DEFINITE INTEGRALS.

FUNDAMENTAL THEOREM OF CALCULUS

If a function f is continuous on a closed interval [a, b] and F is an antiderivative of f, on the interval (a, b) then the

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

$\int_a^b f(x)\,dx$ is called the definite integral of $f(x)$ from a to b.
The numbers a and b are known as the lower limit and upper limit respectively of the integral.

Properties of Definite Integrals.

1. $\int_a^b K\,dx = K(b-a)$

2. $\int_a^b K f(x)\,dx = K\int_a^b f(x)\,dx$

3. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

4. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx \pm \int_c^b f(x)\,dx$

   provided that $a \le c \le b$

Examples

1) $\int_0^3 x^2 dx$

Soln

$$\int_0^3 x^2 dx = \left[\frac{x^3}{3}\right]_0^3 = \left(\frac{3^3}{3} - \frac{0^3}{3}\right) = \frac{27}{3} - 0 = 9$$

2) $\int_0^{\pi/2} \cos x \, dx = \left[\sin x\right]_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0$ $\quad \sin\frac{\pi}{2} = \sin 90^\circ = 1$

$= 1 - 0$

$= 1$

3) $\int_0^1 x^2\sqrt{x^3+1}\, dx$

Soln

Using direct substitution

Let $u = x^3 + 1$

$du = 3x^2 dx \Rightarrow dx = \frac{du}{3x^2}$

Thus,

$$\int x^2\sqrt{x^3+1}\, dx = \int x^2 \sqrt{u}\, \frac{du}{3x^2}$$

$$= \frac{1}{3}\int u^{1/2} du$$

$$= \frac{1}{3}\left[\frac{u^{3/2}}{3/2}\right] + c$$

$$= \frac{1}{3} \times \frac{2}{3} u^{3/2} + c = \frac{2}{9} u^{3/2} + c$$

but $u = x^3 + 1$

$\Rightarrow \int x^2\sqrt{x^3+1}\, dx = \frac{2}{9}(x^3+1)^{3/2} + c$

$$\therefore \int_0^1 x^2\sqrt{x^3+1}\,dx = \left[\frac{2}{9}(x^3+1)^{3/2}\right]_0^1$$
$$= \frac{2}{9}\left((1^3+1)^{3/2} - (0^3+1)^{3/2}\right)$$
$$= \frac{2}{9}\left(2^{3/2} - 1^{3/2}\right)$$
$$= \frac{2}{9}(2\sqrt{2} - 1)$$

$$2^{3/2} = (\sqrt{2})^3 = \sqrt{(2^3)} = \sqrt{8} = \sqrt{4\times 2} = \sqrt{4}\sqrt{2} = 2\sqrt{2}$$

Alternatively

From $\int_0^1 x^2\sqrt{x^3+1}\,dx$

Let $u = x^3+1$

$du = 3x^2dx \Rightarrow dx = \dfrac{du}{3x^2}$

Find the limits in terms of u.

From $u = x^3+1$

Upper limit of $x = 1$

$\Rightarrow u = 1^3+1 = 2$

Lower limit of $x = 0$

$\Rightarrow u = 0^3+1 = 1$

Thus

$$\int_0^1 x^2\sqrt{x^3+1}\,dx = \int_1^2 x^2\cdot\sqrt{u}\cdot\frac{du}{3x^2}$$
$$= \frac{1}{3}\int_1^2 u^{1/2}du$$
$$= \frac{1}{3}\left[\frac{u^{3/2}}{3/2}\right]_1^2$$
$$= \frac{1}{3}\times\frac{2}{3}\left[u^{3/2}\right]_1^2$$
$$= \frac{2}{9}\left[u^{3/2}\right]_1^2 = \frac{2}{9}\left[2^{3/2} - 1^{3/2}\right]$$
$$= \frac{2}{9}(2\sqrt{2} - 1)$$

4) $\int_0^1 \frac{1}{(4-x^2)^{3/2}}\,dx$

Soln Using trigonometric substitution

Let $x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta \Rightarrow$

$$\int \frac{1}{(4-x^2)^{3/2}}\,dx = \int \frac{2\cos\theta\, d\theta}{(4-4\sin^2\theta)^{3/2}} = \int \frac{2\cos\theta\, d\theta}{(4(1-\sin^2\theta))^{3/2}}$$

$4^{3/2} = (\sqrt{4})^3$
$= 2^3$
$= 8$

$$= \int \frac{2\cos\theta\, d\theta}{8(\cos^2\theta)^{3/2}} = \frac{1}{4}\int \frac{\cos\theta\, d\theta}{\cos^3\theta}$$

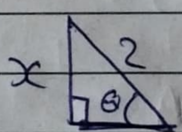
$(\cos^2\theta)^{3/2}$
$= (\sqrt{\cos^2\theta})^3$
$= (\cos\theta)^3$
$= \cos^3\theta$

$$= \frac{1}{4}\int \frac{1}{\cos^2\theta}\, d\theta = \frac{1}{4}\int \sec^2\theta\, d\theta$$

$$= \frac{1}{4}\tan\theta + C$$

but $x = 2\sin\theta$

$\Rightarrow \sin\theta = \frac{x}{2}$, using $\triangle$ triangle



$\sqrt{2^2 - x^2} = \sqrt{4-x^2}$

$\tan\theta = \frac{Opp}{Adj} = \frac{x}{\sqrt{4-x^2}}$

Thus

$$\int \frac{1}{(4-x^2)^{3/2}}\,dx = \frac{1}{4}\frac{x}{\sqrt{4-x^2}} + C$$

$$\therefore \int_0^1 \frac{1}{(4-x^2)^{3/2}}\,dx = \frac{1}{4}\left[\frac{x}{\sqrt{4-x^2}}\right]_0^1 = \frac{1}{4}\left(\frac{1}{\sqrt{4-1^2}} - \frac{0}{\sqrt{4-0^2}}\right)$$

$$= \frac{1}{4\sqrt{3}} - 0 = \frac{1}{4\sqrt{3}}$$

5) $\int_0^{\frac{\pi}{3}} x^2 \sin 3x \, dx$

Soln

Using integration by parts

$$\int u\,dv = uv - \int v\,du$$

Let

$u = x^2$ $\quad dv = \sin 3x\,dx$

$du = 2x\,dx$ $\quad v = -\frac{1}{3}\cos 3x$

Thus

$$\int x^2 \sin 3x\,dx = -\frac{1}{3}x^2\cos 3x + \frac{2}{3}\int x\cos 3x\,dx \cdots (i)$$

For $\int x \cos 3x\,dx$

Let $u = x$ $\quad dv = \cos 3x\,dx$

$du = dx$ $\quad v = \frac{1}{3}\sin 3x$

$$\int x\cos 3x\,dx = \frac{1}{3}x\sin 3x - \frac{1}{3}\int \sin 3x\,dx$$

$$= \frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x + C_1 \cdots (ii)$$

Substitute in (i)

$$\int x^2 \sin 3x\,dx = -\frac{1}{3}x^2\cos 3x + \frac{2}{3}\left[\frac{1}{3}x\sin x + \frac{1}{9}\cos 3x + C_1\right]$$

$$= -\frac{1}{3}x^2\cos 3x + \frac{2}{9}x\sin x + \frac{2}{27}\cos 3x + \frac{2}{3}C_1$$

Thus $\int_0^{\frac{\pi}{3}} x^2 \sin 3x\,dx =$

$$\left[-\frac{1}{3}x^2\cos 3x + \frac{2}{9}x\sin x + \frac{2}{27}\cos 3x + \frac{2}{3}C_1\right]_0^{\frac{\pi}{3}}$$

$\int_0^{\frac{\pi}{3}} x^2 \sin 3x \, dx$

$= \left(-\frac{1}{3} \cdot \frac{\pi^2}{9} \cos \pi + \frac{2}{9} \cdot \frac{\pi}{3} \sin \pi + \frac{2}{27} \cos \pi\right) -$

$\left(-\frac{1}{3} \cdot 0 \cos 0 + \frac{2}{9} \cdot 0 \cdot \sin 0 + \frac{2}{27} \cdot \cos 0\right)$

$= \left[\frac{\pi^2}{27} - \frac{2}{27}\right] - \left[\frac{2}{27}\right]$

$= \frac{1}{27}\left[\pi^2 - 4\right]$

6) $\int_{-1}^{2} \frac{dx}{x^2-9}$

Soln

Using integration by partial fractions

$\int \frac{dx}{x^2-9} = \int \frac{dx}{(x+3)(x-3)}$

$\frac{1}{(x+3)(x-3)} = \frac{A}{(x+3)} + \frac{B}{(x-3)} = \frac{A(x-3)+B(x+3)}{(x+3)(x-3)}$

$\Rightarrow A(x-3) + B(x+3) = 1$

let $x = 3$

$\Rightarrow B(3+3) = 1 \quad \Rightarrow 6B = 1 \quad \Rightarrow B = \frac{1}{6}$

Let $x = -3$

$\Rightarrow A(-3-3) = 1 \Rightarrow -6A = 1 \quad \Rightarrow A = -\frac{1}{6}$

Thus

$\int \frac{dx}{x^2-9} = -\frac{1}{6}\int \frac{1}{x+3} dx + \frac{1}{6}\int \frac{1}{x-3} dx$

$$= -\frac{1}{6}\ln|x+3| + \frac{1}{6}\ln|x-3| + c$$

$$\therefore \int_{-1}^{2} \frac{dx}{x^2-9} = \left[-\frac{1}{6}\ln|x+3| + \frac{1}{6}\ln|x-3|\right]_{-1}^{2}$$

$$= \left[-\frac{1}{6}\ln 5 + \frac{1}{6}\ln 1\right] - \left[-\frac{1}{6}\ln 2 + \frac{1}{6}\ln 4\right]$$

$$= -\frac{1}{6}\ln 5 + \frac{1}{6}\ln 2 - \frac{1}{6}\ln 4 \qquad \ln 1 = 0$$

$$= -0.3838$$

Exercise

Evaluate the following definite integrals

1) $\int_{-5}^{3} \frac{dx}{x^2+4}$

2) $\int_{0}^{\pi/2} \frac{dx}{2+\cos 2x}$

3) $\int_{0}^{1} (2x + 6x^4 + 5)\,dx$

4) $\int_{1/4}^{3/4} \frac{x+1}{x^2(x-1)}\,dx$

5) $\int_{2}^{4} \frac{\sqrt{16-x^2}}{x}\,dx$

6) $\int_{1}^{2} x^3 \ln x\,dx$

7) $\int_{0}^{1} \ln(x^2+1)\,dx$

8) $\int_{0}^{3} \frac{dx}{\sqrt{1+x}}$

9) $\displaystyle\int_{-\frac{1}{2}}^{0} \frac{x^3}{x^2+x+1}\,dx$

10) $\displaystyle\int_{0}^{1} \frac{2x}{x^2-x-2}\,dx$

11) $\displaystyle\int_{2}^{-1} \frac{x-1}{\sqrt{x^2-4x+3}}$

12) $\displaystyle\int_{-2}^{0} (6e^{2x}-3x)\,dx$

13) $\displaystyle\int_{0}^{5} \frac{x}{\sqrt{x+4}}\,dx$

IMPROPER INTEGRALS

When defining definite integrals $\int_a^b f(x)dx$ it was assumed that the integrand $f(x)$ was finite and continuous for all values in the range of integration

$$a \leq x \leq b$$

Lets consider cases when integrands become infinite, when $x=a$ or $x=b$ or at a point within the range of integration

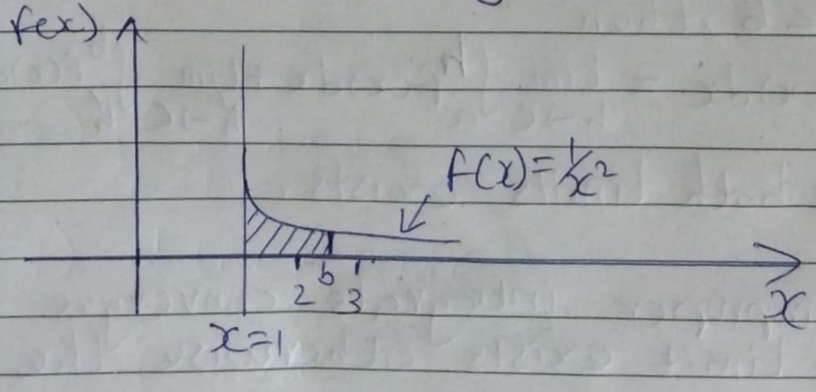
Examples of improper integrals are:-

$$\int_1^\infty \frac{dx}{x} \; ; \; \int_{-\infty}^\infty \frac{dx}{x^2+1} \; ; \; \int_1^5 \frac{dx}{\sqrt{x-1}} \; ; \; \int_{-2}^2 \frac{dx}{(x+1)^2}$$

To get an idea on how to evaluate an improper integral, consider the integral

$$\int_1^b \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_1^b = 1 - \frac{1}{b}$$

which can be interprated, as the Area of the shaded region shown below



Taking the limit as ~~b~~ b tends to infinity ($\lim_{b \to \infty}$), produces

$$\int_1^{\infty} \frac{dx}{x^2} = \lim_{b \to \infty} \int_1^b \frac{dx}{x^2} = \lim_{b \to \infty} \left(1 - \frac{1}{b}\right) = 1$$

This improper integrals can be interpreted as the area of unbounded region between the graph $f(x) = \frac{1}{x^2}$ and the x-axis (to the right of $x = 1$).

Definitions:

1. Given that $x = a$ is the point where $f(x)$ become infinite, we define

$$\int_a^b f(x)\,dx = \lim_{h \to a} \int_h^b f(x)\,dx$$

where this limit exists.

2. Similarly, if $f(x)$ is infinite when $x = b$, then

$$\int_a^b f(x)\,dx = \lim_{h' \to b} \int_a^{h'} f(x)\,dx$$

3. If $f(x)$ is infinite when $x = c$ where $a < c < b$, then

$$\int_a^b f(x)\,dx = \lim_{h \to c} \int_a^h f(x)\,dx + \lim_{h' \to c} \int_{h'}^b f(x)\,dx$$

where both limits exists.

NOTE: Improper integrals converges if the limit exists otherwise the improper integrals diverges.

Examples

1. $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

Soln

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{h\to 0^+} \int_h^1 \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{h\to 0^+} \left[2\sqrt{x}\right]_h^1$$

$$= \lim_{h\to 0^+} \left[2 - 2\sqrt{h}\right]$$

$$= 2 - 0 = \underline{\underline{2}}$$

Since the limit converge, then the improper integral converges and therefore the value of

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \underline{\underline{2}}$$

2. $\int_0^1 \frac{1}{x^2}\,dx$

Soln

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{h\to 0^+} \int_h^1 \frac{1}{x^2}\,dx$$

$$= \lim_{h\to 0^+} \left[-\frac{1}{x}\right]_h^1 = \lim_{h\to 0^+}\left[-\frac{1}{1} - \left(-\frac{1}{h}\right)\right]$$

$$= \lim_{h\to 0^+} \left[\frac{1}{h} - 1\right]$$

$$= \underline{\underline{\infty}} \quad \text{undefined}$$

The limit diverges i.e the integral doesn't converge.

3. $\int_{-1}^{1} \frac{dx}{x^{\frac{2}{3}}}$

<u>Soln</u>

$$\int_{-1}^{1} \frac{dx}{x^{\frac{2}{3}}} = \int_{-1}^{0} \frac{dx}{x^{\frac{2}{3}}} + \int_{0}^{1} \frac{dx}{x^{2/3}}$$

$$= \lim_{h \to 0^-} \int_{-1}^{h} \frac{dx}{x^{2/3}} + \lim_{h \to 0^+} \int_{h'}^{1} \frac{dx}{x^{2/3}}$$

$$= \lim_{h \to 0^-} \left[3x^{\frac{1}{3}}\right]_{-1}^{h} + \lim_{h' \to 0^+} \left[3x^{\frac{1}{3}}\right]_{h'}^{1}$$

$$= \lim_{h \to 0^-} \left[3h^{1/3} - (-1)^{1/3}\right] + \lim_{h' \to 0^+} \left[3(1)^{1/3} - 3(h')^{1/3}\right]$$

$$= \lim_{h \to 0^-} \left[3h^{1/3} + 3(1)\right] + \lim_{h' \to 0^+} \left[-3h'^{1/3} + 3\right]$$

$$= 3 + 3 = \underline{\underline{6}}$$