1. $U = Lo\{(1,1,1),(3,1,-1)\}$

$\mathbb{R}^3$ so skalárnym súčinom

Karolína Vallová

U je podpriestor $\mathbb{R}^3$

$$\begin{array}{cc} y_1 & y_2 \end{array}$$

$U = Lo\{(1,1,1),(3,1,-1)\}$

báza $Z = (\vec{x}_1, \vec{x}_2)$

ortogonálna báza

$$\vec{x}_1 = \vec{y}_1 = (1,1,1)$$

$$\vec{x}_2 = y_2 - proj_{\vec{x}_1}(\vec{y}_2) = y_2 - \frac{\langle \vec{y}_2, \vec{x}_1 \rangle}{\langle \vec{x}_1, \vec{x}_1 \rangle}\vec{x}_1 = \vec{y}_2 - \frac{3}{3}\vec{x}_1 = \vec{y}_2 - \vec{y}_1 \boxed{=}$$

$$\langle \vec{x}_1, \vec{x}_1 \rangle = 3$$

$$\langle \vec{y}_2, \vec{x}_1 \rangle = 3 + 1 - 1 = 3$$

$$\boxed{=} (2,0,-2) = x_2$$

$$\langle x_2, x_2 \rangle = 4 + 4 = 8$$

$$Z = (\overset{x_1}{(1,1,1)}, \overset{x_2}{(2,0,-2)})$$

$$X = \left(\frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|}\right) = \left(\frac{x_1}{\sqrt{\langle x_1, x_1\rangle}}, \frac{x_2}{\sqrt{\langle x_2, x_2\rangle}}\right) = \left(\frac{x_1}{\sqrt{3}}, \frac{x_2}{\sqrt{8}}\right) = \left(\frac{1}{\sqrt{3}}(1,1,1), \frac{1}{\sqrt{8}}(2,0,-2)\right)$$

ortonormálna báza X

$$U^{\perp} = \{\vec{w} \in \mathbb{R}^3 : \vec{w} \perp U\}$$

$$\langle \vec{w}_1, \vec{y}_1 \rangle = 0 \qquad \vec{w}_1 = (-1,1,1)$$

$$\langle \vec{w}_2, \vec{y}_2 \rangle = 0 \qquad \vec{w}_2 = (-1,1,3)$$

$1+1+1$

$-1+1+3$

$$\vec{x}_1' = \vec{w}_1$$

$$X' = \left(\frac{\vec{x}_1'}{\sqrt{3}}, \frac{\vec{x}_2'}{\sqrt{8}}\right) = \left(\frac{1}{\sqrt{3}}(-1,1,1), \frac{1}{\sqrt{8}}(2,0,2)\right)$$

$$\vec{x}_2' = \vec{w}_2 - proj_{\vec{x}_1}(\vec{w}_2) = \vec{w}_2 - \frac{\langle \vec{w}_2, \vec{x}_1' \rangle}{\langle \vec{x}_1, \vec{x}_1 \rangle}\vec{x}_1 = \vec{w}_2 - \frac{3}{3}x_1 = \vec{w}_2 - \vec{w}_1 = (2,0,2) = \vec{x}_2'$$

$$Z' = ((-1,1,1),(2,0,2)) \quad X' = \left(\frac{\vec{x}_1'}{\sqrt{3}}, \frac{\vec{x}_2'}{\sqrt{8}}\right)$$

(2.)

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$$\det(A_\alpha - \lambda I) = \begin{vmatrix} \alpha-\lambda & 1 & 0 \\ 1 & \alpha-\lambda & 1 \\ 0 & 1 & \alpha-\lambda \end{vmatrix} = (\alpha-\lambda)^3 - 2(\alpha-\lambda) = 0$$

$$\begin{matrix} \alpha-\lambda & 1 & 0 \\ 1 & \alpha-\lambda & 1 \end{matrix}$$

$$(\alpha-\lambda)\left[(\alpha-\lambda)^2 - 2\right] = 0$$

$\lambda_1 = \alpha$

$\lambda_2 = \alpha - \sqrt{2}$

$\lambda_3 = \alpha + \sqrt{2}$

$$\operatorname{spec}(A_\alpha) = \{\alpha - \sqrt{2}, \alpha, \alpha + \sqrt{2}\}$$

$$\begin{pmatrix} \alpha & 1 & 0 \\ 1 & \alpha & 1 \\ 0 & 1 & \alpha \end{pmatrix} \qquad \alpha + 2\sqrt{2} = x$$

$\cancel{\alpha = \sqrt{2}}$

$\cancel{\operatorname{Ker}(A_{\sqrt{2}} - \lambda I)}$

$$\cancel{\left(\begin{array}{ccc|c} \sqrt{2}-(\alpha-\sqrt{2}) & 1 & 0 & 0 \\ 1 & \sqrt{2}-(\alpha-\sqrt{2}) & 1 & 0 \\ 0 & 1 & \sqrt{2}-(\alpha-\sqrt{2}) & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} \alpha+2\sqrt{2} & 1 & 0 & 0 \\ 1 & \alpha+2\sqrt{2} & 1 & 0 \\ 0 & 1 & \alpha+2\sqrt{2} & 0 \end{array}\right)} \sim$$

$$\sim \cancel{\left(\begin{array}{ccc|c} x & 1 & 0 & 0 \\ 1 & x & 1 & 0 \\ 0 & 1 & x & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & x & 1 & 0 \\ 0 & 1-x^2 & -x & 0 \\ 0 & 1 & x & 0 \end{array}\right)} \sim$$

$\dim(\mathbb{R}^3) = 3$

ak je ~~sa~~ $\operatorname{spec}(A_\alpha)$ trojprvková, má A_α $\dim(\mathbb{R}^3) = 3$ rôzne vlastné hodnoty, tak je A_α diagonalizovateľná pre $\forall \alpha \in \mathbb{R}$, lebo ~~nikdy~~ $\alpha - \sqrt{2} \neq \alpha \neq \alpha + \sqrt{2}$ nikdy.

$\rightarrow$ A_α je diagonalizovateľná pre $\alpha \in \mathbb{R}$

$\rightarrow$ A_α má vlastnú hodnotu rovnú $\sqrt{2}$ ~~ak~~ $\cancel{\alpha \in \{2\sqrt{2}, 0\}}$

pre $\alpha = 2\sqrt{2}$ alebo $\alpha = 0$

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(3) $\|\vec{x}\|^2 + \|\vec{y}\|^2 = \frac{1}{2}\left(\|\vec{x}+\vec{y}\|^2 + \|\vec{x}-\vec{y}\|^2\right)$

$$\frac{1}{2}\left(\|\vec{x}+\vec{y}\|^2 + \|\vec{x}-\vec{y}\|^2\right) = \frac{1}{2}\left(\langle \vec{x}+\vec{y}, \vec{x}+\vec{y}\rangle + \langle \vec{x}-\vec{y}, \vec{x}-\vec{y}\rangle\right) =$$

$$\cancel{= \frac{1}{2}\langle \vec{x}+\vec{y}, \vec{x}+\vec{y}\rangle + \frac{1}{2}\langle \vec{x}-\vec{y}, \vec{x}-\vec{y}\rangle}$$

$$= \frac{1}{2}\left(\langle \vec{x}, \vec{x}+\vec{y}\rangle + \langle \vec{y}, \vec{x}+\vec{y}\rangle + \langle \vec{x}, \vec{x}-\vec{y}\rangle - \langle \vec{y}, \vec{x}-\vec{y}\rangle\right) =$$

$$= \frac{1}{2}\left(\langle \vec{x}, \vec{x}\rangle + \langle \vec{x}, \vec{y}\rangle + \langle \vec{y}, \vec{x}\rangle + \langle \vec{y}, \vec{y}\rangle + \right.$$

$$\left. + \langle \vec{x}, \vec{x}\rangle - \cancel{\langle \vec{x}, \vec{y}\rangle} - \cancel{\langle \vec{y}, \vec{x}\rangle} + \langle \vec{y}, \vec{y}\rangle\right) =$$

$$= \frac{1}{2}\left(2\langle \vec{x}, \vec{x}\rangle + 2\langle \vec{y}, \vec{y}\rangle\right) = \langle \vec{x}, \vec{x}\rangle + \langle \vec{y}, \vec{y}\rangle = \|\vec{x}\|^2 + \|\vec{y}\|^2$$

(4.)

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$P^2 = P = P^*$ ak $\vec{x} \in Ker(P)$ a $\vec{y} \in Im(P)$, tak $\vec{x} \perp \vec{y}$

$P\vec{x}_q = \lambda \vec{x}$, $\vec{x} \neq \vec{o}$

$P\vec{x} - \lambda \vec{x} = \vec{o}$

$P\vec{x} - \lambda I \vec{x} = \vec{o}$

$(P - \lambda I)\vec{x} = \vec{o}$

$\vec{x} \in ker(P - \lambda I)$

$\lambda = 0$

$\vec{x} \in ker(P)$

$\vec{Y} \in ker(P)$

$Ker(P) = det(P - \lambda I)$

ak $\vec{x} \in ker(P)$

tak $P(\vec{x}) = \vec{o}$

~~ak $\vec{y} \in Im(P)$~~

~~tak $\vec{y} = P(\vec{x})$~~

$P: V \to V$

$P^*: V \to V$

$P^2: V \to V$

ak $\vec{x} \in ker(P)$, $P\vec{x} = \vec{o}$, $\vec{o} \in ker(P)$

ak $\vec{y} \in ker(P)$, $\vec{y} = P(\vec{x}) = P\vec{x}$

$\langle \vec{x}, \vec{y} \rangle = \langle \vec{x}, P\vec{x} \rangle = \langle \vec{x}, \vec{o} \rangle = 0$

$P = P^*$

$ker(P) = ker(P^*)$

$ker(P) = (Im(P))^{\perp}$

tak $ker(P)$ je kolmý na $Im(P)$

teda $\vec{x}$ ~~je~~ $\perp \vec{y}$

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(4.)

$P^2 = P = P^*$ ak $\vec{x} \in Ker(P)$ a $\vec{y} \in Im(P)$, tak $\vec{x} \perp \vec{y}$

$P\vec{x} = \lambda \vec{x}$, $\vec{x} \neq \vec{0}$

$P\vec{x} - \lambda\vec{x} = \vec{0}$

$P\vec{x} - \lambda I \vec{x} = \vec{0}$

$(P - \lambda I)\vec{x} = \vec{0}$

$\vec{x} \in ker(P - \lambda I)$

$\lambda = 0$

$\vec{x} \in ker(P)$

$\vec{x} \in ker(P)$

$Ker(P) = det(P - \lambda I)$

ak $\vec{x} \in ker(P)$

tak $P(\vec{x}) = \vec{0}$

~~ak $\vec{y} \in Im(P)$~~

~~tak $\vec{y} = P(\vec{x})$~~

$P: V \to V$

$P^*: V \to V$

$P^2: V \to V$

ak $\vec{x} \in ker(P)$, $P\vec{x} = \vec{0}$, $\vec{0} \in ker(P)$

ak $\vec{y} \in ker(P)$, $\vec{y} = P(\vec{x}) = P\vec{x}$

$\langle \vec{x}, \vec{y} \rangle = \langle \vec{x}, P\vec{x} \rangle = \langle \vec{x}, \vec{0} \rangle = 0$

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5.

$AB = BA \quad B^2 = 0$

λ je vlastním hodnotou $\underline{A}$ právě tehdy ale λ je vlastním hodnotou $\underline{A+B}$

$A\vec{x} = \lambda\vec{x}, \ \vec{x} \neq \vec{0}$

$\cancel{BA\vec{x} = B\lambda\vec{x}}$

$(A+B)\vec{x} = \lambda\vec{x}, \ \vec{x} \neq \vec{0}$

$A\vec{x} + B\vec{x} = \lambda\vec{x}$

$\lambda\vec{x} + B\vec{x} = \lambda\vec{x}$

$B\vec{x} = \lambda\vec{x} - \lambda\vec{x} = \vec{0}$

$B\vec{x} = \vec{0}$

$B^2\vec{x} = \vec{0}$

$B^2 = 0$

$AB = BA$

$A\vec{x} = \lambda\vec{x}, \ \vec{x} \neq \vec{0}$

$BA\vec{x} = B\lambda\vec{x} = \lambda B\vec{x} = \lambda\vec{0} = \vec{0}$

$B\vec{x} = \vec{0}$

$AB\vec{x} = A\vec{0} = \vec{0}$

$BA\vec{x} = AB\vec{x} = \vec{0}$

$AB = BA$

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