_0 is consumption at date 0 and c_s is consumption in state s at date 1. The VNM utility functions satisfy the usual conditions (positive first derivative and negative second derivative). Let ψ_s be the price of Arrow security s, and denote the state price deflator as by $m_s := \frac{\psi_s}{\pi_s}$. We define agent i's state price deflator as $m_s^i := \frac{u_1^{i\prime}(c_s^i)}{u_0^{i\prime}(c_0^i)}$. The aggregate endowment is e_0 at date 0 and e_s in state s at date 1 Answer the following questions about the properties of an asset market equilibrium with consumption allocation $\left(c_0^i, c^i\right)$.

(a) Using the first order conditions of the agents show that $m_s^i = m_s$, for all agents i and all states s.

The state- s FOC for agent i is

$$\frac{\pi_s u_1^{i\prime}\left(c_s^i\right)}{u_0^{i\prime}\left(c_0^i\right)} = \psi_s$$

i.e. $m_s^i = m_s$. This holds for all s and i.

(b) Show that consumption is decreasing in the state price deflator. More precisely, show that $c_s^i > c_{s'}^i$ if and only if $m_s < m_{s'}$.

We have

$$\frac{u_1^{i\prime}\left(c_s^i\right)}{u_0^{i\prime}\left(c_0^i\right)} = m_s$$

and the result follows from the fact that $u_1^{i\prime}$ is a strictly decreasing due to risk aversion.

(c) Now deduce that the state price deflator is decreasing in aggregate consumption, and that individual consumption is increasing in aggregate consumption. More precisely, show that $m_s > m_{s'}$ if and only if $e_s < e_{s'}$, and $c_s^i > c_{s'}^i$ if and only if $e_s > e_{s'}$.

In a state with higher aggregate consumption, individual consumption must be higher for at least one agent (due to market clearing). From this agent's FOC, the state price deflator must be lower. But the state price deflator m is the same for all agents, implying that consumption is higher for all agents.

(d) We say that agent i has positive time preference if $\frac{u_1^{i\prime}(x)}{u_0^{i\prime}(x)} < 1$, for all $x \geq 0$ Suppose all agents have positive time preference, and $e_s \geq e_0$ for all s. Show that the equilibrium rate of interest r (the implied net rate of return on the riskless asset) is positive.

The price of the riskless asset (which is a portfolio consisting of one unit each of all the Arrow securities) is just the sum of the state prices $\sum_s \psi_s$, i.e.

$$(1+r)^{-1} = \sum_s \pi_s m_s \tag{2}$$

Since $e_s \geq e_0$, we must have, for at least one agent i, $c_s^i \geq c_0^i$. Due to positive time preference and risk aversion, $m_s^i < 1$. Therefore, $m_s = m_s^i < 1$. The same argument applies to all the other states, so that $m_s < 1$ for all s. Hence, from (2), $(1+r)^{-1} < 1$, or $r > 0$.

(e) Now suppose that VNM utility functions are state dependent, i.e. of the form

$$U^i(c_0, c) = u_0^i(c_0) + \sum_{s=1}^{S} \pi_s u_s^i(c_s)$$

For this case we define $m_s^i := \frac{u_s^{i\prime}(c_s)}{u_0^{i\prime}(c_0)}$. Comment on whether the results in parts (a)-(c) still hold.

It is still the case that $m = m^i$, for all i. However, the comonotonicity property for individual or aggregate consumption is lost because $u_s^{i\prime}\left(c_s^i\right) > u_{s'}^{i\prime}\left(c_{s'}^i\right)$ does not tell us whether $c_s^i < c_{s'}^i$ or not.

Week 12 - Arrow Debreu Pricing

1. Consider a two-period economy with a single consumption good, three states of the world, and three securities traded at the initial date. The asset payoff matrix is:

$$\mathbf{D} = \begin{pmatrix} 2 & 1 & 8 \\ 1 & 0 & 3 \\ 0 & 1 & 2 \end{pmatrix}$$

The (i, j) -th element of $\mathbf{D}$ is the payoff, in terms of the consumption good, of the j -th security in the i -th state of the world.

(a) What is the marketed subspace (or the asset span)? Give an example of a 3×3 payoff matrix D', in which none of the securities has the same payoffs as those in D, with the same marketed subspace as D.

Let d^j be the j 'th column of D. We see that

$$\mathrm{d}^3 = 3\,\mathrm{d}^1 + 2\,\mathrm{d}^2$$

i.e. asset 3 is redundant. The marketed subspace A is the subspace of $\mathbb{R}^3$ spanned by d^1 and d^2. It is given by

$$A = \left\{ \begin{pmatrix} 2x_1 + x_2 \\ x_1 \\ x_2 \end{pmatrix} : x_1, x_2 \in \mathbb{R} \right\}$$

For D' we choose three vectors in A, all of which are not collinear. For example,

$$\mathbf{D}' = \begin{pmatrix} 3 & 4 & 5 \\ 1 & 2 & 2 \\ 1 & 0 & 1 \end{pmatrix}$$

(b) What is the dimension of the marketed subspace? Are security markets complete? If not, is it possible to introduce another asset so that markets do become complete? Give an example of such an asset, and show that it does indeed complete the markets.

Clearly the dimension of A is 2, and hence markets are incomplete. We can complete the markets by introducing an asset whose payoff is not in A, for example the riskless asset with payoff 1 in every state.

(c) Characterize the set of arbitrage-free prices for the securities in D. In other words, find all the prices q for the assets such that there does not exist an arbitrage, given $\mathbf{D}$. Give an example of a price vector in this set.

The asset price vector $\mathrm{q} \in \mathbb{R}^3$ is arbitrage-free if and only if there exists a state price vector $\psi \in \mathbb{R}^3_{++}$, such that $\mathbf{q}^\top = \psi^\top \mathbf{D}$, i.e.

$$\begin{aligned} q_1 &= 2\psi_1 + \psi_2 \\ q_2 &= \psi_1 + \psi_3 \\ q_3 &= 8\psi_1 + 3\psi_2 + 2\psi_3 \end{aligned} \tag{1}$$

This holds for some $\psi \in \mathbb{R}^3_{++}$ if and only if

$$q_1 > 0$$

$$\begin{aligned} q_2 &> 0 \\ q_3 &= 3q_1 + 2q_2 \end{aligned}$$

For example,

$$\mathrm{q} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$$

is an arbitrage-free price vector.

(d) Provide an example of (positive) prices for the securities in D, such that there does exist an arbitrage. Exhibit an arbitrage in your example.

Any price vector q that does not satisfy (1) admits an arbitrage. For instance,

$$\mathrm{q} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

For this q

$$\mathbf{x} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

is an arbitrage since

$$\begin{pmatrix} -\mathrm{q}^\top \\ \mathrm{D} \end{pmatrix} \mathrm{x} = \begin{pmatrix} 1 \\ 5 \\ 2 \\ 1 \end{pmatrix}$$

(e) Suppose $(\mathbf{D}, \mathbf{q})$ does not admit arbitrage. Consider the following asset

$$\mathrm{d} = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$$

i. Show that d can be priced by arbitrage. Find two distinct replicating portfolios for d.

The asset d is attainable since $\mathrm{d} = 2\,\mathrm{d}^1 + \mathrm{d}^2$. Hence it can be priced by arbitrage. One replicating portfolio for d is immediate:

$$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

To find another replicating portfolio, consider for example a portfolio of just assets 2 and 3 in D, i.e. a portfolio of the form

$$\mathbf{x} = \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix}$$

In order for this to be a replicating portfolio for d, we must have $\mathrm{Dx} = \mathrm{d}$:

$$x_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 8 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$$

Solving for x_2 and x_3, we get

$$\mathbf{x} = \begin{pmatrix} 0 \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix}.$$

ii. What is the riskfree rate in this economy? Can you find it via no arbitrage?

The riskfree asset is not attainable. Therefore we cannot find the riskfree rate by arbitrage.

2. Consider a two-period economy with a single physical good. Assets are traded at date 0 and pay off at date 1. Suppose there are three states and two securities. The asset payoff matrix is

$$\begin{pmatrix} 64 & 0 \\ 16 & 0 \\ 4 & 1 \end{pmatrix}$$

The market prices of these assets are $q_1 = 32$ and $q_2 = 1$, respectively. In the following you are asked to price a variety of derivative assets. (Note: In this question, the price of an asset at date 1 is its price before the asset pays off. Hence an asset's date 1 price in a particular state is equal to its payoff in that state.)

(a) Consider a derivative asset described as "One unit of this asset confers the right to buy one unit of asset 1 at 75% of its price in period 1 (after the state of the world occurs)." Price this asset.

(b) The situation is the same as in (a) except that the asset is modified to read "One unit of this asset confers the right to buy one unit of asset 1 at 75% of its price in period 1 (after the state of the world occurs), provided this price (i.e. the price of asset 1) is at least 10 "

(c) Suppose that the asset is as in (b) except that "at least 10" is replaced by "at least 19." Argue that this asset cannot be priced by arbitrage with the available primary assets.

(d) How would the analysis in (c) differ if we had in addition an asset with payoff

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

and price equal to one? Compute the arbitrage price of the derivative asset.

(e) Suppose that now the asset is further complicated to read "One unit of this asset confers, at the choosing of the holder, either one unit of consumption in period 1 or the right to buy one unit of asset 1 at 75% of its price in period 1 (after the state of the world occurs)." Price this asset.

(f) The situation is the same as in (e) except that the asset is modified to read "One unit of this asset confers, at the choosing of the holder, either one unit of consumption in period 1 or the right to buy one unit of asset 1 at 75% of its price in period 1 (after the state of the world occurs), provided the price of asset 1 is at least 10."

State prices are a solution to the following equations

$$\begin{aligned} 64\psi_1 + 16\psi_2 + 4\psi_3 &= 32 \\ \psi_3 &= 1 \end{aligned}$$

There are multiple state prices. For example, we can choose

$$\begin{aligned} \psi_1 &= \tfrac{3}{16} \\ \psi_2 &= 1 \\ \psi_3 &= 1 \end{aligned}$$

Since state prices exist, there is no arbitrage, so we can price attainable claims by arbitrage. Note that the price of an asset in period 1, after the state of the world has been realized, must be the payoff of the asset in that state. The derivative assets in parts (a), (b) and (c) are

$$\begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 16 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 16 \\ 0 \\ 0 \end{pmatrix}$$

The prices are 8 and 7 respectively for the first two assets. The third asset is not attainable. In part (d), however, the third asset is attainable, and its price is 3. The derivative asset in parts (e) and (f) is the same as in (a), and hence has the same price, i.e., 8.

Week 13 - Risk-Neutral Pricing

1. Consider an economy in which a stock is traded at date 0 and date 1. The price of the stock at date 1 is either S_1 with probability π or S_2 with probability $(1-\pi)$, and this is the only uncertainty in the economy. In addition, there is a European call and a European put option on the stock, with maturity at date 1. Both options have the same strike price K, and at date 0 they trade at prices C and P respectively. Suppose that $0 < S_1 < K < S_2$. The stock does not pay dividends before date 1. In answering the following questions assume that there is no arbitrage.

(a) Calculate the prices at date 0 (as a function of P, C, K, S_1 and S_2)

i. of the stock

ii. of a zero coupon bond that has face value 1 at date 1.

This is a two-state economy in which the stock price is S_1 in state 1 and S_2 is state 2. Let C_s and P_s be the maturity value of the call and the put in state s. Then

$$\begin{aligned} C_1 &= 0 \\ C_2 &= S_2 - K \\ P_1 &= K - S_1 \\ P_2 &= 0 \end{aligned}$$

Thus the put is collinear with Arrow security 1, and the call is collinear with Arrow security 2. Hence the state prices are

$$\psi_1 = \frac{P}{K - S_1}, \quad \psi_2 = \frac{C}{S_2 - K}$$

We have complete markets: any asset can be priced using these state prices. The price of the bond at date 0 is

$$B = \psi_1 + \psi_2 = \frac{P}{K - S_1} + \frac{C}{S_2 - K}$$

The price of the stock at date 0 is

$$S = \psi_1 S_1 + \psi_2 S_2 = \frac{P S_1}{K - S_1} + \frac{C S_2}{S_2 - K}$$

(b) Consider an investor with utility function $\log(c_0) + \beta \mathbb{E} \log(\tilde{c}_1)$, where c_0 is consumption at date $0, \tilde{c}_1$ is consumption at date 1, and β is a subjective discount factor $(0 < \beta < 1)$. His initial wealth is W_0 and he has no endowment at date 1. Write down his utility maximization problem and calculate his optimal consumption plan. (Do your calculations for arbitrary prices, and substitute the prices that apply in this question only at the very end.)

The investor faces prices ψ_1 and ψ_2 for state-contingent consumption at date 1. His utility maximization problem is

$$\begin{aligned} &\max_{c_0, c_1, c_2} \log(c_0) + \beta\left[\pi \log(c_1) + (1-\pi)\log(c_2)\right] \\ &\text{s.t. } c_0 + \psi_1 c_1 + \psi_2 c_2 = W_0 \end{aligned}$$

We can write this as an unconstrained problem:

$$\max_{c_1, c_2} \log(W_0 - \psi_1 c_1 - \psi_2 c_2) + \beta\left[\pi \log(c_1) + (1-\pi)\log(c_2)\right]$$

The first order conditions are:

$$\begin{aligned} -\frac{\psi_1}{c_0} + \frac{\beta\pi}{c_1} &= 0 \\ -\frac{\psi_2}{c_0} + \frac{\beta(1-\pi)}{c_2} &= 0 \end{aligned}$$

which can be rewritten as:

$$\begin{aligned} \psi_1 c_1 &= c_0 \beta \pi \\ \psi_2 c_2 &= c_0 \beta (1-\pi) \end{aligned}$$

Therefore,

$$\begin{aligned} c_0 &= W_0 - \psi_1 c_1 - \psi_2 c_2 \\ &= W_0 - c_0 \beta \end{aligned}$$

so that

$$c_0 = \frac{W_0}{1+\beta}$$

Substituting this value in the FOCs above, we can solve for c_1 and c_2 :

$$\begin{aligned} c_1 &= \frac{\beta}{1+\beta} \cdot \frac{\pi W_0}{\psi_1} \\ c_2 &= \frac{\beta}{1+\beta} \cdot \frac{(1-\pi) W_0}{\psi_2} \end{aligned}$$

We can now obtain the investor's optimal consumption plan by substituting in the values in the expressions of ψ_1 and ψ_2 from above (see part (a)).

(c) Can you implement the consumption plan in part (b) using a portfolio of

i. just the stock and the bond?

ii. just the call and the put?

In both cases, explain why or why not. Do not calculate any portfolio.

For the case of the stock and the bond, the payoffs of the two assets are not collinear. Therefore, markets are complete given these two assets, and the consumption plan can be implemented via a portfolio of the two assets. The same is true for the put and the call.

Week 4 - Modern Portfolio Theory

1. Let $\mathbf{p}$ be a frontier portfolio, and let $\mathbf{q}$ be any portfolio with the same expected rate of return. Show that $\operatorname{cov}\left(\tilde{R}_{\mathrm{p}}, \tilde{R}_{\mathrm{q}}\right) = \operatorname{var}\left(\tilde{R}_{\mathrm{p}}\right)$.

Since $\mathbf{p}$ and $\mathbf{q}$ have the same expected return, and $\mathbf{p}$ is a frontier portfolio, $w = 0$ is a solution to the following program:

$$\min_{w \in \mathbb{R}} \operatorname{var}\left[w \tilde{R}_{\mathbf{q}} + (1-w)\tilde{R}_{\mathbf{p}}\right]$$

In particular, $w = 0$ must solve the first order condition

$$2w \operatorname{var}\left(\tilde{R}_{\mathbf{q}}\right) - 2(1-w)\operatorname{var}\left(\tilde{R}_{\mathbf{p}}\right) + 2(1-2w)\operatorname{cov}\left(\tilde{R}_{\mathbf{p}}, \tilde{R}_{\mathbf{q}}\right) = 0$$

which implies that $\operatorname{cov}\left(\tilde{R}_{\mathbf{p}}, \tilde{R}_{\mathbf{q}}\right) = \operatorname{var}\left(\tilde{R}_{\mathbf{p}}\right)$ The above proof involves a "trick" which is not completely obvious. A more direct method would be to use the first-order condition for a variance-minimizing portfolio. Try it!

2. Suppose there are two assets with uncorrelated returns. Expected (net) returns are $r_1 = 2$ and $r_2 = 4$, and the variances of returns are $\sigma_1^2 = 1$ and $\sigma_2^2 = 2$. Determine the mean-variance portfolio frontier (in $\sigma - r$ space), the global minimum variance portfolio, and the mean-variance efficient frontier.

Since there are only two assets, the portfolio frontier is the set of all portfolios:

$$\{(w, 1-w); w \in \mathbb{R}\}$$

To derive the portfolio frontier in mean-variance space, we proceed as follows. For any fixed expected return r, the portfolio weight w must solve

$$2w + 4(1-w) = r$$

Hence,

$$w = 2 - \frac{r}{2} \tag{1}$$

The variance of the rate of return of this portfolio is

$$\sigma^2 = w^2 + 2(1-w)^2$$

Substituting from (1), we get the equation of the portfolio frontier:

$$\sigma^2 = \frac{3}{4}r^2 - 4r + 6 \tag{2}$$

The expected return on the global minimum variance portfolio r_g is the r that solves

$$\frac{d\sigma^2}{dr} = \frac{3}{2}r - 4 = 0$$

Hence

$$r_g = 2\frac{2}{3}$$

Using (1)

$$\mathrm{g} = \left(\frac{2}{3}, \frac{1}{3}\right)$$

The efficient frontier is given by (2), with $r \geq 2\frac{2}{3}$.

3. Suppose that an investor has von Neumann-Morgenstern utility function

$$U = -e^{-bW} \quad b > 0$$

and that wealth W is normally distributed. Show that maximizing expected utility of wealth is equivalent to maximizing the following mean-variance criterion:

$$\mathcal{E}(W) := \mathrm{E}(W) - \frac{b}{2}\operatorname{Var}(W)$$

Show that $\mathcal{E}(W)$ is the certainty-equivalent of W. Hint: Use the following result: Suppose $X \sim N(\mu, \sigma^2)$. Then $E\left(e^X\right) = e^{\mu + \frac{\sigma^2}{2}}$.

$$\begin{aligned} E\left[-e^{-bW}\right] &= -\exp\left[E(-bW) + \frac{1}{2}\operatorname{var}(-bW)\right] \\ &= -\exp\left[-bE(W) + \frac{b^2}{2}\operatorname{var}(W)\right] \\ &= -\exp\left[-b\left(E(W) - \frac{b}{2}\operatorname{var}(W)\right)\right] \\ &= -\exp[-b\mathcal{E}(W)] \end{aligned} \tag{3}$$

which is a strictly increasing function of $\mathcal{E}(W)$. Therefore maximizing $E\left[-e^{-bW}\right]$ is equivalent to maximizing $\mathcal{E}(W)$. Furthermore, from (3) it is clear that $\mathcal{E}(W)$ is the certainty-equivalent of W.