

## **Taxman Revisited**

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### **Abstract**

In an earlier article [1], Carmony and Holliday illustrated how the game called Taxman could be used in CS1 to introduce some elementary concepts from Artificial Intelligence. This article will present several strategies that were created (as part of a CS2 assignment) to play Taxman, and evaluate their performance.

### **Introduction**

As described in [1], Diane Resek of San Francisco State University invented the game called Taxman. This is a one person game which can be played alone or, more easily, on a computer. The player selects a positive integer, let's call it  $N$ , and the game then begins with all the integers from 1 to  $N$  being "active". The player can choose any active integer that still has at least one smaller, active integer that evenly divides into it, and the player's score will then be increased by the number chosen while all smaller, active divisors of that chosen number are added to the Taxman's score. (All integers that were added to either score are then marked as inactive.) The player keeps choosing active integers until the only active integers that remain are all relatively prime to each other. At this point the game ends, with the Taxman's score increased by the those unused integers. The object of the game is for the player to acquire a score larger than the Taxman's.

As an example with  $N=8$ , the player could choose 7 and the Taxman would only get 1. The next selection could be 6, giving the 3 and the 2 to the Taxman. If 8 was then chosen, the player's score would now be 21 and the 4 would go to the Taxman, as well as the 5, which remains and can not be selected by the player, giving the Taxman a score of 15.

### **Taxman modifications**

The students in my CS2 class this spring were given two short assignments involving the Taxman game. The first was to write a program to facilitate the playing of the game with one modification. If someone playing the Taxman game chooses a large value for  $N$ , he/she can appear to be winning right up to the final choice. After making the final selection, all remaining integers are added to the Taxman's score and the player usually loses. It seemed reasonable to keep the player better informed as to his/her actual position as the game progressed. (In the example given, as soon as the 7 was selected, it is known that the 5 will eventually be added to the Taxman's score since there are no active divisors for it left and it is not a divisor for any other active integers.) Therefore, the program is to immediately add to the Taxman's score all integers that could neither be selected nor used as a divisor, and then mark them as inactive. (This modification is implicit in all future references to the Taxman game in this article.)

The second assignment was a team project with two students per team. Each team then modified one of their programs so that **it** would choose the integers selected, with a strategy of their own invention. It was pointed out in class that the optimal first choice was always to select the largest prime number but after that, they were on their own. One class was allocated (5 days after the assignment was given) where each team would use my version of their first assignment in our computer lab, and try out the ideas they had come up with over the previous weekend. This also counted as a quiz, where they would be graded on how well their strategy worked with  $N=64$ , in an attempt to have them settle on a reasonable strategy early on in the project. The rest of the paper describes the strategies used and the results that were collected.

### Rules for Selecting Integers

There were two strategies described in [1]: take the largest integer that still had at least one active divisor; and maximize the difference between the next choice and its active divisors. I will refer to these simply as strategies #1 and #2. After thinking about the problem myself and playing the game a few times with large  $N$  values, I wondered if there was a way to come up with a generalized rule that would explain why choosing the largest prime first was optimal, and to see if that strategy would perform well for many different values of  $N$ .

Strategy #3 is the one I implemented (as well as one team of students) and the rule it uses is to always take the largest active integer with only one active divisor (smaller than it). The idea behind this was that if that number, call it  $X$ , has only one divisor left, call it  $D$ , it would behoove one to take  $X$  before choosing another integer which could use up  $D$  and thereby add  $X$  to the Taxman's score. Other strategies that were used by student teams are as follows (assume all references to selected values are from the list of active integers):

#4 - Strategy #3 as described above. (I will modify #3 to handle one special case in the next section.)

#5 - Like #2, but it also includes in the calculated differences, the integers that will be given to the Taxman because they would become unusable if this selection were actually made.

#6 - Examine the active integers, starting with the largest, and choose the first one whose active divisors add up to half that integer or less.

#7 - Uses 3 steps: choose the largest prime; then choose the odd numbers, starting with the largest and working downward until  $N/2$  is reached; and finally choose the even numbers starting at  $N/2+1$  and working upwards towards  $N$ .

#8 - Same as above except that when working downwards with the odds, the stopping point is  $N/4$ .

#9 - Finds the largest integer with the fewest divisors that is larger than  $N/2-1$ . (Similar to #3 and #4 listed above.)

#10 - A "random" strategy which attempted to implement #2 but failed.

Before reading about how these strategies performed, you might ponder these and predict which one you think will be the "best".

### **Some Preliminary Comments**

While the students were testing their strategies during their aforementioned quiz, I manually attempted to play strategy #3 and noticed an interesting "endgame" situation. Using  $N=64$ , I found that there were always the same 3 integers left at the end of the game - 16, 32 and 64. According to the basic premise, 32 should be selected since it is the largest remaining integer with only 1 divisor, but one can easily see that 64 is the optimal choice. Therefore, the difference between strategies #3 and #4 is that with #3, when there are 5 numbers or less left, it will use strategy #2 and maximize the difference between the integer selected and the active divisors of that integer. (There was no difference between changing from strategy #3 to #2 when 5 or 7 integers remained, and it probably would be the same when using 3 as well. The endgame problem seems to arise when there are an odd number of integers at the game's end.)

Another avenue that I investigated was to find out what the optimal scores were for the Taxman game. Due to the exponential growth in the number of possible ways to choose the integers, only the games from  $N=2$  to  $N=32$  were investigated. (With  $N=32$ , it took an RS/6000 one full day to determine what is the optimal way to select the integers after forcing the program to select 31 - the largest prime - first!)

### **Summary of Results**

There are several metrics that could be used to rank the strategies listed: how well they performed for a specific, large value for  $N$ ; what the average player and Taxman scores were over a large range of values; and "head to head" comparison for each value of  $N$  in a given range (for two different strategies), to see which one achieves the larger player score more often. The first two of these metrics can be found in the Appendix along with a few columns comparing the strategies against the optimal values: summing the player's score for all  $N$  in the range 2-32, and how many times the given strategy matched the optimal score. (A few head to head comparisons were made before the class received this assignment, but the metric appeared unnecessary considering the results summarized in the Appendix. As stated in [1], strategy #2 achieved a larger score than the Taxman for all tested values except for  $N=3$ . When comparing strategy #4 against #2 for  $N=2-512$ , #4 had a higher score 486 times, #2 "beat" #4 only 14 times, and there were 11 ties. Strategy #3 does even better, since it will always be at least as large as #4, losing only 7 times to #2, with the largest such  $N$  being 32, and there were 12 ties.)

Just comparing the sums in the 2-32 column against the optimal score of 3789, one can see that strategies #1 and #10 do not really work well in comparison to the other ones listed. The eight others agree with the optimal score from 13 to 20 times out of the 31 possible  $N$  values, and if you also exclude #8, the sums range only from 3652 to 3744, so they all appear to work reasonably well for small values of  $N$ . In fact, strategies #6, #7 and #9 all arrive at the same scores for all  $N \leq 32$ . (#2 also has the same sum - 3652 - but its individual scores are different from these three in this range.) At this point, one would say that #3 and #5 worked the best since

they have the two largest sums, but as the N values get larger this is not true.

When the range is increased to use N values from 2 to 512, one can see by the average and the scores given on N=256, etc. that strategy #7 is clearly the top performer in this group, though it does not seem to really outperform #3 and #4 until somewhere past N=128. (I must give credit to Kevin Purcell and Robert Connelly for having discovered strategy #7 and thereby finding a strategy which is better than mine - #3. The pattern that they discovered while working on this also appears in the choices given by the optimal solution program!) Using the average of the player's score for values of N in the range 2-512, the strategies would be ranked, from best to worst, in the following order: #7, #3, #4, #9, #2, #6, #5, #8, #10, #1. (In head to head comparison when using N in the range 2-512, strategy #7 had a higher score than #3 379 times, #3 was higher 118 times and there were 14 ties. Increasing the range to 2048 only showed that #7 was better for all values > 512.)

Before listing what the integer choices are to achieve the optimal player's score for certain values of N (and to highlight the pattern used by strategy #7), one surprising fact did pop up from studying the numbers given in the Appendix - strategy #5 started out better than #2, but as N increased, #2 did better than it and I am not sure why, and the fact that it did appears to be counter-intuitive!

Here are the optimal choices discovered (and the player's respective scores are given at the end of the Appendix):

N - integers chosen in this order ---->

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24 - 23 9 15 21 14 16 18 20 22 24  
25 - 23 25 15 21 14 16 18 20 22 24  
26 - 23 25 15 21 14 16 20 22 26 27 18 24  
27 - 23 25 15 21 14 16 20 22 26 27 18 24  
28 - 23 25 15 21 14 22 26 27 18 28 16 20 24  
29 - 29 25 15 21 14 22 26 27 18 28 16 20 24  
30 - 29 22 25 15 21 26 27 18 30 20 16 24 28  
31 - 31 22 25 15 21 26 27 18 30 20 16 24 28

The pattern of odds, then evens, does seem to have some underlying benefit, but why that is the case is not the main thrust of this paper, and is only superficially understood by this author.

## Conclusions

The game of Taxman is simple enough to allow students to investigate different strategies to play the game, and yet is complex enough to make them think as to what strategy will perform best. There are certainly many different ways to implement the basic game, as well as the concomitant performance issues. Taxman served my CS2 course well in that several illustrative discussions on these issues were raised during lecture and it provided one of several application themes that were extended throughout the semester.

## Reference

[1] Carmony, Lowell, and Holliday, Robert. *An example from Artificial Intelligence for CSI*. 24th SIGCSE Technical Symposium on Computer Science Education, SIGCSE Bulletin, Volume 25, #1, 1993, pages 1-5.

### Appendix

	Sum 2-32	Opt Agr	Avg 2-512	N= 128	N= 256	N= 512
#1	2365	3	17128	3389	12872	51398
#2	3652	14	25043	4816	18612	74332
#3	3744	20	27106	5289	19816	81035
#4	3684	17	27043	5289	19696	81035
#5	3724	19	24563	4962	18175	75148
#6	3652	19	24772	4842	18439	72879
#7	3652	19	27619	5084	20530	82474
#8	3498	13	23373	4466	17057	72638
#9	3652	19	25744	4945	19213	76446
#10	3141	6	22742	4415	16945	67336

Optimal scores for the player:

2-14 - 2, 3, 7, 9, 15, 17, 21, 30, 40, 44, 50, 52, 66  
 15-23 - 81, 89, 93, 111, 113, 124, 144, 166, 170  
 24-32 - 182, 198, 224, 251, 279, 285, 301, 303, 319