UE23MA141A: Engineering Mathematics – I (4-0-0-4-4)

Unit 3: Differential Equations of Higher Order and Partial Differential Equations

Method of Variation of Parameters, Solution to higher order Linear Differential Equations with variable coefficients - Legendre's Differential Equations, Application problems - LR circuits and Mass Spring Mechanical system (Free undamped oscillations of a spring).

Linear Partial Differential Equations of the first order, Lagrange's Linear Equation, Solution of PDE by the method of Separation of Variables, Solution of Homogeneous Linear Partial Differential Equations with constant co-efficient.

Self-learning component: Cauchy's Differential Equations and Formation of Partial Differential Equations. **14 Hours (18 Sessions)**

Class work problems

Problems on Method of Variation of Parameters

1. Solve $y'' + 4y = sec2x$ by the method of variation of parameters.

   Ans: $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x \log(\cos 2x)$

2. Solve $y'' + 6y' + 9y = \frac{e^{-3x}}{x}$ by the method of variation of parameters.

   Ans: $y = (c_1 x + c_2) e^{-3x} + e^{-3x} x(\log x - 1)$

3. Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$ by the method of variation of parameters.

   Ans: $y = C_1 e^{-x} + C_2 e^{-2x} + e^{e^x} e^{-2x}$

4. Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = \frac{1}{1-e^x}$ by the method of variation of parameters. **(Homework)**

   Ans: $y = C_1 e^x + C_2 e^{-2x} + e^{-x} - 1 + \log(1-e^x)(e^x - e^{-2x}) + xe^x$

5. Solve by the method of variation of parameters $y'' + y = \frac{1}{1+sinx}$. **(Homework)**

   Ans: $y = c_1 \cos x + c_2 \sin x + \sin x \log(1 + \sin x) - x \cos x - 1$

Problems on Legendre's Differential Equations

1. Solve $(2x-1)^2 \frac{d^2y}{dx^2} + (2x-1)\frac{dy}{dx} - 2y = 8x^2 - 2x + 3$.

Ans: $y = C_1(2x-1) + C_2(2x-1)^{\frac{-1}{2}} + \frac{1}{5}(2x-1)^2 + \frac{1}{2}(2x-1)\log(2x-1)$

2. Solve $(2x+1)^2\frac{d^2y}{dx^2} - 6(2x+1)\frac{dy}{dx} + 16y = 8(2x+1)^2$.

   Ans: $y = (1+2x)^2(C_1 + C_2\log(1+2x) + \{\log(1+2x)\}^2)$

3. Solve $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = sin[2log(1+x)]$.

   Ans: $y = C_1\cos(\log(1+x) + C_2\sin(\log(1+x) - \frac{1}{3}\sin[2\log(1+x)]$

4. Solve $(2x+5)^2\frac{d^2y}{dx^2} - 6(2x+5)\frac{dy}{dx} + 8y = 6x$. **(Homework)**

   Ans: $C_1(2x+5)^{2+\sqrt{2}} + C_2(2x+5)^{2-\sqrt{2}} - \frac{3}{4}(2x+5) - \frac{15}{18}$

Application Problems

1. A condenser of capacity C discharged through an inductance L and resistance R in series and the charge q at time t satisfies the equation $L\frac{d^2q}{dt^2}$+R$\frac{dq}{dt} + \frac{q}{C}$=0. Given that L=0.25 Henries, R=250 ohms and C=2*10^{-6} Farads and when t=0 charge q is 0.002 columbs and the current $\frac{dq}{dt}$=0. Obtain the value of q in terms of t.

   Ans: $q = e^{-500t}(0.002cos1323t + 0.0008sin1323t)$

2. A body weighing 10 kg is hung from a spring. A pull of 20kg weight will stretch the spring to10cm. The body is pulled down to 20cm below the static equilibrium position and then released. Find the displacement of the body from its equilibrium position at time t sec., the maximum velocity and the period of oscillation. d2x/dt2+k/m=0

   Ans: displacement =$0.2cos14t\ m$, max velocity=$2.8\ m/s$

Problems on Solution of Lagrange's linear PDE of first order in the form Pp+Qq=R

Solve the following Lagrange's linear PDE's

1. $2yzp + zxq = 3xy$

   Ans: $\phi\left(x^2 - 2y^2, 3y^2 - z^2\right) = 0$

2. $y^2p - xyq = x(z - 2y)$

   Ans $\phi\left(x^2 + y^2, yz - y^2\right) = 0$

3. $\frac{y^2z}{x}p + xzq = y^2$. **(Homework)**

   Ans: $\emptyset(x^3 - y^3, x^2 - z^2) = 0$

4. $2p + q = sin\ (x - 2y)$

Ans: $\phi\left(x-2y, y-\frac{z}{\sin(x-2y)}\right)=0$

5. $xzp + yzq = xy$

   Ans: $\emptyset\left(\frac{x}{y}, xy - z^2\right) = 0$

6. $p + 3q = 5z - \tan(3x - y)$. **(Homework)**

   $Ans:: \emptyset(3x - y, 5x - \log(5z - \tan(3x - y)) = 0$

7. $(x^2 - y^2 - z^2)p + 2xyq = 2xz$

   Ans: $\phi\left(\frac{y}{z}, \frac{x^2+y^2+z^2}{z}\right)=0$

   Multiply each with x,y,z respectively

   group as d(x2+y2+z2)/2

8. $x^2(y - z)p + y^2(z - x)q = z^2(x - y)$.

   $Ans: \emptyset\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}, xyz\right) = 0$

   Case1) take x^2 to numerator

   Case2) Take x to numerator

9. $(x^2 - y^2 - yz)p + (x^2 - y^2 - zx)q = z(x - y)$.

   Ans: $\phi\left(x-y-z, \frac{\left(x^2-y^2\right)}{z^2}\right)=0$

   1. sub dx and dy
   2. multiply x and y respectively group using a3+b3
   3. use your brains MANYAAAA

10. $(y + z)p + (z + x)q = x + y$

    1.Add all and compare with diff of any two

    2. difference of two pairs and equate

    $Ans: \emptyset\left(\frac{x - y}{x - z}, (x + y + z)(x - y)^2\right) = 0$

11. $(x + 2z)p + (4zx - y)q = 2x^2 + y$. **(Homework)**

    Ans: $\phi\left(x^2-y-z, yx-z^2\right)=0$

    1. Multiple to get similar denominators
    2. multiply cross variables and group

    can do ntg but use your brain

Problems on Solution of PDEs by the method of Separation of Variables

1. Solve $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} = 0$ by the method of separation of variables.

   Ans: $u = ce^{\left(-\frac{k}{x}+\frac{c}{y}\right)}$

2. Solve by the method of separation of variables $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$, given that $u(0, y) = 2e^{5y}$. Ans: $u = 2e^{-\frac{x}{2}+5y}$

3. Solve by the method of separation of variables $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$, given that $u(x, 0) = 6e^{-3x}$. Ans: $u = 6e^{-(3x+2t)}$
4. Solve by the method of separation of variables $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2(x + y)u$.
   Ans:$u = ce^{k(x-y)+x^2+y^2}$ X and Y should be pure functions of x and y
5. Find various possible solutions of the one-dimensional heat equation $u_t = c^2 u_{xx}$ by the method of separation of variables.
6. Find various possible solutions of the two-dimensional Laplace equation $u_{xx} + u_{yy} = 0$ by the method of separation of variables.

Problems on homogeneous Linear PDE with constant coefficients

Notations: D =$\boldsymbol{D_x = \frac{\partial}{\partial x}}$; $\boldsymbol{D'}$ =$\boldsymbol{D_y = \frac{\partial}{\partial y}}$; $\boldsymbol{D_x^2 D_y = \frac{\partial^3}{\partial x^2 \partial y}}$; $\boldsymbol{D_x D_y^2 = \frac{\partial^3}{\partial x \partial y^2}}$

1. Solve: $\left(D_x^3 - 6D_x^2 D_y + 11D_x D_y^2 - 6D_y^3\right)z = 0$.
   Ans: $z = f_1(y + x) + f_2(y + 2x) + f_3(y + 3x)$
2. Solve: $\left(4D_x^2 + 12D_x D_y + 9D_y^2\right)z = 0$
   Ans: $z = f_1(2y - 3x) + xf_2(2y - 3x)$ or $z = f_1(y - 1.5x) + xf_2(y - 1.5x)$
3. Solve: $\left(D_x^3 - 3D_x^2 D_y + 4D_y^3\right)z = e^{x+2y}$.
   Ans: $z = f_1(y - x) + f_2(y + 2x) + xf_3(y + 2x) + \frac{e^{x+2y}}{27}$
4. Solve: $\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = e^{2x+y}$
   $\boldsymbol{Ans}$: $z = f_1(y + 2x) + xf_2(y + 2x) + \frac{x^2 e^{2x+y}}{2}$
5. Solve: $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = cos(x + 2y)$
   $\boldsymbol{Ans}$: $z = f_1(y) + f_2(y + x) + cos(x + 2y)$
6. Solve: $(D_x^3 - 4D_x^2 D_y + 4D_x D_y^2)\text{z} = 2\sin(3x + 2\text{y})$
   $\boldsymbol{Ans}$: $z = f_1(y) + f_2(y + 2x) + xf_3(y + 2x) + \frac{2}{3}cos(3x + 2y)$
7. Solve: $(D^2 + DD' - 6D'^2)z = \cos(2x + y)$
   Ans: $z = f_1(y + 2x) + f_2(y - 3x) + \frac{x}{5}sin(2x + y)$
8. **Solve:** $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = x + y.$
   Ans: $z = f_1(y - x) + f_2(y - 2x) + \frac{x^2 y}{2} - \frac{x^3}{3}$
9. Solve: $(D_x^3 - 2D_x^2 D_y)z = 3x^2 y.$

Ans: $z = f_1(y) + xf_2(y) + f_3(y+2x) + \frac{1}{60}(3x^5y + x^6)$

10. Solve: $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = (y-1)e^x$

Ans: $z = f_1(y-x) + f_2(y+2x) + ye^x$

11. *Solve*: $(D - D')^2 z = e^{x+y} \sin(x+2y)$

Ans: $z = f_1(y+x) + f_2(y+2x) + ye^x$

Unit 4: Partial Differential Equations and Special Functions

Solution of non-Homogeneous Linear Partial Differential Equations with constant co-efficient (PI: When $F(x,y) = e^{ax+by}$, $cos(ax+by)$, $\sin(ax+by)$, $x^m y^n$, $e^{ax+by}V(x,y)$).

Definition of Beta and Gamma functions and its properties, relation between Gamma and Beta functions, Duplication formula and problems, Series solution of Bessel's Differential Equation, Recurrence relations (without proof) & related problems, Generating functions, Bessel's Integral formula, and Jacobi series, Orthogonality of Bessel's functions, Problems.

Self-learning component: Proof of Recurrence relations. **14 Hours (19 sessions)**

Problems on non-homogeneous Linear PDE with constant coefficients

1. Solve: $[(D + D' - 2)(D + 4D' - 3)]z = 0.$

   Ans: $z = e^{2x} f_1(y-x) + e^{3x} f_2(y-4x)$

2. Solve: $(D^2 + 2DD' + D'^2 + 2D + 2D' + 1)z = 0.$

   Ans: $z = e^{-x} f_1(y-x) + xe^{-x} f_2(y-x)$

3. Solve: $(2D^2 - DD' - D'^2 + D - D')z = e^{2x+3y}$

   Ans: $z = f_1(y+x) + e^{\frac{x}{2}} f_2(2y-x) - \frac{1}{8}e^{2x+3y}$

4. Solve: $(2D^2 - 5DD' + 3D'^2 + D - D')z = 12e^{x+y}$

   Ans: $z = f_1(y+x) + e^{-\frac{x}{2}} f_2(2y+3x) + 3x^2 e^{x+y}$

5. Solve: $(D^2 + 2DD' + D'^2 - 2D - 2D')z = \sin(x+2y).$

   Ans: $z = f_1(y-x) + e^{2x} f_2(y-2x) + \frac{1}{39}(2cos(x+2y) - 3sin(x+2y)$

6. Solve: $(D^2 - DD' + D' - 1)z = \cos(x+2y)$

   Ans: $\boldsymbol{z} = e^x f_1(y) + e^{-x} f_2(y+x) + \frac{sin(x+2y)}{2}$

7. Solve: $(D^2 - D'^2 + D + 3D' - 2)z = x^2 y$

   Ans:

   $$z = e^{-2x} f_1(y+x) + e^x f_2(y-x) - \frac{1}{2}(x^2 y + \frac{3x^2}{2} + xy + \frac{3y}{2} + 3x + \frac{21}{4})$$

8. Solve: $(D + D' - 1)(D + 2D' - 3)z = 4 + 3x + 6y$
Ans: $z = e^x f_1(y - x) + e^{3x} f_2(y - 2x) + 6 + x + 2y$
9. Solve: $(D - 3D' - 2)^2 z = 2e^{2x}\sin(y + 3x)$
Ans: $z = e^{2x} f_1(y + 3x) + xe^{2x} f_2(y + 3x) + x^2 e^{2x} sin(y + 3x)$

10. **Solve:** $r - 3s + 2t - p + 2q = (2 + 4x)e^{-y}$
Ans: $z = f_1(y + 2x) + e^x f_2(y + x) + x^2 e^{-y}$

Beta and Gamma Functions

Prove the following Standard results:

1. $\beta(m, n) = \beta(n, m)$ (Symmetry Property)
2. $(i)\beta(m, n) = 2\int_0^{\frac{\pi}{2}} sin^{2m-1}\theta cos^{2n-1}\theta d\theta.$ (Beta function in terms of trigonometric functions)
   (ii) $\int_0^{\frac{\pi}{2}} sin^p\theta cos^q\theta d\theta = \frac{1}{2}\beta(\frac{p+1}{2}, \frac{q+1}{2})\ p > -1, q > -1$
3. $\beta(m, n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dy$ (Beta function in terms of improper integral)
4. $\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ (Relation between Beta and Gamma function)
5. $\Gamma(n)\Gamma(1 - n) = \frac{\pi}{sinn\pi}, 0 < n < 1$ (without proof)
6. Use the above formula to prove the result $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \pi\sqrt{2}$
7. **Legendre's duplication formula (Statement only)**
   For Gamma function:
   $$\Gamma(2p)\sqrt{\pi} = 2^{2p-1}\Gamma(p)\Gamma(p + \tfrac{1}{2})$$
   For Beta function:
   $$\beta\left(p, \tfrac{1}{2}\right) = 2^{2p-1}\beta(p, p)$$

Prove the following results.

a) $\int_0^\infty x^p e^{-ax^q}\,dx = \frac{1}{q}\frac{\Gamma\left(\frac{p+1}{q}\right)}{a^{\frac{(p+1)}{q}}}$, where p and q are positive constants

b) $\int_0^1 x^m (\log x)^n\,dx = \frac{(-1)^n\Gamma(n+1)}{(m+n)^{n+1}}$, where n is positive integer and m > - 1

c) $\int_0^1 x^p (1-x^q)^r\,dx = \frac{1}{q}\beta\left(\frac{p+1}{q}, r+1\right)$

d) $\int_0^\infty e^{-ax}x^{n-1}dx = \frac{\Gamma(n)}{a^n}$, where a and n are positive constants.

Evaluate the following integrals using Beta and Gamma functions.

(Note: We can make use of the above results to evaluate some of the integrals given below)

1. $\int_0^\infty \sqrt{y}e^{-y^2}dy$; **Ans:** $\frac{\Gamma(\frac{3}{4})}{2}$
2. $\int_0^\infty (x^2+4)e^{-2x^2}dx$; **Ans:** $\frac{17}{8}\sqrt{\frac{\pi}{2}}$
3. $\int_0^1 \frac{1}{\sqrt{-logx}}dx$; **Ans:**$\sqrt{\pi}$
4. Show that for $m, n > 0$ $\int_0^1 x^{m-1}\left(log\,\frac{1}{x}\right)^{n-1} dx = \frac{\Gamma(n)}{m^n}$
5. $\int_0^\infty 3^{-4x^2}dx$; **Ans:** $\frac{\sqrt{\pi}}{4\sqrt{log3}}$
6. $\int_0^1 x^4(1-x)^3dx$; **Ans:** $\frac{1}{280}$ **(Homework)**
7. $\int_0^1 x^2(1-x^5)^{-1/2}dx$; **Ans:** $\frac{1}{5}\frac{\Gamma(\frac{3}{5})\sqrt{\pi}}{\Gamma(\frac{11}{10})}$
8. $\int_0^1 x^2(1-x^3)^4dx$ **Ans:** $\frac{1}{15}$ **(Homework)**
9. Evaluate a) $\int_0^{\pi/2}\frac{\sqrt[3]{sin8x}}{\sqrt{cosx}}dx$ b) $\int_0^2 (8-x^3)^{-1/3}dx$

   Ans: a) $\frac{60}{13}\frac{\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{12}\right)}$ b) $\frac{2\pi}{3\sqrt{3}}$

10. $\int_0^2 \frac{x^2}{\sqrt{2-x}}dx$; **Ans:** $\frac{64\sqrt{2}}{15}$

11. $\int_0^a x^4\sqrt{a^2-x^2}dx$; **Ans:** $\frac{\pi a^6}{32}$ **(Homework)**

Prove the following results:

1. Show that $\int_0^\infty \sqrt{x}e^{-x^2}dx \ * \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}dx = \frac{\pi}{2\sqrt{2}}$
2. Prove that $\int_0^3 \frac{x^{3/2}}{\sqrt{3-x}}dx \times \int_0^1 \frac{dx}{\sqrt{1-x^{\frac{1}{4}}}} = \frac{432\pi}{35}$

Bessel Functions

1. Give the series solution of Bessel's differential equation $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0$ interms of Bessel's functions $J_n(x)$ and $J_{-n}(x)$. (No proof)
2. Prove that $J_{-n}(x)=(-1)^nJ_n(x)$ where n is a positive integer.
3. Prove that $J_{\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\, sinx$ and hence find $\int_0^{\frac{\pi}{2}}\sqrt{x}\ J_{\frac{1}{2}}(2x)dx$.
4. Prove that $J_{-\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\, cosx$. **(Homework)**
5. Give the following recurrence relations without proof.
   I. $\frac{d}{dx}[x^nJ_n(x)]=x^nJ_{n-1}(x)$
   II. $\frac{d}{dx}[x^{-n}J_n(x)]=-x^{-n}J_{n+1}(x)$
   III. $\frac{d}{dx}[J_n(x)]=J_{n-1}(x)-\frac{n}{x}J_n(x)$
   IV. $J_n'(x)=\frac{n}{x}J_n(x)-J_{n+1}(x)$
   V. $J_n'(x)=\frac{1}{2}[J_{n-1}(x)-J_{n+1}(x)]$
   VI. $J_{n-1}(x)+J_{n+1}(x)=\frac{2n}{x}J_n(x)$
6. Express $J_5(x)$ in terms of $J_0(x)$ and $J_1(x)$.
   Ans: $\boldsymbol{J_5(x)=\left(\frac{384}{x^4}-\frac{72}{x^2}-1\right)J_1+\left(\frac{12}{x}-\frac{192}{x^3}\right)J_0}$
7. Express $J_{\frac{5}{2}}(x)$ in terms of sine and cosine functions.
   Ans: $\boldsymbol{J_{\frac{5}{2}}(x)=\sqrt{\frac{2}{\pi x}}\left[\left(\frac{3-x^2}{x^2}\right)sinx-\frac{3}{x}cosx\right]}$
8. Express $J_{-\frac{5}{2}}(x)$ in terms of sine and cosine functions. **(Homework)**

Ans: $\boldsymbol{J_{\frac{-5}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3}{x} sinx + \frac{(3-x^2)}{x^2} cosx\right]}$

9. Evaluate $a) \int J_3(x)dx \quad b) \int x^4 J_1(x)$
10. Give the expression for the generating function of Bessel's function (without proof)

    i.e. $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)t^n$

11. Establish the Jacobi series:

    $a) \cos(xcos\theta) = J_0 - 2J_2cos2\theta + 2J_4cos4\theta - \cdots$

    b) $\sin(xcos\theta) = 2[J_1cos\theta - J_3cos3\theta + J_5cos5\theta - \cdots]$
12. Prove that $J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - xsin\theta)\, d\theta$ where n is positive integer. **(Bessel's Integral formula)**
13. Using the Jacobi series, show that $J_o^2 + 2J_1^2 + 2J_2^2 + 2J_3^2 + \cdots = 1$
14. **Orthogonality of Bessel function:**

    Prove that $\int_0^{a} xJ_n(\alpha x)J_n(\beta x)dx = \begin{cases} 0 & if\ \alpha \neq \beta \\ \frac{a^2}{2} {J_{n+1}}^2(a\alpha) & if \alpha = \beta \end{cases}$

    Where $\alpha\ and\ \beta$ are the roots of $J_n(ax) = 0$.