Contents

13 Session Thirteen: Applications of Differentiation III

13.1 Session Objectives

By the end of this session, you should be able to:

(i) Apply differentiation in approximation

(ii) Solve rate of change, minima and maxima problems

13.2 Introduction

The last part of application of differentiation is dedicated to Small changes, Related rate of change, Maxima and Minima.

13.3 Small Changes

We saw that $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x}$ (by letting $h = \delta x$). The gradient of the curve at the point $P(x, f(x)) = \frac{dy}{dx}$. If δx is small, then we say that $\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$.
We use this approximation to estimate the value of a function close to a known value. $\delta y \approx \frac{dy}{dx} \cdot \delta x$ and $y + \delta y$ can be approximated if y is known.

Example 13.3.1. *Use $y = \sqrt{x}$ to approximate the value of $\sqrt{1.1}$*

Solution: Let $x = 1$, then $\delta x = 0.1$. $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

$$\begin{aligned} \delta y &= \frac{dy}{dx} \cdot \delta x \\ &= \frac{1}{2\sqrt{1}} \cdot 0.1 \\ &= 0.05 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \sqrt{1.1} &\approx y + \delta y \\ &\approx \sqrt{1} + 0.05 \\ &\approx 1.05 \end{aligned}$$

Example 13.3.2. *Using $y = \ln x$, approximate $\ln 1.1$.*

Solution: $x = 1$, $\delta x = 0.1$. $\frac{dy}{dx} = \frac{1}{x}$, $\delta y = 1 \times 0.1 = 0.1$.

$$\begin{aligned} \ln 1.1 &\approx y + \delta y \\ &\approx \ln x + \delta y \\ &\approx \ln 1 + 0.1 \\ &\approx 0 + 0.1 \\ &\approx 0.1 \end{aligned}$$

Exercise 13.3.1. *By taking $1^\circ = 0.0175$ radians, approximate $\sin 29^\circ$*

13.4 Related rate of change

The identity $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ is useful in solving certain rate of change problems.

$$\begin{aligned} \frac{dy}{dt} &= \text{Rate of change of } y \text{ w.r.t } t. \\ \frac{dx}{dt} &= \text{Rate of change of } x \text{ w.r.t } t. \end{aligned}$$

Example 13.4.1. *How fast does the radius of a spherical soap bubble change when air is blown into it at the rate of* $10\ cm^3/sec$

Solution: We need to find $\frac{dr}{dt}$.

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ \frac{dv}{dt} &= 4\pi r^2 \frac{dr}{dt} \\ \frac{dv}{dt} &= 10 \\ &\Rightarrow \frac{dr}{dt} = \frac{10}{4\pi r^2} \end{aligned}$$

Example 13.4.2. *How fast does the water level drop when a cylindrical tank is drained at the rate of* 3 *litres/sec?*

Solution: The radius is constant but V and h change with time (since the water level is dropping). So V and h are differentiable functions of time (t). We need to find $\frac{dh}{dt}$.

$$\begin{aligned} V &= \pi r^2 h \\ \frac{dv}{dt} &= \pi r^2 \frac{dh}{dt} \\ \frac{dv}{dt} &= -3 \\ &\Rightarrow \frac{dh}{dt} = \frac{-3}{\pi r^2} \end{aligned}$$

13.5 Maxima and Minima

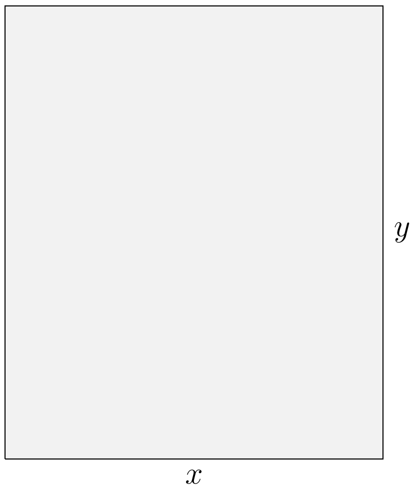
Example 13.5.1. *Find two numbers whose sum is 60 and whose product is maximum*
Solution:

$$\begin{aligned} S = x + y &\qquad P = xy \\ 60 = x + y &\qquad P = x(60 - x) \\ 60 - x = y &\qquad P = 60x - x^2 \\ &\qquad \frac{dP}{dx} = 60 - 2x \end{aligned}$$

$$\textit{For maximum values,} \quad \frac{dP}{dx} = 0 \ \Rightarrow\ 2x = 60 \Rightarrow x = 30,\ y = 30$$
$$\Rightarrow\ P = xy = 30 \cdot 30 = 900$$

Example 13.5.2. *Find the area of a rectangular engineering workshop with perimeter 100m whose area is as large as possible.*

Solution:



$$\begin{aligned} P &= 2(x+y) = 100 \\ y &= 50 - x \\ A &= xy \\ A &= x(50-x) = 50x - x^2 \\ \frac{dA}{dx} &= 50 - 2x \end{aligned}$$

$\frac{dA}{dx} = 0 \Rightarrow x = 25.$ $f''(x) = -2 < 0 \Rightarrow$ $x = 25$ is a maximum value and $y = 25$.

Example 13.5.3. *An open box is to be made by cutting a square from each corner of 12in square piece of metal and then folding up the sides. What size of square should be cut from each corner to produce a box of maximum volume?*

Solution: Let x be the length of a side of the square that is cut from each corner as shown in Figure . $x \geq 0$

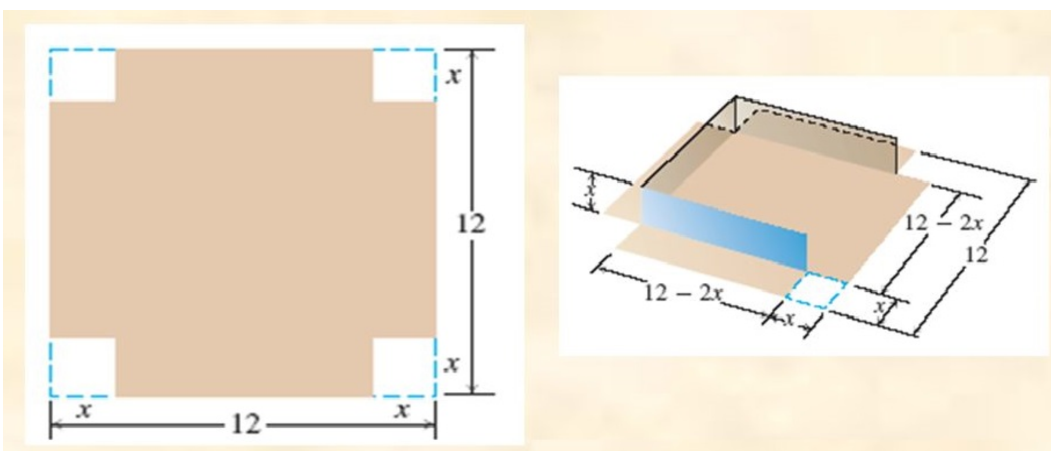


Figure 1: 12×12 box

$V(x) = x(12-2x)(12-2x) = 144x - 48x^2 + 4x^3$

13.6 Session Summary

For more material on this section check out [1, 2] or visit related rate of change, small changes. You can also watch the lecture videos: related rate of change, small changes, maxima and minima.

13.7 Student Activity

Exercise

Small Change

Approximate

1. $\sqrt{101}$
2. $\sqrt[3]{65}$
3. $\sqrt[5]{33}$
4. $\sqrt{82}$

Related Rate of Change

A ladder $20ft$ long leans against a vertical wall. The bottom of the ladder slides away from the wall at the rate of $2ft/sec$. How fast is the ladder sliding down when the top of the ladder is $12ft$ above the ground

Maxima and Minima

1. Find two positive numbers whose product is 400 and whose sum is minimum.
2. What number exceeds its square by a maximum value?
3. A projectile is fired straight upwards with a velocity of 400 m/s. Its altitude above the ground t seconds after bieng fired is given by

$$s(t) = -16t^2 + 400t$$

   (i) Find the time after which the projectile hits the ground.
   (ii) Find the velocity at which the projectile hits the ground.
   (iii) What is the maximum altitude achieved by the projectile?.
   (iv) What is the acceleration at any time t?.

References

[1] E. Purcell D. Varberg and S. Rigdon. *Calculus*. Pearson Education, Inc., 9 edition, 2006. ISBN-13 : 978-0132306331.

[2] J. Stewart. *Calculus*. Cengage Learning 20 Channel Center Street, Boston, MA 02210, USA, 8 edition, 2016. ISBN-13: 978-1-305-27176-0.

