F. Biral - 2023

Vehicle dynamics -racing cars

Longitudinal performance

UNIVERSITY
OF TRENTO

Summing up

Longitudinal dynamics

Equations of longitudinal dynamics

The equations that governs the longitudinal dynamics are:

$$m\,a_x = F_{x_r} + F_{x_f} - F_{A_x}$$

$$F_{z_r} = m\,g\frac{L_f}{L} + F_{A_{z_r}} + m\,a_x\frac{h_G}{L}$$

$$F_{z_f} = m\,g\frac{L_r}{L} + F_{A_{z_f}} - m\,a_x\frac{h_G}{L}$$

Assume that $\mu_x^{\max}$ does not depend on load (we can extend this simplification)
The maximum conditions of adherence for longitudinal forces are:

$$F_{x_r} \le \mu_x^{\max} F_{z_r} = \mu_x^{\max}\left(m\,g\frac{L_f}{L} + m\,a_x\frac{h_g}{L} + F_{A_{z_r}}\right)$$

$$F_{x_f} \le \mu_x^{\max} F_{z_f} = \mu_x^{\max}\left(m\,g\frac{L_r}{L} - m\,a_x\frac{h_g}{L} + F_{A_{z_f}}\right)$$

Maximum traction

Rear driven vehicle

Maximum traction conditions

Maximum acceleration is a combination between what is potentially available and what can be exchanged with road via tyres.

There can be three possible limiting conditions

- **Front wheel lift**: this is a case relevant for motorcycle
- **Maximum engine torque**:
  - At low speed usually the limiting factor is the peace force of the tyre unless the engine torque is not high
  - At high speed the limiting factor is the engine, since power decreases with speed.
- **Maximum tyre peak force**: this is usually the limiting factor at low medium speed

1) Maximum traction: front wheel load >0

Constraint on vertical load to be positive:

$$F_{z_f} = m\,g\frac{L_r}{L} - m\,a_x\frac{h_g}{L} + F_{A_{z_f}} \geq 0$$

Wheel lift condition is for $F_{z_f} = 0$ which let us find the normalised acceleration that satisfies this condition:

$$m\,g\frac{L_r}{L} - m\,a_x\frac{h_g}{L} + F_{A_{z_f}} = 0 \Rightarrow \frac{a_x}{g} = \frac{L_r}{h_G} + \frac{L}{mgh_G}F_{A_{z_f}}$$

To avoid front wheel lift we could increase L_r and decrease h_G. Also downforce helps to keep front wheel in contact with road surface.

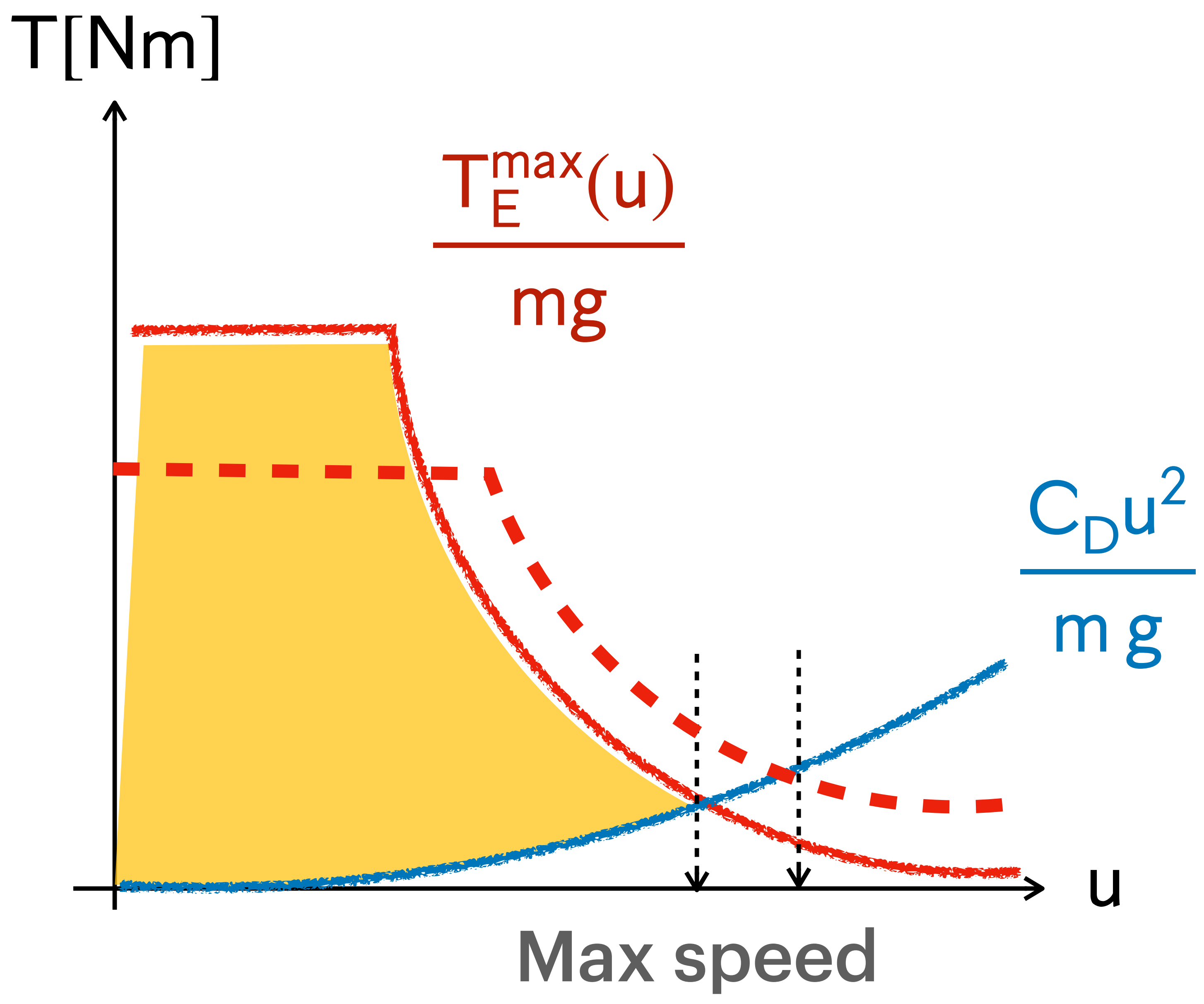
2) maximum available normalised acceleration

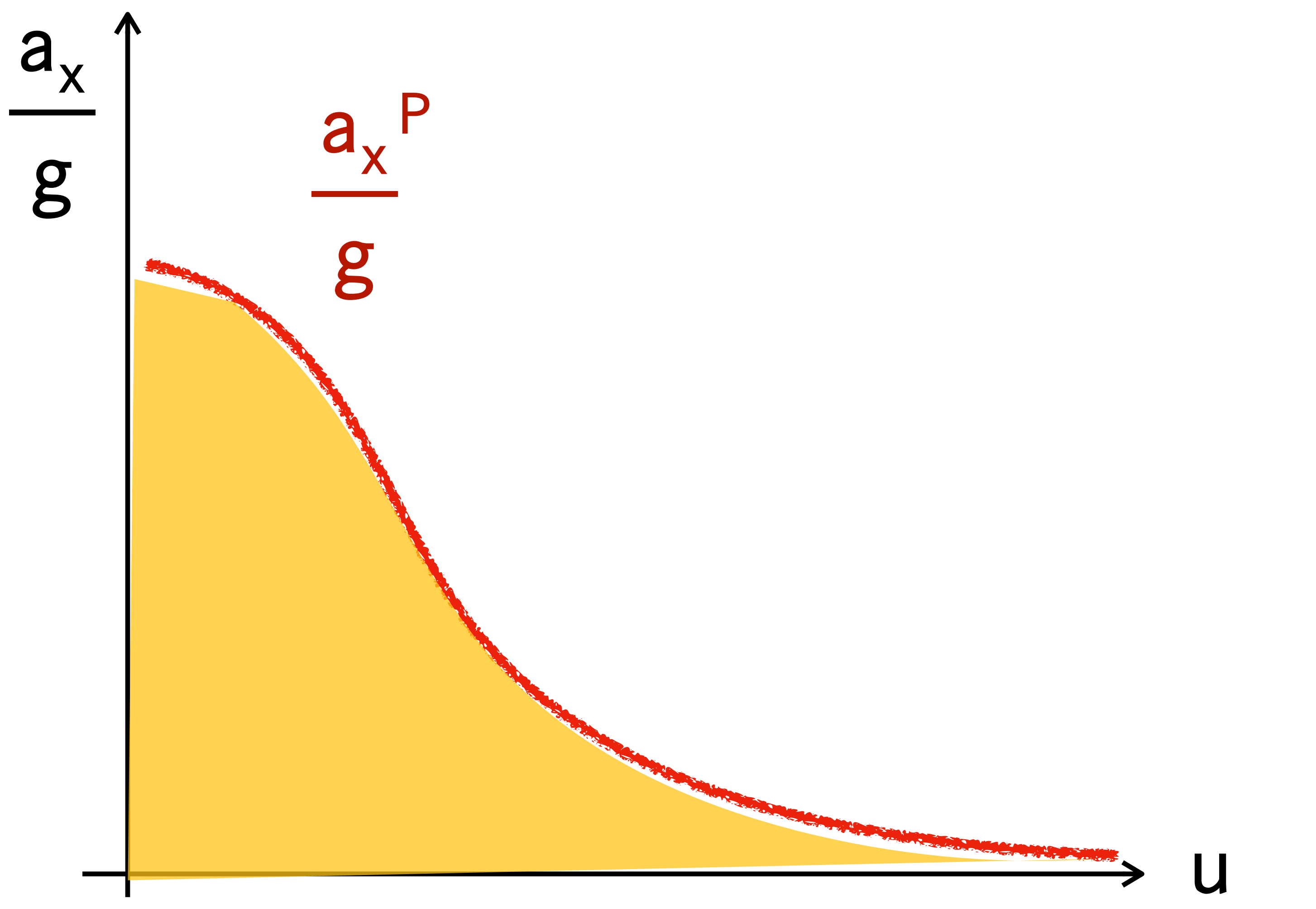
Assuming the tyres can deliver the maximum traction torque $T_E^{max}(u)$ generated by the engine the **maximum potential acceleration** is:

$$\frac{a_x{}^P}{g} = \frac{1}{m\,g}\left(\frac{\tau_g}{r} T_E^{max}(u) - C_D u^2\right)$$

Given the engine envelope torque the maximum potential acceleration depends on:

- **Vehicle mass**: lower mass greater accelerations
- **Aerodynamic drag**: minimise drag increase acceleration also at high speed
- **Gear ratios** can extend acceleration and maximum speed





Maximum traction: case Rear Driven (RD) vehicle

Maximum longitudinal acceleration with traction at front axle and $F_{x_r} = \mu_{x^r_{\max}} F_{z_r}$ and $F_{x_f} = 0$

$$\frac{a_x}{g} = \frac{1}{1 - \frac{h_G}{L}\mu_{x^r_{\max}}} \left(\mu_{x^r_{\max}} \left(\frac{L_f}{L} + \frac{(1-\epsilon_A) C_L}{m\,g} u^2 \right) - \frac{C_D}{m\,g} u^2 \right)$$

If we compare with 4WD:

$$\frac{a_x}{g} = \mu_{x_{\max}} \left(1 + \frac{C_L}{m\,g} u^2 \right) - \frac{C_D}{m\,g} C_D u^2$$

The term $1 - \frac{h_G}{L}\mu_{x^r_{\max}}$ is a reduction factor which is greater than 1 and $\frac{L_f}{L} < 1$.
Therefore normalised acceleration of RWD is less than 4WD but greater than FWD.

NOTE: with RWD with higher CoM we get increase of normalised acceleration.
Also moving CoM rear, L_f increases.

Maximum braking

Optimal braking vs rear only or front only

Assumption

Assume that μ_x^{max} does not depend on load F_z.

$$\mu_{x_r}^{max} = \mu_{x_f}^{max} = \mu_x^{max}$$

Maximum braking torques >> maximum adherence equivalent torque (i.e. $F_x^{max} r_w$ with r_w wheel radius)

Optimum braking: both axles reach the same maximum adherence conditions at the same time

$$F_{x_r}^{max} = -\mu_x^{max} F_{z_r} = -\mu_x^{max}\left(m\,g\frac{L_f}{L} + m\,a_x\frac{h_g}{L} + F_{A_{z_r}}\right)$$

$$F_{x_f}^{max} = -\mu_x^{max} F_{z_f} = -\mu_x^{max}\left(m\,g\frac{L_r}{L} - m\,a_x\frac{h_g}{L} + F_{A_{z_f}}\right)$$

Maximum optimal braking

Both tyre are working at the peak force and peak is the same between front and rear:

$$\frac{a_x}{g} = -\mu_{x_{max}}\left(1 + \frac{C_L u^2}{mg}\right) - \frac{C_D u^2}{mg}$$

- $-\dfrac{C_D u^2}{mg}$ drag deceleration is not affected by adherence $\mu_{x_{max}}$ but it is speed dependent
- $-\mu_{x_{max}}\dfrac{C_L u^2}{mg}$ downforce is speed and adherence dependent
- At low friction $\mu_{x_{max}} \to 0$, $-\dfrac{C_D u^2}{mg}$ is more important (e.g. parachute)
- Effect of drag force reduces with square of speed.
- Wheel lift conditions: $F_{z_r} = 0 \Rightarrow a_x = -\left(g\dfrac{L_f}{h_G} + F_{A_{z_r}}\dfrac{L}{mh_G}\right)$

Maximum braking: only with front axle

Only the front tyre generates braking force:

$$F_{x_r} = 0$$

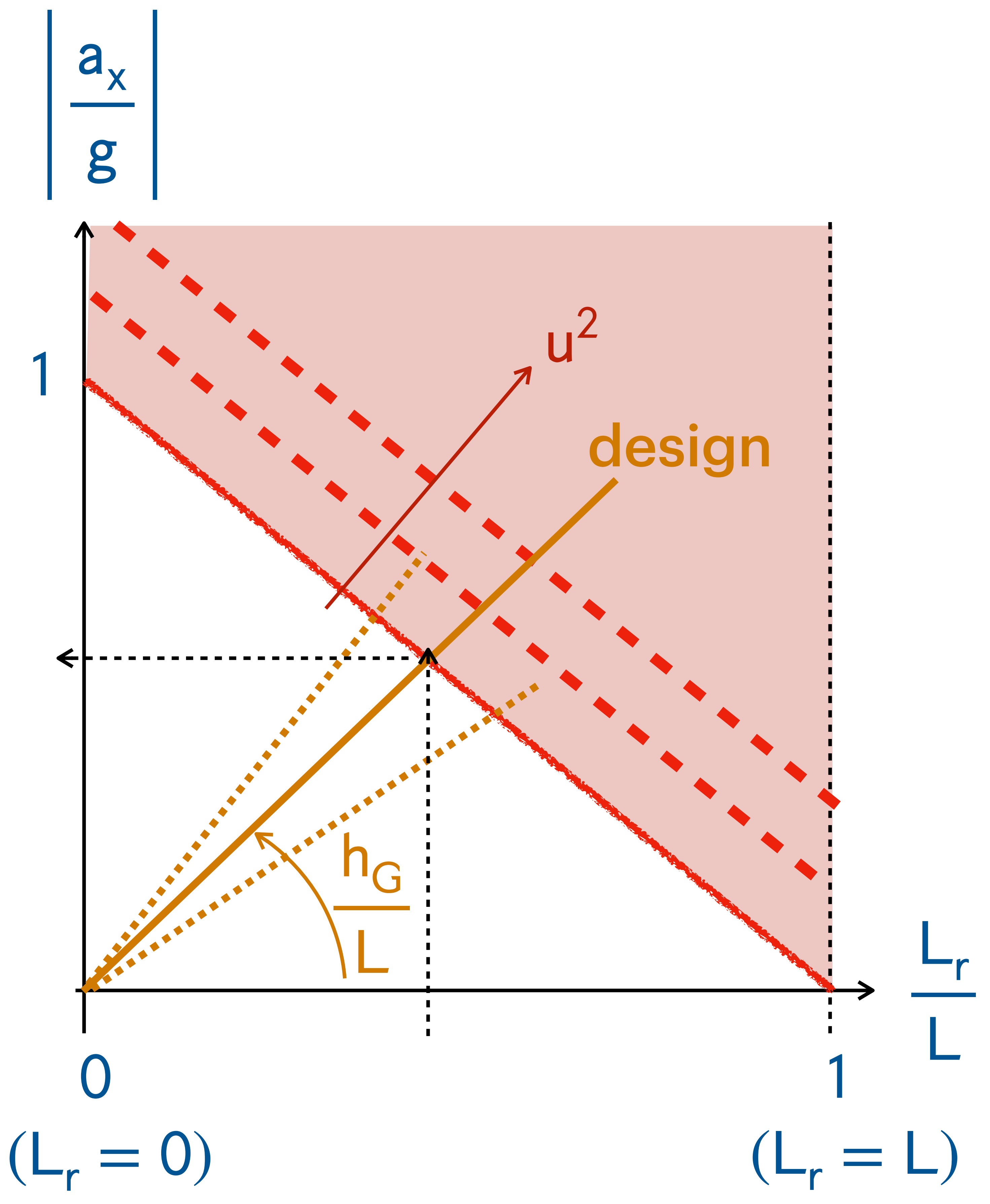
$$F_{x_f}^{\max} = -\mu_x^{\max} F_{z_f} = -\mu_x^{\max}\left(m\,g\frac{L_r}{L} - m\,a_x\frac{h_g}{L} + \hat{C}_L\epsilon_A u^2\right)$$

Solving for acceleration gives the **normalised maximum deceleration**:

$$\frac{a_x}{g} = -\frac{\mu_x^{\max}\frac{L_r}{L}}{1-\frac{h_G}{L}\mu_x^{\max}} - \frac{\mu_x^{\max}\hat{C}_L\epsilon_A + \hat{C}_D}{1-\frac{h_G}{L}\mu_x^{\max}}u^2$$

Lift off condition (rear wheel in braking):

$$\frac{a_x}{g} = \frac{L_f}{L} = 1-\frac{L_r}{L}$$



Maximum braking: only with rear axle

Only the rear tyre generates braking force:

$$F_{x_r}^{\max} = -\mu_x^{\max} F_{z_f} = -\mu_x^{\max}\left(m\,g\frac{L_r}{L} - m\,a_x\frac{h_G}{L} + \hat{C}_L(1-\epsilon_A)u^2\right)$$

$$F_{x_f} = 0$$

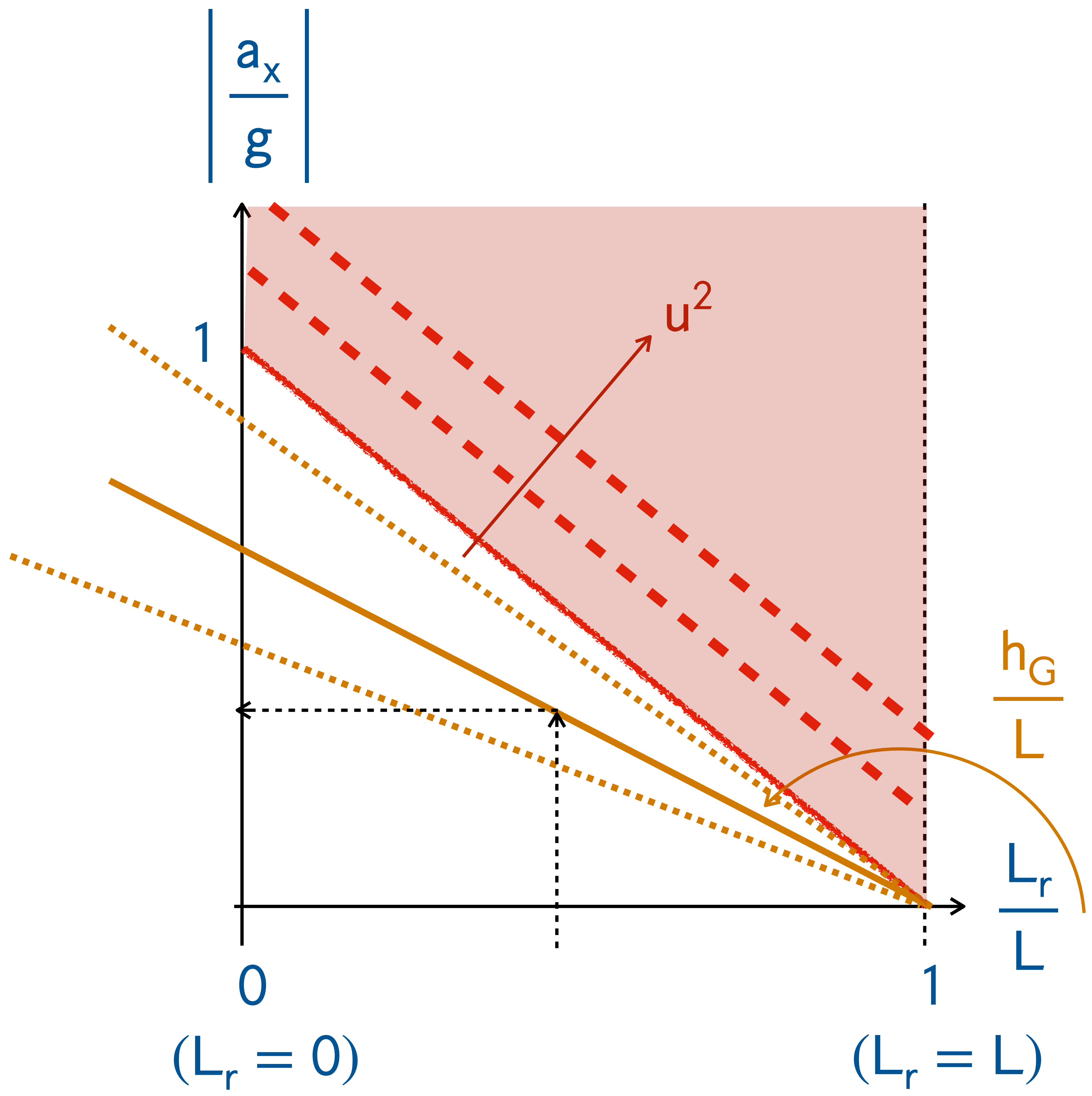
Solving for acceleration gives the **normalised maximum deceleration**:

$$\frac{a_x}{g} = -\frac{\mu_x^{\max}\frac{L_f}{L}}{1+\frac{h_G}{L}\mu_x^{\max}} - \frac{\mu_x^{\max}\hat{C}_L(1-\epsilon_A)+\hat{C}_D}{1+\frac{h_G}{L}\mu_x^{\max}}u^2$$

Or $$\frac{a_x}{g} = -\frac{\mu_x^{\max}\left(1-\frac{L_r}{L}\right)}{1+\frac{h_G}{L}\mu_x^{\max}} - \frac{\mu_x^{\max}\hat{C}_L(1-\epsilon_A)+\hat{C}_D}{1+\frac{h_G}{L}\mu_x^{\max}}u^2$$

Lift off condition (rear wheel in braking):

$$\frac{a_x}{g} = \frac{L_f}{L} = 1-\frac{L_r}{L}$$



Non optimal distribution

Analysis with no-aero

Normalised longitudinal axle forces

We assume we increase the tyre force as a function of a given brake pressure repartition. This means that the longitudinal force generated by the tyre is proportional to the brake repartition.

Assumptions to simplify equations:

- Same peak adherence: $\mu_{x_r}^{max} = \mu_x^{max} = \mu_{x_f}^{max}$
- Neglect aero
- Longitudinal forces are in the range: $F_{x_r} \in [-\mu_x^{max} F_{z_r}, \mu_x^{max} F_{z_r}], \quad F_{x_f} \in [-\mu_x^{max} F_{z_f}, \mu_x^{max} F_{z_f}]$
- Normalised longitudinal forces are in the range: $\dfrac{F_{x_r}}{F_{z_r}} \in [-\mu_x^{max}, \mu_x^{max}], \quad \dfrac{F_{x_f}}{F_{z_f}} \in [-\mu_x^{max}, \mu_x^{max}]$
- With normalised the longitudinal forces we can scale the maximum peak adherence with tyre engagement:

  $\dfrac{F_{x_r}}{F_{z_r}} = \mu_r \mu_x^{max}, \quad \dfrac{F_{x_f}}{F_{z_f}} = \mu_f \mu_x^{max}$ with $\mu_{x_r} \in [-1, 1]$ and $\mu_f \in [-1, 1]$

Normalised acc/deceleration

From force definition (neglecting the downforce) with μ_r and μ_f the longitudinal actual adherence (or **tyre engagement**) we get:

$$\frac{F_{x_r}}{mg} = \mu_r\left(\frac{L_f}{L} + \frac{a_x}{g}\frac{h_g}{L}\right), \quad \frac{F_{x_f}}{mg} = \mu_f\left(\frac{L_r}{L} - \frac{a_x}{g}\frac{h_g}{L}\right)$$

substituting in the longitudinal dynamics we get the normalised acceleration/deceleration:

$$\frac{a_x}{g} = \frac{F_{x_r}}{mg} + \frac{F_{x_f}}{mg} = +\mu_r\frac{L_f}{L} + \mu_f\frac{L_r}{L} + \frac{a_x}{g}\frac{h_g}{L}\left(\mu_r - \mu_f\right) \Rightarrow \frac{a_x}{g} = \frac{\mu_r\frac{L_f}{L} + \mu_f\frac{L_r}{L}}{1 + \frac{h_g}{L}\left(\mu_f - \mu_r\right)}$$

The normalised deceleration depends on axle longitudinal engagement (i.e. μ_r and μ_f).

Optimal brake/traction control

If we rise the axle engagement μ_r and μ_f of the same amount we get the optimal condition reaching the peak at the same instant. On the contrary, we have non optimal braking/traction

Thus, in order to achieve the optimal braking condition:

$$\begin{cases} \mu_r = K_{BT} \\ \mu_f = K_{BT} \end{cases} \text{with } K_{BT} \in [-1, 1]$$

Where $K_{BT} \in [-1, 1]$ is the optimal brake/traction control or axle engagement which (substituting in normalised deceleration) provides the optimal normalised optimal deceleration/acceleration

$$\Rightarrow \frac{a_x}{g} = \mu_x^{max} K_{BT} \quad \Rightarrow \frac{F_{x_r}}{mg} + \frac{F_{x_f}}{mg} = \frac{a_x}{g} = \mu_x^{max} K_{BT}$$

Brake/traction optimal control

We can rewrite the longitudinal forces as a function of the optimal brake/traction control K_{BT}:

$$\frac{F_{x_r}}{mg} = K_{BT}\mu_x^{max}\left(\frac{L_f}{L} + \underbrace{\frac{a_x}{g}}_{=K_{BT}\mu_x^{max}}\frac{h_g}{L}\right), \quad \frac{F_{x_f}}{mg} = K_{BT}\mu_x^{max}\left(\frac{L_r}{L} - \underbrace{\frac{a_x}{g}}_{=K_{BT}\mu_x^{max}}\frac{h_g}{L}\right)$$

Which says that the normalised longitudinal forces are a polynomial function of the optimal brake/traction control K_{BT}:

$$\frac{F_{x_r}}{mg} = K_{BT}\mu_x^{max}\frac{L_f}{L} + \left(K_{BT}\mu_x\right)^2\frac{h_g}{L}, \quad \frac{F_{x_f}}{mg} = K_{BT}\mu_x^{max}\frac{L_r}{L} - \left(K_{BT}\mu_x\right)^2\frac{h_g}{L}$$

Normalised longitudinal forces function of K_{BT}

The normalised longitudinal forces are a polynomial function of the optimal brake/traction control K_{BT}:

$$\frac{F_{x_r}}{mg} = K_{BT}\mu_x^{max}\frac{L_f}{L} + \left(K_{BT}\mu_x^{max}\right)^2\frac{h_g}{L}, \quad \frac{F_{x_f}}{mg} = K_{BT}\mu_x^{max}\frac{L_r}{L} - \left(K_{BT}\mu_x^{max}\right)^2\frac{h_g}{L}$$

With the following conditions:

- Rear wheel lift: $F_{z_r} = 0 \Rightarrow \frac{a_x}{g} = \mu_x^{max}K_{BT} = \frac{L_f}{h_g}$
- Front wheel lift: $F_{z_f} = 0 \Rightarrow \frac{a_x}{g} = \mu_x^{max}K_{BT} = \frac{L_r}{h_g}$
- For $\frac{F_{x_r}}{mg} = 0 \Rightarrow K_{BT}\mu_x^{max} = -\frac{L_f}{h_g} \Rightarrow \frac{F_{x_f}}{mg} = -\left(\frac{L_f}{h_g}\right)$
- For $\frac{F_{x_f}}{mg} = 0 \Rightarrow K_{BT}\mu_x^{max} = +\frac{L_r}{h_g} \Rightarrow \frac{F_{x_r}}{mg} = \left(\frac{L_r}{h_g}\right)$

Braking/traction ratio

Defining the **braking/traction ratio** as:

$$\epsilon_{BT} = \frac{\frac{F_{x_f}}{mg}}{\frac{F_{x_f}}{mg} + \frac{F_{x_r}}{mg}} = \frac{K_{BT}\mu_x^{max}\frac{L_f}{L} + (K_{BT}\mu_x)^2\frac{h_g}{L}}{K_{BT}\mu_x^{max}}$$

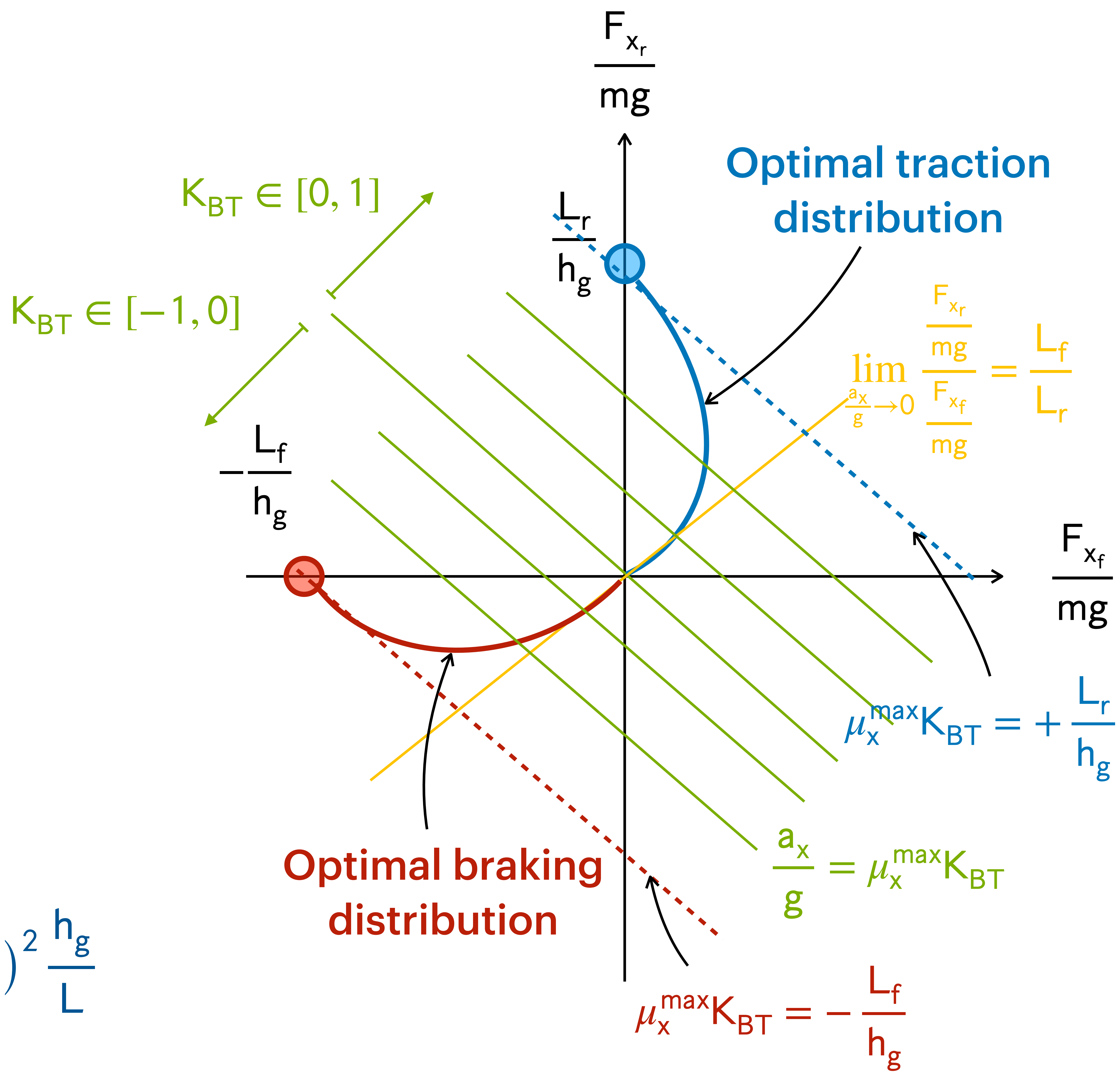
We get: $\epsilon_{BT} = \frac{L_f}{L} - K_{BT}\mu_x^{max}\frac{h_g}{L}$

And optimum braking/traction ratio at rear:

$$1 - \epsilon_{BT} = \frac{L_r}{L} + K_{BT}\mu_x^{max}\frac{h_g}{L}$$

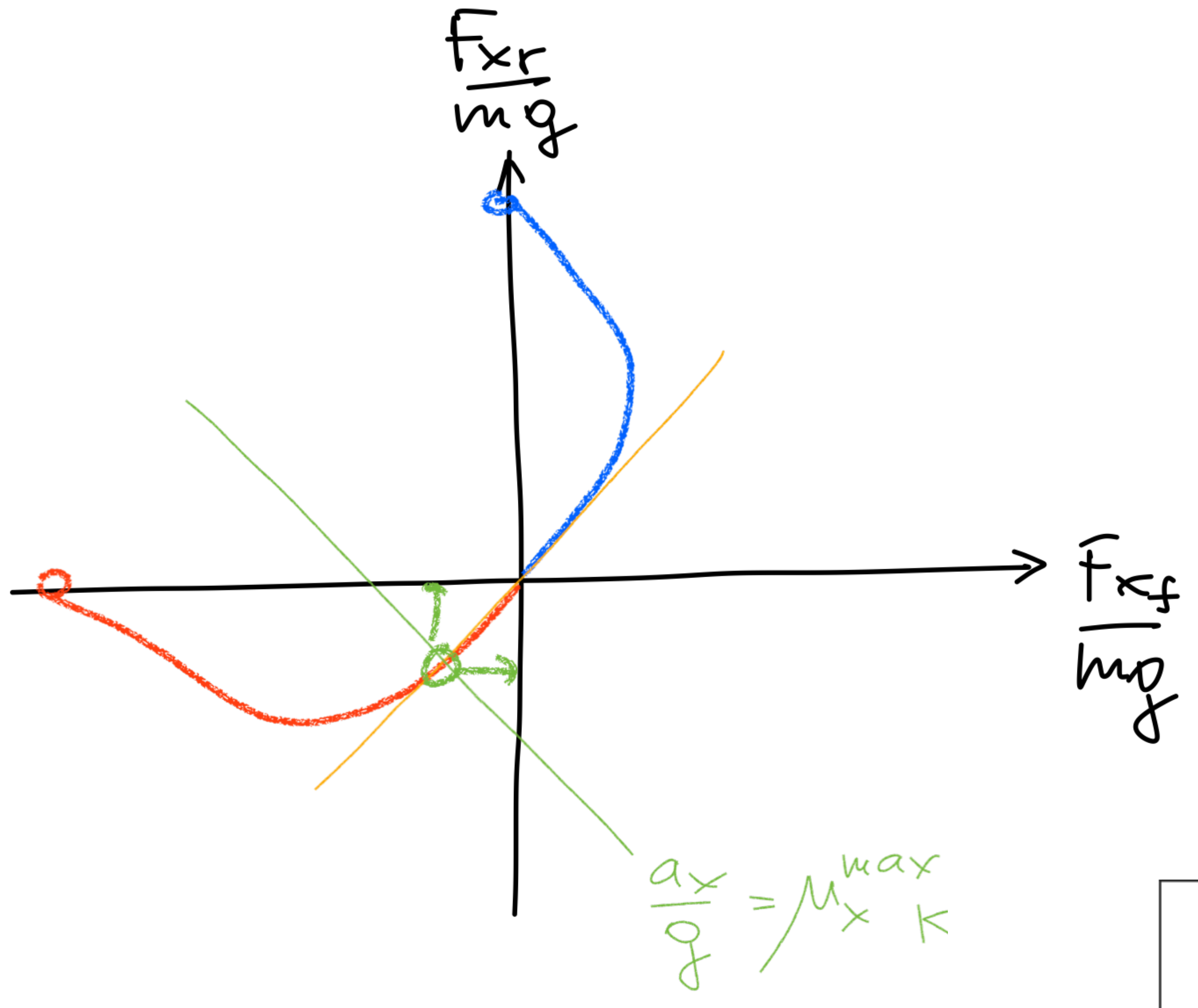
With:

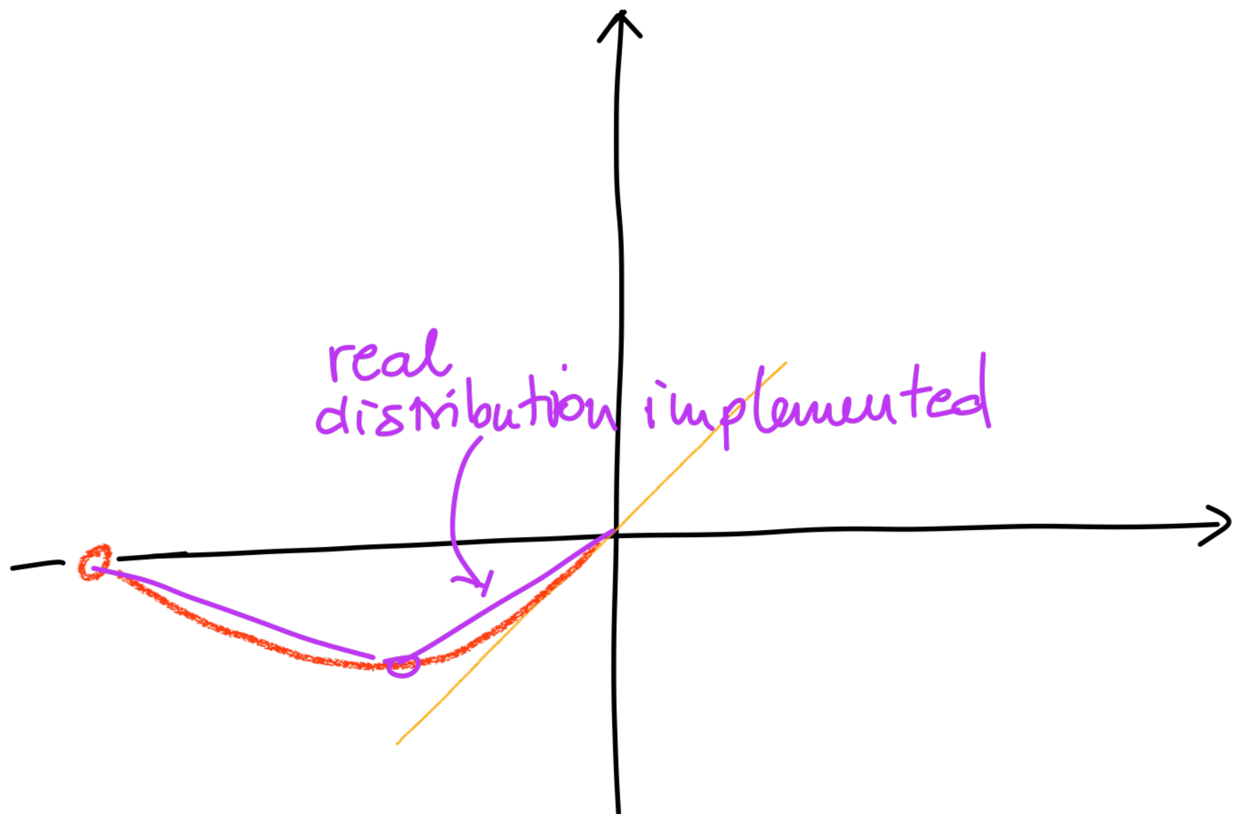
$$\frac{F_{x_r}}{mg} = K_{BT}\mu_x^{max}\frac{L_f}{L} + \left(K_{BT}\mu_x^{max}\right)^2\frac{h_g}{L}, \quad \frac{F_{x_f}}{mg} = K_{BT}\mu_x^{max}\frac{L_r}{L} - \left(K_{BT}\mu_x^{max}\right)^2\frac{h_g}{L}$$



Exact and approximated distribution

In real case the ideal curve is approximated initially with a constant braking ratio and then with variable (or piecewise linear)





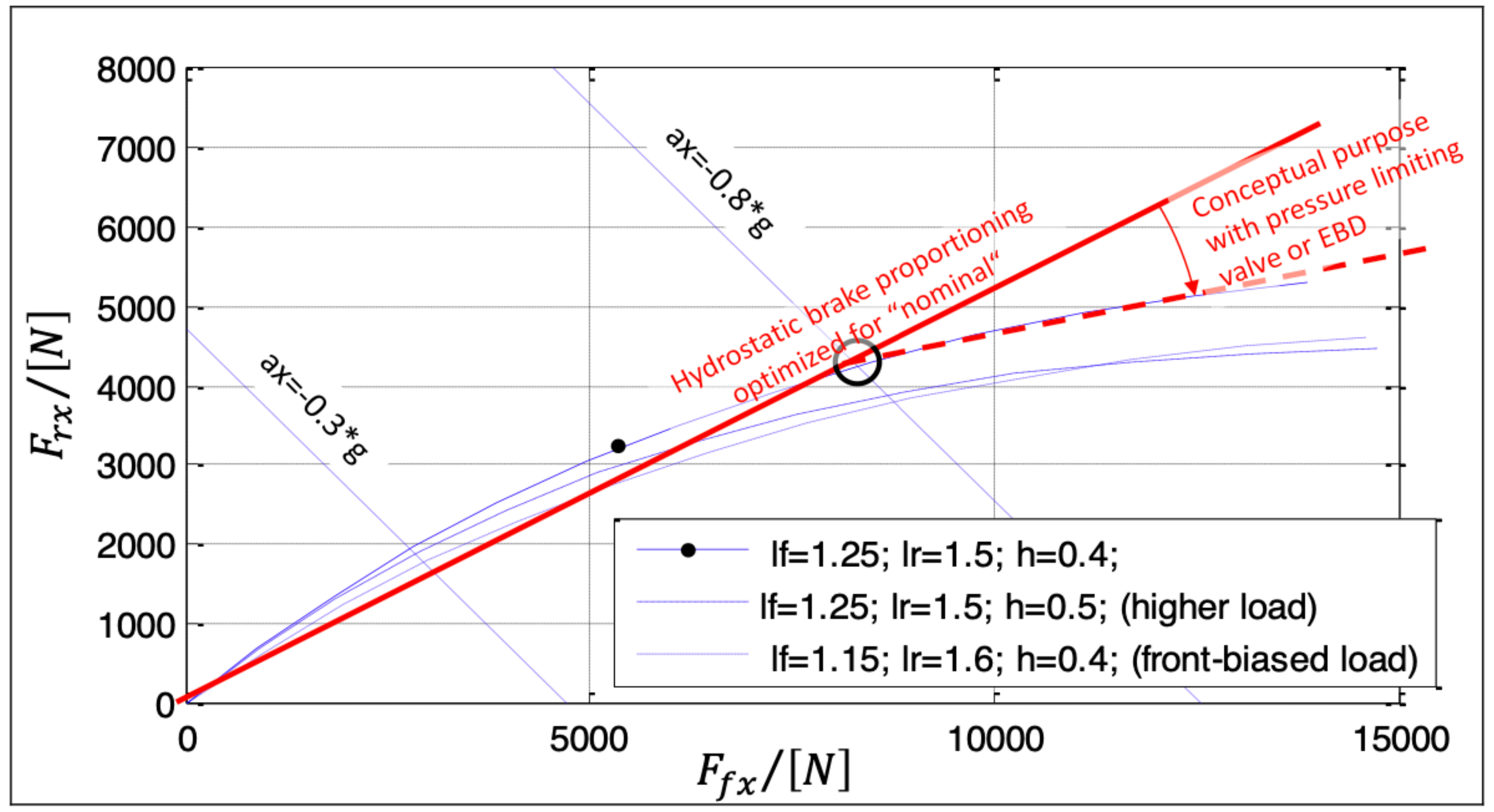


Figure 3-40: Brake Proportioning diagram. The curved curves mark optimal distribution for some variation in position of centre of gravity.

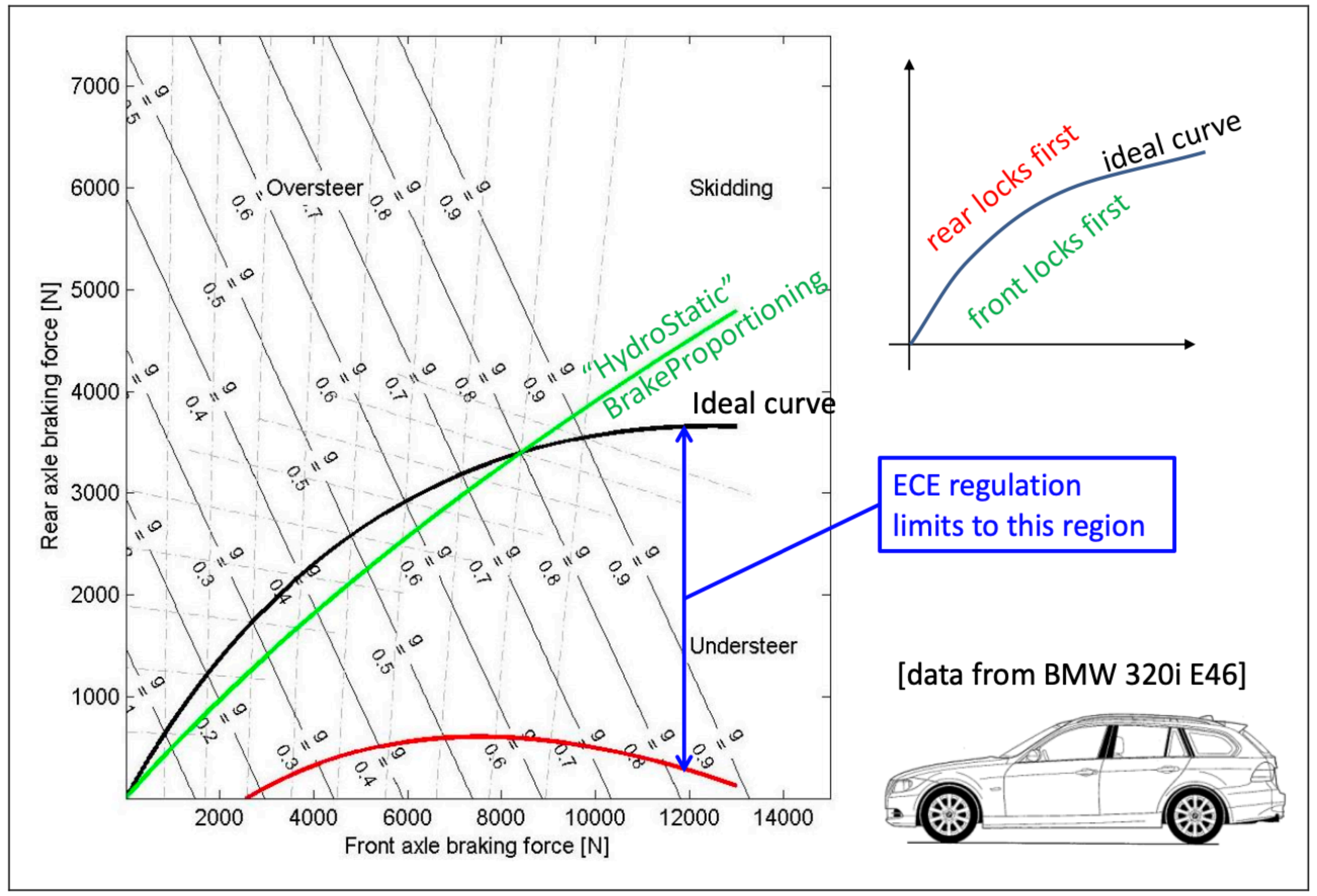


Figure 3-41: Brake Proportioning. From (Boerboom, 2012). If looking carefully, the "HydroStatic" curve is weakly degressive, which is thanks to brake pads with pressure dependent friction coefficient.