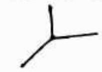
Physical meaning of geometrical proposition. 8 Aug

* Any thing starts with a set of axioms.
  ↓
  (in physics, these are borne out of experiments)

* Logical processing leads to the rest

* But, we also need a 'language' or a method of description
  STR is more about basis of descriptions. How diff^n
  ~~thang~~ descriptions ~~diff~~ are different.

* Every physical event must be associated with a point or 'collection of points' in <u>space</u> of occurrence

* specification of any space-point needs an abstract reference system of rigid bodies → 'coordinate system' 

* The method: (assumes)
  i) Concept of distance via a unique 'straight line' joining the two events with unique 'number' associated with it
     ⇒ not well-defined.
     ↳ distance
  ii) Laws of Euclidean geometry
  iii) As a measurement device, we imagine a rigid body with two marks at <u>fixed distance</u> apart
     unit of length.

Space and Time in classical mechanics.

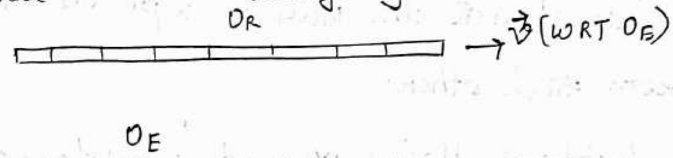
* The purpose of mechanics is to describe how ~~space~~ rigid bodies change their positions in 'space' with 'time'.
* 'Space' is not absolute, we need a 'reference' system.
* Is 'time' absolute?

Origin of time is needed. But whether we need ~~rep~~ any axis?

Origin:

Is time interval' absolute?

? whether ~~the~~ two observers always agree on time measurement.



Thought Experiment

(i) Long Railway carriage travelling uniformly in straight line with $\vec{v}$

(ii) Observer$_{carriage}$ → O_R (iii) observer$_{rest}$ → O_E

the Galilean system of coordinates are ones in which laws of inertia hold.
(coordinate systems)

"Laws of inertia" ⇒ A ~~par~~ body removed sufficiently far from other bodies continues in a state of rest or uniform motion in straight line.

"Principle of relativity" (restricted) → (Discovery of EMT) contradiction. constancy of speed of light

~~Physical~~
All natural phenomena follow the same physical laws in every ~~Galilean coordinate~~ ~~Galilean~~ Galilean coordinate system.

Addition of velocities

The carriage has velocity v.

→ O_R is moving with velocity w w.r.t carriage.

→ O_E is standing still.

The velocity of O_R w.r.t, O_E = $v+w$.

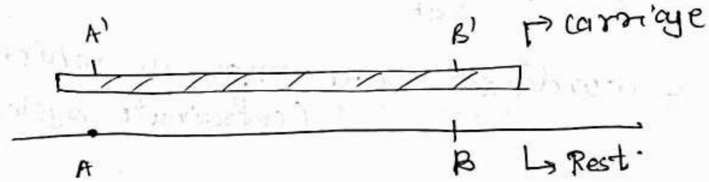
Thought exp (cont^d)

Suppose the following assertions are made

* Lightning has struck the rails at 2 points A and B far distant from each other
* These two lightning flashes occurred 'simultaneously.'

<u>Questions:</u>

How can we ~~define~~ verify that the 2 lightning flashes are simultaneous?

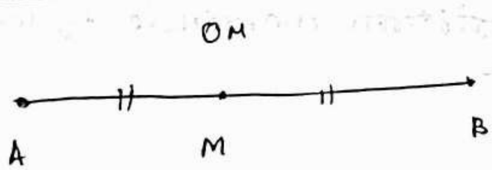


If we have 2 observers O_A and O_B with 2 clocks synchronized
→ How do we confirm that the clocks are really synchronized?
at the ~~same~~ time of lightening

'We do not know whether process of shifting clocks lose the synchronization'.

Thus, simultaneity can never be verified using 2 clocks.

<u>Defination of simultaneity</u> (for 2 distant events)



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contradiction just within the non-relativistic Galilean coordinate transformation.

<u>Defn</u>: Two events, separated in position, will be considered simultaneous, if light rays sourced by these events reach a point, equidistant from both the events exactly at the same time.

<u>Exercise</u>:- If event A and B are simultaneous and event B and C are simultaneous, then A and C must be simultaneous

<u>Defn of time</u>:

→ Defn of simultaneity ⇒ Synchronization of clocks that are spatially separated.

→ <u>Synchronization of 2 separated clocks</u>

① Two 'identical' clocks' arranged at rest in different places of a <u>single</u> reference body.

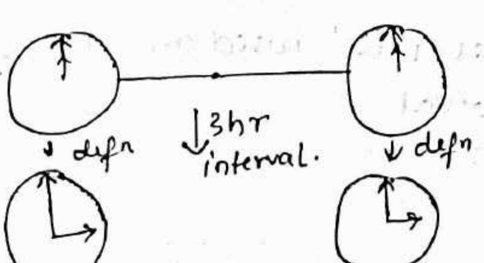
② They are arranged in such a manner that particular position of the pointers of one clock is simultaneous with the same position of pointers of other clock.

③ Then identical 'settings' of the pointers of these 2 clocks are always simultaneous. ↳ effectively a 'defn' of mechanically identical for clocks

⇓

clean defn of time interval for spatially separated events but observed by a single observer.

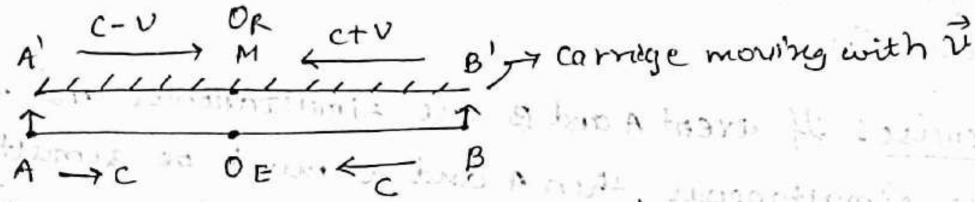
Defn of simultaneity?



Question:

If 2 given events are simultaneous in one galilean system of coordinates, will they remain simultaneous in all other galilean system?

→ Try to answer using 'thought experiment' (yes!, it is back again)



Lightening struck at A, A', $B \& B'$

M is the mid-point between $A \& B$.

M' is the mid-point b/w $A' \& B'$

According to O_E, lightening at A and B are simultaneous

If we use 'addition of velocity' formula in Galilean coordinate system, then if L_A and L_B are simultaneous in frame of O_E, they are not simultaneous in frame of O_R.

But to derive 'addition of velocities' formulae, we need to define time interval across different frames.

Summary

→ 'Addition of velocities' ⟶ based on frame independence of time intervals

→ simultaneity is defined in a way so that it matches our intuition and also uses least number of assumptions about the measuring method & devices.

* We derived 'addition of velocities rule' based on the frame independence of time & interval

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* Simultaneity, defined so that
  (i) it matches with our intuition
  (ii) least # of assumptions about the measuring method & device.

* Now apply 'addition of velocity rule' and see zero time interval becomes non-zero with a frame change $\Rightarrow \Leftarrow$ ↳ contradiction.

Conclusion

To resolve the contradiction, either:

(i) make ~~time intervals~~ time-interval frame dependent in a specific way that fixes the problem ↳ Lorentz transformation

or

(ii) Change the definition of simultaneity appropriately (exercise)

or both

Galilean Relativity

* Invariance of physical laws under a specific space-time transformation G-T

$$x^i \longrightarrow y^i = R^i_j x^j + (V_0^i)t + a^i$$

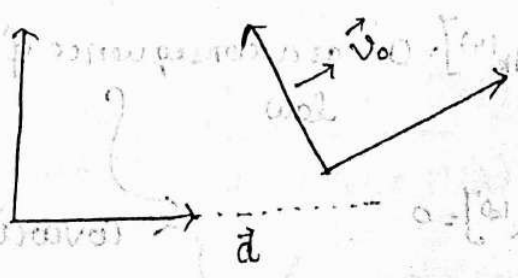
$$t \longrightarrow t' = t + b$$

$R^i_j \rightarrow$ Rotation matrix $\left[\begin{array}{l}\text{3X3 orthogonal} \\ \text{matrices in 3 space-dimensions}\end{array}\right]$

$V_0^i \rightarrow$ Constant 3-D vector

$a^i \rightarrow$ Constant 3-D vector

$b \rightarrow$ real number.



~~Physics~~

What are physical laws

Suppose $A_1, A_2 \ldots A_k$ are a set of observables that could be measured independently. [No correlation]

Let $A_1, A_2 \ldots A_k$ denote the results of a given measurement in a given coordinate system.

A physical law is an universal mathematical relation between these ~~measurements~~ measured values, typically of the form

$$R[A_1, A_2, \ldots A_k] = 0$$

Example Newton's Law of motion:

A relation between 3 observables: Force ~~(F)~~, $(\mathcal{F})$, mass $(\mathcal{M})$ $\downarrow \vec{F}$, $\downarrow m$

acceleration $(\vec{\mathfrak{a}}) \to \vec{a}$

(No absolute frame due to invariance)

Relation:

$$R(\vec{F}, \cancel{m} M, \vec{a}) = 0 \Rightarrow \vec{F} - m\vec{a} = 0$$

One ~~dep~~ independent quantity is now becomes dependent due to a physical law.

Invariance of a Physical Law under GT

Suppose $A_1, A_2 \ldots A_k$ are measured in 2 different ~~systems~~ frames $X = \{t, \vec{x}\}$ and $Y = \{t', \vec{y}\}$ $\because \vec{x} = (x_1, x_2, x_3)$

Let $A_1^{(x)}, \ldots A_k^{(x)}$ are the results in X system.

& $A_1^{(y)}, \ldots A_k^{(y)}$ are the results in Y system.

~~cov~~ If $R[A_1^{(x)}, A_2^{(x)} \ldots A_k^{(x)}] = 0$ as a consequence of a physical law

then: (same relation R)

$$R[A_1^{(y)}, A_2^{(y)} \ldots A_k^{(y)}] = 0$$

Covariance.

Invariance of Newtons Laws

Case 1: $V_0^i = 0, a^i = 0, b = 0$

$$F_i \longrightarrow F_i' = R_{ij} F_j$$

$$a_i = \frac{d^2 x_i}{dt^2} \longrightarrow a_i' = R_{ij} a_j$$

$$m' = m$$

$$\underbrace{F_i' - m a_i'}_{0} = R_{ij} [\underbrace{F_j - m a_j}_{0}]$$; R_{ij} is a non-zero ~~for~~ phase. factor.
"covariance".

law is "invariant" but R is "covariant"

Case 2: $R^{ij} = \delta^{ij}$, $\vec{a} = 0$, $b = 0$

$$x^i \longrightarrow y^i = v_0{}^i t + x^i$$
$$t' = t$$

(1) consider a particle moving along a trajectory (1-D curve)

* Let λ be a real parameter that uniquely specifies a pt on the curve. ↳ "parametraization of curve"

coordinates of any point could be uniquely expressed as $\{x^i(\lambda), t(\lambda)\}$

velocity $$\vec{v}(\lambda) = \lim_{\delta\lambda \to 0} \frac{\vec{x}(\lambda+\delta\lambda) - \vec{x}(\lambda)}{t(\lambda+\delta\lambda) - t(\lambda)}$$

acceleration $$\vec{a}(\lambda) = \lim_{\delta\lambda \to 0} \frac{\vec{v}(\lambda+\delta\lambda) - \vec{v}(\lambda)}{t(\lambda+\delta\lambda) - t(\lambda)}$$

$\{y(\lambda), t'(\lambda)\} \to$ measured in ¥ system of coordinates

$$\vec{v}'(\lambda) = \lim_{\delta\lambda \to 0} \frac{\vec{y}(\lambda+\delta\lambda) - y(\lambda)}{t'(\lambda+\delta\lambda) - t'(\lambda)}$$ λ is property of curve

$$\vec{a}'(\lambda) = \lim_{\delta\lambda \to 0} \frac{\vec{v}'(\lambda+\delta\lambda) - \vec{v}'(\lambda)}{t'(\lambda+\delta\lambda) - t'(\lambda)}$$

$$\vec{V}'(\lambda) = \lim_{\delta\lambda\to 0} \frac{\vec{x}(\lambda+\delta\lambda) + \vec{V}_0\, t(\lambda+\delta\lambda) - \vec{x}(\lambda) - V_0 t(\lambda)}{t(\lambda+\delta\lambda) - t(\lambda)}$$

$$= \lim_{\delta\lambda\to 0} \frac{\vec{x}(\lambda+\delta\lambda) - \vec{x}(\lambda)}{t(\lambda+\delta\lambda) - t(\lambda)} + \left(\lim_{\delta\lambda\to 0} \frac{t(\lambda+\delta\lambda) - t(\lambda)}{t(\lambda+\delta\lambda) - t(\lambda)} \right) V_0$$

$$= \vec{V}(\lambda) + \vec{V}_0$$

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Two frame of references:

$\{X\}$ $\{x^i, t\}$, $\{Y\} = \{y^i, t'\}$

$x^i \to y^i = x^i + V_0 t$

$t' = t$

Invariance of Newton's Laws:

→ A relation between 3 observables.

$\vec{F}, m, \vec{a} = \dfrac{d^2\vec{x}}{dt^2} = \dfrac{d\vec{V}}{dt}$

$R[\vec{F}, m\vec{a}] = \vec{F} - m\vec{a}$

Law: $R[\vec{F}, m, \vec{a}_0] = 0$

we have seen:

$\vec{V}' = \vec{V} + \vec{V}_0$

$$\vec{a} = \lim_{\delta\lambda\to 0} \frac{v(\lambda+\delta\lambda) - v(\lambda)}{t(\lambda+\delta\lambda) - t(\lambda)}$$

$$a' = \lim_{\delta\lambda\to 0} \frac{v'(\lambda+\delta\lambda) - v'(\lambda)}{t'(\lambda+\delta\lambda) - t'(\lambda)}$$

$$= \vec{a}$$

Now, if m and $\vec{F}$ are two observables (independent):

Assumption: $\vec{F}' = \vec{F}$ and $m' = m$.

Therefore, we see invariance.

$\Rightarrow$ Invariance.

Covariance under a transformation

→ Some observable transforms (not invariant).

→ But it transforms in a very specific way

→ And these specific ways of transformation are already classified in mathematics in some universal way without using a ~~mathematic~~ physical input. (Representation Theory).

$\vec{F} \rightarrow$ It transforms like a 'vector'.

$\vec{F}_1 \otimes \vec{F}_2$

Experimental contradiction with electro-magnetic theory.

Electromagnetism:

Maxwell equations

observables $\vec{E}, \vec{B}, \rho, \vec{J}$

The equations:

$$\vec{\nabla}\cdot\vec{E} = k_1 \rho \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -k_2\left(\frac{\partial}{\partial t}B\right) \qquad \vec{\nabla}\times\vec{B} = k_3\left(\vec{J} + \left(\frac{1}{k_1}\right)\frac{\partial\vec{E}}{\partial t}\right)$$

continuity eq^n: $\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$ (divergence of 4th Law)

If $\vec{J}$ is normalized so that continuity eq^n is of the ~~above~~ above form, this coefficient is fixed by consistency.

Force Law (Lorentz force Law)

$$\vec{F} = q\left(\vec{E} + k_L(\vec{v}\times\vec{B})\right)$$

q ↓ charge of charged particle

$\vec{v}$ ↓ velocity of particle.

Transformation of $\vec{E}$ and $\vec{B}$

$\vec{F}' = \vec{F}$

$= q(\vec{E}' + k_L(\vec{v}' \times \vec{B}'))$

$= q(\vec{E} + k_L(\vec{v} \times \vec{B}'))$

Assumptions

① None of the constants transform. (k_1, k_2, k_L)

② q, δ might change as they're property of particle.

or $\vec{E}' + k_L(\vec{v}' \times \vec{B}') = \vec{E} + k_L(\vec{v} \times \vec{B})$ — (A)

" $V_0 \to Y$ velocity WRT X "

or $(\vec{E}' - \vec{E}) = k_L[\vec{v} \times \vec{B} - \vec{v}' \times \vec{B}']$ V_0 is frame dependent

or $(\vec{E}' - \vec{E}) + k_L\{\vec{v} \times (\vec{B}' - \vec{B})\} + k_L \vec{V}_0 \times \vec{B}' = 0$ — (I)

At $\vec{v} = 0$, $\vec{E} = \vec{E}' + k_L(\vec{V}_0 \times \vec{B}')$ — (II) $\vec{E} = \vec{E}' + (\vec{V}_0 \times \vec{B}')k_L$

$\Rightarrow \vec{E}' = \vec{E} - (\vec{V}_0 \times B)k_L$ — (III)   } V_0 being frame dependent

Now assuming $\vec{E}$ transforms like this for all $\vec{v}$; we get:

$\Rightarrow \vec{B}' = \vec{B}$ ⟵ Subtracting (II) & (III): $\vec{V}_0 \times \vec{B} = \vec{V}_0 \times \vec{B}'$

↳ substituting (II) $\vec{E}$ in eq (I)

Transformation of $\vec{E}', \vec{B}'$

$\boxed{\vec{E} = \vec{E}' + k_2(\vec{V}_0 \times \vec{B}')}$

$\boxed{\vec{B}' = \vec{B}}$

we can show: $\vec{B} = \vec{B}' + f_B(\vec{v})$

$\vec{E} = \vec{E}' + k_L(\vec{v} \times \vec{B}) + f_E(\vec{v})$

Both f_B & f_E vanish when $v \to 0$

$\vec{\nabla} = \left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right\}$

$\vec{\nabla}_j' = \frac{\partial}{\partial y_j} = \sum_i \frac{\partial x^i}{\partial y_j}\left(\frac{\partial}{\partial x^i}\right) + \frac{\partial t}{\partial y^j}\partial t$ ↳ 0 as t is not func^n of y^j

$= \sum_i \delta^i_j = \frac{\partial}{\partial x^i}$ | $\vec{\nabla}' = \vec{\nabla}$ $\Rightarrow$ $\boxed{\vec{\nabla} \cdot \vec{B} \text{ is invariant}}$

$\frac{\partial}{\partial t'} = \frac{\partial x^i}{\partial t'}\partial x^i + \frac{\partial t}{\partial t'}\partial_t$; $x^i = y^i - V_0 t$

$= \frac{\partial(y^i - V_0 t)}{\partial t}\partial x^i + \frac{\partial t}{\partial t}\partial_t = \partial_t - V_0^i \partial x_j$

$= \partial_t - \vec{V}_0 \cdot \vec{\nabla}$

$$\vec{\nabla}' \times \vec{E}' + k_2 \frac{\partial \vec{B}'}{\partial t} = 0$$

$$= \vec{\nabla} \times \vec{E}' + k_2 \left\{ \frac{\partial \vec{B}'}{\partial t} - (\vec{v}_0 \cdot \vec{\nabla}) \vec{B}' \right\}.$$

$$= \vec{\nabla} \times \vec{E} - \vec{\nabla} \times k_L (\vec{v} \times \vec{B}) + k_2 \left\{ \frac{\partial \vec{B}'}{\partial t} - (\vec{v}_0 \cdot \vec{\nabla}) \vec{B}' \right\}.$$

$$= \left(\vec{\nabla} \times \vec{E} + k_2 \frac{\partial \vec{B}}{\partial t} \right) + \left\{ -k_L (\vec{v}_0 \times \vec{B}) + k_L (\vec{v}_0 \cdot \vec{\nabla}) \vec{B} \right\}.$$

Transformation of $\vec{B}$ & $\vec{E}$

$$\vec{B} = \vec{B}' + f_B(\vec{v})$$

$$\vec{E} = \vec{E}' + k_L (\vec{v} \times \vec{B}') + f_E(\vec{v})$$

Using these two, substituting in force equation (eq^n (A)).

$$R[\vec{f}_E, \vec{f}_B] = 0 \nearrow \text{Lorentz force law}$$

This relation has a sol^n for $\vec{f}_E = \vec{f}_B = 0$ as they're linear terms anyhow and would cancel.

$$\boxed{\vec{E} = E' + k_L (\vec{v}_0 \times \vec{B}) \quad \& \quad \vec{B}' = \vec{B}}$$

The frame transformation:

$$\vec{x} \rightarrow \vec{y} = \vec{x} + \vec{v}_0 t$$

$$t \rightarrow t' = t$$

$$\vec{\nabla}'_y = \left\{ \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_3} \right\} = \vec{\nabla}_x$$

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial t} - \vec{v} \cdot \vec{\nabla}$$

$\vec{\nabla}' \cdot \vec{B}' = 0$ so, invariant

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$k_2 \frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \vec{E} = 0$$

$$\vec{\nabla}' \times \vec{E}' + k_2 \frac{\partial \vec{B}'}{\partial t}$$

$$= \vec{\nabla} \times \left\{ \vec{E} - k_L (\vec{V_0} \times \vec{B}) \right\} + k_2 \left(\frac{\partial}{\partial t} - \vec{V_0} \cdot \vec{\nabla} \right) \vec{B}$$

$$= \vec{\nabla} \times \vec{E} + \cancel{k_2} \cdot k_2 \frac{\partial \vec{B}}{\partial t} - \left[k_L \, \vec{\nabla} \times (\vec{V_0} \times \vec{B}) + k_2 (\vec{V_0} \cdot \vec{\nabla}) \vec{B} \right] \rightarrow (i)$$

$$\left\{ \vec{\nabla} \times (\vec{V_0} \times \vec{B}) \right\}_i$$

$$(\vec{A} \times \vec{B})_i = \epsilon_{ijk} A_j B_k$$

$$= \epsilon_{ijk} \partial_j (\vec{V_0} \times \vec{B})_k$$

$$= \epsilon_{ijk} \partial_j \left[\epsilon_{klm} (V_0)_l B_m \right]$$

$$= \epsilon_{ijk} \epsilon_{klm} (V_{0l}) (\partial_j B_m)$$

$$= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) (V_0)_l (\partial_j B_m) = (V_0)_i \partial_j B_j - (V_0)_j \partial_j B_i$$

$$= (V_0)_i (\vec{\nabla} \cdot \vec{B}) - (\vec{V_0} \cdot \vec{\nabla}) B_i$$

$$= \left\{ \vec{V_0} (\vec{\nabla} \cdot \vec{B}) - (\vec{V_0} \cdot \vec{\nabla}) \vec{B} \right\}_i \qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$= -\left\{ (\vec{V_0} \cdot \vec{\nabla}) \vec{B} \right\}_i$$

$$\vec{\nabla} \times (\vec{V_0} \times \vec{B}) = -(\vec{V_0} \cdot \vec{\nabla}) \vec{B} \rightarrow (ii)$$

Substituting (ii) in (i):

$$-k_L (-\vec{V_0} \cdot \vec{\nabla}) \vec{B} + k_2 (\vec{V_0} \cdot \vec{\nabla}) \vec{B} = 0$$

or $\boxed{K_2 = K_L}$

When $\vec{\nabla} \cdot \vec{B} = 0$, so invariance Hence; $\boxed{K_2 = K_L}$ → also experimentally true.

In homogenous Maxwell's equations:

$\vec{\nabla}\cdot\vec{E} = k_1 \rho$

we need to know how ρ transforms.

$\vec{\nabla}'\cdot\vec{E}' - k_1\rho'$

$= \vec{\nabla}\cdot[\vec{E} - k_2(\vec{V_0}\times\vec{B})] - k_1\rho'$

$= \vec{\nabla}\cdot\vec{E} - \left\{k_1\rho' + k_2\,\vec{\nabla}\cdot(\vec{V_0}\times\vec{B})\right\}$ (= $k_1\rho$ for invariance)

$\vec{\nabla}\cdot(\vec{V_0}\times\vec{B}) = \partial_i(V_0\times\vec{B})_i = \partial_i \epsilon_{ijk}(V_0)_j B_k$

$= \epsilon_{ijk}(V_0)_j(\partial_i B_k)$

$= -\vec{V_0}\cdot(\vec{\nabla}\times\vec{B})$

If we want this law to be invariant, we have to impose:

$$k_1\rho' - k_2\vec{V_0}(\vec{\nabla}\times\vec{B}) = k_1\rho$$

or $\rho' = \rho + \left(\frac{k_2}{k_1}\right)\vec{V_0}\cdot(\vec{\nabla}\times\vec{B})$

Intuition:

But ρ' shall be invariant, as ρ over volume shall be give total charge and shall be constant.

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Continuity Equation

Expression of $\vec{J}$ in terms of ρ and $\vec{v}$ (velocity of charged particles)

→ Total charge in a volume V : Q(t)
at time t $= \int_V d^3\bar{x}\, \rho(\bar{x},t)$

→ Conservation of total charge ⇒ $\frac{dQ}{dt} = 0$ ⇒ continuity Equation.

As charged particles move, volume element $d^3\bar{x}$ changes:

$$0 = \frac{d}{dt}Q = \frac{d}{dt}\int d^3\bar{x}\, \rho(\bar{x},t)$$

$$= \int \left[\left(\frac{d}{dt} d^3\bar{x}\right)\rho(\bar{x},t) + (d^3\bar{x}) \cdot \frac{\partial \rho(\bar{x},t)}{\partial t} \right]$$

↓
we have to figure out how $d^3\bar{x}$ changes in an infidecimal time interval δt

→ In δt the position of the charged particle at $\bar{x}$ has been shifted by:

$$\delta(\bar{x}) = v(\bar{x})_{,t} \cdot \delta t + O(\delta t^2) + \cdots$$

Define a new instantaneous coordinate system $\{\tilde{x}\}$ s.t

$$\vec{\tilde{x}} = \vec{x} - \delta\vec{x}$$
$$= \vec{x} - v(\bar{x})_{,t}\, \delta t$$

t_0 → instantaneous time where we are using $(\tilde{x})$ coordinates

$$v(x,t) = v(\bar{x},t_0) + \frac{\partial v}{\partial t}(x,t_0)\delta t + \cdots$$
$t = t_0 + \delta t$

→ In $\{\tilde{x}\}$ coordinate system

↳ not galilean transformation

→ In $\{\tilde{x}\}$ each particle ~~past~~ is at rest from $t = t_0$ to $t = t_0 + \delta t$

$$\Rightarrow \frac{d}{dt}(d^3\tilde{x})\Big|_{t=t_0} = 0$$

we have to find $\frac{d}{dt}(d^3\tilde{x})$

So, express $d^3\bar{x}$ in terms of $d^3\tilde{x}$

$$d^3x = \Delta\, d^3\tilde{x} \quad \text{where} \quad \Delta = \mathrm{Det}\left[\frac{\partial x^i}{\partial \tilde{x}^j}\right] \hookrightarrow \text{Jacobian.}$$

<u>Exercise</u>: Show this $\{y^i\} \to \{x^i\}$ (any dimension) let "d" dimension

$$d^m y = \mathrm{Det}\left(\frac{\partial y^i}{\partial x^j}\right) d^m x$$

where $\left(\frac{\partial y^i}{\partial x}\right)$ is a $(m\times m)$ matrix and is invertible

$$\tilde{\bar{x}} = \bar{x} - \bar{V}(x)\cdot\delta t$$

$$\Rightarrow \vec{x} = \tilde{\bar{x}} + \bar{v}(\tilde{x})\delta t + O(\delta t^2)$$

$$\frac{\partial x^i}{\partial x^j} = \frac{\partial(\tilde{x}^i + v^i(\tilde{x})\delta t)}{\partial \tilde{x}^j}$$

$$= \delta^i_j + (\partial_j v^i)\delta t + O(\delta t^2)$$

$$\mathrm{Det}\left[\delta^i_j + (\partial_j v^i)\delta t + O(dt^2)\right] =$$

$$A_{ij} = \delta^i_j + \underbrace{(\partial_j v^i)}_{B_{ij}}\delta t = \delta^i_j + B_{ij}$$

$$\mathrm{Det}(A_{ij}) = \det\left[\hat{1} + \delta t(B_{ij})\right] \to \text{In a basis where } B \text{ is diagonal.}$$

$$= \hat{1} + \delta t \begin{pmatrix} B_{11} & & \\ \vdots & B_{22} & \\ \vdots & & \ddots \end{pmatrix}$$

$$= \det \begin{bmatrix} 1+\delta t B_{11} & \cdots & \\ \vdots & 1+\delta t B_{22} & \\ \vdots & & \ddots \end{bmatrix}_{n \times n}.$$

$\partial_j V^i \quad \dfrac{\partial V^i}{\partial \tilde{x}^j}$

$$= (1+\delta t\, B_{11})(1+\delta t\, B_{22})(\cdots\cdots)(1+\delta t\, B_{nn}) \to \delta t(B_{11}+B_{22}+\cdots)+$$

$$= \delta t\,[\mathrm{Tr}(B)] + 1 + O(\delta t^2)$$

Exercise: Prove the above formulae without assuming that B is diagonizable

$$\text{So, } \det\left[\delta^i_j + (\partial_j V^i)\delta t + O(\delta t^2)\right]$$

$$= 1 + \delta t(\mathrm{Tr}(\vec{\nabla}\cdot\vec{V}) + O(\delta t^2)$$

$$d^3\vec{x} = d^3\vec{\tilde{x}}\,(1+\delta t(\vec{\nabla}\cdot\vec{V})) + O(\delta t^2)$$

$$d^3\vec{x} - d^3\vec{\tilde{x}}$$
$$= \delta t\, d^3\vec{\tilde{x}}(\vec{\nabla}\cdot\vec{V}) + O(t^2)$$
$$= \delta t\, d^3\bar{x}\,(\vec{\nabla}\cdot\vec{V}) + O(\delta t^2)$$

" $d^3\tilde{x} = d^3\bar{x}$ for small t_0 "

$$\frac{d}{dt}(d^3\vec{x}) = \lim_{\substack{\delta t\to 0\\ t\to t_0}} \frac{d^3\vec{x}(t_0+\delta t) - d^3\vec{x}(t_0)}{\delta t}$$

$$= \frac{d^3\bar{x}\,\delta t\,(\vec{\nabla}\cdot\vec{V})}{\delta t} = \vec{\nabla}\cdot\vec{V}\, d^3\bar{x}$$

$$0 = \left(\frac{dQ}{dt}\right) = \frac{d}{dt}\int d^3\bar{x}\, \rho(\bar{x},t)$$

$$= \int \frac{d}{dt}[d^3\bar{x}]\,\rho(\bar{x},t) + \int d^3\bar{x}\, \frac{\partial \rho(\vec{x},t)}{\partial t}$$

$$= \int d^3\bar{x}\,(\vec{\nabla}\cdot\vec{V})\,\rho(\vec{x},t) + \int d^3\bar{x}\left[\frac{\partial\rho}{\partial t} + \left(\frac{\partial\vec{x}}{\partial t}\right)\left(\frac{\partial\rho}{\partial\vec{x}}\right)\right]$$

$$= \int d^3\bar{x}\,(\vec{\nabla}\cdot\vec{V})\,\rho(\vec{x},t)$$

$$= \int d^3\bar{x} \left[\left(\frac{\partial \rho}{\partial t}\right) + (\vec{v} \cdot \vec{\nabla})\rho + \rho (\vec{\nabla} \cdot \vec{v}) \right]$$

$$= \int d^3\bar{x} \left[\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) \right] = 0$$

Hence; $\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0}$

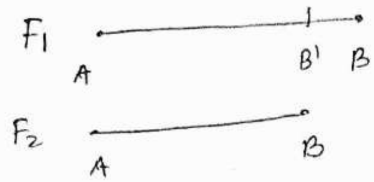
Continuity Eq^n

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0}$$

Hence; $\boxed{\vec{J} = \rho \vec{v}}$

Lorentz Transformation 23 August

→ Magnitude of time interval depends on the choice of galilean forces.
→ The same can be said about space intervals (lengths)



→ The frame independence of speed of light, experimentally observed fact, is theoretically justified by EM theory.
→ There are inconsistencies in galilean transformation
→ The transformation that ~~rotates~~ relates the coordinates between different frames (galilean) has to be reformed,
① Allow space and time transformations as one goes from one frame to other.
② Transformation should be such that the speed of light is frame independent.

In certain approximate limits, the transformation shall reduce to galilean transformation

→ ① Suppose $X = \{\vec{x}, t\}$ and $X' = \{\vec{x}', t'\}$ are the two frames
$\vec{x} = \{x, y, z\}$ and $\vec{x}' = \{x', y', z'\}$

② At $t=0$, the origin of X frame coincides with origin of X' frame.
③ At $t=0$, we set the synchronized clocks in X' system such that they correspond to $t'=0$
④ As viewed from X frame, the origin of X' is moving with velocity v along x axis.
$x' = f(x,t)$, $t' = g(x,t)$
→ Consider a particle moving with uniform velocity "u" along x-axis in X frame

→ Physical Intuition :

From x' frame, the particle's velocity will again be uniform (not u)

$$dx' = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial t} dt$$

$$dt' = \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial t} dt.$$

$$u'(u) = \frac{dx'}{dt'} = \frac{\left(\frac{\partial f}{\partial x}\right)dx + \left(\frac{\partial f}{\partial t}\right)dt}{\left(\frac{\partial g}{\partial x}\right)dx + \left(\frac{\partial g}{\partial t}\right)dt}$$

$$= \frac{\left(\frac{\partial f}{\partial x}\right)\left(\frac{dx}{dt}\right) + \left(\frac{\partial f}{\partial t}\right)}{\left(\frac{\partial g}{\partial x}\right)\left(\frac{dx}{dt}\right) + \left(\frac{\partial g}{\partial t}\right)}$$

$$= \frac{u\left(\frac{\partial f}{\partial x}\right) + \left(\frac{\partial f}{\partial t}\right)}{u\left(\frac{\partial g}{\partial x}\right) + \left(\frac{\partial g}{\partial t}\right)}$$

$$\text{or } u'(u,v) = \frac{u\left(\frac{\partial f}{\partial x}\right) + \frac{\partial f}{\partial t}}{u\left(\frac{\partial g}{\partial x}\right) + \frac{\partial g}{\partial t}} \quad \text{— (I)}$$

claim: (I) is true for all values of u provided $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial t}, \frac{\partial g}{\partial x}$ & $\frac{\partial g}{\partial t}$ are all constants.

⇒ $x' = mx + nt$ & $t' = px + qt$, where m, n, p, q are constants.

→ We shall use the fact that speed of light is a constant.

→ Consider a ray of light emitted from origin at $t = t' = 0$.

→ In x system, the light travelled $x = ct$ distance,

→ In x' system, the light travelled $x' = ct'$ distance.

$\Rightarrow ct' = mct + nt \quad \& \quad t' = pct + qt$

$\Rightarrow c[pct + qt] = mct + nt$

or $\boxed{pc^2 + qc = mc + n}$ —— (A)

→ Now consider the same experiment but ray moves along $-x$ axis starting from origin

$\Rightarrow c \to -c$

or $pc^2 + q(-c) = m(-c) + n$ —— (B)

From (A) & (B), we get :-

$2pc^2 = 2n \quad \Rightarrow pc^2 = n$

$2qc = 2mc. \quad \Rightarrow m = q$

Substituting above in $x' = qx + pc^2t$, $t' = pxt + qt$

Define a new set of coefficients :-

$q = a \;\&\; pc = -b.$

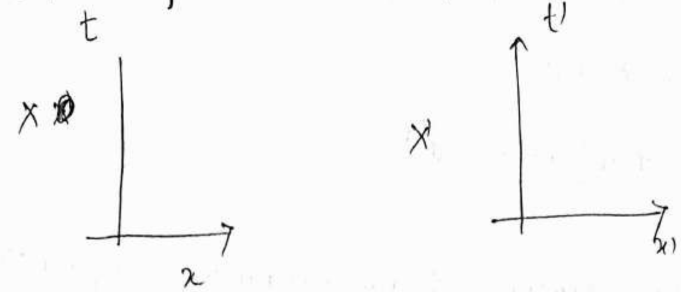
So, we have:

$x' = ax - b(ct)$

$ct' = a(ct) - bx$

Derivation of Lorentz Transformation

Aug 30-



→ $\{t=0, x=0\} \Rightarrow \{t'=0, x'=0\}$

→ Origin of $\vec{x}$ moving with velocity v wrt X frame.

→ Goal: To find $\{x', t'\}$ and $\{x, t\}$ → 1-D motion.

→ After some physical assumptions:-

$$x' = ax - b(ct)$$

$$t' = a(ct) - bx$$

→ The origin of $(\dot{x})$ has velocity v wrt X frame.

→ Suppose $(X'=0)$ has coordinates $\vec{X}(t)$ wrt X frame:

$$\vec{X}(t) = vt$$

$$a\vec{x}(t) - bct = 0$$

$$\text{or } at = bct$$

$$\text{or } \boxed{\frac{b}{a} = \frac{v}{c}}$$

$$\text{or } b = \left(\frac{v}{c}\right)a$$

Thus, we have:

$$X' = a\left[x - \frac{v}{c}(ct)\right]$$

$$t' = a\left[ct - \frac{v}{c}x\right]$$

Reciprocity in frame transformations

① Consider a rod 'R' which is at rest in X' frame with one end coinciding with origin

→ Suppose its length is 1 in X' frame

→ " " " " l' in X frame.

② A rod R is in rest in X frame with 1 end coinciding with origin.

→ R has length "1" as measured in X and length $\tilde{l}$ in X'.

→ Since l doesn't have any direction (scalar), it shall be a funcⁿ of v^2, v^4 or such.

→ Thus, $l = l' = \tilde{l}$.

ⓐ Case of R'

$x_{(1)}' = 0 \qquad x_{(2)}' = 1$

Lets measure length in x frame at $t=0$:

$x_{(2)}' = 1 = a\, x_{(2)} = al \Rightarrow 1 = al.$ (Since length l is known in X frame)

ⓑ Case of R in rest WRT X, lets measure R in x' at $t'=0$:

$$x_{(2)}' = a\left[x_{(2)} - \frac{v}{c}(ct)\right]$$

$$\text{or } l = a\left[1 - \frac{v}{c}(ct)\right]$$

$$\& \quad t' = 0 = a\left[ct - \left(\frac{v}{c}\right)x_{(2)}\right]$$

$$\Rightarrow \left.\begin{aligned} l &= a[1 - vt] \quad \text{—(i)} \\ \& \; 0 &= a\left[t - \frac{v}{c^2}\right] \; \text{(i')} \\ \& \; 1 &= al \quad \text{—(ii)} \end{aligned}\right\} \text{3 eqⁿs}$$

From (ii): $l = 1/a$

From (i'): $t = v/c^2$

Substituting in (i) :-

$$\frac{1}{a} = a\left[1 - v \cdot \frac{v}{c^2}\right]$$

or $$+\frac{1}{a^2} = \left(1 - \frac{v^2}{c^2}\right)$$

or $$a^2 = \left(\sqrt{\frac{1}{1 - v^2/c^2}}\right)^2$$

or $$a = \gamma = \sqrt{\frac{1}{1 - v^2/c^2}}$$

So,

$$\boxed{\begin{aligned} x' &= \gamma\left(x - \left(\frac{v}{c}\right)ct\right) \\ ct' &= \gamma\left(ct - \frac{vx}{c}\right) \end{aligned}}$$ → Always remember these

Exercise (in Assignment (Q-3))

1 Sept

(1) Length ~~Concentration~~ Contraction

(2) Time dilation.

(3) Frame invariance of c.

$$x' = \gamma\left[x - \left(\frac{v}{c}\right)tc\right] \qquad \gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

$$ct' = \gamma\left[ct - \left(\frac{v}{c}\right)x\right]$$

Length of a moving object

We want to measure the length of rod R

→ R is moving with velocity v w.r.t. x.

→ {x'} frame is the rest frame for R'

→ How to measure length of moving object {x' frame}.

• Take a snapshot at any instant t

• Location of the two ends are x_0, $(x_0 + L)$ → length = $(x_0 + L - x_0)$

→ In {x'} frame the two ends are at x_0' and $x_0' + 1$

→ In x' frame, the rod is at rest. So, no need to take snapshot.

$$x_0' = \gamma\left[x_0 - \left(\frac{v}{c}\right)\cdot ct\right] \quad -\text{(I)}$$

$$x_0' + 1 = \gamma\left[x_0 + l - \left(\frac{v}{c}\right)ct\right] - \text{(II)}$$

or $1 = \gamma L$

or $\boxed{L = \frac{1}{\gamma}}$

$$L = \frac{1}{\gamma} = \sqrt{1 - \frac{v^2}{c^2}} < 1.$$

Length of a moving object is less than length of the object at rest.

② Time dilation

→ Suppose at X', we have a clock at rest at x'

→ Two of its successive ticks are at t_0' and $t_0'+1$

These two clocks ticks $\{x_0, t_0\}$, $\{x_0+\delta x, t_0+\Delta\}$ in moving

→ at (x_0', t_0'), $(x_0', t_0'+1)$

Substituting :

$$ct_0' = \gamma\left[ct_0 - \left(\frac{V}{c}\right)x_0\right] \quad \text{— (I)}$$

$$c(t_0'+1) = \gamma\left[c(t_0+\Delta) - \left(\frac{V}{c}\right)(x_0+\delta x)\right] \rightarrow \text{(II)}$$

$$x_0' = \gamma\left[x_0 - \left(\frac{V}{c}\right)ct_0\right] = \gamma\left[(x_0+\delta x) - \left(\frac{V}{c}\right)c(t_0+\Delta)\right] \rightarrow \text{(III)}$$

Subtracting and we have :-

From (III):

$$\delta x - \left(\frac{V}{c}\right)c\Delta = 0$$

$$\text{or } \delta x = V\Delta$$

(I) − (II) :-

$$c = \gamma c\Delta - \gamma\left(\frac{V}{c}\right)\delta x$$

$$= \gamma c\Delta - \gamma\left(\frac{V}{c}\right)\cdot V\Delta$$

$$= \gamma c\Delta\left[1 - \frac{V^2}{c^2}\right]$$

$$= c\Delta\sqrt{1-\frac{V^2}{c^2}}$$

$$\text{or } \quad \Delta = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} > 1 \text{. to}$$

Thus, moving clock goes slower

Addition of velocity

Particle is moving with velocity u wrt $\{X\}$ frame

Goal:- Determine ~~real~~ velocity $\left(\frac{dx'}{dt}\right)$ wrt $\{OX'\}$ frame.

$$dx' = \gamma\left[dx - v\,dt\right]$$

$$dt' = \gamma\left[dt - \left(\frac{v}{c^2}\right)dx\right]$$

$$u' = \frac{dx'}{dt'} = \frac{dx - v\,dt}{dt - \frac{v}{c^2}dx}$$

$$= \frac{\frac{dx}{dt} - v}{1 - \frac{v}{c^2}\left(\frac{dx}{dt}\right)}$$

$$= \frac{u - v}{1 - \frac{vu}{c^2}}$$

In the limit $c \to \infty$, wrt v, u finite, we ~~found~~ find galilean formulae $u' = u - v$

Frame invariance of c

when $u = c$

$$u' = \frac{c - v}{1 - \frac{vc}{c^2}} = \frac{c - v}{(c^2 - vc)} \cdot c^2 = \frac{c^2(c-v)}{c(c-v)} = c$$

Lorentz tranformation as rotation in $\{X, ict\}$ frame.

$$x' = \gamma\left[x - \left(\frac{v}{c}\right)(ct)\right] \quad . \quad ct' = \gamma\left[ct - \left(\frac{v}{c}\right)x\right]$$

Define a space-time vector $\begin{Bmatrix} ct \\ x \end{Bmatrix} = \vec{\gamma}$

$$\vec{\gamma}\,' = \begin{Bmatrix} ct' \\ x' \end{Bmatrix}$$

$$\vec{\gamma} = \gamma \underbrace{\begin{bmatrix} 1 & -v/c \\ -v/c & 1 \end{bmatrix}}_{\{\Lambda \,=\, \text{Lorentz tranformation matrix}\}} \begin{bmatrix} ct \\ x \end{bmatrix}$$

$$\beta = \tanh^{-1}\left(\frac{v}{c}\right)$$

$$\gamma' = \begin{bmatrix} \cosh\beta & -\sinh\beta \\ \sinh\beta & \cosh\beta \end{bmatrix} . \gamma$$

→ We would like to view Λ as rotation matrix by imaginary angle

define $\tau = ict$.

$$S = \begin{bmatrix} -i & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} ct \\ x \end{bmatrix} = S \begin{bmatrix} \tau \\ x \end{bmatrix}$$

$$S \cdot \begin{bmatrix} \tau' \\ x' \end{bmatrix} = \Lambda S \cdot \begin{bmatrix} \tau \\ x \end{bmatrix}$$

$$\begin{bmatrix} \tau' \\ x' \end{bmatrix} = S^{-1} \Lambda S \begin{bmatrix} \tau \\ x \end{bmatrix}$$

Lorentz Transformation (L·T) (Formal Structure) 2^{nd} Sept

→ L-T is a linear transformation of space - time coordinates

→ Physically such transformations is induced by boost and ~~xxx~~ rotation of coordination.

→ 'Boost' ⇒ uniform velocity of one frame wrt other

(A) 'Boost' could be viewed as rotation in 'imaginary time' and spatial coordinates.

$$x' = \gamma \left[x - \left(\frac{v}{c}\right) ct \right]$$

$$ct' = \gamma \left[ct - \left(\frac{v}{c}\right) x \right]$$

Define space - time vector $r = \begin{pmatrix} ct \\ x \end{pmatrix}$ and $r' = \begin{pmatrix} ct' \\ x' \end{pmatrix}$

$\Rightarrow r' = \Lambda \cdot r$ $\quad \Big| \quad \Lambda = \begin{pmatrix} 1 & -v/c \\ -v/c & 1 \end{pmatrix}$ $\Lambda \to$ Lorentz transformation matrix

Define β = rapidity parameter

$= \tanh^{-1}\left(\frac{v}{c}\right)$

$\sim$ 'angle' related to boost

$\gamma = \cosh \beta \qquad \gamma\left(\frac{v}{c}\right) = \sinh \beta$

Thus; $\Lambda = \begin{bmatrix} \cosh\beta & -\sinh\beta \\ -\sinh\beta & \cosh\beta \end{bmatrix}$

Define $\tau = ict$

This is a complex linear transformation on the coordinates

$$\begin{pmatrix} ct \\ x \end{pmatrix} = S \cdot \begin{pmatrix} \tau \\ x \end{pmatrix} \quad ; \quad S = \begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{bmatrix} ct' \\ x' \end{bmatrix} = \Lambda \begin{bmatrix} ct \\ x \end{bmatrix}$$

$$S \begin{bmatrix} \tau' \\ x' \end{bmatrix} = \Lambda \cdot S \cdot \begin{bmatrix} \tau \\ x \end{bmatrix}$$

or $$\begin{bmatrix} \tau' \\ x' \end{bmatrix} = (S^{-1} \Lambda \cdot S) \begin{bmatrix} \tau \\ x \end{bmatrix}$$

The matrix for lorentz transformation in (τ, x) space:

$$\Lambda_I = S^{-1} \cdot \Lambda \cdot S$$

$$= \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cosh\beta & -\sinh\beta \\ -\sinh\beta & \cosh\beta \end{bmatrix} \begin{bmatrix} -i & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -i\cosh\beta & -\sinh\beta \\ -i\sinh\beta & \cosh\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cosh\beta & -i\sinh\beta \\ +i\sinh\beta & \cosh\beta \end{bmatrix}$$

Define $\beta = i\theta$

$$\cosh\beta = \frac{e^{\beta} + e^{-\beta}}{2} = \cos\theta$$

$$\sinh\beta = \frac{e^{\beta} - e^{-\beta}}{2} = i\sin\theta$$

Hence, we would get:

$$\Lambda_I = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \rightarrow \text{Rotation matrix in 2-D}$$

L.T is 'imaginary' rotation in $\{x, ict\}$ plane.

$$\Rightarrow (\tau')^2 + (x')^2 = [\tau', x'] \begin{bmatrix} \tau' \\ x' \end{bmatrix}$$

$$= [\tau, x] \underbrace{\Lambda_I^T \Lambda_I}_{\Uparrow} \begin{bmatrix} \tau \\ x \end{bmatrix}$$

$$= \tau^2 + x^2$$

→ In $\{\tau, x\}$ plane, L.T is defined as a transformation that keeps the norm invariant

→ In $\{ct, x\}$ plane, the norm translates to $\tau^2 + x^2 = (ict)^2 + x^2$

$$= (-ct)^2 + x^2.$$

NOTE:- $x^2 + y^2 + z^2 - c^2t^2 =$ constant for any lorentz transformation.

The 'norm' for a space-time vector $\begin{bmatrix} ct \\ x \end{bmatrix}$ is given as

$$\text{norm} \equiv [ct, x] \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix}$$

$$= [ct', x'] \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} ct' \\ x' \end{bmatrix}$$

$$= [ct, x] \begin{bmatrix} \cosh\beta & -\sinh\beta \\ -\sinh\beta & \cosh\beta \end{bmatrix}^T \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cosh\beta & -\sinh\beta \\ -\sinh\beta & \cosh\beta \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix}$$

$\wedge$

⇒ [One should check by calculation]

$$\Lambda^T \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Extended the defination of LT with 1 time and arbitrary (d) # or space dimensions

LT is a linear transformation on space time vector $\{ct, \vec{x}\}$ s.t

$$\Rightarrow \begin{bmatrix} ct' \\ \vec{x}' \end{bmatrix} = \Lambda \cdot \begin{bmatrix} ct \\ x \end{bmatrix}$$

such that

$$\Lambda^T \cdot \eta \cdot \Lambda = \eta .$$

where $\eta \equiv$ space time metric

$$= \text{Diagonal}\left[-1, 1, 1, \dots 1\right]$$

Compare with def^n of rotation

$R^T R = \mathbb{I}$

In space the equivalent of η is $\mathbb{I}$

$R^T \mathbb{I} R = \mathbb{I}$

$\Rightarrow$ Λ is also a rotation but in a space where 'norm' has a different defination

6/sept

Goal: Formal structure for L·T

→ How to parametrize the most general L·T matrix Λ

As an example we shall do the same for rotation in space.

How ~~to~~ to express general rotation in 3 space-dimensions.

Steps: (i) write the matrix for infinitesimal rotation

(ii) classify the ~~structure~~ generators of the infinitesimal rotation.

(iii) Apply infinitesimal rotation infinite times to get finite rotation.

→ Any continuous transformation which could be realized as matrix multiplication in a linear vector space could be classified using as above.

Use above step in exams for getting transformation matrix

Rotation of 3×3 matrices satisfying $R^T R = \hat{1} \Rightarrow R_{ij} R_{ik} = \delta_{jk}$

Infinitesimal rotation

$\Rightarrow R_{ij} = \delta_{ij} + \epsilon J_{ij}$.

$\epsilon \ll 1$ so that quadratic or higher terms of ϵ would be neglected.

$R_{ij} R_{ik} = \delta_{jk}$

$(\delta_{ij} + \epsilon J_{ij})(\delta_{ik} + \epsilon J_{ik}) = \delta_{jk}$

or $\delta_{ij}\delta_{ik} + \delta_{ij}\epsilon J_{ik} + \delta_{ik}\epsilon J_{ij} + O(\epsilon^2) = \delta_{jk}$

or $\delta_{ij}\delta_{ik} + \epsilon [\cancel{\delta_{ij}} J_{jk} + J_{kj}] = \delta_{jk}$

or $J_{jk} + J_{kj} = 0$

so, J is an antisymmetric real matrix

Infinitdecimal ~~generat~~ rotations $\Rightarrow$ Linear vector space.

~~Suppose~~

Consider 2 infinitdecimal rotation $R_1^{(1)} = \hat{1} + \epsilon J^{(1)}$

$$R_2^{(2)} = \hat{1} + \epsilon J^{(2)}$$

$$R_2 R_1 = (\mathbb{1} + \epsilon J^{(1)})(1 + \epsilon J^{(2)}) = R_3 = 1 + \epsilon J^{(3)}$$

$$= \mathbb{1} + \epsilon (J^{(1)} + J^{(2)}) + O(\epsilon^2)$$

J^1 and J^2 are antisymmetric.

So, $J^{(1)} + J^{(2)}$ is also antisymmetric.

$$J^1 + (J^1)^T = 0$$

$$J^2 + (J^2)^T = 0$$

$$\boxed{(J^1 + J^2)^T + (J^1 + J^2) = 0}$$

So, Any linear combination of generators is also a generator.

Generator of infinidecimal vector space which is equal to the space of 3x3 antisymmetric matrices form a LVS.

The basis for the space of generators

→ To find a basis, we need to know the dimension of the space of generators.

3x3 real anti-symmetric matrix has 3 free parameters

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

$$= a\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

Let us choose a bit differently:

$$\begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix} = a\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

The convenient choice for basis :-

$$[\tilde{J}^i] = \epsilon_{ijk}$$

$$[\tilde{J}^1]_{jk} = \epsilon_{1jk}$$

$$[\tilde{J}^1]_{23} = \epsilon_{123} = 1$$

$$[\tilde{J}^1]_{32} = \epsilon_{132} = -1$$

The most general form of infinitesimal generator:

$$\Rightarrow \tilde{J} = \sum_{i=1}^{3} \alpha_i \tilde{J}^i$$

$$[\tilde{J}]_{jk} = \sum_{i=1} \alpha_i \epsilon_{ijk}$$

Define a unit vector $\hat{n}^i = \dfrac{\alpha^i}{\sqrt{\Sigma(\alpha^i)^2}}$

Define the norm of $\vec{\alpha}$ $\{\alpha^i\}$ as : $\sqrt{\Sigma(\alpha^i)^2}$

$$\tilde{J} = \sum_{i=1}^{3} \alpha_i \tilde{J}^i = \alpha \sum_{i=1}^{3} \hat{n}_i \tilde{J}^i$$

So, $R = \mathbb{1} + \epsilon \sum \alpha_i \tilde{J}^i$

$= \mathbb{1} + \epsilon \alpha_i (\sum \hat{n}_i J^i)$

Axis of Rotation

(For a vector to be axis vector:
$R \hat{v} = \mathbb{1} \hat{v}$)

A vector that remains invariant, i.e an eigen vector of rotation matrix with eigen value 1.

$R_{jk} = \delta_{jk} + \epsilon \alpha \hat{n}_i \epsilon_{ijk}$

$R_{jk} \hat{n}_k = (\delta_{jk} + \epsilon \alpha \hat{n}_i \epsilon_{ijk}) \hat{n}_k$

$= \hat{n}_j + \epsilon \alpha \epsilon_{ijk} \underbrace{\hat{n}_i \hat{n}_k}_{\to 0}$

$\epsilon_{ijk} \hat{n}_i \hat{n}_k$

$= \epsilon_{kji} \hat{n}_k \hat{n}_i = -\epsilon_{ijk} \hat{n}_i \hat{n}_k$

or $2 \epsilon_{kji} \hat{n}_k \hat{n}_i = 0$

or $\boxed{\epsilon_{ijk} \hat{n}_i \hat{n}_k = 0}$

Exercise

- $\delta_{ij} = \delta_{ji}$

$\epsilon_{ij} = -\epsilon_{ji}$

$\sum_{j=1}^{3} \sum_{i=1}^{3} \delta_{ij} \epsilon_{ij}$

So,

$R_{jk} \hat{n}_k = \hat{n}_j$

$\boxed{R \cdot \hat{n} = \hat{n}}$

Infinitesimal rotation 9/09/2022

$$R = \mathbb{1} + \epsilon\alpha \sum_i \hat{n}_i \tilde{J}^i$$

Infinitesimal rotation around axis $\hat{n}_i$ with angle $\epsilon\alpha$ where $\epsilon \ll 1$ s.t any term of $O(\epsilon^2)$ or higher is neglected.

Finite rotation around axis $\hat{n}$ with finite rotation α.

$$R(\alpha, \hat{n}) = \left[R\left(\frac{\alpha}{K}, \hat{n}\right)\right]^K$$

$$\left[R\left(\frac{\alpha}{K}, \hat{n}\right)\right]^K = \lim_{K\to\infty} \left[R\left(\frac{\alpha}{K}, \hat{n}\right)\right]^K$$

when $K \to \infty$, $\frac{1}{K} \to 0$

identify $\frac{1}{K}$ with ϵ of the infinitesimal rotation.

$$\Rightarrow R(\alpha, \hat{n}) = \lim_{K\to\infty}\left[\mathbb{1} + \left(\frac{\alpha}{K}\right)(\hat{n}\cdot\tilde{J})\right]^K$$

$$= \exp\left[\alpha(\hat{n}\cdot\tilde{J})\right]$$

Show that $\lim_{K\to\infty}\left[1 + \frac{A}{K}\right]^K = \exp[A] = \sum_{m=0}^{\infty} \frac{A^m}{m!}$ where A is an $p \times p$ matrix

So, we have expression for finite rotation :-

$$R[\alpha, \hat{n}] = \exp(\underbrace{\alpha}_{\text{Angle}}\ \underbrace{\hat{n}}_{\text{Axis}}\cdot\tilde{J})$$

We need to define range of α:

$\boxed{\text{I is diagonal in any basis}}$

Range of angle α:

$$\tilde{J}^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \tilde{J}^2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \tilde{J}^3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\hat{n}\cdot\tilde{J} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \hat{n}_1 \\ 0 & -\hat{n}_1 & 0 \end{bmatrix} + \hat{n}_2\tilde{J}^2 + \hat{n}_3\tilde{J}^3$$

$$= \begin{bmatrix} 0 & \hat{n}_3 & -\hat{n}_2 \\ -\hat{n}_3 & 0 & \hat{n}_1 \\ \hat{n}_2 & -\hat{n}_1 & 0 \end{bmatrix}$$

If we identify α with physical angles, it might be true that

$R[2\pi, \hat{n}] = \mathbb{1}$ or $R[\alpha+2\pi, \hat{n}] = R[\alpha, \hat{n}]$

we check it in the eigen basis of $[\hat{n}\cdot\tilde{J}]$

eigen values:

$$\det \begin{vmatrix} -\lambda & \hat{n}_3 & -\hat{n}_2 \\ -\hat{n}_3 & -\lambda & \hat{n}_1 \\ \hat{n}_2 & -\hat{n}_1 & -\lambda \end{vmatrix} = 0$$

$$\text{or } -\lambda[\lambda^2 + n_1^2] - \hat{n}_3[\lambda\hat{n}_3 - \underbrace{\hat{n}_1\hat{n}_2}_{\to 0}] - \hat{n}_2[\underbrace{\hat{n}_1\hat{n}_3}_{\to 0} + \lambda\hat{n}_2]$$

$$\text{or } -\lambda[\lambda^2 + \underbrace{n_1^2+n_2^2+n_3^2}_{\to 1}] = 0$$

$$\text{or } -\lambda(\lambda^2+1) = 0$$

$$\text{or } \lambda = 0, \pm i$$

In the E. basis of $[\hat{n}\cdot\tilde{J}]$

$$\alpha[\hat{n}\cdot\tilde{J}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & i\alpha & 0 \\ 0 & 0 & -i\alpha \end{bmatrix}$$

$$R(\alpha,\hat{n}) = \exp\left[\alpha\,(\hat{n}\cdot\vec{J})\right].$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha} & 0 \\ 0 & 0 & e^{-i\alpha} \end{bmatrix}$$

How real rotation has complex ~~bas~~ elements?: we chose complex basis.

$$\underset{\substack{\text{in its} \\ \text{E. basis}}}{R\left[2\pi,\hat{n}\right]} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i} & 0 \\ 0 & 0 & e^{-2\pi i} \end{bmatrix} = \mathbb{1}$$

Thus, range of 'non-identicular α' ~~is~~ = ~~(0,2π)~~ $0 \leq \alpha < 2\pi$

$$A_\phi = \begin{bmatrix} A_1 & & 0 \\ & A_2 & \\ 0 & & \ddots \; A_m \end{bmatrix} ; \quad f(A) = \begin{bmatrix} f(A_1) & 0 & 0 \\ 0 & f(A_2) & 0 \\ & & \ddots \; f(A_m) \end{bmatrix}$$

we want $\hat{n}\cdot\vec{J}$ to have real eigen values. So, we redefine the generators :-

~~Ref~~

- Redefine generators such that they have real eigen values.

$$[J_i]_{jk} = i[\breve{J}^i]_{jk} = i\,\epsilon_{ijk}$$

The new rotation is:

$$R[\alpha,\hat{n}] = \exp\left[\alpha\,(\hat{n}\cdot\breve{J})\right]$$

$$= \exp\left(-i\alpha(\hat{n}\cdot\vec{J})\right)$$

How to specify rotation?

we need to know how rotation matrices combine to give a third rotation.

$$R_1(\alpha_1, \hat{n}_0) \qquad R_2(\alpha_2, \hat{n}_0)$$

$$R_1(\alpha_1, \hat{n}_0) = e^{-i\alpha_1(\hat{n}_0 \cdot J)}$$

$$R_2(\alpha_2, \hat{n}_0) = e^{-i\alpha_2(\hat{n}_2 \cdot J)}$$

$$R_1 R_2 (\alpha_3, \hat{n}_0) = e^{-i\alpha_1(\hat{n} \cdot \vec{J})} e^{-i\alpha_2(\hat{n}_0 \cdot \vec{J})} \quad — (1)$$

using Baker-Hausdorf formulae:

$$\exp[A]\exp[B]$$
$$= \exp\Big[A + B + \underbrace{\frac{1}{2}[A, B] + \frac{1}{6}[A,[A,B]] + \frac{1}{24}[B,[A,B]] + \cdots}_{\text{series of nested commutators.}}\Big]$$

(1) can be computed once we know the commutators of J's, product of Js and so on.

So, if we know $R_1 R_2 = R_3$, some other rotation, commutators of generators must be a generator.

we need to know the commutators of the generators.

$$[J^i, J^j] = \sum_k f^{ijk} J^k \quad \to \text{structure constants.}$$

$f^{ijk} = f^{jik}$.

↳ Intrinsic property of space.

$$[J^i, J^j] = i\epsilon_{ijk} J^k$$

$$[J^i]_{pq} = i\epsilon_{ipq}$$

Finite Rotations :-

12/09

$R[\hat{n},\alpha] = e^{-i\alpha(\hat{n}\cdot\vec{J})}$

$[J^i]_{jk} = -i\epsilon_{ijk}$

$(\hat{n}\cdot\vec{J})$ is the generator of infinitesimal rotation around axis $\hat{n}$.

Generators around the different axes do not commute.

However, $R(\hat{n}_1,\alpha_1)\, R(\hat{n}_2,\alpha_2) =$ Rotation

$R_1 \cdot R_2$ could be computed in terms of nested commutators of $\{J^i\}$s.

$\Rightarrow$ Once we know the commutators of the generators of all $\{J^i\}$, we know how ~~to~~ to combine 2 or more rotations.

~~Combinations of~~

"Successive rotation is also a rotation"

$\Downarrow$

commutator of generators must be a linear combination of generators.

$[J^i, J^j] = \sum_k f^{ijk} J^k$

By construction, $f^{ijk} = -f^{jik}$

$f^{ijk} \rightarrow$ structure constant

$$[J^i]_{jk} = -i\epsilon_{ijk}$$

$$[J^i J^j]_{pq} = [J^i]_{p\cancel{q}m}[J^j]_{mq}$$

$$= (-i\epsilon_{ipm})(-i\epsilon_{jmq})$$

$$= \cancel{+\epsilon}(-i)^2 \epsilon_{ipm}\epsilon_{j\cancel{mq}qm}.$$

$$= +(\delta_{ij}\delta_{pq} - \delta_{iq}\delta_{jp})$$

$$[J^i, J^j] = (\cancel{\delta_{ij}}\delta_{pq} - \delta_{iq}\delta_{jp}) - (\cancel{\delta_{ji}}\delta_{pq} - \delta_{jq}\delta_{ip})$$

$$= \delta_{jq}\delta_{ip} - \delta_{iq}\delta_{jp}$$

$$= -(\delta_{iq}\delta_{jp} - \delta_{jq}\delta_{ip})$$

$$= -\epsilon_{ijk}\epsilon_{qpk}$$

$$= -i\epsilon_{ijk}\,(-i)\,\epsilon_{kqp}$$

$$= i\epsilon_{ijk}\,\underbrace{(\cancel{+}i)\,\epsilon_{kpq}}_{\to [J^k]_{pq}}$$

$$= i\underbrace{\epsilon_{ijk}}_{\to f^{ijk}}[J^k]_{pq}$$

$$[J^i, J^j] = i\epsilon^{ijk}J^k$$

↳ Any nested commutators of generators could be computed from here

Summary

① ~~Any~~ symmetric transformation that could be generated by a set of continuous parametrized parameters can be presented as matrices acting linearly on a vector space of a particular physical observable (exception: wave funcⁿ is non-observable)

Symmetry transformation → laws governing the equation remain invariant.

② In such above cases, we always have the concept of infinitesimal transformation and repeated (infinite times) application of this transformation will lead to finite transformation.

③ The generator of infinitesimal ~~rotation~~ transformation form a linear vector space themselves.

↳ The dimension of the space of generators = # of continuous parameters needed to specify the transfⁿ

& not necessarily = to the dimension of the vector space on which the transformation & therefore its generators are acting on.

④ The finite transformation = exponentiation of the infinitesimal generator

⑤ combination of finite transformation = nested commutator of the generators in the exponent.

⇒ commutators of generators must also be a generator.

⇒ We need to compute the commutators

General structure is:

$$[J^i, J^j] = f^{ijk} J^k.$$

In our particular case, of 3-D rotation :

$$[J^i, J^j] = i\epsilon^{ijk} J^k$$

Generators of L·T

Lorentz Transformation

$\rightarrow$ Group of all matrices that preserves $\eta_{\mu\nu} = \text{Diagonal}\{-1,1,1,1\}$

$$\Lambda^T \cdot \eta \cdot \Lambda = \eta$$

$\Lambda \rightarrow 4\times4$ real matrix

① Pure Rotation is also a lorentz transformation.

Pure rotation in a 4X4 matrix (4 space-time dimension)

$$\begin{array}{c|c|c} t & 1 & 0 \\ \hline x_1 & 0 & \text{rotation} \\ x_2 & & \\ x_3 & & \end{array} = \Lambda_{\text{rot}}$$

$$\Lambda^T_{rot} = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & R^T \end{array}\right]$$

$$\Lambda^T_{rot}\left[\begin{array}{c|c} -1 & \\ \hline & 1_{rot} \end{array}\Lambda^T\right] = \left[\begin{array}{c|c} -1 & 0 \\ \hline 0 & R^T \end{array}\right]$$

Pure rotation has continuous parameters.

All generators of pure rotation are generators of lorentz transformation.

$\Rightarrow$ L·T shall have atleast 3 generators.

Apart from ~~that~~ rotation L·T could mix space and time.

$\hookrightarrow$ rotation between space and imaginary time with imaginary angle. (boost)

$\hookrightarrow$ physically, time could mix with 3 spatial directions

$\Rightarrow$ 3 generators.

# of generators = # of independent continuous parameters.

in L·T: $3+3=6$

$$\underset{4\times4}{\Lambda} = \underset{4\times4}{\mathbb{1}} + \epsilon \omega_{4\times4} + \Theta(\epsilon^2)$$

$$\Lambda^T \eta \Lambda = (1 + \epsilon \omega^T) \ \eta \ (1 + \epsilon \omega) = \eta$$

$$= \eta + \epsilon \underbrace{\left[\omega^T \eta + \eta \cdot \omega \right]}_{\to 0} + O(\epsilon^2)$$

$$\Rightarrow \omega^T \eta + \eta \cdot \omega = 0 \quad \text{—①}$$

$$\boxed{\omega^\mu{}_\nu \ \Big| \ \eta_{\mu\nu}} \qquad \Lambda \to \Lambda^\alpha{}_\mu$$

$$(\omega^T)_\nu{}^\mu .$$

we continue with ①:

$$\omega_\nu{}^\alpha \, \eta_{\alpha\mu} + \eta_{\nu\alpha}$$

$$\Lambda^T \cdot \eta \cdot \Lambda = \Lambda^\alpha{}_\mu \, \eta_{\alpha\beta} \, \Lambda^\beta{}_\nu = \eta_{\mu\nu} \ldots \qquad [\Lambda^T]_\mu{}^\alpha = \Lambda^\alpha{}_\mu .$$

near → row farther → column.

$$\Lambda^T \eta \cdot \Lambda = \eta$$

or $(\Lambda_{\alpha\mu})^T \eta_{\alpha\beta} \Lambda_{\beta\nu} = \eta_{\mu\nu}$

or $(\Lambda^T)_{\mu\alpha} \eta_{\alpha\beta} \Lambda_{\beta\nu} = \eta_{\mu\nu}$

or $\Lambda_{\alpha\mu} \eta_{\alpha\beta} \Lambda_{\beta\nu} = \eta_{\mu\nu}$

$$\Lambda_{\mu\nu} = \delta_{\mu\nu} + \epsilon\omega_{\mu\nu}$$

$$(\delta_{\alpha\mu} + \epsilon\omega_{\alpha\mu}) \eta_{\alpha\beta} (\delta_{\beta\nu} + \epsilon\omega_{\beta\nu}) = \eta_{\mu\nu}$$

$$\Rightarrow \omega_{\alpha\mu}\eta_{\alpha\nu} + \omega_{\beta\nu}\eta_{\mu\beta} = 0$$

New convention

① The components of a vector with indices written as subscript will be considered different from the vector whose components ~~are indices~~ have indices as superscripts.

② A^α and A_α are related by $\eta_{\alpha\beta}$:

$$A_\alpha \neq A^\alpha \quad ; A_\alpha = \eta_{\alpha\beta} A^\beta \qquad \alpha = \{0,1,2,3\}$$

$$\Rightarrow A_0 = -A^0 \quad i = \{1,2,3\}$$

$$A_i = A^i$$ } in matrix

$$\Rightarrow A^\alpha = \eta^{\alpha\beta} A_\beta$$

$$\boxed{\eta^{\alpha\beta}\eta_{\beta\gamma} = \delta^\alpha{}_\gamma}$$

So; $\eta^{\alpha\beta}$ is inverse of $\eta_{\alpha\beta}$

$\eta_{\alpha\beta} = \eta_{\alpha\mu} \eta^{\mu\nu} \eta_{\nu\beta}$ should be true if we want our convention to be consistent

(3) convention of contraction / Modification of einstein summation theorem.

Repeated indices will mean a sum only if one of them is upper and the other is lower.

eg: $A_\alpha B^\alpha = \sum_{\alpha=0}^{3} A_\alpha B^\alpha$.

$A_\alpha B_\alpha \Rightarrow \alpha$ is a free index which could be 0,1, ... but not sum.

$A_{\alpha\beta} \longrightarrow A^{\alpha}{}_{\beta}$

$\Lambda^{\alpha}{}_{\mu}\, \eta_{\alpha\beta}\, \Lambda^{\beta}{}_{\nu} = \eta_{\mu\nu}$

Infinitesimal $\Lambda^{\alpha}{}_{\mu}$

$= \delta^{\alpha}_{\mu} + \epsilon\omega^{\alpha}{}_{\mu}$

$\omega^{\alpha}{}_{\mu}\, \eta_{\alpha\nu} + \omega^{\beta}{}_{\nu}\, \eta_{\mu\beta} = 0$

or $\omega_{\mu\nu} + \omega_{\nu\mu} = 0$

Classifying transformation properties under rotation

Scalars - Do not transform under rotation of coordinates

Vectors - In 3dimensional space, a triplet of real numbers $\{A_i\} = \vec{A}$ transforming as

$$\vec{A} \to \vec{A}' = \{A_i'\} \quad \text{s.t} \quad A_i' = R_{ij}A_j$$

Tensors.

↳ These are physical objects with matrix like representation

↳ Devicing an example :-

consider a machine moving massive bodies with

(i) Input handle where force $\vec{F}$ could be applied

(ii) Output handle displaces the body $\vec{B}$ by $\vec{s}$

(iii) $\vec{s} \nparallel \vec{F}$, but $\vec{s}$ is linear in $\vec{F}$.

$$S_i = M_{ij}F_j$$

M_{ij}, which describes the response of ~~the a two rank~~ the machine is a 2-rank tensor.

Magnitude of any given component of the tensor M will depend on the orientation of the coordinates.

Suppose coordinates are rotated such that:

Linear transformation $F_i' = R_{ij}F_j$

$S_i' = R_{ij}S_j$

So, we can write this:

$S_i' = M_{ij}' F_j'$ ↳ Linearly related in any frame

$\{M_{ij}'\}$ = Response of machine in the rotated coordinates

→ summation convention.

$$R_{ij}S_j = M'_{ij}(R_{jk}F_k)$$

$R\cdot\vec{S} = M'\cdot R\cdot\vec{F}$

R is orthogonal & invertible, so:-

$\vec{S} = (R^{-1}M'R)\cdot\vec{F}$

$= (R^T M' R)\cdot\vec{F}$

But, S was:

$\vec{S} = \vec{M}\,\vec{F}$

& $\vec{S} = (R^T M' R)\vec{F}$

$\Rightarrow M = R^T M' R$

$\Rightarrow \boxed{M' = RMR^T}$

$$M'_{ij} = \sum_{pq} R_{ip} M_{pq} (R)^T_{qj} = \sum_{p,q} R_{ip} R_{jq} M_{pq}$$

we found that:

$$M'_{ij} = \sum_{pq} R_{ip} R_{jp} M_{pq}$$

so, each of the two indices of M transform independently as vectors indices.

Generalize to higher rank tensors.

We could have physical objects whose description requires more than 2 indices. There are vectors with rank > 2.

→ Scalar is a 0 rank tensor.

→ vector is a tensor with rank 1.

→ Rank k tensor

$T_{i_1 i_2 i_3 \cdots i_k}$

$$T'_{i_1 i_2 \cdots i_k} = \sum_{j_1 j_2 \cdots j_k} R_{i_1 j_1} R_{i_2 j_2} \cdots R_{i_k j_k} T_{j_1 j_2 \cdots j_k} \rightarrow \text{Transformation properties}$$

The generators of infinitesimal rotation $[J^i]_{jk} \propto \epsilon_{ijk}$

Show that $\{J^1_{jk}, J^2_{jk}, J^3_{jk}\}$ all form a real space vector for every (jk). (Show how transformation occurs)

Tensors in 1+3 dimensions of space-time.

→ Scalars are invariant of L·T

→ Vectors are 4 component objects.

$$A = \{A^\alpha\} = \begin{Bmatrix} A^0 \\ \vec{A} \end{Bmatrix}$$

A vector written without arrow is 4-Dimensional.

A vector written as $\vec{A}$ is a 3-vector.

$$A^\alpha = \Lambda^\alpha{}_\beta A^\beta$$

where Λ is defined as:

$$\Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \eta_{\alpha\beta} = \eta_{\mu\nu}$$

A rank K tensor will have k indices $T^{\alpha_1\alpha_2\cdots\alpha_k}$

Under L·T :-

$$(T')^{\alpha_1\alpha_2\cdots\alpha_k} = \Lambda^{\alpha_1}{}_{\beta_1}\Lambda^{\alpha_2}{}_{\beta_2}\cdots\Lambda^{\alpha_k}{}_{\beta_k} T^{\beta_1\beta_2\cdots\beta_k}$$

A^α and A_α transform differently under L·T

$$A^\alpha \longrightarrow \tilde{A}^\alpha = \Lambda^\alpha{}_\mu A^\mu$$

suppose $A_\alpha \rightarrow \tilde{A}_\alpha = \tilde{\eta}_{\alpha\beta}\tilde{A}^\beta = \eta_{\alpha\beta}\tilde{A}^\beta$

or $\tilde{A}_\alpha = \eta_{\alpha\beta}\Lambda^\beta{}_\theta A^\theta$

Multiply (and contract) on both sides with $\Lambda^\alpha{}_\nu$

$$\Lambda^\alpha{}_\nu \tilde{A}_\alpha = \underbrace{(\Lambda^\alpha{}_\nu \eta_{\alpha\beta}\Lambda^\beta{}_\theta)}_{\eta_{\nu\theta}} A^\theta$$

$$= \eta_{\nu\theta}A^\theta = A_\nu$$

Thus, we have relation:-

$A_\alpha \longrightarrow \tilde{A}_\alpha$

$\Lambda^\alpha{}_\nu \tilde{A}_\alpha = A_\nu \longrightarrow$ difference....

or $\left(\tilde{A}_\alpha = (\Lambda^\alpha{}_\nu)^{-1} A_\nu\right)$?

$$\boxed{\begin{aligned} A^\alpha &= \Lambda^\alpha{}_\beta{}' A^\beta \\ \Lambda^\alpha{}_\beta \tilde{A}_{\beta\alpha} &= A_\beta \end{aligned}} \longrightarrow \text{2 different transformations.}$$

World Lines: Trajectories of point-particle 20 sept

→ One-dimensional curves in spaces-time

Questⁿ for mid-sem.
World sheet
If you have a line segment moving in space.

→ Any (non-closed) curve could be world line according to our defination

→ But according to classical mechanics (rel or non-rel) only certain world lines are realized in nature.

↓

These satisfy some particular diff eqⁿ called EOM of the particle

→ Action principle → Algorithm to derive EOM.

(No derivation of action occurs)

Action contains more information than EOM.

Action Principle

Action (S) is a functional of world line

Functionals: Take funcⁿ as input
give output as a number

Funcⁿ input ⟶ Functionals ⟶ Number output

Usually, functionals are integrables.

→ A given action will associate a real number with every WL.

→ Action principle says that:

" The WL that extremizes the action is realized in nature."

we cannot have closed curves as if it happens, we are coming back to same time which is not possible.

~~Relativistic~~

Expectation:- Action should be Lorentz scalar

(i) EOMs are physicals laws ~~[illegible]~~

↳ covariant under L-T

(ii) world lines are geometrical ~~lines~~ curves in (3+1) dimensional

↳ exist independent in any coordinate system

(iii) All observers (in different inertial frames) would associate the same value of action for the world lines.

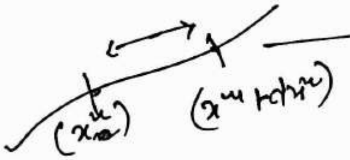
(iv) All observers will ~~go~~ agree on the extrema.

we might need to change axes based on problem.

~~The equations of EOM~~

Action for a ~~r.~~ free relativistic particle.

→ Possible lorentz scalar associated with the world line → invariant length element.



$\longrightarrow \sqrt{dx^\mu dx^\nu \eta_{\mu\nu}} \rightarrow$ proper length.

Total proper length from the starting point (t_1, x^i_1) to end point $\{t_f, x^i_F\}$

$$l = \int_{t_1, x^i_1}^{t_F, x^i_F} \sqrt{|dx^\mu dx^\nu \eta_{\mu\nu}|}$$

→ For a free ~~la~~ extended line, then it would be Area

Action ~~S~~ = $S = Kl$

↳ constant of world lines- across wl, K is constant

$\lambda \rightarrow$ parameter but monotonic.

If it is not, then 2 points in S.T will have same value of λ

$\{x^\mu_i\}$ = initial pt. corresponding to $\lambda = \lambda_I$.

$\{x^\mu_F\}$ = final " . . . to $\lambda = \lambda_F$

$$S = \int_{\lambda_I}^{\lambda_F} d\lambda \sqrt{\left|\left(\frac{dx^\mu}{d\lambda}\right)\left(\frac{dx^\nu}{d\lambda}\right)\right| \eta_{\mu\nu}}$$

Boost along $\hat{n}$ with β.

Generator : $k \cdot \hat{n}$

$\Lambda(\beta, \hat{n}) = e^{\beta(k \cdot \hat{n})}$

Boost along $\hat{n}_1$ by rapidity β_1

" $\hat{n}_2$ by " β_2

⋮

$\hat{n}_k$ by β_k

Successively.

$\Lambda(\beta_1, \beta_2 \cdots \beta_k, n_1 n_2 n_3 \cdots n_k) = \prod_{i=1}^{k} e^{\beta_i (\bar{k} \cdot \hat{n}_k)}$; $\beta = \tanh^{-1}\left(\frac{|V|}{c}\right)$

22 Asept

we found that the action for free relativistic particle :-

suppose :

$$S = K \int_{\lambda_I}^{\lambda_F} d\lambda \sqrt{\left|\left(\frac{dx^\mu}{d\lambda}\right)\left(\frac{dx^\nu}{\partial\lambda}\right)\right| \eta_{\mu\nu}}$$

K : W.L independent constant

where $\{x^\mu(\lambda)\}$ denotes a ~~point on~~ unique point on WL for every λ , λ_F

we have to extremize S ~~over~~ over all W.L such that $x^\mu(\lambda_I) = x_I^\mu$

$$x^\mu(\lambda_F) = x^\mu_F$$

λ_I

⇒ According to action principle :-

⇒ the extrema WL will give the trajectory of the free particle starting from $\{x^\mu_I\}$ & ending at $\{x^\mu_F\}$

→ suppose $\{\bar{x}^\mu(\lambda)\}$ extremizes S.

⇒ consider a world line $x^\mu(\lambda) = \bar{x}^\mu(\lambda) + \epsilon K^\mu(\lambda)$

where $\epsilon \ll 1$ such that any term of $O(\epsilon^2)$ or higher power is neglected

Then, $\bar{x}^\mu(x)$ will be an extrema of s provided

$$S[x^\mu(\lambda) = \bar{x}^\mu(\lambda) + \epsilon K^\mu(\lambda)] = S[\bar{x}^\mu(\lambda)] + O(\epsilon^2)$$

$\forall$ smooth $K^\mu(\lambda)$

satisfying :

$K^\mu(\lambda_I) = K^\mu(\lambda_F) = 0$

→ $O(\epsilon^2)$

f is a func^n & $\bar{x}$ is an extrema, then:

$$f(\vec{x}+\epsilon) = f(\bar{x}) + \underbrace{\frac{df}{dx}\Big|_{\bar{x}}}_{\to 0} \epsilon + \frac{1}{2}\frac{d^2 f}{dx^2}\Big|_{\bar{x}} \epsilon^2 + \cdots$$

$$= f(x) + O(\epsilon^2)$$

For the case :-

$$S[\bar{x}^\mu + \epsilon k^\mu(\lambda)] = \int_{\lambda_I}^{\lambda_F} d\lambda \sqrt{\left(\frac{d\bar{x}^\mu}{d\lambda} + \epsilon \frac{dk^\mu}{d\lambda}\right)\left(\frac{d\bar{x}^\nu}{d\lambda} + \epsilon\frac{dk^\nu}{d\lambda}\right)\eta_{\mu\nu}}$$

$d\bar{x}^\mu dk^\nu \eta_{\mu\nu} + d\bar{x}^\nu dk^\mu \eta_{\mu\nu}$ & rename the indices & $\eta_{\mu\nu} = \eta_{\nu\mu}$ so, we get factor of 2

$$= \int_{\lambda_I}^{\lambda_F} d\lambda \left\{ \eta_{\nu\mu}\left[\frac{d\bar{x}^\mu}{d\lambda}\frac{d\bar{x}^\nu}{d\lambda} + 2\epsilon \frac{dk^\mu}{d\lambda}\frac{d\bar{x}^\nu}{d\lambda} + O(\epsilon^2)\right]\right\}^{1/2}$$

$$= \int_{\lambda_I}^{\lambda_F} d\lambda \left[\left(\frac{d\bar{x}^\mu}{d\lambda}\right)\left(\frac{d\bar{x}_\mu}{d\lambda}\right)\left[1 + 2\epsilon\left(\frac{dk^\nu}{d\lambda}\right)\left(\frac{d\bar{x}_\nu}{d\lambda}\right)\left[\left(\frac{d\bar{x}^\mu}{d\lambda}\right)\left(\frac{d\bar{x}_\mu}{d\lambda}\right)\right]^{-1}\right]\right]^{1/2}$$

$$= \int_{\lambda_I}^{\lambda_F} d\lambda \sqrt{\left(\frac{d\bar{x}^\mu}{d\lambda}\right)\left(\frac{d\bar{x}_\mu}{d\lambda}\right)}\left[1 + 2\epsilon\left(\frac{dk^\nu}{d\lambda}\right)\left(\frac{d\bar{x}_\nu}{d\lambda}\right)\left\{\left(\frac{d\bar{x}^\mu}{d\lambda}\right)\left(\frac{d\bar{x}_\mu}{d\lambda}\right)\right\}^{-1} + O(\epsilon^2)\right]^{1/2}$$

$$\sqrt{1 + 2\epsilon B} = 1 + \epsilon B + O(\epsilon^2)$$

$$= \int_{\lambda_I}^{\lambda_F} d\lambda \sqrt{\left(\frac{d\bar{x}^\mu}{d\lambda}\right)\left(\frac{d\bar{x}_\mu}{d\lambda}\right)}\left[1 + \epsilon \frac{\left(\frac{dk^\nu}{d\lambda}\right)\left(\frac{d\bar{x}_\nu}{d\lambda}\right)}{\left(\frac{d\bar{x}^\mu}{d\lambda}\right)\left(\frac{d\bar{x}_\mu}{d\lambda}\right)} + O(\epsilon^2)\right]$$

multiplying

$$= S[\bar{x}^\mu(\lambda)] + \epsilon \underbrace{\int_{\lambda_I}^{\lambda_F} d\lambda \frac{\left(\frac{dk^\nu}{d\lambda}\right)\left(\frac{d\bar{x}_\nu}{d\lambda}\right)}{\left[\left(\frac{d\bar{x}^\mu}{d\lambda}\right)\left(\frac{d\bar{x}_\mu}{d\lambda}\right)\right]^{1/2}}} + O(\epsilon^2)$$

extremization $\checkmark$

vanish for every $k^\mu(\lambda)$ satisfying of this form: $k^\mu(\lambda_I) = k^\mu(\lambda_F) = 0 \quad \lambda \longleftrightarrow t$

Term linear in ϵ:

$$\int_{\lambda_I}^{\lambda_F} \frac{d}{d\lambda}\left[K^{\mu}(\lambda) \left(\frac{d\bar{x}_{\nu}}{d\lambda}\right)\left[\left(\frac{d\bar{x}^{\mu}}{d\lambda}\right)\left(\frac{d\bar{x}_{\mu}}{d\lambda}\right)\right]^{-1/2}\right.$$

$$-\int_{\lambda_I}^{\lambda_F} d\lambda\, K^{\nu}(\lambda)\, \frac{d}{d\lambda}\left\{ \left(\frac{d\bar{x}_{\nu}}{d\lambda}\right)\left[\left(\frac{d\bar{x}^{\mu}}{d\lambda}\right)\left(\frac{d\bar{x}_{\mu}}{d\lambda}\right)\right]^{-1/2}\right\}$$

Thus, the EOM we get is:-

$$\frac{d}{d\lambda}\left(\left(\frac{d\bar{x}_{\nu}}{d\lambda}\right)\left(\left(\frac{d\bar{x}^{\mu}}{d\lambda}\right)\left(\frac{d\bar{x}_{\mu}}{d\lambda}\right)\right)^{-1/2}\right) = 0$$

Relativistic free particle (cont^{d}) 23 Sept

$$S = k \int_{\lambda_I}^{\lambda_F} d\lambda \sqrt{\left(\frac{dx^\mu}{d\lambda}\right)\left(\frac{dx^\nu}{d\lambda}\right)\eta_{\mu\nu}}$$

Extremizing S over all WL, we get EOM:

$$\frac{d}{d\lambda}\left[\left(\frac{d\bar{x}^\mu}{d\lambda}\right)\left(\frac{d\bar{x}_\mu}{d\lambda}\right)^{-1/2}\cdot\left(\frac{d\bar{x}^\nu}{d\lambda}\right)\right] = 0$$

when $\bar{x}^\mu(\lambda) = \omega \cdot L$, it shall extremize S.

This equation could be solved easily in terms of a new parameter $\tau(\lambda)$ defined as:

$$\frac{d\tau(\lambda)}{d\lambda} = \sqrt{\left(\frac{d\bar{x}^\mu}{d\lambda}\right)\left(\frac{d\bar{x}_\mu}{d\lambda}\right)}$$

or $$\tau(\lambda) = \int d\lambda \sqrt{\left(\frac{d\bar{x}^\mu}{d\lambda}\right)\left(\frac{d\bar{x}_\mu}{d\lambda}\right)}$$

or $$\frac{d}{d\tau} = \left[\left(\frac{d\bar{x}^\mu}{d\lambda}\right)\left(\frac{d\bar{x}_\mu}{d\lambda}\right)\right]^{-1/2}\frac{d}{d\lambda}$$

Thus, the EOM in terms of $\tau(\lambda)$:

$$\left(\frac{d\bar{x}^\mu}{d\lambda}\right)\left(\frac{d\bar{x}_\mu}{d\lambda}\right)^{+1/2}\frac{d}{d\tau(\lambda)}\left[\frac{d\bar{x}^\nu}{d\tau}\right] = 0$$

or $$\frac{d^2\bar{x}^\nu}{d^2\tau(\lambda)} = 0$$

$\Rightarrow \quad \bar{x}^\nu(\tau) = a^\nu \tau + b^\nu \quad$; $\{a^\nu\}$ & $\{b^\nu\}$ are constants. 4 vector.

$ct = x^0 = a^0\tau + b \quad$; $x^i = a^i\tau + b^i$

or $\tau = \frac{ct - b^0}{a^0}$

$$\Rightarrow x^i = a^i\left(\frac{ct - b^0}{a^0}\right) + b^i$$

$$= \frac{a^i}{a^0}ct + \left[b^i - \left(\frac{b^0}{a^0}\right)a^i\right]$$

This is a free particle trajectory in storaight line.

$$x = \vec{v}\hat{t} + \bar{x}_0$$

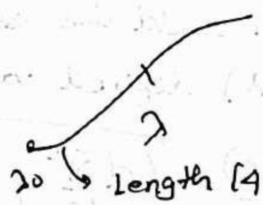
$$v^i = \left(\frac{a^i}{a^0}\right)c$$

$$x_0^i = \vec{b} - \left(\frac{b^0}{a^0}\right)\vec{a}$$

~~Non~~-relati~~vi~~stic

$\tau(\lambda) = l(\lambda) =$ proper length/ time

$$= \int_{\lambda_0}^{\lambda} d\lambda' \sqrt{\left(\frac{dx^\mu}{d\lambda'}\right)\left(\frac{dx^\nu}{d\lambda'}\right)\eta_{\mu\nu}}$$



Length (4-D) up to this λ. This is Lorentz invariant

~~Any~~

Properties τ:

i) Monotonously increasing

ii) Unique at ~~at~~ any point.

The trajectory is a curve in space, parametrized by ~~a~~ time. It can be closed.

Try extremization without squareroot i.e

$$S = \int d\lambda \left(\left(\frac{dx^\mu}{d\lambda}\right)\left(\frac{dx_\mu}{d\lambda}\right)\right)^N \text{ gives free particle?}$$

Time like interval

If ~~x~~ (x^{ν}_1) and (x^{ν}_2) are 2 events in space time such that

$$x^{\nu}_1 - x^{\nu}_2 = (\Delta x)^{\nu}$$

$$(\Delta x)^{\nu} (\Delta x)^{\mu} \eta_{\mu\nu} < 0$$

Then this ~~is called~~ interval $\{\Delta x^{\mu}\}$ is called timelike

If $(\Delta x)^{\nu} (\Delta x)^{\mu} \eta_{\mu\nu} > 0 \Rightarrow$ space trajectory.

$(\Delta x^{\mu})(\Delta x_{\mu}) = 0 \Rightarrow$ null.

① Time like intervals will remain time like in all frames and we can choose a frame where $\Delta x^{\mu} \rightarrow \begin{Bmatrix} \Delta x^0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$ i.e pure ~~time~~ like time.

② Space like intervals remain space like in all frames & one can choose a frame where $\Delta x^{\mu} = \begin{Bmatrix} 0 \\ \Delta x^1 \\ 0 \\ 0 \end{Bmatrix}$ i.e pure space.

Sign of Δx shall not change under L.T for time-like.

③ Null intervals remain null in all frames and one can choose a frame where $\Delta x^{\mu} = \begin{Bmatrix} \Delta x^0 = a \\ \Delta x^1 = a \\ 0 \\ 0 \end{Bmatrix}$

④ For time-like intervals, ~~cause~~ sign of (Δx^0) should not change so that the cause remains the cause in all frames.

Galilean Transformation and schrodinger's eqⁿ. 10 Oct

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2}$$

$\partial_x \psi(x,t) \rightarrow$ keeping t fix, derivative WRT x.

$t = \tau \qquad \vec{y} = \vec{x} - \vec{v}t$

$$\partial_t = \left(\frac{\partial \tau}{\partial t}\right)\partial_\tau + \left(\frac{\partial \vec{y}}{\partial t}\right)\partial_{\vec{y}}$$

$$= \partial_\tau - \vec{v}\cdot\partial_{\vec{y}}$$

$$\partial_x = \left(\frac{\partial y^i}{\partial x^j}\right)\partial y^j + \left(\frac{\partial \tau}{\partial x^j}\right)\partial \tau$$

$$= \delta_{ij}\,\partial y^j = \partial y^i$$

So, $\partial x^2 = \partial y^2$

$$i\hbar\left(\partial_\tau - \vec{V}\partial_{\vec{y}}\right)\psi(y,\tau) = -\frac{\hbar^2}{2m}\partial_y^2\,\psi(y,\tau)$$

$\downarrow$

due to this term, we get non-invariance.

$\psi \rightarrow \tilde{\psi} = \psi e^{-i\phi}$

$\psi = \tilde{\psi} e^{i\phi} \qquad \tilde{\psi}(y,\tau) \;;\; \phi(y,\tau)$

LHS: RHS:

$$i\hbar\left(\partial_\tau - \vec{V}\cdot\partial_y\right)\left(\tilde{\psi}\, e^{i\phi}\right) = -\frac{\hbar^2}{2m}\partial_y^2\left(\tilde{\psi}\, e^{i\phi}\right)$$

or $e^{i\phi}\, i\hbar\left[\left(\partial_\tau \tilde{\psi} - \vec{V}\cdot\partial_{\vec{y}}\tilde{\psi}\right) + i\tilde{\psi}\left[\partial_\tau - \vec{v}\cdot\partial_{\vec{y}}\right]\phi\right]$

$$= -\frac{\hbar^2}{2m}\left[e^{i\phi}\partial_y^2 + 2\left(\partial_{\vec{y}}\tilde{\psi}\right)\partial_y e^{i\phi} + \tilde{\psi}\,\partial_y^2\, e^{i\phi}\right]$$

$$= \left(-\frac{\hbar^2}{2m}\right)\left[\partial_y^2\tilde{\psi} + 2i\left(\partial_y\tilde{\psi}\right)\left(\partial_y\phi\right) + \tilde{\psi}\left[i\partial_y^2\phi - \left(\partial_y\phi\right)^2\right]\right]e^{i\phi}$$

So, we have:

$$\text{LHS} - \text{RHS} = 0$$

$$\Rightarrow \left(i\hbar \partial_\tau \tilde{\psi} + \frac{\hbar^2}{2m} \partial_y^2 \tilde{\psi} \right)$$

$$+ \partial_y \tilde{\psi} \left[-i\hbar \vec{v} + \frac{i\hbar^2}{m} \partial_{\vec{y}} \phi \right] \longrightarrow (\text{I})$$

$$+ \tilde{\psi} \left\{ -\hbar \left(\partial_\tau - \vec{v} \cdot \vec{\partial_y} \right) \phi + \frac{\hbar^2}{2m} \left(i\, \partial_y^2 \phi - (\partial_y \phi)^2 \right) \right\} \text{ --- (II)}$$

$$= 0$$

So;

$$-i\hbar \vec{v} + \frac{i\hbar^2}{m} \partial_{\vec{y}} \phi = 0 \quad \Rightarrow \quad \frac{i\hbar^2}{m} \partial_y \phi = i\hbar v$$

$$\Rightarrow \phi = \left(\frac{m}{\hbar} \right) (\vec{v} \cdot \vec{y}) + f(\tau)$$

$$\partial_y^2 (\phi) = 0$$

$$(\partial_y \phi)^2 = \left(\frac{m}{\hbar} \right)^2 (v)^2$$

$$v \cdot \partial_y \phi = \frac{m}{\hbar} v^2$$

Substituting in (II):-

$$-\hbar \left(f'(\tau) \right) + \cancel{\left(\frac{m}{\hbar} \right)^2 v^2} \cancel{\frac{m}{2}} + m v^2 - \frac{m}{2} v^2 = 0$$

$$\text{or} \quad f'(\tau) = \left(\frac{m}{2\hbar} \right) v^2$$

$$\text{or} \quad f(\tau) = \left(\frac{m}{2\hbar} \right) v^2 \tau$$

$$\text{So;} \quad \phi = \frac{m}{\hbar} \left[\vec{v} \cdot \vec{y} + \frac{v^2 \tau}{2} \right]$$

Four-momentum : conservation for free particle.

Noether's theorem: whenever there is a continuous symmetry, it will result in a conserved quantity on solutions of EOM.

Continuous symmetry: A transformation ~~parametrized~~ specified by 1 or more continuous parameters that leave the EOM invariant

$$S = k\int d\lambda \underbrace{\sqrt{\left(\frac{dx^\mu}{d\lambda}\right)\left(\frac{dx^\nu}{d\lambda}\right)\eta_{\mu\nu}}}_{\mathcal{L}} = k\int d\lambda\, \mathcal{L}(\lambda)$$

$$x^\mu \to x^\mu + a^\mu \Rightarrow \frac{dx^\mu}{d\lambda} \to \frac{dx^\mu}{d\lambda}$$

Action / integrand is trivially invariant

$\Rightarrow$ If we substitute $a^\mu \to \epsilon^\mu$ = infinitesimal ; $\delta\mathcal{L} = 0$

i.e change in $\mathcal{L}$ is 0.

But for $\delta\mathcal{L}$, we know how to get it. we have independent expression for $\delta\mathcal{L}$.

$$S = k\int d\lambda\, \mathcal{L}(\lambda)$$

$$x^\mu(\lambda) = \underbrace{\bar{x}^\mu(\lambda)}_{\text{solves EOM}} + \epsilon^\mu(\lambda)$$

$$S = k\int d\lambda \left[\frac{d(x^\mu + \epsilon^\mu)}{d\lambda}\, \frac{d(x^\nu + \epsilon^\nu)}{d\lambda}\right]^{1/2} \eta_{\mu\nu}{}^{1/2}$$

$$= k\int d\lambda \left[\left(\frac{d\bar{x}^\mu}{d\lambda}\right)\left(\frac{dx^\nu}{d\lambda}\right)\eta_{\mu\nu} + 2\left(\frac{d\epsilon^\mu}{d\lambda}\right)\left(\frac{dx^\nu}{d\lambda}\right)\eta_{\mu\nu} + O(\epsilon^2)\right]^{1/2}$$

$$= k\int d\lambda \left[\left(\frac{d\bar{x}^\mu}{d\lambda}\right)\left(\frac{d\bar{x}^\nu}{d\lambda}\right)\eta_{\mu\nu}\right]^{1/2}\left[1+\frac{\left(2\frac{d\epsilon^\alpha}{d\lambda}\right)\left(\frac{dx^\beta}{d\lambda}\right)\eta_{\alpha\beta}}{\left(\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)\eta_{\mu\nu}}\right]^{1/2}$$

$$= k\int d\lambda \left[\left(\frac{d\bar{x}^\mu}{d\lambda}\right)\left(\frac{d\bar{x}^\nu}{d\lambda}\right)\eta_{\mu\nu}\right]^{-1/2}\left(\frac{d\bar{x}^\beta}{d\lambda}\right)\left(\frac{d\epsilon^\alpha}{d\lambda}\right)\eta_{\alpha\beta} + O(\epsilon^2)$$

$$= S[\bar{x}^\mu(\lambda)] + \int d\lambda \frac{d}{d\lambda}\left[\epsilon^\alpha\left(\frac{d\bar{x}^\beta}{d\lambda}\right)\eta_{\alpha\beta}\left[\left(\frac{d\bar{x}^\mu}{d\lambda}\right)\left(\frac{d\bar{x}^\nu}{d\lambda}\right)\eta_{\mu\nu}\right]^{-1/2}\right]$$

$$- \int d\lambda\, \epsilon^\alpha(\lambda)\underbrace{\frac{d}{d\lambda}\left[\left(\frac{d\bar{x}^\beta}{d\lambda}\right)\eta_{\alpha\beta}\left[\left(\frac{d\bar{x}^\mu}{d\lambda}\right)\left(\frac{d\bar{x}^\nu}{d\lambda}\right)\eta_{\mu\nu}\right]^{-1/2}\right]}_{\to 0}$$

satisfies EOM

$$\Rightarrow \delta\mathcal{L} = \frac{d}{d\lambda}\left\{\epsilon^\alpha\left(\frac{d\bar{x}^\beta}{d\lambda}\right)\eta_{\alpha\beta}\left[\left(\frac{d\bar{x}^\mu}{d\lambda}\right)\left(\frac{d\bar{x}^\nu}{d\lambda}\right)\eta_{\mu\nu}\right]^{-1/2}\right\}$$

↳ True for any $\epsilon^\alpha(\lambda)$, ϵ^α very small.

But, if ϵ^α is independent of λ, (δL) must vanish.

Conserved 4-momentum. 11 oct.

Free particle EOM

- There exists a 4-vector p^{μ} which is conserved along the world line.

$$\frac{d}{d\lambda}\left\{\left[\left(\frac{d\bar{x}^{\mu}}{d\lambda}\right)\left(\frac{d\bar{x}_{\mu}}{d\lambda}\right)\right]^{-1/2}\frac{dx^{\nu}}{d\lambda}\right\}=0$$

where λ is some arbitrary parameter.

$\bar{x}^{\mu}(\lambda)$ is the world line that extremizes the free particle action.

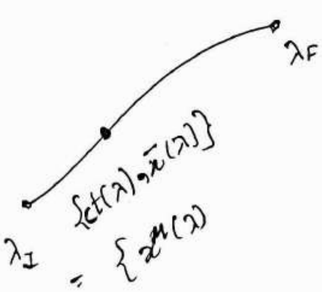
Free EOM $\Rightarrow$ conservation equation

Conserved Quantity is a consequence of the invariance of the action under space-time translation.

$\hookrightarrow$ The conserved quantity ~~EOM~~ $\propto$ Energy and momentum.

~~Conservation~~ on

If we have a conserved quantity Q, what we mean is : $\frac{dQ}{dt}=0$ | $t \to$ coordinate time



we could choose $t(\lambda)$ itself to be a parameter

So, $\frac{d}{d\lambda}=\left(\frac{dt}{d\lambda}\right)\left(\frac{d}{dt}\right)$

$$\left(\frac{dx^{\mu}}{d\lambda}\right)=\left(\frac{dt}{d\lambda}\right)\left(\frac{dx^{\mu}}{dt}\right)$$

$$\sqrt{\left|\left(\frac{d\bar{x}^{\mu}}{d\lambda}\right)\left(\frac{d\bar{x}_{\mu}}{d\lambda}\right)\right|}^{\,-1/2}=\left(\frac{dt}{d\lambda}\right)^{-1}\sqrt{\left(\frac{d\bar{x}^{\mu}}{dt}\right)\left(\frac{d\bar{x}_{\mu}}{dt}\right)}^{\,-1/2}$$

So, we have:

$$\left(\frac{d\bar{x}^{\mu}}{d\lambda}\right)\left(\frac{d\bar{x}_{\mu}}{d\lambda}\right)=\left(\frac{dt}{d\lambda}\right)^{2}\left(\frac{d\bar{x}^{\mu}}{dt}\right)\left(\frac{d\bar{x}_{\mu}}{dt}\right)$$

Thus, we have:

$$0=\frac{d}{d\lambda}\left\{\left[\left|\left(\frac{d\bar{x}^{\mu}}{d\lambda}\right)\left(\frac{d\bar{x}_{\mu}}{d\lambda}\right)\right|\right]^{-1/2}\frac{d\bar{x}^{\nu}}{d\lambda}\right\}=\frac{dt}{d\lambda}\frac{d}{dt}\left\{\underbrace{\left(\frac{dt}{d\lambda}\right)^{-1}\left(\frac{dt}{d\lambda}\right)}\left[\left|\left(\frac{d\bar{x}^{\mu}}{dt}\right)\left(\frac{d\bar{x}_{\mu}}{dt}\right)\right|\right]^{-1/2}\frac{d\bar{x}^{\nu}}{dt}\right\}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{dt}{d\lambda}\right)\frac{d}{dt}\left[\left\{\left|\left(\frac{d\bar{x}^{\mu}}{dt}\right)\left(\frac{d\bar{x}_{\mu}}{dt}\right)\right|\right\}^{-1/2}\left(\frac{d\bar{x}^{\nu}}{dt}\right)\right] = 0$$

$|\,| \to$ mod to get time like trajectory.

$$\Rightarrow \frac{d}{dt}\left\{\left[\left|\left(\frac{d\bar{x}^{\mu}}{dt}\right)\left(\frac{d\bar{x}_{\mu}}{dt}\right)\right|\right]^{-1/2}\left(\frac{d\bar{x}^{\nu}}{dt}\right)\right\} = 0 \quad \left[\text{given that } \frac{dt}{d\lambda} \neq 0\right]$$

$\frac{dt}{d\lambda} = 0 \Rightarrow$ not correct parametrization.

Thus, we get a conserved Quantity - so; p^{ν} (4-momenta):

$$p^{\nu} \propto \left[\left(\frac{d\bar{x}^{\mu}}{dt}\right)\left(\frac{d\bar{x}_{\mu}}{dt}\right)\right]^{-1/2}\left(\frac{d\bar{x}^{\nu}}{dt}\right) \to$$

This should reduce to $p^{\nu}_{\text{non relativistic}} = m_{\text{non rel}}\, \vec{v}_{\text{non rel}}$

The trajectory of free particle

$$x^i(t) = u^i(t) + d^i$$
$$x^0(t) = ct$$

We could have rather have a -ve sign in order to avoid mod function.

$$\frac{dx^{\mu}}{dt} = \{c, u^i\}$$

$$\left(\frac{d\bar{x}^{\mu}}{dt}\right)\left(\frac{dx_{\mu}}{dt}\right) = -\left(\frac{dx^0}{dt}\right)^2 + \left(\frac{dx^i}{dt}\right)^2$$
$$= -(c^2 - u^2)$$

$$\left[\left|\left(\frac{d\bar{x}^{\mu}}{dt}\right)\left(\frac{dx_{\mu}}{dt}\right)\right|\right]^{-1/2} = \frac{1}{\sqrt{c^2-u^2}} = \frac{\gamma}{c}$$

$$p^{\mu} \propto \underbrace{\frac{\gamma}{c}\{c, u^i\}}_{\text{dimension less}}$$

should have momentum units.

Quest^n why can't we ~~use~~ can just use t_i? $\to$ All particles would have same energy, momenta + same actions. Hence, not t_i.

$$[p^{\mu}] = [m_0][\text{velocity}]$$

cannot be particle velocity as p^{μ} it is independent of world line but u^i is dependent on world line.

$\therefore$ m_0 can be the mass of the particle. Mass is particle property and independent of the world line.

Thus, the proportionality constant shall be $(m_0)(c)$

$\Rightarrow$ ~~p^{μ}~~ $p^{\mu} = A \frac{\gamma}{c} \{c, u^i\}$

$A \to$ some property of particle independent of world line and has dimensions of momentum.

$A = [m_0]\, c$

↳ some other property of particle.

We write $A = m_0 c$ since we'll see that m_0 would be related to non-relativistic mass of the particle.

Suppose taking non-relativistic limit:

$p^{\mu} = m_0 c \cdot \frac{\gamma}{c} \{c, u^i\}$

$p^i = m_0 \gamma u^i \qquad ; \gamma = \frac{1}{\sqrt{1 - \frac{u^{i2}}{c^2}}}$

So, $\lim_{\frac{u^i}{c} \to 0} p^i = \frac{m_0 u^i}{\sqrt{1 - \frac{u^{i2}}{c^2}}} \approx m_0 u^i$

↳ mass of particle in non-relativistic limit

Moving mass

If we define mass, to be the ratio of $\frac{\text{spatial momenta}}{\text{spatial velocity}}$.

moving mass $= m_0 \gamma$

↳ rest mass

$= \frac{m_0}{\sqrt{1 - v^2/c^2}}$ $\quad$ v: velocity in the coordinate space

Tutorial

Conservation Law -

In a non-relativistic setup:

$$\frac{d}{dt} Q \,[\text{total charge}] = 0$$

$dt \to$ coordinate time ; Q being total charge and independent of frame as it is volume integral over all space.

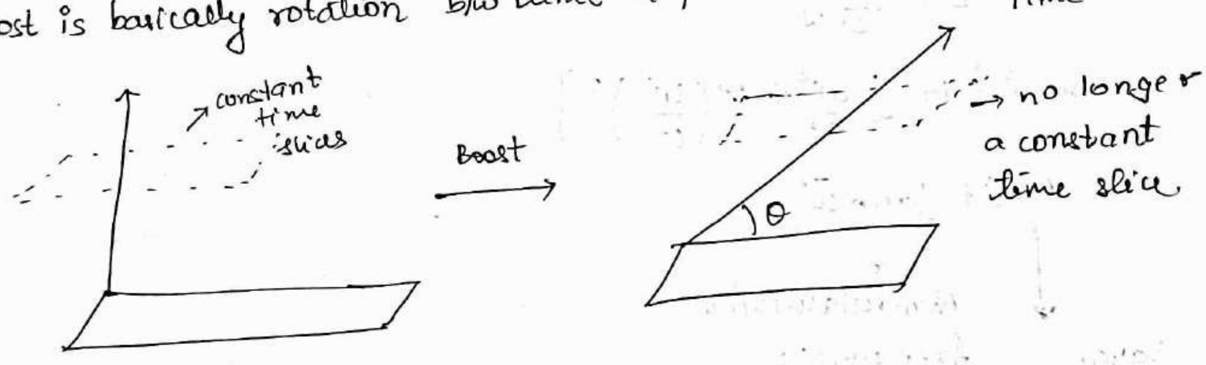
$\Rightarrow$ No Q spatial dependence.

$\Rightarrow$ Non-relativistic limit?
↳ Limit where ~~the~~ all the relevant velocities $<< c$

In some other coordinate transformation, $dt \to dt(t, \tilde{x})$. But still Q is spatial independent & hence $\frac{d}{dt} Q = 0$

But the concept of total space changes under boost

Boost is basically rotation b/w time & space.



On boost, Q changes as well due to "θ" angle b/w s-t.
So, the conservation equation $\left(\frac{dQ}{dt} = 0\right)$ is frame dependent as both Q and t are frame dependent
↳ covariant eq^n.

All conservation equations are of the form:

$$\partial_\mu J^\mu = 0$$

$$\Rightarrow Q = \int J^0 d^3\vec{x}$$
↳ constant time slice.

$$\partial_0 j^0 + \vec{\partial}\cdot\vec{j} = 0$$

$$\Rightarrow \partial_0 \underbrace{\int j^0 d^3x}_{Q} = -\int d^3x (\partial \cdot \vec{j})$$

$$= 0 \quad \left[\text{Assuming no boundaries}\right]$$

All conservation equations for at least extended particle must have the structure $\partial_\mu J^\mu = 0$

(Continued)

$p^{\mu} = m_0 \gamma [c, u^i]$

$p^0 \rightarrow$ energy

$[p^0] \rightarrow$ [momentum]

$c[p^0] =$ [energy]

we expect $c[p^0]$ will reduce to free particle energy in ~~rotation~~ limit

$\left(\frac{u^i}{c}\right) \ll 1$

$$cp^0 = m_0 c^2 \gamma$$

$$= m_0 c^2 \left[1 - \frac{u^{i^2}}{c^2}\right]^{-1/2}$$

$$= m_0 c^2 \left[1 + \frac{1}{2}\frac{u^{i^2}}{c^2} + O\left(\frac{u^{i^2}}{c^2}\right)^2\right]$$

$$= m_0 c^2 + \underbrace{\frac{1}{2} m_0 \bar{u}^2}$$

Large constant. Non-relativistic free particle energy.

$E = m_0 c^2 + \frac{1}{2} m_0 (\bar{u})^2 +$ higher order terms

Does not play any role in dynamics since it is a dynamic independent shift.

$E_0 = m_0 c^2 =$ rest energy $\rightarrow$ can be ignored in non-relativistic limits.

Moving energy [full relativistic formulae]

$E = \underbrace{m\gamma}\cdot c^2 = mc^2$

↳ moving mass from momentum analysis.

Thus; c is like a conversion factor from mass to energy

Doubt

m_{01} $\qquad$ m_2

$\downarrow$

$E^{mov} \Rightarrow cp^0 - m_{0_1}c^2$ $\qquad$ $E^{mov} \Rightarrow cp^0 - m_2c^2$.

$$S = k \int dt \sqrt{\left| \left(\frac{dx^{\mu}}{dt}\right)\left(\frac{d x^{\nu}_{a}}{dt}\right)\eta_{\mu\nu} \right|}$$

$$S_{non-rel} = \int (\text{const} + \tfrac{1}{2} m_0 u^2)\, dt$$

$S_{non-rel}$ = S expanded in $\left(\frac{u^2}{c^2}\right)$ taylor series, keeping upto 1st order non-trivial ~~term~~ U. dependent term.

From non-relativistic analysis, we know:

$$S_{non-rel} = \int \left(\tfrac{1}{2} m u^2\right) dt \quad + \text{constants}.$$

{HW}.

$(A \times B) \cdot (C \times D)$

$= (A \times B)_k (C \times D)_k$

$= \epsilon_{kij} A_i B_j \quad \epsilon_{kmn} C_m D_n$

$= \epsilon_{kij} \epsilon_{kmn} A_i B_j C_m D_n$

$= (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) \; A_i B_j C_m D_n$

$= (A_i C_i)(B_j D_j) - (A_i D_i)(B_j C_j)$

$= (A \cdot C)(B \cdot D) - (A \cdot D)(B \cdot C)$

14 October

Maxwell's eq^n in a manifestly Lorentz Transformation

Fixing of constances of the equations.

① $\vec{\partial}\cdot\vec{E} = k_1 \rho$ ② $\vec{\partial}\cdot\vec{B} = 0$.

③ $\vec{\partial}\times\vec{E} + k_2\left(\frac{\partial \vec{B}}{\partial t}\right) = 0$ ④ $\vec{\partial}\times\vec{B} = k_3\vec{J} + k_4\left(\frac{\partial \vec{E}}{\partial t}\right)$

⑤ continuity equation: $\boxed{\frac{\partial \rho}{\partial t} + \partial\cdot\vec{J} = 0}$ → without constants.

Taking the divergence of ④

$$\vec{\partial}\cdot(\vec{\partial}\times\vec{B}) = 0$$

or $k_3\vec{\partial}\cdot\vec{J} + k_4\frac{\partial}{\partial t}(\vec{\partial}\cdot\vec{E}) = 0$

or $k_3\vec{\partial}\cdot\vec{J} + k_4 k_1\frac{\partial \rho}{\partial t} = 0$

or $-k_3\frac{\partial \rho}{\partial t} + k_4 k_1 = 0$

or $\boxed{k_4 = \frac{k_3}{k_1}}$

From ④ $\partial\times\vec{B} = k_3\vec{J} + \frac{k_3}{k_1}\left(\frac{\partial\vec{E}}{\partial t}\right)$

Relation between k_3, k_1 and k_2 and c; c: velocity of light in vacuum

$\vec{J} = 0, \rho = 0$ | source free EM field

Taking $(\partial\times)$ of ③:

$(\vec{\partial}\times\vec{\partial}\times\vec{E})$

$= \vec{\partial}(\vec{\partial}\cdot\vec{E}) - \partial^2\vec{E}$

So, we have:

$$\vec{\partial}\times(\vec{\partial}\times\vec{E}) + k_2\frac{\partial}{\partial t}(\partial\times\vec{B}) = 0$$

or $\underbrace{\partial_i(\vec{\partial}\cdot\vec{E})}_{\to 0 \text{ due to } \rho=0} - \partial^2 E_i + \frac{k_2 k_3}{k_1}\frac{\partial^2 E_i}{\partial t^2} = 0$ substituting ④

or $-\partial^2 E_i + \frac{k_2 k_3}{k_1}\frac{\partial^2 E_i}{\partial t^2} = 0$

Let us have a trial solution:

$E_i = E_i(x - ct)$ → arguement

$\partial_x^2 E_i = E_i''(x-ct)$

& $\partial_t^2 E_i = c^2 E_i''(x-ct)$

$\partial_t^2 E_i - c^2 \partial_x^2 E_i = 0$

$\Rightarrow \boxed{c^2 = \frac{k_1}{k_2 k_3}}$

→ we onward to get "c" parameter

→ This arguement is done only if eqns have homogenous powers.

Thus, the equations become:

$$\vec{\partial}\cdot\vec{E} = k_1 \rho$$

$$\vec{\partial}\cdot\vec{B} = 0$$

$$\vec{\partial}\times\vec{B} + k_2 \frac{\partial \vec{B}}{\partial t} = 0$$

$$\vec{\partial}\times\vec{B} = \frac{1}{c^2 k_2}\left[k_1 \vec{J} + \frac{\partial \vec{E}}{\partial t}\right]$$

The variables are $\vec{E}, \vec{B}, \vec{J}$ and $\vec{\rho}$

we shall combine them in a way such that they look like tensor components in space-time dimensions.

<u>Continuity</u> <u>Equation</u>: $\frac{\partial \rho}{\partial t} + \partial\cdot\vec{J} = 0$

we can write as:

$$\frac{\partial(c\rho)}{\partial(ct)} + \partial\vec{J} = 0$$

Define $J^\mu = \{c\rho, \vec{J}\}$

Thus we have:

$$\partial_0 J^0 + \partial_i J^i = 0$$

$\Rightarrow \rho$ and $\vec{J}$ could be combined in 4-vector J^μ s.t continuity eqn is like conservation equation $\partial_\mu J^\mu = 0$

$$\left\{\begin{matrix} x^0 = ct \\ \vec{x} \end{matrix}\right\} = \{x^\mu\}$$

$\vec{E}$ and $\vec{B}$ shall mix under L·T because J, ρ mix. And from (3)

$\vec{B}$, $\vec{E}$ shall mix.

⇒ $(\vec{E}, \vec{B})$ together will form a lorentz covariant tensor with 6-independent component.

A natural guess is an anti-symmetric tensor in 4-Dimension called $F_{\mu\nu}$ such that $F_{\mu\nu} = -F_{\nu\mu}$ F: field strength tensor.

(1) We know that the homogenous maxwell equations are automatically solved once we express $\vec{E}$ and $\vec{B}$ in terms of scalar + vector potential forms $\vec{a}$.

$$\vec{\partial} \times \vec{E} + K_2 \frac{\partial \vec{B}}{\partial t} = 0 \quad ; \quad \vec{\partial} \cdot \vec{B} = 0$$

$$\vec{E} = -\vec{\partial}\phi + \beta \left(\frac{\partial \vec{a}}{\partial t}\right) \quad ; \quad \vec{B} = \alpha(\vec{\partial} \times \vec{a})$$

$$\beta \frac{\partial}{\partial t}(\vec{\partial} \times \vec{a}) + K_2 \alpha \frac{\partial}{\partial t}(\vec{\partial} \times \vec{a}) = 0 \quad \text{for every } \vec{a}$$

or $\boxed{\beta = -K_2 \alpha}$

Homogenous equations are solved as identity if $\vec{E} = -\vec{\partial}\phi - K_2 \alpha \frac{\partial \vec{a}}{\partial t}$

& $\vec{B} = \alpha(\vec{\partial} \times \vec{a})$

for any α.

It seems ϕ and $\vec{a}$ ~~form~~ could form a natural combination for a 4-vector

$$a_\mu = \begin{Bmatrix} \phi \\ \vec{a} \end{Bmatrix}$$

Homogenous eq^{n}s: $\vec\nabla\times\vec E + k_2\frac{\partial\vec\beta}{\partial t} = 0$, $\vec\nabla\cdot\vec B = 0$

If we express $\vec E$ and $\vec B$ in terms of scalar and vector potentials: these two eq^{n}s are identically solved

$$\vec E = -\vec\nabla\phi - \frac{k_2\alpha}{\beta}\left(\frac{\partial\vec a}{\partial t}\right) \qquad \vec\beta = \alpha\,\vec\nabla\times\vec a$$

$\left\{\begin{matrix}\phi\\ \vec a\end{matrix}\right\}$ seems a natural combination for 4-vector

choose α so that $(\alpha k_2)\sim\left(\frac{1}{c}\right)$

$$\ni \frac{\partial}{\partial t}\longrightarrow\frac{\partial}{\partial(ct)} = \frac{\partial}{\partial x^0}$$

$\alpha = \frac{1}{ck_2} \ni ck_2\vec\beta = (\vec\nabla\times\vec a)$

$$a_u = \left\{\begin{matrix}\phi\\ \vec a\end{matrix}\right\}$$

Define $F_{uv} = \partial_v a_u - \partial_u a_v$

Identify: $F_{i0} = \partial_0 a_i - \partial_i a_0$

F_{i0} with E_i
$$= \frac{\partial}{\partial(ct)}a_i - \partial_i\phi$$
$$= E_i$$

$F_{ij} = (\partial_i a_j - \partial_j a_i) \longrightarrow$ looks like component of curl.

Define $\vec B_k = \frac{1}{2(ck_2)}\epsilon_{kij}F_{ij}$

$$= \frac{1}{2(ck_2)}\epsilon_{kij}(\partial_i a_j - \partial_j a_i)$$

$$= \frac{1}{2(ck_2)}(\epsilon_{kij}\partial_i a_j - \epsilon_{kij}\partial_j a_i)$$

$$= \frac{1}{2ck_2}\Big(\epsilon_{kij}\partial_i a_j - \underbrace{\epsilon_{kji}\partial_i a_j}_{\text{renaming indices}}\Big)$$

$$= \frac{1}{2ck_2}(\epsilon_{kij}\partial_i a_j + \epsilon_{kij}\partial_i a_j) = \frac{1}{2ck_2}\epsilon_{kij}\partial_i a_j\times 2$$

$$= \frac{1}{(CK_2)} (\vec{\partial}\times\vec{a})_K$$

So, $B_K = \frac{1}{2CK_2} \epsilon_{Kij} F_{ij}$

$$\epsilon_{Kij} B_K = \frac{1}{2CK_2} \epsilon_{Kij} \epsilon_{Klm} F_{lm}$$

$$= \frac{1}{2CK_2}\left[\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right] F_{lm}$$

$$= \frac{1}{2CK_2} \delta_{il}\delta_{jm} F_{lm} - \frac{1}{2CK_2} \delta_{im}\delta_{jl} F_{lm}$$

$$= \frac{1}{2CK_2}\left[F_{ij} - F_{ji}\right]$$

$$= \frac{1}{2CK_2}\left[F_{ij} - (-F_{ij})\right]$$

$$= \frac{1}{CK_2} (2F_{ij})$$

$$= \frac{F_{ij}}{CK_2}$$

$\Rightarrow F_{ij} = (CK_2)\, \epsilon_{Kij} B_K$

Thus:

$\boxed{F_{i0} = E_i}$ & $\boxed{F_{ij} = CK_2 \epsilon_{Kij} B_K}$; $F_{0i} = -E_i$

Bianchi Identity :-

$\partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu} + \partial_\alpha F_{\mu\nu} = 0 \rightarrow$ we have a total of 4 equations | because μ, ν, α have to be independent. Thus only 4 indices we have. Thus, 4 equations

or ~~$\partial_\alpha(\partial_\alpha \partial_\sigma - \partial_\nu \partial$~~ $= \epsilon_{\mu\nu\alpha}$

or $\partial_\mu\left[\partial_\alpha a_\nu - \partial_\nu a_\alpha\right] + \partial_\nu\left[\partial_\mu a_\alpha - \partial_\alpha a_\mu\right] + \partial_\alpha\left[\partial_\nu a_\mu - \partial_\mu a_\nu\right] = 0$

For any (a_μ)

~~Easy to.~~

We have to identify Bianchi identity with the homogenous Maxwell equations.

Case 1: $\mu = i,\ \nu = j,\ \alpha = k$.

$0 = \partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij}$ → we shall get a scalar eqn:

$0 = \epsilon_{ijk}\left(\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij}\right)$

$= 3\,\epsilon_{ijk}(\partial_i F_{jk})$

$= (\epsilon_{ijk})(3K_2 c)(\partial_i \epsilon_{ijk} B_l)$

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{bmatrix}$$

$= 3cK_2\,(\epsilon_{ijk}\,\epsilon_{jkl})(\partial_i B_l)$

$= 3cK_2\; 2\,\delta_{ik}\,\partial_i B_l$

$= \cancel{3cK_2}\; 6\,cK_2\,(\vec{\partial}\cdot\vec{B})$

d dimensions:

(d-1) order tensor → vector of (d-1) order

Contract by d levi civita tensor

Case 2: $\mu = 0;\ \nu = i;\ \alpha = j$

$0 = \partial_0 F_{ij} + \partial_i F_{j0} + \partial_j F_{0i} = \epsilon_{0ij}$ → just a name.

$= \partial_0\left[(cK_2)\,\epsilon_{kij} B_k\right] + \partial_i E_j + (-\partial_i E_j)$

$= (cK_2)\,\partial_0 \epsilon_{kij} B_k + (\partial_i E_j - \partial_j E_i)$

$= \epsilon_{ijm}\left[(\partial_0 B_k)\,\epsilon_{kij}\,K_2 c\right] + \epsilon_{ijm}\left[\partial_i E_j - \partial_j E_i\right]$

$\epsilon_{ijm}\,\epsilon_{0ij} = 0$

$= (K_2 c)\,\epsilon_{kij}\,\epsilon_{ijm}\,(\partial_0 B_k) + \epsilon_{ijm}\,\partial_i E_j - \epsilon_{ijm}\,\partial_j E_i$

$= 2(K_2 c)\,\delta_{km}\,(\partial_0 B_k) + 2\,\epsilon_{ijm}\,\partial_i E_j$

$= 2\left\{(K_2 c)(\partial_0 B_m) + (\vec{\partial}\times\vec{E})_m\right\}$

$= 2\left\{K_2 c\,\dfrac{\partial B_m}{\partial(ct)} + (\vec{\partial}\times\vec{E})_m\right\}$

↳ our original equation

~~Electromagnetic eqn~~: 18 Oct

we had defined

$F_{\mu\nu} = \partial_\nu a_\mu - \partial_\mu a_\nu$

But then used $F_{ij} = \partial_i a_j - \partial_j a_i$

So, inorder to resolve this;

① $\vec{\nabla} \cdot \vec{B} = 0$

② $\vec{\nabla} \times \vec{E} + k_2 \left(\frac{\partial \vec{B}}{\partial t} \right) = 0$

↳ Relative sign matters

In the last class we had

$F_{i0} = \partial_0 a_i - \partial_i a_0 = E_i$

* But, we redefine $\vec{E}$ and $\vec{B}$ as:

$E_i = F_{0i} = \partial_i a_0 - \partial_0 a_i$

$F_{ij} = \partial_j a_i - \partial_i a_j$

$c k_2 \vec{B}_k = \frac{1}{2} \epsilon_{kij} F_{ij}$

Inhomogenous Equations

$\vec{\nabla}\cdot\vec{E} = k_1\rho$ $\qquad\qquad$ $\vec{\nabla}\times\vec{B} = \left(\frac{1}{c^2 k_2}\right)\left[k_1\vec{J} + \frac{\partial E}{\partial t}\right]$

$$\Rightarrow c^2 k_2(\vec{\nabla}\times\vec{B}) = \left(\frac{k_1}{c}\right)\vec{J} + \frac{\partial\vec{E}}{\partial(ct)}$$

$A_0 = \eta_{0\mu}A^{\mu}$
$= \eta_{00}A^{0}$
$= -A^{0}$

$$\partial_i E_i = \partial_i F_{0i} = +\partial_i F^{0i}$$

$$\left.\begin{aligned}\partial_i F^{i0} &= \left(\frac{k_1}{c}\right)c\rho \\ \partial_\mu F^{\mu 0} &= \left(\frac{k_1}{c}\right)(c\rho)\end{aligned}\right\} \quad \partial_\mu J^{\mu} = \partial_t\rho + \vec{\partial}\cdot\vec{J}$$

$$= \frac{\partial}{\partial(ct)}(c\rho) + \vec{\partial}\cdot\vec{J}$$

$$= \left(\frac{k_1}{c}\right)j^0$$

$$j^0 = c\rho \;,\; j^i = \vec{J}$$

$$\vec{\nabla}\cdot\vec{E} = k_1\rho \longrightarrow \partial_\mu F^{\mu 0}\Big|_{0\text{component}} = \left[\left(\frac{k_1}{c}\right)j^{\mu}\right]_{0\text{component}}$$

$$\partial_\mu F^{\mu 0} = \left(\frac{k_1}{c}\right)j^{\mu}$$

$$\Rightarrow \partial_\mu F^{\mu i} = \left(\frac{k_1}{c}\right)j^{i}$$

$$ck_2(\vec{\nabla}\times\vec{B}) - \frac{\partial\vec{E}}{\partial(ct)} = \left(\frac{k_1}{c}\right)j^{i}$$

$$\frac{\partial\vec{E}_i}{\partial(ct)} = \partial_0 F_{0i} = -\partial_0 F^{0i}$$

$$\underbrace{ck_2(\vec{\nabla}\times\vec{B})}_{} + \partial_0 F_{0i} = \left(\frac{k_1}{c}\right)j^{i}$$

↳ should be $\partial_j F^{ji}$

Checking:

$$ck_2(\vec{\nabla}\times\vec{B}) = \epsilon_{ijk}\partial_j(ck_2 B_k)$$

$$= \frac{1}{2}\epsilon_{ijk}\epsilon_{klm}\partial_j F_{lm}$$

$$= \frac{1}{2}(\epsilon_{ijk}\epsilon_{klm})\partial_j F_{lm}$$

$$= \frac{1}{2}\left[\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right]\partial_i \cancel{\delta} F_{lm}$$

$$= \frac{1}{2}\left[\cancel{F}\,\partial_j F_{ij} - \partial_j F_{ji}\right]$$

$$= \left(\frac{1}{2}\times 2\right) + \left(\partial_j F_{ij}\right)$$

$$= -\,\partial_j F_{ji}$$

$$\Big\}\quad \partial_0 F^{0i} \ominus \partial_j F^{ji} = \left(\frac{k_1}{c}\right) j^i$$

$\Rightarrow$ we get the eqn.

$$\partial_0 F^{0i} \ominus \partial_j F^{ji} = \left(\frac{k_1}{c}\right) j^i$$

$\hookrightarrow$ $\vec{B}$ should be defined appropriately so that this sign is $\pm$

Exercise: Correct everything ☺

The equations (which work out), we get:

$$\left.\begin{aligned} \partial_\mu F^{\mu\nu} &= \left(\frac{k_1}{c}\right) j^{\nu} \\ F_{\mu\nu} &= \partial_\nu a_\mu - \partial_\mu a_\nu \end{aligned}\right\}$$

All contents of maxwell's equations

Here we have 4 unknowns + 4 equations
$4\{a_\mu\}$

we had a total of

$$\vec{\nabla}\cdot\vec{B} = 0\ ;\quad \vec{\nabla}\cdot\vec{E} = k_1\rho\ ;\quad \frac{\partial B}{\partial t} + \vec{\nabla}\wedge\vec{E} = 0\ ;\quad ck_2(\vec{\nabla}\wedge\vec{B}) = \left(\frac{k_1}{c}\right)J + \frac{1}{c}\left(\frac{\partial E}{\partial t}\right)$$

(1) (1) (3) (3)

we have 8 equations, but only 6 variables; 3 for $\vec{E}$ and 3 for $\vec{B}$

$\Rightarrow$

suppose we have a soln:

$$a_\mu = a_\mu^{sol}(x)$$

Define a new a_μ

$$a_\mu(new) = a_\mu^{(sol)}(x) + \partial_\mu \Lambda$$

$\hookrightarrow$ any function

$$\Bigg\}\ \partial_\mu a_\nu^{(new)} - \partial_\nu a_\mu^{(new)} = \partial_\nu a_\mu^{sol} - \partial_\mu a_\nu^{sol} - (\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)\Lambda$$

→ Fluid dynamics: Description of near eq^{l} dynamics

→ Unknowns/basic variables : Conserved quantities or corresponding chemical potential.

→ Basic eq^n : conservation eq^{n}s: $\frac{\partial}{\partial t}\begin{pmatrix}\text{total}\\ \text{charge}\end{pmatrix} = 0$

local version → same form of continuity eq^n ⇒ $\partial_0 j^0 + \partial_i j^i = 0$

$$\Rightarrow \partial_\mu j^\mu = 0$$

→ Continuity equations are consequences of continuous symmetries present in system.

→ A global translation: shift of origin of coordinates must be a continuous symmetry in all physical systems

→ All systems must have continuity eq^{n}s corresponding to this symmetry.

→ The symmetry transformation is generated by $x^\mu \to x^\mu + a^\mu$ ↳ constant 4 vector.

Thus, we shall get 4 continuity equations, one corresponding to each component.

The continuity equation will have the structure :

$$[\partial_\mu T^{\mu\nu}] = 0$$

↳ 'ν' tells which component is used to get the conservation eq^n.

Case 1 $\nu = 0$: continuity eqⁿ corresponding to time translation

$$\partial_0 T^{00} + \partial_i T^{0i} = 0$$

$$\Rightarrow \underbrace{\int d^3\bar{x}\, T^{00}}_{E} + \int d^3\bar{x}\, \partial_i T^{\cancel{0i}\, i0} = \text{constant}$$

$\Rightarrow E =$ constant

$\int d^3\bar{x}\,\partial_i T^{i0} \to 0$ because of boundary terms expands & tends to 0.

↳ boundaries are ∞ of space & we just ignore as we think we do not take this.

conserved charged corresponding to time translation symmetry. $E = \int d^3\bar{x}\, T^{00}$

$\Rightarrow T^{00}(x) \sim$ energy density $= \epsilon(x)$

Case 2 $\nu = i$

$$\partial_0 T^{0i} + \partial_j T^{ji} = 0$$

$$\text{or } \int d^3\bar{x}\, T^{0i} + \int d^3\bar{x}\, \partial_j T^{ji} = 0$$

$\int d^3\bar{x}\,\partial_j T^{ji} \to 0$, ↳ boundary term

or $\int d^3\bar{x}\, T^{0i} = p^i$ = conserved Quantity corresponding to spatial translation along x^i

Eqⁿ for an uncharged fluid

$$\partial_\mu T^{\mu\nu} = 0$$

(Variables of fluid dynamics)

$T^{\mu\nu}$ (Stress tensore) is defined through conserved quantities equation. It is not necessary $T^{\mu\nu}$ is symmetric when we solve it from differential equation. We may add identically "0" terms to make it symmetric.

Here, $T_{\mu\nu}$ is defined through a conservation eq^n

→ $T_{\mu\nu}$ is ambiguous upto addition of terms that are identically vanishing, once ∂_μ is applied to it.

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} + \partial_\alpha x^{[\alpha\mu]\nu} \qquad \text{s.t. } x^{\alpha\mu\nu} = -x^{\mu\alpha\nu}$$

$$\partial_\mu \tilde{T}^{\mu\nu} = \partial_\mu T^{\mu\nu} + \underbrace{\underbrace{\partial_\mu \partial_\alpha}_{\text{symmetric}} \underbrace{x^{[\alpha\mu]\nu}}_{\text{anti symmetric}}}_{\text{will identically vanish}}$$

Such terms, ($x^{\alpha\mu\nu}$), are called improvement terms.

→ One can add this improvement term to make $T^{\mu\nu}$ symmetric.

We will assume $T^{\mu\nu}$ to be symmetric

Variables of Fluid Dynamics

$T^{\mu\nu}$ (symmetric) has 10 components in 4-D $\partial_\mu T^{\mu\nu}$

$\partial_\mu T^{\mu\nu} = 0 \Rightarrow$ A set of 4 eq^n

So, we have to write it in terms of 4 fluid variables, so that this eq^n is enough to determine FD completely.

→ FD is near eq^l physics
equilibrium is characterized by (in its rest frame) energy density or temperature (T) } eq^n of state (expresses all thermodynamic quantity in terms of single variable)

$e(T), e(p), e(s)$ etc.
entropy (s)

→ Outside rest frame
↳ Global velocity $\vec{v}$

Equation of state varies from fluid to fluid.

In arbitrary frame, equillibrium is characterized by

① $e(T)$ and ② $\vec{v}$ → All are constant over space & time.

Relativistic fluid dynamics, choose units such that $\boxed{c=1}$

Uncharged fluid in eq^l is characterized by uniform velocity $\vec{v}$, uniform temperature T

According to our defination, T is a scalar.

$\vec{v}$ has to be covariantized into a 4-velocity (some Lorentz vector)

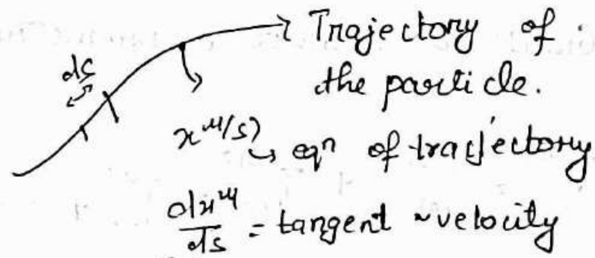
$$\vec{V} = \frac{dx^i}{dt}$$

Construct a '4-vector' out of $\vec{V}$, ~~that~~ whose spatial components should reduce to $\vec{v}$ in non-relativistic limit. ~~or~~ $c \to \infty$.

$$u^\mu = \left(\frac{dx^\mu}{ds}\right) \equiv \text{proper velocity}$$

$\to$ 4 vector

ds ↳ line element (scalar by construction)



Any point can be uniquely defined from a point

$$ds = \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu}$$

"$-\eta$" as time-like trajectory

$$u^\mu = \frac{dx^\mu}{ds}$$

$$= \frac{dx^\mu/dt}{ds/dt} \equiv \frac{\{1, v^i\}}{\sqrt{1-v^2}}$$

$$\frac{ds}{dt} = \frac{1}{dt}\sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu}$$

$$= \sqrt{-\eta_{\mu\nu}\left(\frac{dx^\mu}{dt}\right)\left(\frac{dx^\nu}{dt}\right)}$$

$$\frac{dx^0}{dt} = 1 \; ; \; \frac{dx^i}{dt} = v^i$$

$$-\eta_{\mu\nu}\left(\frac{dx^\mu}{dt}\right)\left(\frac{dx^\nu}{dt}\right) = \left(\frac{dx}{dt}\right)^2 - \sum_{i=1}^{3}\left(\frac{dx^i}{dt}\right)\left(\frac{dx^i}{dt}\right) = 1 - \vec{v}^2$$

u^μ is unit normalized

$$u^\mu u^\nu \eta_{\mu\nu} = \frac{dx^\mu dx^\nu \eta_{\mu\nu}}{ds^2}$$

$$= \frac{dx^\mu dx^\nu \eta_{\mu\nu}}{-(dx^\mu dx^\nu \eta_{\mu\nu})} = -1$$

Thus; $\boxed{u^\mu u_\mu = -1}$

Goal: To construct a symmetric $T^{\mu\nu}$ out of $\{u^\mu, T, \partial_\mu\}$

$$T^{\mu\nu} = T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)} + T^{\mu\nu}_{(2)} + \cdots$$ → derivative expansion

$T^{\mu\nu}_{(0)}$ = ideal fluid stress tensor with zero derivatives

$T^{\mu\nu}_{(1)}$ = " with 1 derivative

⋮

$T^{\mu\nu}_{(n)}$: has n derivatives acting on u and T

$$T_0^{\mu\nu} = A(T)\, u^\mu u^\nu + B(T)\, \eta^{\mu\nu}$$

Can we constrain $A(T)$ and $B(T)$:

→ $T^{\mu\nu}_{(0)}$ is the stress tensor for fluid in equillibrium.

So, $A(T)$, $B(T)$ must be some thermodynamic variables.

$\int T_{(0)}^{00}\, d^3x$ = Total energy in ~~equilibrium~~ for rest frame

In rest frame, $\vec{V} = 0$

$$u^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$T_{(0)}^{00} = A(u^0)^2 + B\eta^{00}$$
$$= A(T) - B(T)$$

So, $A(T) - B(T) = e(T) \longrightarrow$ energy density

T_0^{ij} : related to pressure of fluid.

Matching with thermodynamics ~~of~~ one can further see that $B(T) \simeq$ Pressure.

Thus: we have : Matching can be done only for $T_0^{\mu\nu}$

$$T_{(0)}^{\mu\nu} = (e+p)u^\mu u^\nu + p\eta^{\mu\nu}$$

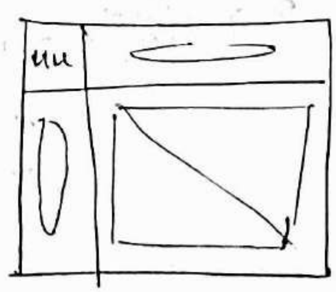
$e(T)$ or $p(T)$ or $e(p)$ are fluid specific functions and are called equation of states & have to be provided.

Constructing $T_{(1)}^{\mu\nu}$ (General structure)

$$T_{(1)}^{\mu\nu} = \overset{\nearrow S_{(1)}}{\mathrm{Scalar}_{(1)}}\, u^\mu u^\nu + \overset{\nearrow (S_2)}{\mathrm{Scalar}_{(2)}}\, \eta^{\mu\nu} + (V^\mu u^\nu + u^\mu V^\nu) \left\{ \text{s.t } V^\mu u_\mu = 0 \right.$$
$$+ \gamma^{\mu\nu} \left(\text{s.t } u_\mu \gamma^{\mu\nu} = 0\right), \left. tr(\gamma^{\mu\nu}) = 0 \right\}$$
$$\text{i.e } \gamma^{\mu\nu}\eta_{\mu\nu} = 0$$

$S_{(1)}$, $S_{(2)}$ are scalars.

Possible structures $(u^\mu \partial_\mu)T$, $(\partial_\mu u^\mu)$



V^μ is a vector $\perp^r$ to u^μ.

$(u^\nu \partial_\nu) u^\mu \qquad (\eta^{\mu\nu}\partial_\nu) T \rightarrow$ 2 possible candidates

$$u^\mu (u^\nu \partial_\nu) u_\mu = \frac{1}{2}(u^\nu\partial_\nu)(\underbrace{u^\mu u_\mu}_{-1})$$

$$= \frac{1}{2}(u^\nu \partial_\nu)(-1)$$

$$= 0$$

$(u_\mu \eta^{\mu\nu}\partial_\nu) T \qquad \{ V^\mu$ is a vector $V^\mu u_\mu = V_\mu u^\mu = 0 \}$

$$= (u^\nu \partial_\nu) T \neq 0$$

~~$\partial_\nu T$~~

Define

$$\bar{\partial}_\mu T = \partial_\mu T + u_\mu (u^\nu \partial_\nu) T$$

$$u^\mu \bar{\partial}_\mu T = u^\mu \partial_\mu T + (\underbrace{u^\mu u_\mu}_{-1})(u^\nu \partial_\nu T)$$

$$= u^\mu \partial_\mu T - u^\nu \partial_\nu T = 0$$

$$\bar{\partial}_\mu T = \left(\delta_\mu^{\nu} + u^\nu u_\mu\right)(\partial_\nu T)$$

$P_\mu^\nu \;\rightarrow$ projector in direction $\perp^r$ to u^μ.

$$\gamma^{(1)}_{\mu\nu} = (\partial_\mu u_\nu + \partial_\nu u_\mu)/2$$

$$\gamma^{(1)}_{\mu\nu} \eta^{\mu\nu} = \partial \cdot u$$

$$\gamma^{(2)}_{\mu\nu} = \text{Traceless version of } \gamma^{(1)}_{\mu\nu}$$

$$= \frac{(\partial_\mu u_\nu + \partial_\nu u_\mu)}{2} - \left(\frac{\eta^{\mu\nu}}{4}\right)(\partial \cdot u)$$

$$\eta^{\mu\nu} \gamma^{\mu\nu}_{\mu\nu} = 0$$

Projecting with both indices ⊥ to u^μ:

$$P^{\mu}{}_{\alpha} P^{\nu}{}_{\beta} \gamma^{(2)} = \gamma_{\alpha\beta}$$ check whether all conditions are satisfied

$$T^{\mu\nu}_{(1)} = [a_1(T)(\partial \cdot u) + b_1(T)\, u(\partial T)]\, u^\mu u^\nu$$

$$+ [a_2(T)(\partial \cdot u) + b_2(T)\, u \cdot (\partial T)]\, \eta^{\mu\nu}$$

$$+ \left\{ [c_1(T)(u \cdot \partial) u^\mu + c_2(T)\, P^{\mu\alpha}(\partial_\alpha T)]\, u^\nu \right\}_{\mu \leftrightarrow \nu}$$

$$+ c(T)\, T^{\mu\nu}_{(2)}$$

$a_1, a_2, \ldots c_1, c_2 \ldots$
$b_1, b_2 \ldots$

$a_i(T)$, $b_i(T)$, $c_i(T)$ are called transport coefficients & are measured through experiments.

We can reduce 7 eqns to 2 eqns. using change in temp, velocity & symmetries. These 2 constants would be bulk & shear modulus

27 October

Relativistic fluid dynamics (uncharged)

Ideal fluid → no viscosity

Eqⁿ : $\partial_\mu T^{\mu\nu} = 0$

Variables: u^μ satisfying $u^\mu u_\mu = -1$
T temperature

Fluid regime: $u^\mu(\{x^\nu\})$ $T(\{x^\nu\})$
But these are slowly varying function compared to some microscopic length scale λ

$T \gg \lambda \partial T \gg \lambda^2 \partial^2 T$

$u \gg \lambda \partial u \gg \lambda^2 \partial^2 u$

$T^{\mu\nu} = T_{(0)}^{\mu\nu} + T_{(1)}^{\mu\nu}$

zero derivative on fluid variables

1 derivative on fluid variables.

Goal: To get /construct $T_{(0)}^{\mu\nu}$, $T_{(1)}^{\mu\nu}$ from (1) symmetry and (2) thermodynamic equilibrium.

$T_0^{\mu\nu}$ is symmetric by our choice.

u^μ: velocity of fluid.

(1) $T^{\mu\nu}_{(0)} \rightarrow$ A symmetric tensor constructed out of u^μ and $\eta^{\mu\nu}$

The 2 structures possible :- $u^\mu u^\nu$ and $\eta^{\mu\nu}$ and T

$$T_0^{\mu\nu} = A(T)\, u^\mu u^\nu + B(T)\, \eta^{\mu\nu} \quad \text{—(1)}$$

$T_0^{\mu\nu} \rightarrow$ Ideal fluid stress tensor

$\hookrightarrow$ non-vanishing at equilibrium as others have derivatives but this has no derivative, ∴ for fluid in eq^l in 'flat surface'.

u^μ and $\eta^{\mu\nu}$ are only 2 available symmetric tensors.

Thus, we get eq (1)

Uniformly rotating fluids have a pressure which is caused due to temperature gradient and it balances the centrifugal force.

On non-flat surface, $T_0^{\mu\nu}, T_1^{\mu\nu}$ etc can contribute

Detour: Eq^l fluid in rest frame

- $e(T) \rightarrow$ energy density
- Go to frame with constant velocity $\vec{v}$
- From an arbitrary frame eq^l fluid has 2 variables (e, T, p), $\vec{v}$

~~In other case, fluid is slowly~~

we go to a frame & find the eq of states near eq^l.

In the rest frame; $u^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

energy density $\sim T^{00}$

$$= A(T)\,(u^0)^2 + B(T)\,\eta^{00}$$

$$= [A(T) - B(T)]$$

For eq^l fluid : $B(T) \equiv$ pressure ; using partition function in arbitrary basis, vary it using space-time tensor and then compare. It can be found that

$B(T) \equiv$ pressure.

⑥ (II) For $T^{\mu\nu}_{(1)}$:

we have to construct symmetric tensors out of u^μ, T, exactly 1 derivative.

$T^{\mu\nu}_{(1)} =$ u^μ is a special tensor. $T^{\mu\nu}_{(1)}$ shall be in u^μ direction or $\perp^r$ to it.

So, define a projector that project any vector index ~~pe~~ perpendicular to u_μ. (let be $P^{\mu\nu}$)

So, $P^{\mu\nu} u_\mu = 0$

(V_μ be any vector).

$V_\mu = V_\mu \cdot u_\mu + \bar{V}_\mu \rightarrow \perp^r$ to u^μ i.e. $V_\mu u^\mu = 0$

$P^{\mu\nu} V_\mu = 0 + P^{\mu\nu} \bar{V}_\mu = \bar{V}_\nu$

So, $\boxed{\bar{V}_\nu = P^{\mu\nu} V_\mu} \rightarrow$ construction of a projector that gives $\bar{V}_\nu$ $\perp^r$ to u^μ.

Idea of projector.

$P^{\mu\nu} = \eta^{\mu\nu} + u^\mu u^\nu$; $P^{\mu\nu} u_\nu = \eta^{\mu\nu} u_\nu + u^\mu u^\nu u_\nu$

$= u^\mu + u^\mu(-1)$

$= 0$ → Thus checked that it satisfies projection.

$P^{\mu\nu}$ ~~to~~ ~~non~~-derivative is 0 and gives identity in $\perp^r$ direction.

$T^{\mu\nu}_{(1)} = S_{(1)} u^\mu u^\nu + S_{(2)} P^{\mu\nu} + (V^\mu u^\nu + V^\nu u^\mu) \rightarrow$ with condition that $V^\mu u_\mu = 0$

$+ T^{\mu\nu}$ with $u_\mu \otimes T^{\mu\nu} = 0$

2 index in u direction, 2 index in $\perp^r$ direction, $1\perp^r$ 1 in u direction

$S_{(1)} u^\mu u^\nu + S_2 P^{\mu\nu}$ +

$= [S_{(1)} + S_{(2)}] u^\mu u^\nu + S_{(2)} \eta^{\mu\nu}$

28 Oct

Relativistic fluid dynamics

Constructing most general $T_{(1)}^{\mu\nu}$.

$T_{(1)}^{\mu\nu}$ has (u^μ, T and exactly 1 ∂_μ) $\hookrightarrow$ derivative

$$T_{(1)}^{\mu\nu} = S_1 u^\mu u^\nu + S_2 P^{\mu\nu} + (V^\mu u^\nu + V^\nu u^\mu) + \tau^{\mu\nu}$$

with condition: $u_\mu V^\mu = 0$

$u_\mu \tau^{\mu\nu} = 0$

$P^{\mu\nu} = u^\mu u^\nu + \eta^{\mu\nu} \equiv$ projector $\perp^r$ to u^μ.

$P^{\mu\nu} \tau_{\mu\nu} = 0$

uu	uv
vu	$P_{ij} + \tau_{ij}$

$S_{(1)}$ and $S_{(2)}$ are scalars constructed out of U, T, ∂:

→ ∂ operator is a vector operator

→ $S_{(1,2)}$ has exactly 1 ∂ operator

→ Free index in ∂_μ operator must be contracted with some upper index to make $S_{(1,2)}$ scalar

→ Only such available vector ~~is~~ for constraction with ∂_μ is u^μ.

$$(u^\mu \partial_\mu) T \quad , \quad (\partial_\mu u^\mu)$$

$$S = a(T)(u\cdot\partial)T + b(T)(\partial\cdot u)$$

$$S_1 = a_1(T)(u\cdot\partial)T + b_1(T)(\partial\cdot u)$$

$$S_2 = a_2(T)(u\cdot\partial)T + b_2(T)(\partial\cdot u)$$

Most general form of V^μ:

<u>Strategy</u>: a) construct vectors out of ∂_μ, u, T

b) Project them $\perp^r$ to u^μ using $P^{\mu\nu}$

We have 2 vectors: ∂_μ, ($u_\mu = \eta_{\mu\nu} u^\nu$)

If we want free index of vector ~~is~~ being generated from ∂_μ, then ∂_μ must act on a scalar $\to \partial_\mu T$

$Q_\mu \equiv (\partial_\mu T) \to$ 1 candidate for Q_μ.

→ The free index Q_μ is related to u_μ.

$Q_\mu = (u^\nu \partial_\nu) u_\mu$

$Q_\mu = q_1(T)(\partial_\mu T) + q_2(T)(u^\nu \partial_\nu) u_\mu$

$$V^\mu = P^{\mu\nu} Q_\nu$$

$$= q_1(T)\left[P^{\mu\nu} \partial_\nu T\right] + q_2(T)\left[P^{\mu\nu}(u^\nu \partial_\nu) u_\mu\right]$$

$u^\nu (u\cdot\partial) u_\nu$

$= \frac{1}{2}(u\cdot\partial)(u^\nu u_\nu)$

$= \frac{1}{2}(u\cdot\partial)(-1)$

$= 0$

$u^\nu \partial_\alpha u_\nu = K_\alpha$

$= u^\nu \partial_\alpha u^\mu \eta_{\mu\nu}$

$\partial_\alpha [u^\mu u^\nu \eta_{\mu\nu}]$

$= u^\nu \partial_\alpha u^\mu \eta_{\mu\nu} + u^\mu \partial_\alpha \eta_{\mu\nu}$

$P^{\mu\nu}(u\cdot\partial) u_\nu$

$= \eta^{\mu\nu}(u\cdot\partial) u_\nu + u^\mu u^\nu \cancel{(u\cdot\partial) u_\nu}^{\,0}$

$= (u\cdot\partial) u^\mu$.

$\partial_\alpha u^\mu \equiv \perp^r$ to u^μ

$V^\mu = q_1(T)$

$$V^{\mu} = q_1(T)\, P^{\mu\nu}(\partial_\nu T) + q_2(T)\,(u\cdot\partial)\, u^{\mu}$$

◉ <u>Construction of $T_{\mu\nu}$</u>

- Strategy: 1) Construct a symmetric tensor out of ∂, u, T
  2) Project both indices $\perp^r$ to u using $P^{\mu\nu}$
  3) Subtract off the trace

<u>step 1</u>

$$Q^{(1)}_{\mu\nu} = \frac{(u_\nu \partial_\mu T + u_\mu \partial_\nu T)}{2} +$$

$$Q^{(2)}_{\mu\nu} = \frac{(\partial_\mu u_\nu + \partial_\nu u_\mu)}{2}$$

<u>Step 2</u>

$$\tilde{Q}_{\mu\nu} = P^{\alpha}_{\mu} P^{\beta}_{\nu}\, Q_{\alpha\beta}$$

$$\tilde{Q}^{(1)}_{\mu\nu} = P^{\alpha}_{\mu} P^{\beta}_{\nu}\, Q^{(1)}_{\mu\nu}$$

$$= (\delta^{\nu}_{\mu} + u^{\alpha} u_{\mu})(\delta^{\beta}_{\nu} + u^{\beta} u_{\nu})(u_\alpha \partial_\beta T + u_\beta \partial_\alpha T)$$

$$= 0$$

Relativistic fluid dynamics 31 October

$T^{\mu\nu} = T^{\mu\nu}_{(0)} + T^{(\mu\nu)}_{(1)} + \cdots$

$T_{(0)}^{\mu\nu} = (e+p)u^\mu u^\nu + p\eta^{\mu\nu}$ e: energy density

p: pressure

$T_{(1)}^{\mu\nu} = s_1 u^\mu u^\nu + s_2 p^{\mu\nu} + (v^\mu u^\nu + v^\nu u^\mu) + \tau^{\mu\nu}$

s.t $\tau^{\mu\nu}$ $p^{\mu\nu} \cdot \tau_{\mu\nu} = 0$

$p^{\mu\nu} = \eta^{\mu\nu} + u^\mu u^\nu$ $\quad V^\mu u_\mu = 0$ $\quad \tau^{\mu\nu} u_\mu = 0$

$S_{(1)} = a_1 (u\cdot\partial T) + b_1 (\partial u)$

$S_{(2)} = a_2 (u\cdot\partial T) + b_2 (\partial\cdot u)$

$V^\mu = q_1 p^{\mu\nu}(\partial_\nu T) + q_2 (u\cdot\partial) u^\mu$

Step 2 $\tau^{\mu\nu}_{(1)}$ ① $u^\mu(\partial^\nu T) + u^\nu(\partial^\mu T)$

② $u^\mu (u\cdot\partial)u^\nu + u^\nu (u\cdot\partial)u^\mu$

③ $(\partial^\mu u^\nu) + (\partial^\nu u^\mu)$

$\tau_{(1)}$ does not satisfy $u_\mu \tau^{\mu\nu}_{(1)} = 0$

Project both indices of $\tau^{\mu\nu}_{(1)}$ to the space $\perp^r$ to u_μ

① & ② vanishes for projection & only ③ survives as $p^{\mu\nu}u_\nu = 0$

So; $\tau^{\mu\nu}_{(2)} = p^{\mu\alpha} p^{\nu\beta} \left(\partial_\alpha u_\beta + \partial_\beta u_\alpha\right)$

step 3 Subtract off the trace to make $\tau^{\mu\nu}_{(2)}$ traceless.

$\tau = p_{\mu\nu} [\tau_{(2)}]^{\mu\nu}$ → gives trace because $P_{\mu\nu} \equiv \delta_{ij} A_{ij}$

$= P_{\mu\nu} p^{\mu\alpha} p^{\nu\beta} \left(\partial_\alpha u_\beta + \partial_\beta u_\alpha\right)$

$= (\eta_{\mu\nu} + u_\mu u_\nu)(\eta^{\mu\alpha} + u^\mu u^\alpha)$

$= \delta_\nu^\alpha + u_\nu u^\alpha = p_\nu^\alpha$

$$\gamma = P^{\alpha}_{u} P^{\nu\beta} (\partial_\alpha u_\beta + \partial_\beta u_\alpha)$$

$$= P^{\alpha\beta} (\partial_\alpha u_\beta + \partial_\beta u_\alpha)$$

$$= (\eta^{\alpha\beta} + u^\alpha u^\beta)(\partial_\alpha u_\beta + \partial_\beta u_\alpha)$$

$$= 2(\partial \cdot u)$$

To make γ traceless, we have:

$$\gamma_{\mu\nu} = P^{\mu\alpha} P^{\nu\beta} [\partial_\alpha u_\beta - \partial_\beta u_\alpha - 2\underset{\text{arbitrary constant}}{k} (\partial \cdot u) \eta_{\alpha\beta}]$$

$$P_{\mu\nu} \gamma^{\mu\nu} = 0$$

or $P_{\mu\nu} P^{\mu\alpha} P^{\nu\beta} [\partial_\alpha u_\beta \ \partial_\beta u_\alpha] - 2k(\partial\cdot u)(P_{\mu\nu} P^{\mu\alpha} P^{\nu\beta} \eta_{\alpha\beta})$

$$= 2(\partial\cdot u) - 2k(\partial\cdot u)(\underbrace{P^{\alpha\beta}\eta_{\alpha\beta}}_{3})$$

$$P^{\alpha\beta}\eta_{\alpha\beta} = (\eta^{\alpha\beta} + u^\alpha u^\beta)\eta_{\alpha\beta}$$

$$= 4 - 1 = 3$$

$$\Rightarrow \boxed{k = \frac{1}{3}}$$

After subtracting off the trace appropriately

$$\gamma^{\mu\nu} = \eta_{(T)} P^{\mu\alpha} P^{\nu\beta} [\partial_\alpha u_\beta + \partial_\beta u_\alpha - \tfrac{2}{3}\eta_{\alpha\beta}(\partial\cdot u)] \longrightarrow \text{shear of fluid}$$

$$= \underset{\text{viscosity}}{\eta_{(T)}} \sigma^{\mu\nu} : \text{(shear Tensor)}$$

$$T^{\mu\nu}_{(1)} = [a_1(\partial\cdot u) + b_1(u\partial T)]u^\mu u^\nu + P^{\mu\nu}[a_2(\partial\cdot u) + b_2(u\partial T)]$$

$$+ [q_1(T) P^{\mu\alpha}(\partial_\alpha T) + q_2(T)(u\cdot\partial)u^\mu]u^\nu + \mu \leftrightarrow \nu$$

$$+ \eta(T)\sigma^{\mu\nu}$$

At first order, T and u^{μ} are not uniquely defined.

Suppose one claims to have some def^n of $T(x)$, $u^{\mu}(x)$ and another person redefines it $\tilde{u}^{\mu}(x) = \tilde{u}^{\mu}(x) + \delta u^{\mu}(x)$

or $\tilde{T}(x) = T(x) + \delta T(x)$ $\longrightarrow$ terms containing atleast 1 derivative of u^{μ}, T

Representation of a ~~loore~~ lorentz group

In any ~~theory~~ 'physical theory:

① set of dynamical variables which will evolve according to the equations generated out of theory.

② some 'non-dynamical' variables that might also evolve but by some external source and not governed by the theory.

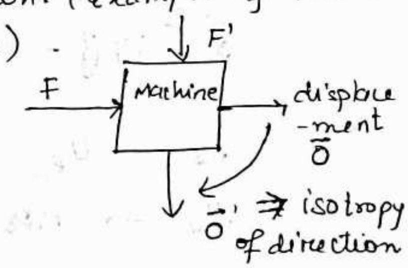
The measurements are done and ~~then~~ then checked whether equations are right. Thus, we can observe and make restrictions to some variables & invariance

③ Dynamical variables and external sources are identified using "our physical intuition".

(i) Look for symmetries in the system

~~(ii)~~ · Transformations on variables (both dy and non-dy) such that if we know the result of some experiment ~~once~~, we can predict what the result should be after transformation.

} System shall transform in a way that shall be known. (Example of box & force).



(ii) Theory has variables which transform under the 'symmetry transformation' being considered → but the transformations are linear → assumption cannot occur

Earlier, we thought transformation ~~can~~ ~~force~~ But, we allowed transformations but only of linear ones.

→ Variables could be grouped in vector spaces (finite or infinite dimensional)

↳ $\left\{\begin{matrix} v_1 \\ v_2 \\ v_3 \end{matrix}\right\} : \mathcal{V}$

→ Transformations would act as linear operators on the vector space of variables

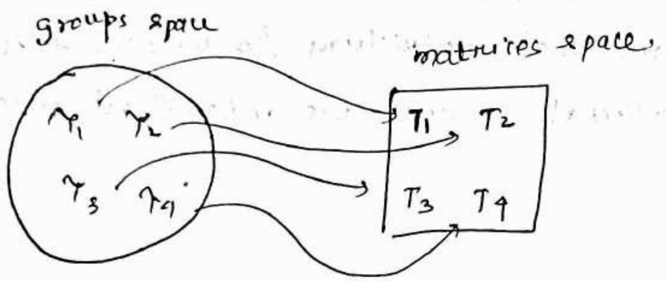
$T_i \mathcal{V} \Rightarrow \quad v_i \to v_i' = T_{ij} v_j$

$$\gamma^{(1)}\gamma^{(2)} v_i = T\, T^{(2)}_{i'j} V_j$$
$$= T^{(1)}_{ki'} \left(T^{(2)}_{i'j} V_j\right)$$
$$= \tilde{V}_k$$
$$\longrightarrow \gamma^{(3)}, v = T^{(3)}_{ki'} v_i$$

A map from the Transformations to $(n \times n)$ matrices so that

$\gamma^{(1)} \longrightarrow T^{(1)}_{ij}$ $\quad \gamma^{(2)} \longrightarrow T^{(2)}_{ij}$ $\quad \gamma^{(3)} = \gamma^{(2)}_{i'j}\gamma^{(1)}_{jk} \longrightarrow \left[T^{(2)}_{i} \; T^{(1)}\right]_{ik}$

simple matrix multiplication



Example: space of wavefunctions in Q.M. $\{|\psi\rangle\}$

We apply a rotation on system:

$$\overset{\text{rotations}}{R}\{|\psi\rangle\} \longrightarrow U_R\{|\psi\rangle\}$$

where U_R is set of (2×2) unit matrices

$|\psi\rangle$ is the space of spin-$\frac{1}{2}$ particle

$$\left.\begin{array}{l} R_1 \longrightarrow U_{R_1} \\ R_2 \longrightarrow U_{R_2} \end{array}\right\} R_1 R_2 \longrightarrow U_{R_1} U_{R_2}$$

U_R can have any dimension when rotation is just 3-D.

"Product" is important rule.

Doubt session

$$S = m_0 \int d\lambda \sqrt{\left(\frac{dx^\mu}{d\lambda}\right)\left(\frac{dx^\nu}{d\lambda}\right) - \eta_{\mu\nu}}$$

what would be the action when in the limit $m_0 \to 0$

(1) m_0 is an overall constant identified with mass for massive particles.

For "massless particles", it is not the mass but some other constant

$$\frac{d}{d\lambda}\left|\frac{(dx^\alpha/d\lambda)}{\sqrt{-\eta_{\mu\nu}\left(\frac{dx^\mu}{d\lambda}\right)\left(\frac{dx^\nu}{d\lambda}\right)}}\right| = 0.$$

↳ diverging as $\eta_{\mu\nu}\left(\frac{dx^\mu}{d\lambda}\right)\left(\frac{dx^\nu}{d\lambda}\right) = 0$

In such case, nothing is well defined as extremization is not defined as well.

In such cases, what we do is :-

For massless particles we need to devise an action that reduces it to some EOM as the above one ~~for~~ is for massive particles and works for massless particles.

$$S = \frac{1}{2}\int d\lambda \left[\frac{1}{e}\left(\frac{dx^\mu}{d\lambda}\right)\left(\frac{dx^\nu}{d\lambda}\right)\eta_{\mu\nu} - \overset{\text{mass}}{m^2} e\right] \to \text{Introducing an auxillary variable } e(\lambda)$$

↳ This is a trick and we remove the square root, which posed problem in EOM. Thus, we remove it.

$e(\lambda)$ transforms appropriately

EOM: $\frac{\delta S}{\delta e} = 0$ and $\frac{\delta S}{\delta (x^\mu)} = 0$

This is kind of similar to langrange multipliers and is related to curvature in 1-D space (or something. I need to look up)

$$\frac{\delta S}{\delta e} : \quad \frac{-1}{e^2}\left(\frac{dx^\mu}{d\lambda}\right)\left(\frac{dx^\nu}{d\lambda}\right)\eta_{\mu\nu} = m_0^2$$

$$\text{or} \quad e^2 = \frac{1}{m_0^2}\left[\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right]\eta_{\mu\nu}$$

Thus S becomes:

$$S = \frac{1}{2}\int d\lambda \left[\frac{\left(\frac{dx^\mu}{d\lambda}\right)\left(\frac{dx^\nu}{d\lambda}\right)\eta_{\mu\nu}\, m_0}{\sqrt{-\left(\frac{dx^\nu}{d\lambda}\right)\left(\frac{dx^\mu}{d\lambda}\right)\eta_{\mu\nu}}} + m_0\sqrt{-(dx^\mu/d\lambda)(dx^\nu/d\lambda)} \right]$$

$$= m_0 \int d\lambda \sqrt{\left(\frac{dx^\mu}{d\lambda}\right)\left(\frac{dx^\nu}{d\lambda}\right)\eta_{\mu\nu}}$$

For massless particle $m_0 = 0$ ::

$$\frac{\delta S}{\delta e} = 0 \Rightarrow \left(\frac{dx^\mu}{d\lambda}\right)\left(\frac{dx^\nu}{d\lambda}\right)\eta_{\mu\nu} = 0$$

$$\frac{\delta S}{\delta x^\nu} \longrightarrow \frac{d}{d\lambda}\left(\frac{1}{e(\lambda)}\left(\frac{dx^\mu}{d\lambda}\right)\right) = 0$$

In 1D space, metric has only 1 component

$$ds^2 = e(\lambda)(d\lambda)^2$$

$$\text{or } \left(\frac{1}{e}\right)\frac{\partial}{\partial\lambda}\frac{\partial}{\partial\lambda}$$

At every λ, what is the length element.

Representation of Lorentz Group in 4-D

7 Nov, 2022

In 4 dimensions, it simplifies
→ like rotation group commutation rules simplifies in 3-D.

We want a map from lorentz transformations to matrices such that:
→ Multiplication preserves the group multiplication laws.

Strategy
(I) Generators generates infinitesimal transformations
(II) we get finite transformation by exponentiation of the generators.
(III) we can compute the product of two consecutive transformations if we know the commutators of all the generators. The commutator shall be closed for generators.

(I), (II) + (III) must be true at the level of representations (reps).
↓
If a transformation (symmetry group element) mapped to some $n\times n$ matrix T, then T can be expressed as:

$$\underset{n\times n}{T} = \exp\left\{-i\sum_{a=1}^{\text{dim of group}} \alpha_a \, {J_a}_{n\times n}\right\}$$

$n\times n$ matrix

↳ $n\times n$ matrix, the map of the corresponding generators to the set of $n\times n$ matrices.

Different concepts associated with word 'dimensions': –

① Lorentz group in 4-D
Defining reps of Lorentz group.
→ Physically characterizeds how lorentz transformations act on the 4 space-time coordinates $\{\Lambda^{\mu}{}_{\nu}\}$

② Dimension of lorentz group in 4 space time dimension = 6
= # of independent generators

Intuition:
Behind 'dimensions of a group'
→ All the group members (parametrized by a set of continuous parameters)

together form a hypersurface or 'manifolds'.
Thus, no. of ways we can move far from $\hat{1}$ on manifold is equal to no. of generators. The movement of on these manifolds is characterized by generators along the 'geometric surface' of the group. Generators are analogous to coordinates for the hypersurface.

$\Rightarrow$ # of independent generators = dimension of the 'group manifold'.

n

③ <u>Dimension of Reps</u>

→ Dimension of the rep linear vector space on which the group elements are acting as matrices "n"

<u>Goal:</u>

To figure out set of 6 $(n\times n)$ matrices that satisfy the same commutation relation as the boost and rotation generators

So, we need to compute the commutation relations

$J_i \rightarrow$ Generators of rotation

$K_i \rightarrow$ Generators of boost

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$

$$[J_i, K_j] = i\epsilon_{ijk} K_k$$

$$[K_i, K_j] = i\epsilon_{ijk} J_k$$

There is something special in 4-D and 6 generators because there is stark difference in rotation and boost.

2011, 212

choose a new basis of Generators

$$N_i = \frac{1}{2}\left(J_i - i k_i\right)$$

$$N_i^{+} = \frac{1}{2}\left(J_i + i k_i\right)$$

$[N_i, N_j] = i\epsilon_{ijk} N_k$ } → like the commutators of 3d rotation

$[N_i^{+}, N_j^{+}] = i\epsilon_{ijk} N_k^{+}$ } Another set of 3-D rotation.

$[N_i, N_j^{+}] = 0$

This is special to 4-D. Higher dimensions can't be decoupled like this.

For 3-d rotations, reps are labelled by half integers.

→ For 4-D ~~rotation~~ LT ≈ 3 d rotation ⊗ 3d rotation

reps are labelled by pair of half integers.

→ $(0,0)$ → scalar [Higg's particle]

→ $(\frac{1}{2},0)$ or $(0,\frac{1}{2})$ → chiral & antichiral fermions (2-d), (2-d)

→ $(\frac{1}{2},\frac{1}{2})$ → Gauge bosons / electromagnetic field/gauge field. (4-d)

$$\underbrace{\frac{1}{2} \otimes \frac{1}{2}}_{0 \quad 1}$$

Representation of symmetry Groups

Lie algebra for p-physics
Havard Georgie

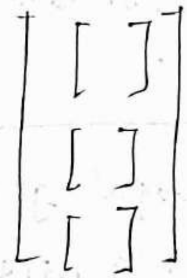
Review of $SU(2)$ → Related to angular momentum algebra

↳ Special unitary (2×2) : det = 1

$[J_i, J_j] = i\epsilon_{ijk}J_k$ — $SU(2)$: Group of 2×2 unitary matrices with det = 1

Rotation groups in 3-D space

Constructing reps: Reduce the Hilbert space of the world (any space which transforms linearly under $SU(2)$) in a block ~~dia~~ diagonal form so that action of $SU(2)$ never mixes states from 2 different blocks.



Step 1: Diagonalize one of the generators. We call it J_3

Step 2: $J^+ = \frac{1}{\sqrt{2}}(J_1 + iJ_2)$

$J^- = \frac{1}{\sqrt{2}}(J_1 - iJ_2)$

$\Big\}$ $[J_3, J^\pm] = \pm J^\pm$

$[J^+, J^-] = J_3$

If $J_3|m\rangle = m|m\rangle$

$J_3 J^\pm|m\rangle = (m\pm1)(J^\pm|m\rangle)$

Assumption: The rep is finite dimensional. → (each block diagonal is finite dimensional but there can be ∞ no. of blocks.)

⇒ must have some state(s) with highest J_3 eigen value

↳ ~~point where~~ True when J_3 has real eigen values or J_3 is hermitian

Let highest eigen value be j.

Let the state with highest eigen value is $|j,\alpha\rangle$.

↳ α is degeneracies of j eigen values.

Step 3: Apply lowering operator

$J^-|j,\alpha\rangle = N_j(\alpha)|j-1,\alpha\rangle$

$(\nabla^2 - E)\psi = 0$

$= D_r^2 \psi + \frac{1}{r^2} \nabla \cdot \psi_{(\theta,\phi)}$; $[\psi] = \psi_r(r)\psi_{\theta\phi}(\theta,\phi)$

$\nabla^2_{\theta,\phi} \psi(\theta,\phi) = e\psi(\theta,\phi)$

$\psi(\theta,\phi) = a_{lm} Y_{l,m}(\theta,\phi)$

$[\psi] = \sum_{lm} \psi_{l,m}(r)\psi_{l,m}(\theta,\phi)$

$l=0, m=0 \; [\;] \longrightarrow$ finite dimensional blocks but ∞ in number.

$[\psi]$, infinite dimensional

$l=1, \; m=-1, \; m=0, \; m=1 \Big\} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}$

Computing $N_j(\alpha)$:

$N_j(\alpha) N_j^*(\beta) \langle j-1, \beta | j-1, \alpha \rangle$ $\qquad (J^-)^+ = J^+ \rightarrow$ hermitian properties of J, J', J^2

$= \langle j, \beta | J^+ J^- | j, \alpha \rangle$

$= \langle j, \beta | J^+ J^- | j, \alpha \rangle$

$= \langle j, \beta | [J^+, J^-] + J^- J^+ | j, \alpha \rangle$

$= \langle j, \beta | J_3 + J^- J^+ | j, \alpha \rangle$ highest state with a ladder operator would vanish

$= \langle j, \beta | J_3 | j, \alpha \rangle$

$= j \langle j, \beta | j, \alpha \rangle$ $\longrightarrow$ choose a basis such that $\langle j \beta | j \alpha \rangle = \delta_{\alpha\beta}$

$= j \, \delta_{\alpha\beta}$

Thus

$|N_j(\alpha)|^2 \langle j-1, \beta | j-1, \alpha \rangle = j \, \delta_{\alpha\beta}$ $\qquad$ eigenvector

If we choose an orthonormal basis for highest sector $\{|j, \alpha\rangle\}$ the state $\{|j-1, \alpha\rangle\}$ will continue to be orthogonal choose $N_j(\alpha) = \sqrt{j}$

$\Rightarrow$ The states $\{|j-1, \alpha\rangle\}$ are orthonormal

$$\langle j-1, \alpha | j-1, \beta\rangle = \delta_{\alpha\beta}$$

Reps in SU(2)

11 Nov

① $[J_j, J_k] = i\epsilon_{jkl} J_l$

② $J^{\pm} = \frac{1}{\sqrt{2}}[J_1 \pm J_2]$

$[J_3, J^{\pm}] = \pm J^{\pm}$

$[J^+, J^-] = J_3$

$\Rightarrow$ If $J_3|\psi\rangle = m|\psi\rangle$

$J_3(J^{\pm}|\psi\rangle) = (m \pm 1)|\psi\rangle$

③ We are studying finite dimensional reps.

Finite no. of states

$\hookrightarrow$ If we diagonalize J_3 (J_3 being hermitian operator) there will be states with largest e-value $|j, \alpha\rangle$

$\hookrightarrow$ denotes the other labels.

$\Rightarrow J_3 |j,\alpha\rangle = j\,|j,\alpha\rangle$

$J^+|j,\alpha\rangle = 0 \quad \forall \alpha.$

$J^-|j,\alpha\rangle = N_j(\alpha)\,|j-1,\alpha\rangle$

If we choose a basis such that

$\langle j,\beta | j,\alpha\rangle = \delta_{\alpha\beta}.$

$\Rightarrow N_j(\alpha) = N_j = \sqrt{j}$

We recursively use the same argument to show (exercise) that if we choose $\{|j,\alpha\rangle\}$ to be orthogonal to an orthonormal collection of states, the same would be true for $\{|j-k,\alpha\rangle\}$

$\Rightarrow$ we need not track the α ~~[illegible]~~ labels for a given fixed α, we can build the whole reps.

~~Find~~

(1) Fixing N_{j-k}: (Different from than quantum mech)

$J^- | N_{j-k}\rangle =$

$J^- | j-k \rangle = N_{j-k} | j-k-1 \rangle$

$J^+ | j-k-1 \rangle = \tilde{N}_{j-k} | j-k \rangle$

$|N_{j-k}|^2 = \langle j-k | J^+ J^- | j-k \rangle$

$= \langle j-k | [J^+, J^-] | j-k \rangle +$

$\langle j-k | J^+ = N^*_{j-k} \langle j-k-1 |$

$|N_{j-k}|^2 = \langle j-k | J^+ J^- | j-k \rangle$

we shall find $|N_{j-k}|^2$.

we shall assume that $\langle j-k | j-k \rangle = 1 \ \forall k$.

(2) To show that $N_{j-k} = \tilde{N}_{j-k}$

we shall show it by method of induction

Assume that it is true for ~~all~~ $\tilde{N}_{j-(k-1)} = N_{j-(k-1)}$

If it is true for $\tilde{N}_{j-k} = N_{j-k}$, then we're done.

$|N_{j-k}|^2 = \langle j-k | J^+ J^- | j-k \rangle$

$= \langle j-k | [J^+, J^-] + J^- J^+ | j-k \rangle$

$= \langle j-k | J_3 | j-k \rangle + \langle j-k | J^- J^+ | j-k \rangle$

$= ~~\langle j-k | J_3 | j-k \rangle~~ + N$

$(j-k) + |N_{j-(k+1)}|^2$

$J^+ | j-k \rangle = \tilde{N}_{j-k+1} | j-k+1 \rangle$

$$J^+|j-k-1\rangle = \tilde{N}_{j-k}|j-k\rangle$$

$$\Rightarrow J^+\left(\frac{J^-|j-k\rangle}{N_{j-k}}\right) = \tilde{N}_{j-k}(j-k)$$

we choose everything to be real i.e $N_{j-k} \in \mathbb{R}\ \forall k$

$$\text{or } \frac{[J^+,J^-]\,|j-k\rangle + (J^-J^+|j-k\rangle)}{N_{j-k}} = \tilde{N}_{j-k}|j-k\rangle$$

$$\text{or } \frac{(j-k) + |N_{j-k+1}|^2}{N_{j-k}}|j-k\rangle = \tilde{N}_{j-k}|j-k\rangle$$

$$\Rightarrow N_{j-k} \cdot \tilde{N}_{j-k} = N_{j-k}^2 = (j-k) + (N_{j-k+1})^2$$

$$\Rightarrow \boxed{\tilde{N}_{j-k} = N_{j-k}}$$

$$N_{j-k}^2 - N_{j-k+1}^2 = (j-k)$$

$$N_j^2 = j$$

$$N_{j-1}^2 - N_j^2 = j-1$$

$$N_{j-2}^2 - N_{j-1}^2 = j-2$$

$$\vdots$$

$$N_{j-k}^2 + N_{j-k+1}^2 = j-k$$

$$\Rightarrow N_{j-k}^2 = \sum_{p=0}^{k}(j-p)$$

$$= (k+1)j - \frac{k(k+1)}{2}$$

$$= (k+1)\left(j-\frac{k}{2}\right) = \frac{1}{2}(2j-k)(k+1)$$

$$\left.\begin{array}{l} j-k=m \\ k=j-m \end{array}\right\} \quad N_m^2 = \frac{1}{2}(j-m+1)(2j-j+m)$$

$$= \frac{1}{2}(j-m+1)(j+m)$$

$J_3^{a} |m\rangle = m|m\rangle$

$J^{\pm}$ ~~states~~

$$J^{+} |m-1\rangle = \frac{1}{\sqrt{2}}\sqrt{(j+m)(j-m+1)}\ |m\rangle$$

$$J^{-}|m\rangle = \frac{1}{\sqrt{2}}\sqrt{(j+m)(j-m+1)}\ |m-1\rangle$$

(weight sector)

Highest wt st = st. with maximum J_3 evalue. when $m-1=j$

When $m=j+1$, then $J^{+}|j\rangle$ vanishes.

Lowest wt state ~~at~~ $(m=-j)$ vanishes for $(j+m)$ factor

Thus, $J^{-}|j\rangle = 0$

→ SU(2) reps are labelled by j

→ # of states in a rep $= (2j+1)$

with J_3 evalue ranging from $-j$ to $+j$

$2j+1$ is an integer

∴ j = half integers

$\boxed{\text{Adjoint reps?}}$

Adjoint Representation

14 Nov

Space of generators ~ linear vector space where the group elements & generators act like linear operators.

Ex: Generator of ~~momenta~~ rotation – Angular momenta.

Angular momenta transforms like vectors under rotation.

Space of generators form a representation, also transform in a representation and this group is called adjoint representation

Ex: Spin-1 matrices are adjoint representation.

Dimension of adjoint reps = Dimension of the group

If X_a is a general generator:

$$[X_a, X_b] = i f_{abc} X_c$$

$\hookrightarrow$ structure constant

# of generators = d → (also dimension)

Adjoint reps = A set of d×d matrices T_a so that $[T_a, T_b] = i f_{abc} T_c$

$[T_a]_{bc} = -i f_{abc}$: If we define T_a like this, then as a consequence of Jacobi identity, the required commutation rule will be satisfied

From jacobi identity:

$$[X_a, [X_b, X_c]] + [X_b, [X_c, X_a]] + [X_c, [X_a, X_b]] = 0$$

$$\Rightarrow i\{f_{bcm}[X_a, X_m] + f_{cam}[X_b, X_m] + f_{abm}[X_c, X_m]\} = 0$$

$$\Rightarrow (i)^2\{f_{amp} f_{bcm} + f_{bmp} f_{cam} + f_{cmp} f_{abm}\} = 0$$

Now,

$$[T_a, T_b]_{pq} = (-i)^2 \{ [f_{apm} f_{bmq} - f_{bpm} f_{amq}]$$

$$= (-i)^2 [f_{amp} f_{bqm} + f_{bmp} f_{amq}]$$

we ~~can~~ should do it again as antisymmetry of 2-3 is not ~~antisymmetry everywhere~~ & Relabelling would work

"Assume complete antisymmetry for f_{abc} which is yet to be proved in the previous step"

$= f_{qmp} f_{abm}$

$= i f_{qmp} (-i) f_{abm}$

$= i f_{abm} (-i f_{mpq})$

$= i f_{abm} [T_m]_{pq}$

Therefore,

$[T_a]_{bc} = -i f_{abc}$

constitute the adjoint rep of any group.

Define innerproduct

In adjoint reps : $X_a \sim |X_a\rangle$
$$\downarrow$$
$$T_a$$

$$\langle X_b | X_a \rangle = Tr[T_a^{+} T_b]$$

When we take generators to be hermitian :

$$g_{ab} = Tr[T_a T_b] = g_{ba}$$ = real symmetric matrix

(g_{ab} is a component of a matrix)

$\Rightarrow$ The space of generators $\rightarrow$ metric space with concept of distance.

we want the simplest g_{ab}

$$X_a \longrightarrow X_a' = L_{ab} X_b$$

$$[X_a', X_b'] = L_{ac} L_{bd} [X_c, X_d]$$

$$= i L_{ac} L_{bd} f_{cdq} X_q$$

$$= i \underbrace{L_{ac} L_{bd} f_{cdq} \cancel{X_a} [L^{-1}]_{qp}}_{f'_{abp}} X_p' = f'_{abp}$$

$$= i f'_{abp} X_p'$$

$$L_{ac} L_{bd} f_{cdq} [L^{-1}]_{qp} = f'_{abp}$$

$$\Rightarrow \boxed{[T_a']_{bp} = L_{ac} L_{bd} [T_c]_{dq} [L^{-1}]_{qp}}$$

$$\Rightarrow \boxed{L_{ac} [L \; T_c \; L^{-1}] = T_a'}$$

$$[\cancel{Tr}(g_a')] \Rightarrow L_{a_1 c_1} L_{a_2 c_2}$$

$$Tr[T_{a_1}', T_{a_2}'] = L_{a_1 c_1} L_{a_2 c_2} Tr[L \cdot T_{c_1} \overset{1}{L^{-1} L} \cdot T_{c_2} L]$$

$$= L_{a_1 c_1} L_{a_2 c_2} Tr[T_{c_1}, T_{c_2}]$$

$$\Rightarrow g'_{a_1 a_2} = L_{a_1 c_1} L_{a_2 c_2} g_{c_1 c_2}$$

$$g' = L \cdot g \cdot L^T$$

By construction, g is real symmetric matrix

we could choose L to be orthagonal matrices and diagonalize g_{ab}

$\cancel{g_{ab} = k_a \delta_{ij}}$

$g_{ab} = k_a \delta_{ab}$ [not summed over a]

$$\begin{bmatrix} k_1 & & & \\ & k_2 & & \\ & & \ddots & \\ & & & k_n \end{bmatrix}$$

15 Nov

Adjoint rep (cont^{d})

Generators of the group satisfy

$$[X_a, X_b] = i f_{ab}{}^{c} X_c$$

$$[X_a, [X_b, X_c]] + [X_b, [X_c, X_a]] + [X_c, [X_a, X_b]] = 0 \quad \{\text{Jacobi}\}$$

Adjont rep: $X_a \longrightarrow T_a$ = A $d \times d$ matrix where
$d = \#$ of generators.

$$[T_a]_{bc} = -i f_{abc}$$

$$[X_a, [X_b, X_c]] + [X_b, [X_c, X_a]] + [X_c, [X_a, X_b]]$$

$$= i f_{bcm} [X_a, X_m] + i f_{cam} [X_b, X_m] + i f_{abm} [X_c, X_m]$$

$$= (i)^2 \left[f_{bcm} f_{amp} + f_{cam} f_{bmp} + f_{abm} f_{cmp} \right] X_p$$

$$= ((-1)) \Big[\qquad\qquad \Big\} \; f_{abc} = i [T_a]_{bc}$$

$$f_{bcm} f_{amp} + f_{cam} f_{bmp} + f_{abm} f_{cmp} = 0$$

or $(i [T_b]_{cm}) (i [T_a]_{mp}) - f_{acm} f_{bmp} + f_{abm} f_{cmp} = 0$

or $(i^2) [T_b]_{cm} [T_a]_{mp} - (i)^2 [T_a]_{cm} [T_b]_{mp} + f_{abm} f_{mcp} = 0$

or $+[T_a, T_b]_{cp} - f_{abm}\, i [T_m]_{cp} = 0$

or $[T_a, T_b]_{cp} = + i f_{abm} [T_m]_{cp}$

Inner product: X_a is generator

In adj rep $X_a \rightarrow |X_a\rangle \rightarrow T_a$ (vectorial state) (group element (matrix))

Inner product is defined as

$$\langle X_a | X_b \rangle = \mathrm{Tr}\left[X_a^{\dagger} X_b \right]$$

when $X_a^{\dagger} = X_a$ (In a choice ~~of~~ of basis where the generators are represented by hermitian matrices) like T_a.

$$g_{ab} = \mathrm{Tr}\left[T_a T_b \right]$$

$T_a \to T_a'$

$|X_a\rangle \to |X_a'\rangle = L_{ab} |X_b\rangle$

$$\Rightarrow [T_a']_{bp} = L_{ab} L_{bd} [T_c]_{de} [L^{-1}]_{ep}$$

$$= L_{ac} (L \cdot T_c \cdot L^{-1})$$

$$\Rightarrow g'_{a_1 a_2} = L_{a_1 c_1} L_{a_2 c_2} g_{c_1 c_2}$$

$$\Rightarrow g' = L \cdot g \cdot L^T$$

choose L such that g (symmetric matrix) becomes diagonal.

$g'_{ab} = k_a \delta_{ab}$ [No summation rule]

$$g' = \begin{bmatrix} k_1 & & & 0 \\ & k_2 & & \\ & & \ddots & \\ 0 & & & k_d \end{bmatrix}$$

~~choose another L~~

Scale each direction so that $g' \to g'' = \delta_{ab}$

choose $L = \begin{bmatrix} \frac{1}{\sqrt{k_1}} & & 0 \\ & \frac{1}{\sqrt{k_2}} & \\ 0 & & \ddots \end{bmatrix}$

$$\Rightarrow L \cdot g' \cdot L^T = \hat{\mathbb{1}}$$

$[T_a, T_b] = i f_{abc} T_c$

$Tr(T_m [T_a, T_b]) = i f_{abc} \, Tr[T_m T_c]$

$= i f_{abc} \delta_{mc}$

$= i f_{abm}.$

$Tr(T_m [T_a, T_b])$

$= Tr[T_m T_a T_b - T_m T_b T_a]$

$= Tr[T_b T_m T_a - T_m T_b T_a]$ } cyclic property of trace.

$= Tr[[T_b, T_m] T_a]$

$= Tr[T_a [T_b, T_m]] = i f_{bma}.$

Thus f_{abc} is antisymmetric,

[ab antisymmetry + cyclic property $\rightarrow$ all a, b, c antisymmetry]

How generators act in adj rep

$$\hat{X}_b |X_a\rangle = \sum_c |X_c\rangle \underbrace{\langle X_c | X_b | X_a \rangle}_{[T_b]_{ca}}$$

$$= |X_c\rangle [T_b]_{ca} = -i f_{bca} |X_c\rangle$$

$$= i f_{bac} |X_c\rangle = |i f_{bac} X_c\rangle$$

$$= |[X_b, X_a]\rangle$$

$$X_a \longrightarrow |X_a\rangle$$
$$\hookrightarrow [T_a]_{bc}$$

$X_a|X_b\rangle = [X_a, X_b]$

$[T_a]_{bc} = -if_{abc}$

~~This rep. all T_a's are hou~~

In this choice of basis, all T_a's are hermitian

$\langle X_a|X_b\rangle = Tr[X_a^{+} X_b]_{adj}$

$= \delta_{ab}$ (choice for the basis of ~~τ~~ $|X_a\rangle$)

Commutation Relations

$$\boxed{[X_a, X_b] = if_{abc}\ X_c}$$

① Cartans $\{H_i\}$ $i = 1, 2, \cdots R < d$, $d = \#$ of generators.

Generators that mutually commute.

$[H_i, H_j] = 0$

set : $\{H_i\} \subset \{X_a\}$

Cartans do not span the group.

$(SO(4)) \sim SO(3,1)$

Rotation in 4-Dimensions

generators would be identified : small rotations in (ij) place

$i \neq j$, $i = 1, 2 \cdots 4$
$j = 1, 2 \cdots 4$

Rotation in 1-2 plane and 3-4 plane

must commute (as they're distinct)

$[M^{12}, M^{34}] - \delta^2_4 M^{13} + \cdots$

Cartans of $SO(4)$

So in any rep, cartans could be simultaneously diagonalized

Eigen states: $|\vec{\mu}\rangle = \begin{Bmatrix} \mu_1 \\ \vdots \\ \mu_R \end{Bmatrix}$.

$\underbrace{H_i}_{n\times n} |\vec{\mu}\rangle = \mu_i |\vec{\mu}\rangle$ {Assuming no degeneracy}

True in any space.

$\begin{Bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_R \end{Bmatrix} \rightarrow$ weight vectors

Assume each wt vector corresponds to an unique state in a given reps

Roots: weights in adjoint reps

$H_i \longrightarrow |H_i\rangle$

$[H_i, H_j] = 0 \Rightarrow |H_i| H_j\rangle = 0$

Cartans are also states in adj Representation. But all have 0 wts.

Suppose $|E_\alpha\rangle$ is a state with non-zero weight vector $\alpha = \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_R \end{Bmatrix}$

$H_i |E_\alpha\rangle = \alpha_i |E_\alpha\rangle$ - Any reps

Now choose a basis where all {H_i's} are hermitian

$\Rightarrow \alpha_i$ is real

$\Rightarrow [H_i, E_\alpha] = \alpha_i E_\alpha \rightarrow$ speciality of adj reps

$[H_i, E_{\vec{\alpha}}] = \alpha_i E_{\vec{\alpha}}$

$[H_i, E_\alpha]^+ = \alpha_i^* \cdot E_{\vec{\alpha}}^+ = \alpha_i E_{\vec{\alpha}}^+$

$[E_{\vec{\alpha}}^+, H_i] = \alpha_i E_{\vec{\alpha}}^+$

$\Rightarrow [H_i, E_{\vec{\alpha}}^+] = -\alpha_i E_{\vec{\alpha}}^+$

Thus; $E_{\vec{\alpha}}^+ = E_{-\vec{\alpha}}$. $[J_3, J^+] = J^+$; $[J_3, J^-] = -J^-$

$E_{\vec{\alpha}}^+$ is also a eigen state

We have a pair of generators.

In any reps,

$$H_i |\mu\rangle = \mu_i |\mu\rangle$$

$$H_i (E_\alpha |\mu\rangle) = [H_i, E_\alpha] |\mu\rangle + E_\alpha H_i |\mu\rangle$$

$$= \alpha_i E_\alpha |\mu\rangle + \mu_i E_\alpha |\mu\rangle$$

$$= (\alpha + \mu_i)(E_\alpha |\mu\rangle)$$

$$E_\alpha |\mu\rangle = |\vec{\mu} + \vec{\alpha}\rangle$$

we shall see similarly :

$$(E_{-\vec{\alpha}}) |\vec{\mu}\rangle = |\vec{\mu} - \alpha\rangle$$

weight ~~vector~~ vector corresponding to $|H_i\rangle = 0$

In adj rep: follows from here

$$E_{\vec{\alpha}} |E_{-\vec{\alpha}}\rangle = |\vec{0}\rangle = \beta_i |H_i\rangle$$

$$\Rightarrow [E_\alpha, E_{-\alpha}] = \beta_i H_i$$

$$\mathrm{Tr}\,[H_j, [E_\alpha, E_{-\alpha}]] = \mathrm{Tr}[\beta_i H_i, H_j]$$

$$= \beta_i \delta_{ij} = \beta_j$$

$$\mathrm{Tr}[H_j, [E_\alpha, E_{-\vec{\alpha}}]]$$

$$= \mathrm{Tr}\,(H_j, E_\alpha E_{-\alpha} - H_j E_{-\alpha} E_\alpha)$$

$$= \mathrm{Tr}\left(-E_{\vec{\alpha}} H_j E_{-\vec{\alpha}} + H_j E_{\vec{\alpha}} E_{-\vec{\alpha}}\right)$$

$$= \mathrm{Tr}\,([H_j, E_{\vec{\alpha}}], E_{-\vec{\alpha}})$$

$$= \alpha_j \,\mathrm{Tr}\,[E_\alpha E_{-\alpha}] = \alpha_j$$

$$\Rightarrow [E_\alpha, E_{-\alpha}] = \alpha_j H_j$$