QComp Homework 2

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1 Problem 1

$$|\psi\rangle = \frac{1}{\sqrt{2}} |0\rangle |f(0)\rangle + \frac{1}{\sqrt{2}} |1\rangle |f(1)\rangle = \begin{pmatrix} 1 - f(0) \\ 1 - f(1) \\ f(0) \\ f(1) \end{pmatrix}$$

Applying Hadamard's gate to each qbit, we get

$$(H_0 \otimes H_1) |\psi\rangle = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 - f(0) \\ 1 - f(1) \\ f(0) \\ f(1) \end{pmatrix} =$$

$$\frac{1}{2\sqrt{2}} \begin{pmatrix} 2 \\ 0 \\ 2(1 - f(0) - f(1)) \\ 2(f(1) - f(0)) \end{pmatrix} = \frac{1}{\sqrt{2}} \left(|00\rangle + (1 - f(0) - f(1)) |10\rangle + (f(1) - f(0)) |11\rangle \right).$$

If $f(0) = f(1)$, the resulting state is $\frac{1}{\sqrt{2}} \left(|00\rangle + (1 - 2f(0)) |10\rangle \right)$, so when measured, the state collapses to $|00\rangle$ or $|10\rangle$ equiprobably.

In the other case, when $f(1) = 1 - f(0)$, the resulting state is $\frac{1}{\sqrt{2}} \left(|00\rangle + (1 - 2f(0)) |11\rangle \right)$, so when measured, the state collapses to $|00\rangle$ or $|11\rangle$ equiprobably.

Thus, with 50% probability the measured state is not $|00\rangle$ and it is possible to distinguish cases $f(0) = f(1)$ and $f(0) \neq f(1)$.

2 Problem 2

2.1 a

Let's prove the identity by induction. First,

$$H |0\rangle_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2^1}} \left((-1)^{\mathbf{0} \cdot \mathbf{0}} |0\rangle + (-1)^{\mathbf{0} \cdot \mathbf{1}} |1\rangle \right),$$

because $\mathbf{0} \cdot \mathbf{x} = 0$ for any vector $\mathbf{x}$.

$$H|1\rangle_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{\sqrt{2^1}}\left((-1)^{\mathbf{1}\cdot\mathbf{0}}|0\rangle + (-1)^{\mathbf{1}\cdot\mathbf{1}}|1\rangle\right),$$

because $\mathbf{1}\cdot\mathbf{1} = 1$.

Thus the identity holds for $n = 1$.

Now suppose for any $k \leq n$ the identity is true. Then

$$H^{\otimes(n+1)}|x\rangle_n|0\rangle = \frac{1}{\sqrt{2^n}}\sum_{z\in\{0,1\}^n}(-1)^{\mathbf{x}\cdot\mathbf{z}}|z\rangle H|0\rangle =$$

$$\frac{1}{\sqrt{2^{n+1}}}\sum_{z\in\{0,1\}^n}(-1)^{\mathbf{x}\cdot\mathbf{z}}|z\rangle(|0\rangle+|1\rangle) = \frac{1}{\sqrt{2^{n+1}}}\sum_{z\in\{0,1\}^{n+1}}(-1)^{\overline{\mathbf{x}0}\cdot\mathbf{z}}|z\rangle,$$

because $\overline{\mathbf{x}0}\cdot\overline{\mathbf{z}t} = \mathbf{x}\cdot\mathbf{z}$ for $t \in \{0,1\}$.

$$H^{\otimes(n+1)}|x\rangle_n|1\rangle = \frac{1}{\sqrt{2^n}}\sum_{z\in\{0,1\}^n}(-1)^{\mathbf{x}\cdot\mathbf{z}}|z\rangle H|1\rangle =$$

$$\frac{1}{\sqrt{2^{n+1}}}\sum_{z\in\{0,1\}^n}(-1)^{\mathbf{x}\cdot\mathbf{z}}|z\rangle(|0\rangle-|1\rangle) = \frac{1}{\sqrt{2^{n+1}}}\sum_{z\in\{0,1\}^{n+1}}(-1)^{\overline{\mathbf{x}1}\cdot\mathbf{z}}|z\rangle,$$

because $\overline{\mathbf{x}1}\cdot\overline{\mathbf{z}0} = \mathbf{x}\cdot\mathbf{z}$, and $\overline{\mathbf{x}1}\cdot\overline{\mathbf{z}1} = 1 - \mathbf{x}\cdot\mathbf{z}$ for $t \in \{0,1\}$.

Thus the identity holds for any $n \geq 1$.

2.2 b

$$\frac{1}{\sqrt{2}}H^{\otimes n}(|\mathbf{x}\rangle + |\mathbf{y}\rangle) = \frac{1}{\sqrt{2^{n+1}}}\sum_{z\in\{0,1\}^n}((-1)^{\mathbf{x}\cdot\mathbf{z}} + (-1)^{\mathbf{y}\cdot\mathbf{z}})|\mathbf{z}\rangle$$

Now, $((-1)^{\mathbf{x}\cdot\mathbf{z}} + (-1)^{\mathbf{y}\cdot\mathbf{z}})$ can take three values 0, 1, and -1. It is 0 when $\mathbf{x}\cdot\mathbf{z} \neq \mathbf{y}\cdot\mathbf{z}$, otherwise it is $2\cdot(-1)^{\mathbf{x}\cdot\mathbf{z}}$.

$$\mathbf{x}\cdot\mathbf{z} \neq \mathbf{y}\cdot\mathbf{z} \Leftrightarrow \mathbf{x}\cdot\mathbf{z}\oplus\mathbf{y}\cdot\mathbf{z} = 1$$

$$\mathbf{x}\cdot\mathbf{z}\oplus\mathbf{y}\cdot\mathbf{z} = (\mathbf{x}\oplus\mathbf{y})\cdot\mathbf{z}$$

Thus, whenever $\mathbf{z} \notin \mathbf{s}^{\perp}$, $((-1)^{\mathbf{x}\cdot\mathbf{z}} + (-1)^{\mathbf{y}\cdot\mathbf{z}}) = 0$. So,

$$\frac{1}{\sqrt{2}}H^{\otimes n}(|\mathbf{x}\rangle + |\mathbf{y}\rangle) = \frac{1}{\sqrt{2^{n-1}}}\sum_{\mathbf{z}\in\mathbf{s}^{\perp}}(-1)^{\mathbf{x}\cdot\mathbf{z}}|\mathbf{z}\rangle,$$

And it seems there is a typo in normalization coefficient in the task description.