QComp Homework 1

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1 Problem 3

For an arbitrary 1-qbit gate W, 2-qbit matrix for cW_{01} is obtained using

$$cW_{01} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes I + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes W.$$

Also, 2-qbit matrix for W_0 (which only applies W to the first qbit) is obtained using

$$W_0 = W \otimes I.$$

1.1 1

$$HXH = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = Z$$

1.2 2

$$cZ_{01} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = cZ_{10}$$

1.3 3

$$H_0H_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$H_0H_1cX_{01}H_0H_1 = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} =$$

$$\frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} =$$

$$\frac{1}{4}\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \end{pmatrix} = cX_{10}$$

1.4 4

$$c(e^{i\alpha})_{01} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i\alpha} & 0 \\ 0 & 0 & 0 & e^{i\alpha} \end{pmatrix} = U_1(\alpha)_{(0)}$$

2 Problem 4

Separable states should allow representation in the form

$$(a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle) = ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle .$$

2.1 1

If the state is separable, then

$$ac = \frac{2}{3}, \quad ad = \frac{1}{3}, \quad bc = 0, \quad bd = -\frac{2}{3}.$$

It is clear that none of a, b, c, d are 0, so $\frac{c}{d} = \frac{ac}{ad} = 2$. Thus $bc = bd\frac{c}{d} = 2bd = -\frac{4}{3}$. Contradiction, so the state is entangled.

2.2 2

$$\frac{1}{2}(|00\rangle - i|01\rangle + i|10\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - i|i\rangle)$$

Separable state.

2.3 3

If the state is separable, then (omitting the normalization coefficient)

$$ac = 1, \quad ad = -1, \quad bc = 1, \quad bd = 1.$$

It is clear that none of a, b, c, d are 0, so $\frac{c}{d} = \frac{ac}{ad} = -1$. Thus $bc = bd\frac{c}{d} = -bd = -1$. Contradiction, so the state is entangled.

3 Problem 5

We can represent $|\psi\rangle$ using Shmidt decomposition:

$$|\psi\rangle = \sum_{i=1}^{2} \lambda_i |\varphi_i^A\rangle |\varphi_i^B\rangle .$$

To get λ_i, φ_i^A, φ_i^B, we can apply SVD to matrix A:

$$A = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} = U\Sigma V^T,$$

where U and V are unitary matrices, and Σ is diagonal with non-negative values:

$$\Sigma = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad U = \begin{pmatrix} \varphi_{1,0}^A & \varphi_{2,0}^A \\ \varphi_{1,1}^A & \varphi_{2,1}^A \end{pmatrix}, \quad V = \begin{pmatrix} \varphi_{1,0}^B & \varphi_{2,0}^B \\ \varphi_{1,1}^B & \varphi_{2,1}^B \end{pmatrix}, \quad \text{where } |\varphi_i^\alpha\rangle = \begin{pmatrix} \varphi_{i,0}^\alpha \\ \varphi_{i,1}^\alpha \end{pmatrix}.$$

Since U and V are unitary, they can be used as 1-qbit gate; we can also construct a 1-qbit gate B as follows:

$$B = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_2 & -\lambda_1 \end{pmatrix}$$

It is unitary because $\lambda_i \geq 0$ and $\lambda_1^2 + \lambda_2^2 = 1$.

Now,

$$U|0\rangle = |\varphi_1^A\rangle, \quad U|1\rangle = |\varphi_2^A\rangle, \quad V|0\rangle = |\varphi_1^B\rangle, \quad V|1\rangle = |\varphi_2^B\rangle,$$

thus

$$U_1 V_0 cX_{01} B_0 |00\rangle = U_1 V_0 cX_{01}(\lambda_1 |00\rangle + \lambda_2 |01\rangle) = U_1 V_0(\lambda_1 |00\rangle + \lambda_2 |11\rangle) = (\lambda_1 |\varphi_1^A\rangle |\varphi_1^B\rangle + \lambda_2 |\varphi_2^A\rangle |\varphi_2^B\rangle).$$

So to prepare an arbitrary state, one needs to obtain Shmidt decomposition and then compute $U_1 V_0 cX_{01} B_0 |00\rangle$.

3.1 a

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle) = \frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{1}{\sqrt{2}}|1\rangle|1\rangle.$$

$$B = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = H, \quad U = I, \quad B = I$$

3.2 b

$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle) = \frac{1}{\sqrt{2}}|0\rangle|1\rangle + \frac{1}{\sqrt{2}}|1\rangle|0\rangle.$$

$$B = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = H, \quad U = I, \quad B = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = X$$

3.3 c

Let $A = \sqrt{\frac{1}{2} + \frac{1}{\sqrt{5}}}$, $B = \sqrt{\frac{1}{2} - \frac{1}{\sqrt{5}}}$.

$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{12}}(3|00\rangle + |01\rangle + |10\rangle - |11\rangle) = \frac{1}{\sqrt{6}}\sqrt{3+\sqrt{5}}(A|0\rangle + B|1\rangle)(A|0\rangle + B|1\rangle) + \frac{1}{\sqrt{6}}\sqrt{3-\sqrt{5}}(A|0\rangle - B|1\rangle)(-A|0\rangle + B|1\rangle).$$

$$B = \begin{pmatrix}\frac{1}{\sqrt{6}}\sqrt{3+\sqrt{5}} & \frac{1}{\sqrt{6}}\sqrt{3-\sqrt{5}}\\ \frac{1}{\sqrt{6}}\sqrt{3-\sqrt{5}} & -\frac{1}{\sqrt{6}}\sqrt{3+\sqrt{5}}\end{pmatrix}, \quad U = \begin{pmatrix}A & B\\ B & -A\end{pmatrix}, \quad B = \begin{pmatrix}A & -B\\ B & A\end{pmatrix}$$

The Qiskit code is in `.ipynb` part.

4 Problem 6

Let $l = n - m - k$. We can represent the states $|\Phi_x\rangle$ as

$$|\Phi_x\rangle_{n-m} = \sum_y b_{x,y}|y\rangle_k |\Omega_{x,y}\rangle_l.$$

4.1 a

After the measurement of m qbits, the state is $|x\rangle_m |\Phi_x\rangle_{n-m} = \sum_y b_{x,y}|x\rangle_m |y\rangle_k |\Omega_{x,y}\rangle_l$ with probability $|a_x|^2$. When another k qbits are measured, any vector y, for which $b_{x,y} \neq 0$, can be result of measurement, and the state collapses to $|x\rangle_m |y\rangle_k |\Omega_{x,y}\rangle$ with probability $|b_{x,y}|^2$. So the possible final states are $|x\rangle_m |y\rangle_k |\Omega_{x,y}\rangle_l$ with probability $|a_x|^2 |b_{x,y}|^2$.

4.2 b

In general, after the measurement of $m+k$ qbits, the state could be one of $\sum_z c_z |x\rangle_m |y\rangle_k |z\rangle_l$ for some x and y. The probability amplitudes c_z are given by

$$c_z = \langle x|_m \langle y|_k \langle z|_l |\Psi\rangle_n = \langle x|_m \langle y|_k \langle z|_l \sum_{x'} a_{x'} |x'\rangle_m |\Phi_{x'}\rangle_{n-m} =$$

$$\langle x|_m \langle y|_k \langle z|_l \sum_{x'} a_{x'} \sum_{y'} b_{x',y'} |x'\rangle_m |y'\rangle_k |\Omega_{x',y'}\rangle_l =$$

$$\sum_{x'}\sum_{y'} a_{x'} b_{x',y'} \langle x|_m \langle y|_k \langle z|_l |x'\rangle_m |y'\rangle_k |\Omega_{x',y'}\rangle_l.$$

It is clear that the only possibly non-zero summand is when $x' = x$, and $y = y'$, so $c_z = a_x b_{x,y} \langle z|_l |\Omega_{x,y}\rangle_l$. Now,

$$\sum_z c_z |x\rangle_m |y\rangle_k |z\rangle_l = a_x b_{x,y} \sum_z \langle z|_l |\Omega_{x,y}\rangle_l |z\rangle_l = a_x b_{x,y} |x\rangle_m |y\rangle_k |\Omega_{x,y}\rangle_l$$

Thus, the possible final states are $|x\rangle_m |y\rangle_k |\Omega_{x,y}\rangle_l$ with probability $|a_x|^2 |b_{x,y}|^2$, the same result as in (a).