g) $y = e^{2x}\cos 3x$

2) Find the derivative of each of the following functions with respect to x.

a) $y = 8^x - \frac{1}{4}x^4 + 9$

b) $g(x) = \frac{\ln x^3}{\cos(x)}$

c) $x^2y^2 - 3x^2 + y^4 = 5$

## Lecture Seven

### Learning Objectives

By the end of this lecture the learner is expected to do the following:

- Define the inverse trigonometric functions and compute their derivatives
- Describe and sketch curves of functions that are given parametrically
- Compute the first and second derivatives of functions given parametrically

### 3.6 Differentiation of inverse trigonometric functions

The inverse trigonometric functions comprises of the inverse functions of the trigonometric functions together with their reciprocal functions. In this case we have the inverse functions to the sine, cosine, tangent, cosecant, secant and cotangent of an angle A given respectively as
$Sin^{-1}A,\ Cos^{-1}A,\ \tan^{-1}A,\ Co\sec^{-1}A,\ Sec^{-1}A\ \ and\ \ Cot^{-1}A$. These are also named by writing the word arc before the trigonometric function. The inverse trigonometric functions listed above can therefore be represented respectively as arcsinA, arccosA, arctanA, arccosecA, arcsecA and arccotA. Let us consider the derivatives of each of these inverse functions by way of example.

**Example 1**

Differentiate the function $y = Sin^{-1}x$.

**Solution**

Given the function $y = Sin^{-1}x$ then we have $x = Sin\,y$.

Differentiating the function x = sin y with respect to x we have

$$\frac{d}{dx}x = \frac{d}{dx}Sin\,y$$

$$1 = Cos\,y\frac{dy}{dx} \quad or \quad \frac{dy}{dx} = \frac{1}{Cos\,y}$$

From the pythagorean identity $Sin^2y + Cos^2y = 1$ we have $Cos\,y = \sqrt{1 - Sin^2y}$ and therefore

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-Sin^2 y}}$$

$$= \frac{1}{\sqrt{1-x^2}}$$

i.e. $\frac{d}{dx} Sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

**Example 2**

Differentiate the function $y = Cos^{-1}x$

**Solution**

Given the function $y = Cos^{-1}x$ then we have $x = Cos\, y$

Differentiating the function $x = Cos\, y$ with respect to x we have

$$\frac{d}{dx}x = \frac{d}{dx}Cos\, y$$

$$1 = -Sin\, y\frac{dy}{dx} \quad or \quad \frac{dy}{dx} = -\frac{1}{Sin\, y}$$

From the identity $Sin^2 y + C\,os^2\, y = 1$ we have $Sin\, y = \sqrt{1 - C\,os^2\, y}$ and therefore

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-Cos^2 y}}$$

$$= -\frac{1}{\sqrt{1-x^2}}$$

i.e. $\frac{d}{dx} Cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$

**Example 3**

Differentiate the function $y = \tan^{-1} x$

**Solution**

Given the function $y = \tan^{-1} x$ then we have $x = Tan\, y$

Differentiating the function $x = Tan\, y$ with respect to x we have

$$\frac{d}{dx}x = \frac{d}{dx}Tan\, y$$

$$1 = Sec^2\, y\frac{dy}{dx} \quad or \quad \frac{dy}{dx} = \frac{1}{Sec^2 y}$$

From the Pythagorean identity $Sec^2 y = 1 + T\,an^2\, y$ we have

$$\frac{dy}{dx} = \frac{1}{1+Tan^2 y}$$

$$= \frac{1}{1+x^2}$$

$$i.e. \quad \frac{d}{dx} Tan^{-1} x = \frac{1}{1+x^2}$$

### 3.7 Parametric differentiation

Some relationships between two quantities or variables may be complicated such that a third quantity or variable may be necessary to simplify the representation.
This third quantity is called a parameter. When a parameter is introduced, we will have two equations instead of one equation relating two variables say x and y. One equation will relate x and the parameter t and another equation will relate y and the parameter t.
A curve is defined parametrically as follows. Let f and g be continuous functions of t on an interval I. Then the equations x = g(t) and y = h(t) are called the parametric equations for the curve c generated by the ordered pair (x(t), y(t)).

**Example 4**
Plot the curve represented by the following parametric equations.
x=cost
y=sint  for $0 \leq t \leq 2\pi$

**Solution**
We can plot the curve on the xy-plane by getting values of x and y for given values of the parameter t (in radians) as presented in the following table.

| t | X | y |
|---|---|---|
| 0 | 1 | 0 |
| 1 | 0.54 | 0.84 |
| π/2 | 0 | 1 |
| 2 | -0.416 | 0.91 |
| Π | -1 | 0 |
| 3π/2 | 0 | -1 |

Plotting the points given by the x and y coordinates in the table above and joining them with a smooth curve we obtain the graph. On doing this we find out that the parametric equations define a circle centred at the origin and having a radius of 1 unit. Alternatively, we can also plot the curve by first eliminating the parameter. The process of eliminating a given parameter depends on the nature of the equations under consideration. In this example we use the Pythagorean identity $Sin^2 x + Cos^2 x = 1$. From the equations representing the curve we have $x^2 = Cos^2 t$, $y^2 = Sin^2 t$ and therefore $x^2 + y^2 = Cos^2 t + Sin^2 t = 1$ i.e. $x^2 + y^2 = 1$. This is an equation of a circle centred at the origin (0,0) and having a radius of 1 unit. We say that x = cos t, y = sin t is a parametrization of the curve $x^2 + y^2 = 1$. It should be noted that the parametrization of an equation is not unique. In other words, parametric equations of curves are not unique.

**Example 5**

Describe and sketch the curve represented by the parametric equations

$x = 2t,\ \ y = t^2 + 1$ for $-1 \le t \le 2$

**Solution**

We can plot the curve on the xy-plane by getting values of x and y for given values of the parameter t (in radians) as presented in the following table.

| t | x | y |
|---|---|---|
| -1 | -2 | 0 |
| -1/2 | -1 | -3/4 |
| ½ | 1 | 3/4 |
| 1 | 2 | 0 |
| 3/2 | 3 | 5/4 |
| 2 | 4 | 3 |

Plotting the points given by the x and y coordinates in the table above and joining them with a smooth curve we obtain a part of a parabola opening upwards and having a vertex (0,-1).

Eliminating the parameter we have $t = \frac{x}{2}$ and $y = \left(\frac{x}{2}\right)^2 - 1$ *or* $y + 1 = \frac{1}{4}x^2$

Generally the conic sections are represented parametrically by the following equations:

- A circle is represented by $x = r\,Cos\ \theta,\ \ y = r\,Sin\ \theta$
- An ellipse is represented by: $x = a\,Cos\ \theta,\ \ y = b\,Sin\ \theta$
- A hyperbola is represented by $x = a\,Sec\ \theta,\ \ y = b\tan\ \theta$

**Example 6**

The curve of $x = t^2 - 9,\ \ y = \frac{1}{3}t\ \ for\ \ -3 \le t \le 2$ is the same as that for

$x = 9\left(9t^2 - 1\right),\ \ y = 3t\ \ for -\frac{1}{3} \le t \le \frac{2}{9}$

**First derivative**

If a function is defined parametrically we use the chain rule of differentiation.

From the chain rule we have $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$.

On rearrangement we have $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ provided that $\frac{dx}{dt} \neq 0$.

**Example 7**

Compute $\frac{dy}{dx}$ given that x = cost, y = sint.

**Solution**

$$\frac{dy}{dt} = Cos\,t$$
$$\frac{dx}{dt} = -Sin\,t$$
$$\frac{dy/dt}{dx/dt} = \frac{Cos\,t}{-Sin\,t} = -Cot\,t$$

**Example 8**

Find $\frac{dy}{dx}$ given that $x = t^3 - t,\ y = 4 - t^2$.

**Solution**

$$\frac{dy}{dt} = -2t$$
$$\frac{dx}{dt} = 3t^2 - 1$$
$$\frac{dy}{dx} = \frac{-2t}{3t^2 - 1}$$

insert example on stationary points

**Example 9**

**Second derivative**

Given that x = x(t) and y = y(t) we can compute the second derivative of y with respect to x. Using the chain rule of differentiation we have

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}$$

**Example 10**

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point where t = 3 for a curve represented parametrically by x = 7t + 2 and $y = t^3 - 12t$.

**Solution**

$$\frac{dy}{dt} = 3t^2 - 12$$

$$\frac{dx}{dt} = 7$$

$$\frac{dy}{dx} = \frac{3t^2 - 12}{7}$$

$$\left.\frac{dy}{dx}\right|_{t=3} = \frac{27-12}{7} = \frac{15}{7}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{\frac{d}{dt}\left(\frac{3t^2-12}{7}\right)}{7}$$

$$= \frac{6t}{49}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=3} = \frac{18}{49}$$

**Example 11**

Find $\frac{d^2y}{dx^2}$ when $x = t^2$ *and* $y = t^3$ .

**Solution**

$$\frac{dy}{dt} = 3t^2$$

$$\frac{dx}{dt} = 2t$$

$$\frac{dy}{dx} = \frac{3t^2}{2t} = \frac{3t}{2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{\frac{d}{dt}\left(\frac{3t}{2}\right)}{2t}$$

$$= \frac{3}{4t}$$

**Arc length of a curve described parametrically**

If a curve is described parametrically by the equations x = x(t) and y = y(t) on an interval [a,b] where $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are continuous and c does not intersect itself (it is a simple curve) then the arc length of c is $\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

Recall that for a function defined as y = f(x), the arc length is $s=\int_a^b\sqrt{1+\left(\frac{df(x)}{dx}\right)^2}\,dx$

If y is a continuous function of x on [a,b] and that x = x(t), y = y(t) for $t_1 \le t \le t_2$ where x(t1) = a and x(t2) = b for a simple closed curve then x'(t) is continuous on [t1, t2] and the area under a curve described parametrically by x = x(t) and y = y(t) is given by $A=\int_a^b y\,dx=\int_{t_1}^{t_2}\left[y(t)\frac{dx}{dt}\right]dt$

**Exercise 7**

1. Differentiate the following functions with respect to x.
   a. $f(x)=4Cos^{-1}x-10\tan^{-1}x$
   b. $y=\sqrt{x}\;Sin^{-1}x$
2. Prove that
   a. $\frac{d}{dx}\sec^{-1}x=\frac{1}{x\sqrt{x^2-1}}$
   b. $\frac{d}{dx}Csc^{-1}x=-\frac{1}{x\sqrt{x^2-1}}$
   c. $\frac{d}{dx}Cot^{-1}x=-\frac{1}{1+x^2}$
3. Differentiate each of the following functions with respect to x.
   a. $y=Cos^{-1}\left(Sin^{-1}x\right)$
   b. $y=Cos^{-1}(3x)$
4. Determine dy/dx given the following functions hence identify the stationary points.
   a. $x=t^2+1\ and\ y=t^3-1$
   b. $x=3Cost\ and\ y=3Sint$
   c. $x=te^{-t}\ and\ y=2t^2+1$
5. Compute $\frac{d^2y}{dx^2}$ in each of the following functions
   a. $x=t^3+3t^2,\ y=t^4-8t^2$
   b. $x=Sin\,t\ and\ y=Cost$
   c. $x=\frac{1}{2}t^2+2,\ y=Sin(t+1)$
   d. $x=e^{-t},\ y=t^3+t+1$
   e. $x=3t^2+4t,\ y=Sin2t$

**Lecture Eight**

**Learning Objectives**

By the end of this lecture the learner is expected to do the following:

- Discuss the applications of differentiation in geometry and mechanics.

### 3.9 Applications of differentiation

We discuss the various applications to differentiation in relation to the following:

(i) Geometry

(ii) Mechanics in particular velocity and acceleration

(iii) Turning points

### 3.9.1 Application to Geometry

The applications considered here include the following:

Application to tangent and normal lines

Application to maximization of area

Application to maximization of volume

**a) Tangent and normal lines**

A tangent to a curve at a point is a straight line touching the curve at that point. The gradient to a curve at a point is equal to the gradient of the tangent to the curve at that point.

**Example 29**

Find the equation of the tangent line to the curve $y = x^3 + 2x$ at the point where $x = 1$.

**Solution**

When $x = 1$, $y = 1^3 + 2(1) = 3$, we are therefore interested with the gradient of the curve at the point $(1,3)$.

The gradient of the curve at any point is $\frac{dy}{dx} = \frac{d}{dx}\left(x^3 + 2x\right) = 3x^2 + 2$.

The gradient of the curve at the point $(1,3)$ is $\left.\frac{dy}{dx}\right|_{x=1} = 3x^2 + 2\Big|_{x=1} = 3(1)^2 + 2 = 5$.

The gradient of the tangent to the curve at the point $(1,3)$ is therefore $m = 5$. Writing the equation of the tangent to the curve to be $y = mx + c$ and since we have $m = 5$ then the equation of the tangent line is $y = 5x + c$. The point $(1,3)$ is common both to the curve and to the tangent line and therefore its coordinates must satisfy the equations of both curves. Substituting these coordinates in the equation of the tangent line we have $3 = 5(1) + c$ and solving for c we have $c = 3 - 5$, $c = -2$. Therefore the equation of the tangent line to the curve is $y = 5x - 2$.

The normal line and the tangent line are perpendicular to each other. From geometry, it is known that if two straight lines are perpendicular to each other then the product of their gradients is negative one. Therefore if $m_T$ is the gradient of the tangent line and $m_n$ is the gradient of the normal line then $m_T m_n = -1$. Formulating we have $m_T = -\dfrac{1}{m_n}$ and $m_n = -\dfrac{1}{m_T}$. Let us demonstrate how to get the equations of the tangent lines and equations of the normal lines using some examples.

**Example 30**

Find the equation of the tangent and the normal lines to the curve $y = x^2 - 3x$ at the point where $x = 2$.

**Solution**

Let $m_T$ be the gradient of the tangent line and $m_n$ be the gradient of the normal line to the curve, then

$$m_T = \frac{d}{dx}\left(x^2 - 3x\right) = 2x - 3$$
$$m_T\big|_{x=2} = 2(2) - 3 = 1$$

The equation of the tangent line is therefore $y = x + c$. Since $m = 1$ is substituted in the equation $y = mx + c$.

When $x = 2$, $y = x^2 - 3x = 2^2 - 3(2) = -2$ and thus the point $(2,-2)$ is on the tangent line and must satisfy the equation of the tangent line i.e. $y = x + c$. Substituting the values of x and y into the equation we have $-2 = 2 + c \Rightarrow c = -4$ and the equation of the tangent line is therefore $y = x - 4$.

Since $m_T = 1$, then $m_n = -\frac{1}{1} = -1$ and equation of normal line is $y = -x + c$. The point $(2,-2)$ must also be on the normal line. Actually this point is the intersection of the curve, the tangent line and the normal line. At the point $(2,-2)$, we have $-2 = -2 + c$ or $c = 0$ and the equation of normal line is $y = -x + 0$, *i.e.* $y = -x$.

### 3.9.2 Application to Velocity and acceleration

Velocity is the rate of change of displacement with time. This is denoted by $v = \frac{dS}{dt}$ and is the derivative of displacement with respect to time. Acceleration is the rate of change of velocity with time. This is denoted by $a = \frac{dv}{dt}$ and is the derivative of velocity with respect to time. Since $v = \frac{dS}{dt}$ and $a = \frac{dv}{dt}$ then

$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dS}{dt}\right) = \frac{d^2S}{dt^2}$. That is the second derivative of the displacement with respect to time also gives the acceleration.

**Example 31**

Find the velocity of a body moving in a straight line with a displacement given in meters by $S = t^3 - 2t + 4$ at $t = 5$ seconds.

**Solution**

$$v(t) = \frac{dS}{dt} = \frac{d}{dt}\left(t^3 - 2t + 4\right) = 3t^2 - 2$$
$$v(5) = 3(5)^2 - 2 = 73m/s$$

**Example 32**

Calculate the tangential velocity and acceleration of a particle when $t = 2$ given that $S(t) = t^3 + 3t^2 + 4t + 2$.

**Solution**

$$v = \frac{dS}{dt} = \frac{d}{dt}\left(t^3 + 3t^2 + 4t + 2\right) = 3t^2 + 6t + 4$$
$$v(2) = 3(2)^2 + 6(2) + 4$$
$$= 28m/s$$

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(3t^2 + 6t + 4\right) = 6t + 6$$
$$a(2) = 6(2) + 6$$
$$= 18m/s^2$$

**Lecture Nine**

**Learning Objectives**

By the end of this lecture the learner is expected to do the following:

- Identify and classify stationary points using the gradient and the second derivative test.
- Discuss the applications of differentiation in optimization: maximizing area and volume, minimizing cost.

### 3.9.3 Application to turning points

For any function $f(x)$, a turning point is also called a stationary point. A stationary value is the value of the dependent variable y at the stationary point. In particular, if the stationary point p has co-ordinates (a, b) then the stationary value of the function at point p is $y = b$. At a stationary point we have the rate of change of the function with respect to the independent variable being equal to zero. For the function $f(x)$ we therefore have $\frac{d}{dx}f(x) = 0$. Geometrically, this can be explained by the fact that the tangent to a curve at a turning point is a horizontal straight line. Further, the gradient of any horizontal straight line is always zero.

**Example 33**

Find the stationary values and stationary points of the following functions.

a) $y = x^3 - 3x^2 + 2$

b) $y = (x-2)(x+3)$

**Solution**

a) $$\frac{dy}{dx} = \frac{d}{dx}\left(x^3 - 3x^2 + 2\right)$$
$$= 3x^2 - 6x$$

For stationary points we have $\frac{dy}{dx} = 0$ and therefore

$$3x^2 - 6x = 0$$
$$3x(x-2) = 0$$
$$x(x-2) = 0$$

solving for x we have either $x = 0$ or $x = 2$. Solving for y from the given equation we have, when $x=0$, $y=0^3-3(0)^2+2=2$ and when $x=2$, $y=2^3-3(2)^2+2=-2$. The stationary points are therefore $(0,2)$ and $(2,-2)$.

b) For stationary points we have $\frac{d}{dx}\{(x-2)(x+3)\}=0$. To differentiate this function you can expand the function and then differentiate the resulting polynomial or you can differentiate the function as a product function. In either case we have $\frac{d}{dx}\{(x-2)(x+3)\}=2x+1$. ( the reader is left to work this out and show that this is true. For the stationary point therefore $2x+1=0$ or $x=-\frac{1}{2}$. For this value of x we have the stationary value as $y=\left(-\frac{1}{2}-2\right)\left(-\frac{1}{2}+3\right)=-\frac{5}{2}\times\frac{5}{2}=-\frac{25}{4}$ and the stationary point is therefore $\left(-\frac{1}{2},-\frac{25}{4}\right)$.

Stationary points fall into any one of the three cases namely:

(i) Maximum point
(ii) Minimum point
(iii) Point of inflection

Besides identifying a stationary point, it may be necessary to classify the particular stationary point. Among the methods used in investigating the nature of stationary points are

(i) the method of gradient and
(ii) the second derivative test method.

Let us consider each of these methods.

**The gradient method**

This method depends on the graphical representation of each of the curve around the turning point.

**Diagram max, min & inflection pt**

Considering each turning point at a time we can classify each point using the gradient method as follows:

(i) The curve around a **maximum turning point** has the gradient of the tangent to the curve being positive to the left of the point, the gradient is zero at the turning point and the gradient to the tangent to the right of the point is negative. If we describe this property in terms of the function we say that the curve is increasing to the left of the turning point and the curve is decreasing to the right of the point.

(ii) The curve around a **minimum turning point** has the gradient of the tangent to the curve being negative to the left of the point, the gradient is zero at the turning point and the gradient to the tangent to the right of the point is positive. If we describe this property in terms of the function we say that the curve is decreasing to the left of the turning point and the curve is increasing to the right of the point.

(iii) The point of inflection can take either of the forms shown in the figure given above. For the first case, we have the gradient of the tangent to the curve being positive to the left of the point, the gradient is zero at the turning point and the gradient to the tangent to the right of the point is also positive. If we describe this property in terms of the function we say that the curve is increasing both to the left and the right of the turning point. For the second case, we have the gradient of the tangent to the curve being negative to the left of the point, the gradient is zero at the turning point and the gradient to the tangent to the right of the point is also negative. If we describe this property in terms of the function we say that the curve is decreasing both to the left and the right of the turning point.

The idea in this method is to consider a small neighbourhood about the turning point and then evaluate the gradient of the curve at these points as illustrated in the examples that follow.

**Example 34**

Use the method of gradient to classify the turning points of the function

$f(x)=\frac{1}{4}\left(x^4-4x^3\right)$.

**Solution**

We first identify the stationary point or points of the function and then classify the points.

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left\{\frac{1}{4}\left(x^4-4x^3\right)\right\}$$
$$=\frac{1}{4}\left(4x^3-12x^2\right)$$
$$=x^3-3x^2$$

For stationary values, $x^3-3x^2=0$. On factoring we have $x^2(x-3)=0$ which means that either $x=0$ or $x=3$. The stationary values are $f(0)=\frac{1}{4}\left(0^4-4(0)^3\right)=0$ and $f(3)=\frac{1}{4}\left(3^4-4(3)^3\right)=-\frac{27}{4}$. The stationary points are therefore $(x,y)=(0,0)$ and $(x,y)=\left(3,-\frac{27}{4}\right)$.

After identifying the points, we can now classify each of them.

Consider the point $(x,y)=(0,0)$.

To the left of this point where $x=0$ we consider the point where $x=-\frac{1}{2}$ and we have $\left.\frac{df}{dx}\right|_{x=-\frac{1}{2}}=\left(-\frac{1}{2}\right)^3-3\left(-\frac{1}{2}\right)^2=-\frac{1}{8}-\frac{3}{4}=\frac{-1-6}{8}=-\frac{7}{8}$. i.e. $\left.\frac{df}{dx}\right|_{x=-\frac{1}{2}}=-\frac{7}{8}$. But $-\frac{7}{8}<0$.

To the right of this point where $x=0$ we consider the point where $x=\frac{1}{2}$ and we have $\left.\frac{df}{dx}\right|_{x=\frac{1}{2}}=\left(\frac{1}{2}\right)^3-3\left(\frac{1}{2}\right)^2=\frac{1}{8}-\frac{3}{4}=\frac{1-6}{8}=-\frac{5}{8}$. i.e. $\left.\frac{df}{dx}\right|_{x=\frac{1}{2}}=-\frac{5}{8}$. $-\frac{5}{8}$ is also negative. Since the gradient of the curve both to the left and the right of the point $(x,y)=(0,0)$ is negative then the point $(0,0)$ is a point of inflection.

Consider the point $(x,y)=\left(3,-\frac{27}{4}\right)$.

To the left of this point where $x = 3$ we consider the point where $x = 2\frac{1}{2}$ and we have $\left.\frac{df}{dx}\right|_{x=\frac{5}{2}} = \left(\frac{5}{2}\right)^3 - 3\left(\frac{5}{2}\right)^2 = \frac{125}{8} - \frac{75}{4} = -\frac{25}{8}$. i.e. $\left.\frac{df}{dx}\right|_{x=-\frac{1}{2}} = -\frac{25}{8}$. $-\frac{25}{8}$ is negative.

To the right of this point where $x = 3$ we consider the point where $x = 3\frac{1}{2}$ and we have $\left.\frac{df}{dx}\right|_{x=\frac{7}{2}} = \left(\frac{7}{2}\right)^3 - 3\left(\frac{7}{2}\right)^2 = \frac{343}{8} - \frac{147}{4} = \frac{49}{8}$. i.e. $\left.\frac{df}{dx}\right|_{x=\frac{7}{2}} = \frac{49}{8}$. $\frac{49}{8}$ is positive.

Since the gradient of the curve to the left of the point $\left(3, -\frac{27}{4}\right)$ is negative and the gradient of the curve to the right of the point is positive then the point $\left(3, -\frac{27}{4}\right)$ is a minimum point.

**Second derivative test**

This is another method used in classifying stationary points. The stationary points for the function $y = f(x)$ are sought by considering the first derivative $\frac{dy}{dx} = 0$ . By taking the second derivative $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$, then the stationary point can be classified. The method classifies the stationary points as follows:

a) The stationary point is a minimum point if the second derivative is positive.

   i.e. $\frac{d^2y}{dx^2} > 0$.

b) The stationary point is a maximum point if the second derivative is negative.

   i.e. $\frac{d^2y}{dx^2} < 0$.

c) If $\frac{d^2y}{dx^2} = 0$ then compute the third derivative such that $\frac{d^3y}{dx^3} \neq 0$, otherwise if $\frac{d^3y}{dx^3} = 0$ then compute the fourth derivative and so on until a point where you can use the conditions in part (a) or (b).

Let us consider some examples to clarify how this method is applied.

**Example 35**

Use the second derivative test to classify the stationary points of the following functions.

a) $y=(x-3)(2x+1)$

b) $y=x^2(3-x)$

c) $y=2x^3-15x^2+24x+19$

**Solution**

a)

$\frac{dy}{dx}=2(x-3)+2x+1=4x-5$ for the stationary point we have

$4x-5=0 \Rightarrow x=\frac{5}{4}$

The stationary value is $y\left(\frac{5}{4}\right)=\left(\frac{5}{4}-3\right)\left(2\left\{\frac{5}{4}\right\}+1\right)=-\frac{49}{8}$ and the stationary point is therefore the point $\left(\frac{5}{4},\frac{-49}{8}\right)$.

To classify this point we take the second derivative of the function.

$\frac{d^2y}{dx^2}=\frac{d}{dx}(4x-5)=4$. Since $\frac{d^2y}{dx^2}=4$ and $4>0$, then the point $\left(\frac{5}{4},\frac{-49}{8}\right)$ is a minimum point.

b)

$\frac{dy}{dx}=2x(3-x)-x^2=6x-3x^2$. For the stationary point we have $\frac{dy}{dx}=0$ and therefore $6x-3x^2=3x(2-x)=0$.

Solving this equation we have either $x=0$ or $x=2$. The stationary values are $y(0)=0^2(3-0)=0$ and $y(2)=2^2(3-2)=4$ and the stationary points are $(0,0)$ and $(2,4)$.

Taking the second derivative we have $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(6x-3x^2\right)=6-6x$. At the point $(0,0)$ we have $\left.\frac{d^2y}{dx^2}\right|_{x=0} = 6-0=6$. Since $6>0$ then the point $(0,0)$ is a minimum point.

At the point $(2,4)$ we have $\left.\frac{d^2y}{dx^2}\right|_{x=2} = 6-12=-6$. Since $-6<0$, then the point $(2,4)$ is a maximum point.

c) We take the first derivative and equate it to zero so that we can get the stationary point.

$$\frac{dy}{dx} = 6x^2-30x+24$$
$$6x^2-30x+24=0,\ \ x^2-5x+4=0$$
$$(x-4)(x-1)=0$$

either $x=1$ or $x=4$. The stationary values are

$y(1)=2(1)^3-15(1)^2+24(1)+19=30$ and

$y(4)=2(4)^3-15(4)^2+24(4)+19=3$. The stationary points are therefore the points $(1,30)$ and $(4,3)$.

Taking the second derivative helps us to classify these points and doing this we have

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(6x^2-30x+24\right)=12x-30.$$

at the point $(1,30)$ we have $\left.\frac{d^2y}{dx^2}\right|_{x=1} = 12-30=-18$. At the point $(4,3)$ we have $\left.\frac{d^2y}{dx^2}\right|_{x=4} = 48-30=18$. From the values obtained from the second derivatives that is $-18<0$ and $18>0$ respectively then we classify the point $(1,30)$ as a maximum point and the point $(4,3)$ as a minimum point.

### 3.9.4 Applications to optimization

**a) Maximizing area using trigonometry**

Another application of differentiation is found in trigonometry. In particular we can use derivatives to look for a trigonometric figure that will give the maximum area as revealed in the following example.

**Example 36**

Two sides of a triangle are 4 cm long. What should the angle between these sides be to make the area of the triangle as large as possible?

**Solution**

**Diagram**

For a triangle with sides of length b and h, the area $A=\frac{1}{2}bh$. Given that $b=4cm$,

$$\sin\theta=\frac{h}{4}\Rightarrow h=4\sin\theta$$

$$A=\frac{1}{2}\times4\times4\sin\theta=8\sin\theta$$

For the domain we have $0\le\theta\le\pi$. In other words the least value of the angle $\theta$ is 0 and the largest value is $180^0$ or $\pi$ radians. The derivative $\frac{dA}{d\theta}$ is defined throughout this interval since $\frac{d}{d\theta}(8\sin\theta)=8\cos\theta$ and the cosine function is continuous for all values of the angle $\theta$.

The critical points are found when $\frac{dA}{d\theta}=0$

i.e.

$8\cos\theta=0$ and solving this equation we have $\theta=\frac{\pi}{2}$.

The value of the area at the critical points is given by

$A\left(\frac{\pi}{2}\right)=8\sin\frac{\pi}{2}=8$ and at the end points of the domain we have

$$A(0)=8\sin 0=0$$

$$A(\pi)=8\sin\pi=0$$

Therefore the maximum area is found when $\theta = \frac{\pi}{2}$ i.e. when the triangle is a right triangle with sides 4 cm and $A = \frac{1}{2} \times 4 \times 4 = 8cm^2$. We can generalize these results and state that in general, the maximum area of an isosceles triangle is found when the triangle is a right triangle.

**b) Maximizing a volume**

Another application is found in computing a volume. The property of differentiation can be used in optimization. We take two examples where one is used to show how you can maximize the use of a given material and the other example shows how to minimize the cost of some material.

**Example 37**

A box is to be made from a rectangular piece of cardboard 15cm by 24cm by cutting equal squares out of the corners and folding up the flaps. Find the dimensions for which the box will have greatest volume.

**Solution**

**Figure**

The length for the fold lines are $(15-2x)cm$ and $(24-2x)cm$. The dimensions for the required box are $(15-2x)\times(24-2x)\times x$ and the volume of the box is

$$\begin{aligned} V = f(x) &= (15-2x)\times(24-2x)\times x \\ &= 4x^3 - 78x^2 + 360x \end{aligned}$$

There are restrictions to the domain as follows:

The box cannot have zero dimensions and therefore the height $x \geq 0$ and the length $(15-2x) \geq 0$ or $x \leq \frac{15}{2}$ and combining these two we have the domain defined as $0 \leq x \leq \frac{15}{2}$. Considering the third side we have the restriction $(24-2x) \geq 0 \Rightarrow x \leq 12$ but $\frac{15}{2} < 12$ and therefore we take $x \leq \frac{15}{2}$.

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left(4x^3 - 78x^2 + 360x\right)$$
$$= 12x^2 - 156x + 360$$
$$= 12(x-3)(x-10)$$

For the turning points

$$f'(x) = 0$$
$$12(x-3)(x-10) = 0$$

Solving gives x as either 3 or 10. But $x = 10$ does not give meaningful sense since $x = 10$ is outside the domain so we take $x = 3$. Taking the second derivative we have:

$$f''(x) = 24x - 156$$
$$f''(3) = 72 - 156 = -84$$

Since $-84 < 0$ then we have $f(3) = 486$ as a relative maximum. Since $f(x)$ is differentiable throughout, it is possible that the greatest value of $f(x)$ occurs at the end point of the domain. Now $f(0) = 0$, $f\left(\frac{15}{2}\right) = 0$ and therefore this is not the case.

Therefore the dimensions of the greatest volume are $(15-2x)\times(24-2x)\times x$ when $x = 3$, i.e $3cm,\ 9cm\ and\ 18cm$. And the volume is $V = 3cm \times 9cm \times 18cm = 486cm^3$.

**c) Minimizing the cost**

**Example 38**

A box of volume $32m^3$ having a square base and no top is to be constructed from material costing sh 80 per square meter. Find the dimensions of such a box for which the cost of the material is least.

**Solution**

Let the sides of the box be x, x and y meters.

**Diagram**

The surface area is computed as $A = x^2 + 4xy$ $\qquad (i)$

The volume $V = x^2 y$ and we are given that $x^2 y = 32 \Rightarrow \; y = \frac{32}{x^2}$. Substituting this in equation (i) we have

$$A = f(x) = x^2 + 4x\left(\frac{32}{x^2}\right) = x^2 + \frac{128}{x}.$$

The domain of f is $\{x : x > 0\}$

$$\frac{dA}{dx} = f'(x) = 2x + 128\left(-\frac{1}{x^2}\right)$$

$$= \frac{2(x^3 - 64)}{x^2}$$

$f'(x) = 0$ when $x^3 = 64$ or $x = 4$.

Using the second derivative test we have

$$f''(x) = \frac{d}{dx}\left(\frac{2x^3 - 128}{x^2}\right)$$

$$= \frac{x^2(6x^2) - 2x\left(2x^3 - 128\right)}{x^4}$$

$$= \frac{2x^4 + 256x}{x^4}$$

$$f''(x)\Big|_{x=4} = \frac{2(4)^4 + 256(4)}{4^4} = 6$$

Since $6 > 0$, then $f(4)$ is a relative minimum. The function $A = f(x) = x^2 + \frac{128}{x}$ is differentiable everywhere in its domain and the domain has no end points. Since $f(x)$ has only one relative minimum then $f(4)$ is an absolute minimum and for $x = 4,\; y = \frac{32}{4^2} = 2$ and the required dimensions are therefore $4m \times 4m \times 2m$.

**Exercise 3**

1) Find the equation of the tangent and normal line to the curve $f(x) = 2x^3 + 4x^2 + 18$ at the point where $x = -1$.
2) Find the equation of the tangent and normal lines to the curve $y = 5x^2 - 2x + 8$ at the point where $x = 1$.

3) The function $y = ax^2 + bx + c$ has a gradient function $6x + 2$ and a stationary value of 1. Find the values of the constants a, b and c.
4) Find the stationary point of the function $f(x) = x^2 + 4x + 4$ and hence classify the nature of this point.
5) An open box is made from a rectangular sheet of cardboard by cutting out equal squares out of the corners and folding up the flaps. Find the dimensions for which the box will have the greatest volume given that the cardboard measures 18 cm by 30 cm.
6) An open box is made from a square sheet of cardboard with sides 50 cm long by cutting out equal squares out of the corners and folding up the flaps. Find the dimensions for which the box will have the greatest volume.

# 4. Introduction to Integral Calculus

**Lecture Eleven**

**Learning Objectives**

By the end of this lecture the learner is expected to do the following:

- Define integration as the reverse process of differentiation

- State and apply the fundamental theorem of calculus
- State and apply some basic integration formulas
- Discuss properties of definite integrals

**Introduction**

Calculus II is dedicated to integral calculus and this lecture aims at introducing the learner to integral Calculus as it relates to differential Calculus.

The process of integration can be viewed as the reverse process of differentiation. Integration aims at getting the function whose derived function is given.

**4.1 Anti-derivatives**

Anti differentiation is the reverse process to differentiation. A function $F(x)$ is called an anti derivative of the function $f(x)$ written as $\int f(x)dx = F(x)+c$ if the derivative of this function i.e. $\frac{dF(x)}{dx} = f(x)$ for all x in the domain of $f(x)$. Here, c is an arbitrary constant. In this case $\int f(x)dx$ is called the indefinite integral of the function $f(x)$. In simple language we can say that the anti derivative of a function $f(x)$ is the function that will be differentiated to yield the function $f(x)$. The process of finding indefinite integrals is called indefinite integration. Let us consider a few examples.

**Example 1**

Evaluate the following integrals from definition.

a) $\int 5x^3 dx$

b) $\int \sec^2 x\, dx$

**Solution**

a) We know that $\frac{d}{dx}x^4 = 4x^3$ and thus $\frac{d}{dx}\left(\frac{5}{4}x^4\right) = \frac{5}{4}\frac{d}{dx}x^4 = \frac{5}{4}\times 4x^3 = 5x^3$ and therefore the indefinite integral $\int 5x^3 dx = \frac{5}{4}x^4 + c$.

b) The derivative $\frac{d}{dx}\tan x = \sec^2 x$ and therefore $\int \sec^2 x\,dx = \tan x + c$

**The fundamental theorem of calculus**

This theorem states that if a function $f(x)$ is continuous on the interval $[a,b]$ and $F(x)$ is any function that satisfies the condition $\frac{dF(x)}{dx} = f(x)$ (i.e. $F(x)$ is an anti derivative of $f(x)$) throughout this interval then $\int_a^b f(x)dx = F(b) - F(a)$. The integral $\int_a^b f(x)dx$ is called a definite integral. The difference $F(b) - F(a)$ is denoted by $F(b) - F(a) = F(x)\big|_a^b$. In computing definite integrals the limits of integration are given. In this case "a" and "b" are the limits of integration. Unlike the indefinite integrals where a constant of integration is given, we do not give this constant in the case of a definite integral.

**Example 2**

Evaluate the integral $\int_{-2}^{1} 3xdx$.

**Solution**

$\int_{-2}^{1} 3xdx = F(1) - F(-2)$ where F(x) is an anti derivative of $f(x) = 3x$. We can write the constant outside the integral and we therefore write the integral $\int_{-2}^{1} 3xdx$ as $3\int_{-2}^{1} xdx$.

In this case we have $\int_{-2}^{1} 3xdx = 3\int_{-2}^{1} xdx = 3[F(1) - F(-2)]$ such that F(x) is an anti derivative of $f(x) = x$. Since we know that $\frac{d}{dx}x^2 = 2x$ then we have $\frac{1}{2}\frac{d}{dx}x^2 = \frac{1}{2}\times 2x = x$ or $\frac{d}{dx}\left(\frac{1}{2}x^2\right) = x$ and we say that $F(x) = \frac{1}{2}x^2$ is an anti derivative of $f(x)=x$ and the integral $\int_{-2}^{1} xdx = \frac{1}{2}(1^2)-\frac{1}{2}(-2)^2 = \frac{1}{2}-\frac{4}{2} = -\frac{3}{2}$ and therefore the required integral is given as $\int_{-2}^{1} 3xdx = 3\times\left(-\frac{3}{2}\right) = -\frac{9}{2}$.

### 4.3 Integration formulas

The procedure followed when computing integrals depends on the nature of the given function. For this reason we give a general rule (or formula) for a particular group of functions. These are derived from the differentiation formulas as follows

(i) Constant rule $\int 0du = c$ since $\frac{d}{dx}(c)=0$.

(ii) Power rule $\int u^n du = \frac{u^{n+1}}{n+1}+c$ where $n \neq -1$. This is because $\frac{d}{dx}(u^n) = nu^{n-1}$.

(iii) Trigonometric rules

$$\int \sin u\, du = -\cos u + c. \quad \frac{d}{du}(\cos u) = -\sin u.$$

$$\int \cos u\, du = \sin u + c. \quad \frac{d}{du}(\sin u) = \cos u.$$

$$\int \sec^2 u\, du = \tan u + c. \quad \frac{d}{du}(\tan u) = \sec^2 u.$$

$$\int \sec u\, \tan u\, du = \sec u + c. \quad \frac{d}{du}(\sec u) = \sec u\, \tan u.$$

(iv) Exponential rule

$$\int e^u\, du = e^u + c$$

$$\int a^u\, du = \frac{a^u}{\ln a} + c, \qquad a>0,\ a\neq -1$$

$$\int \frac{1}{u} du = \ln|u| + c$$

(v) Logarithmic rule $\int \ln u\, du = u\ln u - u + c, \quad u>0$

Let us consider a few integrals by way of example.

**Example 3**

Evaluate the following integrals.

a) $\int_{2}^{5} 7dx$

b) $\int_{-3}^{4} x^2 dx$

c) $\int_{1}^{5} \left(x^3 + 3x^4\right) dx$

d) $\int_{0}^{180^0} \sin\theta d\theta$

e) $\int_{1}^{2} \left(x + \frac{1}{x}\right) dx$

**Solution**

a) $\int_{2}^{5} 7dx = 7x\Big|_2^5 = 7(5) - 7(2) = 21$

b) $\int_{-3}^{4} x^2 dx = \frac{x^3}{3}\Big|_3^4 = \frac{1}{3}\left(4^3 - 3^3\right) = \frac{37}{3} = 12\frac{1}{3}$

c)

$$\int_{1}^{5} \left(x^3 + 3x^4\right) dx = \frac{x^4}{4}\Big|_1^5 + 3\frac{x^5}{5}\Big|_1^5 = \left(\frac{5^4}{4} - \frac{1^4}{4}\right) + \frac{3}{5}\left(5^5 - 1^5\right)$$

$$= \frac{624}{4} + \frac{3}{5} \times 3124$$

$$= 156 - 1874\frac{2}{5} = 1718\frac{2}{5}.$$

d) $\int_{0}^{180^0} \sin\theta d\theta = -\cos\theta\Big|_{0^0}^{180^0} = -\left(\cos 180^0 - \cos 0^0\right) = -(-1-1) = 2$

e) $\int_{1}^{2} \left(x + \frac{1}{x}\right) dx = \int_{1}^{2} xdx + \int_{1}^{2} \frac{1}{x} dx = \frac{x^2}{2}\Big|_1^2 + \ln x\Big|_1^2 = 2 - \frac{1}{2} + \ln 2 =$

It may happen that a function is a combination of a sum, difference or even product with a constant. Let us consider how to handle such cases.

## 4.4 Properties of definite integrals

Consider constants a, b, c and d, then we define the following definite integrals

a) $\int_a^b cf(x)dx = c\int_a^b f(x)dx$

b) $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

c) $\int_a^b [cf(x)+dg(x)]dx = \int_a^b cf(x)dx + \int_a^b dg(x)dx = c\int_a^b f(x)dx + d\int_a^b g(x)dx$. This property is a combination of properties a and b given above.

**Velocity**

Velocity is the rate of change of displacement with time or the derivative of the displacement with respect to time given by $v(t) = \frac{ds}{dt}$. By the process of reverse differentiation we have the displacement given by $s = \int_a^b v(t)dt$.

**Example 4**

An object moves along a straight line with velocity $v(t) = t^2$ for $t > 0$. How far does the object travel between the times $t = 1$ and $t = 2$ seconds?

**Solution**

$s = \int_1^2 t^2\,dt = \left.\frac{t^3}{3}\right|_1^2 = \frac{2^3}{3} - \frac{1^3}{3} = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$. Therefore the displacement is $s = \frac{7}{3}$ units.

**Example 5**

The acceleration of a particle is given by $a(t) = 3t + 4$. Find the velocity and the distance moved by the particle between the times $t = 2$ and $t = 5$ seconds.

**Solution**

We stated earlier that the acceleration $a(t) = \frac{dv}{dt}$ and therefore by reverse differentiation we have the velocity given as $v = \int_a^b a(t)dt$.

$$a = \int_2^5 (3t+4)\,dt = \left.\left(\frac{3t^2}{2} + 4t\right)\right|_2^5 = \left.\frac{3t^2}{2}\right|_2^5 + 4t|_2^5 = \frac{3(5)^2}{2} - \frac{3(2)^2}{2} + 4(5) - 4(1)$$

$$= \frac{75}{2} - \frac{12}{2} + 20 - 4 = 47\frac{1}{2}$$

By reverse differentiation, the displacement is given by $s = \int_a^b v dt$ and therefore we have

$$s = \int_2^5 \left( \frac{3t^2}{2} + 4t \right) dt = \left( \frac{t^3}{2} + 2t^2 \right) \Bigg|_2^5 = \frac{5^3}{2} - \frac{2^3}{2} + 2(5)^2 - 2(2)^2$$

$$= \frac{125 - 8}{2} + 50 - 8 = 100\frac{1}{2}$$

**Exercise 4**

1) Simplify the following integrals.

   a) $\int \left( x^2 + \frac{1}{x} \right) dx$

   b) $\int S\text{in}\, x\, dx$

   c) $\int 2^x \ln 2\, dx$

2) Evaluate each of the following

   a) $\int_{-3}^{7} (4x^3 - 2x)\, dx$

   b) $\int_{-5}^{2} (4x^3 + x)\, dx$

   c) $\int_0^4 \sqrt{x} dx$