Contents

12 Session Twelve: Applications of Differentiation II

12.1 Session Objectives

By the end of this session, you should be able to:

(i) Discuss concavity of graphs of functions

(ii) Identify relative and absolute extreme points of a graph

(iii) Find asymptotes of a graph

12.2 Introduction

In this second part of applications of differentiation, we discuss: Concavity, Absolute extrema, Horizontal and Verticle Asymptotes.

12.3 Concavity of a graph

A function is a **concave up** on (a, b) if the the graph of the function lies above its tangent line at each point of (a, b).
A function is a **concave down** on (a, b) if the the graph of the function lies below its tangent line at each point of (a, b).

Definition **12.3.1** (Inflection point). *A point where a graph changes its concavity is called an inflection point (see Figure 1).*

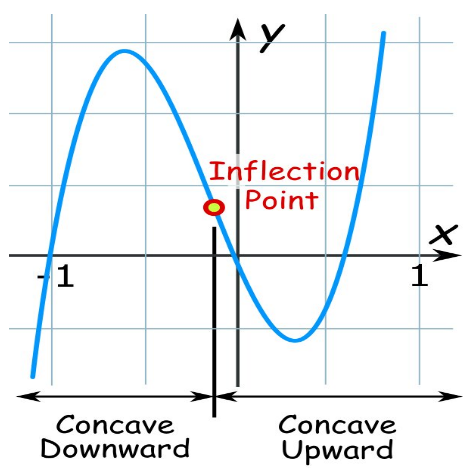

Figure 1: Inflection point

Test for concavity
Let f be a function with derivatives f' and f'' existing at all points in an interval (a, b). Then

1. f is concave up on (a, b) if $f''(x) > 0$ for all x in (a, b)

2. f is concave down on (a, b) if $f''(x) < 0$ for all x in (a, b)

Example 12.3.1. *Find all intervals where $f(x) = x^4 - 8x^3 + 18x^2$ is concave up or down and find all inflection points.*

Solution: $f'(x) = 4x^3 - 24x^2 + 36x$, $f''(x) = 12x^2 - 48x + 38x = 12(x-1)(x-3)$

Table 1: Signs of $f'(x)$

x	0	1	2	3	4
Sign of f'	+		-		+
Concavity	up		down		up

Concave up: $(-\infty, 1)$, $(3, \infty)$ Concave down: $(1, 3)$

Example 12.3.2. *Find all intervals where $f(x) = x^2 + 10x - 9$ is concave up or down and find all inflection points.*

Solution: $f'(x) = 2x + 10$, $f''(x) = 2 \Rightarrow f''(x) > 0 \ \forall \ x \Rightarrow f$ is a concave up everywhere. f has no points of inflection.

Note 12.3.1. *At an inflection point for a function f, $f''(x) = 0$ or $f''(x)$ does not exist.*

The converse of of Note 12.3.1 is not true in general i.e., $f''(x) = 0$ does not imply x is a point of inflection.

Second derivative test

Let f'' exist on some open interval containing c (except possibly at c itself) and let $f'(c) = 0$.

1. If $f''(c) > 0$, then $f(c)$ is a relative minimum.
2. If $f''(c) < 0$, then $f(c)$ is a relative maximum.
3. If $f''(c) = 0$ or $f''(c)$ does not exist, then the test gives no information about extrema, therefore, use the first derivative test.

Note 12.3.2. *The second derivative test only applies for critical numbers c for which $f'(c) = 0$ but not where $f'(c)$ does not exist (since $f''(c)$ will not exist either)*

Example 12.3.3. *Use the second derivative test to distinguish between the critical numbers in $f(x) = 3x^3 - 3 + 1$.*

Solution: $f'(x) = 9x^2 - 6x = 3x(3x - 2) = 0 \Rightarrow x = 0, \frac{2}{3}$ are critical numbers. $f''(x) = 18x - 6$

$f''(0) = -6 < 0 \Rightarrow$ at $x = 0$ we have a relative maximum.

$f''(\frac{2}{3}) = 6 > 0 \Rightarrow$ at $x = \frac{2}{3}$ we have a relative minimum.

12.4 Absolute Extrema

Definition **12.4.1** (Absolute maximum and absolute minimum)**.** *Let f be a function defined on some interval I. Let c be a number in I. Then*

1. *$f(c)$ is the absolute (or global) maximum of f on I if $f(x) \leq f(c)$ for every x in I.*
2. *$f(c)$ is the absolute (or global) minimum of f on I if $f(x) \geq f(c)$ for every x in I.*

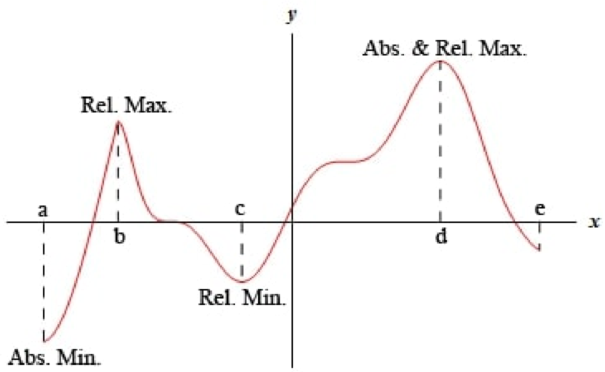

Figure 2: Absolute extrema

A function has an absolute extrenum (plural: extrema) at c if it has either an absolute maximum or absolute minimum there.

Remark **12.4.1.** *1. Absolute extrema may occur at the end points or at relative extrema*

2. *Although a function can have only one absolute minimum (or maximum) value, it can have many points where these values occur e.g $f(x) = 2$ has abs min=2 and abs max=2 and both occur at every real number x.*
3. *A continuous function on an open interval may or may not have an absolute maximum (or minimum)*

Theorem **12.4.1** (Extreme Value Theorem)**.** *A function f that is continuous on a closed interval $[a, b]$ will have both an absolute maximum and an absolute minimum.*

Caution: Just like a relative extrenum, an absolute extrenum is a y-value not an x-value.

Finding absolute extrema

To find absolute extrema for a function f continuous on a closed interval $[a, b]$

1. Find all critical numbers for f in (a, b)
2. Evaluate f for all critical numbers in (a, b)
3. Evaluate f for the endpoints a and b of the interval $[a, b]$

4. The largest value in 2 or 3 is the absolute maximum for f on $[a, b]$ and the smallest value is the value is the absolute minimum for f on $[a, b]$.

Example 12.4.1. *Find the absolute extrema of the function* $f(x) = x^{\frac{8}{3}} - 16x^{\frac{2}{3}}$ *on* $[-1, 8]$

Solution:

$$\begin{aligned} f'(x) &= \frac{8}{3}x^{\frac{5}{3}} - \frac{32}{3}x^{-\frac{1}{3}} \\ &= \frac{8}{3}x^{-\frac{1}{3}}(x^2 - 4) \\ &= \frac{8}{3}\left(\frac{x^2 - 4}{x^{\frac{1}{3}}}\right) \end{aligned}$$

$f'(\pm 2) = 0$ and $f'(0)$ does not exist $\Rightarrow x = \pm 2,\ 0$ are critical numbers but $-2 \notin [-1, 8]$ so ignore it. The Table 2 gives the absolute extrema candidates

Table 2: Absolute extrema for $f(x) = x^{\frac{8}{3}} - 16x^{\frac{2}{3}}$

x-value	value of $f(x)$
-1	-15
0	0
2	-19.05
8	192

Therefore, the absolute maximum=192 and absolute minimum=-19.05.

Exercise 12.4.1. *Find the absolute extrema of the functions* $f(x) = 3x^4 - 4x^3 - 12x^2 + 2$.

12.5 Horizontal, Verticle and Oblique Asymptotes

Definition 12.5.1 (Horizontal and Verticle Asymptotes)**.** *A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either*

$$\lim_{x\to\infty} f(x) = b \ \textit{or} \ \lim_{x\to-\infty} f(x) = b$$

*A line $x = a$ is a **vertical asymptote** of the graph if either*

$$\lim_{x\to a^+} f(x) = \pm\infty \ \textit{or} \ \lim_{x\to a^-} f(x) = \pm\infty$$

Example 12.5.1. *Find the asymptotes of* $y = \dfrac{x}{x-1}$

Solution: $y = 1 + \dfrac{1}{x-1}$, $\lim\limits_{x\to+\infty} \dfrac{x}{x-1} = 1$ and $\lim\limits_{x\to-\infty} \dfrac{x}{x-1} = 1 \Rightarrow y = 1$ is an horizontal asymptote.
Similarly, $\lim\limits_{x\to 1^-} \dfrac{x}{x-1} = -\infty$ and $\lim\limits_{x\to 1^+} \dfrac{x}{x-1} = \infty \Rightarrow x = 1$ is a vertical asymptote.

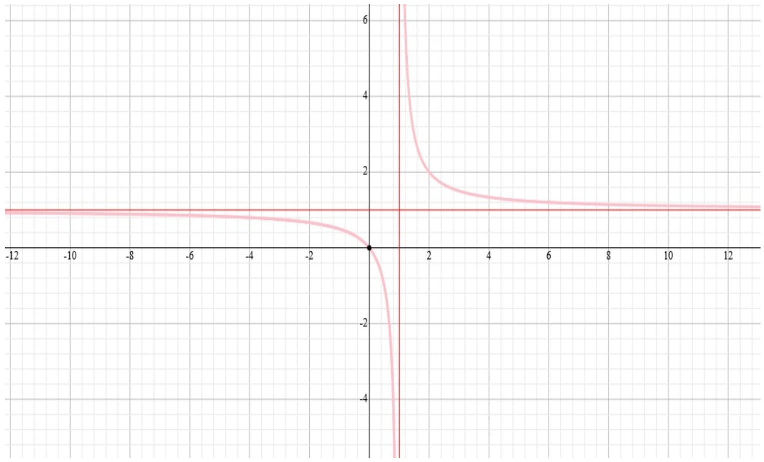

Figure 3: $y = \frac{x}{x-1}$

Example 12.5.2. *The following are examples that explain vertical asymptotes.*

1. *The function* $f(x) = \dfrac{5x-1}{x-3}$ *has the line* $x = 3$ *as a vertical asymptote.* $\lim_{x\to 3^+} f(x) = +\infty$ *and* $\lim_{x\to 3^-} f(x) = -\infty$. *(see Figure 4)*

2. *Similarly, the function* $\dfrac{x(x-1)}{(x-5)(x+2)}$ *has the lines* $x = 5$ *and* $x = -2$ *as vertical asymptotes.*

3. *The function* $\dfrac{x-1}{x^2+1}$ *has no vertical asymptotes as* $x^2+1 \neq 0$ *for any real number* x.

4. *The function* $f(x) = \tan x = \dfrac{\sin x}{\cos x}$ *has infinitely many vertical asymptotes* $x = \pm\frac{3}{2}\pi, \pm\frac{5}{2}\pi, \pm\frac{7}{2}\pi, \cdots$ *since* $\cos x = 0$ *at* $x = \pm\frac{3}{2}\pi, \pm\frac{5}{2}\pi, \pm\frac{7}{2}\pi, \cdots$

5. *The function* $f(x) = \dfrac{x^2-9}{x-3}$ *has no vertical asymptote since* $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = 6$. *The graph has a "hole" at* $x = 3$

Example 12.5.3. *Find the horizontal asymptote in the functions*

1. $f(x) = \dfrac{5x-1}{x-3}$

   Solution: $\lim_{\pm\infty} \dfrac{5x-1}{x-3} = \lim_{\pm\infty} 5 + \dfrac{14}{x-3} = 5$. *So* $y = 5$ *is the horizontal asymptote. (see Figure 4)*

2. $f(x) = \dfrac{5x^2-7}{9x^2+3}$

   Solution: $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \dfrac{5-\frac{7}{x^2}}{9+\frac{3}{x^2}} = \dfrac{5}{9} = \lim_{x\to+\infty} \dfrac{5-\frac{7}{x^2}}{9+\frac{3}{x^2}}$ *(see Figure 5)*

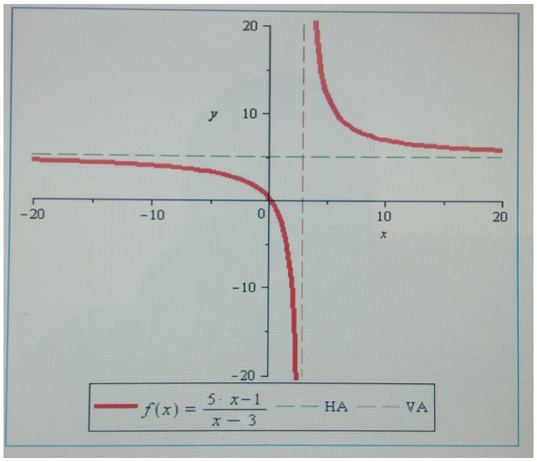

Figure 4

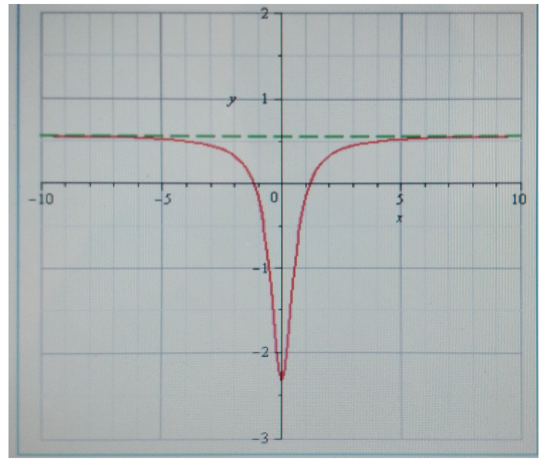

Figure 5

Example 12.5.4. *Find the horizontal and vertical asymptotes of* $f(x) = \dfrac{x^2-1}{x^3}$

Solution: $\dfrac{x^2-1}{x^3} = \dfrac{1}{x} - \dfrac{1}{x^3}$. $\lim\limits_{x\to\pm\infty} \dfrac{1}{x} - \dfrac{1}{x^3} = 0 \Rightarrow y = 0$ is a horizontal asymptote.

Similarly, $\lim\limits_{x\to 0^-} \dfrac{1}{x} - \dfrac{1}{x^3} = +\infty$ and $\lim\limits_{x\to 0^+} \dfrac{1}{x} - \dfrac{1}{x^3} = -\infty \Rightarrow x = 0$ is a vertical asymptote.

Note 12.5.1. *A rational function* $f(x) = \dfrac{p(x)}{q(x)}$ *has:*

1. *A vertical asymptote when $q(x) = 0$ (with $f(x)$ in its simplest form).*
   Example: $f(x) = \dfrac{2x-1}{x^2-1} = \dfrac{2x-1}{(x-1)(x+1)}$. *The denominator is zero when $x = -1$ and $x = 1$. Therefore, $x = -1$ and $x = 1$ are vertical asymptotes.*

2. *A horizontal asymptote when the deg $p(x) \leq$ deg $q(x)$.*

   - *If deg $p(x) <$ deg $q(x)$, then the horizontal asymptote is $y = 0$ (x-axis)*
     Example: $f(x) = \dfrac{2x-1}{x^2-1}$, *deg $p(x) = 1$, deg $q(x) = 2$. So $y = 0$ is a horizontal asymptote.*
   - *If deg $p(x) =$ deg $q(x)$, the horizontal asymptote is $y = \frac{a}{b}$ where a is the leading coefficient of the numerator $p(x)$ and b is the leading coefficient of the denominator $q(x)$.*
     Example: $\dfrac{2x^2-1}{x^2-1}$, *deg $p(x) =$ deg $q(x) = 2$. So the horizontal asymptote is $y = \frac{2}{1} = 2$.*

Definition 12.5.2 (Oblique Asymptote)**.** *The line $y = mx + c$ is an oblique (or slant) asymptote to the graph $y = f(x)$ if $f(x) = mx + c$* $\lim\limits_{x\to\pm\infty} g(x) = 0$

$\lim\limits_{x\to\pm\infty} g(x) = 0$ means that the graph of $f(x)$ approaches the graph of $y = mx + c$ as x approaches ∞.

Note 12.5.2. *1. When $m = 0$, in $y = mx + b$, this becomes a horizontal asymptote.*

2. For a rational function $f(x) = \dfrac{p(x)}{q(x)}$, an oblique asymptote occurs when the degree of the numerator $p(x)$ is exactly 1 greater that of the denominator $q(x)$.

Example 12.5.5. *The function $f(x) = \dfrac{x^2+1}{x-1} = x + 1 + \dfrac{2}{x-1}$ has the line $y = x + 1$ oblique asymptote since $\lim\limits_{x\to\pm\infty} g(x) = \lim\limits_{x\to\pm\infty} \frac{2}{x=1} = 0$. (Figure 6)*

Example 12.5.6. *Find the oblique asymptote in $g(x) = \dfrac{x^2+2x-3}{x+1}$.*

Solution: *By long division $g(x) = x + 1 + \dfrac{-4}{x+1}$. Therefore, $y = x + 1$ is an oblique asymptote. (see Figure 7)*

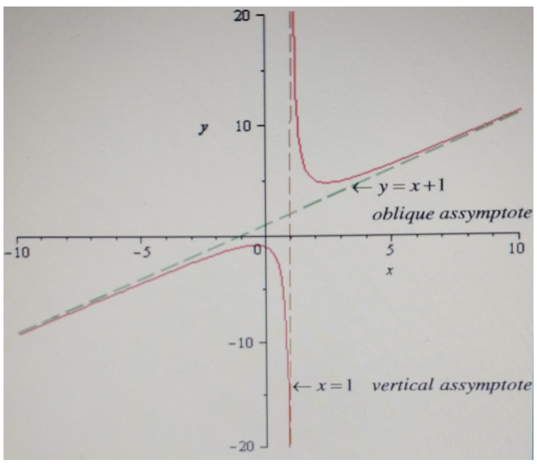

Figure 6

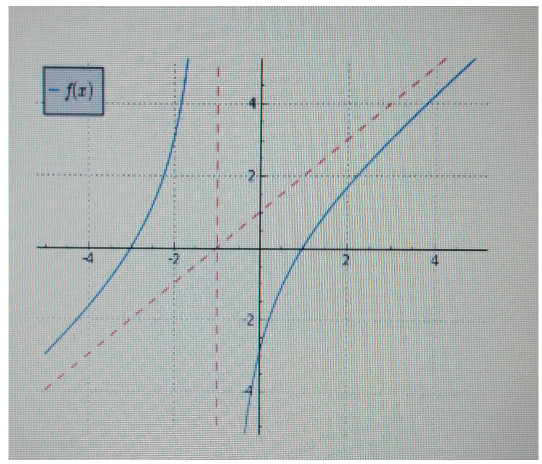

Figure 7

Note 12.5.3. *1. You can never have both horizontal and obliqye asymptotes on the same graph. (why?)*

2. Graphs can cross both horizontal and oblique asymptotes but will never cross a verticel asymptote.

12.6 Session Summary

We emphasize that getting the first derivative right is very key since subsequent derivatives depend on it. For more material on this section check out [1, 2] or visit concavity, absolute extrema, asymptotes 1, asymptotes 2. You can also watch the lecture videos: Concavity, absolute extrema, asymptotes.

12.7 Student Activity

Exercise

Concavity of a Graph

In questions 1–4, Find all intervals where $f(x) = x^2 + 10x - 9$ is concave up or down and find all inflection points.

1. $f(x) = (x-1)^4$
2. $f(x) = x^{\frac{1}{3}}$
3. $f(x) = x^{\frac{8}{3}} - 4x^{\frac{5}{3}}$
4. $f(x) = -2x^3 + 9x^2 + 168x - 3$

Second Derivative Test

Use the second derivative test to distinguish between the critical numbers f.

1. $f(x) = 4x^3 + 7x^2 - 10x + 8$
2. $f(x) = x^2 + 10x - 9$

Horizontal, Verticle and Oblique Asymptotes

Find the horizontal, verticle and oblique asymptotes in the following functions and sketch the graphs.

1. $f(x) = x + \dfrac{1}{x}$
2. $f(x) = \dfrac{2x}{x-3}$
3. $f(x) = \dfrac{x-1}{x-2}$
4. $f(x) = \dfrac{x^3 - 2x^2 - x + 2}{x^2 - x - 6}$
5. $f(x) = \dfrac{5x^3 + 2x^2 + 3}{x^2}$
6. $f(x) = \dfrac{3x^2 + x - 2}{2x + 6}$

References

[1] E. Purcell D. Varberg and S. Rigdon. *Calculus.* Pearson Education, Inc., 9 edition, 2006. ISBN-13 : 978-0132306331.

[2] J. Stewart. *Calculus.* Cengage Learning 20 Channel Center Street, Boston, MA 02210, USA, 8 edition, 2016. ISBN-13: 978-1-305-27176-0.

