Abstract nonsense?

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Abstract nonsense?

in any abelian category, the following statements are equivalent for a short exact sequence

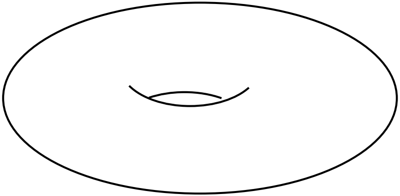
$$0 \longrightarrow A \xrightarrow{q} B \xrightarrow{r} C \longrightarrow 0.$$

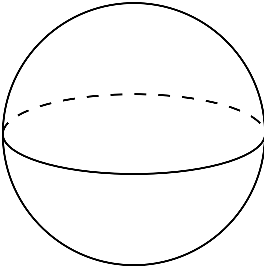
Then the following statements are equivalent:

- There exists a morphism $u\colon C \to B$ such that ru is the identity on C, id_C,
- There is an isomorphism h from B to the direct sum of A and C, such that hq is the natural monomorphism of A in the direct sum, and rh^{-1} is the natural projection of the direct sum onto C.

Origins in topology

How do we know these spaces are not homeomorphic formally?





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In "regular" mathematics one basically always assumes that our category is **locally small** (as in this presentation) meaning, the morphisms between any two objects $\hom(A, B)$ are of set size.

Categories 2

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We can form a category from any directed multigraph and it's subgraphs will be subcategories

Diagrams of morphisms example

Given two commutative squares,

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "g"'] & B \arrow[d, "f"] \\ C \arrow[r, "a'"'] & D \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "b"] \arrow[d, "f"'] & F \arrow[d, "g'"] \\ D \arrow[r, "b'"'] & E \end{tikzcd} \quad \leadsto \quad \begin{tikzcd} A \arrow[r, "a"] \arrow[d, "g"'] & B \arrow[r, "b"] \arrow[d, "f"] & F \arrow[d, "g'"] \\ B \arrow[r, "a'"'] & D \arrow[r, "b'"'] & E \end{tikzcd}$$

they make up a commutative rectangle.

Familiar examples of categories

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- A **poset** $(P, \leq)$ is a category where $A \to B$ if $A \leq B$ when $(P, \leq)$ is viewed as partially ordered set. For example $\mathrm{Op}(X)$ the category of open sets of a space X, with inclusions as morphisms.

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- **Cat** the category of small categories (category is **small** if it has set amount of objects). We will soon find out what the morphisms in this category are.

Example of duality

$C \overset{a}{\underset{b}{\rightrightarrows}} A \xrightarrow{f} B$	f is a monomorphism if for all C, a, b
$C \overset{b}{\underset{b}{\leftleftarrows}} A \xleftarrow{f} B$	f is an epimorphism if for all C, a, b

Now if f is a monomorphism in a category **C** iff it's associated morphism is an epimorphism in the opposite category $\mathbf{C}^{\mathrm{op}}$.

Diagram chasing example

Suppose the outer most square below commutes.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "g"'] & B \arrow[r, "b"] \arrow[d, "f"] & F \arrow[d, "g'"] \\ B \arrow[r, "a'"'] & D \arrow[r, "b'"'] & E \end{tikzcd}$$

This data defines a commutative rectangle if either the RHS square commutes and b' is a monomorphism or LHS square commutes and a is an epimorphism.

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An object T in a category $\mathbf{C}$ is **inital**, if there exists a unique morphism $T \to C$ for any other object C in $\mathbf{C}$. The dual object of terminal object is called **terminal**. An object satisfies an **universal property** if it is initial (or dually terminal) in some ambient category.

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Theorem

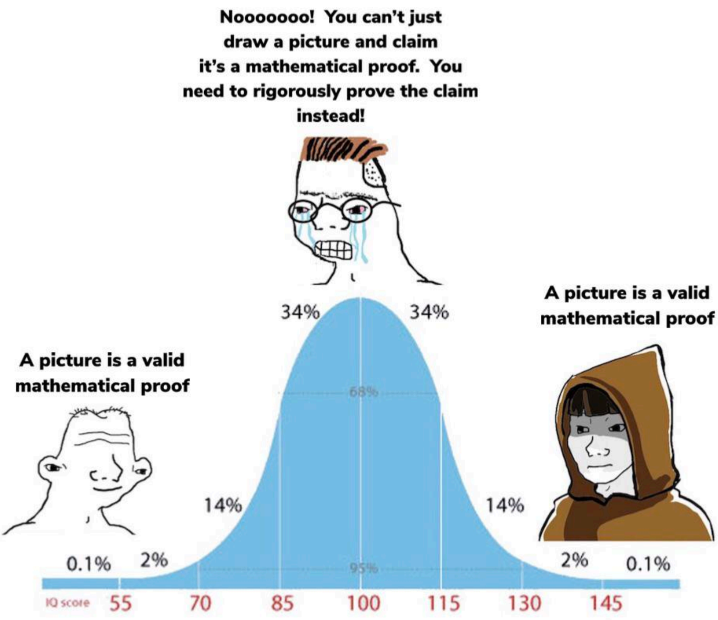
$\mathbf{C}$ has an initial object iff $\mathbf{C}^{op}$ has a terminal object.

Theorem

Intial objects are unique up to a unique isomorphism.

Visual proof of the second theorem

$$\begin{tikzcd}
T' \arrow[r, "\exists!"'] \arrow[rr, bend left, "1_{T'}"] & T \arrow[r, "\exists!"'] \arrow[rr, bend right, "1_T"'] & T' \arrow[r, "\exists!"'] & T
\end{tikzcd}$$


Nooooooo! You can't just draw a picture and claim it's a mathematical proof. You need to rigorously prove the claim instead!
A picture is a valid mathematical proof
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34%
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IQ score
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85
100
115
130
145

Functors

Definition

A (covariant) **functor** $F \colon \mathbf{C} \to \mathbf{D}$ takes an object in C in $\mathbf{C}$ and returns an object $F(C)$ in $\mathbf{D}$. For any morphism $f \colon C \to C'$ there is an associated morphism $F(f) \colon F(C) \to F(C')$ where F preserves the composition of morphisms $F(f \circ g) = F(f) \circ F(g)$ and $F(1_A) = 1_{F(A)}$.

It is easy to see from the composition laws that functors preserve isormophisms. Functors preserve commutative diagrams up to variance.

Examples of functors

- The dual space functor on vector spaces $(-)^* \colon \mathbf{Vect}_{\mathbb{K}}^{\mathrm{op}} \to \mathbf{Vect}_{\mathbb{K}}$, where $V \mapsto V^*$ and $L \colon V \to W \mapsto L^* \colon W^* \to V^*$ where the associated morphism is the precomposition by L. Not that $(gf)^* = f^* g^*$ because the domain of the functor is an opposite category.

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- A forgetful functor (literally forgets structure). Example $\text{Forget} \colon \mathbf{Ring} \to \mathbf{Ab}$ forgets the multiplicative structure.

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- The hom functors $\hom(A, -)\colon \mathbf{C} \to \mathbf{Set}$ with post composition and $\hom(-, A)\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ with pre composition are one of the most imporant ones where. For example, the dual space functor is $\hom(-, \mathbb{K})\colon \mathbf{Vect}_{\mathbb{K}}^{\mathrm{op}} \to \mathbf{Set}$ if we do not equip the dual space with a vector space structure.

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- **Presheaf** on a topological space X is a functor $\mathrm{Op}(X)^{\mathrm{op}} \to \mathbf{Set}$.

Natural transformations

Definition

Given a categories **C**, **D** and pair of functors $F, G\colon \mathbf{C} \rightrightarrows \mathbf{D}$ a natural transformation $\eta\colon F \Rightarrow G$ between the functors is collection of morphisms $\eta_C\colon F(C) \to G(C)$ for each object C in **C** such that the following square commutes

$$\begin{tikzcd} F(C) \arrow[r, "\eta_C"] \arrow[d, "F(f)"'] & G(C) \arrow[d, "G(f)"] \\ F(D) \arrow[r, "\eta_D"'] & F(D) \end{tikzcd}$$

If each of the components of the natural transformation is an isormorphism we call it a natural isormophism. Heuristicly this is an isomorphism with "lack of arbituary choices".

Natural transformations

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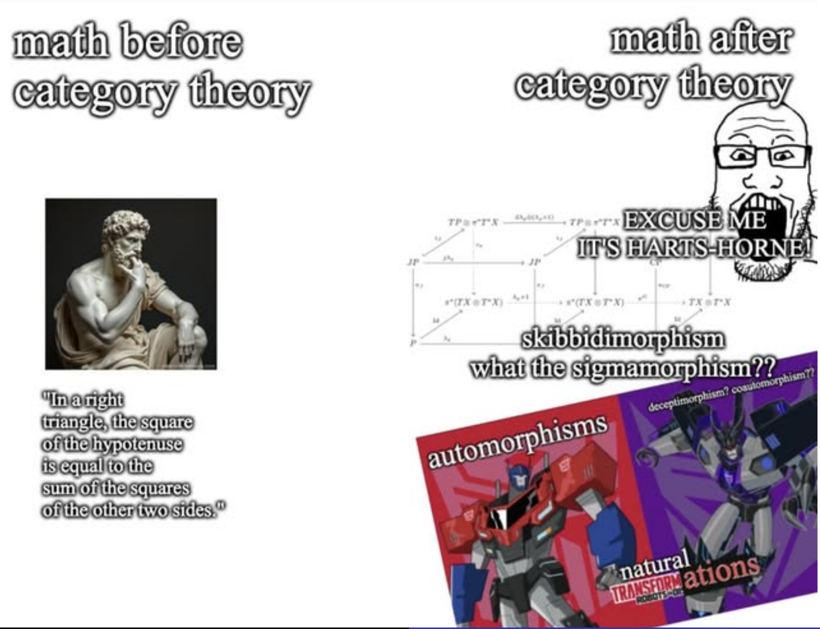
- Hurewicz homomorphism gives that for path connected spaces, $\pi_n(-) \Rightarrow H_n(-)$ and $\mathrm{Ab}\,\pi_1(-) \cong H_1(-)$.
- Two categoris are said to be **equivalent**, if there exists pair of functors $F\colon \mathbf{C} \rightleftarrows \mathbf{D} : G$ with natural isormophisms $GF \cong 1_{\mathbf{C}}$ and $FG \cong 1_{\mathbf{D}}$. If we make the symbols $\cong$ into equalities, then we will talk about isomorphism of categories. Equivalences preserve many of essential categorical properties. The ones not preserved are called **evil**.

Facebook meme



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- The representations of a finite group as a category is isormorphic to representations of it's group algebra.
- Every category is equivalent to it's **skeleton** which has objects as one representative of an isomorphism class of the original category.
- Category of totally disconnected (singletons are the connected components) Hausdorff spaces is oppositely equivalent to the category of Boolean algebras.

Composition

Theres an obvious **vertical** composition of natural transformations.

$$
\begin{tikzcd}
F(A) \arrow[r, "\eta_A"] \arrow[d, "Ff"'] & G(A) \arrow[r, "\mu_B"] \arrow[d, "Gf"'] & K(A) \arrow[d, "Kf"] \\
F(B) \arrow[r, "\eta_B"] & G(B) \arrow[r, "\mu_B"] & K(B)
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbf{C} \arrow[rr, bend left=60, "F"{name=U}] \arrow[rr, "G"{name=M}] \arrow[rr, bend right=60, "K"'{name=L}] & & \mathbf{D}
\arrow[Rightarrow, from=U, to=M, "\eta"']
\arrow[Rightarrow, from=M, to=L, "\mu"']
\end{tikzcd}
$$

Composition 2

There is also something called the **horizontal** composition.

$$
\begin{tikzcd}
\mathbf{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathbf{D} \arrow[r, bend left, "F'"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\mu"] & \mathbf{E}
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
\mathbf{C} \arrow[rr, bend left, "F'F"] \arrow[rr, bend right, "G'G"'] \arrow[rr, phantom, "\mu * \eta \Downarrow"] & & \mathbf{E}
\end{tikzcd}
$$

$$
\begin{tikzcd}
F'F(C) \arrow[r, "\mu_{F(C)}"] \arrow[d, "F'\eta_C"'] \arrow[dr, dotted, "(\mu * \eta)_C"] & G'F(C) \arrow[d, "G'\eta_C"] \\
F'G(C) \arrow[r, "\mu_{G(C)}"'] & G'G(C)
\end{tikzcd}
$$

Equivalences in locally small categories

Definition

A functor is **full** if it is surjective on homsets. Simularly a functor is **faifthful** if it is injective on homsets. Combining both definitions we get **fully faithful** functors i.e. bijective on level of homsets. A functor is **essentially surjective** if every object in the image has an object isomorphic to it. An **embedding** is faifthful functor that acts "injectively" on objects.

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Theorem

A functor defining an equivalence is fully faifthful and essentially surjective. Converse holds assuming the axiom of choice.

Theorem

(The Yoneda lemma) Given a functor $F\colon \boldsymbol{C} \to \boldsymbol{Set}$ with $\boldsymbol{C}$ locally small, there is a bijection

$$\mathrm{Nat}(\hom(c,-),F) \cong Fc$$

that is natural is both F (here F is viewed as an object in the functor category $\boldsymbol{Set}^{\boldsymbol{C}}$) and c given by $\eta \mapsto \eta_c(1_c)$.

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Corollary

(Yoneda embeddings) The bifunctor $\hom(-,-)\colon \boldsymbol{C}^{op} \times \boldsymbol{C} \to \boldsymbol{Set}$. Defines fully faithful embeddings $\boldsymbol{C} \hookrightarrow \boldsymbol{Set}^{\boldsymbol{C}^{op}}$ and $\boldsymbol{C}^{op} \hookrightarrow \boldsymbol{Set}^{\boldsymbol{C}}$.

Definition

The category of elements $\int F$ of a set valued functor $F\colon \mathbf{C} \to \mathbf{Set}$ consists of pairs (C, x) with $x \in F(C)$. A morphism $(C, x) \to (C', x')$ is a morphism $f\colon C \to C'$ with $Ffx = x'$.

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Proofs of these rely on the proof of Yoneda lemma.

Example of initial object in the category of elements

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Example of initial object in the category of elements

Let $\mathrm{Bilin}_{\mathbf{Vect}_{\mathbb{K}}}(V, W; -) \colon \mathbf{Vect}_{\mathbb{K}} \to \mathbf{Set}$ the functor of bilinear maps. This functor acts on linear maps by precomposition. Then $\int \mathrm{Bilin}_{\mathbf{Vect}_{\mathbb{K}}}(V, W; -)$ consists of pairs (E, L) with E being a $\mathbb{K}-$vectorspace and $L \colon V \times W \to E$ a bilinear map. A morphism $M \colon (E, L) \to (E', L')$ is a linear map with

$$\begin{tikzcd} V \times W \arrow[r, "L"] \arrow[d, "L'"'] & E \arrow[dl, "M"] \\ E' & \end{tikzcd} .$$

The universal property of the tensor product says that $- \otimes - \colon V \times W \to V \otimes_{\mathbb{K}} W$ defines an inital object in the category of elements (hence the name universal property). Therefore $V \otimes_{\mathbb{K}} W$ also represents the functor $\mathrm{Bilin}_{\mathbf{Vect}_{\mathbb{K}}}(V, W; -)$.

Limits

Definition

A small **diagram** of shape $\mathbf{J}$ in $\mathbf{C}$, is a functor $F\colon \mathbf{J} \to \mathbf{C}$ with $\mathbf{J}$ being a small category.

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A **cone** over diagram $F\colon \mathbf{J} \to \mathbf{C}$ with a **summit** C is a natural transformation $\lambda\colon c \Rightarrow F$ (thinking of C as a constant natural transformation). So a family of morphisms $(\lambda_j\colon C \to Fj)_{j\in\mathbf{J}}$ that satisfies

$$\begin{tikzcd}
& C \arrow[dl, "\lambda_i"'] \arrow[dr, "\lambda_j"] & & j \arrow[d, "f"] \\
Fj \arrow[rr, "Ff"] & & Fi & i
\end{tikzcd}$$

Limits

Definition

Dually there are cones **under** or a **cocone** of a diagram, with a **nadir** C and a natural transformation $\lambda \colon F \Rightarrow C$ where

$$\begin{tikzcd} Fj \arrow[rr, "Ff"] \arrow[dr, "\lambda_j"'] & & Fi \arrow[dl, "\lambda_i"] & j \arrow[d, "f"'] \\ & C & & i \end{tikzcd}$$

Limits

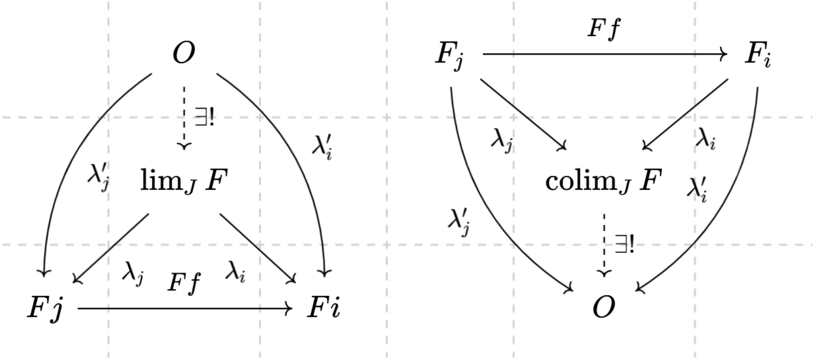
Both cones and cocones form a category with the morphism being m such that

$$
\begin{tikzcd}
 & O \arrow[d, "m"] \arrow[ddl, bend right, "\lambda'_j"'] \arrow[ddr, bend left, "\lambda'_i"] & \\
 & C \arrow[dl, "\lambda_j"] \arrow[dr, "\lambda_i"'] & \\
Fj \arrow[rr, "Ff"] & & Fi
\end{tikzcd}
\qquad
\begin{tikzcd}
F_j \arrow[rr, "Ff"] \arrow[dr, "\lambda_j"'] \arrow[ddr, bend right, "\lambda'_j"'] & & F_i \arrow[dl, "\lambda_i"] \arrow[ddl, bend left, "\lambda'_i"] \\
 & C \arrow[d, "m"] & \\
 & O &
\end{tikzcd}
$$

Limits

Definition

A limit $\lim_{\mathbf{J}} F$ over a diagram $F\colon \mathbf{J} \to \mathbf{C}$ is a universal object in the category of cones. Alternative one could define limits as representing objects of so called cone functors.



We would also like to study (co)continuous functors so ones that preserve (co)limits and also functors that reflect (co)limits.

Examples of limits

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- A **product** is limit over diagram with **J** discrete. Examples are cartesian product product topology. Dually it is a called a coproduct in **Set** this is for example the disjoint union. In many familiar categories like **Ab**, finite coproducts and products are isomorphic.

Biproduct

$$\begin{tikzcd}
 & C \arrow[dl, "\lambda'_i"'] \arrow[d, dashed, "\exists!"] \arrow[dr, "\lambda'_j"] & \\
X_i & X_i \prod X_j \arrow[l, "\lambda_i"'] \arrow[r] & X_j
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- An **(co)equalizer** (example: (co)kernel is an equalizer of a homomorphism and the zero map)

$F\colon \{\bullet \rightrightarrows \bullet\} \to \mathbf{C}$

$$\begin{tikzcd}
E \arrow[r, "eq"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
O \arrow[u, dashed, "\exists!"] \arrow[ur, "m"'] & &
\end{tikzcd}$$

Examples of limit

- A direct limit is a colimit indexed by a directed set (such as $1 \to 2 \to 3 \to \dots$). Dually inverse limits are limits over directed sets.
- **Pullback** (dually pushforward) is a limit of a diagram indexed by the category $\bullet \to \bullet \leftarrow \bullet$.

$$
\begin{tikzcd}
O \arrow[dr, dashed, "\exists!"'] \arrow[drr, bend left, "\lambda_1'"] \arrow[ddr, bend right, "\lambda_2'"'] & & \\
& P \arrow[r, "\lambda_1"] \arrow[d, "\lambda_2"'] & C \arrow[d, "f"] \\
& A \arrow[r, "g"'] & B
\end{tikzcd}
$$

Existance of limits and continuity

A limit may not exist. For example the category of fields has no initial or terminal object. Neither does the category of manifolds have arbituary products.

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A functor is **(co)continuous** if it preserves (co)limits.

Adjunctions

A fundamental fact is that equivalenses preserve limits and colimits. But finding an equivalence is not always possible. Hence we have a weaker notion of equivalence called an adjunction of functors which preserve limits in some sense.

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Definition

If for pair of functors $F \colon \mathbf{C} \rightleftarrows \mathbf{D} : G$ there is a natural transformation (in both C and D) $\hom(F(C), D) \cong \hom(C, G(D))$, then F is **left adjoint** to G. Simularly G is **right adjoint** to F. Denote this relation by $F \dashv G$.

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Theorem

Right adjunctions preserve limits, and left adjunctions preserve colimits.

- Tensor-Hom adjunction in **Ab** (or $\mathbb{Z}-\mathbf{Mod}$)
  $\hom(A \otimes B, C) \cong \hom(A, \hom(B, C))$.

Examples and non examples of adjunctions

- Tensor-Hom adjunction in **Ab** (or $\mathbb{Z}-$**Mod**) $\hom(A \otimes B, C) \cong \hom(A, \hom(B, C))$.
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- The Free $\dashv$ Forget adjunction (free vector space, free group, free Abelian group, Free category on a directed multigraph etc... satisfy a certain universal property)
- Theres no right or left adjoint to Forget : **Field** $\to$ **Set**, since theres no inital or terminal object in the category of fields. Therefore it does not preserve limits, meaning theres no free field.

Category theory in programming

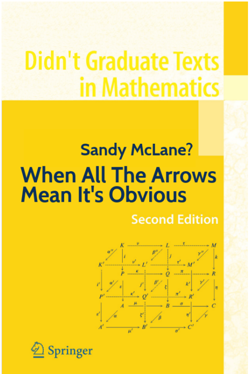
In functional programming one deals with computational side effects by monads. These arise from the units of adjunctions. Moggi's paper "Notions of computation and monads". Functional programming 2 MOOC (which is free) at the University of Helsinki is a good introduction to monads in programming.

Something for the physics students to think about

Definition

A (2d) **TQFT** is a symmetric monoidal functor $Z\colon \mathbf{2COB} \to \mathbf{fdVect}_{\mathbb{K}}$. The first category has morphisms between closed 1-manifolds (disjoint union of circles) as cobordisms, composition by combining cobordism and monoidal operation of disjoint union. Think about pair of pants being a cobordism between circle and disjoint union of two circles. The target category is finite dimensional vector spaces over $\mathbb{K}$ with tensor product.

Books and sources



Category theory for a working mathematician by Saunders MacLane, Category theory in context by Emily Riehl, Wikipedia.com, https://www.dtubbenhauer.com for definition of TQFT. Memes from facebook, diagrams screenshots from q.uiver, since tikz-cd and beamer don't like each other.