APPLICATIONS OF INTEGRATION.

① AREAS UNDER CURVE.

(i) AREA BOUNDED PARTLY BY A CURVE

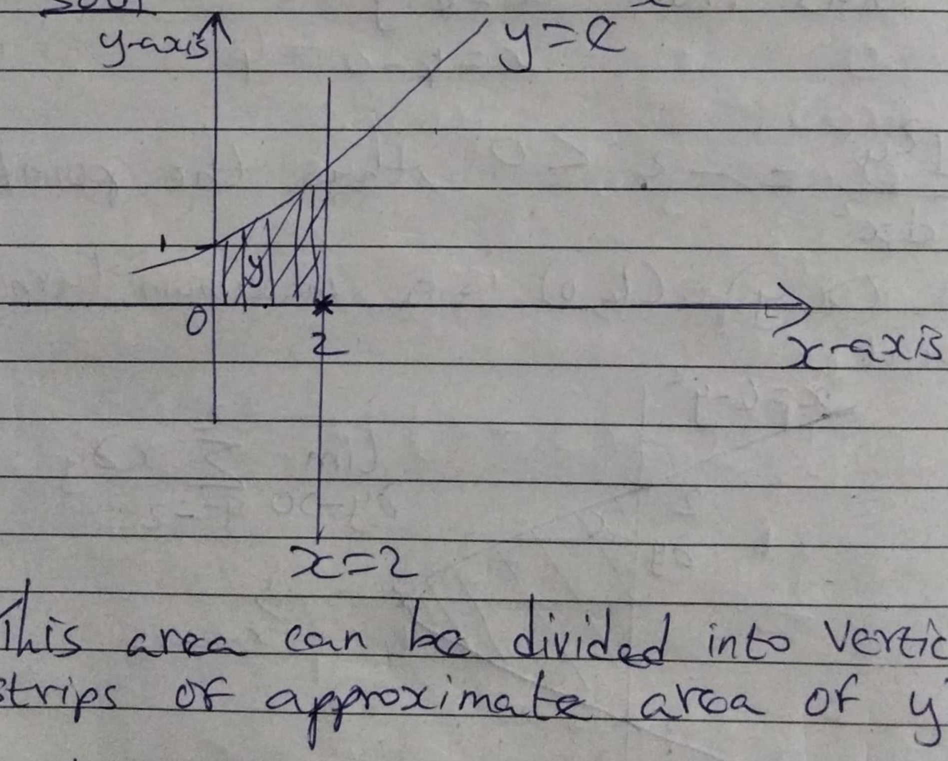
If the area A between two specified Values of x, ($x=a$, $x=b$) is bounded by the x-axis and a curve $y=f(x)$ then

$$\text{Area} = \int_a^b f(x)\,dx$$

Examples

1. Find the Area bounded by x-axis, y-axis, the curve $y=e^x$ and the line $x=2$.

soln



This area can be divided into verticle strips of approximate area of $y\delta x$

∴ The required area will be

$$A = \lim_{\delta \to 0} \sum_{x=0}^{x=2} y\,\delta x$$

$$A = \int_0^2 y\,dx = \int_0^2 e^x\,dx = \left[e^x\right]_0^2$$

$A = \underline{\underline{e^2 - 1}}$ square units.

2. Find the area between the curve $x = 4 - y^2$ and the y-axis

Soln

$x = 4 - y^2$

x - intercept ; $y = 0$; $x = 4$

y - intercept ; $x = 0$; $y = \pm 2$

Turning point

$\frac{dx}{dy} = -2y$; $\frac{dx}{dy} = 0$

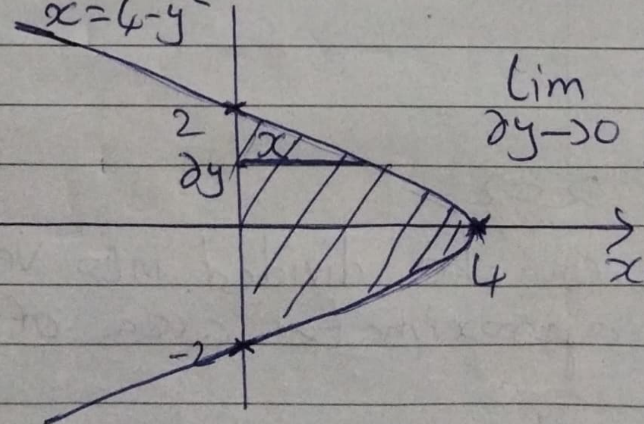
$\Rightarrow -2y = 0$

$y = 0$

Thus from $x = 4 - y^2$

$x = 4 - 0 = 4$

$\frac{d^2y}{dx^2} = -2 < 0$ thus the point $(x, y) = (4, 0)$ is a maximum turning point.



$$\lim_{\partial y \to 0} \sum_{y=-2}^{2} x \partial y$$

$$A = \int_{-2}^{2} x\,dy = \int_{-2}^{2} (4 - y^2)\,dy = \left[4y - \frac{y^3}{3}\right]_{-2}^{2}$$

$$= 8 - \frac{8}{3} + 8 - \frac{8}{3} = 16 - \frac{16}{3} = \frac{32}{3}$$

$$= 10\tfrac{2}{3} \text{ square units}$$

(ii) Area of a region btw Intersecting curves.

Definition:- If $f(x)$ and $g(x)$ are continuous on a closed interval a, b and $f(x) \geq g(x)$ for all x is $[a, b]$ then the area of the region bounded by the graphs $f(x)$ and $g(x)$ and the interval lines $x=a$ and $x=b$ is given by

$$A = \int_a^b [f(x) - g(x)]\, dx$$

Example 1

Find the area of region bounded by the graphs $y = 2-x^2$ and $y = x$.

Soln

Find the point of intersection

$$2-x^2 = x$$

$$-x^2 - x + 2 = 0$$

$$x^2 + x - 2 = 0$$

$$(x+2)(x-1) = 0$$

$x = -2 \quad ; y = -2$

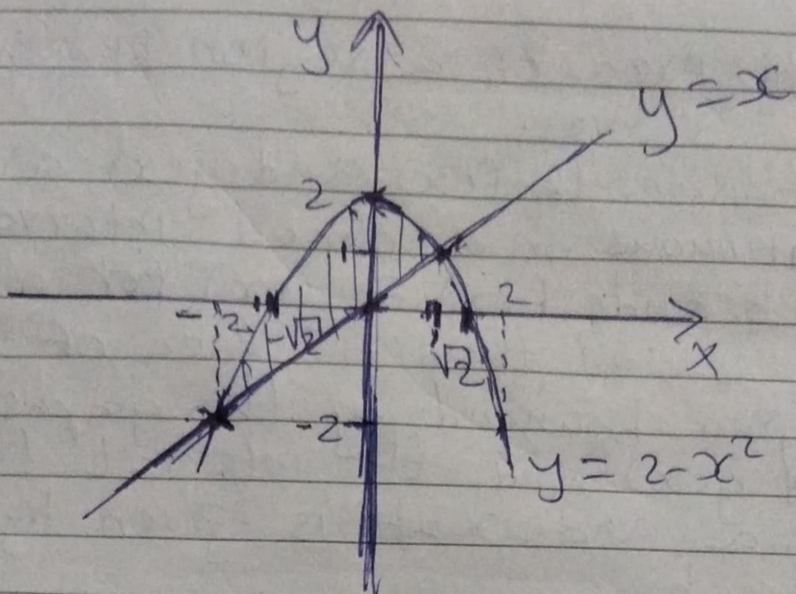
$x = 1 \quad ; y = 1$

So, the points of intersection are $(-2,-2)$ \$ $(1,1)$

The curve $y = 2-x^2$

intercepts $(0,2)$, $(\sqrt{2},0)$, $(-\sqrt{2},0)$

maximum at $(0,2)$



but $f(x) \geq g(x)$
at the end points i.e $[-2, 1]$
$f(x) = g(x)$

at 0
$f(0) = 2 \quad ; g(0) = 0$
$\therefore f(x) > g(0)$

$$A = \int_{-2}^{1} [f(x) - g(x)]\,dx$$

$$= \int_{-2}^{1} [(2-x^2) - x]\,dx$$

$$= \int_{-2}^{1} [2 - x^2 - x]\,dx$$

$$= \left[2x - \frac{x^3}{3} - \frac{x^2}{2}\right]_{-2}^{1}$$

$$= 2 - \frac{1}{3} - \frac{1}{2} + 4 - \frac{8}{3} + 2$$

$$= \frac{9}{2} \text{ square units.}$$

Example 2

Find the area bounded by the curves $y^2 = 4x$ and $x^2 = 4xy$

Soln

Point of intersection

$$y^2 = 4x \Rightarrow y = 2\sqrt{x}$$

$$x^2 = 4y \Rightarrow y = \frac{x^2}{4}$$

At point of intersection

$$2\sqrt{x} = \frac{x^2}{4}$$

$$\Rightarrow 4x = \frac{x^4}{16}$$

$$\Rightarrow 64x - x^4 = 0$$

$$\Rightarrow x(64 - x^3) = 0$$

$$x = 0 \quad ; \quad x^3 = 64 \quad \Rightarrow x = 4$$

$$y = 0 \qquad\qquad y = 4$$

The curve

$$y^2 = 4x$$

Intercepts $(0,0)$, $(0,0)$

Turning points:

$$y^2 = 4x$$

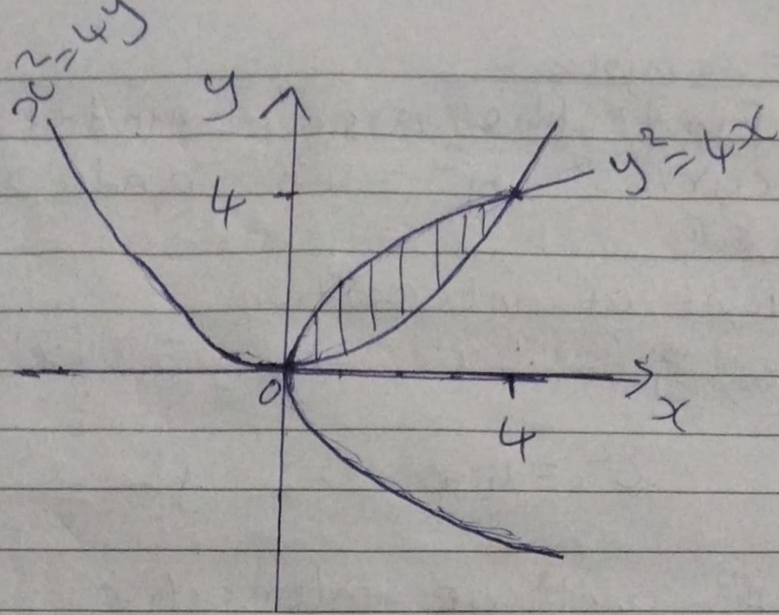
$$\Rightarrow 2y\frac{dy}{dx} = 4$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{y}$$

$$\Rightarrow \frac{dx}{dy} = \frac{y}{2} = 0$$

$$\Rightarrow y = 0, \; x = 0$$

Minimum $(0,0)$



$$A = \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx$$

$$= \left[2 \cdot \frac{2}{3} x^{\frac{3}{2}} - \frac{x^3}{12}\right]_0^4$$

$$= \frac{4}{3} \times 8 - \frac{64}{12}$$

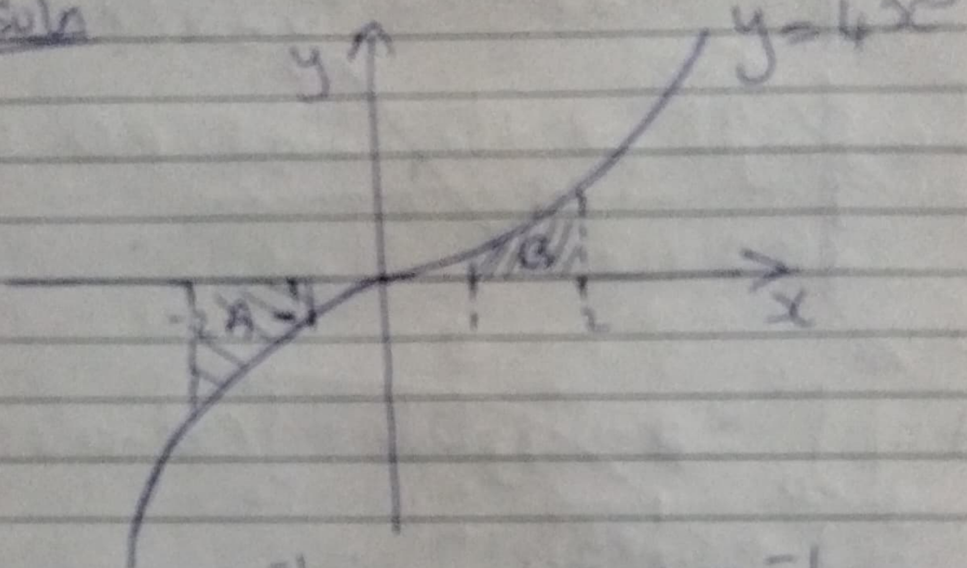
$$= \frac{32}{3} - \frac{64}{12} = \frac{16}{3} \text{ sq units}$$

The Meaning of Negative Result

Example

1. Find the area under the curve $y = 4x^3$ and x-axis from a) $x = -2$ to $x = -1$. b) from $x = 1$ and $x = 2$

Soln



a) $A = \int_{-2}^{-1} 4x^3 dx = [x^4]_{-2}^{-1}$

$= 1 - 16 = -15$ square units

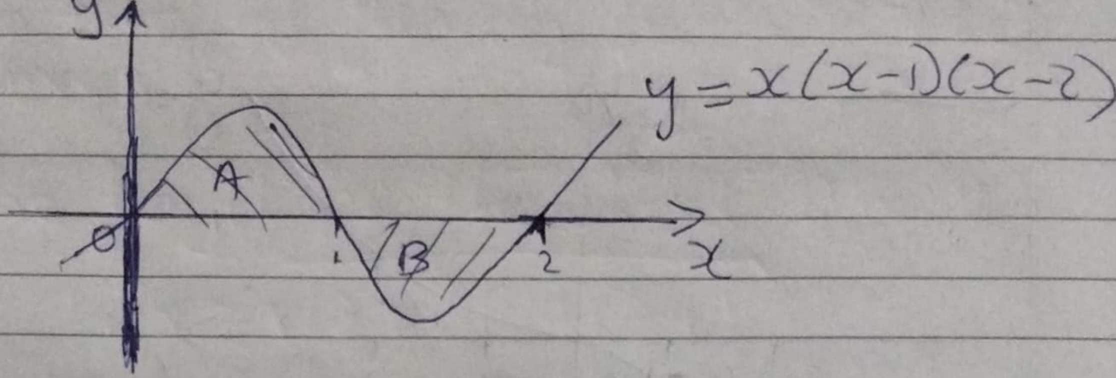
b) $B = \int_{1}^{2} 4x^3 dx = [x^4]_{1}^{2}$

$= 16 - 1 = 15$ square units

The magnitudes of the two areas are equal, the result for the area of A which is below x-axis is negative.

2. Find the area enclosed between the curves.
$y = x(x-1)(x-2)$ and x-axis

Soln



$$A = \int_0^1 x(x-1)(x-2)\,dx = \int_0^1 (x^3 - 3x^2 + 2x)\,dx$$
$$= \left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 = \frac{1}{4}$$

$$B = \int_1^2 (x^3 - 3x^2 + 2x)\,dx = \left[\frac{x^4}{4} - x^3 + x^2\right]_1^2 = -\frac{1}{4}$$

$\therefore$ The Required Area $= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ sq units

Exercise

Find the areas bounded by the specified lines and curves in the following questions

① x and y-axis, the line $x=3$ and the curve $y=x^2+1$ (Ans: 12 sq units)

② The x-axis, lines $x=1$ and $x=4$, and the curve $xy=2$
(Ans: $2\ln 4$ or $\ln 16$ sq units

③ The line $y=2$ and the curve $y=3x-x^2$
(Ans: $\frac{1}{6}$ sq units)

④ The curve $y=9-x^2$ and the x-axis
(Ans: 36 sq units)

⑤ The curve $y=2+x-x^2$ and the x-axis (Ans: $4\frac{1}{2}$ sq units)

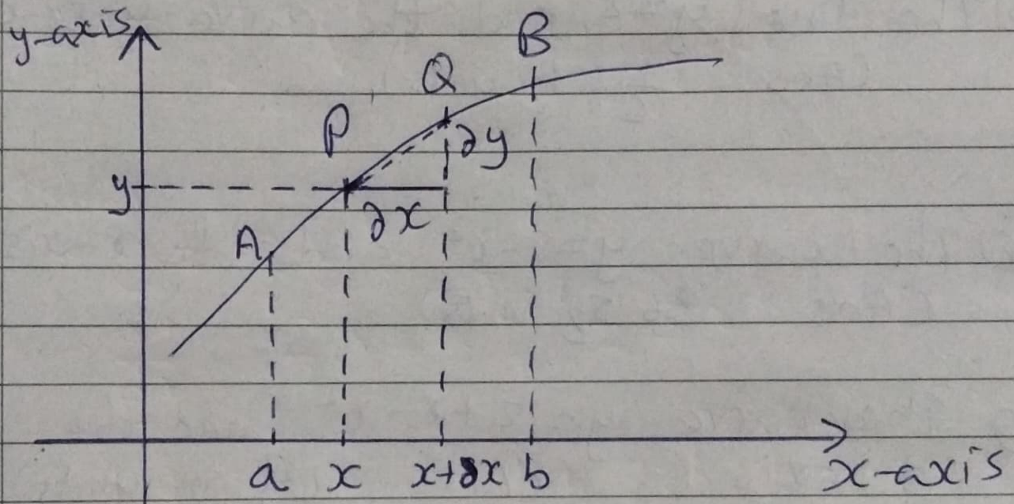
THE LENGTH OF CURVE

Let A B be an arc of a curve $y = f(x)$, the points A and B are at $x = a$ and $x = b$ respectively.

Let $P(x, y)$ and $Q(x + \partial x, y + \partial y)$ be two neighbouring points in the curve.

The length of the chord PQ is

$$PQ = \sqrt{(\partial x)^2 + (\partial y)^2}$$



From $PQ = \sqrt{(\partial x)^2 + (\partial y)^2}$

$$= \sqrt{\left(1 + \left(\frac{\partial y}{\partial x}\right)^2\right)(\partial x)^2}$$

$$= \sqrt{\left(1 + \left(\frac{\partial y}{\partial x}\right)^2\right)}\, \partial x$$

Divide the interval $x = a$ and $x = b$ into n equal portions each of length ∂x. The length of the portion of the inscribed polygon between A and B is then given by

$$\sum_{x=a}^{x=b} \sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}\,\partial x$$

then by definition, the length S of the curve between A and B is given by

$$S = \lim_{\partial x \to 0} \sum_{x=a}^{x=b} \sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}\,\partial x$$

as $\partial x \to 0$, $\frac{\partial y}{\partial x} \to \frac{dy}{dx}$

$$\therefore\ S = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

S is the length of a curve from $x=a$ to $x=b$ and it is also referred as length of a arc or simply <u>Arc Length</u> of the curve $y=f(x)$.

If the expression is given in the form $x=g(y)$ then

$$S = \int_{y_1}^{y_2} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

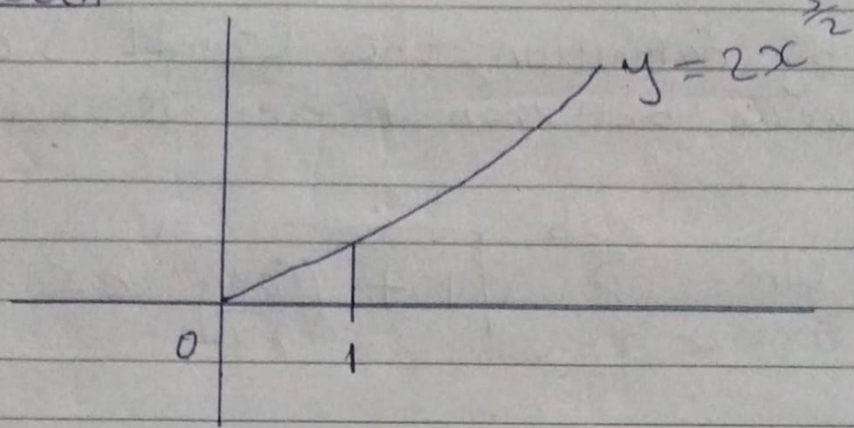
When the equation of the curve is given in parametric form say $X = f(\theta)$, $y = g(\theta)$ then the length of the curve

$$S = \int_A^B \sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

Example 1

Find the length of the curve $y = 2x^{\frac{3}{2}}$ between $x = 0$ and $x = 1$

Soln



$$S = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

From

$$y = 2x^{\frac{3}{2}}$$

$$\frac{dy}{dx} = \frac{3}{2} \times 2x^{\frac{3}{2}-1} = 3x^{\frac{1}{2}}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(3x^{\frac{1}{2}}\right)^2 = 1 + 9x$$

So,

$$S = \int_0^1 \sqrt{1 + 9x}\, dx$$

Let $U = 1 + 9x$

$du = 9\,dx$

$dx = \frac{1}{9}\, du$

$$S = \frac{1}{9} \int U^{\frac{1}{2}}\, du = \frac{1}{9} \times \frac{2}{3} U^{\frac{3}{2}} = \frac{2}{27} U^{\frac{3}{2}}$$

$$= \frac{2}{27} \left[(1 + 9x)^{\frac{3}{2}}\right]_0^1 = \frac{2}{27} \left[10^{\frac{3}{2}} - 1\right]$$

Example 2

Find the length of the curve

$x = \ln(\cos y)$ from $y=0$ to $y=\frac{\pi}{4}$

Soln

$$S = \int_{y=0}^{y=\frac{\pi}{4}} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$; since the interval is in terms of y

From $x = \ln(\cos(y))$; using chain rule i.e $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$

$$\frac{dx}{dy} = \frac{-\sin y}{\cos y}$$

$$= -\tan y$$

$$1+\left(\frac{dx}{dy}\right)^2 = 1+(-\tan y)^2 = 1+\tan^2 y$$

$$= \sec^2 y$$

Thus

$$S = \int_0^{\pi/4} \sqrt{\sec^2 y}\,dy = \int_0^{\pi/4} \sec y\,dy$$

$$= \Big[\ln|\sec y + \tan y|\Big]_0^{\pi/4}$$

$$= \ln|\sqrt{2}+1| - \ln|1+0|$$

$$= \ln|\sqrt{2}+1| - \ln 1$$ $\ln 1 = 0$

$$= \underline{\ln|\sqrt{2}+1|}$$

Example 3
Find the Length of the curve
$x = 3\cos t$, $y = 3\sin t$ between
$t = 0$ and $t = \pi$

Soln

$$S = \int_{t=0}^{t=\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\frac{dx}{dt} = -3\sin t, \quad \frac{dy}{dt} = 3\cos t$$

$$\left(\frac{dx}{dt}\right)^2 = 9\sin^2 t \ ; \ \left(\frac{dy}{dt}\right)^2 = 9\cos^2 t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9\sin^2 t + 9\cos^2 t$$
$$= 9(\sin^2 t + \cos^2 t) = 9(1)$$
$$= 9$$

$$\therefore S = \int_{t=0}^{t=\pi} \sqrt{9}\, dt = \int_0^{\pi} 9^{\frac{1}{2}} dt = \left[9^{\frac{1}{2}} t\right]_0^{\pi}$$
$$= 3\,[t]_0^{\pi} = \underline{\underline{3\pi}}$$

Example 4
A curve has the equation
$$y = \frac{1}{2}\ln x - \frac{x^2}{4}$$
show that
$$1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{1}{2x} + \frac{x}{2}\right)^2$$

hence find the length of the curve between $x=1$ and $x=4$.

Soln

From $y = \frac{1}{2}\ln x - \frac{x^2}{4}$

$$\frac{dy}{dx} = \frac{1}{2x} - \frac{2x}{4} = \frac{1}{2x} - \frac{x}{2}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{1}{2x} - \frac{x}{2}\right)^2$$

$$= 1 + \frac{1}{4x^2} - \frac{1}{2} + \frac{x^2}{4}$$

$$= \frac{1}{4x^2} + \frac{1}{2} + \frac{x^2}{4}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{1}{2x} + \frac{x}{2}\right)^2$$

$$S = \int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^4 \sqrt{\left(\frac{1}{2x} + \frac{x}{2}\right)^2}\, dx$$

$$= \int_1^4 \left(\frac{1}{2x} + \frac{x}{2}\right) dx = \frac{1}{2}\left[\ln x + \frac{x^2}{2}\right]_1^4$$

$$= \frac{1}{2}\left[\ln 4 + 8 - \left(\ln 1 + \frac{1}{2}\right)\right] = \frac{1}{2}\left[\ln 4 + 8 - \ln 1 - \frac{1}{2}\right]$$

$$= \frac{1}{2}\left[\ln 4 + \frac{15}{2}\right]$$

$$= \frac{1}{2}\ln 4 + \frac{1}{2} \times \frac{15}{2} = \ln 4^{\frac{1}{2}} + \frac{15}{4}$$

$$= \underline{\ln 2 + 3\tfrac{3}{4}}$$