Gromov-Gallot upper bound for first Betti number via the Bochner technique

Heinonen, Kalle

Abstract

This is a small project for the Riemannian geometry course lectured in the fall semester 2024 at the University of Helsinki. We will present basics of Hodge theory and the Bochner technique to show certain lower bounds on the Ricci curvature gives upper bounds for the first Betti number. We will assume that the reader has knowledge about basic Riemannian geometry and de Rham cohomology. Knowledge of functional analysis will also be helpful.

1 Introduction

Betti numbers are one of the first invariants observed topology leading to the development of algebraic topology. Namely Euler's famous observation for convex polyhedra that $V - E + F = 2$. This is the classical formulation of phenomena leading to the Euler characteristic. For a modern definition we have the following. For any topological space X we can define it's *Euler characteristic* by the sum $\chi(M) := \sum_{i=0}^{\infty}(-1)^i \operatorname{rank} H_n(M;\mathbb{Z})$ assuming the sum converges somehow. The *nth Betti number* is the rank of the nth integral homology group. For a space with the homotopy type of a convex polyhedra, for example the $2-$sphere as we have that $\chi(S^2) = 2$. In the context of closed manifolds[1] the sum always converges which will be sufficient enough for this project. Infact it is a famous theorem by de Rham that for smooth manifolds, singular cohomology groups and de Rham cohomology groups are isomorphic. Then it follows from some homological algebra that $\operatorname{rank} H_n(M;\mathbb{Z}) = \dim H^n(M;\mathbb{R}) = \dim H^n_{dR}(M)$ when M is a closed manifold. In the context of smooth manifolds, de Rham cohomology is usually nicer to work with. Thus we will define the first Betti number $b_1(M)$ as the dimension of the first de Rham cohomology group $H^1_{dR}(M)$. So the measurement of how much closed $1-$forms fail to be exact or equivalently count of 1 dimensional "circular" holes. We will see that depending on certain conditions on the Ricci curvature of a Riemannian manifold, we can get nice upper bounds for the first Betti number using a technique called the Bochner technique.

Classically for harmonic functions in Euclidean domains we have the following identity $\Delta\frac{1}{2}|\nabla u|^2 = |\operatorname{Hess} u|^2$. It was then Bochner who realized that doing this

[1]During this project every manifold will be assumed to possess smooth structure, so the word "manifold" will mean smooth manifold.

trick for Riemannian manifolds yields us a term depending on the Ricci curvature

$$\Delta\frac{1}{2}|\nabla u|^2 = |\mathrm{Hess}\, u|^2 + \mathrm{Ric}(\nabla u).$$

After this realization he developed a way of going from harmonic functions to so-called harmonic differential forms[2], which yield simular results. This will gives us purely topological data [3] by the de Rham cohomology of forms.

We will use the convention that $\mathrm{Ric} \geq k$ if all the eigenvalues of $\mathrm{Ric}(v) := \mathrm{Ric}(v, v)$ are $\geq k$. In the language of $(0, 2)-$tensors this means $\mathrm{Ric}(v, v) \geq kg(v, v)$ for all v. Here g denotes the Riemannanian metric. Lower bounds of Ric allows one to often derive global geometric information of the space. We will call Ric the *Ricci curvature* of the space. Additionally for functions the symbol ∇ denotes the gradient, $\nabla f = df^\sharp$, and for other types of tensor fields the covariant derivative.

The covariant derivative of a $1-$form along a vector field Y will be defined as the unique $1-$form satisfying

$$(\nabla\omega)(X, Y)_p := (\nabla_Y\omega)(X)_p := \nabla_Y\omega(X)|_p - \omega\nabla_X Y|_p$$

with respect to the Levi-Civita connection on a given Riemannian manifold.

This project will mainly follow [3] and [1]. But we will also pick up small things from [4] and [2]. For de Rham cohomology and the reader is suggested to take a look into [5] chapter 7.

The structure of this project goes in order as follows. Hodge theory, Bochner formula, Gromov-Gallot upper bounds, analysis type estimates.

2 Hodge theory

Assume that M is closed and oriented Riemannian $n-$manifold. Then the pairing

$$\Omega^k(M) \times \Omega^{n-k}(M) \to \mathbb{R}, \quad (\omega, \tau) \mapsto \int_M \omega \wedge \tau.$$

This also induces a nondegenerate pairing $H^k_{dR}(M) \times H^{n-k}_{dR}(M) \to \mathbb{R}$ on the level of cohomology groups also known as the Poincaré duality (assuming M is connected). We would like to have this kind of pairing for forms of same degree. For example two $k-$forms wedged together, might get degree more or less than n. Let's introduce the so called *Hodge star* operator $\star$. The operator takes a $k-$form on an $n-$manifold and returns a $(n-k)-$form. Let V be finite dimensional oriented $\mathbb{R}-$inner product space. We define an operator for $k-$forms $\alpha, \beta \in \bigwedge^k V$ as

$$\alpha \wedge \star\beta = \langle\langle \alpha, \beta \rangle\rangle \,\mathrm{Vol}\,. \tag{1}$$

Here the $\mathrm{Vol} := e_1 \wedge \cdots \wedge e_n$ with respect to some orthonormal basis and $\langle\langle \cdot, \cdot \rangle\rangle$ is the Gram inner product. The *Gram inner product* is defined on the elements of type

[2]or more generally $(k, l)-$tensor fields

[3]Homotopy invariance of de Rham cohomology requires no differentiability assumptions for the maps.

$\alpha = \alpha_1 \wedge \cdots \wedge \alpha_k, \beta = \beta_1 \wedge \cdots \wedge \beta_k$ as $\det\big(\langle \alpha_i, \beta_j \rangle_{i,j=1}^k\big)$ since we can write $k-$forms as linear combinations of those types of objects. Note that the inner product of $1-$forms here is the induced dual inner product by musical isomorphisms. For differential $k-$forms on Riemannian $n-$manifold (M, g) that is oriented we can do everything obviously pointwise, so we get that the Hodge star becomes the operator that gives $\alpha \wedge \star\beta = \langle\langle \alpha, \beta \rangle\rangle_g \operatorname{Vol}_g$ with respect to the Riemannian volume form and Gram inner product in the tangent spaces. Hence on the space of $k-$forms $\Omega^k(M)$ the pairing

$$\langle \cdot, \cdot \rangle_{L^2} \colon \Omega^k(M) \times \Omega^k(M) \to \mathbb{R}, \quad (\omega, \tau) \mapsto \int_M \omega \wedge \star\tau = \int_M \langle\langle \omega, \tau \rangle\rangle \operatorname{Vol}_g$$

becomes the L^2-*inner product of $k-$forms* when $k \leq n$. For the integrability atleast one of the wedged forms should have compact support [4]. With the help of the Hodge star we can define the adjoint of the exterior derivative as the linear map $\delta := d^* \colon \Omega^{k+1}(M) \to \Omega^k(M)$. I.e

$$\int_M \langle\langle \alpha, d\beta \rangle\rangle_g \operatorname{Vol}_g = \int_M \langle\langle \delta\alpha, \beta \rangle\rangle_g \operatorname{Vol}_g . \tag{2}$$

This adjoint (assuming it exists) lives in direct sum $\bigoplus_{k=0}^n \Omega^k(M)$ since we can extend the L^2-inner product to be inner product in the whole direct summand by making each $\Omega^p(M) \perp \Omega^q(M)$ for $p \neq q$. The reader can check that the Hodge star operator satisfies the following conditions: $\star\star = (-1)^{k(n-k)}$ and that $\delta = (-1)^{n(k+1)+1} \star d\star$ with help of the Leibniz rule and Stokes' theorem.

Now we are ready to define a type of Laplacian operator for differential forms called the *Hodge-Laplacian* (or *de Rham-Laplace operator*)

$$\Delta \colon \Omega^k(M) \to \Omega^k(M), \quad \Delta := \delta d + d\delta.$$

With respect to the L^2-norm this is self-adjoint operator. We call the differential forms for which $\Delta\omega = 0$ *Harmonic forms* like we call functions that vanish of the regular Laplace operator harmonic functions. Indeed if we consider zero forms, this reduces to the regular Laplace operator of a function.

Proposition 1. *$\Delta\omega = 0$ iff $d\omega = 0$ and $\delta\omega = 0$.*

Proof. Left to right implication is clear. Assume that $\Delta\omega = 0$. Thus

$$0 = \langle \Delta\omega, \omega \rangle_{L^2} = \langle (\delta d + d\delta)\omega, \omega \rangle_{L^2} = \underbrace{\langle \delta\omega, \delta\omega \rangle_{L^2}}_{=0} + \underbrace{\langle d\omega, d\omega \rangle_{L^2}}_{=0} .$$

□

Next we will show that each de Rham cohomology class has a harmonic representative. Denote $\mathcal{H}^k(M) := \ker \Delta \subset \Omega^k(M)$ has the subspace of harmonic $k-$forms. For example if M is closed, connected and oriented, the harmonic $0-$forms are all constant functions by the previous proposition (locally constant if more components). But now we get to the big result

[4] Assuming compactness every form is compactly supported

Proposition 2. *If M is compact and oriented Riemannian $n-$manifold. There is a natural isomorphism of $\mathbb{R}-$vector spaces*

$$\mathcal{H}^k(M) \cong H^k_{dR}(M).$$

Proof. Let ι denote the map that sends the harmonic form to it's equivalence class in the cohomology group. Then by previous proposition $\Delta\omega = 0$ implies $d\omega = 0$, so $\omega \in \ker(d\colon \Omega^k(M) \to \Omega^{k+1}(M))$. Also same proposition implies that $\delta\omega = 0$. Thus we have that ω is not exact since if we had that for for some τ, $d\tau = \omega$, then

$$\langle \omega, \omega \rangle_{L^2} = \langle d\tau, d\tau \rangle_{L^2} = \langle \tau, \delta d\tau \rangle_{L^2} = 0$$

since $\delta d\tau = \delta\omega = 0$. This shows that $\ker \iota = 0$ i.e. injective because a nonzero harmonic form is not exact. For surjectivity we need tougher machinery called the Hodge theorem. $\square$

2.1 The Hodge theorem

In this subsection M be closed and oriented Riemannian manifold. Even thought the Hodge Laplacian is self-adjoint for a later purpose denote Δ^* to be the adjoint with respect to the L^2-norm of $k-$forms on $\Omega^k(M)$. We want to study necessary and sufficient conditions for when the equation $\Delta\omega = \alpha$ has a solution in the space $k-$forms.

In this subsection denote $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{L^2}$ and the induced norm by $\|\cdot\|$. Suppose that ω is a solution of $\Delta\omega = \alpha$, then we have that for all $\varphi \in \Omega^k(M)$ that

$$\langle \Delta\omega, \varphi \rangle = \langle \alpha, \varphi \rangle \text{ and } \langle \omega, \Delta^*\varphi \rangle = \langle \alpha, \varphi \rangle.$$

This might suggest us that the study of the solution ω could be viewing it as a linear functional $l = \langle \omega, \cdot \rangle \colon \Omega^k(M) \to \mathbb{R}$. Now this means that $l(\Delta^*\varphi) = \langle \alpha, \varphi \rangle$. We will call l a *weak solution* of $\Delta\omega = \alpha$, if for all $\varphi \in \Omega^k(M)$ the equality $l(\Delta^*\varphi) = \langle \alpha, \varphi \rangle$ holds. The study of functionals will allow us to make us of functional analysis machinery. From now on we have seen that an ordinary solution of the above equation is also a weak solution. The goal from now on is to prove that the converse is also true for the operator Δ. This is in a sense that if l is a functional that is a weak solution, then l has a representative form in $\omega \in \Omega^k(M)$. Meaning

$$\langle \Delta\omega, \varphi \rangle = \langle \omega, \Delta^*\varphi \rangle = l(\Delta^*\varphi) = \langle \alpha, \varphi \rangle$$

for all $\varphi \in \Omega^k(M)$.

Proposition 3. *(Regularity) Let α be a $k-$form and l a weak solution of $\Delta\omega = \alpha$. Then there exists representative $k-$form for l.*

Proposition 4. *Let $\{\alpha_n\}$ be a sequence of $k-$forms on M with the constant bounds $\|\alpha_n\| \leq c$ and $\|\Delta\alpha_n\| \leq c$ for all n. Then $\{\alpha_n\}$ has a Cauchy subsequence in $\Omega^k(M)$.*

Proofs of these two propositions require machinary from functional analysis, Sobolev spaces and elliptic pde and we will take them as given. Formal treatment can be found in [1, p. 227-251]. Before proving the Hodge theorem, we show that the space of forms can be expressed in a nice direct summand way.

Proposition 5. *(The Hodge decomposition) For all $0 \leq k \leq n$ we have a direct sum decomposition*

$$\Omega^k(M) = \Delta(\Omega^k(M)) \oplus \mathcal{H}^k(M) \tag{1}$$
$$= d\delta(\Omega^k(M)) \oplus \delta d(\Omega^k(M)) \oplus \mathcal{H}^k(M) \tag{2}$$
$$= d(\Omega^{k-1}(M)) \oplus \delta(\Omega^{k+1}(M)) \oplus \mathcal{H}^k(M). \tag{3}$$

In addition $\mathcal{H}^k(M)$ is finite dimensional and the equation $\Delta\omega = \alpha$ has a solution iff the $k-$form $\alpha \perp \mathcal{H}^k(M)$.

Proof. Assuming the decompoistion holds we first check the finite dimensionality. Suppose contrary that $\mathcal{H}^k(M)$ is not finite dimensional. Then there is an orhtonormal sequence of harmonic forms $\{\alpha_n\}$ where $\|\alpha_n\| = 1$ for all n and $\|\Delta\alpha_n\| = 0$ obviously for all n. Thus by previous proposition, it contains a Cauchy subsequence. But such sequence cannot exist since $\|\alpha_n - \alpha_m\| \geq \sqrt{2}$ when $n \neq m$. Now to prove the decomposition.

Line (2) follows from the fact that $\langle d\delta\omega, \delta d\omega\rangle = 0$ for non harmonic components. Line (3) follows from the following fact. Firstly we only have to care about the non harmonic summand for non trivial part. Let be a non zero form $\beta = d\theta + \delta\tau \in d(\Omega^{k-1}(M)) \oplus \delta(\Omega^{k+1}(M))$. By proposition 1 then either $d(d\theta + \delta\tau) = d\delta\tau \neq 0$ or $\delta(d\theta + \delta\tau) = \delta d\tau \neq 0$. Again since $\langle d\delta\tau, \delta d\tau\rangle = 0$, we have a non zero representation for β in $d\delta(\Omega^k(M)) \oplus \delta d(\Omega^k(M))$. Other way around ignore the harmonic part and pick two nonzero forms $\alpha, \beta \in d\delta(\Omega^k(M)) \oplus \delta d(\Omega^k(M))$. Thus they can be written as $\alpha = d\delta\omega + \delta d\tau, \beta = d\delta\omega' + \delta d\tau'$. Now assuming $0 = \alpha - \beta$, we get that $\Delta(0) = 0 = \alpha - \beta = d\delta(\omega - \omega') + \delta d(\tau - \tau')$. This happens only if $\tau = \tau'$ and $\omega = \omega'$. Hence the direct summand representation goes both ways. Now the line (1) is much more non trivial. We can write by finite dimensionality that $\Omega^k(M) = \mathcal{H}^k(M)^\perp \oplus \mathcal{H}^k(M)$. So it suffices to show that $H^k(M)^\perp = \Delta(\Omega^k(M))$. Firstly if $\beta \in \Delta(\Omega^k(M))$ and $\alpha \in \mathcal{H}^k(M)$, then

$$\langle \Delta\beta, \alpha\rangle = \langle \beta, \Delta\alpha\rangle = 0 \tag{4}$$

so $\Delta(\Omega^k(M)) \subset H^k(M)$. For reverse inclusion we want to find a constant $c > 0$ for which

$$\|\beta\| \leq c\|\Delta\beta\| \quad \text{for all } \beta \in \mathcal{H}^k(M)^\perp. \ (*)$$

Suppose contrary that there a sequence $\{\beta_j\} \subset \mathcal{H}^k(M)^\perp$ for which $\|\beta_j\| = 1$, but $\|\Delta\beta_j\| \to 0$. By proposition 4 it has a Cauchy subsequence $\{\beta_i'\} \subset \{\beta_j\}$. Assuming we have topology induced by the L^2-norm, the map $\langle\cdot, \omega\rangle\colon \Omega^k(M) \to \mathbb{R}$ is continuous. Thus we get by completeness of $\mathbb{R}$ that $\lim_{j\to\infty}\langle\beta_j', \varphi\rangle$ exists for all $\varphi \in \Omega^k(M)$. Thus define a functional

$$l(\psi) = \lim_{j\to\infty}\langle\beta_j', \psi\rangle, \quad \psi \in \Omega^k(M).$$

This is a bounded linear functional by Cauchy-Schwartz. In addition by self-adjointness of the Hodge Laplacian we get $l(\Delta\psi) = 0$ meaning it is a weak solution of the equation $\Delta\beta' = 0$. Here β' is the representative given by regularity theorem and thus we need to have that $\beta'_j \to \beta'$ converges in norm. Also this means that $\|\beta'\| = 1$ and $\beta' \in \mathcal{H}^k(M)^\perp$. But we know that $\Delta\beta' = 0$, so we arrive at a contradiction since β' cannot be the zero vector.

Now we can use the fact $(*)$ to finally show the other inclusion. Take $\alpha \in \mathcal{H}^k(M)^\perp$ and define a functional on $\Delta(\Omega^k(M))$ by $l(\Delta\varphi) = \langle \alpha, \varphi \rangle$ for all $\varphi \in \Omega^k(M)$. This is welldefined, since if $\Delta\varphi = \Delta\psi$, then $\varphi - \psi \in \mathcal{H}^k(M)$ so $\langle \alpha, \varphi - \psi \rangle = 0$. The functional l is bounded. Let $\varphi \in \Omega^k(M)$ and ψ be the non harmonic part of φ so

$$|l(\Delta\varphi)| = |l(\Delta\psi)| = |\langle \alpha, \psi \rangle| \overset{CS}{\leq} \|\alpha\|\|\psi\| \overset{(*)}{\leq} c\|\alpha\|\|\Delta\psi\| = c\|\alpha\|\|\Delta\varphi\|.$$

By a version of Hahn-Banach theorem [4, theorem 3.6], we have a bounded extension of l to all of $\Omega^k(M)$. Hence l is a weak solution of $\Delta\omega = \alpha$ and by regularity there exists a suitable representative $\omega \in \Omega^k(M)$ such that $\Delta\omega = \alpha$. Thus $\mathcal{H}^k(M)^\perp = \Delta(\Omega^k(M))$ and this concludes the proof.

□

Now the Hodge theorem follows from the following observation. Assuming the Hodge decomposition a $k-$form can be written as $\omega = d\delta\tau + \delta d\tau + \alpha$, where α is harmonic. Assume that ω is a closed form. Then we get that $\Delta d\tau = d\delta d\tau = 0$, so $d\tau$ is harmonic. So by proposition 1 also $\delta d\tau = 0$. Hence $\omega = d\delta\tau + \alpha$ so by definition ω is cohomologous to α. If ω was exact it would be cohomologous to zero which is harmonic obviously.

This type of Hodge decomposition can also be done for Dolbeault operators i.e. the complex manifold analogue of the exterior differentiation of differential forms. Also the Hodge theorem can be used to prove Poincaré-Duality for closed, connected and oriented manifolds.

3 The Bochner formula

Now we want to turn our focus to Harmonic $1-$forms. Let (M, g) be closed and oriented Riemannian manifold and let θ be a harmonic $1-$form. Define a function $f = \frac{1}{2}|\theta|^2$, where by the Riemannian metric applied on a $1-$form means that we apply it to the corresponding vector field X (i.e. $\theta(v) = g(X, v)$). To open up the structure of f, we can write

$$f = \frac{1}{2}|\theta|^2 = \frac{1}{2}|X|^2 = \frac{1}{2}\theta(X).$$

Now recall the Riemannian geometry analogue of divergence of a vector field $\mathrm{Div}\, V = \mathrm{tr}(X \mapsto \nabla_V X)$ where our connection is now the Levi-Civita connection.

Proposition 6. *If V is a vector field on (M, g), then*

$$\mathrm{Div}\, V = -\delta\omega$$

where ω is the corresponding dual $1-$form of V.

Proof. By definition $\langle df, \omega \rangle = \langle f, \delta\omega \rangle$ with respect to the L^2-inner product where f is a smooth function. Now the RHS of the previous equation is just the classic L^2-inner product of functions. Meaning

$$\langle f, \delta\omega \rangle = \int_M f \cdot \delta\omega \operatorname{Vol}.$$

The LHS has the following property by definition

$$\langle df, \omega \rangle = \int_M g(df^\sharp, \omega^\sharp) \operatorname{Vol} = \int_M g(df^\sharp, V) \operatorname{Vol} = \int_M df(V) \operatorname{Vol}.$$

The divergence of a volume form can be written via the Lie derivative $L_V \operatorname{Vol} = \operatorname{Div} V \operatorname{Vol}$. So far we have that

$$-\langle f, \delta\omega \rangle = -\int_M f \cdot \delta\omega \operatorname{Vol} = -\int_M df(V) \operatorname{Vol} = -\int_M (L_V f) \operatorname{Vol}.$$

By Leibniz formula taking Lie derivative of tensor fields

$$\begin{aligned} L_V(f \operatorname{Vol}) &= (L_V f) \operatorname{Vol} + f(L_V \operatorname{Vol}) \\ &= df(V) \operatorname{Vol} + f(L_V \operatorname{Vol}) \end{aligned}$$

If we show that $\int_M L_V(f \operatorname{Vol}) = 0$, then we get that

$$\int_M f \cdot \delta\omega \operatorname{Vol} = \int_M df(V) \operatorname{Vol} = -\int_M f(L_V \operatorname{Vol}) = -\int_M f \operatorname{Div} V \operatorname{Vol}.$$

If $L_V(f \operatorname{Vol})$ is exact, this is then true by Stokes' theorem. Exactness follows by applying Cartan's magic formula. Thus $-\delta\omega = \operatorname{Div} V$ by the fundamental lemma of calculus of variations on Riemannian manifolds. □

Proposition 7. *Let V, ω and M be as above. Then $v \mapsto \nabla_v X$ is symmetric iff ω is closed.*

Proof. Recall the formula $d\omega(Y, Z) + L_X(V, W) = 2g(\nabla_V X, W)$. Since $L_X g$ is symmetric and $d\omega$ is skew-symmetric we get the result. □

This gives that for harmonic $1-$forms ω, the corresponding vector field X has vanishing divergence and ∇X is symmetric $(1,1)-$tensor field. In addition this implies that X is a killing field ($L_X g \equiv 0$) iff ω is closed.

Proposition 8. *(Bochner formula for $1-$forms) Let X be a vector field such that ∇X is symmetric (the corresponding $1-$form is closed). If $f = \frac{1}{2}|X|^2$ and X is the gradient of some function u in a neighbourhood of p, then*

1.

$$\nabla f = \nabla_X X.$$

2.

$$\begin{aligned}\operatorname{Hess} f(V,V) &= \operatorname{Hess}^2 u(V,V) + (\nabla_X \operatorname{Hess} u)(V,V) + R(V,X,X,V)\\ &= |\nabla_V X|^2 + g(\nabla^2_{X,V}X, V) + R(V,X,X,V).\end{aligned}$$

3.

$$\begin{aligned}\Delta f &= |\operatorname{Hess} u|^2 + D_X \Delta u + \operatorname{Ric}(X)\\ &= |\Delta X|^2 + D_X \operatorname{Div} X + \operatorname{Ric}(X).\end{aligned}$$

Proof. For (1) $g(\nabla f, X) = d(\frac{1}{2}|X|^2)(X) = g(\nabla_V X, X) = g(\nabla_X X, V)$. (2) and (3) are longer computations so we refer them to be found in [3, 9.2.2.]. □

Proposition 9. *(Bochner 1948) If M is a closed Riemannian manifold with non negative* $\operatorname{Ric} \geq 0$*, then every harmonic $1-$form is parallel (i.e. $\nabla\omega \equiv 0$).*

Proof. Let ω be harmonic $1-$form and X it's dual vector field. By plugging in $f = \frac{1}{2}|X|^2$ to the Bochner formula, we by previous theorem that

$$\Delta f = |\nabla X|^2 + \operatorname{Ric}(X) \geq 0,$$

because $\operatorname{Div} X = \Delta u = 0$. By Green's theorem

$$0 = \int_M \Delta f \operatorname{Vol} = \int_M |\nabla X|^2 + \operatorname{Ric}(X) \operatorname{Vol} \geq \int_M |\nabla X|^2 \operatorname{Vol} \geq 0$$

so $|\nabla X| = 0$. Because musical isomorphisms commute with Levi-Civita connection, we get that $|\nabla X| = g(\nabla X, \nabla X) = g((\nabla X)^\flat, (\nabla X)^\flat) = g(\nabla\omega, \nabla\omega) = |\nabla\omega|$. □

Corollary 1. *If M is as in the previous proposition and has positive Ricci curvature at one point, then all harmonic $1-$forms vanish everywhere.*

Proof. Assume that $\operatorname{Ric}_p > 0$ in the tangent space T_pM. In a same way as above we can deduce that, $\operatorname{Ric}(X) \equiv 0$ so we must have $X_p = 0$. By previous theorem X is parallel so we must have that $X \equiv 0$, since $0 = \Delta f(p) = \operatorname{Ric}(X)_p > 0$ is a contradiction unless X is zero. □

Corollary 2. *(Upper bound for first Betti number) If M is closed Riemannian $n-$manifold with* $\operatorname{Ric} \geq 0$*, then $b_1(M) \leq n$. In addition $b_1(M) = n$ iff M is a flat torus.*

Proof. By Hodge theory $b_1(M) = \dim \mathcal{H}^1(M)$. Let ω be a harmonic, form. Then the evaluation map $\omega \mapsto \omega_p \in T_p^*M$ injective. Just like before if $\omega_p = 0$, then for the corresponding vectorfield $X_p = 0$. So just like above, X is zero everywhere meaning $\omega \equiv 0$. Hence $b_1(M) \leq \dim T_p^*M = n$. If the equality holds, then there is n linearly independent parallel vector fields on M. So M must be flat. With small Riemannian covering theory, one can show that M is a flat torus. Notably the fundamental group can be identified as the group of deck transformations of the universal cover. □

4 More general result for Ricci curvature of negative lower bound

To generalize the previous upper bound for larger class of Riemannian manifolds, we need some general analysis type results. Most of these can be found under this in subsection 4.1. The reader is recommend to skim through these first and then come back later to see the nature of some of the upper bounds. We start very generally considering a vector bundle $E \to M$ over a closed riemannian $n-$manifold M, where each fiber has a smoothly varying inner product $\langle \cdot, \cdot \rangle$ and each fiber E_p has dimension m. We can know equip sections of this vector bundle with several norms

$$\|s\|_\infty := \max_{x \in M} |s(x)| \;\; \& \;\; \|s\|_p := \left(\frac{1}{\operatorname{Vol} M} \int_M |s|^p \operatorname{Vol} \right)^{1/p}.$$

Note that the now $|\cdot|$ is induced by the smoothly varying inner product. This could be now called the *normalized* L^p-*norm* where we have that $\|\cdot\|_p \le \|\cdot\|_q$ for all $0 < p \le q \le \infty$. For L^2 we have a natural innerproduct given by the integral $(v, w) = \frac{1}{\operatorname{Vol} M} \int_M \langle v, w \rangle \operatorname{Vol}$. Now if we fix a finite dimensional subspace $V \subset \Gamma(E)$[5], all of the norms above will be equivalent in this subspace. Now we can define a constant

$$C(V) := \max_{s \in V \setminus \bar{0}} \frac{\|s\|_\infty}{\|s\|_2}$$

where the dimension of V can be estimated by the above constant and the dimension of the fibers E_p.

Lemma 1. *(Li) With the above conventions* $\dim V \le mC(v)$.

Proof. Let $e_1, \ldots, e_d$ be a basis for V and define $f(x) = \sum_{i=1}^d |e_i(x)|^2$. Here f does not depend on the chosen basis, and

$$\frac{1}{\operatorname{Vol} M} \int_M f \operatorname{Vol} = d = \dim V.$$

Let x_0 the maximum of f. Consider the evaluation map $\operatorname{ev}: V \to E_{x_0}$, where $g \mapsto g(x_0)$. Assume that the basis is chosen in a way that the last $d - k$ elements span $\ker \operatorname{ev}$. We have then that $k \le m$ and $\dim V \le f(x_0) \le kC(V) \le mC(V)$ since each section has unit L^2-norm. ☐

The proof of our main result will rely on the following estimation.

Proposition 10. *(Moser iteration) Let M be a closed Riemannian manifold for which*

$$\|u\|_{2\nu} \le S\|\nabla u\|_2 + \|u\|_2$$

[5] $\Gamma(E)$ denotes the vector space of sections of E.

for all smooth functions u, where $\nu > 1$. If $f\colon M \to \mathbb{R}_{\geq 0}$ is continuous, smooth on $\{f > 0\}$, and $\Delta f \geq -\lambda f$, then

$$\|f\|_\infty \leq \exp\left(\frac{S\sqrt{\lambda\nu}}{\sqrt{\nu}-1}\right)\|f\|_2.$$

Proof. The the set f is not supported in either has derivative zero or it is measure zero point, so we can ignore it when integrating. Additionally at the points of vanishing Δf is non negative in the vanishing set (in a distributional sense). Thus we can use Green's formula and chain rule to get $(f^{2q-1}, \Delta f) = (d(f^{2q-1}), df) = -(2q-1)(f^{2q-2}df, df)$. This leads to

$$\begin{aligned}
\|d(f^q)\|_2^2 &= q^2(f^{2q-2}df, df) \\
&= -\frac{q^2}{2q-1}(f^{2q-1}, \Delta f) \\
&\leq \frac{q^2\lambda}{2q-1}(f^{2q-1}, f) \\
&= \frac{q^2\lambda}{2q-1}\|f^q\|_2^2.
\end{aligned}$$

By assumption (and by ($|df| = |\nabla f|$ for all functions) and above deduced inequality

$$\|f^q\|_{2v} \leq S\|d(f^q)\|_2 + \|f^q\|_2 \leq \left(Sq\left(\frac{\lambda}{2q-1}\right)^{1/2} + 1\right)\|f^q\|_2.$$

Taking qth root form both sides of the above equation one gets that

$$\|f\|_{2vq} \leq \left(Sq\left(\frac{\lambda}{2q-1}\right)^{1/2} + 1\right)^{1/q}\|f\|_{2q}.$$

Choose $q = v^k$ to get

$$\|f\|_{2v^{k+1}} \leq \left(Sv^k\left(\frac{\lambda}{2v^k-1}\right)^{1/2} + 1\right)^{v^{-k}}\|f\|_{2v^k}.$$

By recursive reasoning starting at $k = 0$ we get that

$$\begin{aligned}
\|f\|_{2v^{k+1}} &\leq \left(Sv^k\left(\frac{\lambda}{2v^k-1}\right)^{1/2} + 1\right)^{v^{-k}}\|f\|_{2v^k} \\
&\leq \left(Sv^k\left(\frac{\lambda}{2v^k-1}\right)^{1/2} + 1\right)^{v^{-k}}\left(Sv^{k-1}\left(\frac{\lambda}{2v^{k-1}-1}\right)^{1/2} + 1\right)^{v^{-(k-1)}} \leq \|f\|_{2v^{k-1}} \\
&\leq \dots \\
&\leq \prod_{i=0}^{k}\left(Sv^i\left(\frac{\lambda}{2v^i-1}\right)^{1/2} + 1\right)^{v^{-i}}\|f\|_2.
\end{aligned}$$

Taking the limit as $k \to \infty$, we get

$$\|f\|_\infty \leq \prod_{k=0}^{\infty} \left(Sv^k \left(\frac{\lambda}{2v^k - 1} \right)^{1/2} + 1 \right)^{v^{-k}} \|f\|_2.$$

To see that this product can be estimated by a constant, we use the log inequality $\log(1+x) \leq x$. That is

$$\sum_{k=0}^{\infty} \log \left(Sv^k \left(\frac{\lambda}{2v^k - 1}^{1/2} \right) + 1 \right) \leq S\sqrt{\lambda} \sum_{k=0}^{\infty} \left(\frac{1}{2v^k - 1} \right)^{1/2} \leq \frac{S\sqrt{\lambda v}}{\sqrt{v} - 1}.$$

Taking exp now yields the desired result. $\square$

Theorem 1. *(Gromov 1980, Gallot 1981). If M is a closed Riemannian $n-$manifold, such that $\mathrm{Ric} \geq (n-1)k$ and $\mathrm{Diam}(M) \leq D$, then there is a function $C(n, kD^2)$ for which*

$$b_1(M) \leq C(n, kD^2).$$

In addition $\lim_{\varepsilon \to 0} C(n, \varepsilon) = n$. In particular there is $\varepsilon(n) > 0$ such that when $kD^2 \geq -\varepsilon(n)$, then $b_1(M) \leq n$.

Gromov's original proof relies on theory of Riemannian covering spaces to control the Betti number where first homology group can be seen as the abelinization of the fundamental group. Instead we will prove the Gallot's version which has applications in wider context see [3, 9.3.1]. Gallot's contribution was in part to the Sobolev constant in proposition 13 where the techniques for the proof were developed by Li in late 70's.

Proof. Since we can represent the first cohomology class by harmonic $1-$forms, we can look at $\dim \mathcal{H}(M)$ and $1-$forms $\omega \in \mathcal{H}(M)$. We want to study the ratio $\frac{\|\omega\|_\infty}{\|\omega\|_2}$. Hence let $f = |\omega|$, where f is smooth other than at possibly the places where ω vanishes (minimums of f). By Leibniz rule, metric compatability, Cauchy-Schwartz and musical isomorphisms commuting with the connection we have that $2f df = d(f^2) = 2g(\nabla\omega, \omega) \leq 2|\nabla\omega| f$, so $df \leq |\nabla\omega|$ (Kato's inequality). Hence the Bochner formula implies

$$\begin{aligned} |df|^2 + d\Delta f &= \frac{1}{2}\Delta f^2 \\ &= |\nabla\omega|^2 + \mathrm{Ric}(X) \\ &\geq |\omega|^2 + (n-1)kf^2. \end{aligned}$$

$\square$

By Kato's inequality, this gives that $\Delta f \geq (n-1)kf$. Applying now Moser iteration

$$\|f\|_\infty \leq \exp\left(\frac{S\sqrt{-(n-1)kv}}{\sqrt{v} - 1} \right) \|f\|_2,$$

where $S = DC(c, kD^2)$ is estimated in the analysis type results section propositions 14 and 13. Since $\|f\|_\infty/\|f\|_2 = \|\omega\|_\infty/\|\omega\|_2$, by Li's lemma

$$b_1(M) = \dim \mathcal{H}^1(M) \leq nC(\mathcal{H}^1(M)) \leq n \exp\left(\frac{S\sqrt{-(n-1)kv}}{\sqrt{v}-1}\right).$$

Nature of these relying on S bounds show the condition $\lim_{\varepsilon\to 0} C(n,\varepsilon) = n$ (note that this is not same C as what S depends on).

4.1 Analysis type estimates on $C(n, kD^2)$ of the Gromov-Gallot bound

Some analysis type results from [3, 7.1.5] to give vision on how the upper bound $C(n, kD^2)$ looks like. So rather than going with the full proves we want to see the estimates. Throught out all of this section M is closed and oriented Riemannian $n-$manifold with $\mathrm{Ric} \geq (n-1)k$, $k \leq 0$, $\mathrm{Diam}(M) \leq D$. Let B be a domain, where we define the avaraging L^p-norm by

$$\|u\|_{p,B} := \left(\frac{1}{\mathrm{Vol}\, B}\int_B |u|^p \,\mathrm{Vol}\right)^{1/p}.$$

Also denote the average of a function u with respect to a domain by $u_B := \frac{1}{\mathrm{Vol}\, B}\int_B u \,\mathrm{Vol}$.

We call a curve $c_{x,y}\colon [a,b] \to M$ a *segment*, if it is arclength parametrized, $\mathrm{length}(c_{x,y}) = d(x,y)$ and $c_{x,y}(a) = x, c_{x,y}(b) = y$.

Proposition 11. *(Cheeger-Colding Segment inequality) Let $f\colon M \to \mathbb{R}_{\geq 0}$ and $A, B \subset W \subset M$. Further select segments $c_{x,y}\colon [0,1] \to M$ between points $x, y \in M$. If $c_{x,y}(t) \in W$ for all $x \in A, y \in B, t \in [0,1]$ and $\mathrm{Diam}(W) \leq D$, then*

$$\int_{A\times B}\int_0^1 f \circ c_{x,y}(t)dt \,\mathrm{Vol}_x \wedge \mathrm{Vol}_y \leq C(\mathrm{Vol}\, A + \mathrm{Vol}\, B)\int_W f \,\mathrm{Vol}\,.$$

Proof. Full proof can be found in Petersen as listed above, but the bound C can be explicitly expressed as

$$C = \max_{R\leq D} \frac{\mathrm{sn}_k^{n-1}(R)}{\mathrm{sn}_k^{n-1}(R/2)}.$$

Here $\mathrm{sn}_k(R)$ denotes the unique solution to differential eqution

$$\ddot{x}(t) + kx(t) = 0, x(0) = 0, \dot{x}(0) = 1$$

(this is the differential equation appearing in constant curvature spaces).

When $k = 0, C = 2^{n-1}$ and otherwise

$$C = \frac{\sinh^{n-1}(\sqrt{-k}D)}{\sinh^{n-1}\sqrt{-k}D/2}.$$

□

Corollary 3. *(Cheeger-Colding inequality) Let $u\colon M \to \mathbb{R}_{\geq 0}$ be smooth. Then*

$$\left\|u - u_{B(p,R)}\right\|_{1,B(p,R)} \leq 4C^2 R\|du\|_{1,B(p,2R)}.$$

Proof. Once again full proof found in Petersen. Here we use the previous proposition with $A = B = B(p,R), W = B(p,2R), f = |du|$. □

This corollary holds for any measurable u with a function G *(upper gradient)* in place of $|du|$.

Proposition 12. *(Weak Sobolev-Poicaré inequality) Let $u\colon M \to \mathbb{R}_{\geq 0}$ be smooth. Then there is a weak Poincaré-Sobolev inequality.*

$$t^{\frac{n}{n-1}} \operatorname{Vol}\{\left|u - u_{B(x,R)}\right| > 0\} \leq C(n, kD^2) R^{\frac{n}{n-1}} \||du|\|_{1,B(c,R)}^{\frac{n}{n-1}}.$$

.

Proof. Full proof in Peterson. For simplicity we show $R = D$. Fix any $x \in M$ and define $R_i = 2^{-i}D$, $B_i = B(x, R_i)$, where $B_0 = M$. For smooth funtions the Lebesgue differentiation theorem holds everywhere, so $u(x) = \lim u_{B_i}$. This gives us estimates

$$\begin{aligned}
|u - u_{b_0}| &\leq \sum_{i=0}^{\infty} \left|u_{B_i} - u_{B_{i+1}}\right| \leq \sum_{i=0}^{\infty} \|u - u_{B_i}\|_{1,B_{i+1}} \\
&\leq \sum_{i=0}^{\infty} \frac{\operatorname{Vol} B_i}{\operatorname{Vol} B_{i+1}} \|u - u_{B_i}\|_{1,B_i} \overset{\text{Cheeger-Colding}}{\leq} 2C^3 \sum_{i=0}^{\infty} R_i \||du|\|_{1,B_{i-1}}.
\end{aligned}$$

It then suffices to show that

$$t^{n/(n-1)} \operatorname{Vol}\left\{\sum_{i=0}^{\infty} R_i \||du|\|_{1,B_{i-1}} > t\right\} \leq C D^{n/(n-1)} \operatorname{Vol} M \||du|\|_1^{n/(n-1)}.$$

□

Proposition 13. *(Stronger Sobolev-Poicaré inequality) For all smooth $u\colon M \to \mathbb{R}_{\geq 0}$ and $v \in [1, n/(n-1)]$, we have*

$$\left\|u - u_{B(x,R)}\right\|_{v,B(x,R)} \leq C(n, kD^2) R \||du|\|_{1,B(x,R)},$$

where $R \leq D$.

Proof. Full proof in Petersen. First notice two elementary results that for any $c \in \mathbb{R}$, $\|u - u_M\|_p \leq 2\|u - c\|_p$ and $\inf_{c\in\mathbb{R}} \|u - c\|_p \leq \|u - u_M\|_p$. We want to approximate $\|u - c\|$ for a suitable c. For a general u, we want to find some m, such that $\operatorname{Vol}\{u \geq m\} \geq \operatorname{Vol} M/2$ and $\operatorname{Vol}\{u \leq m\} \geq \operatorname{Vol} M/2$. We split u into two functions $v^+ = \max\{u - m, 0\}$, $v^- = \max\{m - u, 0\}$. Both of these satisfy $\operatorname{Vol}\{v^\pm = 0\} \geq \operatorname{Vol} M$. Whilst $v^\pm$ might not be smooth, we can use Radamacher's theorem type reasoning to define $|dv^{\pm}| = 0$ at points of vanishment. Thus one needs just show that $\|v^{\pm}\|_{n/(n-1)} \leq C(n, kD^2)D\||dv^{\pm}|\|_1$. Then one defines a truncated function

$$v^{\pm}_{a,b} = \begin{cases} b - a, & \text{if } v^{\pm}(x) \geq b \\ v^{\pm}(x) - a, & \text{if } a < v^{\pm}(x) \leq b, \\ 0, & \text{if } v^{\pm}(x) \leq a. \end{cases}$$

The weak Poincaré-Sobolev inequality is then applied to $v^{\pm}_{a,b}$ with the upper gradient $|dv^{\pm}| \cdot \chi_{a<v^{\pm}\leq b}$, ($\chi$ is an indicator function). So one gets

$$t^{n/(n-1)} \operatorname{Vol}\{v^{\pm}_{a,b} > t\} \leq CD^{n/(n-1)} 2^{n/(n-1)+1} \operatorname{Vol} \left\| |dv^{\pm}| \cdot \chi_{a<v^{\pm}\leq b} \right\|_1^{n/(n-1)}$$

where C depends on the weak Poincaré-Sobolev inequality. With final estimates one gets that

$$\int_M (v^{\pm})^{n/(n-1)} \leq 2^{3n/(n-1)+1} CD^{n/(n-1)} \operatorname{Vol} M \left\| |dv^{\pm}| \right\|_1^{n/(n-1)}.$$

□

Proposition 14. *Assume that $u \in C^{\infty}(M)$ satisfies*

$$\|u - u_M\|_{s/(s-1)} \leq S\|du\|_1,$$

with $s > 1$. Then for $1 \leq p < s$

$$\|u\|_{sp/(s-p)} \leq \frac{p(s-1)}{s-p} S\|du\|_p + \|u\|_p.$$

Proof. If $p = 1$, then

$$\begin{aligned} \|u - u_M\|_{s/(s-1)} &\geq \|u\|_{s/(s-1)} - \|u_M\|_{s/(s-1)} \\ &= \|u\|_{s/(s-1)} - |u_M| \\ &\geq \|u\|_{s/(s-1)} - \|u_M\|_1. \end{aligned}$$

For $p > 1$

$$\begin{aligned} \|u\|^q_{qs/(s-1)} &= \|u^q\|_{s/(s-1)} \\ &\leq S\|d(u^q)\|_1 + \|u^q\|_1 \\ &= Sq\left\|u^{q-1} du\right\|_1 + \|u_{q-1}u\|_1 \\ &\leq \|u\|^{q-1}_{p(q-1)/(p-1)} (Sq\|du\|_p + \|u\|_p). \end{aligned}$$

Choosing $q = p(s-q)/(s-p)$ yields the result. □

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