Spectral theory notes for Spring 2024 Course at University of Helsinki

Heinonen, Kalle

References: Hall's Quantum theory for mathematicians, Eero Saksman's hand written lecture notes.

We always assume that H is a infinite dimenisonal, complex and separable Hilbert space. So it always has countable orthonormal Hilbert basis. All these types of spaces are isometrically isormorphic to each other, but we are more interested in studying operators represented in certain spaces like L^2, ℓ^2 etc... We denote $(e_n)_n$ be as an countable orthonormal basis of H. We will mainly study operators $H \to H$, so we might aswell just assume every operator is like that unless mentioned otherwise. These notes are not meant as comprehensive study on subject, but rather revision (for me) of main results without presenting much of the proofs.

The goal of spectral theory is to classify linear operators like we have done in linear algebra. In Banach space theory it can already be an impossible task, so thats why from application stand point and from the standpoint of the theory, it is nice to restrict ourselves to study infinite dimensional, complex and separable Hilbert spaces. In linear algebra we are able to solve this classification by studying eigenvalues, eigenvectors, minimal and characteristic polynomials. We want to extend this type of study of Hilbert spaces aswell. For examples in PDE one is very interested in the eigenfunctions of Laplace operator. We build the study all the way up to unbounded operators, since many applicable (for example differential) operators are unbounded.

Contents

1 Spectral theorem for compact operators

We motivate the spectral theory of compact operators by linear algebra, since we can view compact operators are limits of finite rank operators (essentially matrices).

Recap of linear algebra spectral theorem. Let $A \in \mathcal{M}^{d\times d}(\mathbb{C})$. We say that $0 \neq g \in \mathbb{C}^d$ is an **eigenvalue** if there exists some scalar λ for which $Ag = \lambda g$. Finding the eigenvalues corresponds to solving the polynomial equation $\det(A - \lambda I) = 0$. By fundamental theorem of calculus there exists atleast one eigenvalue.

Theorem 1 (Spectral theorem for self-adjoint matrices)**.** *Let A be as above and self-adjoint. Then there exists an orthonormal basis consting of eigenvalues of A.*

It turns out that all the eigenvalues of a self-adjoint matrix are real. This allows us the so called spectral decomposition $A = U\Lambda U^{-1}$, where U is a unitary matrix ($U^{-1} = U^*$) and Λ is a diagonal matrix of eigenvalues.

1.1 Compact operators

Definition 1. A linear operator T on H is **bounded if** if the **operator norm** is bounded,

$$\|T\| := \sup_{x\in H, x\neq 0} \|Tx\|/\|x\| < \infty.$$

There are multiple equivalent definitions of bounded linear operator. Indeed a linear operator is continuous iff it is bounded (up to certain topological vector spaces). We denote the set of all bounded linear operators as $\mathcal{B}(H)$, which forms a Banach space with respect to the operator norm. It also forms a Banach algebra by the inequality $\|AB\| \leq \|A\|\|B\|, A, B \in \mathcal{B}(H)$.

Definition 2. The **adjoint** of an operator $A \in \mathcal{B}(H)$, is defined by

$$\langle Ax, y\rangle = \langle x, A^*y\rangle .$$

The adjoint exists, is unique and $(A^*)^* = A$. Also $\|A\| = \|A^*\|$ and $\|AA^*\| = \|A^*A\| = \|A\|^2$.

Definition 3. Let $(e_n)_n$ be the countable orthonormal basis of H and A a bounded operator on H. Then the **matrix** of A with respect to the basis $(e_n)_n$, is

$$(a_{kl}), \;\; a_{kl} := \langle Ae_k, e_l\rangle .$$

The matrix determines A uniquely. Recall that A is invertible if it has a settheoretical inverse. By linearity the inverse is also linear and bounded by open mapping theorem or closed graph theorem.

Definition 4. The **spectrum** of the operator A is defined as

$$\sigma(A) := \{\lambda \in \mathbb{C} | A - \lambda I \text{ is not invertible}\}.$$

The complement of the spectrum is called **resolvent**, which is denoted $\rho(A)$.

Lemma 1. *Let A be ounded operator with $\|A\| \leq 1$. Then $I - X$ is invertible and $(I - A)^{-1} = I + A + A^2 + A^3 + \dots$. Here the Neumann series converges in operator norm.*

Lemma 2. *Let A be invertible bounded operator. Then if $h \in \mathbb{C}$ and $|h| < \|A^{-1}\|$, also $A - hI$ is invertible and $(A - hI)^{-1} = A^{-1} \sum_{k=0}^{\infty} h^k (A^{-1})^k$.*

Theorem 2. *Let A be bounded operator, then*

1. *$\rho(A)$ is open*
2. *$\sigma(A) \subset \{z \in \mathbb{C} | |z| \leq \|A\|\}$*
3. *$\sigma(A)$ is non empty and closed.*

Definition 5. Bounded operator A is self-adjoint if $A^* = A$.

Definition 6. If A is bounded, we set $R(A) := \mathrm{Ran}(A)$ and $\ker(A)$ being the nullity.

For bounded operators, the kernel is a closed subspace.

Lemma 3. *For bounded operator A, we have $R(A)^{\perp} = \ker(A^*)$.*

Definition 7. Se say a sequence $(x_k)_k$ converges weakly if

$$\langle x_k, y \rangle \to \langle x, y \rangle$$

for all $y \in H$. Denoted $x_k \xrightarrow{w} x$.

Theorem 3. *Assume that $x_k \xrightarrow{w} x, z_k \xrightarrow{w} z$ as $k \to \infty$ in H. Then*

1. *If $x_k \xrightarrow{w} x'$, $x = x'$*
2. *$ax_k + bz_k \xrightarrow{w} ax + bz$*
3. *If A is bounded, then $Ax_n \xrightarrow{w} Ax$ as $k \to \infty$*
4. *if $\|x_n\| \leq c$ for all n, then $\|x\| \leq c$.*

Theorem 4. *Let $(x_n)_n \subset H$ be a sequence for which $\|x_n\| \leq 1$ for all n. THen there exists a subsequence $(x'_n)_n$ which converges weakly in H.*

Definition 8. (Compact operator) Operator A on H is compact if $A\overline{B(0,1)}$ is relatively compact (closure is compact).

Theorem 5. *The following are equivelent for a bounded operator A on H,*

1. *A is compact.*

2. *If $\|x_n\| \leq 1$ for all n, then one may pick a subsequence such that (Ax'_n) converges in norm.*

3. *if $x_k \xrightarrow{w} x$ then $\left\|Tx_k \xrightarrow{w} Tx\right\| \to 0$.*

4. *The image $A(\overline{B(0,1)})$ is compact.*

Corollary 1. *If A is bounded and compact, there is $x \in \overline{B(0,1)}$ such that $\|T\| = \|Tx\|$.*

The previous theorem and corollary does not hold in general for example in Banach spaces.

Definition 9. We denote by $\mathcal{K}(H)$ the set of all compact operators on H.

This is naturally a Banach subspace of $\mathcal{B}(H)$.

Theorem 6. *Operator A on H is compact iff it belongs to the closure of the space of finite rank operators.*

Indeed every finite rank operator is compact and we can essentially think of compact operators as "infinite limits of matrices".

Corollary 2. *The adjoint of a compact operator is compact.*

Definition 10. A bounded operator is called **Hilbert-Schmidt** if the **Hilbert-Schmidt norm** of A is finite

$$\|A\|_{HS}^2 := \sum_{n=1}^{\infty} \|Ae_n\|^2 < \infty.$$

We denote set of these operators by $\mathcal{HS}(H)$.

Theorem 7. 1. *For every sequence of vectors $f_n \in H$ with $\sum_{n=1}^{\infty} \|f_n\|^2 < \infty$ there is a unique $A \in \mathcal{HS}(H)$ with $Ae_n = f_n$ for all n.*

2. *If $(a_{kl})_{k,l=1}^{\infty}$ is the matrix of A, then*

$$\|A\|_{HS}^2 = \sum_{k,l=1}^{\infty} |a_{kl}|^2.$$

3. *$\|A\| \leq \|A\|_{HS}$.*

Theorem 8.
$$\mathcal{HS}(H) \subset \mathcal{K}(H).$$

Definition 11. Let $\Omega \subset \mathbb{R}^d$ be a subdomain and assume that $K : \Omega \times \Omega \to \mathbb{C}$ is Lebesgue measurable, such that for any given $f \in L^2(\Omega)$ the function

$$y \mapsto K(x, y)f(y)$$

is integrable on Ω for a.e. $x \in \Omega$. Then we define for $f \in L^2(\Omega)$ the action of the integral operator with the kernel K by setting

$$Tf(x) = \int_\Omega K(x, y)f(y)dy.$$

Theorem 9. *Let T be as above with $K \in L^2(\Omega \times \Omega)$, then T is Hilbert-Schmidt integral operator on $L^2(\Omega)$, and*

$$\|T\|_{HS}^2 = \int_\Omega \int_\Omega |K(x, y)|^2 dxdy.$$

It is useful to observe that $\mathcal{HS}$ is a very easy class of compact operators to deal with. Especially for an integral operator it is often easy to verify that a given operator is in $\mathcal{HS}$ by the previous thing. We have some important examples like the Green's operator (inverse of laplacian) on a domain in $\mathbb{R}^2$ or $\mathbb{R}^3$.

Theorem 10.
1. $\|A + B\|_{HS} \leq \|A\|_{HS} + \|B\|_{HS}$ *for all* $A, B \in \mathcal{HS}(H)$.
2. $\|ABC\|_{HS} \leq \|A\|\|C\|\|B\|_{HS}$ *for all* $B \in \mathcal{HS}(H), A, C \in \mathcal{B}(H)$.

1.2 Spectral theorem for compact operators

Proof is pretty simular to linear algebra one. We will list here lemmas used for the proof.

Lemma 4. *Assume that A is self-adjoint and bounded operator on H*

1. *If $Au = \lambda u$ with $u \neq 0$, we have that $\lambda \in \mathbb{R}$, and $\langle x, u\rangle = 0$ implies that $\langle Ax, u\rangle = 0$.*
2. *$\langle Ax, x\rangle = 0$ for all $x \in H$ implies that $A = 0$ (Note that this only holds in $\mathbb{C}-$vector space because of polarization identity), In fact $\|A\| = \sup_{\|x\|=1} |\langle Ax, x\rangle|$.*

Lemma 5. *Let $A \in \mathcal{K}(H)$ be self-adjoint and non zero. Then there is $H \ni x \neq 0$ and $\lambda \in \mathbb{R}$ so that $|\lambda| = \|A\|$ and $Ax = \lambda x$.*

Theorem 11 (Spectral theorem for compact operators by Hilbert in 1906)**.** *Let T be a compact self-adjoint operator. There there is an orthogonal countable Hilbert basis $(g_n)_n$ consisting of eigenvectors of T,*

$$Tg_n = \lambda_n g_n$$

where $\lambda_n \in \mathbb{R}$ for all n and $\lambda_n \to 0$ as n grows larger.

This fundamental result can be stated like in the prectral decomposition for matrices. Here T equals a diagonal operator T_λ, $\lambda = (\lambda_n)_n$ that acts on the orthonormal eigenvector Hilbert basis $T_\lambda g_n = \lambda_n g_n$.

Definition 12. A bounded operator is unitary if $UU^* = I = U^*U$.

Theorem 12. *Let U be bounded operator on H. The following are equivalent.*

1. *$\|Ux\| = \|x\|$ for all $x \in H$ and U has dense range.*
2. *U is unitary*
3. *U maps any orthonormal basis to another orthonormal basis of H (Hilbert basis).*

Theorem 13 (Spectral theorem for compact operators 2nd form)**.** *Let T be compact and self-adjoint operator on H. Then there is a **diagonal operator** of eigenvalues of T denoted Λ_λ and unitary operator U, such that $T = U\Lambda_\lambda U^*$.*

1.3 Application of the spectral theorem for compact operators

Definition 13. Bounded operator T is **positive** if $\langle Tx, x\rangle \geq 0$ for all $x \in H$.

We write this by $T \geq 0$. For two bounded operators A, B the notation $A \geq B$ means that $A - B \geq 0$.

Theorem 14. *Let T be positive bounded operator, then T is self-adjoint.*

As a remark in some sources the above condition is called "non-negative" rather than positive. As first application of the spectral theorem, a positive compact operator has a unique square root.

Theorem 15. *For compact operator $T \geq 0$, there exists a unique positive compact operator A such that $A^2 = T$. Denote $A := \sqrt{T} := T^{1/2}$.*

Theorem 16. *Let T be compact operator. Set $|T| := (T^*T)^{1/2}$. Then there is a bounded operator U, such that $U|\overline{R(|T|)}$ is an isometry, $U|\overline{R(|T|)}^{\perp} = 0$ and $T = U|T|$ (polar decomposition).*

Here $|T|$ is called **absolute value** of the operator, and U is called a **partial isometry**. Also note that $|T| \in \mathcal{K}(H)$.

Definition 14. Let T be compact. Order the eigenvalues of $|T|$ as a sequence

$$s_1 \geq s_2 \geq \cdots \geq s_n \to 0.$$

One denotes $s_n =: s_n(T)$, and the numbers $s_n(T)$ are called the **singular values** of T.

As a remark, the eigenvalues of $|T|$ are repeated as many times in the sequence as is their geometric multiplicity i.e. number of accurances in the diagonal in the spectral decomposition. For λ this is $\dim\ker(|T|-\lambda I)$.

Let T be compact operator on H, then we have the polar decomposition $T = U|T|$. One can thus write

$$|T|x = \sum_{n\geq 1} s_n \langle x, \varphi_n\rangle \varphi_n$$

where sum is over nonzero singularvalues and corresponding eigenvectors φ_n. Since U is an isometry on $\overline{R(T)}$, we get another orthonormal sequence $\psi_n = \underline{T\varphi_n}.$ Thus we obtain if $UU^* = P_M$, then U is a partial isometry equivalently $U|\overline{R(T)}$ is an isometry and $U|\overline{R(T)}^{\perp} = 0$.

Theorem 17. *A compact operator T cna be written as*

$$Tx = \sum_{n\geq 1} s_n \langle x, \varphi_n\rangle \psi_n$$

where $(\varphi_n)_n, (\psi_n)_n$ are orthonormal sequences.

Definition 15. A compact operator belongs to the **Schatten class $\mathcal{S}_p$**, if

$$\|T\|_{S_p} := \left(\sum_{n=1} s_n(T)^p\right)^{1/p} < \infty.$$

We call the above **Schatten $p-$norm**. Case $p=2$ equivalent to the Hilbert-Schmidt class

$$\|T\|_{HS} = \sum_{n\geq 1} s_n^2.$$

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Definition 16. The class $p=1$ is called **trace class**.

Theorem 18. *Let $T \in \mathcal{S}_1(H)$, then*

$$\|T\|_{S_1} = \sup_{(e_n)_n,(f_n)_n} \sum_{n\geq 1} |\langle Te_n, f_n\rangle|.$$

Here the supremum is over all ortonormal sequences $(e_n)_n, (f_n)_n$ of same length.

Theorem 19. *Let T be compact, then*

1. $\|T^*\|_{S_1} = \|T\|_{S_1}$.
2. $\|ATB\|_{S_1} \leq \|A\|\|B\|\|T\|_{S_1}$ *for all A, B bounded.*
3. $\|T\|_{S_1} = \min\{\|A\|_{HS}\|B\|_{HS} | A, B \in \mathcal{HS}(H), AB = T\}$.
4. $\|T+U\|_{S_1} \leq \|T\|_{S_1} + \|S\|_{S_1}$ *for $S \in \mathcal{S}_1$.*

5. *Finite rank operators are dense in* $\mathcal{S}_1(H)$ *with respect to the Schatten* $1-$*norm.*

Definition 17. The **trace** of $T \in \mathcal{S}_1(H)$ is the quantity

$$\operatorname{tr}(T) := \sum_{n=1}^{\infty} \langle Te_n, e_n \rangle .$$

Theorem 20. *The trace is independent of the orthonormal basis used.*

Theorem 21. *Let* T *be in trace class*

1. $|\operatorname{tr}(T)| \leq \|T\|_{S_1}$.
2. $T \mapsto \operatorname{tr}(T)$ *is a bounded linear functional of* $\mathcal{S}_1(H)$.
3. $\operatorname{tr}(AT) = \operatorname{tr}(TA)$ *for bounded* A *(or for both* $A, B \in \mathcal{HS}(H)$*).*
4. $\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}$.
5. $\operatorname{tr}(A) = \|A\|_{S_1}$ *if* $A \geq 0$.

It is often easy to check when a given integral operator is $\mathcal{HS}$ just by checking if the kernel is L^2-integrable. However, checking membership of the trace class can be more elusive.

Theorem 22. *Let* $T \in \mathcal{HS}(L^2(0,1))$. *Then there is a function* $K : (0,1)^2 \to \mathbb{C}$ *such that* $K \in L^2([0,1]^2)$ *and* T*is the integral operator defined by the kernel* K.

Theorem 23 (Mercer)**.** *Assume that* $K \in C([0,1]^2)$ *is realvalued and symmetric (*$K(x,y) = K(y,x)$*) and the operator* $T \in \mathcal{B}(L^2(0,1))$ *defined by the kernel* K *is positive. Then* T *is trace class and*

$$\operatorname{tr}(T) = \int_0^1 k(x,x)dx.$$

Also if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \to 0$ *are the eigenvalues (with respect to eigenfunctions* $\varphi_1, \varphi_2, \dots$*), we may write for every* $x, y \in [0,1]$

$$K(x,y) = \sum_{n \geq 1} \lambda_n \varphi_n(x) \varphi_n(y)$$

with absolute and uniform convergence on $[0,1]^2$.

The proof follows from few lemmas we list down below.

Lemma 6. $\varphi_n \in C([0,1])$ *for all* $n \geq 1$.

Lemma 7. *Let* $f \in C([0,1])$. *Then (here the inner product is* L^2*)*

$$Tf(s) = \sum_{n=1}^{\infty} \lambda_n \langle f, \varphi_n \rangle \varphi(s) \ \forall s \in [0,1],$$

where the convergence is absoolute and uniform.

Lemma 8. *$k(t,t) \geq 0$ for all $t \in [0,1]$.*

Lemma 9 (Dini's theorem)**.** *Let $f_1, f_2, \dots \in C([0,1])$ and assume that $\lim_{n \nearrow \infty} f_n(t) = f(t)$ monotonically for every $t \in [0,1]$. Then the convergence is uniform.*

As our final application we sketch how to reat the spectrum of Laplace operator on bounded smooth domain $\Omega \subset \mathbb{R}^d$. For the sake of clarity the exposition we will assume first that $d = 2$ and often only look at the case $\Omega = \mathbb{D} = \{z \in \mathbb{C} | |z| = 1\}$.

Definition 18. Let $\Omega \subset \mathbb{R}^2$ be bounded, smooth and simply connected domain. Then Green's function (with Dirichlet boundary conditions) is the function $G = G_\Omega : \Omega \times \Omega \to \mathbb{R}$ such that

$$G(z,w) := \frac{1}{2\pi} \log \left| \frac{1 - f(z)\overline{f(w)}}{f(z) - f(w)} \right|,$$

where $f : \Omega \to \mathbb{D}$ is a conformal map.

Equivalently for any $w \in \Omega$, $z \mapsto G(z,w)$ is integrable function that is smooth in $\Omega \setminus \{w\}$, satisfies

$$\lim_{z \to z_0} G(z,w) = 0$$

for every $z_0 \in \partial\Omega$,

$$\Delta_z(G(z,w)) = \delta_w,$$

where $\Delta = \partial_x^2 + \partial_y^2$ in the sense of distributions. We are using the complex notation for $\mathbb{R}^2$.

One always has $G(z,w) = G(w,z)$. We will mainly consider the case when $G_{\mathbb{D}}(z,w) = \frac{1}{2\pi} \log \left| \frac{1 - z\overline{w}}{z-w} \right|$.

Definition 19. Green's operator $G_\Omega : L^2(\Omega) \to L^2(\Omega)$ is the integral operator with kernel G_Ω, thus

$$G_\Omega f(z) := \int_\Omega G_\Omega(z,w) \mathcal{L}(A) \text{ (two dimensional Lebesgue measure)}.$$

Well definedness of Green's operator follows from following lemma

Lemma 10. *G_Ω is Hilbert-Schmidt.*

Lemma 11. *Let $\Omega \subset \mathbb{R}^2$ be as before.*

1. *If $f \in L^2(\Omega)$, we have that $G_\omega f \in C^\alpha(\overline{\Omega})$ (Hölder continuous functions) for some $\alpha > 0$ with $G_\omega f | \partial\Omega = 0$.*
2. *If $f \in C^\alpha(\overline{\Omega})$, we have that $-\Delta(G_\Omega f) = f$ on Ω.*

Theorem 24. *Let $\Omega \subset \mathbb{R}^2$ be as before*

1. *$G_\Omega : L^2(\Omega) \to L^2(\Omega)$ is positive Hilbert-Schmidt operator with all its eigenvalues*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \to 0$$

strictly positive.

2. *An eigenfunction φ_n with eigenvalue λ_n satisfies*

$$-\Delta \varphi_n = \frac{1}{\lambda_n} \varphi_n, \quad \varphi_n | \partial\Omega$$

and $\varphi_n \in C^1(\overline{\Omega}) \cap C^{2+\alpha}(\Omega)$ for some $\alpha > 0$. Conversly if $\varphi \in C^1(\overline{\Omega})$ with $\varphi | \partial\Omega = 0$ and $\varphi \in C^{2+\alpha}(\Omega)$ and

$$-\Delta \varphi = \mu \varphi$$

on Ω, then φ is an eigenfunction of G_Ω and $1/\mu$ is an eigenvalue of G_Ω. Especially $\mu > 0$.

All in all, the previous shows that the Laplace eigenfunctions can be recovered via eigenfunctions of a compat self-adjoint operator G_Ω at least in case of smooth, simply connected planar domain.

Corollary 3. *Laplace eigenfunctions in a simply connected bdd smooth domain $\Omega \subset \mathbb{R}^2$ are a basis for $L^2(\Omega)$ and the eigenvalues form a sequence $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \to \infty$.*

This result also holds in all dimensions and some less smooth domains.

2 Spectral theorem for bounded operators

2.1 Couple analysis results

We start by recalling some analysis results needed in this section. These results appear in the proof for spectral therem of bounded operators. First thing is general measures on compact subsets of $\mathbb{R}^d$. If not specificly indicated, every measure that we use will be a Radon measure on a borel measurable subset of $\mathbb{R}^d$ for some $d \geq 1$.

Definition 20. A **Radon measure** is a non-negative measure on the $\sigma-$algebra of Borel subsets of $\mathbb{R}^d$ that is finite on all compact sets, and is inner and outer regular on all Borel subsets.

Theorem 25. *Every locally finite borel measure on $\mathbb{R}^d$ is Radon.*

Local finiteness here means that each point has a neighbourhood that has finite measure and each compact set has finite measure. Also remark that a measure μ is **outer regular** on a borel set $A \subset \mathbb{R}^d$ if $\mu(A) = \inf\{\mu(U) | A \subset U \Subset \mathbb{R}^d\}$ and **inner regular** if $\mu(A) = \sup\{\mu(K) | K \subset A, K \text{ is compact}\}$. Note thatevery Radon measure on $\mathbb{R}^d$ is $\sigma-$compact since $\mathbb{R}^d = \cup_{n\in\mathbb{N}} \overline{B(0,n)}$ and $\mu(\overline{B(0,n)}) < \infty$ for all n.

Agreement! from now on (in not stated else) every measure on $\mathbb{R}^d$ will be a Radon measure.

Lemma 12. *A Radon measure μ restricted to any opensubset U of $\mathbb{R}^d$ is determined by the integrals*

$$\int_\Omega f(x)\mu(dx),$$

where $f \in C_c(\Omega)$ (continuous functions with compact support).

Theorem 26. *If μ is a measure supported on $(a,b]$, define $F(x) = \mu(a,x]$ for $x \in [a,b]$. Then F satisfies: F is increasing and right continuous, $F(a) = 0$.*

Conversely, given any F on $[a,b]$ with these properties, there is a unique measure μ supported on $(a,b]$ such that $\mu((a,x]) = F(x)$ for all $x \in [a,b]$.

Remark: $F = F_\mu$ is called the **distribution function** of the measure μ.

Theorem 27 (Riesz)**.** *Assume that $\Lambda : C([0,1]) \to \mathbb{R}$ is a positive and linear functional. Then there is a unique measure μ on $[0,1]$ such that*

$$\Lambda(f) = \int_0^1 f(x)\mu(dx)$$

for all $f \in C(0,1)$.

the last "real analysis" result we need is the density of polynomials in $C(0,1)$ or more generally $C(K)$ for compact K.

Theorem 28. *Let $f : [-1,1] \to \mathbb{R}$ be continuous and $\varepsilon > 0$. Then there is a polynomial p such that*

$$|p(x) - f(x)| \leq \varepsilon \quad \textit{for all } x \in [-1,1].$$

Stone-Weierstarss generalizes this for any compact Hausdorff space K with maps $C(K, \mathbb{R})$.

Theorem 29. *1. Let $K \subset \mathbb{R}$ be compact. Then $f \in C(K)$ may be uniformly approximated (by restrictions to K of) polynomials.*

2. Let $K \subset \mathbb{R}^2$ be compact. Then any $f \in C(K)$ may be uniformly approximated by polynomials in two variables simularly.

We finish our real-analysis recap by the very useful monotone class theorem with typical application.

Theorem 30 (Monotone class theorem)**.** *Let $\mathcal{A}$ be an algebra of subsets of fixed set E (thus it is closed under complementation and finite union). Assume that $\mathcal{B} \subset \mathcal{P}(E)$ is a set of subsets of E such that it contains $\mathcal{A}$ and is a **monotone class**:*

1. *If $A_1 \subset A_2 \subset \cdots$ and $A_j \in \mathcal{B}$ for all $j \geq 1$, then $\cup_{j=1}^{\infty} A_j \in \mathcal{B}$.*
2. *If $A_1 \supset A_2 \supset \cdots$ and $A_j \in \mathcal{B}$ for all $j \geq 1$, then $\cap_{j=1}^{\infty} A_j \in \mathcal{B}$.*

Then $\mathcal{B}$ contains the $\sigma-$algebra generated by $\mathcal{A}$, or in otherwords $\sigma(\mathcal{A}) \subset \mathcal{B}$.

The formulation for bounded functions can be done in various ways, we'll pick an odten used one. Recall that a $\pi-$**system** of subsets of E means that it is nonempty and closed under intersections.

Theorem 31 (Monotone class theorem for functions)**.** *let $\mathcal{A}$ be a $\pi-$system on E with $E \in \mathcal{A}$, and let $\mathcal{F}$ be a collection of bounded functions $E \to \mathbb{R}$ such that:*

1. *If $A \in \mathcal{A}$, then $\chi_A \in \mathcal{F}$.*
2. *If $f, g \in \mathcal{F}$ and $a, b \in \mathbb{R}$, then $af + bg \in \mathcal{F}$.*
3. *If $0 \leq f_1(x) \leq f_2(x) \leq \cdots \leq f_n(x) \leq c$ for all $n \geq 1$, $x \in E$, then $f \in \mathcal{F}$, where $f(x) = \lim_{n\to\infty} f_n(x)$.*

Then $\mathcal{F}$ contains all bounded functions on E that are measurable with respect to $\sigma(\mathcal{A})$.

Corollary 4. *Let $K \subset \mathbb{R}^d$ be compact. let $\mathcal{F}$ be a class of bounded functions $K \to \mathbb{R}$ (or $\mathbb{C}$) such that*

1. *$\mathcal{F}$ is a real (complex) vector space.*
2. *$\mathcal{F}$ contains $C(K)$ (complex valued $C(K)$).*
3. *$\mathcal{F}$ is closed under monotone pointwise convergence of uniformlybounded elements from $\mathcal{F}$.*

Then $\mathcal{F}$ contains all bounded (complex valued) Boreal measurable functions on E.

Theorem 32. *Let $K \subset \mathbb{R}^d$ be compact and let μ be a finite Borel measure on K. Then $C(K)$ is dense in $L^p(K, d\mu)$ for all $1 \leq p < \infty$.*

Basic example. Let $H = L^2(0,1)$ and define the multiplication operator

$$Mf(x) = xf(x)$$

for all $f \in H$. We would like to find a spectral decomposition for M. Clearly M is bounded and self-adjoint with $\|M\| = 1$. However, there is no eigenvectors for this operator. If $Mf = \lambda f$, for some complex scalar λ, we should ahve $(\lambda - x)f(x) = 0$ for a.e. $x \in (0,1)$. This implies that $f \equiv 0$ a.e. One checks $\sup_{\|f\|_{L^2(0,1)}=1} \langle Mf, f\rangle = 1$, but there is no function for which the sup is achieved. How to deal with M? Note that for the compact diagonal operator $D : \ell^2 \to \ell^2$ with $Dx = (x_1, x_2/2, x_3/3, \dots)$ we may find the eigenvectors using the spectral projections $P_\lambda, \lambda \in (0,1)$:

$$P_\lambda := \sum_{\lambda_n \le \lambda} \langle x, e_n\rangle e_n, \quad \lambda_n = \frac{1}{n}.$$

Simply $Mx = \sum_{n=1}^\infty \lambda_n (P_{\lambda_n} - P_{\lambda_{n-1}})x = \sum_{n=1}^\infty \lambda_n \langle x, e_n\rangle e_n$. If $f \in L^2(0,1)$ is supported on $[a,b]$ $(0 \le a < b \le 1)$, then $a\|f\|^2 \le \langle Mf, f\rangle \le b\|f\|^2$.

Define $M_\lambda = \{f \in L^2(0,1) | f(x) = 0 \text{ if } x \notin (0,\lambda)\}$ and let P_λ be the orthogonal projection on M_λ. Then one may check easily that

$$\left\| M - \sum_{k=1}^n \frac{k}{n}(P_{\frac{k}{n}} - P_{\frac{k-1}{n}}) \right\| \to 0$$

as n grows large. Thus a natural replacement for a spectral theorem might be something like

$$M = \int_0^1 \lambda dP_\lambda.$$

This turns out to be ture for all bounded self-adjoint operators. Note that in our situation, if $Nf(x) = \sqrt{x} f(x)$, then $N^2 = M$. Simularly we might write $N = \int_0^1 \sqrt{x} dP_\lambda$. Turns out this can be done for all bounded Borel functions in place of $\sqrt{x}$.

We star by defining in general spectral resolutions of the type $\{P_\lambda\}_{\lambda \in [a,b]}$ as in above. Recall first

Definition 21. Let $T_1, T_2, \dots, T \in \mathcal{B}(H)$. We say that $T_n \xrightarrow{s} T$ as n grows large if $\|T_n x - Tx\| \to 0$ for all $x \in H$ as n grows large. This is called convergence in the strong operator topology.

Theorem 33. *1.* $T_n \to T \Rightarrow T_n \xrightarrow{s} T \Rightarrow T_n \xrightarrow{w} T$

2. Let $A, B, T_1, T_2, \dots, T \in \mathcal{B}(H)$. *If* $T_n \xrightarrow{s} T \Rightarrow T_n \xrightarrow{s} T$. *Also if* $T_n \xrightarrow{w} T \Rightarrow AT_nB \to ATB$.

2.2 Projection valued measures

Motivation. Unlike with compact operators, we have hard time expressing the bounded and unbounded spectral theorem in terms of diagonal operator and unitary operator. If $A \in \mathcal{B}(H)$ we would like to associate each borel set $E \subset \sigma(A)$ a closed subspace V_E of H, where we think intuitively that V_E is the closed span of the generalized eigenvectors for A with eigenvalues in E. So basically we would like to have the following properties

1. $V_{\sigma(A)} = H$ & $V_\varnothing = \{0\}$.
2. If $E \cap F = \varnothing$, then $V_E \perp V_F$.
3. For any E and F, $V_{E\cap F} = V_E \cap V_F$.
4. If $E_1, E_2, \dots$ are disjoint and $E = \cup_{j\geq 1} E_j$, then
$$V_E = \bigoplus_j V_{E_j}.$$
5. For any E, V_E is invariant under A.
6. If $E \subset [\lambda - \varepsilon, \lambda + \varepsilon]$ and $\psi \in V_E$, then
$$\|(A - \lambda I)\psi\| \leq \varepsilon \|\psi\|.$$

We call V_E's **spectral subspaces**. Condition 1. captures the idea that generalized eigenvectors should span H. Propoerty 2. captures the idea that our generalized eigenvectors should posess some sort of orthogonality for distinct eigenvalues even if they are not actually in the Hilbert space. In 4. the direct sum is in sense of Hilbert space. Properties 5. and 6. capture the idea that V_E is made up of generalized eigenvectors for A with eigenvalues in E.

It is convinient to assume that our subspaces are closed Hilbert subspaces so that our projection operators in infact unique. Also we have that projection operators are infact self-adjoint. Our above given properties remind us little bit of measures.

Definition 22 (**Projection-valued measure**)**.** Let X be a set and Ω the $\sigma-$algebra in X. A map $\mu : \Omega \to \mathcal{B}(H)$ is called a **projection-valued measure** if

1. For each $E \in \Omega$, $\mu(U)$ is an orthogonal projection.
2. $\mu(\varnothing) = 0$ & $\mu(X) = I$.
3. If $E_1, E_2, \dots$ in Ω are disjoint, then for all $v \in H$, we have
$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) v = \sum_{j=1}^{\infty} \mu(E_j) v,$$
where the convergence of the sum is in norm of H (strong operator topology).
4. For all $E_1, E_2 \in \Omega$, we have $\mu(E_1 \cap E_2) = \mu(E_1)\mu(E_2)$.

Note a great example to keep in mind is $E = (0,1), \Omega = \{\text{Borel sets of } (0,1)\}$, $H = L^2(0,1)$ and $\mu_A = \chi_A f$ for all $A \in \Omega$.

Directly from the definition we can deduce for disjoint E_1, E_2, that $\mu(E_1)\mu(E_2) = 0$. Also in addition one can verify that $\mu(E_1)\mu(E_2)$ is projection onto the intersection. If we define $V_E := R(\mu(E))$, this measure definition thus satisfies our first four properties of the motivation.

Lemma 13. *Let μ be projection valued measure. Then for all sets beloning to Ω*

1. *If $A \cap B = \varnothing$, then $R(\mu(A)) \perp R(\mu(B))$ and $\mu(A \cup B) = \mu(A) + \mu(B)$.*
2. *in general $R(\mu(A \cap B)) = R(\mu(A)) + R(\mu(B))$.*

Next we will try to assosiate any bounded operator A a projection-valued measure μ^A. Here $\mu^A(E)$ will be thought of as a projeciton onto the spectral subspace corresponding to E. With these projection-valued measures we can introduced operator valued integration with respect the the projection-valued measure. In the case of projection-valued measure μ^A associated with A, this operator valued integral will be the **functional calculus** for A.

As an observation for any projection-valued measure, we can get an ordinary positive real-valued measure μ_ψ, $\psi \in H$ by assigning $\mu_\psi(E) = \langle \mu(E)\psi, \psi \rangle$. This links the operator-valued integration to real-valued measure integration.

Theorem 34. *Let Ω be $\sigma-$algebra in a set X and let $u : \Omega \to \mathcal{B}(H)$ be a projection-valued measure. Then there exists a unique linear map, denoted $f \mapsto \int_\Omega f d\mu$, from the space of bounded, measurable, complex-valued functions on Ω into $\mathcal{B}(H)$ with the property that*

$$\left\langle \psi, \left(\int_X f d\mu \right) \psi \right\rangle = \int_X f d\mu_\psi$$

for all f and $\psi \in H$, where μ_ψ is given as above. This integral has additional properties

1. *For all $E \in \Omega$, we have*
$$\int_X \chi_E d\mu = \mu(E)$$
where in particular the integral of constant 1 has value I.
2. *For all f, we have*
$$\left\| \int_X f d\mu \right\| \leq \sup_{\lambda \in X} |f(\lambda)|.$$
3. *Integration is multiplicative*
$$\int_X f d\mu \int_X f d\mu = \int_X fg d\mu.$$
4. *We have an adjoint property*
$$\int_X \overline{f} d\mu = \left(\int_X f d\mu \right)^*.$$

Note: one may ofcourse write $\int_E f d\mu = \int_E f(\lambda)\mu(d\lambda)$.

Lemma 14 (Polarization indentities)**.** *For any $x, y \in H$ and $A \in \mathcal{B}(H)$*

1. $$\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 - i\|x+iy\|^2 + i\|x-iy\|^2)$$

2. $$\langle x, Ay\rangle = \frac{1}{4}(\langle x+y, A(x+y)\rangle + \langle x-y, A(x-y)\rangle + i\,\langle x+iy, A(x+iy)\rangle + \langle (x-iy), A(x-iy)\rangle).$$

3. $$\|A\| \leq 2 \sup_{\|\psi\|=1} |\langle \psi, A\psi\rangle| \leq 2\|A\|.$$

This is wikipedia version which assumes that antilineariy is on the first argument of bracket.

2.3 Sesquilinear forms and quadratic forms

The following theory will be used in the proof for spectral theorem of unbounded operators.

Definition 23. A **sesquilinear form** on H is a map $L : H \otimes \overline{H} \to \mathbb{C}$.

So linear in first argument and sesquilinear in second argument (order can be flipped around like the inner product just depends on the source). L is bounded if $|L(x, y)| \leq C\|x\|\|y\|$. The smalles of such C is the operator norm $\|L\|$.

Theorem 35. *If L is sesquilinear form, we have some $A \in \mathcal{B}(H)$*

$$L(x, y) = \langle Ax, y\rangle$$

for all $x, y \in H$ and $\|A\| = \|L\|$.

Weak operator topology has some nice compactness properties.

Theorem 36. *Let $T_n \in \mathcal{B}(H)$ satisfy $\|T\| \leq 1$ for all $n \geq 1$. Then there is a weakly convering subsequence $(T_n)_n$.*

Corollary 5.
1. *If L_n's are sesquilinear forms with $\|L_n\| \leq c$, one may pick a converging subsequence L'_n where the limit is sesquilinear.*
2. *If L_n's are uniformly bounded sesquilinear forms, and $L_n(x, y) \to L(x, y)$ for all $x, y \in H$, then L is a bounded sesquilinear form.*

Definition 24. A bounded *quodratic form* on H is a function

$$x \mapsto Q(x) = L(x, x)$$

where L is some bounded sesquilinear form.

The norm Q is naturally the smallest constant for which $|Q(X)| \leq c\|x\|^2$.

Remark: Thus quadratic forms are restrictions of sesquilinear form on the diagonal. They obviously satisfy $Q(\lambda x) = |\lambda|^2 Q(x)$ for all $x \in H, \lambda \in \mathbb{C}$, but this alone does not make Q a quadratic form.

Lemma 15. *Let $Q : H \to \mathbb{C}$ be a quadratic form and assume that $Q(x) = L(x, x)$, where $L : H \otimes \overline{H} \to C$ is a sesquilinear form. Then $\mathbb{Q}$ determines L uniquely. Moreover, Q and L are simultaneously bounded and we have*

$$\|Q\| \leq \|L\| \leq 6\|Q\|.$$

This is because by sesquilinearity we can expree L only with $Q's$.

Lemma 16. *Assume that Q_n is a quadratic form on H for $n \in \mathbb{N}$ such that the finite limit*

$$Q(x) = \lim_{n\to\infty} Q_n(x)$$

exists for all $x \in H$. Then

1. *Q is a quadratic form.*
2. *If additionally $\|Q_n\| \leq c$ for all x, then also $\|Q\| \leq c$.*
3. *The corresponding sesquilinear forms also converge.*

2.4 Spectral theorem for unbounded operators

Lemma 17. *Let A be a bounded self-adjoint operator on H. Then*

1. *if $\lambda = a + ib \in \mathbb{C}$ $(a, b \in \mathbb{R})$, we have that*

   $$\|(A - \lambda I)f\|^2 \geq b^2\|f\|^2$$

   for all $f \in H$.
2. *$\sigma(A) \subset \mathbb{R}$ and it is non empty and compact.*
3. *The spectral radius $r_\rho(A) := \sup\{|\lambda| : \lambda \in \sigma(A)\}$ satisfies $r_\rho(A) = \|A\|$.*

Lemma 18. *For all bounded A on H and any polynomial p it holds that*

$$\sigma(p(A)) = p(\sigma(A)).$$

Theorem 37 (Spectral theorem for bounded self-adjoint operators)**.** *Let A be bounded and self-adjoint on H. Then there exists a unique projection valued measure μ^A on Borel sets of $\sigma(A)$ (where $\sigma(A) \subset \mathbb{R}$ is compact and non-empty) such that*

$$A = \int_{\sigma(A)} \lambda d\mu^A(\lambda).$$

Proof uses following reulsts. We start by considering polynomials $p(A)$ and first a version called "**continuous functional calculus** for A".

Theorem 38. *There exists a unique linear isometry $f \mapsto f(A)$ from $C_{\mathbb{R}}(\sigma(A)) \to \mathcal{B}(H)$ such that $f(A) = A^m$ if $f(t) = t^m|\sigma(A)$ for integer $m \geq 1$.*

Here we used the result of density of polynomials and the spectral radius theorem for norm.

Theorem 39. *The continuous functional calculus defined in previous therem has the following properties for all $f, g \in C_{\mathbb{R}}(\sigma(A))$:*

1. $(fg)(A) = f(A)g(A)$.
2. $f(A)^* = f(A)$.
3. *if* $f \geq 0$, *then* $f(A) \geq 0$.
4. $\|f(A)\| = \|f\|_{C_{\mathbb{R}}(\sigma(A))} = \sup_{\lambda \in \sigma(A)} |f(\lambda)|$.
5. $\sigma(f(A)) = f(\sigma(A))$.
6. $f \equiv 1 \Rightarrow f(A) = I$.

Next we consider suitable quadratic forms Q_f for any $f \in C(\sigma(A))$.

Definition 25. Let $f \in C(\sigma(A))$. The quadratic form Q_f is defiend by setting

$$Q_f(\psi) := \langle f(A)\psi, \psi \rangle .$$

Note that Q_f is a quadratic form, since it is a restriction of the sesquilinear form $(\psi, \phi) \mapsto \langle f(A)\psi, \phi \rangle$ on the diagonal.

Lemma 19. *For each $\psi \in H$ there is a unique Borel measure μ_ψ on $\sigma(A)$ such that*

$$Q_f(\psi) = \int_{\sigma(A)} f(\lambda) d\mu_\psi(\lambda)$$

for all $f \in C(\sigma(A))$. We then have $\mu_\psi(\sigma(A)) = \|\psi\|^2$.

Existance and uniqueness comes from Riesz reprentation theorem for measures.

Definition 26. Let f be a bounded Borel function (complex valued) on $\sigma(A)$. We define the map

$$Q_f : H \to \mathbb{C}$$

by setting

$$Q_f(\psi) := \int_{\sigma(A)} f(\lambda) d\mu_\psi(\lambda).$$

The main property of thus defined f is contained in our next theorem.

Theorem 40. *For any bounded and Borel function f on $\sigma(A)$, the map Q_f is a bounded quadratic form satisfying*

$$\|Q_f\| \leq \sup_{\lambda \in \sigma(A)} |f(\lambda)|.$$

Theorem 41. *For any bounded Borel function f on $\sigma(A)$ it holds that*

$$Q_f(\psi) = \langle f(A)\psi, \psi \rangle \, ,$$

where $f(A) \in \mathcal{B}(H)$ is uniquely defined by the above equality. Moreover, we have for any bounded Borel function f and g on $\sigma(A)$ that $f(A)g(A) = (fg)(A)$.

Previous results rely heavily on monotone class theorem.
The proof of the spectral theorem is finished via the final result

Theorem 42. *Define for are Borel set $E \subset \sigma(A)$ the operator $\mu^A(E) := \chi_E(A)$. Then μ^A is a projection valued measure on the Borel sets of $\sigma(A)$ and it satisfies*

$$A = \int_{\sigma(A)} \lambda \mu^A(\lambda).$$

And more generally

$$f(A) = \int_{\sigma(A)} f(\lambda)\mu^A(\lambda)$$

for any bounded borel function $f : \sigma(A) \to \mathbb{C}$, where the left hand side was defined via 41.

This finishes the proof of the spectral theorem.

2.5 Borel spectral calculus for self-adjoint bounded operators

After the specral theorem we may now define the Borel spectral calculus for self-adjoint operators by simply recalling back to the section about projection valued measures and integration.

Theorem 43. *Let A be bounded self-adjoint operator on H have the spectral representation*

$$A = \int_{\sigma(A)} \lambda \mu^A(\lambda).$$

For any bounded Borel function $f : \sigma(A) \to \mathbb{C}$ set

$$f(A) := \int_{\sigma(A)} f(\lambda)\mu^A(\lambda).$$

Then the map $f \mapsto f(A)$ is linear, multiplicative, $f(A)^* = \overline{f}(A), id(A) = A, 1(A) = I, f(A) \geq 0$ if $f \geq 0$.

Theorem 44. *Let A be bounded and positive (hence self-adjoint). Theb there is a unique bounded linear positive square root of A denoted B. Moreover B commutes with every bounded operator that commutes with A.*

This comes directly from Borel functional calculus and spectral theorem.

Theorem 45. *Let A be bounded and self-adjoint on H. Let $f : \sigma(A) \to \mathbb{C}$ be Borel function. Then*

$$\sigma(f(A)) \subset \overline{f(\sigma(A))}.$$

Note if A is self-adjoint, one usually defines the measure $\mu^A(E) = \mu^A(E \cap \sigma(A))$ for general Borel sets of $\mathbb{C}$ rather than just Borel subsets of $\sigma(A)$.

Theorem 46. *Let A be bounded and self-adjoint on H. Let μ^A be the associated projection-valued measure. For any Borel set $E \subset \mathbb{R}$, set $V_E := R(\mu^A(E))$ is the spectral subspace associated to E. Then*

1. *$AV_E \subset V_E$.*
2. *$\sigma(A|V_E) \subset \overline{E}$.*
3. *If U is open, $U \cap \sigma(A) \neq \varnothing$, then $V_U \neq \{0\}$.*
4. *If $E \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$, then for all $\psi \in V_E$ one has*

$$\|(A - \lambda_0 I)\psi\| \leq \varepsilon \|\psi\|. \quad (*)$$

Definition 27. If vector $\psi \in H$ satisfies $(*)$, then we call ψ an $\varepsilon-$almost eigenvector for A with eigenvalue λ_0.

2.6 Spectral theorem for normal operators

Our aim is next to generalize the spectral theorem to normal operators.

Definition 28. A bounded operator T on H is normal if $TT^* = T^*T$.

For example every self-adjoint operator is normal.

Definition 29. Let T be bounded on H. Real part of T is defined as

$$\operatorname{Re} T := \frac{1}{2}(T + T^*)$$

and imaginary part

$$\operatorname{Im} T := \frac{1}{2i}(T - T^*).$$

*) Point λ is an **approximate eigenvalue** if there is $\varepsilon-$almost eigenvector with eigenvalue λ for all $\varepsilon > 0$ (i.e. $(T - \lambda I)$ is not bounded from below). Such λ's consitute the **aproximate spectrum**, denoted $\sigma_{ap}(T) := \{\lambda \in \mathbb{C} | T - \lambda I \text{ not bdd from below}\}$.

Our aim next is to establish a polynomial spectral mapping theorem for normal operators. We need a couple of lemmas for that purpose.

Lemma 20. *Assume that T is normal. Then*

1. *If ψ is $\varepsilon-$almost eigenvector for T, for eigenvalue λ, then it is also $\varepsilon-$almost eigenvector for T^*, for eigenvalue $\overline{\lambda}$.*

2. $\sigma(T) = \sigma_{ap}(T)$.

Lemma 21. *Let T be bounded normal operator on H and let p be a complex polynomial in two variables. Then*

$$\{p(\lambda,\overline{\lambda})|\lambda \in \sigma(T)\} \subset \sigma(p(T,T^*)).$$

If for examples $p(\lambda,\overline{\lambda}) = \lambda^2\overline{\lambda}^3$, then $p(A,A^*) = A^2(A^*)^3$.

Theorem 47. *Let A be bounded and self-adjoint. Then every bounded B that commutes with A, commutes with every $f(A)$, where $f : \sigma(A) \to \mathbb{C}$ is bounded Borel function.*

Lemma 22. *Let T be bounded normal operator and p a polynomial in two complex variables. Then for any $\varepsilon > 0$ there exists a nontrivial closed subspace $M_\varepsilon \subset H$ such that $TM_\varepsilon \subset M_\varepsilon$ and $T^*M_\varepsilon \subset M_\varepsilon$ and if $x \in M_\varepsilon$, $x \neq 0$ and $\mu \in \sigma(p(T,T^*))$, then x is an $\varepsilon-$almost eigenvector for $p(T,T^*)$.*

Lemma 23. *Let T be bounded and normal on H and p a compelx polynomial in two variables. Then*

$$\sigma(p(T,T^*)) \subset p(\sigma(T)).$$

This leads to the fact that

Theorem 48. *If T is bounded normal operator on H and p two varibale complex polynomial, then*

$$\sigma(p(T,T^*)) \subset p(\sigma(T)).$$

Theorem 49. *If T is bounded normal operator on H, then the spectral radius $r_\rho(T) = \|T\|$.*

Corollary 6. *If T is bounded normal operator on H and p complex polynomial in two variables, then*

$$\|p(T,T^*)\| = \sup_{\lambda\in\sigma(T)} \left|p(\lambda,\overline{\lambda})\right|.$$

We are now ready for the spectral theorem for normal operators.

Theorem 50 (Specral theorem for bounded normal operators)**.** *Let T be bounded and normal. Then there is a unique projection valued measure μ^T on the Borel $\sigma-$algebra of $\sigma(T)$, with values in $\mathcal{B}(H)$ such that*

$$\int_{\sigma(T)} \lambda d\mu^T(\lambda) = T.$$

One has that if $E \subset \sigma(T)$ is Borel, then $R(\mu^T(E))$ is invariant both under T and T^*.

Theorem 51. *If $U \in \mathcal{B}(H)$ is unitary, then $\sigma(U) \subset \mathbb{T} := \{z \in \mathbb{C}||z| = 1\}$.*

We finally sketch a "multiplication operator" version of the spectral theorem. The proof applies the notion of "cyclic vector".

Definition 30. Let $A \in \mathcal{B}(H)$. The vector $\psi \in H$ is **cyclic** for A if $\overline{\text{span}}\{A^k\psi, k = 0, 1, 2, 3, \dots\} = H$.

Lemma 24. *Let A be bounded and self-adjoint, and assume that it has a cyclic vector. Then there is a unitary map*

$$U : H \to L^2(\sigma(A), \mu_\psi)$$

such that

$$(UAU^{-1}f)(\lambda) = \lambda f(\lambda)$$

for all $f \in L^2(\sigma(A), \mu_\psi)$.

Note that above μ_ψ is the spectral measure of A corresponding to vector ψ. A unitary map $U : H_1 \to H_2$ is surjective isometry (equivalently U has adjoint as its inverse).

Lemma 25. *If A is bounded and self-adjoint we may decompose H into orthogonal sum of closed subspaces W_j where each W_j is invariant under A, and $A|W_j : W_j \to W_j$ has a cyclic vector ψ_j for each $j \geq 1$. Then number of W_j's is either finite or countably infinite.*

Theorem 52. *Assume that A is boudned and self-adjoint on H. Then there exists a Borel measure and a bounded Borel function h on $[0, 1]$ and a unitary map $U : H \to L^2([0, 1], d\mu)$ such that*

$$UAU^{-1}\psi(\lambda) = h(\lambda)\psi(\lambda).$$

For all $\psi \in L^2([0, 1], d\mu)$.

Remark: Simular representation is true for normal operators. For families of commuting operators (as well as normal operators) one may construct a common spectral resolution, i.e. all the operators can be expressed via functional calculus using the same projection value dmeasure. This is most conviniently done by using the basic theory of C^*-algebras.

3 Spectral theorem for unbounded operators

3.1 Unbounded operators

Many operators that appear in dealing with e.g. probability, PDE or quantum meachanics are linear but not bounded. The fundamental examples include

- $\frac{d}{dx}$ acting on let say $L^2(\mathbb{R})$ is not even well defined for all of L^2. However it is linear on a dense subset like C_0^1 or C_0^∞. For two smooth compactly supported functions we can copute

$$\left\langle \frac{d}{dx} f, g \right\rangle = \int_{\mathbb{R}} \overline{f'} g = - \int_{\mathbb{R}} \overline{f} g' = - \left\langle f, \frac{d}{dx} g \right\rangle .$$

  Thus the operator is anti symmetric. Though $f \mapsto i\frac{d}{dx} f$ is symmetric. Could it share some properties as one has for self-adjoint ones? E.e. where is the spectrum (can it be defined?) located? Is there even a spectral theorem?

- The multiplication operator M_x on $L^2(\mathbb{R})$, defined by $M_x f(x) = x f(x)$. Clearly it is unbounded and not defined for all of L^2. However we can easily see that the elements for which M_x makes sennse are exactly the set

$$D(M_x) := \{ f \in L^2(\mathbb{R}) | \int_{\mathbb{R}} (1 + |x|^2) |f(x)|^2 dx < \infty \}.$$

  Then if $f, g \in D(M_x)$ one clearly checks

$$\langle M_x f, g \rangle = \langle f, M_x g \rangle .$$

  We will later see that this operator is very easy to write down in a conrete resolvent., the spectrum lies on $\mathbb{R}$ and one may define a spectral calculus as before. In fact, M_x will be kind of a **model operator** one could keep in mind while we develop the theory of unbounded operators.

- Δ on $\mathbb{R}^d$ or $\Delta q(x)$ (Laplace + potential), or more general differential operators.

- Generators of bounded semigroups. A typical property of bounded semigroups is that they are continuous in the strong operator topology but not in norm topology. However, if U is unitary semigroup, i.e.

$$\begin{cases} U(t)U(s) = U(s+t), \ \forall s, t \geq 0 \\ U(t) \in \mathcal{B}(H), U(t)^* U(t) = U(t) U(t)^* = I, \ \forall t \geq 0 \\ t \mapsto U(t) \ \text{ strongly continuous} \end{cases}$$

  thejn it turns out that $U(t) = \exp(itA)$ where A is (typically unbounded) self-adjoint operator.

Definition 31. An **unbounded operator** on H is a linear map

$$T : D(T) \to H,$$

where $D(T) \subset H$ is a dense domain of definition.

Note that $D(T)$ is part of the definition. For examples the previously seen derivative operators

$$T_1 : C_0^\infty(\mathbb{R}) \to L^2(\mathbb{R}), \quad T_1 f = f',$$

and

$$T_2 : C_0^\infty(\mathbb{R}) \to L^2(\mathbb{R}), \quad T_2 f = f',$$

are different operators. Actually T_2 is an extension of T_1.

Definition 32. T_2 is an extension of T_1 if $D(T_1) \subset D(T_2)$ and $T_2 x = T_1 x$ for $x \in D(T_1)$. This is denoted by $T_1 \subset T_2$.

- Often one denotes $D(T) = \mathrm{Dom}(T)$.
- Every bounded operator T is also an unbounded operator but with domain $D(T) = H$

For us hence definition of unbounded operators include the bounded ones.

Definition 33. Let T be an operator on H. We define the adjoint T^* as the operator with $\varphi \in D(T^*)$ iff the linear functional

$$\psi \to \langle T\psi, \varphi \rangle\,, \quad \psi \in D(T^*)$$

is bounded. Then we set $T^*\varphi = \theta$, where $\theta \in H$ is the unique vector such that

$$\langle T\psi, \varphi \rangle = \langle \psi, \theta \rangle \quad (*)$$

for all $\psi \in D(T)$.

Basically we apply the condition on the linear side of the bracket not the antilinear. Since $D(T)$ is dense, the functional $\psi \to \langle T\psi, \varphi \rangle$ extends to a unique element in the dual of H, is.e it can be uniqualy written in the form $(*)$. It follows that

$$\langle T\psi, \varphi \rangle = \langle \psi, T^*\varphi \rangle \quad \text{for all } \psi \in D(T), \varphi \in D(T^*).$$

For examples we have the domain of definition for the multiplication operator as before

$$D(M_x) := \{f \in L^2(\mathbb{R}) | \int_{\mathbb{R}} (1 + |x|^2)|f(x)|^2 dx < \infty\}.$$

To find it's adjoints domain of definition we have to look when

$$\psi \mapsto \int_{\mathbb{R}} x\psi(x)\overline{\varphi(x)} dx$$

is bounded. It is necessary and sufficient that

$$D(M_x^*) = \{f \in L^2(\mathbb{R}) | x\varphi(x) \in L^2\} = D(M_x).$$

which means that M_x is self-adjoint according to the following definition.

Definition 34. Operator is self-adjoint iff $T^* = T$. Note that this means that the domain of definition coinside.

Definition 35. Operator is symmetric if

$$\langle T\varphi, \psi \rangle = \langle \varphi, T\psi \rangle$$

for all $\varphi, \psi \in D(T)$.

A central question is when is a symmetric operator self-adjoint, or better when does a symmetric operator have a (possibly unique?) self-adjoint extension? A good example is $i\frac{d}{dx}$ on $L^2(\mathbb{R})$ (or $L^2(I)$ where I is an interval). We shall obtain later on some useful answers to this question.

Lemma 26. *Operator A is symmetric iff $A \subset A^*$*

Definition 36. Operator A on H is **closed** iff it's graph $\Gamma(A) \subset H \times H$ is closed. In addition we say that A is **closable** if the closure of it's graph is a graph of a function. Then A^{cl} denotes the operator with graph $\overline{\Gamma(A)}$.

Lemma 27.

1. *T is closable iff for two sequence $(x_n), (y_n)$ from $D(T)$ if $x_n \to x_0$ and $y_n \to x_0$ and $Tx_n \to x_0$ and $Ty_n \to y_0$, then $x_0 = y_0$.*
2. *If T is closable, then T^{cl} is unbounded operator i.e $D(T^{cl}$ is dense linear subspace and T^{cl} well-defined.*

Theorem 53. *Let A be a symmetric operator on H. Then*

1. *A^* is closed*
2. *A is closable*

Definition 37. An operator A on H is **essentially self-adjoint**, if A is symmetric and A^{cl} is self-adjoint.

Theorem 54. *If A is essentially self-adjoint, then A^{cl} is the unique self-adjoint extension of A.*

Lemma 28.

1. *Let A be an operator on H and B bounded operator. Then*

$$D((A+B)^*) = D(A^*).$$

2. *If in (1) both A and B are self-adjoint, then $A + B$ is self-adjoint.*

One has to be much more careful in summing (or taking products) of unbounded operators, and we do not developt any general formalism for it.

Lemma 29. *If A is a closed on operator on H and $\lambda \in \mathbb{C}$, then the inequality $(\varepsilon_0 > 0)$*

$$\|(A - \lambda I)\psi\| \geq \varepsilon_0 \|\psi\|, \ \ \psi \in D(A)$$

implies that $R(A - \lambda I)$ is closed.

We are ready to define the invertability and hence the spectrum of unbounded operators

Definition 38. Let T be an operator on H. Then

1. T is **invertible** if there exists a bounded $B \in \mathcal{B}(H)$ such that $R(B) \subset D(T)$ and
$$TB\psi = \psi \quad \text{for all } \psi \in H$$
and
$$BT\varphi = \varphi \quad \text{for all } \varphi \in D(T).$$

2. $\lambda \in \rho(T)$ if $T - \lambda I$ is invertible and $\sigma(T) := \mathbb{C} \setminus \rho(T)$.

Lemma 30. *If A is an operator on H, then*
$$(R(A))^{\perp} = \ker(A^{*}).$$

Theorem 55. *For any operator T on H.*

1. *If T is invertible, then the inverse is unique*

2. *$\sigma(T)$ is closed subset of $\mathbb{C}$ (possibly empty or even all of $\mathbb{C}$) .*

3. *The map $\lambda \mapsto (T - \lambda)^{-1}$ is analytic on $\rho(T)$.*

Theorem 56. *Let A be symmetric operator on H. Then*

1. *If $R(A)$ is dense, then A is injective.*

2. *If $D(A) = H$, then $A = A^{*}$ and A is bounded.*

3. *If A is closed with $R(A) = H$, then $A^{*} = A$, and A is invertible.*

Corollary 7. *If $S \in \mathcal{B}(H)$ has a dense range and is injective, then its* ***formal inverse*** *T defined by $D(T) = R(S)$ ($R : H \to H$ is well posed by injectivity of S) and and $T(Rx) = x$ for H, is a closed operator. If in addition S is self-adjoint, then T is also self-adjoint.*

Remarks: The spectrum of an unbounded operator can behave in surprising ways. For examples, it can be empty. Also the spectrum of an extension of a symmetric operator may depend very strongly on the extension.

The previous corollary is sometimes useful in checking that an operator is self-adjoint (or in constructing self-adjoint operators).

Part (2) of last theorem is Hellinger-Toepliz theorem. It explains why for unbounded operators the naturla domain of definition is usually smaller than H.

Theorem 57. *Let T be symmetric operator on H. Then*

1. $\|Tx + ix\|^2 = \|Tx - ix\|^2 = \|x\|^2 + \|Tx\|^2$ *for all $x \in D(T)$.*

2. *T is closable iff $R(I + iI)$ is closed.*

3. $T + ix$ *is injective.*

4. statements (2) and (3) remain true if i *is replaced by* $-i$.

Theorem 58. *If operator* A *is self-adjoint, then* $\sigma(A) \subset \mathbb{R}$.

Theorem 59. *If* A *is self-adjoint operator on* H*, then* $\lambda \in \sigma(A)$ *iff there exists a sequence* (x_n) *of vectors in* $D(A)$ *so that* $\|(A - \lambda I)x_n\|/\|x_n\| \to 0$ *when* n *grows large.*

We next give some useful conditions for essential self-adjointness.

Lemma 31. *Assume that* A *is closable operator on* H*. Then*

1. $A^{cl*} = A^*$.

2. If A *is symmetric, then* A^{cl} *is symmetric.*

Theorem 60. *Let* A *be symmetric operator on* H*. The following are equivalent:*

1. A *is essentially self-adjoint.*

2. Both $R(A - iI)$ *and* $R(A + iI)$ *are dense in* H.

3. $\ker(A^* - iI) = \ker(A^* + iI) = \{0\}$.

Theorem 61. *Let* A *be symmetric on* H *and positive in the sense that*

$$\langle Ax, x\rangle \geq 0 \forall x \in D(A).$$

Then the followin are equivalent:

1. A *is essentially self-adjoint.*

2. $A + I$ *has dense range*

3. $\ker(A^* + I) = \{0\}$.

Corollary 8. *Assume that* A *is symmetric operator. Then the following are equivalent:*

1. A *is self-adjoint.*

2. $R(A + iT) = R(A - iI) = H$.

3. $\sigma(A) \subset \mathbb{R}$.

Theorem 62. *Assume that* A *is symmetric, and there is a sequence of subspace* $M_n \subset H$ *such that* $M_n \subset D(A)$ *for all* n*,* $AM_n \subset M_n$*,* $\|Ax\| \leq c_n\|x\|$ *for all* $x \in M_n$*, and* $\overline{\cup_{n\geq 1} M_n} = H$*. Then* A *is essentially self-adjoint.*

Theorem 63. *Let* A *be symmetric operator on* H*, which has eigenfunctions* $\varphi_n \in D(A)$ *so that* φ_n*'s form an orthonormal Hilbert basis of* H*. Then* A *is self-adjoint.*

Example: Let $H = L^2([0,2\pi)$ and consider the operator $D(T) = C^\infty([0,2\pi]), Tf(x) = f''(x)$, where $C^\infty([0,2\pi))$ consists of smooth functions f on $[0,2\pi]$ such that $f^{(k)}(0) = f^{(k)}(2\pi)$ for all $k \geq 0$ (i.e. restrictions of smooth $2\pi-$periodic functions on $[0,2\pi)$). Then T is symmetric since (by the periodicity), if $f, g \in D(T)$, then

$$\begin{aligned}
\langle Tg, f\rangle &= \int_0^{2\pi} g''(t)\overline{f}(t)dt \\
&= \underbrace{|_0^{2\pi} g'(t)\overline{f(t)}dt}_{=0} - \underbrace{\int_0^{2\pi} g'(t)\overline{f'}(t)dt}_{(*)} \\
&= -|_0^{2\pi} g(t)\overline{f'(t)}dt \int_0^{2\pi} g(t)\overline{f''}(t)dt = \langle g, Tf\rangle\,.
\end{aligned}$$

We claim that T is essentially self-adjoint (use the previous theorems). In fact $D(T^{cl}) = H^2([0,2\pi))$ (the Sobolev space). So another way to define sobolev space and weak differentiation.

Consider the same operator but restricted to $C_0^\infty(0,2\pi)$. Then it is not essentially self-adjoint.

Lesson: Essential self-adjointness may depend in a delicate way on the choice of $D(T)$ in case of some (e.g. differential) operators.

3.2 Spectral theorem for unbounded operators

Not that the map

$$t \mapsto \frac{t-i}{t+i}$$

is a homeomorphism between $\mathbb{R}$ and $\mathbb{T} \setminus \{1\}$, where $\mathbb{T}$ is the unit circle (complex notation).

Definition 39. Let A be a symmetric operator on H. It's **Cayley transform** is the isometric operator

$$U : R(A+iT) \to R(A-iT)$$

defined by $U(Ax+ix) = Ax - ix$ for $x \in D(A)$.

Note: U is well defined since $A + iI$ is injective on $D(A)$. Then bijectivity of U immidiately and the isometry property follows also easily (look back previous theorems on unbounded operators).

Formally $U = (T-iI)(T+iI)^{-1}$. This will make perfect sense if T is self-adjoint, but good to mave in mind all the time as a guiding idea.

Lemma 32. *Let U be an operator between subspaces of H such that it's action*

$$U : D(U) \to R(U)$$

is an isometry (i.e. $\|Ux\| = \|x\| \forall x \in D(U)$). Then

1. $\langle Ux, Ux\rangle = \langle x, y\rangle$ *for $x, y \in D(U)$.*

2. *If $R(I-U)$ is dense in H, then $I-U$ is injective.*

3. *If any of the three spaces $D(U)$, $R(U)$ or $\Gamma(U)$ is closed, then so are other two.*

We proceed to show that there is a useful $1-1$ correspondence between symmetric operators and (partial) isometries U with $R(I-U)$ dense and $\ker(I-U)=\{0\}$.

Theorem 64. *Assume that U is the Cayley transform of a symmetric operator A on H. Then*

1. *A is closed iff $D(U)$ is closed iff $R(U)$ is closed.*

2. *$R(I-U)=D(A)$ and operator $I-U$ is injective (on $D(U)$).*

3. *$T=i(I+U)(I-U)^{-1}$ (RHS defined in an obvious way on $D(A)$)*

4. *U is unitary iff A is self-adjoint*

5. *Conversely, if V is an isometry between subspace of H such that $R(I-V)$ is dense and $\ker(I-V)$ is trivial, then V is the cayley transform of a symmetric operator on H.*

It is easy to understand when isometries between subspaces can be extended to unitary operators of the whole space. Also it is clear that for symmetric operators A_1, A_2 we have $A_1 \subset A_2$ iff the Cayley transforms satisfy $U_1 \subset U_2$. Thus it will be useful to define

Definition 40. Let A be closed symmetric operator on H. The **deficiency subspaces** of A are

$$\begin{cases} \mathcal{L}^+ := \ker(A^* - iI) = (R(A+iI))^\perp \\ \mathcal{L}^- := \ker(A^* + iI) = (R(A-iI))^\perp. \end{cases}$$

The deficiency indices of operator A are defined as the dimensions of the above spaces $n_+ := \dim \mathcal{L}^+, n_- := \dim \mathcal{L}^-$.

The following theorem is now evident after the previous results and discussion

Theorem 65. *Let A be a closed symmetric operator on H with deficiency indices $n_\pm$.*

1. *A is self-adjoint iff $n_+ = n_- = 0$*

2. *A has self-adjoint extension iff $n_+ = n_-$. The extensions can be naturally parametrized by unitary operators between $\mathcal{L}^+$ and $\mathcal{L}^-$.*

Theorem 66. *Assume that $H = \bigoplus_{j=1}^\infty M_j$, which means that M_j's are mutually orthogonal closed subspaces of H such that $\overline{span \cup_{j=1}^\infty M_j} = H$. Let $T_j \in \mathcal{B}(M_j)$ be self-adjoint on M_j, $j \geq 1$. Define the operator A_0 on H by setting*

$$D(A_0) = span \cup_{j=1}^\infty M_j$$

and

$$A_0(z_1 + \cdots + z_m) = \sum_{j=1}^{m} T_j z_j, \ \ m \geq 1$$

where $z_j \in M_j$ *for* $1 \leq j \leq m$. *Then* A_0 *is essentially self-adjoint, and if* $A := A^{cl}$, *then we have*

$$A(\sum_{j=1}^{\infty} z_j) = \sum_{j=1}^{\infty} T_j z_j$$

for

$$z = \sum_{j=1}^{\infty} z_j \in D(A) := \{\sum_{j=1}^{\infty} z_j | z_j \in M_j, \sum_{j=1}^{\infty} \|z_j\|^2 + \|T_j z_j\|^2 < \infty\}.$$

Moreover, any symmetric A_1 *with* $A_0 \subset A_1$ *is essentially self-adjoint and* $A_1^{cl} = A$.

Examples:

- If $\lambda = (\lambda_n)_n$ is a sequence of real numbers. Then the diagonal operator M_λ, $M_\lambda e_n = \lambda_n e_n$ is self-adjoint as soon as we set

  $$D(M_\lambda) = \{x = \sum_{n=1}^{\infty} x_n e_n \in H | \sum_{n=1}^{\infty} (1 + \lambda_n^2) x_n^2 < \infty\}.$$

  Further, $\sigma(M_\lambda) = \overline{\lambda_n | n \geq 1}$.

- If $g : (0,1) \to \mathbb{R}$ is measurable, the multiplication map $f \mapsto gf$, $f \in L^2$ yields as well a self-adjoint operator as soona s the domain is chosen correctly.

- We give as third example the (possibly unbounded) operator $\int f d\mu$, where μ is a spectral resolution and (contraty to what we assumed before) f may be unbounded.

Theorem 67. *Let* $\mu : \mathcal{A} \to \mathcal{B}(H)$ *be a projection valued measure on set* E *(*$\mathcal{A}$ $\sigma-$*algebra on* E*). Let* $f : E \to \mathbb{R}$ *be a* $\mathcal{A}-$*measurable function. THen there is a (possibly unbounded) self-adjoint operator*

$$A = \int_E f d\mu,$$

which is uniquely determined by the condition

$$(*) \quad A|M_n = \int_E \chi_{E_n} f d\mu, \ \ n \geq 1$$

where $M_n = \mu(E_n)H$ *(note that* $\chi_{E_n} f$ *is bounded for each* $n \geq 1$*). The domain of* A *can be written in the form*

$$D(A) = \{x \in H | \int_E |f(\lambda)|^2 d\mu_x(\lambda) < \infty\}.$$

We are now ready to state the spectral theorem.

Theorem 68 (Spectral theorem for unbounded operators)**.** *Let A be self-adjoint operator on H. Then there is a unique projection-valued measure μ^A on the Borel sets of $\sigma(A) \subset \mathbb{R}$ such that*

$$A = \int_{\sigma(A)} \lambda \mu^A(d\lambda).$$

The proof uses all of the stuff we have dealt with by far previously.
As a consiquence

- One may define a unique positive square root for positive operators.
- There is a multiplication operator version of the spectral theorem also in the unbounded case.
- One may develop a spectral calculus and define operator $f(A)$ for any real valued Borel function $f : \sigma(A) \to \mathbb{R}$. However, one has to be careful in formulating statements like

$$f_1(A)f_2(A) = (f_2 f_2)(A),$$

since this holds only on

$$\{x \in D(f_2(A)) | f_2(x) \in D(f_1(A))\}.$$

- As a corollary of the theorem 67, we obtain the important formula

$$D(A) = \{x \in H | \int_{\sigma(A)} \lambda^2 \mu_x(d\lambda) < \infty\}.$$

4 Strongly continuous semigroups and Stone's theorem

Definition 41. A **one-parameter unitary semigroup** is a family $\{U(t)|t \geq 0\}$ of unitary operators with the properties:

- $U(0) = I$.
- $U(t+s) = U(t)U(s)$ for all $t, s \geq 0$.

Definition 42. A one-parameter unitary semigroup $(U_t)_{t\geq 0}$ is **strongly continuous** if for every $x \in H$ we have

$$\lim_{t \searrow 0} \|U(t)x - x\| = 0.$$

Lemma 33. *1. If $(U_t)_{t\geq 0}$ is a one-parameter unitary semigroup, it can be extended to a one-parameter unitary group $(U_t)_{t\in\mathbb{R}}$ by setting $U_{-t} := U_t^*$.*

2. If $(U_t)_{t\geq 0}$ in (1) is strongly continuous, then the extension is strongly continuous at every time $s \in \mathbb{R}$:

$$\lim_{t\to s} \|U(t)x - U(s)x\|$$

for all $x \in H$.

Note that we can thus consider $(U_t)_{t\in\mathbb{R}}$ without loss of generality.

- All practically interesting (semi)groups are strongly continuous.
- A an example of strongly continuous one-parameter semigroup, where it was shown that the **translation**

$$U_t f(\cdot) = f(\cdot + t), f \in L^2(\mathbb{R})$$

is strongly continuous (obviously unitary) semigroup.

Definition 43. For a strongly continuous one-parameter group $(U_t)_{t\in\mathbb{R}}$ the **infinitesimal generator** is defined by setting

$$Ax = \lim_{t\to 0} \frac{1}{i} \frac{U_t x - x}{t}, \ \ x \in D(A)$$

where $D(A)$ consists of all $x \in H$ such that the above limit exists in the norm topology of H.

Theorem 69. *If A is self-adjoint on H and $u_t = \int_{\mathbb{R}} \exp(it\lambda)\mu^A(d\lambda)$ (i.e. $A = \exp(itA)$, then $(U_t)_{t\in R}$ is a unitary strongly continuous one-parameter subgroup whose infinitesimal generator equals A.*

Lemma 34. *Let $(U_t)_{t\in\mathbb{R}}$ be a strongly continuous one-parameter unitary group with infinitesimal generator A. Then*

1. *If* $t \in \mathbb{R}$ *and* $x \in D(A)$*, then* $U_t x \in D(A)$ *and* $A(U_t x) = U_t(Ax)$*.*

2. $D(A)$ *is dense in* H*.*

Theorem 70 (Stone's theorem)**.** *Assume that* $(U_t)_{t\in R}$ *is strongly continuous one-parameter unitary group on* H*. Then it's infinitesimal generator* A *is a self-adjoint operator on* H*, and* $U_t = \exp(itA)$ *for* $t \in \mathbb{R}$*. Conversely, for any self-adjoint* A *on* H $(\exp(itA))_{t\in\mathbb{R}}$ *defines such a group.*

Remarks:

- Stone's theorem is a fundamental statement in the key structures of quantum mechanics. It "shows" that natural Hamiltonian's must be self-adjoint, and is related to the preservation of "probability mass".
- There are number of important applications in other fields of mathematics also, including PDE and probability.
- Hille–Yosida theorem generalizes Stone's characterization of generators to strongly continuous (not necessarily norm-presenving) semigroups of operators on any Banach space.

5 Exercises

To be added...

6 Appendix for linear algebra type stuff

The algebraic multiplicity of an eigenvalue is the number of times it appears as a root of the characteristic polynomial (i.e., the polynomial whose roots are the eigenvalues of a matrix). The geometric multiplicity of an eigenvalue is the dimension of the linear space of its associated eigenvectors (i.e., its eigenspace).

We might not always have spectral decomposition since it is only formulated for self-adjoint matrices i.e. the whole space can be spanned by eigenvector. Meaning if the geometric multiplicity is greater than algebraic multiplicity, then we might wanna think about generalized eigenvectors and the Jordan normal form (almost diagonal matrix decomposition). Its a theorem that every square matrix A over $\mathbb{C}$ is simular to a Jordan block matrix ($A = P^{-1}JP$).