

# Lesson 1: Integer Sequences—Should You Believe in Patterns?

## Classwork

### Opening Exercise

Mrs. Rosenblatt gave her students what she thought was a very simple task:

What is the next number in the sequence 2, 4, 6, 8, ...?

*Cody: I am thinking of a plus 2 pattern, so it continues 10, 12, 14, 16, ....*

*Ali: I am thinking of a repeating pattern, so it continues 2, 4, 6, 8, 2, 4, 6, 8, ....*

*Suri: I am thinking of the units digits in the multiples of two, so it continues 2, 4, 6, 8, 0, 2, 4, 6, 8, ....*

- a. Are each of these valid responses?
- b. What is the hundredth number in the sequence in Cody's scenario? Ali's? Suri's?
- c. What is an expression in terms of  $n$  for the  $n^{\text{th}}$  number in the sequence in Cody's scenario?

### Example 1

Jerry has thought of a pattern that shows powers of two. Here are the first six numbers of Jerry's sequence:

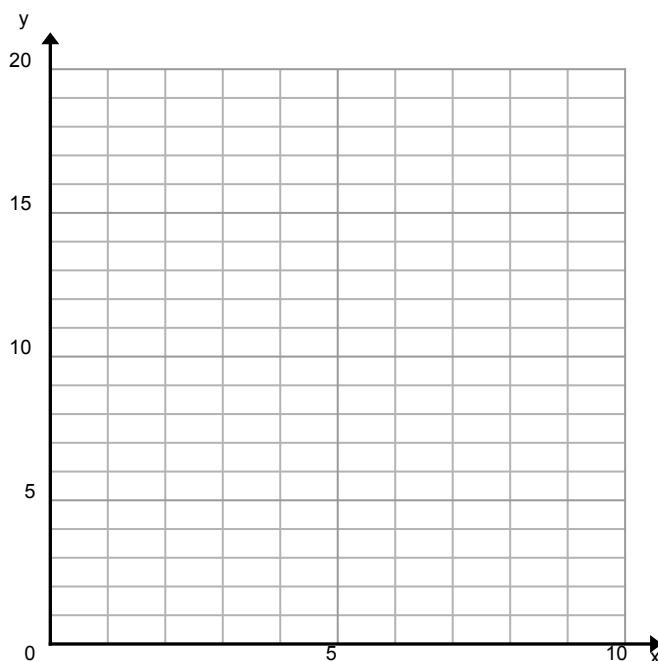
1, 2, 4, 8, 16, 32, ....

Write an expression for the  $n^{\text{th}}$  number of Jerry's sequence.

**Example 2**

Consider the sequence that follows a plus 3 pattern: 4, 7, 10, 13, 16, ....

- a. Write a formula for the sequence using both the  $a_n$  notation and the  $f(n)$  notation.
- b. Does the formula  $f(n) = 3(n - 1) + 4$  generate the same sequence? Why might some people prefer this formula?
- c. Graph the terms of the sequence as ordered pairs  $(n, f(n))$  on the coordinate plane. What do you notice about the graph?

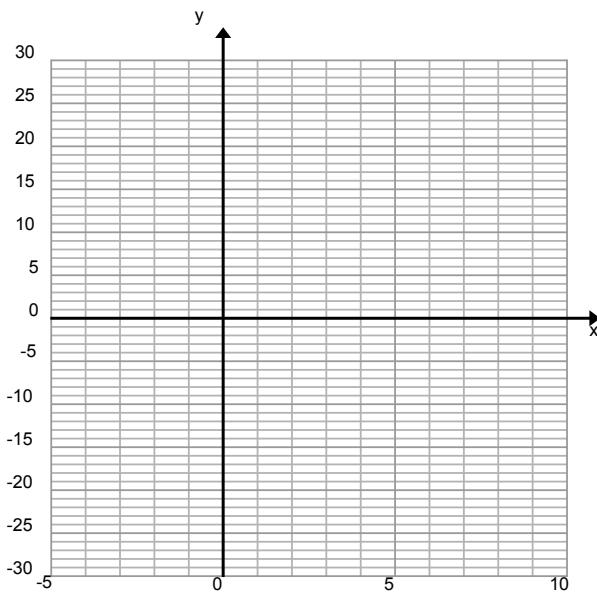


**Exercises**

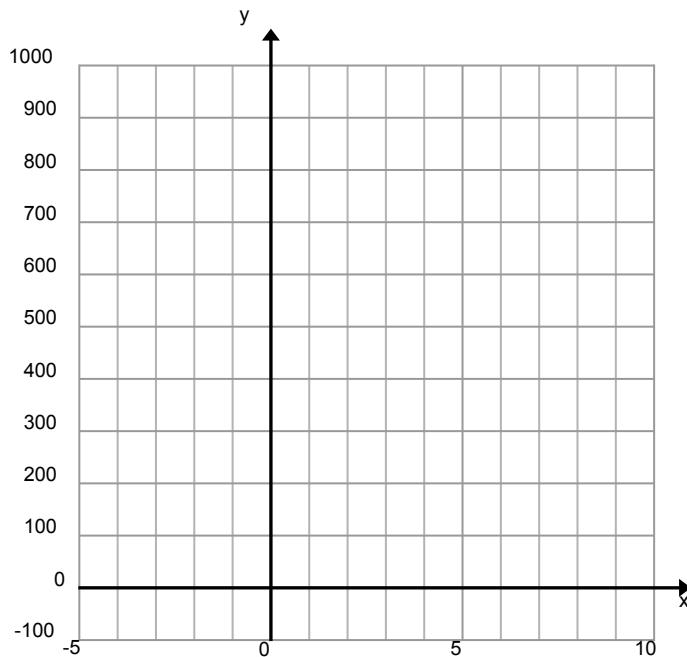
1. Refer back to the sequence from the Opening Exercise. When Mrs. Rosenblatt was asked for the next number in the sequence 2, 4, 6, 8, ..., she said "17." The class responded, "17?"

Yes, using the formula,  $f(n) = \frac{7}{24}(n-1)^4 - \frac{7}{4}(n-1)^3 + \frac{77}{24}(n-1)^2 + \frac{1}{4}(n-1) + 2$ .

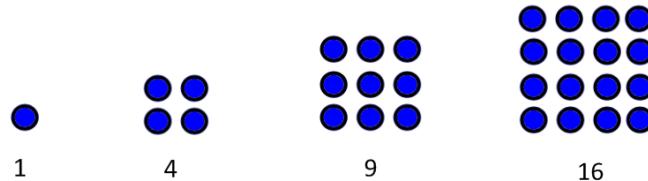
- Does her formula actually produce the numbers 2, 4, 6, and 8?
  - What is the 100<sup>th</sup> term in Mrs. Rosenblatt's sequence?
2. Consider a sequence that follows a minus 5 pattern: 30, 25, 20, 15, ....
- Write a formula for the  $n^{\text{th}}$  term of the sequence. Be sure to specify what value of  $n$  your formula starts with.
  - Using the formula, find the 20<sup>th</sup> term of the sequence.
  - Graph the terms of the sequence as ordered pairs  $(n, f(n))$  on a coordinate plane.



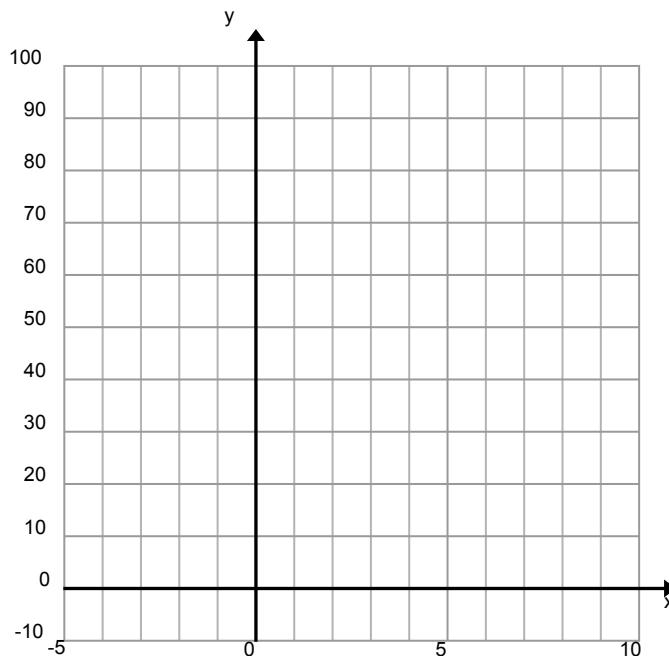
3. Consider a sequence that follows a times 5 pattern: 1, 5, 25, 125, ....
- Write a formula for the  $n^{\text{th}}$  term of the sequence. Be sure to specify what value of  $n$  your formula starts with.
  - Using the formula, find the  $10^{\text{th}}$  term of the sequence.
- c. Graph the terms of the sequence as ordered pairs  $(n, f(n))$  on a coordinate plane.



4. Consider the sequence formed by the square numbers:



- a. Write a formula for the  $n^{\text{th}}$  term of the sequence. Be sure to specify what value of  $n$  your formula starts with.
- b. Using the formula, find the 50<sup>th</sup> term of the sequence.
- c. Graph the terms of the sequence as ordered pairs  $(n, f(n))$  on a coordinate plane.



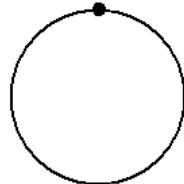
5. A standard letter-sized piece of paper has a length and width of 8.5 inches by 11 inches.
- Find the area of one piece of paper.
  - If the paper were folded completely in half, what would be the area of the resulting rectangle?
  - Write a formula for a sequence to determine the area of the paper after  $n$  folds.
  - What would the area be after 7 folds?

**Lesson Summary**

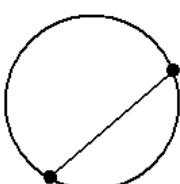
Think of a sequence as an ordered list of elements. Give an explicit formula to define the pattern of the sequence. Unless specified otherwise, find the first term by substituting 1 into the formula.

**Problem Set**

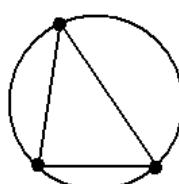
- Consider a sequence generated by the formula  $f(n) = 6n - 4$  starting with  $n = 1$ . Generate the terms  $f(1)$ ,  $f(2)$ ,  $f(3)$ ,  $f(4)$ , and  $f(5)$ .
- Consider a sequence given by the formula  $f(n) = \frac{1}{3^{n-1}}$  starting with  $n = 1$ . Generate the first 5 terms of the sequence.
- Consider a sequence given by the formula  $f(n) = (-1)^n \times 3$  starting with  $n = 1$ . Generate the first 5 terms of the sequence.
- Here is the classic puzzle that shows that patterns need not hold true. What are the numbers counting?



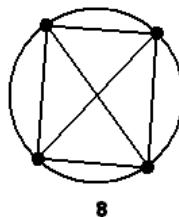
1



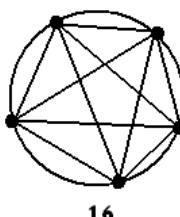
2



4



8



16



??

- Based on the sequence of numbers, predict the next number.
- Write a formula based on the perceived pattern.
- Find the next number in the sequence by actually counting.
- Based on your answer from part (c), is your model from part (b) effective for this puzzle?

For each of the sequences in Problems 5–8:

- a. Write a formula for the  $n^{\text{th}}$  term of the sequence. Be sure to specify what value of  $n$  your formula starts with.
  - b. Using the formula, find the  $15^{\text{th}}$  term of the sequence.
  - c. Graph the terms of the sequence as ordered pairs  $(n, f(n))$  on a coordinate plane.
5. The sequence follows a plus 2 pattern: 3, 5, 7, 9, ....
6. The sequence follows a times 4 pattern: 1, 4, 16, 64, ....
7. The sequence follows a times  $-1$  pattern: 6,  $-6$ , 6,  $-6$ , ....
8. The sequence follows a minus 3 pattern: 12, 9, 6, 3, ....

## Lesson 2: Recursive Formulas for Sequences

### Classwork

#### Example 1

Consider Akelia's sequence 5, 8, 11, 14, 17, ....

- If you believed in patterns, what might you say is the next number in the sequence?
- Write a formula for Akelia's sequence.
- Explain how each part of the formula relates to the sequence.
- Explain Johnny's formula.

#### Exercises 1–2

- Akelia, in a playful mood, asked Johnny: "What would happen if we change the '+' sign in your formula to a '-' sign? To a '×' sign? To a '÷' sign?"
  - What sequence does  $A(n + 1) = A(n) - 3$  for  $n \geq 1$  and  $A(1) = 5$  generate?
  - What sequence does  $A(n + 1) = A(n) \cdot 3$  for  $n \geq 1$  and  $A(1) = 5$  generate?
  - What sequence does  $A(n + 1) = A(n) \div 3$  for  $n \geq 1$  and  $A(1) = 5$  generate?

2. Ben made up a recursive formula and used it to generate a sequence. He used  $B(n)$  to stand for the  $n^{\text{th}}$  term of his recursive sequence.
- What does  $B(3)$  mean?
  - What does  $B(m)$  mean?
- If  $B(n + 1) = 33$  and  $B(n) = 28$ , write a possible recursive formula involving  $B(n + 1)$  and  $B(n)$  that would generate 28 and 33 in the sequence.
- What does  $2B(7) + 6$  mean?
  - What does  $B(n) + B(m)$  mean?
- Would it necessarily be the same as  $B(n + m)$ ?
- What does  $B(17) - B(16)$  mean?

**Example 2**

Consider a sequence given by the formula  $a_n = a_{n-1} - 5$ , where  $a_1 = 12$  and  $n \geq 2$ .

- a. List the first five terms of the sequence.

- b. Write an explicit formula.

- c. Find  $a_6$  and  $a_{100}$  of the sequence.

**Exercises 3–6**

3. One of the most famous sequences is the Fibonacci sequence:

1, 1, 2, 3, 5, 8, 13, 21, 34, ....

$f(n+1) = f(n) + f(n-1)$ , where  $f(1) = 1$ ,  $f(2) = 1$ , and  $n \geq 2$ .

How is each term of the sequence generated?

4. Each sequence below gives an explicit formula. Write the first five terms of each sequence. Then, write a recursive formula for the sequence.

- a.  $a_n = 2n + 10$  for  $n \geq 1$

b.  $a_n = \left(\frac{1}{2}\right)^{n-1}$  for  $n \geq 1$

5. For each sequence, write either an explicit or a recursive formula.

a.  $1, -1, 1, -1, 1, -1, \dots$

b.  $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$

6. Lou opens a bank account. The deal he makes with his mother is that if he doubles the amount that was in the account at the beginning of each month by the end of the month, she will add an additional \$5 to the account at the end of the month.
- Let  $A(n)$  represent the amount in the account at the beginning of the  $n^{\text{th}}$  month. Assume that he does, in fact, double the amount every month. Write a recursive formula for the amount of money in his account at the beginning of the  $(n + 1)^{\text{th}}$  month.
  - What is the least amount he could start with in order to have \$300 by the beginning of the third month?

**Lesson Summary**

**RECURSIVE SEQUENCE:** An example of a *recursive sequence* is a sequence that (1) is defined by specifying the values of one or more initial terms and (2) has the property that the remaining terms satisfy a recursive formula that describes the value of a term based upon an expression in numbers, previous terms, or the index of the term.

An explicit formula specifies the  $n^{\text{th}}$  term of a sequence as an expression in  $n$ .

A recursive formula specifies the  $n^{\text{th}}$  term of a sequence as an expression in the previous term (or previous couple of terms).

**Problem Set**

For Problems 1–4, list the first five terms of each sequence.

1.  $a_{n+1} = a_n + 6$ , where  $a_1 = 11$  for  $n \geq 1$
2.  $a_n = a_{n-1} \div 2$ , where  $a_1 = 50$  for  $n \geq 2$
3.  $f(n+1) = -2f(n) + 8$  and  $f(1) = 1$  for  $n \geq 1$
4.  $f(n) = f(n-1) + n$  and  $f(1) = 4$  for  $n \geq 2$

For Problems 5–10, write a recursive formula for each sequence given or described below.

5. It follows a plus one pattern: 8, 9, 10, 11, 12, ....
6. It follows a times 10 pattern: 4, 40, 400, 4000, ....
7. It has an explicit formula of  $f(n) = -3n + 2$  for  $n \geq 1$ .
8. It has an explicit formula of  $f(n) = -1(12)^{n-1}$  for  $n \geq 1$ .
9. Doug accepts a job where his starting salary is \$30,000 per year, and each year he receives a raise of \$3,000.
10. A bacteria culture has an initial population of 10 bacteria, and each hour the population triples in size.

## Lesson 3: Arithmetic and Geometric Sequences

### Classwork

#### Exercise 2

Think of a real-world example of an arithmetic or a geometric sequence. Describe it, and write its formula.

#### Exercise 3

If we fold a rectangular piece of paper in half multiple times and count the number of rectangles created, what type of sequence are we creating? Can you write the formula?

**Lesson Summary**

Two types of sequences were studied:

**ARITHMETIC SEQUENCE:** A sequence is called *arithmetic* if there is a real number  $d$  such that each term in the sequence is the sum of the previous term and  $d$ .

**GEOMETRIC SEQUENCE:** A sequence is called *geometric* if there is a real number  $r$  such that each term in the sequence is a product of the previous term and  $r$ .

**Problem Set**

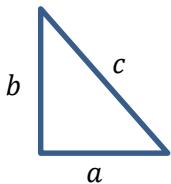
For Problems 1–4, list the first five terms of each sequence, and identify them as arithmetic or geometric.

1.  $A(n + 1) = A(n) + 4$  for  $n \geq 1$  and  $A(1) = -2$
2.  $A(n + 1) = \frac{1}{4} \cdot A(n)$  for  $n \geq 1$  and  $A(1) = 8$
3.  $A(n + 1) = A(n) - 19$  for  $n \geq 1$  and  $A(1) = -6$
4.  $A(n + 1) = \frac{2}{3}A(n)$  for  $n \geq 1$  and  $A(1) = 6$

For Problems 5–8, identify the sequence as arithmetic or geometric, and write a recursive formula for the sequence. Be sure to identify your starting value.

5. 14, 21, 28, 35, ...
6. 4, 40, 400, 4000, ...
7.  $49, 7, 1, \frac{1}{7}, \frac{1}{49}, \dots$
8.  $-101, -91, -81, -71, \dots$
9. The local football team won the championship several years ago, and since then, ticket prices have been increasing \$20 per year. The year they won the championship, tickets were \$50. Write a recursive formula for a sequence that models ticket prices. Is the sequence arithmetic or geometric?

10. A radioactive substance decreases in the amount of grams by one-third each year. If the starting amount of the substance in a rock is 1,452 g, write a recursive formula for a sequence that models the amount of the substance left after the end of each year. Is the sequence arithmetic or geometric?
11. Find an explicit form  $f(n)$  for each of the following arithmetic sequences (assume  $a$  is some real number and  $x$  is some real number).
- $-34, -22, -10, 2, \dots$
  - $\frac{1}{5}, \frac{1}{10}, 0, -\frac{1}{10}, \dots$
  - $x + 4, x + 8, x + 12, x + 16, \dots$
  - $a, 2a + 1, 3a + 2, 4a + 3, \dots$
12. Consider the arithmetic sequence 13, 24, 35, ....
- Find an explicit form for the sequence in terms of  $n$ .
  - Find the 40<sup>th</sup> term.
  - If the  $n^{\text{th}}$  term is 299, find the value of  $n$ .
13. If  $-2, a, b, c, 14$  forms an arithmetic sequence, find the values of  $a$ ,  $b$ , and  $c$ .
14.  $3 + x, 9 + 3x, 13 + 4x, \dots$  is an arithmetic sequence for some real number  $x$ .
- Find the value of  $x$ .
  - Find the 10<sup>th</sup> term of the sequence.
15. Find an explicit form  $f(n)$  of the arithmetic sequence where the 2<sup>nd</sup> term is 25 and the sum of the 3<sup>rd</sup> term and 4<sup>th</sup> term is 86.
16. Challenge: In the right triangle figure below, the lengths of the sides  $a$  cm,  $b$  cm, and  $c$  cm of the right triangle form a finite arithmetic sequence. If the perimeter of the triangle is 18 cm, find the values of  $a$ ,  $b$ , and  $c$ .



17. Find the common ratio and an explicit form in each of the following geometric sequences.
- $4, 12, 36, 108, \dots$
  - $162, 108, 72, 48, \dots$
  - $\frac{4}{3}, \frac{2}{3}, \frac{1}{3}, \frac{1}{6}, \dots$
  - $xz, x^2z^3, x^3z^5, x^4z^7, \dots$

18. The first term in a geometric sequence is 54, and the 5<sup>th</sup> term is  $\frac{2}{3}$ . Find an explicit form for the geometric sequence.
19. If  $2, a, b, -54$  forms a geometric sequence, find the values of  $a$  and  $b$ .
20. Find the explicit form  $f(n)$  of a geometric sequence if  $f(3) - f(1) = 48$  and  $\frac{f(3)}{f(1)} = 9$ .

## Lesson 4: Why Do Banks Pay YOU to Provide Their Services?

### Classwork

#### Example 1

Kyra has been babysitting since sixth grade. She has saved \$1,000 and wants to open an account at the bank so that she earns interest on her savings. Simple Bank pays simple interest at a rate of 10%. How much money will Kyra have after 1 year? After 2 years, if she does not add money to her account? After 5 years?

Raoul needs \$200 to start a snow cone stand for this hot summer. He borrows the money from a bank that charges 4% simple interest per year.

- a. How much will he owe if he waits 1 year to pay back the loan? If he waits 2 years? 3 years? 4 years? 5 years?
  
- b. Write a formula for how much he will owe after  $t$  years.

**Example 2**

Jack has \$500 to invest. The bank offers an interest rate of 6% compounded annually. How much money will Jack have after 1 year? 2 years? 5 years? 10 years?

**Example 3**

If you have \$200 to invest for 10 years, would you rather invest your money in a bank that pays 7% simple interest or in a bank that pays 5% interest compounded annually? Is there anything you could change in the problem that would make you change your answer?

**Lesson Summary**

**SIMPLE INTEREST:** Interest is calculated once per year on the original amount borrowed or invested. The interest does not become part of the amount borrowed or owed (the principal).

**COMPOUND INTEREST:** Interest is calculated once per period on the current amount borrowed or invested. Each period, the interest becomes a part of the principal.

**Problem Set**

1. \$250 is invested at a bank that pays 7% simple interest. Calculate the amount of money in the account after 1 year, 3 years, 7 years, and 20 years.
2. \$325 is borrowed from a bank that charges 4% interest compounded annually. How much is owed after 1 year, 3 years, 7 years, and 20 years?
3. Joseph has \$10,000 to invest. He can go to Yankee Bank that pays 5% simple interest or Met Bank that pays 4% interest compounded annually. At how many years will Met Bank be the better choice?

## Lesson 5: The Power of Exponential Growth

### Classwork

#### Opening Exercise

Two equipment rental companies have different penalty policies for returning a piece of equipment late.

Company 1: On day 1, the penalty is \$5. On day 2, the penalty is \$10. On day 3, the penalty is \$15. On day 4, the penalty is \$20, and so on, increasing by \$5 each day the equipment is late.

Company 2: On day 1, the penalty is \$0.01. On day 2, the penalty is \$0.02. On day 3, the penalty is \$0.04. On day 4, the penalty is \$0.08, and so on, doubling in amount each additional day late.

Jim rented a digger from Company 2 because he thought it had the better late return policy. The job he was doing with the digger took longer than he expected, but it did not concern him because the late penalty seemed so reasonable. When he returned the digger 15 days late, he was shocked by the penalty fee. What did he pay, and what would he have paid if he had used Company 1 instead?

Company 1	
Day	Penalty
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
13	
14	
15	

Company 2	
Day	Penalty
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
13	
14	
15	

### Example 1

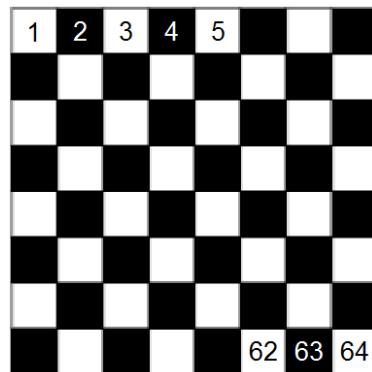
Folklore suggests that when the creator of the game of chess showed his invention to the country's ruler, the ruler was highly impressed. He was so impressed, he told the inventor to name a prize of his choice. The inventor, being rather clever, said he would take a grain of rice on the first square of the chessboard, two grains of rice on the second square of the chessboard, four on the third square, eight on the fourth square, and so on, doubling the number of grains of rice for each successive square. The ruler was surprised, even a little offended, at such a modest prize, but he ordered his treasurer to count out the rice.

- a. Why is the ruler surprised? What makes him think the inventor requested a modest prize?

The treasurer took more than a week to count the rice in the ruler's store, only to notify the ruler that it would take more rice than was available in the entire kingdom. Shortly thereafter, as the story goes, the inventor became the new king.

- b. Imagine the treasurer counting the needed rice for each of the 64 squares. We know that the first square is assigned a single grain of rice, and each successive square is double the number of grains of rice of the previous square. The following table lists the first five assignments of grains of rice to squares on the board. How can we represent the grains of rice as exponential expressions?

Square #	Grains of Rice	Exponential Expression
1	1	
2	2	
3	4	
4	8	
5	16	



- c. Write the exponential expression that describes how much rice is assigned to each of the last three squares of the board.

Square #	Exponential Expression
62	
63	
64	

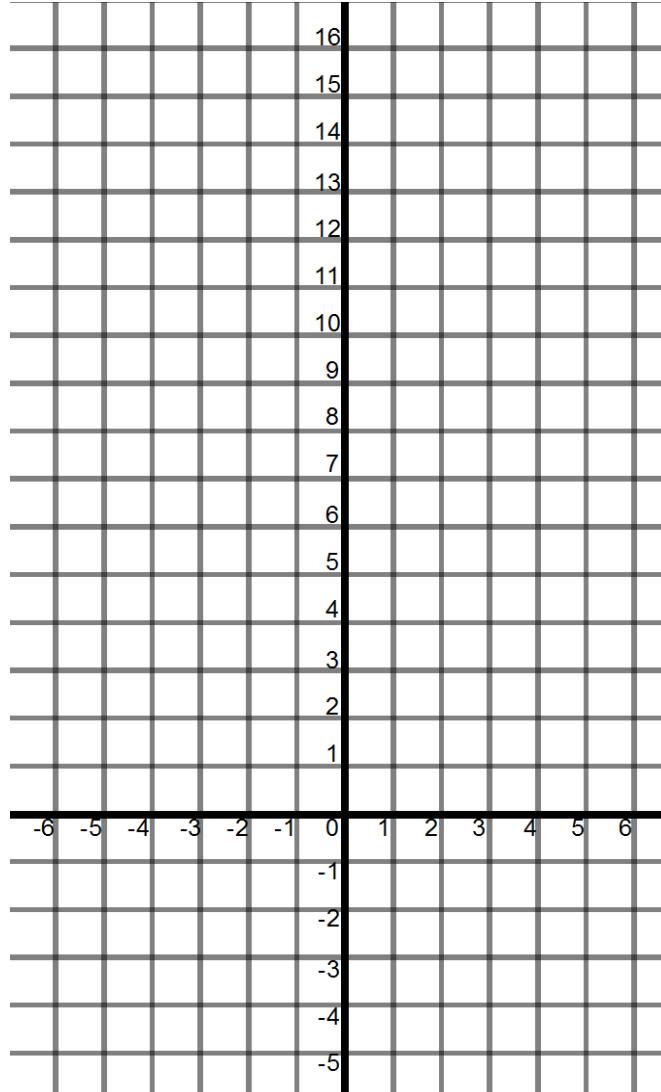
**Example 2**

Let us understand the difference between  $f(n) = 2n$  and  $f(n) = 2^n$ .

- a. Complete the tables below, and then graph the points  $(n, f(n))$  on a coordinate plane for each of the formulas.

$n$	$f(n) = 2n$
-2	
-1	
0	
1	
2	
3	

$n$	$f(n) = 2^n$
-2	
-1	
0	
1	
2	
3	



- b. Describe the change in each sequence when  $n$  increases by 1 unit for each sequence.

**Exercise 1**

A typical thickness of toilet paper is 0.001 inch. This seems pretty thin, right? Let's see what happens when we start folding toilet paper.

- a. How thick is the stack of toilet paper after 1 fold? After 2 folds? After 5 folds?
  
  
  
  
  
- b. Write an explicit formula for the sequence that models the thickness of the folded toilet paper after  $n$  folds.
  
  
  
  
  
- c. After how many folds does the stack of folded toilet paper pass the 1-foot mark?
  
  
  
  
  
- d. The moon is about 240,000 miles from Earth. Compare the thickness of the toilet paper folded 50 times to the distance from Earth.

Watch the following video *How folding paper can get you to the moon* (<http://www.youtube.com/watch?v=AmFMC45f1Q>).

**Exercise 2**

A rare coin appreciates at a rate of 5.2% a year. If the initial value of the coin is \$500, after how many years will its value cross the \$3,000 mark? Show the formula that models the value of the coin after  $t$  years.

**Problem Set**

1. A bucket is put under a leaking ceiling. The amount of water in the bucket doubles every minute. After 8 minutes, the bucket is full. After how many minutes is the bucket half-full?
2. A three-bedroom house in Burbville sold for \$190,000. If housing prices are expected to increase 1.8% annually in that town, write an explicit formula that models the price of the house in  $t$  years. Find the price of the house in 5 years.
3. A local college has increased its number of graduates by a factor of 1.045 over the previous year for every year since 1999. In 1999, 924 students graduated. What explicit formula models this situation? Approximately how many students will graduate in 2014?
4. The population growth rate of New York City has fluctuated tremendously in the last 200 years, the highest rate estimated at 126.8% in 1900. In 2001, the population of the city was 8,008,288, up 2.1% from 2000. If we assume that the annual population growth rate stayed at 2.1% from the year 2000 onward, in what year would we expect the population of New York City to have exceeded ten million people? Be sure to include the explicit formula you use to arrive at your answer.
5. In 2013, a research company found that smartphone shipments (units sold) were up 32.7% worldwide from 2012, with an expectation for the trend to continue. If 959 million units were sold in 2013, how many smartphones can be expected to sell in 2018 at the same growth rate? (Include the explicit formula for the sequence that models this growth.) Can this trend continue?
6. Two band mates have only 7 days to spread the word about their next performance. Jack thinks they can each pass out 100 fliers a day for 7 days, and they will have done a good job in getting the news out. Meg has a different strategy. She tells 10 of her friends about the performance on the first day and asks each of her 10 friends to each tell a friend on the second day and then everyone who has heard about the concert to tell a friend on the third day, and so on, for 7 days. Make an assumption that students are not telling someone who has not already been told.
  - a. Over the first 7 days, Meg's strategy will reach fewer people than Jack's. Show that this is true.
  - b. If they had been given more than 7 days, would there be a day on which Meg's strategy would begin to inform more people than Jack's strategy? If not, explain why not. If so, on which day would this occur?
  - c. Knowing that she has only 7 days, how can Meg alter her strategy to reach more people than Jack does?

7. On June 1, a fast-growing species of algae is accidentally introduced into a lake in a city park. It starts to grow and cover the surface of the lake in such a way that the area it covers doubles every day. If it continues to grow unabated, the lake will be totally covered, and the fish in the lake will suffocate. At the rate it is growing, this will happen on June 30.
- When will the lake be covered halfway?
  - On June 26, a pedestrian who walks by the lake every day warns that the lake will be completely covered soon. Her friend just laughs. Why might her friend be skeptical of the warning?
  - On June 29, a cleanup crew arrives at the lake and removes almost all of the algae. When they are done, only 1% of the surface is covered with algae. How well does this solve the problem of the algae in the lake?
  - Write an explicit formula for the sequence that models the percentage of the surface area of the lake that is covered in algae,  $a$ , given the time in days,  $t$ , that has passed since the algae was introduced into the lake.
8. Mrs. Davis is making a poster of math formulas for her students. She takes the 8.5 in.  $\times$  11 in. paper she printed the formulas on to the photocopy machine and enlarges the image so that the length and the width are both 150% of the original. She enlarges the image a total of 3 times before she is satisfied with the size of the poster. Write an explicit formula for the sequence that models the area of the poster,  $A$ , after  $n$  enlargements. What is the area of the final image compared to the area of the original, expressed as a percent increase and rounded to the nearest percent?

## Lesson 6: Exponential Growth—U.S. Population and World Population

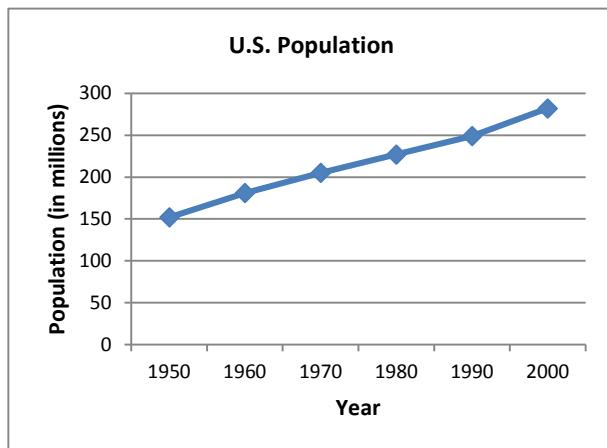
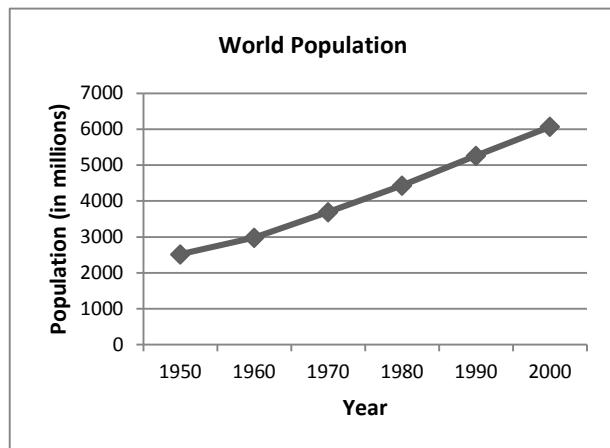
### Classwork

#### Mathematical Modeling Exercise 1

Callie and Joe are examining the population data in the graphs below for a history report. Their comments are as follows:

Callie: It looks like the U.S. population grew the same amount as the world population, but that can't be right, can it?

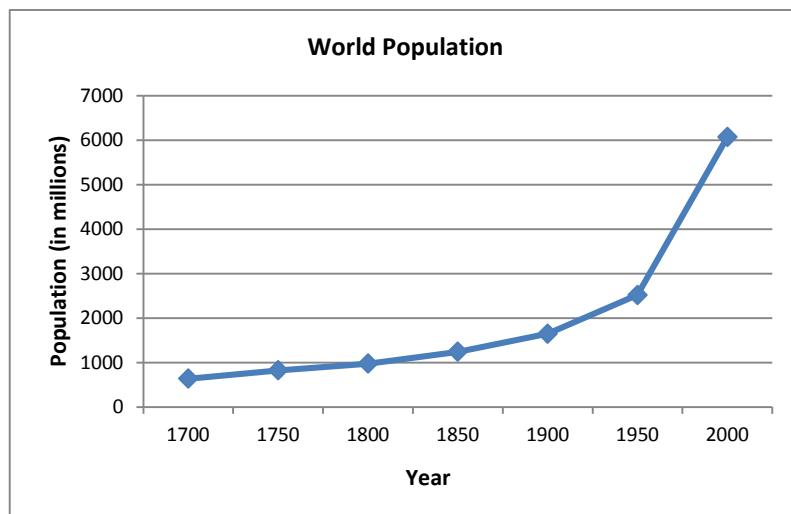
Joe: Well, I don't think they grew by the same *amount*, but they sure grew at about the same rate. Look at the slopes.



a. Is Callie's observation correct? Why or why not?

b. Is Joe's observation correct? Why or why not?

- c. Use the World Population graph to estimate the percent increase in world population from 1950 to 2000.
- d. Now, use the U.S. Population graph to estimate the percent increase in the U.S. population for the same time period.
- e. How does the percent increase for the world population compare to that for the U.S. population over the same time period, 1950 to 2000?
- f. Do the graphs above seem to indicate linear or exponential population growth? Explain your response.
- g. Write an explicit formula for the sequence that models the world population growth from 1950 to 2000 based on the information in the graph. Assume that the population (in millions) in 1950 was 2,500 and in 2000 was 6,000. Use  $t$  to represent the number of years after 1950.

**Mathematical Modeling Exercise 2**

- How is this graph similar to the World Population graph in Mathematical Modeling Exercise 1? How is it different?
- Does the behavior of the graph from 1950 to 2000 match that shown on the graph in Mathematical Modeling Exercise 1?
- Why is the graph from Mathematical Modeling Exercise 1 somewhat misleading?

- d. An exponential formula that can be used to model the world population growth from 1950 through 2000 is as follows:

$$f(t) = 2519(1.0177^t)$$

where 2,519 represents the world population in the year 1950, and  $t$  represents the number of years after 1950. Use this equation to calculate the world population in 1950, 1980, and 2000. How do your calculations compare with the world populations shown on the graph?

- e. The following is a table showing the world population numbers used to create the graphs above.

Year	World Population (in millions)
1700	640
1750	824
1800	978
1850	1,244
1900	1,650
1950	2,519
1960	2,982
1970	3,692
1980	4,435
1990	5,263
2000	6,070

How do the numbers in the table compare with those you calculated in part (d) above?

- f. How is the formula in part (d) above different from the formula in Mathematical Modeling Exercise 1, part (g)? What causes the difference? Which formula more closely represents the population?

### Exercises 1–2

1. The table below represents the population of the United States (in millions) for the specified years.

Year	U.S. Population (in millions)
1800	5
1900	76
2000	282

- a. If we use the data from 1800 to 2000 to create an exponential equation representing the population, we generate the following formula for the sequence, where  $f(t)$  represents the U.S. population and  $t$  represents the number of years after 1800.

$$f(t) = 5(1.0204)^t$$

Use this formula to determine the population of the United States in the year 2010.

- b. If we use the data from 1900 to 2000 to create an exponential equation that models the population, we generate the following formula for the sequence, where  $f(t)$  represents the U.S. population and  $t$  represents the number of years after 1900.

$$f(t) = 76(1.013)^t$$

Use this formula to determine the population of the United States in the year 2010.

- c. The actual U.S. population in the year 2010 was 309 million. Which of the above formulas better models the U.S. population for the entire span of 1800–2010? Why?
- d. Complete the table below to show projected population figures for the years indicated. Use the formula from part (b) to determine the numbers.

Year	World Population (in millions)
2020	
2050	
2080	

- e. Are the population figures you computed reasonable? What other factors need to be considered when projecting population?
2. The population of the country of Oz was 600,000 in the year 2010. The population is expected to grow by a factor of 5% annually. The annual food supply of Oz is currently sufficient for a population of 700,000 people and is increasing at a rate that will supply food for an additional 10,000 people per year.
- a. Write a formula to model the population of Oz. Is your formula linear or exponential?

- b. Write a formula to model the food supply. Is the formula linear or exponential?
- c. At what point does the population exceed the food supply? Justify your response.
- d. If Oz doubled its current food supply (to 1.4 million), would shortages still take place? Explain.
- e. If Oz doubles both its beginning food supply and doubles the rate at which the food supply increases, would food shortages still take place? Explain.

**Problem Set**

1. Student Friendly Bank pays a simple interest rate of 2.5% per year. Neighborhood Bank pays a compound interest rate of 2.1% per year, compounded monthly.
- Which bank will provide the largest balance if you plan to invest \$10,000 for 10 years? For 20 years?
  - Write an explicit formula for the sequence that models the balance in the Student Friendly Bank account  $t$  years after a deposit is left in the account.
  - Write an explicit formula for the sequence that models the balance in the Neighborhood Bank account  $m$  months after a deposit is left in the account.
  - Create a table of values indicating the balances in the two bank accounts from year 2 to year 20 in 2-year increments. Round each value to the nearest dollar.

Year	Student Friendly Bank (in dollars)	Neighborhood Bank (in dollars)
0		
2		
4		
6		
8		
10		
12		
14		
16		
18		
20		

- Which bank is a better short-term investment? Which bank is better for those leaving money in for a longer period of time? When are the investments about the same?
- What type of model is Student Friendly Bank? What is the rate or ratio of change?
- What type of model is Neighborhood Bank? What is the rate or ratio of change?

2. The table below represents the population of the state of New York for the years 1800–2000. Use this information to answer the questions.

Year	Population
1800	300,000
1900	7,300,000
2000	19,000,000

- a. Using the year 1800 as the base year, an explicit formula for the sequence that models the population of New York is  $P(t) = 300\,000(1.021)^t$ , where  $t$  is the number of years after 1800.  
 Using this formula, calculate the projected population of New York in 2010.
- b. Using the year 1900 as the base year, an explicit formula for the sequence that models the population of New York is  $P(t) = 7\,300\,000(1.0096)^t$ , where  $t$  is the number of years after 1900.  
 Using this formula, calculate the projected population of New York in 2010.
- c. Using the Internet (or some other source), find the population of the state of New York according to the 2010 census. Which formula yielded a more accurate prediction of the 2010 population?

## Lesson 7: Exponential Decay

### Classwork

#### Example 1

- a. Malik bought a new car for \$15,000. As he drove it off the lot, his best friend, Will, told him that the car's value just dropped by 15% and that it would continue to depreciate 15% of its current value each year. If the car's value is now \$12,750 (according to Will), what will its value be after 5 years?

Complete the table below to determine the car's value after each of the next five years. Round each value to the nearest cent.

Number of years, $t$ , passed since driving the car off the lot	Car value after $t$ years	15% depreciation of current car value	Car value minus the 15% depreciation
0	\$12,750.00	\$1,912.50	\$10,837.50
1	10,837.50		
2			
3			
4			
5			

- b. Write an explicit formula for the sequence that models the value of Malik's car  $t$  years after driving it off the lot.
- c. Use the formula from part (b) to determine the value of Malik's car five years after its purchase. Round your answer to the nearest cent. Compare the value with the value in the table. Are they the same?
- d. Use the formula from part (b) to determine the value of Malik's car 7 years after its purchase. Round your answer to the nearest cent.

**Exercises 1–6**

1. Identify the initial value in each formula below, and state whether the formula models exponential growth or exponential decay. Justify your responses.

a.  $f(t) = 2\left(\frac{2}{5}\right)^t$

b.  $f(t) = 2\left(\frac{5}{3}\right)^t$

c.  $f(t) = \frac{2}{3}(3)^t$

d.  $f(t) = \frac{2}{3}\left(\frac{1}{3}\right)^t$

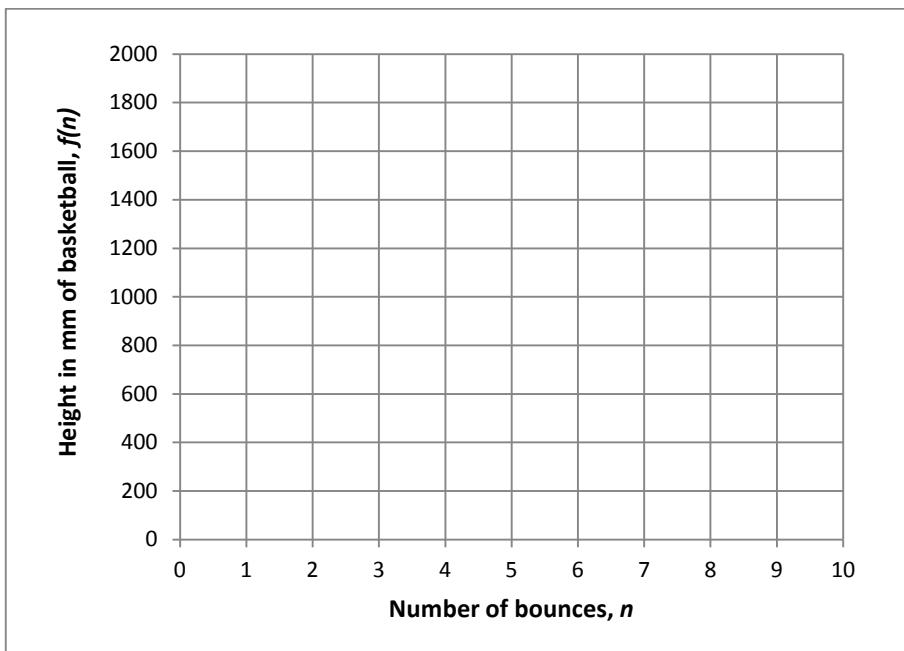
e.  $f(t) = \frac{3}{2}\left(\frac{2}{3}\right)^t$

2. If a person takes a given dosage  $d$  of a particular medication, then the formula  $f(t) = d(0.8)^t$  represents the concentration of the medication in the bloodstream  $t$  hours later. If Charlotte takes 200 mg of the medication at 6:00 a.m., how much remains in her bloodstream at 10:00 a.m.? How long does it take for the concentration to drop below 1 mg?

3. When you breathe normally, about 12% of the air in your lungs is replaced with each breath. Write an explicit formula for the sequence that models the amount of the original air left in your lungs, given that the initial volume of air is 500 ml. Use your model to determine how much of the original 500 ml remains after 50 breaths.
4. Ryan bought a new computer for \$2,100. The value of the computer decreases by 50% each year. When will the value drop below \$300?
5. Kelli's mom takes a 400 mg dose of aspirin. Each hour, the amount of aspirin in a person's system decreases by about 29%. How much aspirin is left in her system after 6 hours?
6. According to the International Basketball Association (FIBA), a basketball must be inflated to a pressure such that when it is dropped from a height of 1,800 mm, it rebounds to a height of 1,300 mm. Maddie decides to test the rebound-ability of her new basketball. She assumes that the ratio of each rebound height to the previous rebound height remains the same at  $\frac{1300}{1800}$ . Let  $f(n)$  be the height of the basketball after  $n$  bounces. Complete the chart below to reflect the heights Maddie expects to measure.

$n$	$f(n)$
0	1,800
1	
2	
3	
4	

- a. Write the explicit formula for the sequence that models the height of Maddie's basketball after any number of bounces.
- b. Plot the points from the table. Connect the points with a smooth curve, and then use the curve to estimate the bounce number at which the rebound height drops below 200 mm.



**Lesson Summary**

The explicit formula  $f(t) = ab^t$  models exponential decay, where  $a$  represents the initial value of the sequence,  $b < 1$  represents the growth factor (or decay factor) per unit of time, and  $t$  represents units of time.

**Problem Set**

1. From 2000 to 2013, the value of the U.S. dollar has been shrinking. The value of the U.S. dollar over time ( $v(t)$ ) can be modeled by the following formula:

$$v(t) = 1.36(0.9758)^t, \text{ where } t \text{ is the number of years since 2000}$$

- a. How much was a dollar worth in the year 2005?
  - b. Graph the points  $(t, v(t))$  for integer values of  $0 \leq t \leq 14$ .
  - c. Estimate the year in which the value of the dollar fell below \$1.00.
2. A construction company purchased some equipment costing \$300,000. The value of the equipment depreciates (decreases) at a rate of 14% per year.
- a. Write a formula that models the value of the equipment each year.
  - b. What is the value of the equipment after 9 years?
  - c. Graph the points  $(t, v(t))$  for integer values of  $0 \leq t \leq 15$ .
  - d. Estimate when the equipment will have a value of \$50,000.
3. The number of newly reported cases of HIV (in thousands) in the United States from 2000 to 2010 can be modeled by the following formula:

$$f(t) = 41(0.9842)^t, \text{ where } t \text{ is the number of years after 2000}$$

- a. Identify the growth factor.
  - b. Calculate the estimated number of new HIV cases reported in 2004.
  - c. Graph the points  $(t, f(t))$  for integer values of  $0 \leq t \leq 10$ .
  - d. During what year did the number of newly reported HIV cases drop below 36,000?
4. Doug drank a soda with 130 mg of caffeine. Each hour, the caffeine in the body diminishes by about 12%.
- a. Write a formula to model the amount of caffeine remaining in Doug's system each hour.
  - b. How much caffeine remains in Doug's system after 2 hours?
  - c. How long will it take for the level of caffeine in Doug's system to drop below 50 mg?

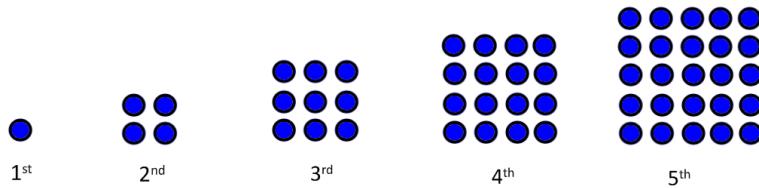
5. 64 teams participate in a softball tournament in which half the teams are eliminated after each round of play.
  - a. Write a formula to model the number of teams remaining after any given round of play.
  - b. How many teams remain in play after 3 rounds?
  - c. How many rounds of play will it take to determine which team wins the tournament?
  
6. Sam bought a used car for \$8,000. He boasted that he got a great deal since the value of the car two years ago (when it was new) was \$15,000. His friend, Derek, was skeptical, stating that the value of a car typically depreciates about 25% per year, so Sam got a bad deal.
  - a. Use Derek's logic to write a formula for the value of Sam's car. Use  $t$  for the total age of the car in years.
  - b. Who is right, Sam or Derek?

## Lesson 8: Why Stay with Whole Numbers?

### Classwork

#### Opening Exercise

The sequence of perfect squares {1,4,9,16,25, ... } earned its name because the ancient Greeks realized these quantities could be arranged to form square shapes.



If  $S(n)$  denotes the  $n^{\text{th}}$  square number, what is a formula for  $S(n)$ ?

#### Exercises

1. Prove whether or not 169 is a perfect square.

2. Prove whether or not 200 is a perfect square.

3. If  $S(n) = 225$ , then what is  $n$ ?

4. Which term is the number 400 in the sequence of perfect squares? How do you know?

Instead of arranging dots into squares, suppose we extend our thinking to consider squares of side length  $x$  cm.

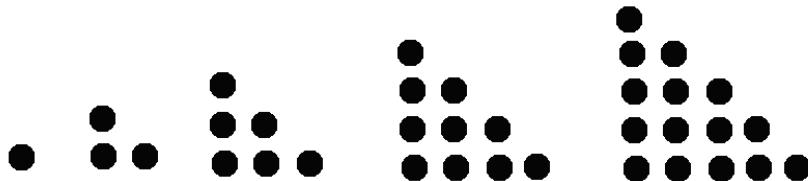
5. Create a formula for the area  $A(x)$  cm<sup>2</sup> of a square of side length  $x$  cm:  $A(x) = \underline{\hspace{2cm}}$ .

6. Use the formula to determine the area of squares with side lengths of 3 cm, 10.5 cm, and  $\pi$  cm.

7. What does  $A(0)$  mean?

8. What does  $A(-10)$  and  $A(\sqrt{2})$  mean?

The triangular numbers are the numbers that arise from arranging dots into triangular figures as shown:



9. What is the 100<sup>th</sup> triangular number?
  10. Find a formula for  $T(n)$ , the  $n^{\text{th}}$  triangular number (starting with  $n = 1$ ).
  11. How can you be sure your formula works?
  12. Create a graph of the sequence of triangular numbers  $(n) = \frac{n(n+1)}{2}$ , where  $n$  is a positive integer.

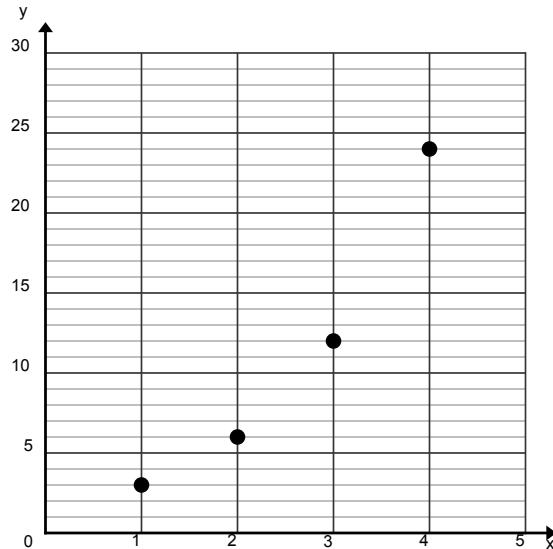
13. Create a graph of the triangle area formula  $T(x) = \frac{x(x+1)}{2}$ , where  $x$  is any positive real number.

14. How are your two graphs alike? How are they different?

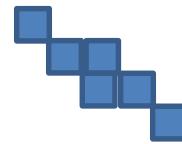
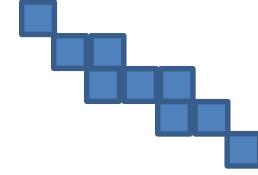
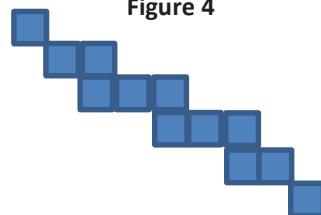
**Problem Set**

1. The first four terms of two different sequences are shown below. Sequence *A* is given in the table, and sequence *B* is graphed as a set of ordered pairs.

<b><i>n</i></b>	<b><i>A(n)</i></b>
1	15
2	31
3	47
4	63

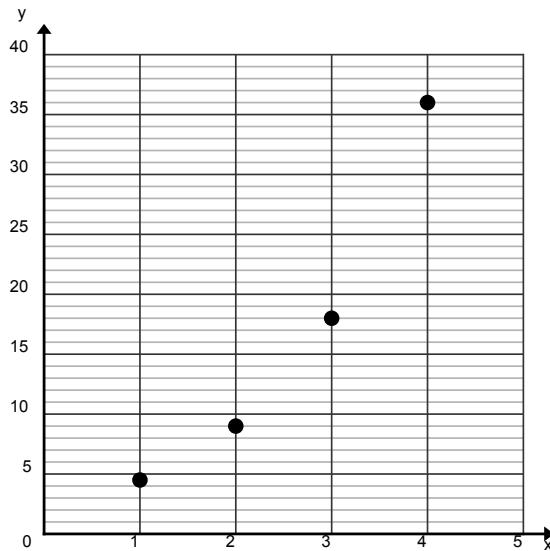


- a. Create an explicit formula for each sequence.  
 b. Which sequence will be the first to exceed 500? How do you know?
2. A tile pattern is shown below.

**Figure 1****Figure 2****Figure 3****Figure 4**

- a. How is this pattern growing?  
 b. Create an explicit formula that could be used to determine the number of squares in the  $n^{\text{th}}$  figure.  
 c. Evaluate your formula for  $n = 0$ , and  $n = 2.5$ . Draw Figure 0 and Figure 2.5, and explain how you decided to create your drawings.

3. The first four terms of a geometric sequence are graphed as a set of ordered pairs.



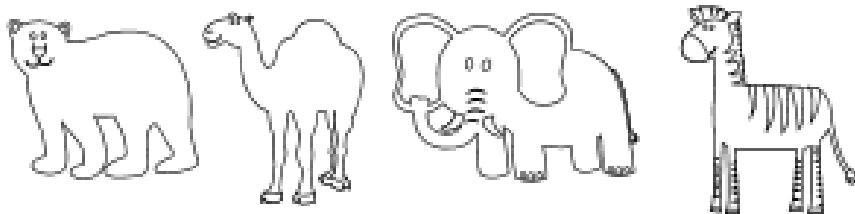
- What is an explicit formula for this sequence?
- Explain the meaning of the ordered pair  $(3, 18)$ .
- As of July 2013, Justin Bieber had over 42,000,000 Twitter followers. Suppose the sequence represents the number of people that follow your new Twitter account each week since you started tweeting. If your followers keep growing in the same manner, when will you exceed 1,000,000 followers?

## Lesson 9: Representing, Naming, and Evaluating Functions

### Classwork

#### Opening Exercise

Match each picture to the correct word by drawing an arrow from the word to the picture.



Elephant
Camel
Polar Bear
Zebra

**FUNCTION:** A *function* is a correspondence between two sets,  $X$  and  $Y$ , in which each element of  $X$  is matched to one and only one element of  $Y$ . The set  $X$  is called the *domain of the function*.

The notation  $f: X \rightarrow Y$  is used to name the function and describes both  $X$  and  $Y$ . If  $x$  is an element in the domain  $X$  of a function  $f: X \rightarrow Y$ , then  $x$  is matched to an element of  $Y$  called  $f(x)$ . We say  $f(x)$  is the value in  $Y$  that denotes the *output* or *image* of  $f$  corresponding to the *input*  $x$ .

The *range (or image)* of a function  $f: X \rightarrow Y$  is the subset of  $Y$ , denoted  $f(X)$ , defined by the following property:  $y$  is an element of  $f(X)$  if and only if there is an  $x$  in  $X$  such that  $f(x) = y$ .

#### Example 1

Define the Opening Exercise using function notation. State the domain and the range.

**Example 2**

Is the assignment of students to English teachers an example of a function? If yes, define it using function notation, and state the domain and the range.

**Example 3**

Let  $X = \{1, 2, 3, 4\}$  and  $Y = \{5, 6, 7, 8, 9\}$ .  $f$  and  $g$  are defined below.

$$f: X \rightarrow Y \qquad \qquad g: X \rightarrow Y$$

$$f = \{(1,7), (2,5), (3,6), (4,7)\} \qquad g = \{(1,5), (2,6), (1,8), (2,9), (3,7)\}$$

Is  $f$  a function? If yes, what is the domain, and what is the range? If no, explain why  $f$  is not a function.

Is  $g$  a function? If yes, what is the domain and range? If no, explain why  $g$  is not a function.

What is  $f(2)$ ?

If  $f(x) = 7$ , then what might  $x$  be?

**Exercises**

1. Define  $f$  to assign each student at your school a unique ID number.

$f: \{\text{students in your school}\} \rightarrow \{\text{whole numbers}\}$

Assign each student a unique ID number.

- a. Is this an example of a function? Use the definition to explain why or why not.

- b. Suppose  $f(\text{Hilda}) = 350123$ . What does that mean?

- c. Write your name and student ID number using function notation.

2. Let  $g$  assign each student at your school to a grade level.

- a. Is this an example of a function? Explain your reasoning.

- b. Express this relationship using function notation, and state the domain and the range.

$g: \{\text{students in the school}\} \rightarrow \{\text{grade level}\}$

Assign each student to a grade level.

3. Let  $h$  be the function that assigns each student ID number to a grade level.

$h: \{\text{student ID number}\} \rightarrow \{\text{grade level}\}$

Assign each student ID number to the student's current grade level.

- a. Describe the domain and range of this function.

- b. Record several ordered pairs  $(x, f(x))$  that represent yourself and students in your group or class.

- c. Jonny says, "This is not a function because every ninth grader is assigned the same range value of 9. The range only has 4 numbers {9, 10, 11, 12}, but the domain has a number for every student in our school." Explain to Jonny why he is incorrect.

**Problem Set**

1. Which of the following are examples of a function? Justify your answers.
  - a. The assignment of the members of a football team to jersey numbers.
  - b. The assignment of U.S. citizens to Social Security numbers.
  - c. The assignment of students to locker numbers.
  - d. The assignment of the residents of a house to the street addresses.
  - e. The assignment of zip codes to residences.
  - f. The assignment of residences to zip codes.
  - g. The assignment of teachers to students enrolled in each of their classes.
  - h. The assignment of all real numbers to the next integer equal to or greater than the number.
  - i. The assignment of each rational number to the product of its numerator and denominator.
  
2. Sequences are functions. The domain is the set of all term numbers (which is usually the positive integers), and the range is the set of terms of the sequence. For example, the sequence 1, 4, 9, 16, 25, 36, ... of perfect squares is the function:

*Let  $f: \{\text{positive integers}\} \rightarrow \{\text{perfect squares}\}$*

Assign each term number to the square of that number.

- a. What is  $f(3)$ ? What does it mean?
- b. What is the solution to the equation  $f(x) = 49$ ? What is the meaning of this solution?
- c. According to this definition, is  $-3$  in the domain of  $f$ ? Explain why or why not.
- d. According to this definition, is  $50$  in the range of  $f$ ? Explain why or why not.
  
3. Write each sequence as a function.
  - a.  $\{1, 3, 6, 10, 15, 21, 28\}$
  - b.  $\{1, 3, 5, 7, 9, \dots\}$
  - c.  $a_{n+1} = 3a_n$ ,  $a_1 = 1$ , where  $n$  is a positive integer greater than or equal to 1.

## Lesson 10: Representing, Naming, and Evaluating Functions

### Classwork

#### Opening Exercise

Study the 4 representations of a function below. How are these representations alike? How are they different?

TABLE:

Input	0	1	2	3	4	5
Output	1	2	4	8	16	32

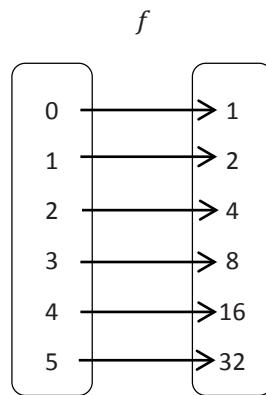
FUNCTION:

Let  $f: \{0, 1, 2, 3, 4, 5\} \rightarrow \{1, 2, 4, 8, 16, 32\}$  such that  $x \mapsto 2^x$ .

SEQUENCE:

Let  $a_{n+1} = 2a_n, a_0 = 1$  for  $0 \leq n \leq 4$  where  $n$  is an integer.

DIAGRAM:



**Exercise 1**

Let  $X = \{0, 1, 2, 3, 4, 5\}$ . Complete the following table using the definition of  $f$ .

$$f: X \rightarrow Y$$

Assign each  $x$  in  $X$  to the expression  $2^x$ .

$x$	0	1	2	3	4	5
$f(x)$						

What are  $f(0), f(1), f(2), f(3), f(4)$ , and  $f(5)$ ?

What is the range of  $f$ ?

**Exercise 2**

The squaring function is defined as follows:

Let  $f: X \rightarrow Y$  be the function such that  $x \mapsto x^2$ , where  $X$  is the set of all real numbers.

What are  $f(0), f(3), f(-2), f(\sqrt{3}), f(-2.5), f\left(\frac{2}{3}\right), f(a)$ , and  $f(3 + a)$ ?

What is the range of  $f$ ?

What subset of the real numbers could be used as the domain of the squaring function to create a range with the same output values as the sequence of square numbers  $\{1, 4, 9, 16, 25, \dots\}$  from Lesson 9?

### Exercise 3

Recall that an equation can either be true or false. Using the function defined by  $f: \{0, 1, 2, 3, 4, 5\} \rightarrow \{1, 2, 4, 8, 16, 32\}$  such that  $x \mapsto 2^x$ , determine whether the equation  $f(x) = 2^x$  is true or false for each  $x$  in the domain of  $f$ .

$x$	Is the equation $f(x) = 2^x$ true or false?	Justification
0	True	<p>Substitute 0 into the equation.  <math>f(0) = 2^0</math>  <math>1 = 2^0</math></p> <p>The 1 on the left side comes from the definition of <math>f</math>, and the value of <math>2^0</math> is also 1, so the equation is true.</p>
1		
2		
3		
4		
5		

If the domain of  $f$  were extended to all real numbers, would the equation still be true for each  $x$  in the domain of  $f$ ? Explain your thinking.

#### Exercise 4

Write three different polynomial functions such that  $f(3) = 2$ .

#### Exercise 5

The domain and range of this function are not specified. Evaluate the function for several values of  $x$ . What subset of the real numbers would represent the domain of this function? What subset of the real numbers would represent its range?

$$\text{Let } f(x) = \sqrt{x - 2}$$

**Lesson Summary**

**ALGEBRAIC FUNCTION:** Given an algebraic expression in one variable, an *algebraic function* is a function  $f: D \rightarrow Y$  such that for each real number  $x$  in the domain  $D$ ,  $f(x)$  is the value found by substituting the number  $x$  into all instances of the variable symbol in the algebraic expression and evaluating.

The following notation will be used to define functions going forward. If a domain is not specified, it is assumed to be the set of all real numbers.

For the squaring function, we say      Let  $f(x) = x^2$ .

For the exponential function with base 2, we say      Let  $f(x) = 2^x$ .

When the domain is limited by the expression or the situation to be a subset of the real numbers, it must be specified when the function is defined.

For the square root function, we say      Let  $f(x) = \sqrt{x}$  for  $x \geq 0$ .

To define the first 5 triangular numbers, we say      Let  $f(x) = \frac{x(x+1)}{2}$  for  $1 \leq x \leq 5$  where  $x$  is an integer.

Depending on the context, one either views the statement " $f(x) = \sqrt{x}$ " as part of defining the function  $f$  or as an equation that is true for all  $x$  in the domain of  $f$  or as a formula.

**Problem Set**

1. Let  $f(x) = 6x - 3$ , and let  $g(x) = 0.5(4)^x$ . Find the value of each function for the given input.
 

a. $f(0)$	j. $g(0)$
b. $f(-10)$	k. $g(-1)$
c. $f(2)$	l. $g(2)$
d. $f(0.01)$	m. $g(-3)$
e. $f(11.25)$	n. $g(4)$
f. $f(-\sqrt{2})$	o. $g(\sqrt{2})$
g. $f\left(\frac{5}{3}\right)$	p. $g\left(\frac{1}{2}\right)$
h. $f(1) + f(2)$	q. $g(2) + g(1)$
i. $f(6) - f(2)$	r. $g(6) - g(2)$

2. Since a variable is a placeholder, we can substitute letters that stand for numbers in for  $x$ . Let  $f(x) = 6x - 3$ , and let  $g(x) = 0.5(4)^x$ , and suppose  $a$ ,  $b$ ,  $c$ , and  $h$  are real numbers. Find the value of each function for the given input.
- a.  $f(a)$
  - b.  $f(2a)$
  - c.  $f(b + c)$
  - d.  $f(2 + h)$
  - e.  $f(a + h)$
  - f.  $f(a + 1) - f(a)$
  - g.  $f(a + h) - f(a)$
  - h.  $g(b)$
  - i.  $g(b + 3)$
  - j.  $g(3b)$
  - k.  $g(b - 3)$
  - l.  $g(b + c)$
  - m.  $g(b + 1) - g(b)$
3. What is the range of each function given below?
- a. Let  $f(x) = 9x - 1$ .
  - b. Let  $g(x) = 3^{2x}$ .
  - c. Let  $f(x) = x^2 - 4$ .
  - d. Let  $h(x) = \sqrt{x} + 2$ .
  - e. Let  $a(x) = x + 2$  such that  $x$  is a positive integer.
  - f. Let  $g(x) = 5^x$  for  $0 \leq x \leq 4$ .
4. Provide a suitable domain and range to complete the definition of each function.
- a. Let  $f(x) = 2x + 3$ .
  - b. Let  $f(x) = 2^x$ .
  - c. Let  $C(x) = 9x + 130$ , where  $C(x)$  is the number of calories in a sandwich containing  $x$  grams of fat.
  - d. Let  $B(x) = 100(2)^x$ , where  $B(x)$  is the number of bacteria at time  $x$  hours over the course of one day.
5. Let  $f: X \rightarrow Y$ , where  $X$  and  $Y$  are the set of all real numbers, and  $x$  and  $h$  are real numbers.
- a. Find a function  $f$  such that the equation  $f(x + h) = f(x) + f(h)$  is not true for all values of  $x$  and  $h$ . Justify your reasoning.
  - b. Find a function  $f$  such that equation  $f(x + h) = f(x) + f(h)$  is true for all values of  $x$  and  $h$ . Justify your reasoning.
  - c. Let  $f(x) = 2^x$ . Find a value for  $x$  and a value for  $h$  that makes  $f(x + h) = f(x) + f(h)$  a true number sentence.

6. Given the function  $f$  whose domain is the set of real numbers, let  $f(x) = 1$  if  $x$  is a rational number, and let  $f(x) = 0$  if  $x$  is an irrational number.

- Explain why  $f$  is a function.
- What is the range of  $f$ ?
- Evaluate  $f$  for each domain value shown below.

$x$	$\frac{2}{3}$	0	-5	$\sqrt{2}$	$\pi$
$f(x)$					

- List three possible solutions to the equation  $f(x) = 0$ .

## Lesson 11: The Graph of a Function

### Classwork

In Module 1, you graphed equations such as  $y = 10 - 4x$  by plotting the points in the Cartesian plane by picking  $x$ -values and then using the equation to find the  $y$ -value for each  $x$ -value. The number of ordered pairs you plotted to get the general shape of the graph depended on the type of equation (linear, quadratic, etc.). The graph of the equation was then a representation of the solution set, which could be described using set notation.

In this lesson, we extend set notation slightly to describe the graph of a function. In doing so, we explain a way to think about set notation for the graph of a function that mimics the instructions a tablet or laptop might perform to draw a graph on its screen.

### Exploratory Challenge 1

Computer programs are essentially instructions to computers on what to do when the user (you!) makes a request. For example, when you type a letter on your smart phone, the smart phone follows a specified set of instructions to draw that letter on the screen and record it in memory (as part of an email, for example). One of the simplest types of instructions a computer can perform is a *for-next loop*. Below is code for a program that prints the first 5 powers of 2:

```
Declare x integer
For all x from 1 to 5
    Print 2x
Next x
```

The output of this program code is

2  
4  
8  
16  
32

Here is a description of the instructions: First,  $x$  is quantified as an integer, which means the variable can only take on integer values and cannot take on values like  $\frac{1}{3}$  or  $\sqrt{2}$ . The *For* statement begins the loop, starting with  $x = 1$ . The instructions between *For* and *Next* are performed for the value  $x = 1$ , which in this case is just to *Print 2*. (*Print* means “print to the computer screen.”) Then the computer performs the instructions again for the next  $x$  ( $x = 2$ ), that is, *Print 4*, and so on until the computer performs the instructions for  $x = 5$ , that is, *Print 32*.

**Exercise 1**

Perform the instructions in the following programming code as if you were a computer and your paper was the computer screen.

```
Declare x integer
For all x from 2 to 8
    Print 2x + 3
Next x
```

**Exploratory Challenge 2**

We can use almost the same code to build a set: First, we start with a set with zero elements in it (called the *empty set*), and then we increase the size of the set by appending one new element to it in each for-next step.

```
Declare x integer
Initialize G as {}
For all x from 2 to 8
    Append 2x + 3 to G
    Print G
Next x
```

Note that  $G$  is printed to the screen after each new number is appended. Thus, the output shows how the set builds:

```
{7}
{7, 9}
{7, 9, 11}
{7, 9, 11, 13}
{7, 9, 11, 13, 15}
{7, 9, 11, 13, 15, 17}
{7, 9, 11, 13, 15, 17, 19}.
```

**Exercise 2**

We can also build a set by appending ordered pairs. Perform the instructions in the following programming code as if you were a computer and your paper were the computer screen (the first few are done for you).

```

Declare x integer
Initialize G as {}
For all x from 2 to 8
    Append (x, 2x + 3) to G
Next x
Print G
  
```

Output:

{(2,7), (3,9), \_\_\_\_\_}

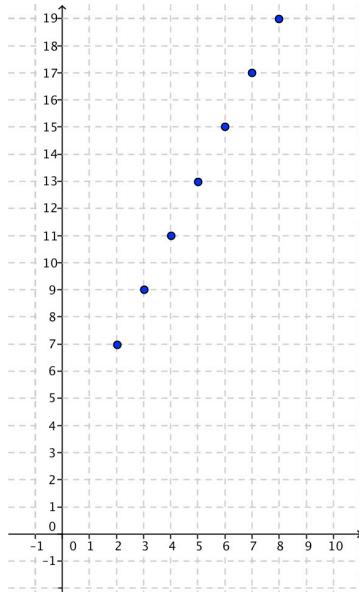
**Exploratory Challenge 3**

Instead of Printing the set  $G$  to the screen, we can use another command, *Plot*, to plot the points on a Cartesian plane.

```

Declare x integer
Initialize G as {}
For all x from 2 to 8
    Append (x, 2x + 3) to G
Next x
Plot G
  
```

Output:



In mathematics, the programming code above can be compactly written using set notation, as follows:

$$\{(x, 2x + 3) \mid x \text{ integer and } 2 \leq x \leq 8\}.$$

This set notation is an abbreviation for “The set of all points  $(x, 2x + 3)$  such that  $x$  is an integer and  $2 \leq x \leq 8$ .” Notice how the set of ordered pairs generated by the for-next code above,

$$\{(2, 7), (3, 9), (4, 11), (5, 13), (6, 15), (7, 17), (8, 19)\},$$

also satisfies the requirements described by  $\{(x, 2x + 3) \mid x \text{ integer, } 2 \leq x \leq 8\}$ . It is for this reason that the set notation of the form

$$\{\text{type of element} \mid \text{condition on each element}\}$$

is sometimes called *set-builder notation*—because it can be thought of as building the set just like the for-next code.

## Discussion

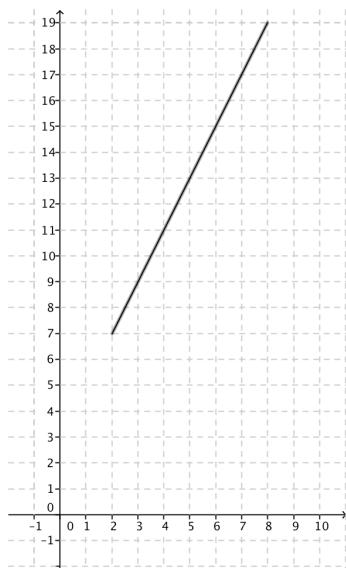
We can now upgrade our notion of a for-next loop by doing a thought experiment: Imagine a for-next loop that steps through *all* real numbers in an interval (not just the integers). No computer can actually do this—computers can only do a finite number of calculations. But our human brains are far superior to that of any computer, and we can easily imagine what that might look like. Here is some sample code:

```

Declare x real
Let f(x) = 2x + 3
Initialize G as {}
For all x such that 2 ≤ x ≤ 8
    Append (x, f(x)) to G
Next x
Plot G

```

The output of this thought code is the graph of  $f$  for all real numbers  $x$  in the interval  $2 \leq x \leq 8$ :

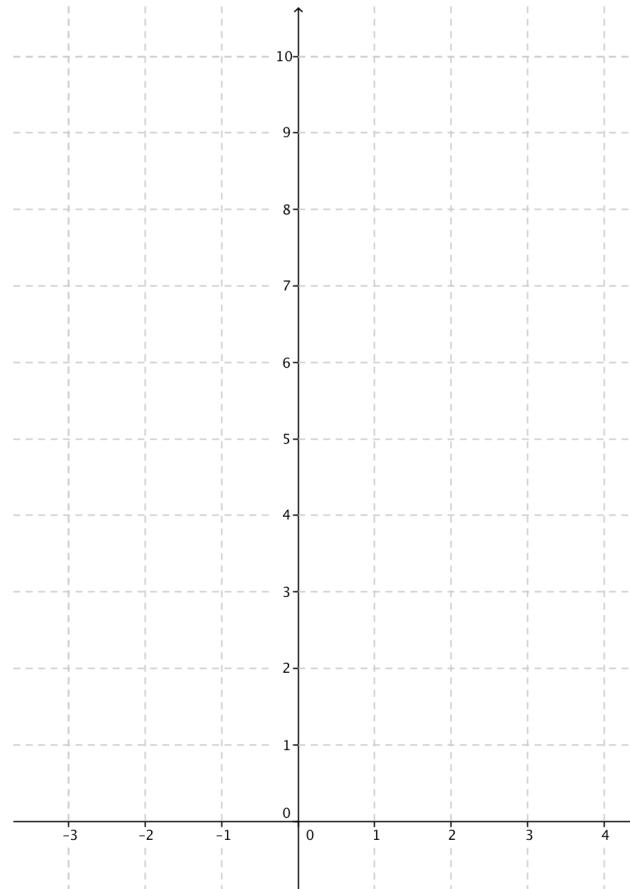


**Exercise 3**

- a. Plot the function  $f$  on the Cartesian plane using the following for-next thought code.

```

Declare x real
Let f(x) = x2 + 1
Initialize G as {}
For all x such that -2 ≤ x ≤ 3
    Append (x, f(x)) to G
Next x
Plot G
  
```



- b. For each step of the for-next loop, what is the input value?
- c. For each step of the for-next loop, what is the output value?
- d. What is the domain of the function  $f$ ?
- e. What is the range of the function  $f$ ?

**Closing**

The set  $G$  built from the for-next thought code in Exercise 4 can also be compactly written in mathematics using set notation:

$$\{(x, x^2 + 1) \mid x \text{ real}, -2 \leq x \leq 3\}.$$

When this set is thought of as plotted in the Cartesian plane, it is the same graph. When you see this set notation in the Problem Set and/or future studies, it is helpful to imagine this set-builder notation as describing a for-next loop.

In general, if  $f: D \rightarrow Y$  is a function with domain  $D$ , then its *graph* is the set of all ordered pairs,

$$\{(x, f(x)) \mid x \in D\},$$

thought of as a geometric figure in the Cartesian coordinate plane. (The symbol  $\in$  simply means “in.” The statement  $x \in D$  is read, “ $x$  in  $D$ .“)

**Lesson Summary**

**GRAPH OF  $f$ :** Given a function  $f$  whose domain  $D$  and range are subsets of the real numbers, the graph of  $f$  is the set of ordered pairs in the Cartesian plane given by

$$\{(x, f(x)) \mid x \in D\}.$$

**Problem Set**

1. Perform the instructions for each of the following programming codes as if you were a computer and your paper was the computer screen.

a.

```
Declare x integer
For all x from 0 to 4
    Print 2x
Next x
```

b.

```
Declare x integer
For all x from 0 to 10
    Print 2x + 1
Next x
```

c.

```
Declare x integer
For all x from 2 to 8
    Print x2
Next x
```

d.

```
Declare x integer
For all x from 0 to 4
    Print 10 · 3x
Next x
```

2. Perform the instructions for each of the following programming codes as if you were a computer and your paper were the computer screen.

a.

```

Declare x integer
Let f(x) = (x + 1)(x - 1) - x2
Initialize G as {}
For all x from -3 to 3
    Append (x, f(x)) to G
Next x
Plot G

```

b.

```

Declare x integer
Let f(x) = 3-x
Initialize G as {}
For all x from -3 to 3
    Append (x, f(x)) to G
Next x
Plot G

```

c.

```

Declare x real
Let f(x) = x3
Initialize G as {}
For all x such that -2 ≤ x ≤ 2
    Append (x, f(x)) to G
Next x
Plot G

```

3. Answer the following questions about the thought code:

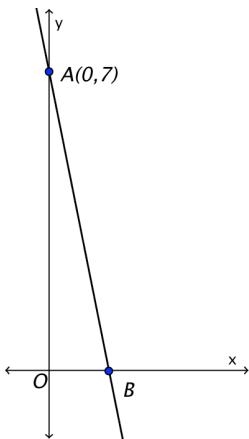
```

Declare x real
Let f(x) = (x - 2)(x - 4)
Initialize G as {}
For all x such that 0 ≤ x ≤ 5
    Append (x, f(x)) to G
Next x
Plot G

```

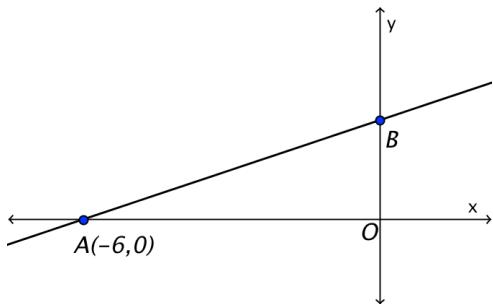
- What is the domain of the function  $f$ ?
  - Plot the graph of  $f$  according to the instructions in the thought code.
  - Look at your graph of  $f$ . What is the range of  $f$ ?
  - Write three or four sentences describing in words how the thought code works.
4. Sketch the graph of the functions defined by the following formulas, and write the graph of  $f$  as a set using set-builder notation. (Hint: Assume the domain is all real numbers unless specified in the problem.)
- $f(x) = x + 2$
  - $f(x) = 3x + 2$
  - $f(x) = 3x - 2$
  - $f(x) = -3x - 2$
  - $f(x) = -3x + 2$
  - $f(x) = -\frac{1}{3}x + 2, -3 \leq x \leq 3$
  - $f(x) = (x + 1)^2 - x^2, -2 \leq x \leq 5$
  - $f(x) = (x + 1)^2 - (x - 1)^2, -2 \leq x \leq 4$

5. The figure shows the graph of  $f(x) = -5x + c$ .



- Find the value of  $c$ .
- If the graph of  $f$  intersects the  $x$ -axis at  $B$ , find the coordinates of  $B$ .

6. The figure shows the graph of  $f(x) = \frac{1}{2}x + c$ .



- Find the value of  $c$ .
- If the graph of  $f$  intersects the  $y$ -axis at  $B$ , find the coordinates of  $B$ .
- Find the area of triangle  $AOB$ .

## Lesson 12: The Graph of the Equation $y = f(x)$

### Classwork

In Module 1, you graphed equations such as  $4x + y = 10$  by plotting the points on the Cartesian coordinate plane that corresponded to all of the ordered pairs of numbers  $(x, y)$  that were in the solution set. We called the geometric figure that resulted from plotting those points in the plane the *graph of the equation in two variables*.

In this lesson, we extend this notion of the graph of an equation to the graph of  $y = f(x)$  for a function  $f$ . In doing so, we use computer thought code to describe the process of generating the ordered pairs in the graph of  $y = f(x)$ .

### Example 1

In the previous lesson, we studied a simple type of instruction that computers perform called a for-next loop. Another simple type of instruction is an *if-then statement*. Below is example code of a program that tests for and prints “True” when  $x + 2 = 4$ ; otherwise it prints “False.”

```

Declare x integer
For all x from 1 to 4
    If x + 2 = 4 then
        Print True
    else
        Print False
    End if
Next x

```

The output of this program code is

False  
True  
False  
False

Notice that the if-then statement in the code above is really just testing whether each number in the loop is in the solution set.

**Example 2**

Perform the instructions in the following programming code as if you were a computer and your paper were the computer screen.

```

Declare x integer
Initialize G as {}
For all x from 0 to 4
  If  $x^2 - 4x + 5 = 2$  then
    Append x to G
  else
    Do NOT append x to G
  End if
Next x
Print G
  
```

Output: {1, 3}

**Discussion**

Compare the for-next/if-then code above to the following set-builder notation we used to describe solution sets in Module 1:

$$\{x \text{ integer} \mid 0 \leq x \leq 4 \text{ and } x^2 - 4x + 5 = 2\}.$$

Check to see that the set-builder notation also generates the set {1, 3}. Whenever you see set-builder notation to describe a set, a powerful way to interpret that notation is to think of the set as being generated by a program like the for-next or if-then code above.

**Exploratory Challenge 1**

Next we write code that generates a graph of a *two-variable equation*  $y = x(x - 2)(x + 2)$  for  $x$  in  $\{-2, -1, 0, 1, 2\}$  and  $y$  in  $\{-3, 0, 3\}$ . The solution set of this equation is generated by testing each ordered pair  $(x, y)$  in the set,

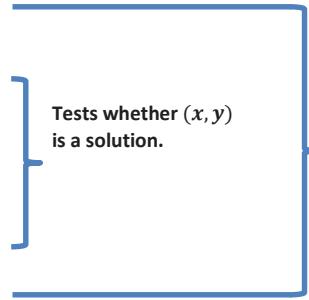
$$\{(-2, -3), (-2, 0), (-2, 3), (-1, -3), (-1, 0), (-1, 3), \dots, (2, -3), (2, 0), (2, 3)\},$$

to see if it is a solution to the equation  $y = x(x - 2)(x + 2)$ . Then the graph is just the plot of solutions in the Cartesian plane. We can instruct a computer to find these points and plot them using the following program.

```

Declare x and y integers
Initialize G as {}
For all x in {-2, -1, 0, 1, 2}
    For all y in {-3, 0, 3}
        If y = x(x - 2)(x + 2) then
            Append (x, y) to G
        else
            Do NOT append (x, y) to G
        End if
    Next y
Next x
Print G
Plot G

```



Loops through each  $y$  for  $x = -2$ , then for  $x = -1$ , and so on (see arrows in table below).

- a. Use the table below to record the decisions a computer would make when following the program instructions above. Fill in each cell with “Yes” or “No” depending on whether the ordered pair  $(x, y)$  would be appended or not. (The step where  $x = -2$  has been done for you.)

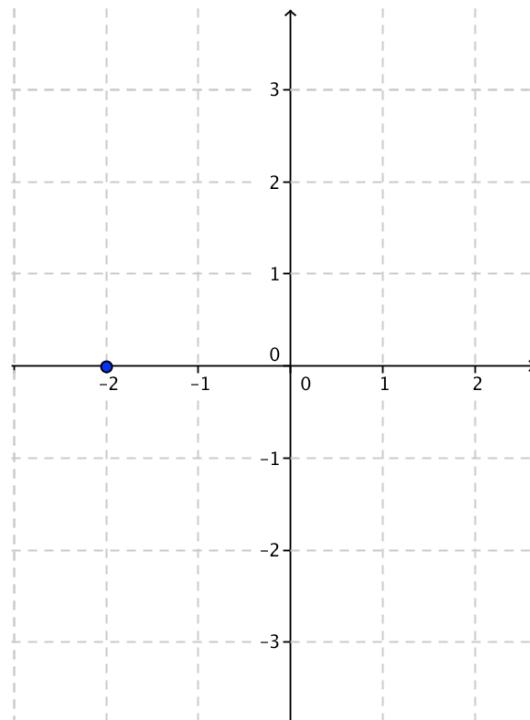
	$x = -2$	$x = -1$	$x = 0$	$x = 1$	$x = 2$
$y = 3$	No				
$y = 0$	Yes				
$y = -3$	No				

- b. What would be the output to the Print  $G$  command? (The first ordered pair is listed for you.)

Output:

{  $(-2, 0)$  , \_\_\_\_\_, \_\_\_\_\_, \_\_\_\_\_, \_\_\_\_\_ }

- c. Plot the solution set  $G$  in the Cartesian plane. (The first ordered pair in  $G$  has been plotted for you.)



### Exploratory Challenge 2

The program code in Exercise 3 is a way to imagine how set-builder notation generates solution sets and figures in the plane. Given a function  $f(x) = x(x - 2)(x + 2)$  with domain and range all real numbers, a slight modification of the program code above can be used to generate the graph of the equation  $y = f(x)$ :

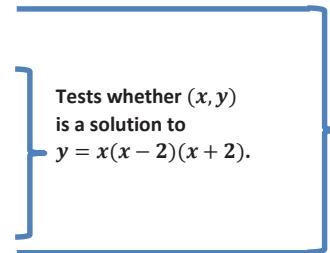
$$\{(x, y) \mid x \text{ real} \text{ and } y = f(x)\}.$$

Even though the code below cannot be run on a computer, students can run the following thought code in their minds.

```

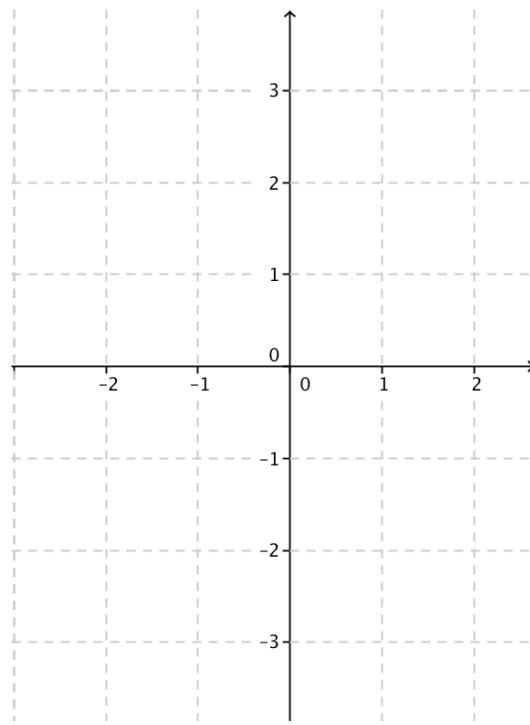
Declare x and y real
Let f(x) = x(x - 2)(x + 2)
Initialize G as {}
For all x in the real numbers
  For all y in the real numbers
    If y = f(x) then
      Append (x, y) to G
    else
      Do NOT append (x, y) to G
    End if
  Next y
Next x
Plot G

```



For each  $x$ -value, the code loops through all  $y$ -values.

- a. Plot  $G$  on the Cartesian plane (the figure drawn is called the graph of  $y = f(x)$ ).



- b. Describe how the thought code is similar to the set-builder notation  $\{(x, y) \mid x \text{ real and } y = f(x)\}$ .
- c. A *relative maximum* for the function  $f$  occurs at the  $x$ -coordinate of  $(-\frac{2}{3}\sqrt{3}, \frac{16}{9}\sqrt{3})$ . Substitute this point into the equation  $y = x(x^2 - 4)$  to check that it is a solution to  $y = f(x)$ , and then plot the point on your graph.

- d. A *relative minimum* for the function  $f$  occurs at the  $x$ -coordinate of  $\left(\frac{2}{3}\sqrt{3}, -\frac{16}{9}\sqrt{3}\right)$ . A similar calculation as you did above shows that this point is also a solution to  $y = f(x)$ . Plot this point on your graph.
- e. Look at your graph. On what interval(s) is the function  $f$  decreasing?
- f. Look at your graph. On what interval(s) is the function  $f$  increasing?

**Lesson Summary**

- **GRAPH OF  $y = f(x)$ :** Given a function  $f$  whose domain  $D$ , and the range are subsets of the real numbers, the *graph of  $y = f(x)$*  is the set of ordered pairs  $(x, y)$  in the Cartesian plane given by

$$\{(x, y) \mid x \in D \text{ and } y = f(x)\}.$$

When we write  $\{(x, y) \mid y = f(x)\}$  for the graph of  $y = f(x)$ , it is understood that the domain is the largest set of real numbers for which the function  $f$  is defined.

- The graph of  $f$  is the same as the graph of the equation  $y = f(x)$ .
- **INCREASING/DECREASING:** Given a function  $f$  whose domain and range are subsets of the real numbers, and  $I$  is an interval contained within the domain, the function is called *increasing on the interval  $I$*  if

$$f(x_1) < f(x_2) \text{ whenever } x_1 < x_2 \text{ in } I.$$

It is called *decreasing on the interval  $I$*  if

$$f(x_1) > f(x_2) \text{ whenever } x_1 < x_2 \text{ in } I.$$

**Problem Set**

1. Perform the instructions in the following programming code as if you were a computer and your paper were the computer screen.

```

Declare x integer
For all x from 1 to 6
    If  $x^2 - 2 = 7$  then
        Print True
    else
        Print False
    End if
Next x
  
```

2. Answer the following questions about the computer programming code.

```

Declare x integer
Initialize G as {}
For all x from -3 to 3
  If  $2^x + 2^{-x} = \frac{17}{4}$  then
    Append x to G
  else
    Do NOT append x to G
  End if
Next x
Print G

```

- a. Perform the instructions in the programming code as if you were a computer and your paper were the computer screen.  
 b. Write a description of the set  $G$  using set-builder notation.
3. Answer the following questions about the computer programming code.

```

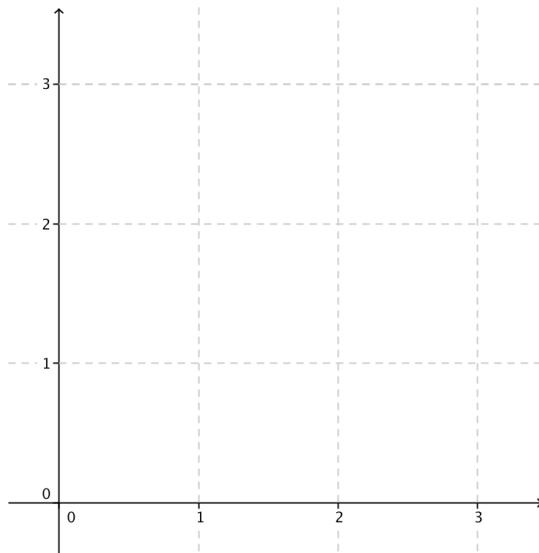
Declare x and y integers
Initialize G as {}
For all x in {0, 1, 2, 3}
  For all y in {0, 1, 2, 3}
    If  $y = \sqrt{4 + 20x - 19x^2 + 4x^3}$  then
      Append (x, y) to G
    else
      Do NOT append (x, y) to G
    End if
  Next y
Next x
Plot G

```

- a. Use the table below to record the decisions a computer would make when following the program instructions above. Fill in each cell with “Yes” or “No” depending on whether the ordered pair  $(x, y)$  would be appended or not.

	$x = 0$	$x = 1$	$x = 2$	$x = 3$
$y = 3$				
$y = 2$				
$y = 1$				
$y = 0$				

- b. Plot the set  $G$  in the Cartesian plane.



4. Answer the following questions about the thought code.

```

Declare x and y real
Let  $f(x) = -2x + 8$ 
Initialize G as {}
For all x in the real numbers
  For all y in the real numbers
    If  $y = f(x)$  then
      Append (x,y) to G
    else
      Do NOT append (x,y) to G
    End if
  Next y
Next x
Plot G
  
```

- What is the domain of the function  $f(x) = -2x + 8$ ?
- What is the range of the function  $f(x) = -2x + 8$ ?
- Write the set  $G$  generated by the thought code in set-builder notation.
- Plot the set  $G$  to obtain the graph of the function  $f(x) = -2x + 8$ .
- The function  $f(x) = -2x + 8$  is clearly a decreasing function on the domain of the real numbers. Show that the function satisfies the definition of decreasing for the points 8 and 10 on the number line; that is, show that since  $8 < 10$ , then  $f(8) > f(10)$ .

5. Sketch the graph of the functions defined by the following formulas, and write the graph of  $y = f(x)$  as a set using set-builder notation. (Hint: For each function below, you can assume the domain is all real numbers.)

- $f(x) = -\frac{1}{2}x + 6$
- $f(x) = x^2 + 3$
- $f(x) = x^2 - 5x + 6$
- $f(x) = x^3 - x$
- $f(x) = -x^2 + x - 1$
- $f(x) = (x - 3)^2 + 2$
- $f(x) = x^3 - 2x^2 + 3$

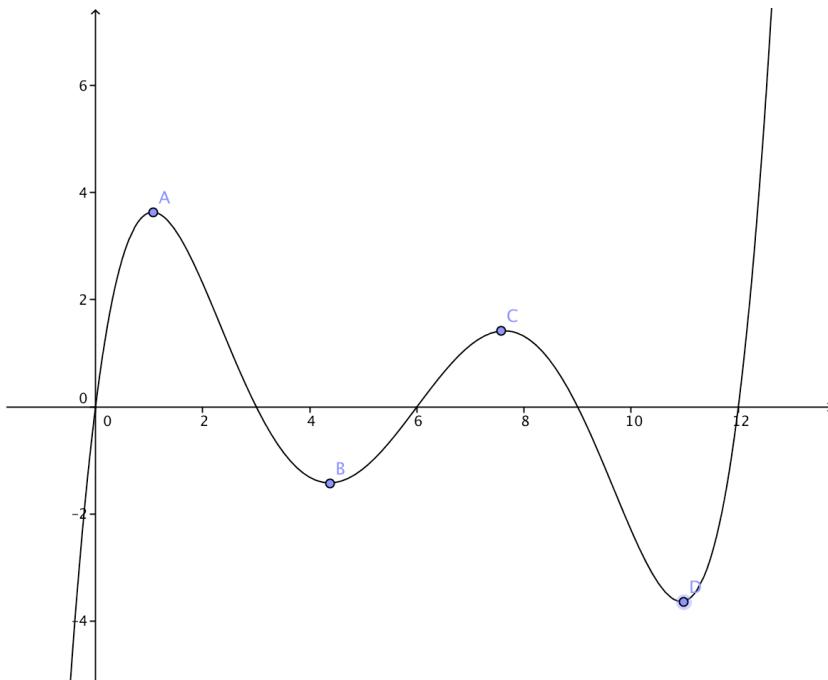
6. Answer the following questions about the set:

$$\{(x, y) \mid 0 \leq x \leq 2 \text{ and } y = 9 - 4x^2\}.$$

- The equation can be rewritten in the form  $y = f(x)$  where  $f(x) = 9 - 4x^2$ . What are the domain and range of the function  $f$  specified by the set?
  - Domain:
  - Range:
- Write thought code such as that in Problem 4 that will generate and then plot the set.

7. Answer the following about the graph of a function below.

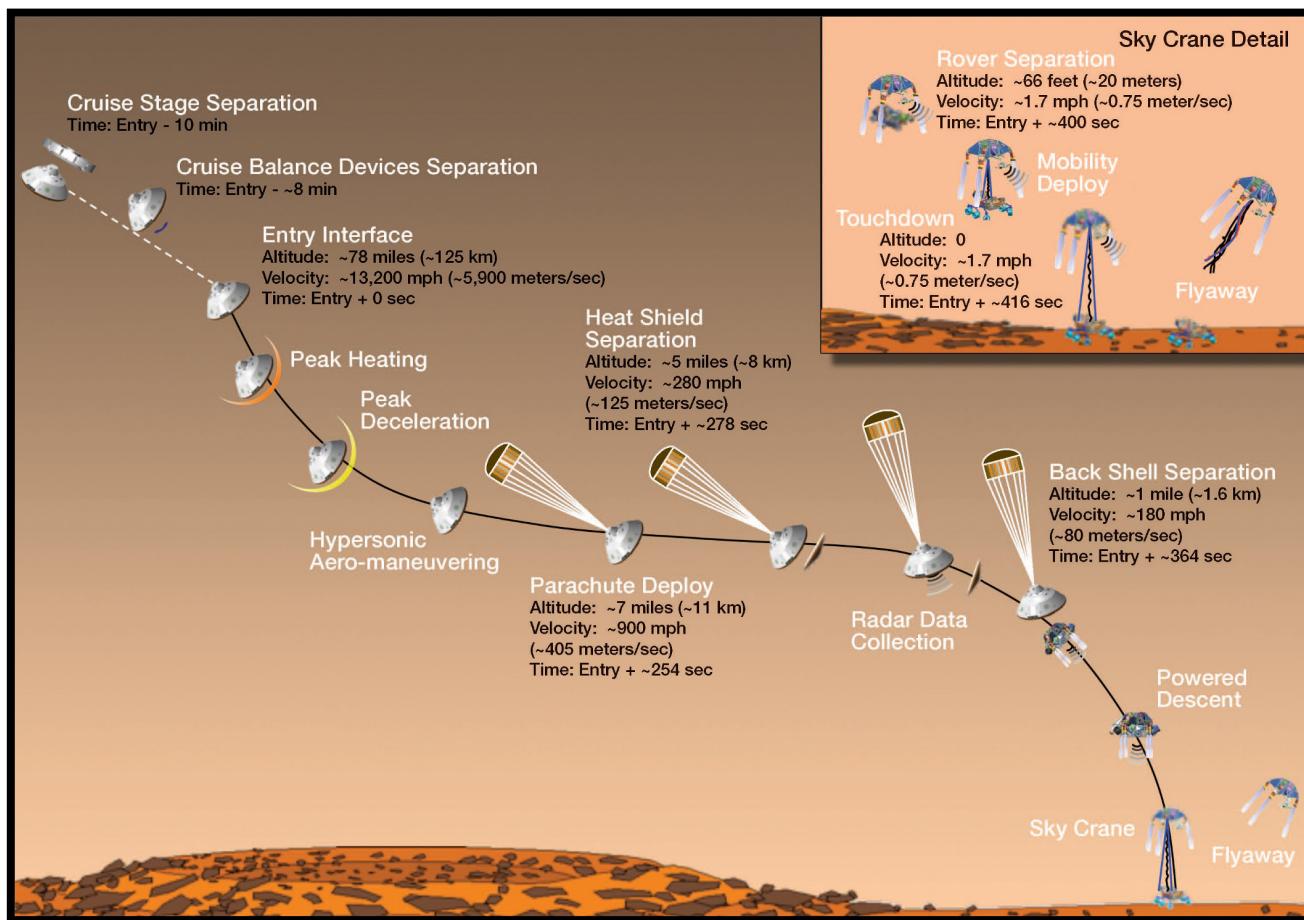
- Which points (A, B, C, or D) are relative maxima?
- Which points (A, B, C, or D) are relative minima?
- Name any interval where the function is increasing.
- Name any interval where the function is decreasing.



## Lesson 13: Interpreting the Graph of a Function

### Classwork

This graphic was shared by NASA prior to the Mars Curiosity Rover landing on August 6, 2012. It depicts the landing sequence for the Curiosity Rover's descent to the surface of the planet.



Courtesy NASA/JPL-Caltech

Does this graphic really represent the landing path of the Curiosity Rover? Create a model that can be used to predict the altitude and velocity of the Curiosity Rover 5, 4, 3, 2, and 1 minute before landing.

**Mathematical Modeling Exercise**

Create a model to help you answer the problem and estimate the altitude and velocity at various times during the landing sequence.

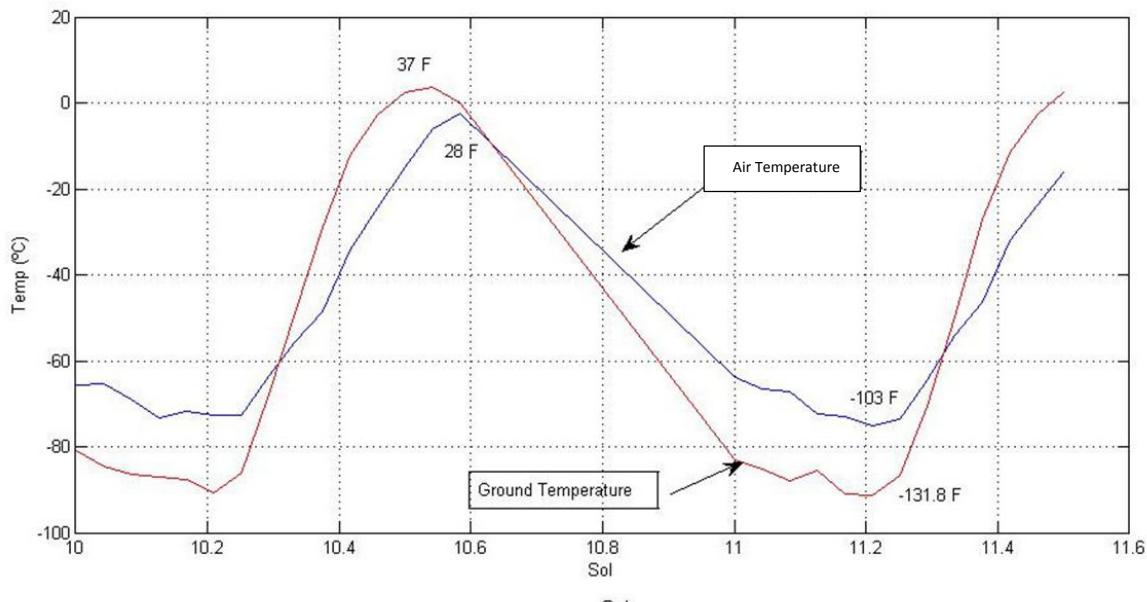
## Exercises

1. Does this graphic really represent the landing path of the Curiosity Rover?
  2. Estimate the altitude and velocity of the Curiosity Rover 5, 4, 3, 2, and 1 minute before landing. Explain how you arrived at your estimate.

3. Based on watching the video/animation, do you think you need to revise any of your work? Explain why or why not, and then make any needed changes.
  4. Why is the graph of the altitude function decreasing and the graph of the velocity function increasing on its domain?
  5. Why is the graph of the velocity function negative? Why does this graph not have a  $t$ -intercept?
  6. What is the meaning of the  $t$ -intercept of the altitude graph? The  $y$ -intercept?

A Mars rover collected the following temperature data over 1.6 Martian days. A Martian day is called a sol. Use the graph to answer the following questions.

# GROUND AND AIR TEMPERATURE SENSOR



7. Approximately when does each graph change from increasing to decreasing? From decreasing to increasing?
  
  8. When is the air temperature increasing?

9. When is the ground temperature decreasing?

10. What is the air temperature change on this time interval?

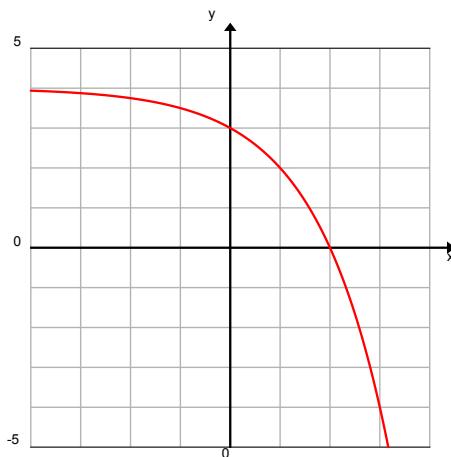
11. Why do you think the ground temperature changed more than the air temperature? Is that typical on earth?

12. Is there a time when the air and ground were the same temperature? Explain how you know.

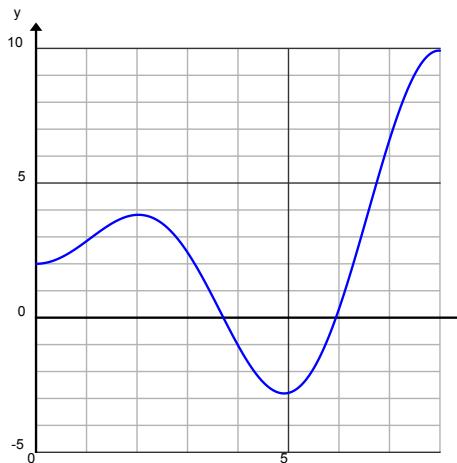
**Problem Set**

1. Create a short written report summarizing your work on the Mars Curiosity Rover Problem. Include your answers to the original problem questions and at least one recommendation for further research on this topic or additional questions you have about the situation.
2. Consider the sky crane descent portion of the landing sequence.
  - a. Create a linear function to model the Curiosity Rover's altitude as a function of time. What two points did you choose to create your function?
  - b. Compare the slope of your function to the velocity. Should they be equal? Explain why or why not.
  - c. Use your linear model to determine the altitude one minute before landing. How does it compare to your earlier estimate? Explain any differences you found.
3. The exponential function  $g(t) = 125(0.99)^t$  could be used to model the altitude of the Curiosity Rover during its rapid descent. Do you think this model would be better or worse than the one your group created? Explain your reasoning.
4. For each graph below, identify the increasing and decreasing intervals, the positive and negative intervals, and the intercepts.

a.



b.



## Lesson 14: Linear and Exponential Models—Comparing Growth

### Rates

#### Classwork

##### Example 1

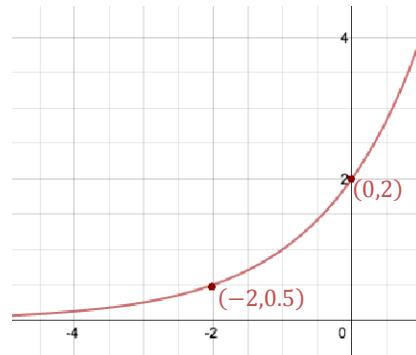
###### Linear Functions

- a. Sketch points  $P_1 = (0,4)$  and  $P_2 = (4,12)$ . Are there values of  $m$  and  $b$  such that the graph of the linear function described by  $f(x) = mx + b$  contains  $P_1$  and  $P_2$ ? If so, find those values. If not, explain why they do not exist.
  
- b. Sketch  $P_1 = (0,4)$  and  $P_2 = (0, -2)$ . Are there values of  $m$  and  $b$  so that the graph of a linear function described by  $f(x) = mx + b$  contains  $P_1$  and  $P_2$ ? If so, find those values. If not, explain why they do not exist.

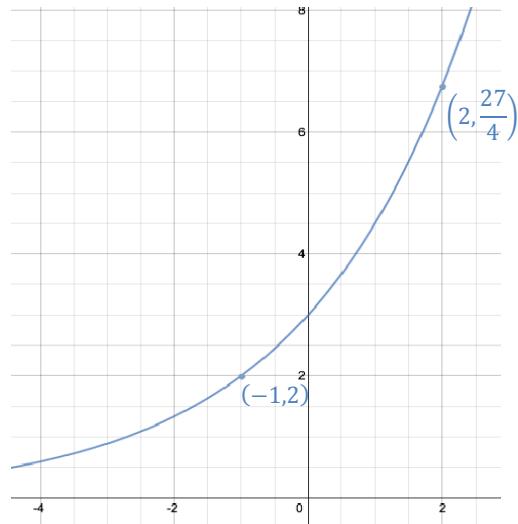
Exponential Functions

Graphs (c) and (d) are both graphs of an exponential function of the form  $g(x) = ab^x$ . Rewrite the function  $g(x)$  using the values for  $a$  and  $b$  that are required for the graph shown to be a graph of  $g$ .

c.  $g(x) =$



d.  $g(x) =$

**Example 2**

A lab researcher records the growth of the population of a yeast colony and finds that the population doubles every hour.

- a. Complete the researcher's table of data:

<b>Hours into study</b>	0	1	2	3	4
<b>Yeast colony population (thousands)</b>	5				

- b. What is the exponential function that models the growth of the colony's population?
- c. Several hours into the study, the researcher looks at the data and wishes there were more frequent measurements. Knowing that the colony doubles every hour, how can the researcher determine the population in half-hour increments? Explain.

- d. Complete the new table that includes half-hour increments.

<b>Hours into study</b>	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
<b>Yeast colony population (thousands)</b>	5						

- e. How would the calculation for the data change for time increments of 20 minutes? Explain.

- f. Complete the new table that includes 20-minute increments.

<b>Hours into study</b>	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$	$\frac{5}{3}$	2
<b>Yeast colony population (thousands)</b>	5						

- g. The researcher's lab assistant studies the data recorded and makes the following claim:

Since the population doubles in 1 hour, then half of that growth happens in the first half hour, and the other half of that growth happens in the second half hour. We should be able to find the population at  $t = \frac{1}{2}$  by taking the average of the populations at  $t = 0$  and  $t = 1$ .

Is the assistant's reasoning correct? Compare this strategy to your work in parts (c) and (e).

### Example 3

A California Population Projection Engineer in 1920 was tasked with finding a model that predicts the state's population growth. He modeled the population growth as a function of time,  $t$  years since 1900. Census data shows that the population in 1900, in thousands, was 1,490. In 1920, the population of the state of California was 3,554 thousand. He decided to explore both a linear and an exponential model.

- Use the data provided to determine the equation of the linear function that models the population growth from 1900–1920.
- Use the data provided and your calculator to determine the equation of the exponential function that models the population growth.

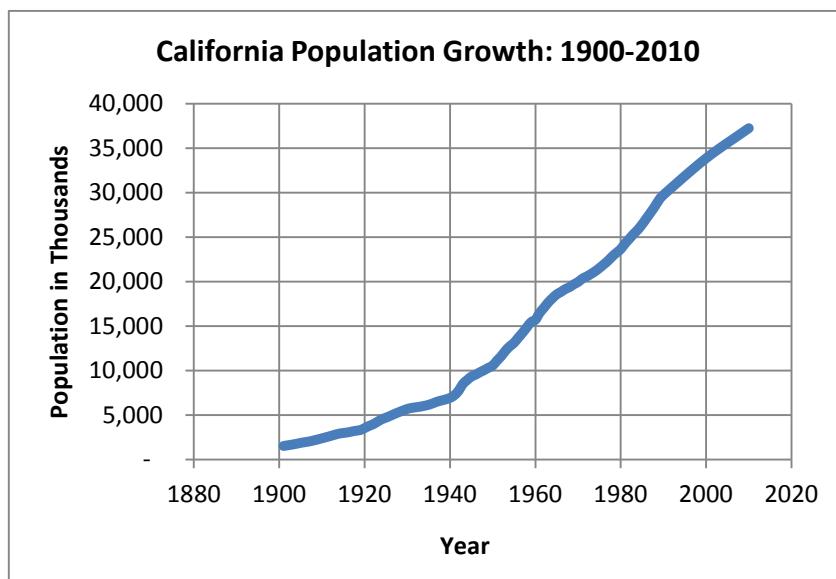
- c. Use the two functions to predict the population for the following years:

	Projected Population Based on Linear Function, $f(t)$ (thousands)	Projected Population Based on Exponential Function, $g(t)$ (thousands)	Census Population Data and Intercensal Estimates for California (thousands)
1935			6175
1960			15717
2010			37253

Courtesy U.S. Census Bureau

- d. Which function is a better model for the population growth of California in 1935 and in 1960?

- e. Does either model closely predict the population for 2010? What phenomenon explains the real population value?



**Lesson Summary**

- Given a linear function of the form  $L(x) = mx + k$  and an exponential function of the form  $E(x) = ab^x$  for  $x$  a real number and constants  $m$ ,  $k$ ,  $a$ , and  $b$ , consider the sequence given by  $L(n)$  and the sequence given by  $E(n)$ , where  $n = 1, 2, 3, 4, \dots$ . Both of these sequences can be written recursively:

$$\begin{aligned}L(n+1) &= L(n) + m \text{ and } L(0) = k, \text{ and} \\E(n+1) &= E(n) \cdot b \text{ and } E(0) = a.\end{aligned}$$

The first sequence shows that a linear function grows additively by the same summand  $m$  over equal length intervals (i.e., the intervals between consecutive integers). The second sequence shows that an exponential function grows multiplicatively by the same factor  $b$  over equal-length intervals (i.e., the intervals between consecutive integers).

- An increasing exponential function eventually exceeds any linear function. That is, if  $f(x) = ab^x$  is an exponential function with  $a > 0$  and  $b > 1$ , and  $g(x) = mx + k$  is a linear function, then there is a real number  $M$  such that for all  $x > M$ , then  $f(x) > g(x)$ . Sometimes this is not apparent in a graph displayed on a graphing calculator; that is because the graphing window does not show enough of the graphs for us to see the sharp rise of the exponential function in contrast with the linear function.

**Problem Set**

- When a ball bounces up and down, the maximum height it reaches decreases with each bounce in a predictable way. Suppose for a particular type of squash ball dropped on a squash court, the maximum height,  $h(x)$ , after  $x$  number of bounces can be represented by  $h(x) = 65\left(\frac{1}{3}\right)^x$ .
  - How many times higher is the height after the first bounce compared to the height after the third bounce?
  - Graph the points  $(x, h(x))$  for  $x$ -values of 0, 1, 2, 3, 4, and 5.
- Australia experienced a major pest problem in the early 20<sup>th</sup> century. The pest? Rabbits. In 1859, 24 rabbits were released by Thomas Austin at Barwon Park. In 1926, there were an estimated 10 billion rabbits in Australia. Needless to say, the Australian government spent a tremendous amount of time and money to get the rabbit problem under control. (To find more on this topic, visit Australia's Department of Environment and Primary Industries website under Agriculture.)
  - Based only on the information above, write an exponential function that would model Australia's rabbit population growth.
  - The model you created from the data in the problem is obviously a huge simplification from the actual function of the number of rabbits in any given year from 1859 to 1926. Name at least one complicating factor (about rabbits) that might make the graph of your function look quite different than the graph of the actual function.

3. After graduating from college, Jane has two job offers to consider. Job A is compensated at \$100,000 a year but with no hope of ever having an increase in pay. Jane knows a few of her peers are getting that kind of an offer right out of college. Job B is for a social media start-up, which guarantees a mere \$10,000 a year. The founder is sure the concept of the company will be the next big thing in social networking and promises a pay increase of 25% at the beginning of each new year.
- Which job will have a greater annual salary at the beginning of the fifth year? By approximately how much?
  - Which job will have a greater annual salary at the beginning of the tenth year? By approximately how much?
  - Which job will have a greater annual salary at the beginning of the twentieth year? By approximately how much?
  - If you were in Jane's shoes, which job would you take?
4. The population of a town in 2007 was 15,000 people. The town has gotten its fresh water supply from a nearby lake and river system with the capacity to provide water for up to 30,000 people. Due to its proximity to a big city and a freeway, the town's population has begun to grow more quickly than in the past. The table below shows the population counts for each year from 2007–2012.
- Write a function of  $x$  that closely matches these data points for  $x$ -values of 0, 1, 2, 3, 4, and 5.

Year	Years Past 2007	Population of the town
2007	0	15,000
2008	1	15,600
2009	2	16,224
2010	3	16,873
2011	4	17,548
2012	5	18,250

- Assume the function is a good model for the population growth from 2012–2032. At what year during the time frame 2012–2032 will the water supply be inadequate for the population?

## Lesson 15: Piecewise Functions

### Classwork

#### Opening Exercise

For each real number  $a$ , the *absolute value* of  $a$  is the distance between 0 and  $a$  on the number line and is denoted  $|a|$ .

1. Solve each one variable equation.

a.  $|x| = 6$       b.  $|x - 5| = 4$       c.  $2|x + 3| = -10$

2. Determine at least five solutions for each two-variable equation. Make sure some of the solutions include negative values for either  $x$  or  $y$ .

a.  $y = |x|$

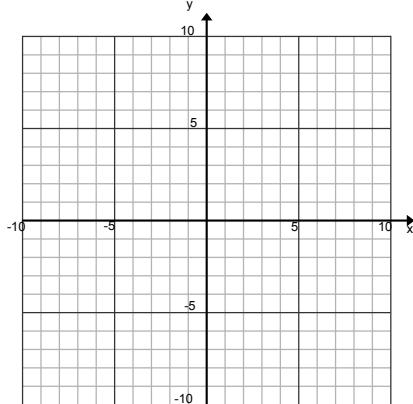
b.  $y = |x - 5|$

c.  $x = |y|$

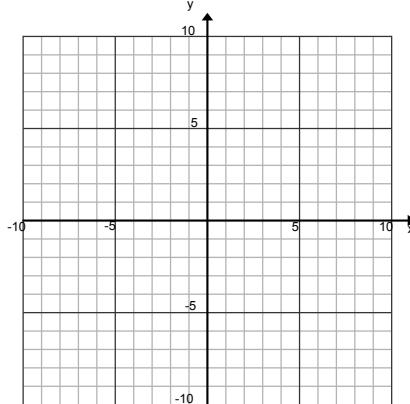
#### Exploratory Challenge 1

For parts (a)–(c) create graphs of the solution set of each two-variable equation from Opening Exercise 2.

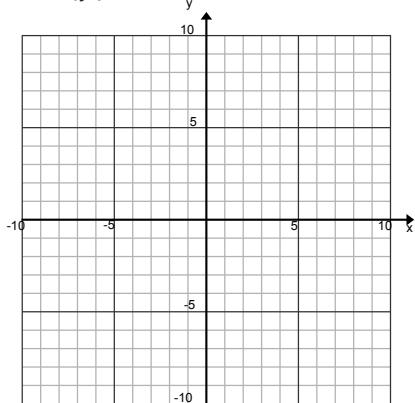
a.  $y = |x|$



b.  $y = |x - 5|$



c.  $x = |y|$



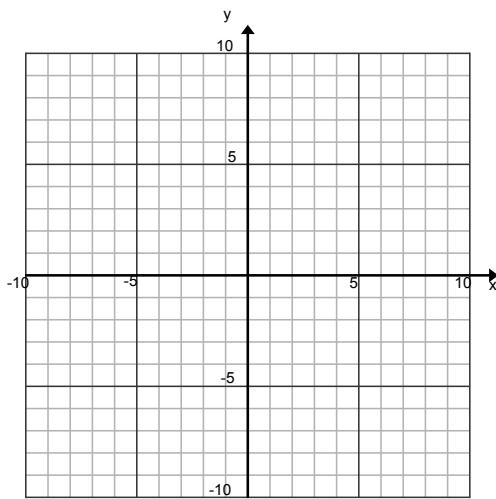
- d. Write a brief summary comparing and contrasting the three solution sets and their graphs.

For parts (e)–(j), consider the function  $f(x) = |x|$ , where  $x$  can be any real number.

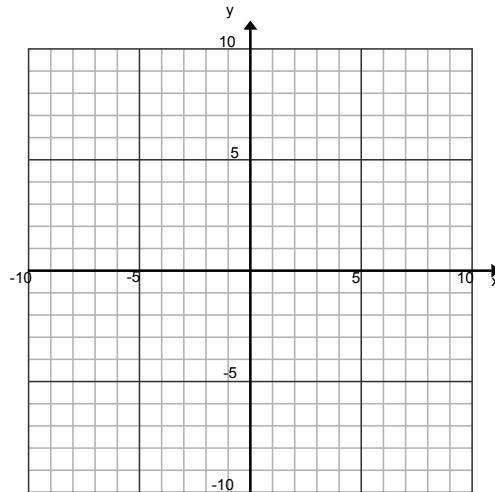
- e. Explain the meaning of the function  $f$  in your own words.

- f. State the domain and range of this function.

- g. Create a graph of the function  $f$ . You might start by listing several ordered pairs that represent the corresponding domain and range elements.



- h. How does the graph of the absolute value function compare to the graph of  $y = |x|$ ?
- i. Define a function whose graph would be identical to the graph of  $y = |x - 5|$ .
- j. Could you define a function whose graph would be identical to the graph of  $x = |y|$ ? Explain your reasoning.
- k. Let  $f_1(x) = -x$  for  $x < 0$ , and let  $f_2(x) = x$  for  $x \geq 0$ . Graph the functions  $f_1$  and  $f_2$  on the same Cartesian plane. How does the graph of these two functions compare to the graph in part (g)?



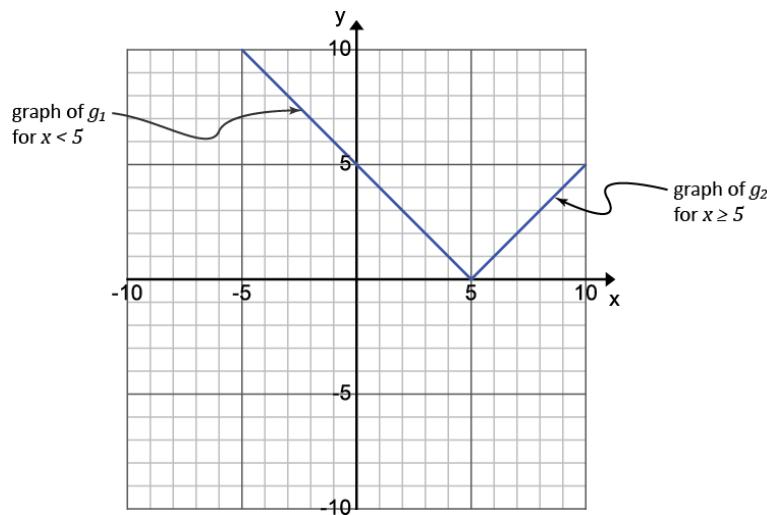
Definition:

The *absolute value function*  $f$  is defined by setting  $f(x) = |x|$  for all real numbers. Another way to write  $f$  is as a piecewise linear function:

$$f(x) = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$$

**Example 1**

Let  $g(x) = |x - 5|$ . The graph of  $g$  is the same as the graph of the equation  $y = |x - 5|$  you drew in Exploratory Challenge 1, part (b). Use the redrawn graph below to rewrite the function  $g$  as a piecewise function.



Label the graph of the linear function with negative slope by  $g_1$  and the graph of the linear function with positive slope by  $g_2$ , as in the picture above.

Function  $g_1$ : The slope of  $g_1$  is  $-1$  (why?), and the  $y$ -intercept is  $5$ ; therefore,  $g_1(x) = -x + 5$ .

Function  $g_2$ : The slope of  $g_2$  is  $1$  (why?), and the  $y$ -intercept is  $-5$  (why?); therefore,  $g_2(x) = x - 5$ .

Writing  $g$  as a piecewise function is just a matter of collecting all of the different “pieces” and the intervals upon which they are defined:

$$g(x) = \begin{cases} -x + 5 & x < 5 \\ x - 5 & x \geq 5 \end{cases}$$

**Exploratory Challenge 2**

The *floor* of a real number  $x$ , denoted by  $\lfloor x \rfloor$ , is the largest integer not greater than  $x$ . The *ceiling* of a real number  $x$ , denoted by  $\lceil x \rceil$ , is the smallest integer not less than  $x$ . The *sawtooth* number of a positive number is the *fractional part* of the number that is to the right of its floor on the number line. In general, for a real number  $x$ , the sawtooth number of  $x$  is the value of the expression  $x - \lfloor x \rfloor$ . Each of these expressions can be thought of as functions with the domain being the set of real numbers.

- a. Complete the following table to help you understand how these functions assign elements of the domain to elements of the range. The first and second rows have been done for you.

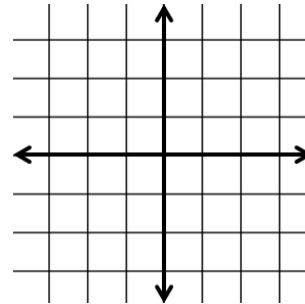
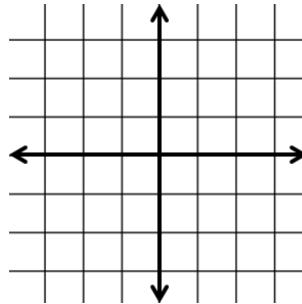
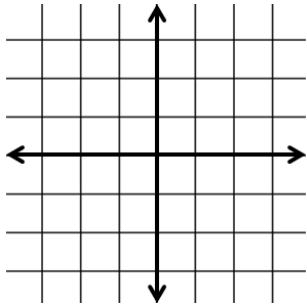
$x$	$\text{floor}(x) = \lfloor x \rfloor$	$\text{ceiling}(x) = \lceil x \rceil$	$\text{sawtooth}(x) = x - \lfloor x \rfloor$
4.8	4	5	0.8
-1.3	-2	-1	0.7
2.2			
6			
-3			
$-\frac{2}{3}$			
$\pi$			

- b. Create a graph of each function.

$$\text{floor}(x) = \lfloor x \rfloor$$

$$\text{ceiling}(x) = \lceil x \rceil$$

$$\text{sawtooth}(x) = x - \lfloor x \rfloor$$



- c. For the floor, ceiling, and sawtooth functions, what would be the range values for all real numbers  $x$  on the interval  $[0,1)$ ? The interval  $(1,2]$ ? The interval  $[-2, -1)$ ? The interval  $[1.5, 2.5]$ ?

**Relevant Vocabulary**

**PIECEWISE LINEAR FUNCTION:** Given a number of nonoverlapping intervals on the real number line, a (*real*) *piecewise linear function* is a function from the union of the intervals to the set of real numbers such that the function is defined by (possibly different) linear functions on each interval.

**ABSOLUTE VALUE FUNCTION:** The absolute value of a number  $x$ , denoted by  $|x|$ , is the distance between 0 and  $x$  on the number line. The *absolute value function* is the piecewise linear function such that for each real number  $x$ , the value of the function is  $|x|$ .

We often name the absolute value function by saying, “Let  $f(x) = |x|$  for all real numbers  $x$ .”

**FLOOR FUNCTION:** The *floor* of a real number  $x$ , denoted by  $\lfloor x \rfloor$ , is the largest integer not greater than  $x$ . The *floor function* is the piecewise linear function such that for each real number  $x$ , the value of the function is  $\lfloor x \rfloor$ .

We often name the floor function by saying, “Let  $f(x) = \lfloor x \rfloor$  for all real numbers  $x$ .”

**CEILING FUNCTION:** The *ceiling* of a real number  $x$ , denoted by  $\lceil x \rceil$ , is the smallest integer not less than  $x$ . The *ceiling function* is the piecewise linear function such that for each real number  $x$ , the value of the function is  $\lceil x \rceil$ .

We often name the ceiling function by saying, “Let  $f(x) = \lceil x \rceil$  for all real numbers  $x$ .”

**SAWTOOTH FUNCTION:** The *sawtooth function* is the piecewise linear function such that for each real number  $x$ , the value of the function is given by the expression  $x - \lfloor x \rfloor$ .

The sawtooth function assigns to each positive number the part of the number (the non-integer part) that is to the right of the floor of the number on the number line. That is, if we let  $f(x) = x - \lfloor x \rfloor$  for all real numbers  $x$ , then

$$f\left(\frac{1}{3}\right) = \frac{1}{3}, f\left(1\frac{1}{3}\right) = \frac{1}{3}, f(1,000.02) = 0.02, f(-0.3) = 0.7, \text{etc.}$$

**Problem Set**

- Explain why the sawtooth function,  $sawtooth(x) = x - \lfloor x \rfloor$  for all real numbers  $x$ , takes only the fractional part of a number when the number is positive.
- Let  $g(x) = \lceil x \rceil - \lfloor x \rfloor$ , where  $x$  can be any real number. In otherwords,  $g$  is the difference between the ceiling and floor functions. Express  $g$  as a piecewise function.
- The Heaviside function is defined using the formula below.

$$H(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

Graph this function, and state its domain and range.

- The following piecewise function is an example of a step function.

$$S(x) = \begin{cases} 3 & -5 \leq x < -2 \\ 1 & -2 \leq x < 3 \\ 2 & 3 \leq x \leq 5 \end{cases}$$

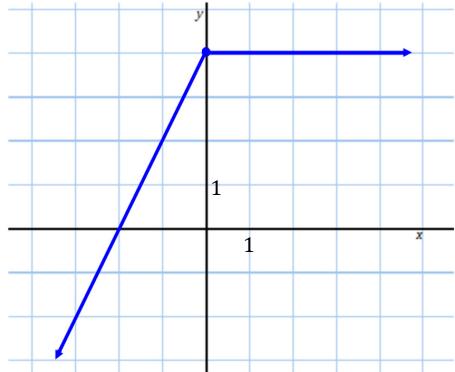
- Graph this function, and state the domain and range.
  - Why is this type of function called a step function?
- Let  $f(x) = \frac{|x|}{x}$ , where  $x$  can be any real number except 0.
    - Why is the number 0 excluded from the domain of  $f$ ?
    - What is the range of  $f$ ?
    - Create a graph of  $f$ .
    - Express  $f$  as a piecewise function.
    - What is the difference between this function and the Heaviside function?

- Graph the following piecewise functions for the specified domain.

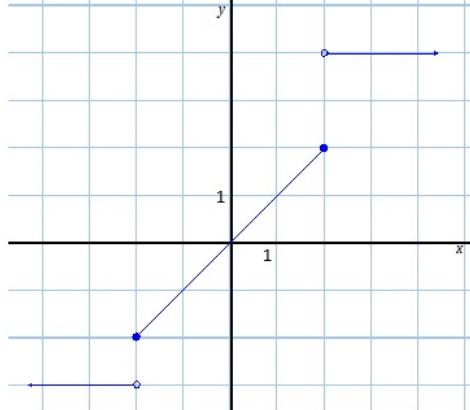
- $f(x) = |x + 3|$  for  $-5 \leq x \leq 3$
- $f(x) = |2x|$  for  $-3 \leq x \leq 3$
- $f(x) = |2x - 5|$  for  $0 \leq x \leq 5$
- $f(x) = |3x + 1|$  for  $-2 \leq x \leq 2$
- $f(x) = |x| + x$  for  $-5 \leq x \leq 3$
- $f(x) = \begin{cases} x & \text{if } x \leq 0 \\ x + 1 & \text{if } x > 0 \end{cases}$
- $f(x) = \begin{cases} 2x + 3 & \text{if } x < -1 \\ 3 - x & \text{if } x \geq -1 \end{cases}$

7. Write a piecewise function for each graph below.

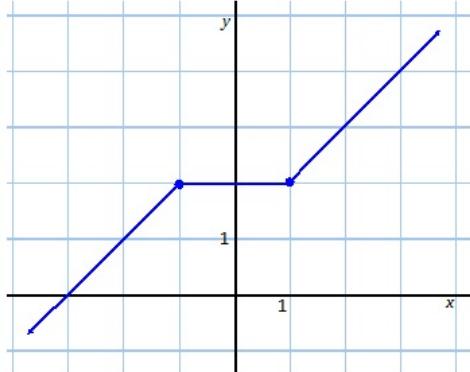
a.

Graph of  $b$ 

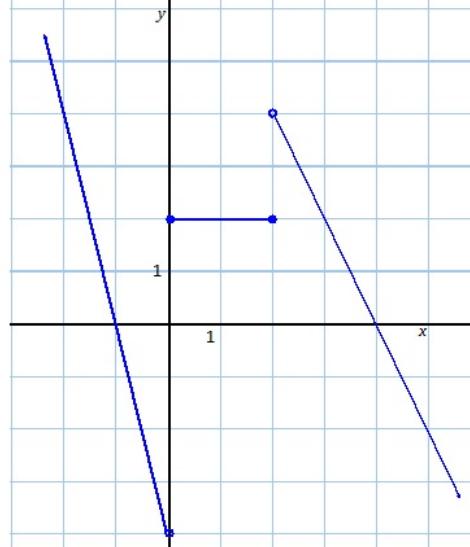
b.

Graph of  $p$ 

c.

Graph of  $k$ 

d.

Graph of  $h$

## Lesson 16: Graphs Can Solve Equations Too

### Classwork

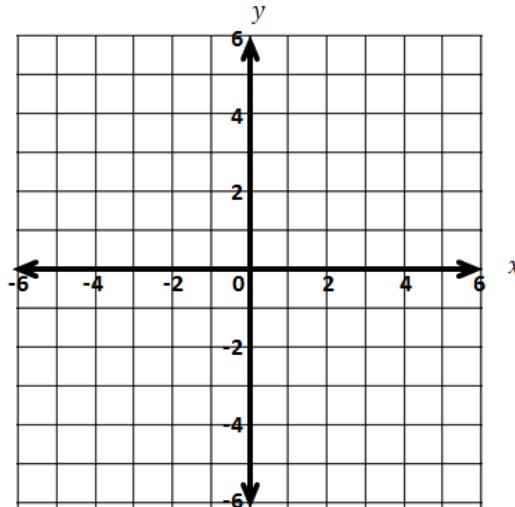
#### Opening Exercise

1. Solve for  $x$  in the following equation:  $|x + 2| - 3 = 0.5x + 1$ .

2. Now, let  $f(x) = |x + 2| - 3$  and  $g(x) = 0.5x + 1$ .

When does  $f(x) = g(x)$ ?

- Graph  $y = f(x)$  and  $y = g(x)$  on the same set of axes.
- When does  $f(x) = g(x)$ ? What is the visual significance of the points where  $f(x) = g(x)$ ?



- Is each intersection point  $(x, y)$  an element of the graph  $f$  and an element of the graph of  $g$ ? In other words, do the functions  $f$  and  $g$  really have the same value when  $x = 4$ ? What about when  $x = -4$ ?

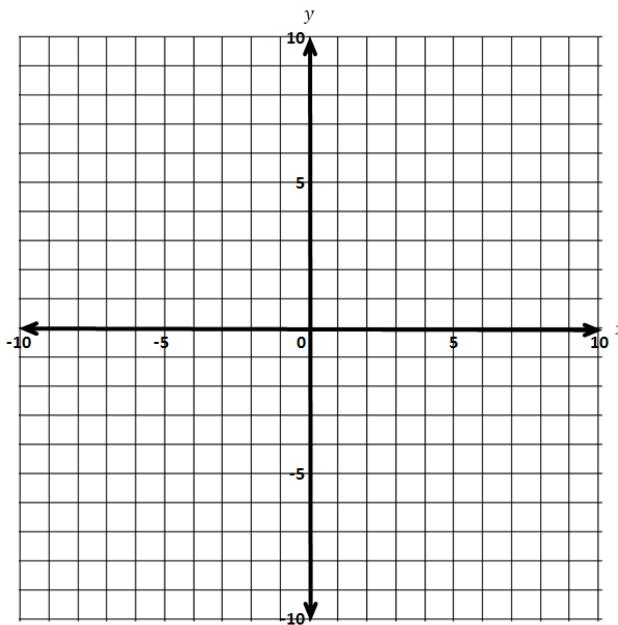
**Example 1**

Solve this equation by graphing two functions on the same Cartesian plane:  $-|x - 3| + 4 = |0.5x| - 5$ .

Let  $f(x) = -|x - 3| + 4$ , and let  $g(x) = |0.5x| - 5$ , where  $x$  can be any real number.

We are looking for values of  $x$  at which the functions  $f$  and  $g$  have the same output value.

Therefore, we set  $y = f(x)$  and  $y = g(x)$ , so we can plot the graphs on the same coordinate plane:



From the graph, we see that the two intersection points are \_\_\_\_\_ and \_\_\_\_\_.

The fact that the graphs of the functions meet at these two points means that when  $x$  is \_\_\_\_\_, both  $f(x)$  and  $g(x)$  are \_\_\_\_\_, or when  $x$  is \_\_\_\_\_, both  $f(x)$  and  $g(x)$  are \_\_\_\_\_.

Thus, the expressions  $-|x - 3| + 4$  and  $|0.5x| - 5$  are equal when  $x =$  \_\_\_\_\_ or when  $x =$  \_\_\_\_\_.

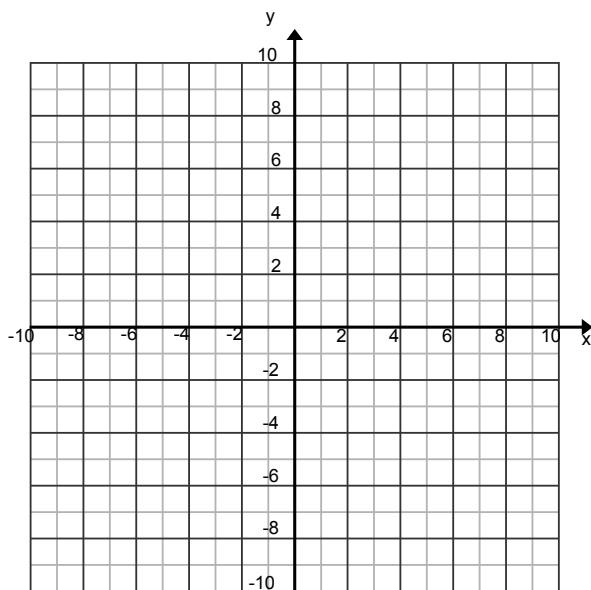
Therefore, the solution set to the original equation is \_\_\_\_\_.

**Example 2**

Solve this equation graphically:  $-|x - 3.5| + 4 = -0.25x - 1$ .

- a. Write the two functions represented by each side of the equation.

- b. Graph the functions in an appropriate viewing window.



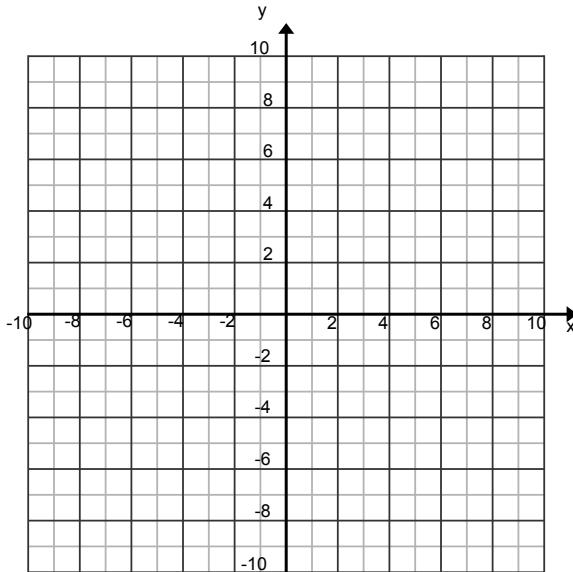
- c. Determine the intersection points of the two functions.

- d. Verify that the  $x$ -coordinates of the intersection points are solutions to the equation.

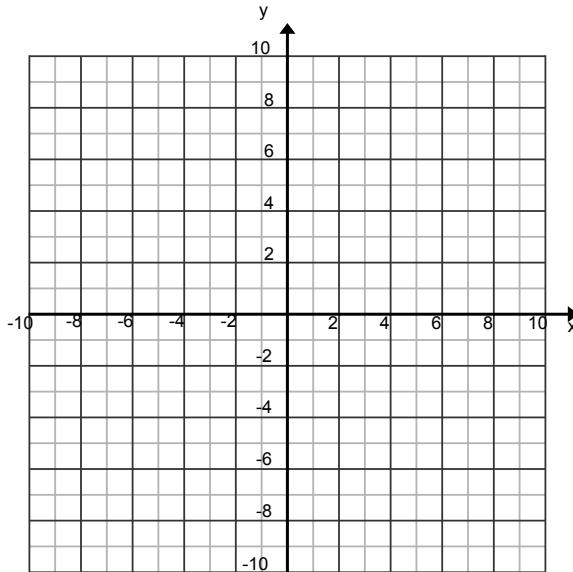
**Exercises 1–5**

Use graphs to find approximate values of the solution set for each equation. Use technology to support your work. Explain how each of your solutions relates to the graph. Check your solutions using the equation.

1.  $3 - 2x = |x - 5|$



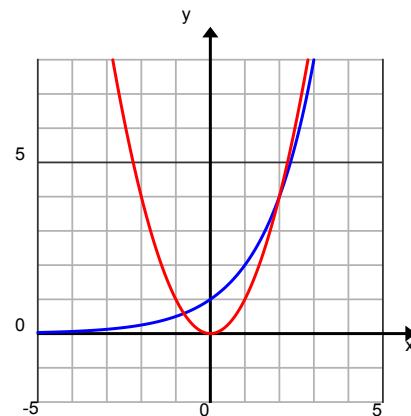
2.  $2(1.5)^x = 2 + 1.5x$



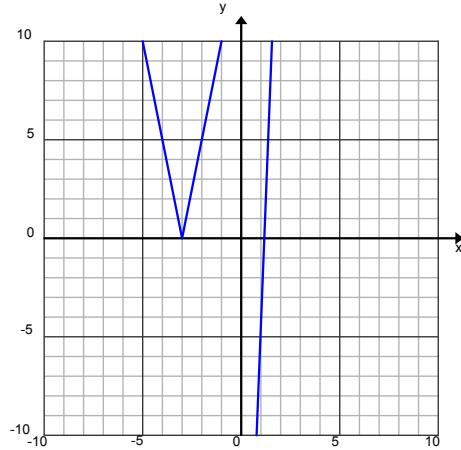
3. The graphs of the functions
- $f$
- and
- $g$
- are shown.

- a. Use the graphs to
- approximate*
- the solution(s) to the equation
- $f(x) = g(x)$
- .

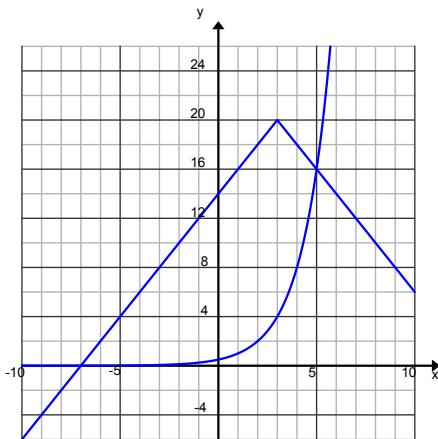
- b. Let
- $f(x) = x^2$
- , and let
- $g(x) = 2^x$
- . Find
- all*
- solutions to the equation
- $f(x) = g(x)$
- . Verify any exact solutions that you determine using the definitions of
- $f$
- and
- $g$
- . Explain how you arrived at your solutions.



4. The graphs of  $f$ , a function that involves taking an absolute value, and  $g$ , a linear function, are shown to the right. Both functions are defined over all real values for  $x$ . Tami concluded that the equation  $f(x) = g(x)$  has no solution. Do you agree or disagree? Explain your reasoning.



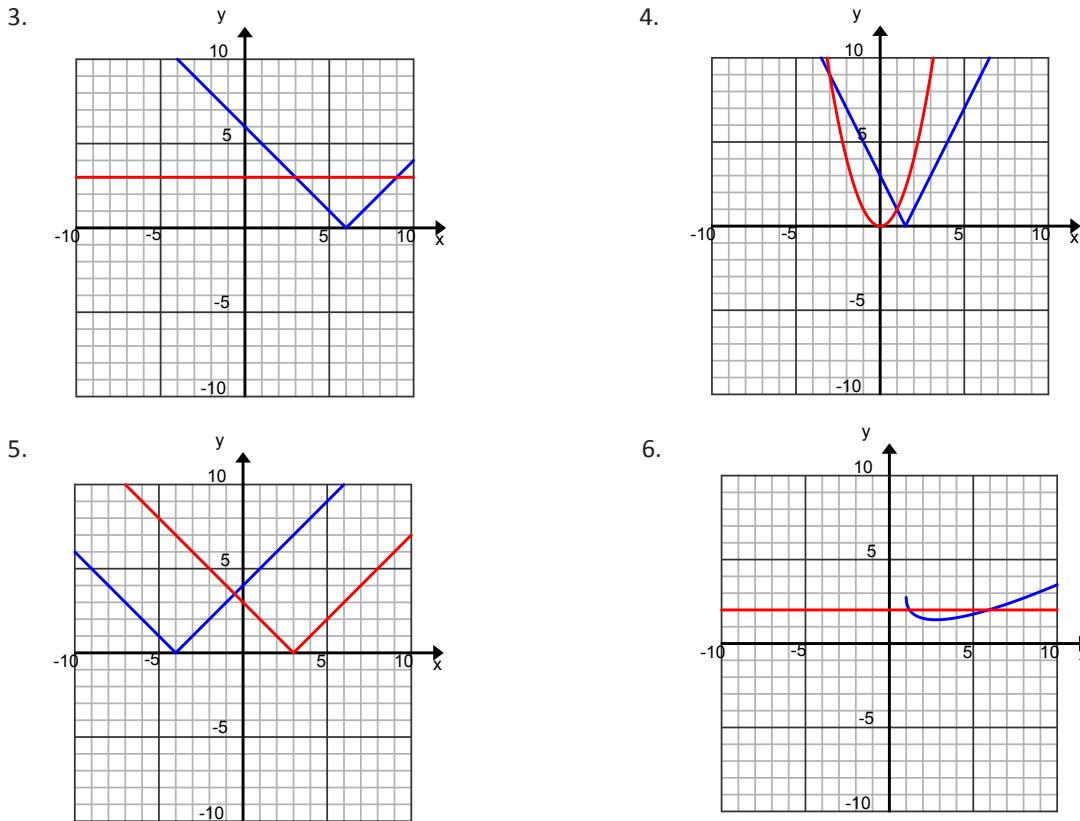
5. The graphs of  $f$  (a function that involves taking the absolute value) and  $g$  (an exponential function) are shown below. Sharon said the solution set to the equation  $f(x) = g(x)$  is exactly  $\{-7, 5\}$ . Do you agree or disagree with Sharon? Explain your reasoning.



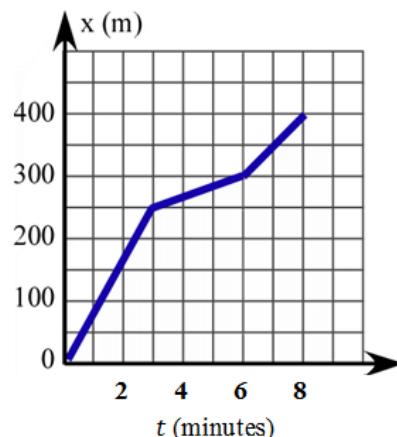
**Problem Set**

- Solve the following equations graphically. Verify the solution sets using the original equations.
  - $|x| = x^2$
  - $|3x - 4| = 5 - |x - 2|$
- Find the approximate solution(s) to each of the following equations graphically. Use technology to support your work. Verify the solution sets using the original equations.
  - $2x - 4 = \sqrt{x + 5}$
  - $x + 2 = x^3 - 2x - 4$
  - $0.5x^3 - 4 = 3x + 1$
  - $6\left(\frac{1}{2}\right)^{5x} = 10 - 6x$

In each problem, the graphs of the functions  $f$  and  $g$  are shown on the same Cartesian plane. Estimate the solution set to the equation  $f(x) = g(x)$ . Assume that the graphs of the two functions intersect only at the points shown on the graph.



7. The graph shows Glenn's distance from home as he rode his bicycle to school, which is just down his street. His next-door neighbor Pablo, who lives 100 m closer to the school, leaves his house at the same time as Glenn. He walks at a constant velocity, and they both arrive at school at the same time.
- Graph a linear function that represents Pablo's distance from Glenn's home as a function of time.
  - Estimate when the two boys pass each other.
  - Write piecewise linear functions to represent each boy's distance, and use them to verify your answer to part (b).



## Lesson 17: Four Interesting Transformations of Functions

### Classwork

#### Exploratory Challenge 1

Let  $f(x) = |x|$ ,  $g(x) = f(x) - 3$ , and  $h(x) = f(x) + 2$  for any real number  $x$ .

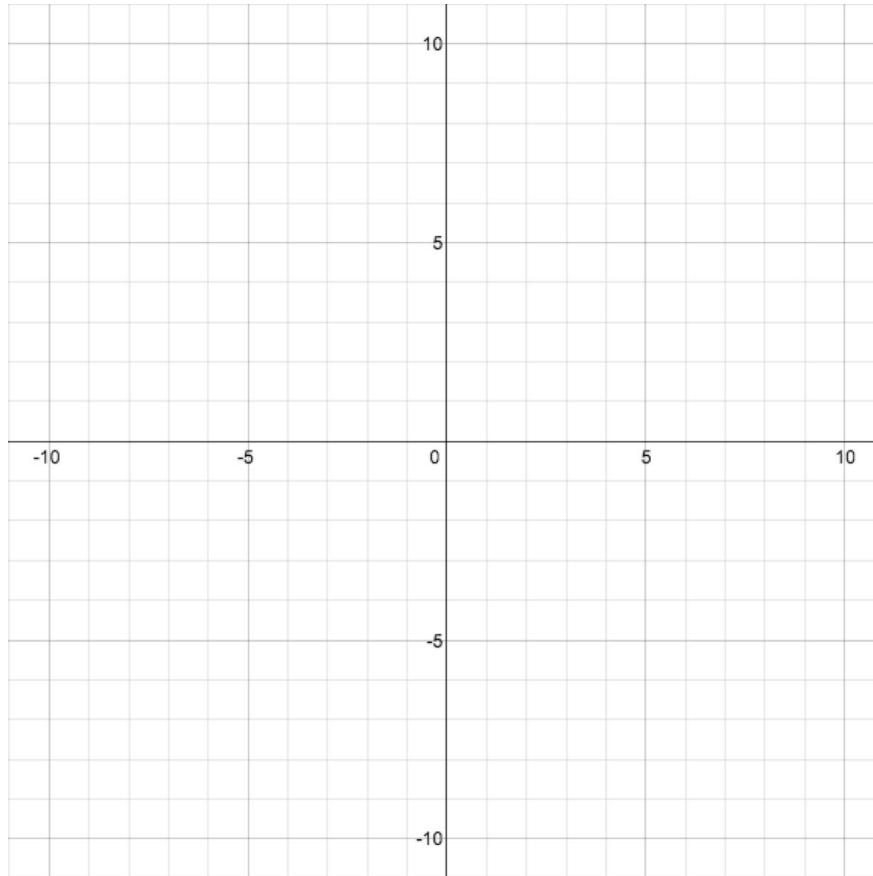
- a. Write an explicit formula for  $g(x)$  in terms of  $|x|$  (i.e., without using  $f(x)$  notation).

- b. Write an explicit formula for  $h(x)$  in terms of  $|x|$  (i.e., without using  $f(x)$  notation).

- c. Complete the table of values for these functions.

$x$	$f(x) =  x $	$g(x) = f(x) - 3$	$h(x) = f(x) + 2$
-3			
-2			
-1			
0			
1			
2			
3			

- d. Graph all three equations:  $y = f(x)$ ,  $y = f(x) - 3$ , and  $y = f(x) + 2$ .



- e. What is the relationship between the graph of  $y = f(x)$  and the graph of  $y = f(x) + k$ ?

- f. How do the values of  $g$  and  $h$  relate to the values of  $f$ ?

**Exploratory Challenge 2**

Let  $f(x) = |x|$ ,  $g(x) = 2f(x)$ , and  $h(x) = \frac{1}{2}f(x)$  for any real number  $x$ .

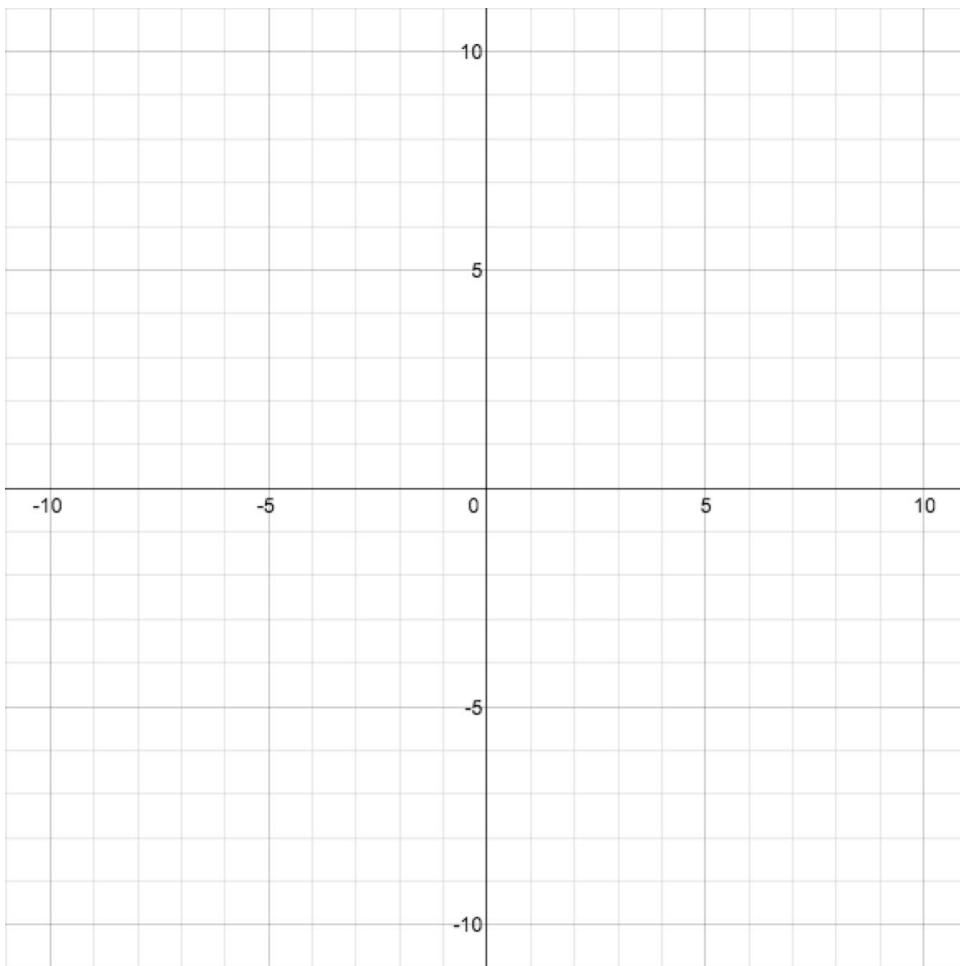
a. Write a formula for  $g(x)$  in terms of  $|x|$  (i.e., without using  $f(x)$  notation).

b. Write a formula for  $h(x)$  in terms of  $|x|$  (i.e., without using  $f(x)$  notation).

c. Complete the table of values for these functions.

$x$	$f(x) =  x $	$g(x) = 2f(x)$	$h(x) = \frac{1}{2}f(x)$
-3			
-2			
-1			
0			
1			
2			
3			

- d. Graph all three equations:  $y = f(x)$ ,  $y = 2f(x)$ , and  $y = \frac{1}{2}f(x)$ .



Given  $f(x) = |x|$ , let  $p(x) = -|x|$ ,  $q(x) = -2f(x)$ , and  $r(x) = -\frac{1}{2}f(x)$  for any real number  $x$ .

- e. Write the formula for  $q(x)$  in terms of  $|x|$  (i.e., without using  $f(x)$  notation).

- f. Write the formula for  $r(x)$  in terms of  $|x|$  (i.e., without using  $f(x)$  notation).

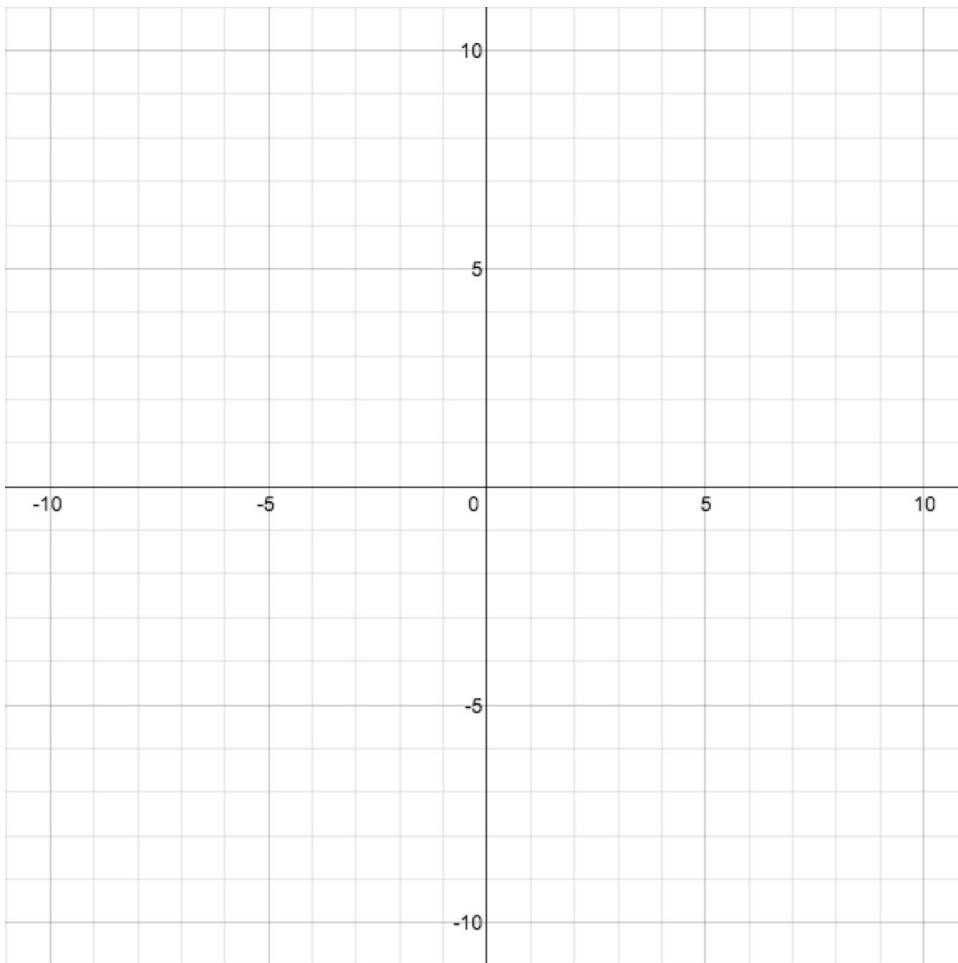
- g. Complete the table of values for the functions  $p(x) = -|x|$ ,  $q(x) = -2f(x)$ , and  $r(x) = -\frac{1}{2}f(x)$ .

$x$	$p(x) = - x $	$q(x) = -2f(x)$	$r(x) = -\frac{1}{2}f(x)$
-3			
-2			
-1			
0			
1			
2			
3			

- h. Graph all three functions on the same graph that was created in part (d). Label the graphs as  $y = p(x)$ ,  $y = q(x)$ , and  $y = r(x)$ .
- i. How is the graph of  $y = f(x)$  related to the graph of  $y = kf(x)$  when  $k > 1$ ?
- j. How is the graph of  $y = f(x)$  related to the graph of  $y = kf(x)$  when  $0 < k < 1$ ?
- k. How do the values of functions  $p$ ,  $q$ , and  $r$  relate to the values of functions  $f$ ,  $g$ , and  $h$ , respectively? What transformation of the graphs of  $f$ ,  $g$ , and  $h$  represents this relationship?

**Exercise**

Make up your own function  $f$  by drawing the graph of it on the Cartesian plane below. Label it as the graph of the equation  $y = f(x)$ . If  $b(x) = f(x) - 4$  and  $c(x) = \frac{1}{4}f(x)$  for every real number  $x$ , graph the equations  $y = b(x)$  and  $y = c(x)$  on the same Cartesian plane.



**Problem Set**

Let  $f(x) = |x|$  for every real number  $x$ . The graph of  $y = f(x)$  is shown below. Describe how the graph for each function below is a transformation of the graph of  $y = f(x)$ . Then, use this same set of axes to graph each function for Problems 1–5. Be sure to label each function on your graph (by  $y = a(x)$ ,  $y = b(x)$ , etc.).

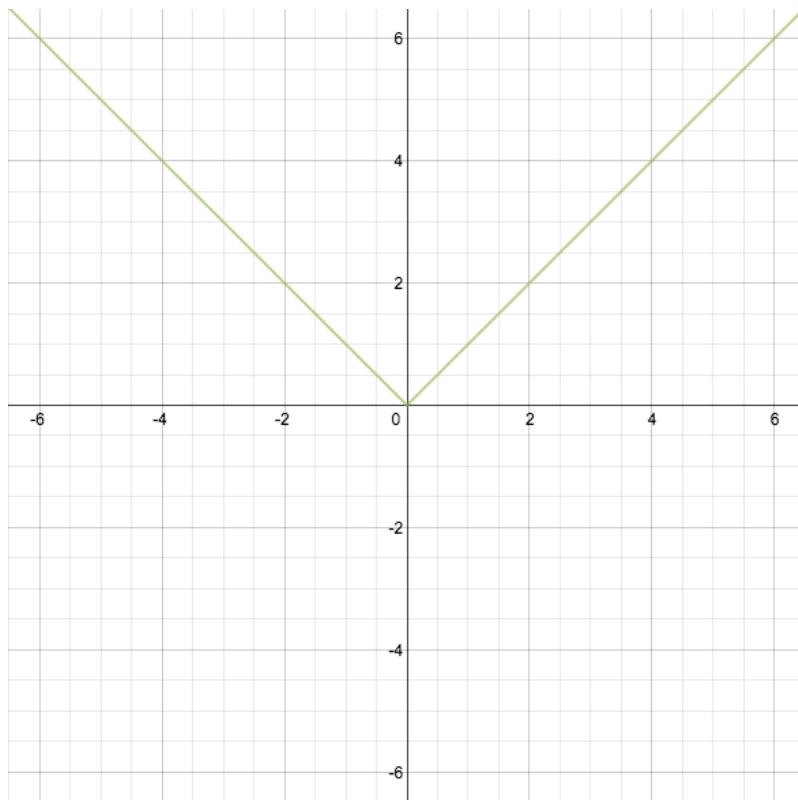
1.  $a(x) = |x| + \frac{3}{2}$

2.  $b(x) = -|x|$

3.  $c(x) = 2|x|$

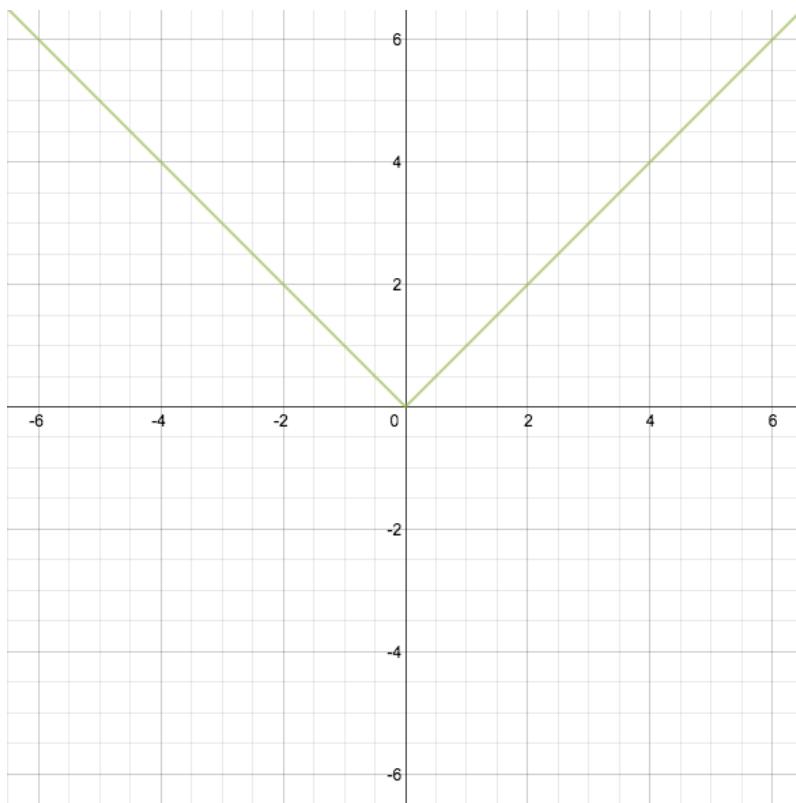
4.  $d(x) = \frac{1}{3}|x|$

5.  $e(x) = |x| - 3$

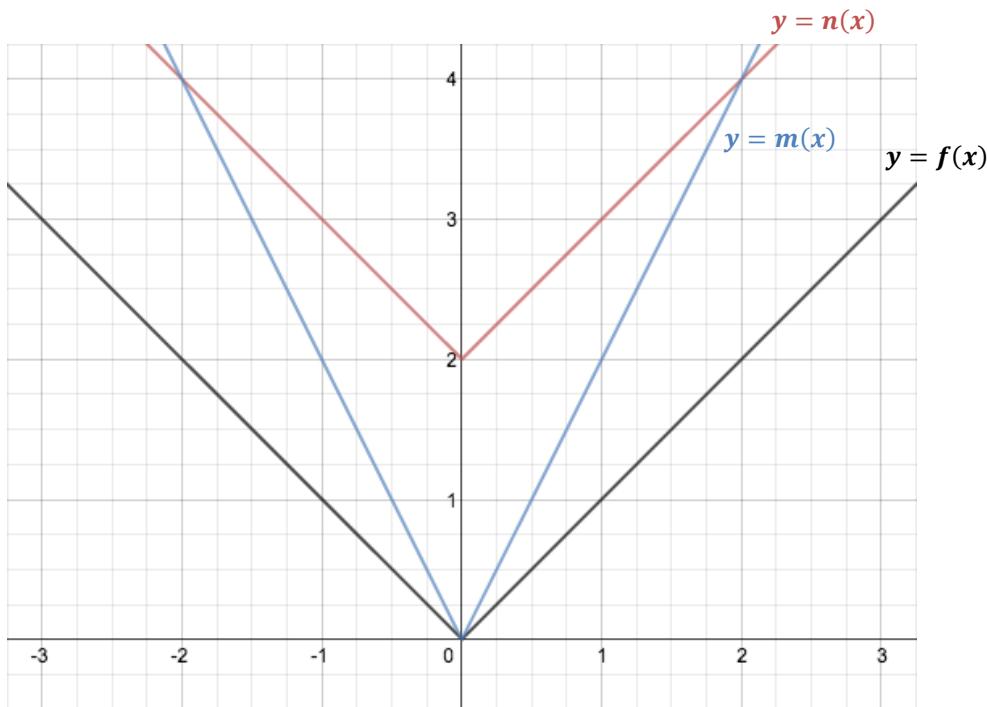


6. Let  $r(x) = |x|$  and  $t(x) = -2|x| + 1$  for every real number  $x$ . The graph of  $y = r(x)$  is shown below. Complete the table below to generate output values for the function  $t$ , and then graph the equation  $y = t(x)$  on the same set of axes as the graph of  $y = r(x)$ .

$x$	$r(x) =  x $	$t(x) = -2 x  + 1$
-2		
-1		
0		
1		
2		



7. Let  $f(x) = |x|$  for every real number  $x$ . Let  $m$  and  $n$  be functions found by transforming the graph of  $y = f(x)$ . Use the graphs of  $y = f(x)$ ,  $y = m(x)$ , and  $y = n(x)$  below to write the functions  $m$  and  $n$  in terms of the function  $f$ . (Hint: What is the  $k$ ?)



## Lesson 18: Four Interesting Transformations of Functions

### Classwork

#### Example

Let  $f(x) = |x|$ ,  $g(x) = f(x - 3)$ , and  $h(x) = f(x + 2)$ , where  $x$  can be any real number.

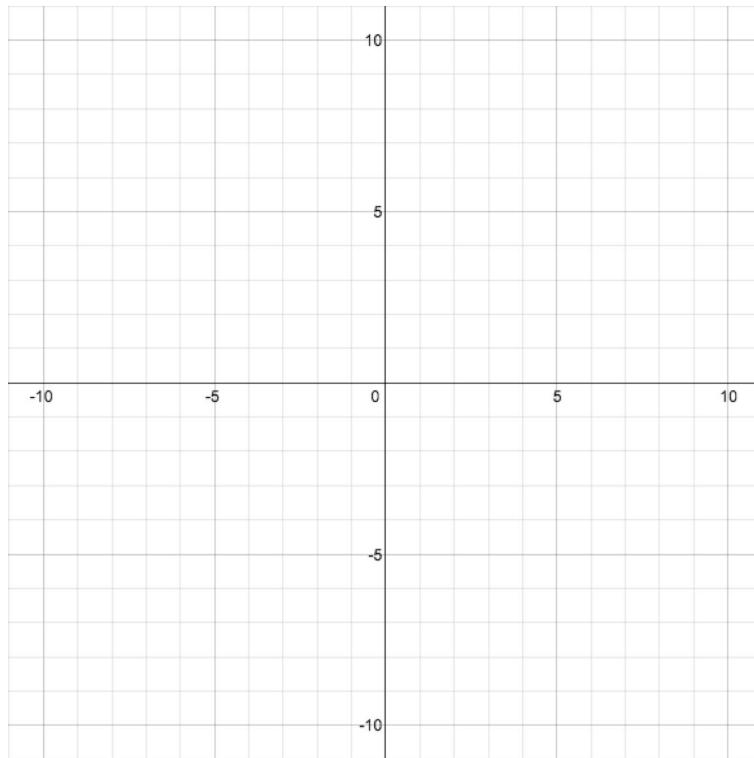
- a. Write the formula for  $g(x)$  in terms of  $|x|$  (i.e., without using  $f(x)$  notation).

- b. Write the formula for  $h(x)$  in terms of  $|x|$  (i.e., without using  $f(x)$  notation).

- c. Complete the table of values for these functions.

$x$	$f(x) =  x $	$g(x) =$	$h(x) =$
-3			
-2			
-1			
0			
1			
2			
3			

- d. Graph all three equations:  $y = f(x)$ ,  $y = f(x - 3)$ , and  $y = f(x + 2)$ .



- e. How does the graph of  $y = f(x)$  relate to the graph of  $y = f(x - 3)$ ?
- f. How does the graph of  $y = f(x)$  relate to the graph of  $y = f(x + 2)$ ?
- g. How do the graphs of  $y = |x| - 3$  and  $y = |x - 3|$  relate differently to the graph of  $y = |x|$ ?
- h. How do the values of  $g$  and  $h$  relate to the values of  $f$ ?

**Exercises**

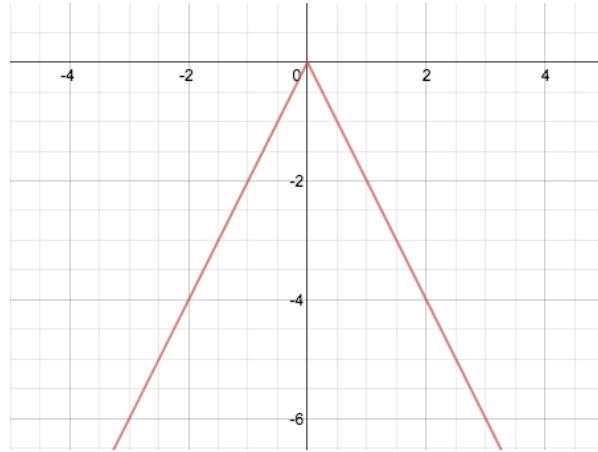
1. Karla and Isamar are disagreeing over which way the graph of the function  $g(x) = |x + 3|$  is translated relative to the graph of  $f(x) = |x|$ . Karla believes the graph of  $g$  is “to the right” of the graph of  $f$ ; Isamar believes the graph is “to the left.” Who is correct? Use the coordinates of the vertex of  $f$  and  $g$  to support your explanation.
  
2. Let  $f(x) = |x|$ , where  $x$  can be any real number. Write a formula for the function whose graph is the transformation of the graph of  $f$  given by the instructions below.
  - a. A translation right 5 units
  
  - b. A translation down 3 units
  
  - c. A vertical scaling (a vertical stretch) with scale factor of 5
  
  - d. A translation left 4 units
  
  - e. A vertical scaling (a vertical shrink) with scale factor of  $\frac{1}{3}$

3. Write the formula for the function depicted by the graph.

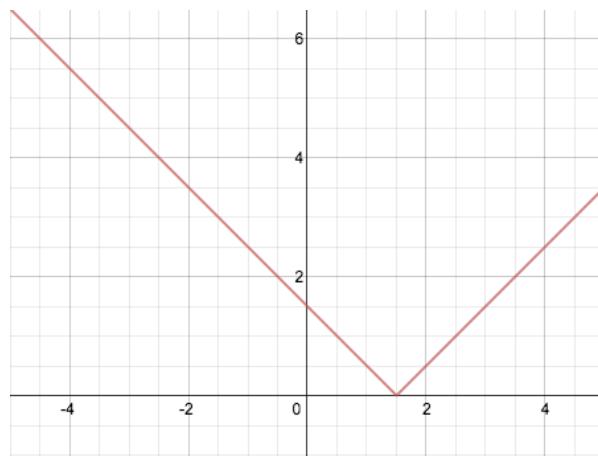
a.  $y =$



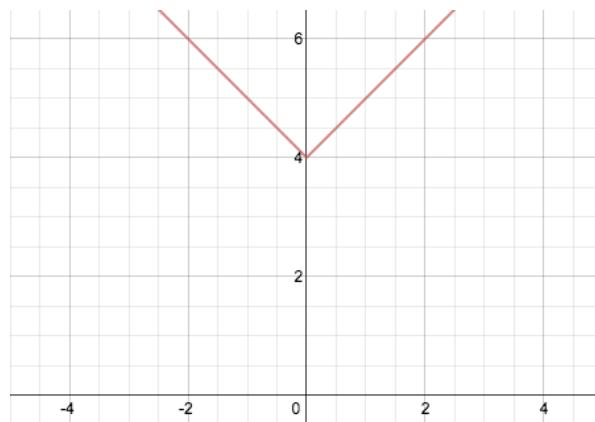
b.  $y =$



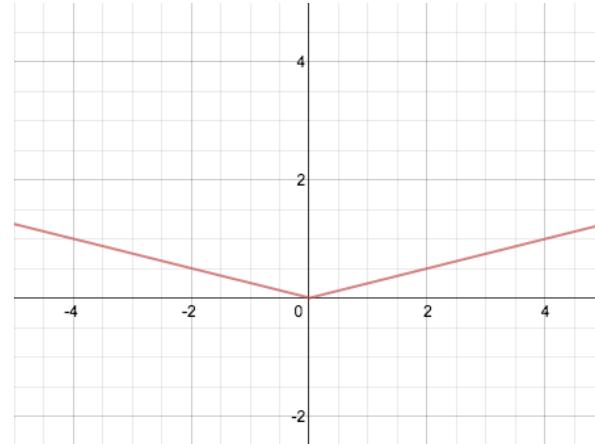
c.  $y =$



d.  $y =$



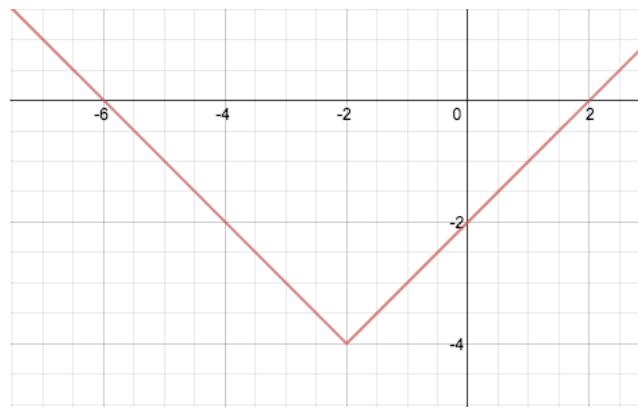
e.  $y =$



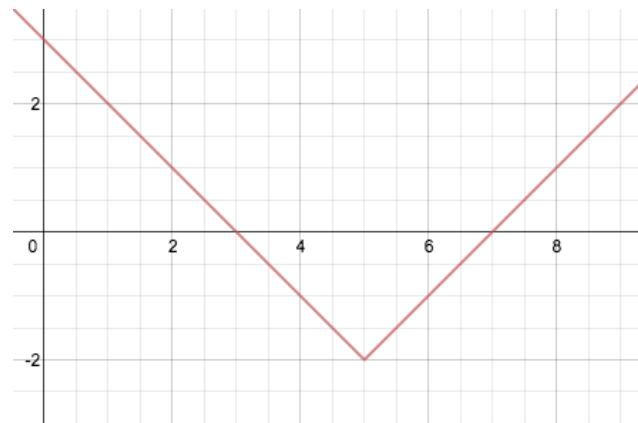
4. Let  $f(x) = |x|$ , where  $x$  can be any real number. Write a formula for the function whose graph is the described transformation of the graph of  $f$ .
- A translation 2 units left and 4 units down
  - A translation 2.5 units right and 1 unit up
  - A vertical scaling with scale factor  $\frac{1}{2}$  and then a translation 3 units right

- d. A translation 5 units right and a vertical scaling by reflecting across the  $x$ -axis with vertical scale factor  $-2$
5. Write the formula for the function depicted by the graph.

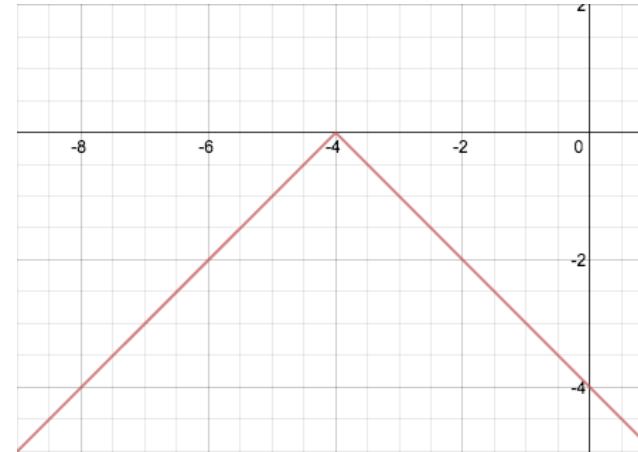
a.  $y =$

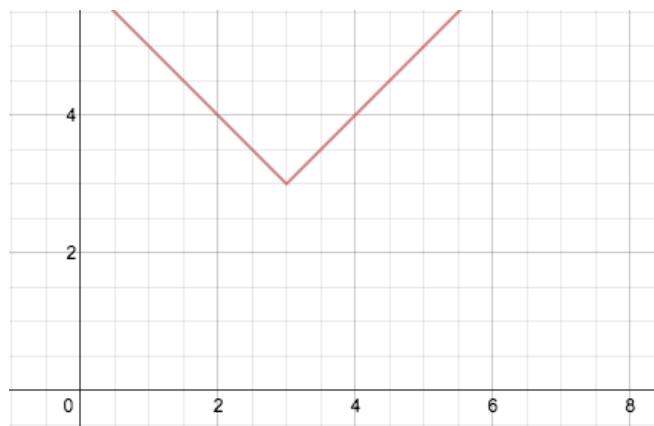


b.  $y =$



c.  $y =$



d.  $y =$ 

## Problem Set

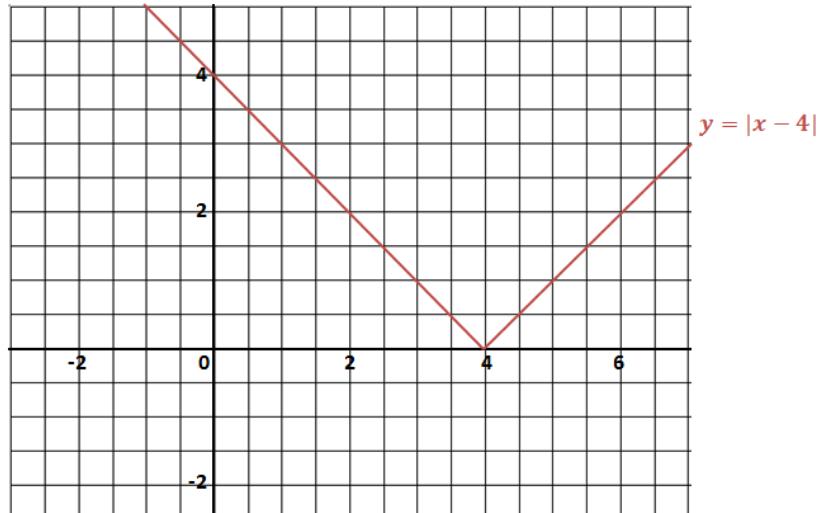
## 1. Working with quadratic functions:

- a. The vertex of the quadratic function  $f(x) = x^2$  is at  $(0,0)$ , which is the minimum for the graph of  $f$ . Based on your work in this lesson, to where do you predict the vertex will be translated for the graphs of  $g(x) = (x - 2)^2$  and  $h(x) = (x + 3)^2$ ?
- b. Complete the table of values, and then graph all three functions.

$x$	$f(x) = x^2$	$g(x) = (x - 2)^2$	$h(x) = (x + 3)^2$
-3			
-2			
-1			
0			
1			
2			
3			

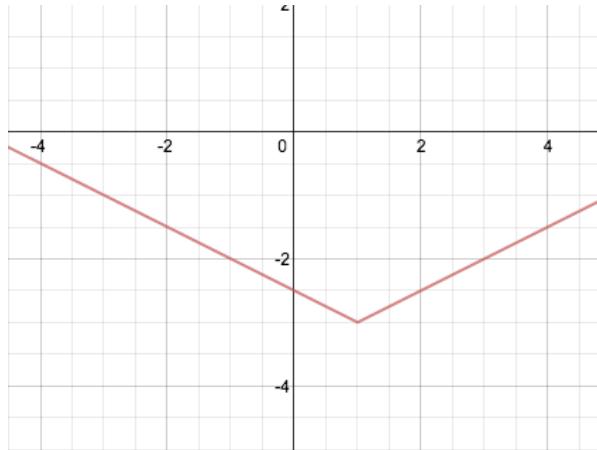
2. Let  $f(x) = |x - 4|$  for every real number  $x$ . The graph of the equation  $y = f(x)$  is provided on the Cartesian plane below. Transformations of the graph of  $y = f(x)$  are described below. After each description, write the equation for the transformed graph. Then, sketch the graph of the equation you write for part (d).

- a. Translate the graph left 6 units and down 2 units.
- b. Reflect the resulting graph from part (a) across the  $x$ -axis.
- c. Scale the resulting graph from part (b) vertically by a scale factor of  $\frac{1}{2}$ .
- d. Translate the resulting graph from part (c) right 3 units and up 2 units. Graph the resulting equation.

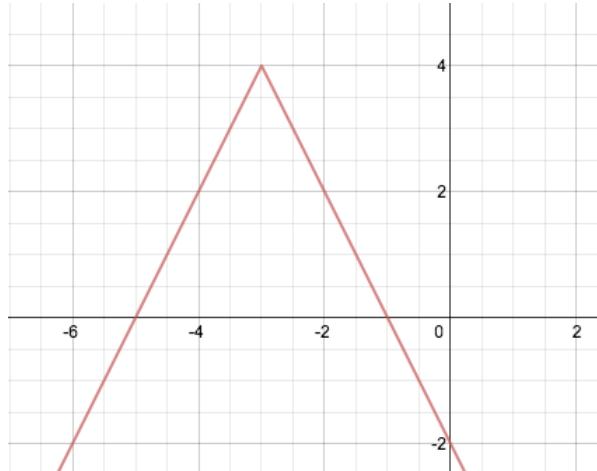


3. Let  $f(x) = |x|$  for all real numbers  $x$ . Write the formula for the function represented by the described transformation of the graph of  $y = f(x)$ .
- First, a vertical stretch with scale factor  $\frac{1}{3}$  is performed, then a translation right 3 units, and finally a translation down 1 unit.
  - First, a vertical stretch with scale factor 3 is performed, then a reflection over the  $x$ -axis, then a translation left 4 units, and finally a translation up 5 units.
  - First, a reflection across the  $x$ -axis is performed, then a translation left 4 units, then a translation up 5 units, and finally a vertical stretch with scale factor 3.
  - Compare your answers to parts (b) and (c). Why are they different?
4. Write the formula for the function depicted by each graph.

a.  $a(x) =$



b.  $b(x) =$



## Lesson 19: Four Interesting Transformations of Functions

### Classwork

#### Exploratory Challenge 1

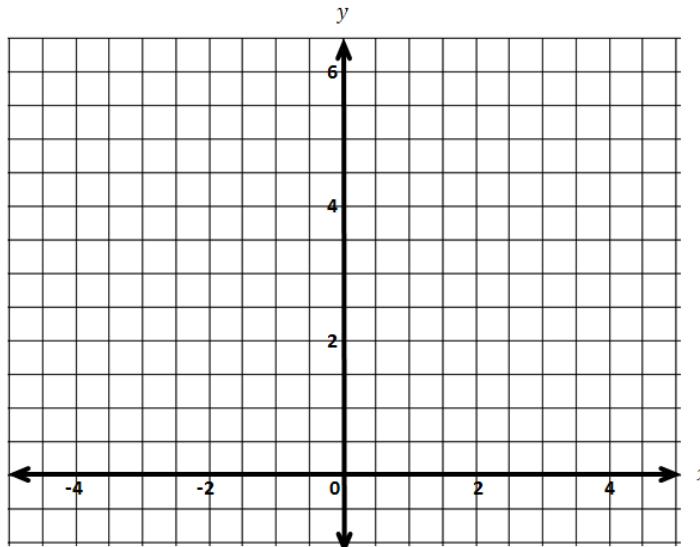
Let  $f(x) = x^2$  and  $g(x) = f(2x)$ , where  $x$  can be any real number.

- a. Write the formula for  $g$  in terms of  $x^2$  (i.e., without using  $f(x)$  notation).

- b. Complete the table of values for these functions.

$x$	$f(x) = x^2$	$g(x) = f(2x)$
-3		
-2		
-1		
0		
1		
2		
3		

- c. Graph both equations:  $y = f(x)$  and  $y = f(2x)$ .



d. How does the graph of  $y = g(x)$  relate to the graph of  $y = f(x)$ ?

e. How are the values of  $f$  related to the values of  $g$ ?

### Exploratory Challenge 2

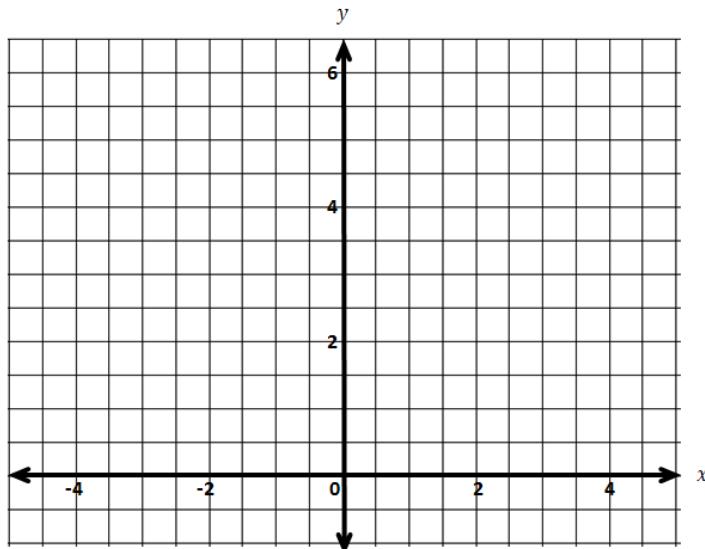
Let  $f(x) = x^2$  and  $h(x) = f\left(\frac{1}{2}x\right)$ , where  $x$  can be any real number.

a. Rewrite the formula for  $h$  in terms of  $x^2$  (i.e., without using  $f(x)$  notation).

b. Complete the table of values for these functions.

$x$	$f(x) = x^2$	$h(x) = f\left(\frac{1}{2}x\right)$
-3		
-2		
-1		
0		
1		
2		
3		

- c. Graph both equations:  $y = f(x)$  and  $y = f\left(\frac{1}{2}x\right)$ .



- d. How does the graph of  $y = f(x)$  relate to the graph of  $y = h(x)$ ?

- e. How are the values of  $f$  related to the values of  $h$ ?

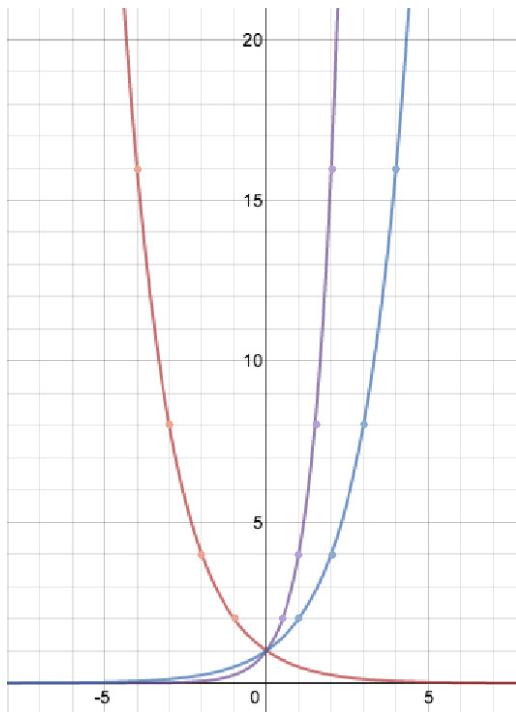
### Exercise

Complete the table of values for the given functions.

a.

$x$	$f(x) = 2^x$	$g(x) = 2^{(2x)}$	$h(x) = 2^{(-x)}$
-2			
-1			
0			
1			
2			

- b. Label each of the graphs with the appropriate functions from the table.

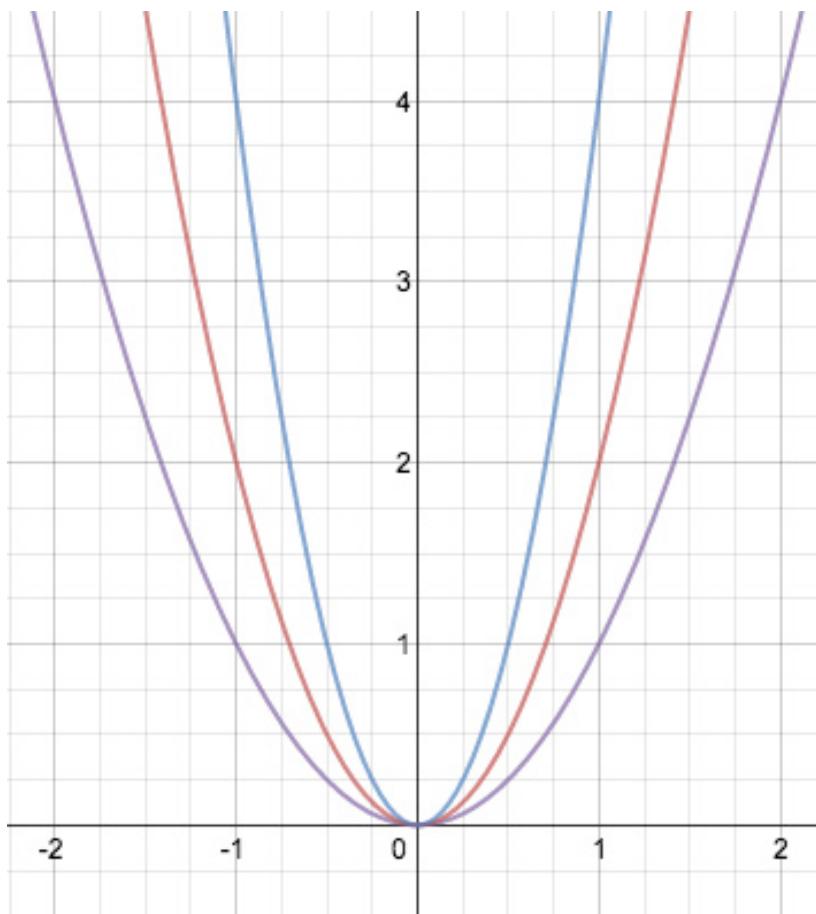


- c. Describe the transformation that takes the graph of  $y = f(x)$  to the graph of  $y = g(x)$ .
- d. Consider  $y = f(x)$  and  $y = h(x)$ . What does negating the input do to the graph of  $f$ ?
- e. Write the formula of an exponential function whose graph would be a horizontal stretch relative to the graph of  $g$ .

## Exploratory Challenge 3

- a. Look at the graph of  $y = f(x)$  for the function  $f(x) = x^2$  in Exploratory Challenge 1 again. Would we see a difference in the graph of  $y = g(x)$  if  $-2$  were used as the scale factor instead of  $2$ ? If so, describe the difference. If not, explain why not.
  - b. A reflection across the  $y$ -axis takes the graph of  $y = f(x)$  for the function  $f(x) = x^2$  back to itself. Such a transformation is called a *reflection symmetry*. What is the equation for the graph of the reflection symmetry of the graph of  $y = f(x)$ ?
  - c. Deriving the answer to the following question is fairly sophisticated; do this only if you have time. In Lessons 17 and 18, we used the function  $f(x) = |x|$  to examine the graphical effects of transformations of a function. In this lesson, we use the function  $f(x) = x^2$  to examine the graphical effects of transformations of a function. Based on the observations you made while graphing, why would using  $f(x) = x^2$  be a better option than using the function  $f(x) = |x|$ ?

## Problem Set



Let  $f(x) = x^2$ ,  $g(x) = 2x^2$ , and  $h(x) = (2x)^2$ , where  $x$  can be any real number. The graphs above are of the functions  $y = f(x)$ ,  $y = g(x)$ , and  $y = h(x)$ .

1. Label each graph with the appropriate equation.
2. Describe the transformation that takes the graph of  $y = f(x)$  to the graph of  $y = g(x)$ . Use coordinates to illustrate an example of the correspondence.
3. Describe the transformation that takes the graph of  $y = f(x)$  to the graph of  $y = h(x)$ . Use coordinates to illustrate an example of the correspondence.

## Lesson 20: Four Interesting Transformations of Functions

### Classwork

#### Opening Exercise

Fill in the blanks of the table with the appropriate heading or descriptive information.

Graph of $y = f(x)$	Vertical			Horizontal		
Translate	$y = f(x) + k$	$k > 0$	Translate up by $ k $ units		$k > 0$	Translate right by $ k $ units
			Translate down by $ k $ units		$k < 0$	
Scale by scale factor $k$		$k > 1$		$y = f\left(\frac{1}{k}x\right)$		Horizontal stretch by a factor of $ k $
		$0 < k < 1$	Vertical shrink by a factor of $ k $		$0 < k < 1$	
			Vertical shrink by a factor of $ k $ and reflection over $x$ -axis		$-1 < k < 0$	Horizontal shrink by a factor of $ k $ and reflection across $y$ -axis
		$k < -1$			$k < -1$	Horizontal stretch by a factor of $ k $ and reflection over $y$ -axis

**Exploratory Challenge 1**

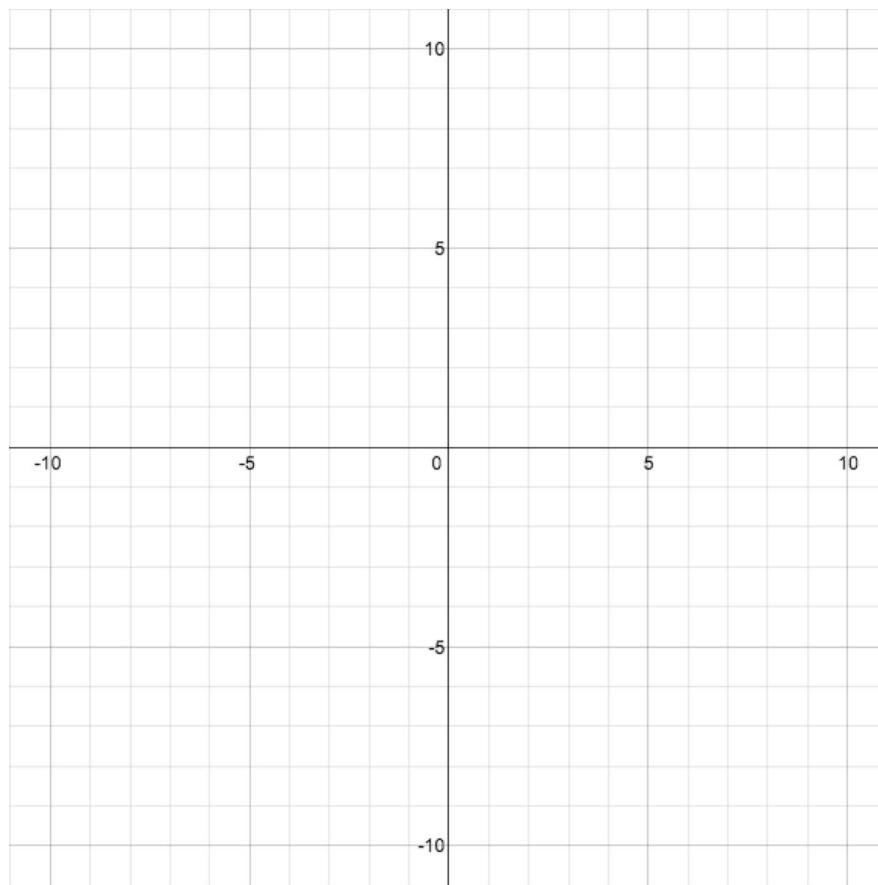
A transformation of the absolute value function  $f(x) = |x - 3|$  is rewritten here as a piecewise function. Describe in words how to graph this piecewise function.

$$f(x) = \begin{cases} -x + 3, & x < 3 \\ x - 3, & x \geq 3 \end{cases}$$

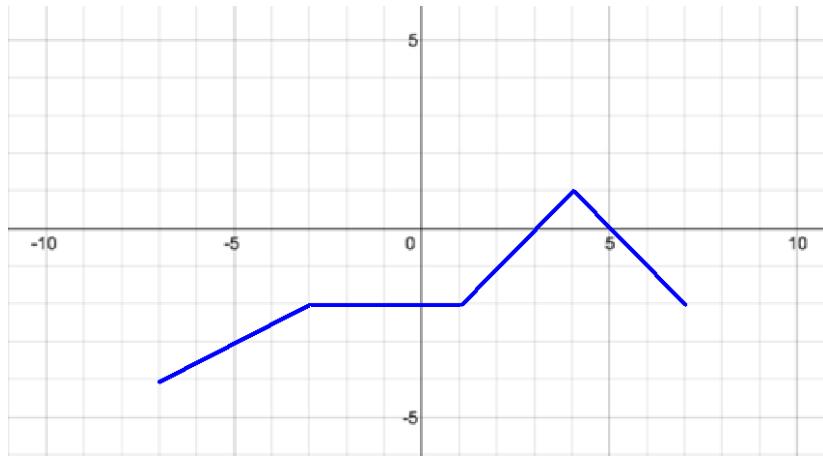
**Exercises 1–2**

1. Describe how to graph the following piecewise function. Then, graph  $y = f(x)$  below.

$$f(x) = \begin{cases} -3x - 3, & x \leq -2 \\ 0.5x + 4, & -2 < x < 2 \\ -2x + 9, & x \geq 2 \end{cases}$$



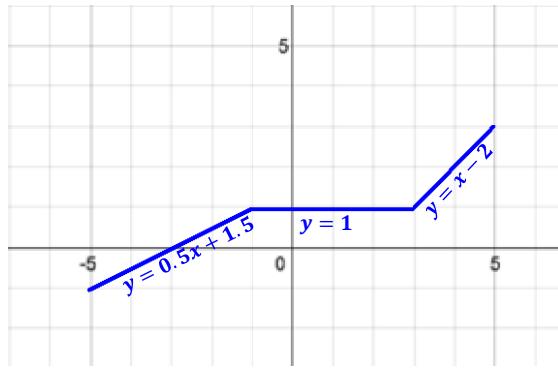
2. Using the graph of  $f$  below, write a formula for  $f$  as a piecewise function.



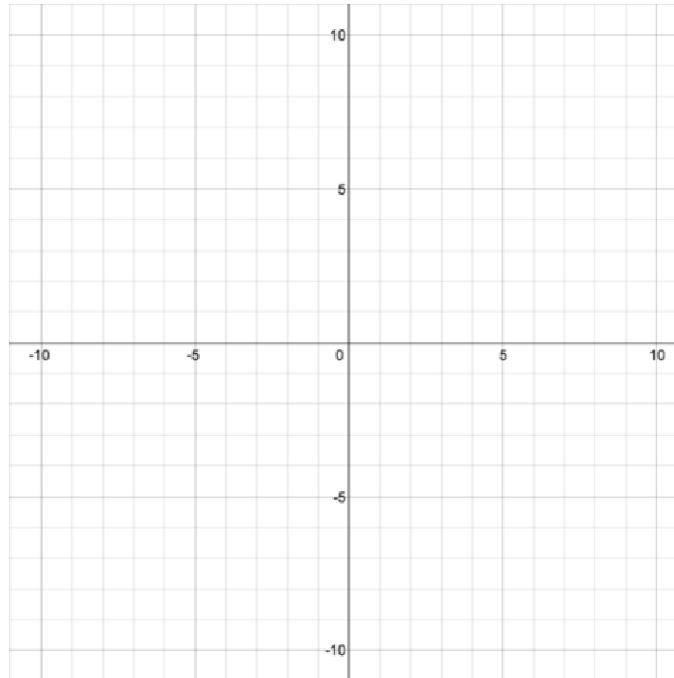
### Exploratory Challenge 2

The graph  $y = f(x)$  of a piecewise function  $f$  is shown. The domain of  $f$  is  $-5 \leq x \leq 5$ , and the range is  $-1 \leq y \leq 3$ .

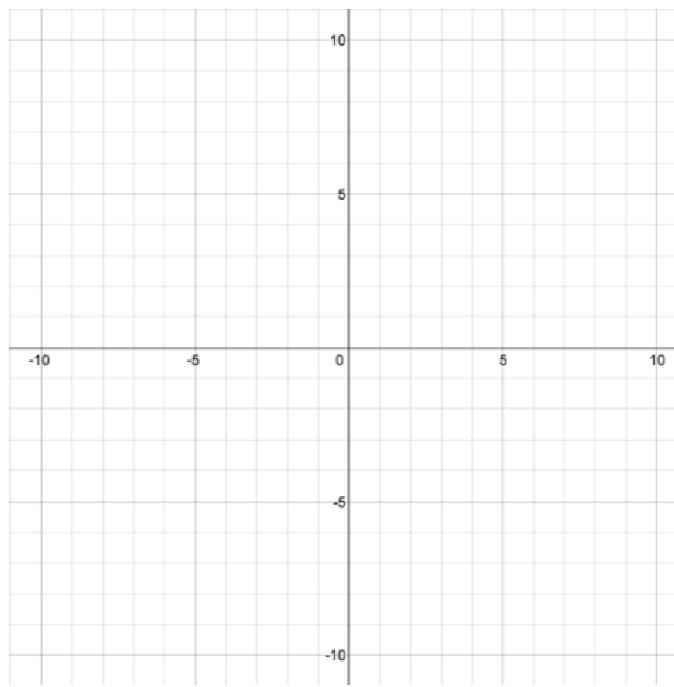
- a. Mark and identify four strategic points helpful in sketching the graph of  $y = f(x)$ .



- b. Sketch the graph of  $y = 2f(x)$ , and state the domain and range of the transformed function. How can you use part (a) to help sketch the graph of  $y = 2f(x)$ ?



- c. A horizontal scaling with scale factor  $\frac{1}{2}$  of the graph of  $y = f(x)$  is the graph of  $y = f(2x)$ . Sketch the graph of  $y = f(2x)$ , and state the domain and range. How can you use the points identified in part (a) to help sketch  $y = f(2x)$ ?



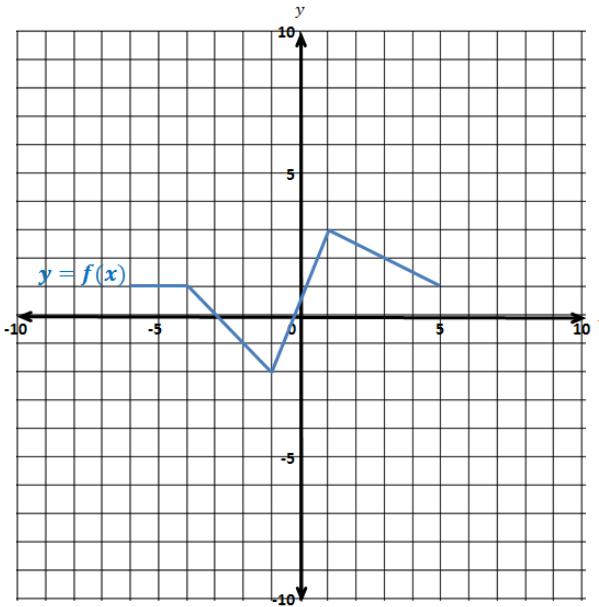
**Exercises 3–4**

3. How does the range of  $f$  in Exploratory Challenge 2 compare to the range of a transformed function  $g$ , where  $g(x) = kf(x)$ , when  $k > 1$ ?
4. How does the domain of  $f$  in Exploratory Challenge 2 compare to the domain of a transformed function  $g$ , where  $g(x) = f\left(\frac{1}{k}x\right)$ , when  $0 < k < 1$ ? (Hint: How does a graph shrink when it is horizontally scaled by a factor  $k$ ?)

**Problem Set**

1. Suppose the graph of  $f$  is given. Write an equation for each of the following graphs after the graph of  $f$  has been transformed as described. Note that the transformations are not cumulative.
  - a. Translate 5 units upward.
  - b. Translate 3 units downward.
  - c. Translate 2 units right.
  - d. Translate 4 units left.
  - e. Reflect about the  $x$ -axis.
  - f. Reflect about the  $y$ -axis.
  - g. Stretch vertically by a factor of 2.
  - h. Shrink vertically by a factor of  $\frac{1}{3}$ .
  - i. Shrink horizontally by a factor of  $\frac{1}{3}$ .
  - j. Stretch horizontally by a factor of 2.
2. Explain how the graphs of the equations below are related to the graph of  $y = f(x)$ .
  - a.  $y = 5f(x)$
  - b.  $y = f(x - 4)$
  - c.  $y = -2f(x)$
  - d.  $y = f(3x)$
  - e.  $y = 2f(x) - 5$

3. The graph of the equation  $y = f(x)$  is provided below. For each of the following transformations of the graph, write a formula (in terms of  $f$ ) for the function that is represented by the transformation of the graph of  $y = f(x)$ . Then, draw the transformed graph of the function on the same set of axes as the graph of  $y = f(x)$ .



- A translation 3 units left and 2 units up
  - A vertical stretch by a scale factor of 3
  - A horizontal shrink by a scale factor of  $\frac{1}{2}$
4. Reexamine your work on Exploratory Challenge 2 and Exercises 3 and 4 from this lesson. Parts (b) and (c) of Exploratory Challenge 2 asked how the equations  $y = 2f(x)$  and  $y = f(2x)$  could be graphed with the help of the strategic points found in part (a). In this problem, we investigate whether it is possible to determine the graphs of  $y = 2f(x)$  and  $y = f(2x)$  by working with the piecewise linear function  $f$  directly.
- Write the function  $f$  in Exploratory Challenge 2 as a piecewise linear function.
  - Let  $g(x) = 2f(x)$ . Use the graph you sketched in Exploratory Challenge 2, part (b) of  $y = 2f(x)$  to write the formula for the function  $g$  as a piecewise linear function.
  - Let  $h(x) = f(2x)$ . Use the graph you sketched in Exploratory Challenge 2, part (c) of  $y = f(2x)$  to write the formula for the function  $h$  as a piecewise linear function.
  - Compare the piecewise linear functions  $g$  and  $h$  to the piecewise linear function  $f$ . Did the expressions defining each piece change? If so, how? Did the domains of each piece change? If so how?

## Lesson 21: Comparing Linear and Exponential Models Again

### Classwork

#### Opening Exercise

	Linear Model	Exponential Model																								
General Form																										
Meaning of Parameters $a$ and $b$																										
Example																										
Rule for Finding $f(x + 1)$ from $f(x)$																										
Table	<table border="1" style="display: inline-table; vertical-align: middle;"> <tr><td></td><td></td></tr> <tr><td></td><td></td></tr> <tr><td></td><td></td></tr> <tr><td></td><td></td></tr> <tr><td></td><td></td></tr> <tr><td></td><td></td></tr> </table>													<table border="1" style="display: inline-table; vertical-align: middle;"> <tr><td></td><td></td></tr> <tr><td></td><td></td></tr> <tr><td></td><td></td></tr> <tr><td></td><td></td></tr> <tr><td></td><td></td></tr> <tr><td></td><td></td></tr> </table>												
Graph																										
Story Problem Example																										

**Exercises**

1. For each table below, assume the function  $f$  is defined for all real numbers. Calculate  $\Delta f = f(x + 1) - f(x)$  in the last column in the tables below, and show your work. (The symbol  $\Delta$  in this context means *change in*.) What do you notice about  $\Delta f$ ? Could the function be linear or exponential? Write a linear or an exponential function formula that generates the same input–output pairs as given in the table.

$x$	$f(x)$	$\Delta f = f(x + 1) - f(x)$
1	-3	
2	1	
3	5	
4	9	
5	13	

$x$	$f(x)$	$\Delta f = f(x + 1) - f(x)$
0	2	
1	6	
2	18	
3	54	
4	162	

2. Terence looked down the second column of the table and noticed that  $\frac{3}{1} = \frac{9}{3} = \frac{27}{9} = \frac{81}{27}$ . Because of his observation, he claimed that the input-output pairs in this table could be modeled with an exponential function. Explain why Terence is correct or incorrect. If he is correct, write a formula for the exponential function that generates the input-output pairs given in the table. If he is incorrect, determine and write a formula for a function that generates the input-output pairs given in the table.

$x$	$T(x)$
0	1
1	3
4	9
13	27
40	81

3. A river has an initial minnow population of 40,000 that is growing at 5% per year. Due to environmental conditions, the amount of algae that minnows use for food is decreasing, supporting 1,000 fewer minnows each year. Currently, there is enough algae to support 50,000 minnows. Is the minnow population increasing linearly or exponentially? Is the amount of algae decreasing at a linear or an exponential rate? In what year will the minnow population exceed the amount of algae available?

4. Using a calculator, Joanna made the following table and then made the following conjecture:  $3x$  is always greater than  $(1.02)^x$ . Is Joanna correct? Explain.

$x$	$(1.02)^x$	$3x$
1	1.02	3
2	1.0404	6
3	1.0612	9
4	1.0824	12
5	1.1041	15

**Lesson Summary**

- Suppose that the input-output pairs of a bivariate data set have the following property: For every two inputs that are a given difference apart, the difference in their corresponding outputs is constant. Then, an appropriate model for that data set could be a linear function.
- Suppose that the input-output pairs of a bivariate data set have the following property: For every two inputs that are a given difference apart, the quotient of their corresponding outputs is constant. Then, an appropriate model for that data set could be an exponential function.
- An increasing exponential function will eventually exceed any linear function. That is, if  $f(x) = ab^x$  is an exponential function with  $a > 0$  and  $b > 1$ , and  $g(x) = mx + k$  is any linear function, then there is a real number  $M$  such that for all  $x > M$ , then  $f(x) > g(x)$ . Sometimes this is not apparent in a graph displayed on a graphing calculator; that is because the graphing window does not show enough of the graph to show the sharp rise of the exponential function in contrast with the linear function.

**Problem Set**

For each table in Problems 1–6, classify the data as describing a linear relationship, an exponential growth relationship, an exponential decay relationship, or neither. If the relationship is linear, calculate the constant rate of change (slope), and write a formula for the linear function that models the data. If the function is exponential, calculate the common quotient for input values that are distance one apart, and write the formula for the exponential function that models the data. For each linear or exponential function found, graph the equation  $y = f(x)$ .

<b>1.</b>	<b><math>x</math></b>	<b><math>f(x)</math></b>
	1	$\frac{1}{2}$
	2	$\frac{1}{4}$
	3	$\frac{1}{8}$
	4	$\frac{1}{16}$
	5	$\frac{1}{32}$

2.

$x$	$f(x)$
1	1.4
2	2.5
3	3.6
4	4.7
5	5.8

3.

$x$	$f(x)$
1	-1
2	0
3	2
4	5
5	9

4.

$x$	$f(x)$
1	20
2	40
3	80
4	160
5	320

5.

$x$	$f(x)$
1	-5
2	-12
3	-19
4	-26
5	-33

6.

$x$	$f(x)$
1	$\frac{1}{2}$
2	$\frac{1}{3}$
3	$\frac{1}{4}$
4	$\frac{1}{5}$
5	$\frac{1}{6}$

7. Here is a variation on a classic riddle: Jayden has a dog-walking business. He has two plans. Plan 1 includes walking a dog once a day for a rate of \$5 per day. Plan 2 also includes one walk a day but charges 1 cent for 1 day, 2 cents for 2 days, 4 cents for 3 days, and 8 cents for 4 days and continues to double for each additional day. Mrs. Maroney needs Jayden to walk her dog every day for two weeks. Which plan should she choose? Show the work to justify your answer.
8. Tim deposits money in a certificate of deposit account. The balance (in dollars) in his account  $t$  years after making the deposit is given by  $T(t) = 1000(1.06)^t$  for  $t \geq 0$ .
- Explain, in terms of the structure of the expression used to define  $T(t)$ , why Tim's balance can never be \$999.
  - By what percent does the value of  $T(t)$  grow each year? Explain by writing a recursive formula for the sequence  $T(1), T(2), T(3)$ , etc.
  - By what percentages does the value of  $T(t)$  grow every two years? (Hint: Use your recursive formula to write  $T(n + 2)$  in terms of  $T(n)$ .)

9. Your mathematics teacher asks you to sketch a graph of the exponential function  $f(x) = \left(\frac{3}{2}\right)^x$  for  $x$ , a number between 0 and 40 inclusively, using a scale of 10 units to one inch for both the  $x$ - and  $y$ -axes.
- What are the dimensions (in feet) of the roll of paper needed to sketch this graph?
  - How many more feet of paper would you need to add to the roll in order to graph the function on the interval  $0 \leq x \leq 41$ ?
  - Find an  $m$  so that the linear function  $g(x) = mx + 2$  is greater than  $f(x)$  for all  $x$  such that  $0 \leq x \leq 40$ , but  $f(41) > g(41)$ .

## Lesson 22: Modeling an Invasive Species Population

### Classwork

#### Mathematical Modeling Exercise

The lionfish is a fish that is native to the western Pacific Ocean. The lionfish began appearing in the western Atlantic Ocean in 1985. This is probably because people bought them as pets and then dumped them in waterways leading to the ocean. Because it has no natural predators in this area, the number of lionfish grew very quickly and now has large populations throughout the Caribbean as well as along the eastern coastline of the United States and the Gulf of Mexico. Lionfish have recently been spotted as far north as New York and Rhode Island.

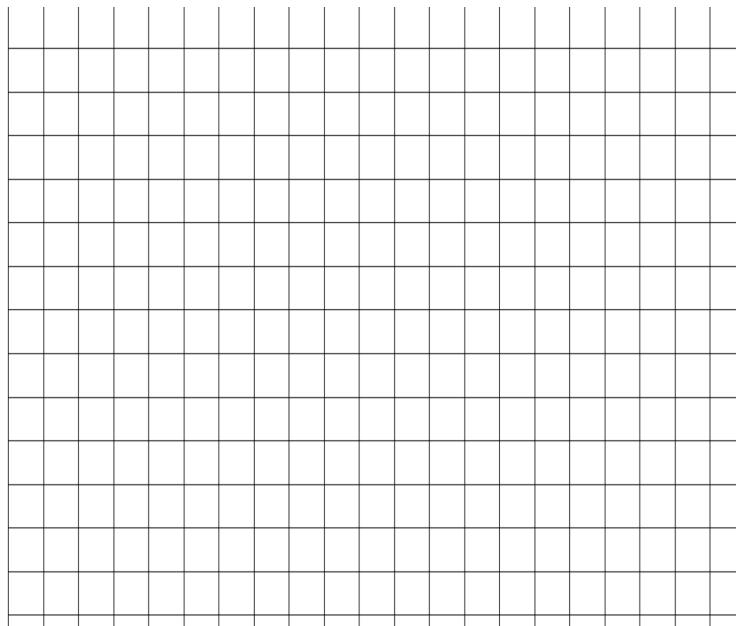
The table below shows the number of new sightings by year reported to NAS (Nonindigenous Aquatic Species), which is a branch of the U.S. Geological Survey Department.

Year	Number of New Sightings	Total Number of Sightings
1985	1	
1992	1	
1995	3	
1996	1	
2000	6	
2001	25	
2002	50	
2003	45	
2004	57	
2005	43	
2006	51	
2007	186	
2008	173	
2009	667	
2010	1,342	

1. Complete the table by recording the total number of sightings for each year.

2. Examine the total number of sightings data. Which model appears to be a better fit for the data—linear or exponential? Explain your reasoning.

3. Make a scatter plot of the year versus the total number of sightings.



4. Based on the scatter plot, either revise your answer from Exercise 2 or explain how the scatter plot supports your answer from Exercise 2.
5. On the scatter plot, draw a smooth curve that best fits the data.

6. From your table, calculate the average rate of change in the total number of sightings for each of the following time intervals.
- a. 1995–2000      b. 2000–2005      c. 2005–2010
7. How do the average rates of change help to support your argument of whether a linear or an exponential model is better suited for the data?
8. Use the regression feature of a graphing calculator to find an equation that models the number of lionfish sightings each year.
9. Use your model to predict the total number of lionfish sightings by the end of 2013.
10. The actual number of sightings as of July 2013 was 3,776. Does it seem that your model produced an accurate prediction? Explain.

## Problem Set

### Another Invasive Species Problem: Kudzu

Kudzu, a perennial vine native to Southeast Asia, now covers a large area of the southern United States. Kudzu was promoted as a forage crop and an ornamental plant when it was introduced to the U.S. at the Philadelphia Centennial Exposition in 1876. Many southern farmers were encouraged to plant kudzu for erosion control from the mid-1930s to the mid-1950s. In 1953, kudzu was removed from the U.S. Department of Agriculture's list of permissible cover plants due to its recognition as an invasive species.

Look up information about kudzu in the U.S. on Wikipedia, and write a short (1- to 2-page) report on the growth of kudzu since its introduction. In your report, choose a function (linear or exponential) to model and graph the growth of kudzu (in hectares) in the U.S. per year over the past half century or so. Remember to cite your sources!

## Lesson 23: Newton's Law of Cooling

### Classwork

#### Opening Exercise

A detective is called to the scene of a crime where a dead body has just been found. He arrives at the scene and measures the temperature of the dead body at 9:30 p.m. After investigating the scene, he declares that the person died 10 hours prior, at approximately 11:30 a.m. A crime scene investigator arrives a little later and declares that the detective is wrong. She says that the person died at approximately 6:00 a.m., 15.5 hours prior to the measurement of the body temperature. She claims she can prove it by using Newton's law of cooling:

$$T(t) = T_a + (T_0 - T_a) \cdot 2.718^{-kt},$$

where:

$T(t)$  is the temperature of the object after a time of  $t$  hours has elapsed,

$T_a$  is the ambient temperature (the temperature of the surroundings), assumed to be constant, not impacted by the cooling process,

$T_0$  is the initial temperature of the object, and

$k$  is the decay constant.

Using the data collected at the scene, decide who is correct: the detective or the crime scene investigator.

$T_a = 68^{\circ}\text{F}$  (the temperature of the room)

$T_0 = 98.6^{\circ}\text{F}$  (the initial temperature of the body)

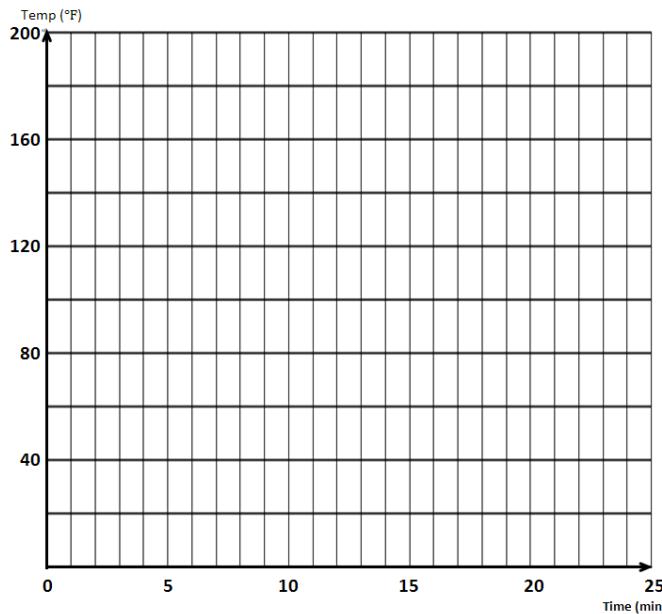
$k = 0.1335$  (13.35% per hour—calculated by the investigator from the data collected)

The temperature of the body at 9:30 p.m. is  $72^{\circ}\text{F}$ .

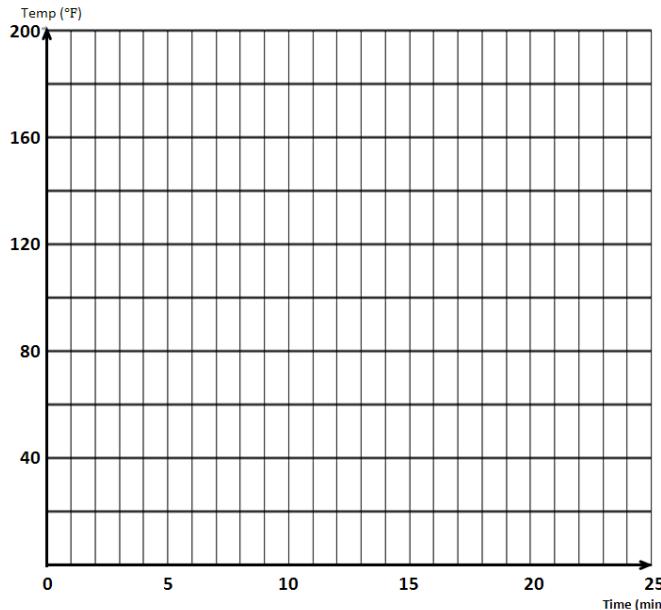
**Mathematical Modeling Exercise**

Two cups of coffee are poured from the same pot. The initial temperature of the coffee is 180°F, and  $k$  is 0.2337 (for time in minutes).

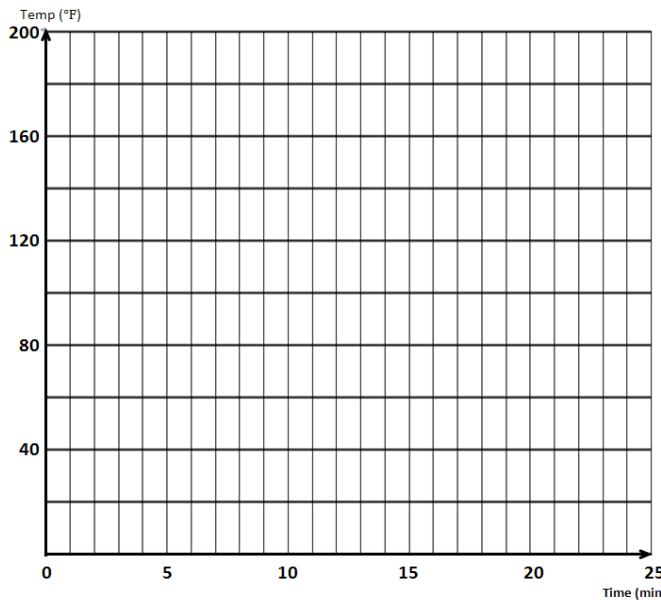
1. Suppose both cups are poured at the same time. Cup 1 is left sitting in the room that is 75°F, and Cup 2 is taken outside where it is 42°F.
  - a. Use Newton's law of cooling to write equations for the temperature of each cup of coffee after  $t$  minutes has elapsed.
  - b. Graph and label both on the same coordinate plane, and compare and contrast the two graphs.



- c. Coffee is safe to drink when its temperature is below  $140^{\circ}\text{F}$ . Estimate how much time elapses before each cup is safe to drink.
2. Suppose both cups are poured at the same time, and both are left sitting in the room that is  $75^{\circ}\text{F}$ . But this time, milk is immediately poured into Cup 2, cooling it to an initial temperature of  $162^{\circ}\text{F}$ .
- Use Newton's law of cooling to write equations for the temperature of each cup of coffee after  $t$  minutes has elapsed.
  - Graph and label both on the same coordinate plane, and compare and contrast the two graphs.



- c. Coffee is safe to drink when its temperature is below  $140^{\circ}\text{F}$ . How much time elapses before each cup is safe to drink?
3. Suppose Cup 2 is poured 5 minutes after Cup 1 (the pot of coffee is maintained at  $180^{\circ}\text{F}$  over the 5 minutes). Both are left sitting in the room that is  $75^{\circ}\text{F}$ .
- Use the equation for Cup 1 found in Exercise 1, part (a) to write an equation for Cup 2.
  - Graph and label both on the same coordinate plane, and describe how to obtain the graph of Cup 2 from the graph of Cup 1.



## Problem Set

Use the Coffee Cooling demonstration on Wolfram Alpha to write a short report on the questions that follow.

<http://demonstrations.wolfram.com/TheCoffeeCoolingProblem/>

(Note that Wolfram's free CDF player needs to be downloaded ahead of time in order to be able to run the demonstration.)

1. If you want your coffee to become drinkable as quickly as possible, should you add cream immediately after pouring the coffee or wait? Use results from the demonstration to support your claim.
  
2. If you want your coffee to stay warm longer, should you add cream immediately after pouring the coffee or wait? Use results from the demonstration to support your claim.

## Lesson 24: Piecewise and Step Functions in Context

### Classwork

#### Opening Exercise

Here are two different parking options in the city.

1-2-3 Parking	Blue Line Parking
\$6 for the first hour (or part of an hour) \$5 for the second hour (or part of an hour) \$4 for each hour (or part of an hour) starting with the third hour	\$5 per hour up to 5 hours \$4 per hour after that

The cost of a 2.75-hour stay at 1-2-3 Parking is  $\$6 + \$5 + \$4 = \$15$ . The cost of a 2.75-hour stay at Blue Line Parking is  $\$5(2.75) = \$13.75$ .

Which garage costs less for a 5.25-hour stay? Show your work to support your answer.

### Mathematical Modeling Exercise

Helena works as a summer intern at the Albany International Airport. She is studying the parking rates and various parking options. Her department needs to raise parking revenues by 10% to help address increased operating costs. The parking rates as of 2008 are displayed below. Your class will write piecewise linear functions to model each type of rate and then use those functions to develop a plan to increase parking revenues.

#### Parking Rates (Effective October 28, 2008)

##### Short Term Rates

*Located on first floor of parking garage and front of the terminal*

First Half Hour:	FREE
Second Half Hour:	\$2.00
Each Additional Half Hour:	\$1.00
Maximum Daily Rate:	\$24.00

##### Long Term Parking Rates

*Located behind the parking garage*

First Hour:	\$2.00
Each Additional Hour:	\$1.00
Maximum Daily Rate:	\$9.00
Five Days:	\$36.00
Seven Days:	\$45.00

##### Garage Parking Rates

*Located on floors two, three, four and five of the parking garage*

First Hour:	\$2.00
Each Additional Hour:	\$2.00
Maximum Daily Rate:	\$12.00
Five Consecutive Days:	\$50.00
Seven Consecutive Days:	\$64.00

##### Economy Remote Lot E - Shuttle to and from Terminal

First Hour:	\$1.00
Hourly Rate:	\$1.00
Maximum Daily Rate:	\$5.00

1. Write a piecewise linear function using step functions that models your group's assigned parking rate. As in the Opening Exercise, assume that if the car is there for any part of the next time period, then that period is counted in full (i.e., 3.75 hours is counted as 4 hours, 3.5 days is counted as 4 days, etc.).

Helena collected all the parking tickets from one day during the summer to help her analyze ways to increase parking revenues and used that data to create the table shown below. The table displays the number of tickets turned in for each time and cost category at the four different parking lots.

**Parking Tickets Collected on a Summer Day at the Albany International Airport**

Short Term			Long Term			Parking Garage			Economy Remote		
Time on Ticket (hours)	Parking Cost (\$)	Number of Tickets	Time on Ticket (hours)	Parking Cost (\$)	Number of Tickets	Time on Ticket (hours)	Parking Cost (\$)	Number of Tickets	Time on Ticket (hours)	Parking Cost (\$)	Number of Tickets
0.5	0	400	1	2	8	1	2	8	1	1	
1	2	600	2	3	20	2	4	12	2	2	
1.5	3	80	3	4	24	3	6	8	3	3	
2	4	64	4	5		4	8	4	4	4	
2.5	5	8	5	6		5	10	0	5	5	
3	6	24	6	7		6	12	16	5 to 24 hrs	5	84
3.5	7	4	7	8	60	6 to 24	12	156	2 days	10	112
4	8		8	9	92	2 days	24	96	3 days	15	64
4.5	9		8 to 24	9	260	3 days	36	40	4 days	20	60
5	10		2 days	18	164	4 days	48	12	5 days	25	72
5.5	11		3 days	27	12	5-6 days	50	8	6 days	30	24
6	12		4 days	36	8	7 days	64	4	7 days	35	76
6.5	13		5 days	36	20				8 days	40	28
7	14		6 days	36	36				9 days	45	8
7.5	15		7 days	45	32				10 days	50	4
8	16	4							14 days	70	8
8.5	17								18 days	90	4
9	18	8							21 days	105	4
9.5	19										
10	20										
10.5	21										
11	22										
11.5	23										
12 to 24	24	8									

For example, there were 600 short term 1-hr tickets charged \$2 each. Total revenue for that type of ticket would be \$1200.

2. Compute the total revenue generated by your assigned rate using the given parking ticket data.

3. The Albany International Airport wants to increase the average daily parking revenue by 10%. Make a recommendation to management of one or more parking rates to change to increase daily parking revenue by 10%. Then, use the data Helena collected to show that revenue would increase by 10% if they implement the recommended change.

**Problem Set**

1. Recall the parking problem from the Opening Exercise.
  - a. Write a piecewise linear function  $P$  using step functions that models the cost of parking at 1-2-3 Parking for  $x$  hours.
  - b. Write a piecewise linear function  $B$  that models the cost of parking at Blue Line parking for  $x$  hours.
  - c. Evaluate each function at 2.75 and 5.25 hours. Do your answers agree with the work in the Opening Exercise? If not, refine your model.
  - d. Is there a time where both models have the same parking cost? Support your reasoning with graphs and/or equations.
  - e. Apply your knowledge of transformations to write a new function that would represent the result of a \$2 across-the-board increase in hourly rates at 1-2-3 Parking. (Hint: Draw its graph first, and then use the graph to help you determine the step functions and domains.)
2. There was no snow on the ground when it started falling at midnight at a constant rate of 1.5 inches per hour. At 4:00 a.m., it started falling at a constant rate of 3 inches per hour, and then from 7:00 a.m. to 9:00 a.m., snow was falling at a constant rate of 2 inches per hour. It stopped snowing at 9:00 a.m. (Note: This problem models snow falling by a constant rate during each time period. In reality, the snowfall rate might be very close to constant but is unlikely to be perfectly uniform throughout any given time period.)
  - a. Write a piecewise linear function that models the depth of snow as a function of time since midnight.
  - b. Create a graph of the function.
  - c. When was the depth of the snow on the ground 8 inches?
  - d. How deep was the snow at 9:00 a.m.?
3. If you earned up to \$113,700 in 2013 from an employer, your social security tax rate was 6.2% of your income. If you earned over \$113,700, you paid a fixed amount of \$7,049.40.
  - a. Write a piecewise linear function to represent the 2013 social security taxes for incomes between \$0 and \$500,000.
  - b. How much social security tax would someone who made \$50,000 owe?
  - c. How much money would you have made if you paid \$4,000 in social security tax in 2013?
  - d. What is the meaning of  $f(150,000)$ ? What is the value of  $f(150,000)$ ?

4. The function  $f$  gives the cost to ship  $x$  lb. via FedEx standard overnight rates to Zone 2 in 2013.

$$f(x) = \begin{cases} 21.50 & 0 < x \leq 1 \\ 23.00 & 1 < x \leq 2 \\ 24.70 & 2 < x \leq 3 \\ 26.60 & 3 < x \leq 4 \\ 27.05 & 4 < x \leq 5 \\ 28.60 & 5 < x \leq 6 \\ 29.50 & 6 < x \leq 7 \\ 31.00 & 7 < x \leq 8 \\ 32.25 & 8 < x \leq 9 \end{cases}$$

- a. How much would it cost to ship a 3 lb. package?
  - b. How much would it cost to ship a 7.25 lb. package?
  - c. What is the domain and range of  $f$ ?
  - d. Could you use the ceiling function to write this function more concisely? Explain your reasoning.
5. Use the floor or ceiling function and your knowledge of transformations to write a piecewise linear function  $f$  whose graph is shown below.

