Technical University of Mombasa

Section One: Functions

1.0 Session objectives

By the end of the session, you should be able to:

i. Define a function
ii. Evaluate the value of a function
iii. Find the domain and the range of a function
iv. Evaluate the composition of a function
v. Find the inverse of a function

1.1 Definition of a function

An equation will be a function if, for any x in the domain of the equation (the domain is all the x's that can be plugged into the equation), the equation will yield exactly one value of y when we evaluate the equation at a specific x.

Example 1: Determine if each of the following are functions.

(a) $y = x^2 + 1$

(b) $y^2 = x + 1$

Solution

(a) Is a function. Given an x, there is only one way to square it and then add 1 to the result. So, no matter what value of x you put into the equation, there is only one possible value of y when we evaluate the equation at that value of x.

(b) The only difference between this equation and the first is that we moved the exponent off the x and onto the y. This small change is all that is required, in this case, to change the equation from a function to something that isn't a function.

To see that this isn't a function is fairly simple. Choose a value of x, say $x = 3$ and plug this into the equation. $y^2 = 3 + 1 = 4$

Now, there are two possible values of y that we could use here. We could use $y = 2$ or $y = -2$. Since there are two possible values of y that we get from a single x this equation isn't a function.

1.2 Function notation

Function notation is a fancy way of writing the y in a function that will allow us to simplify notation and some of our work a little.

Let's take a look at the following function

$$y = 2x^2 - 5x + 3$$

Using function notation, we can write this as any of the following

$f(x) = 2x^2 - 5x + 3$ $\qquad$ $g(x) = 2x^2 - 5x + 3$ $\qquad$ $h(x) = 2x^2 - 5x + 3$

To get the value of the function at $x = -3$. Using function notation we represent the value of the function at $x = -3$ as $f(-3)$.

Function notation gives us a nice compact way of representing function values.

To evaluate the function, we simply substitute whatever is in the parenthesis on the left side everywhere we see an *x* on the right side we will.

For our function this gives,

$$f(-3) = 2(-3)^2 - 5(-3) + 3$$

$$= 2(9) + 15 + 3$$

$$= 36$$

Example 2 Given $f(x) = -x^2 + 6x - 11$ find each of the following.

(a) $f(2)$

(b) $f(-10)$

(c) $f(t)$

(d) $f(t-3)$

(e) $f(4x-1)$

Solution

(a) $f(2) = -(2)^2 + 6(2) - 11 = -4 + 12 - 11 = -3$

(b) $f(-10) = -(-10)^2 + 6(-10) - 11 = -100 - 60 - 11 = -171$ Be careful when

(c) $f(t) = -(t)^2 + 6(t) - 11 = -t^2 + 6t - 11$

(d) $f(t-3) = -(t-3)^2 + 6(t-3) - 11 = -(t^2 - 6t + 9) + 6t - 18 - 11$

$$= -t^2 + 12t - 38$$

(e) $f(4x-1) = -(4x-1)^2 + 6(4x-1) - 11 = -(16x^2 - 8x + 1) + 24x - 6 - 11$

$$= -16x^2 + 32x - 18$$

1.3 The Domain and Range of a function

The **domain** of a function is the set of all values that can be plugged into a function and have the function exist and have a real number for a value. For the domain we need to avoid division by zero, square roots of negative numbers, logarithms of zero and logarithms of negative numbers. The **range** of a function is simply the set of all possible values that a function can take.

Example 4 Find the domain and range of each of the following functions:

(a) $f(x) = 5x - 3$

(b) $g(t) = \sqrt{4 - 7t}$

(c) $h(x) = -2x^2 + 12x + 5$

(d) $f(z) = |z - 6| - 3$

(e) $g(x) = 8$

Solution

(a) $f(x) = 5x - 3$

We know that this is a line and that it's not a horizontal line (because the slope is 5 and not zero). This means that this function can take on any value and so the range is all real numbers.

Range: $(-\infty, \infty)$

This is more generally a polynomial and we know that we can plug any value into a polynomial and so the domain in this case is also all real numbers or,

Domain: $-\infty < x < \infty$ or $(-\infty, \infty)$

(b) $g(t) = \sqrt{4 - 7t}$

This is a square root and we know that square roots are always positive or zero. We know then that the range will be,

Range: $[0, \infty)$

For the domain we need to make sure that we don't take square roots of any negative numbers, so we need to require that,

$$4 - 7t \geq 0$$

$$4 \geq 7t \Rightarrow t \leq \frac{4}{7}$$

The domain is then,

Domain: $t \leq \frac{4}{7}$ or $\left(-\infty, \frac{4}{7}\right]$

(c) $h(x) = -2x^2 + 12x + 5$

Here we have a quadratic, which is a polynomial, so we again know that the domain is all real numbers or,

Domain: $-\infty < x < \infty$ or $(-\infty, \infty)$

In this case the range, we know that the graph of this will be a **parabola** that opens down (because the coefficient of the x^2 is negative) and so the vertex will be the highest point on the graph. If we know the vertex we can then get the range. The vertex is then,

$x = -\frac{12}{2(-2)} = 3$ $\quad\quad$ $y = h(3) = -2(3)^2 + 12(3) + 5 = 23 \Rightarrow (3,23)$

We know that this will be the highest point on the graph or the largest value of the function and the parabola will take all values less than this, so the range is then,

Range: $(-\infty, 23]$

(d) $f(z) = |z - 6| - 3$

This function contains an absolute value and we know that absolute value will be either positive or zero. In this case the absolute value will be zero if $z = 6$ and so the absolute value portion of this function will always be greater than or equal to zero. We are subtracting 3 from the absolute value portion and so we then know that the range will be, Range: $[-3, \infty)$

We can plug any value into an absolute value and so the domain is once again all real numbers or,

Domain: $-\infty < z < \infty$ or $(-\infty, \infty)$

(e) $g(x) = 8$

This is a constant function and so any value of *x* that we plug into the function will yield a value of 8. This means that the range is a single value or,

Range: 8

The domain is all real numbers,

Domain: $-\infty < x < \infty$ or $(-\infty, \infty)$

Example 5 Find the domain of each of the following functions.

(a) $f(x) = \frac{x-4}{x^2-2x-15}$

(b) $g(t) = \sqrt{6 + t - t^2}$

(c) $h(x) = \frac{x}{\sqrt{x^2-9}}$

Solution

(a) $f(x) = \frac{x-4}{x^2-2x-15}4$

With this problem we need to avoid division by zero, so we need to determine where the denominator is zero which means solving,

$x^2 - 2x - 15 = 0 \Rightarrow (x - 5)(x + 3) = 0 \Rightarrow x = -3, x = 5$

So, these are the only values of *x* that we need to avoid and so the domain is,

Domain: All real numbers except $x = -3$ & $x = 5$

(b) $g(t) = \sqrt{6 + t - t^2}$

In this case we need to avoid square roots of negative numbers and so need to require that,

$$6 + t - t^2 \geq 0 \Rightarrow t^2 - t - 6 \leq 0$$

The first thing that we need to do is determine where the function is zero

$$t^2 - t - 6 = 0 \Rightarrow (t - 3)(t + 2) = 0$$

So, the function will be zero at $t = -2$ and $= 3$.

Thus $t \leq -2$ and $t \leq 3$

So the domain for this function is then,

Domain: $-2 \leq t \leq 3$ or $[-2,3]$

(c) $h(x) = \frac{x}{\sqrt{x^2-9}}$

In this case we have to worry about division by zero and square roots of negative numbers. We can cover both issues by requiring that, $x^2 - 9 > 0$

The domain is this case is,

Domain: $x < -3$ & $x > 3$ or $(-\infty, -3)$ & $(3, \infty)$

1.4 Function Composition

The composition of $f(x)$ and $g(x)$ is

$$(f \circ g)(x) = f(g(x))$$

In other words, compositions are evaluated by plugging the second function listed into the first function listed. Note that order is important here. Interchanging the order will more often than not result in a different answer.

Example 6 Given $f(x) = 3x^2 - x + 10$ and $g(x) = 1 - 20x$ find each of the following.

(a) $(f \circ g)(5)$

(b) $(f \circ g)(x)$

(c) $(g \circ f)(x)$

(d) $(g \circ g)(x)$

Solution

(a) $(f \circ g)(5)$

In this case we've got a number instead of an *x* but it works in exactly the same way.

$$(f \circ g)(5) = f\big(g(5)\big) = f(-99) = 29512$$

(b) $(f \circ g)(x)$

$$(f \circ g)(x) = f\big(g(x)\big) = f(1 - 20x)$$

$$= 3(1 - 20x)^2 - (1 - 20x) + 10 = 3(1 - 40x + 400x^2) - 1 + 20x + 10$$

$$= 1200x^2 - 100x + 12$$

Compare this answer to the next part and notice that answers are NOT the same. The order in which the functions are listed is important!

(c) $(g \circ f)(x)$

$(g \circ f)(x) = g\big(f(x)\big) = g(3x^2 - x + 10) = 1 - 20(3x^2 - x + 10)$

$= -60x^2 + 20x - 199$

(d) $(g \circ g)(x)$

$$(g \circ g)(x) = g\big(g(x)\big) = g(1 - 20x) = 1 - 20(1 - 20x) = 400x - 19$$

Example 7 Given $f(x) = 3x - 2$ and $g(x) = \frac{1}{3}x + \frac{2}{3}$ find each of the following.

(a) $(f \circ g)(x)$

(b) $(g \circ f)(x)$

Solution

(a) $(f \circ g)(x)$

$$(f \circ g)(x) = f\big(g(x)\big)$$

$$= f\left(\frac{1}{3}x + \frac{2}{3}\right) = 3\left(\frac{1}{3}x + \frac{2}{3}\right) - 2$$

$$= x + 2 - 2$$

$$= x$$

(b) $(g \circ f)(x)$

$$(g \circ f)(x) = g\big(f(x)\big)$$

$$= g(3x - 2) = \frac{1}{3}(3x - 2) + \frac{2}{3}$$

$$= x - \frac{2}{3} + \frac{2}{3}$$

$$= x$$

1.5 Inverse Functions

In **example 7** we looked at two functions $f\,(x) = 3x - 2$ and $g(x) = \frac{1}{3}x + \frac{2}{3}$ and saw that

$(f \circ g)(x)$=$(g \circ f)(x) = x$

this means that there is a nice relationship between these two functions.

Let's see just what that relationship is. Consider the following evaluations.

$f\,(-1) = 3(-1) - 2 = -5$ $\Rightarrow g(-5) = -\frac{5}{3} + \frac{2}{3} = \frac{-3}{3} = -1$

$g(2) = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$ $\Rightarrow f\left(\frac{4}{3}\right) = 3\left(\frac{4}{3}\right) - 2 = 4 - 2 = 2$

In the first case we plugged $x = -1$ into $f\,(x)$ and got a value of -5. We then turned around and plugged $x = -5$ into $g\,(x)$ and got a value of -1, the number that we started off with.

In the second case we did something similar. Here we plugged $x = 2$ into $g\,(x)$ and got a value of $\frac{4}{3}$, we turned around and plugged this into $f\,(x)$ and got a value of 2, which is again the number that we started with.

Note that the first case is really,

$$(g \circ f)(-1) = g[f(-1)] = g(-5) = -1$$

and the second case is really,

$$(f \circ g)(2) = f[g(2)] = f\left(\frac{4}{3}\right) = 2$$

Note that we get back out of the function evaluation the number that we originally plugged into the composition.

Function pairs that exhibit this behavior are called **inverse functions**.

Before formally defining inverse functions and the notation that we're going to use for them we need to define a **one to one** function.

A function is called **one-to-one** if no two values of *x* produce the same *y*. Mathematically, this is the same as saying,

$f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$

So, a function is one-to-one if whenever we plug different values into the function we get different function values.

Sometimes it is easier to understand this definition if we see a function that isn't one-to-one. Let's take a look at a function that isn't one-to-one. The function $f(x) = x^2$ is not one-to-one because both $f(-2) = 4$ and $f(2) = 4$. In other words, there are two different values of *x* that produce the same value of *y*. Note that we can turn $f(x) = x^2$ into a one-to-one function if we restrict ourselves to $0 \leq x < \infty$. This can sometimes be done with functions.

Now, let's formally define **inverse functions**. Given two one-to-one functions $f(x)$ and $g(x)$ if $(f \circ g)(x) = x$ AND $(g \circ f)(x) = x$ then we say that *f* (*x*) and *g* (*x*) are **inverses** of each other.

More specifically we will say that *g* (*x*) is the **inverse** of *f* (*x*) and denote it by

$$g(x) = f^{-1}(x)$$

Likewise, we could also say that $f(x)$ is the **inverse** of $g(x)$ and denote it by

$$f(x) = g^{-1}(x)$$

The notation that we use really depends upon the problem. In most cases either is acceptable.

For the two functions that we started off this section with we could write either of the following two sets of notation.

$f(x) = 3x - 2$ $\qquad$ $f^{-1}(x) = \frac{1}{3}x + \frac{2}{3}$

$g(x) = \frac{1}{3}x + \frac{2}{3}$ $\qquad$ $g^{-1}(x) = 3x - 2$

When dealing with inverse functions we've got to remember that

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

Finding the Inverse of a Function

Given the function $f(x)$ we want to find the inverse function, $f^{-1}(x)$.

1. First, replace $f(x)$ with y. This is done to make the rest of the process easier.
2. Replace every x with a y and replace every y with an x.
3. Solve the equation from Step 2 for y.
4. Replace y with $f^{-1}(x)$
5. Verify your work by checking that $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$ are both true.

Example 1 Given $f(x) = 3x - 2$ find $f^{-1}(x)$.

Solution

First replace $f(x)$ with y.

$y = 3x - 2$

Next, replace all x's with y and all y's with x.

$x = 3y - 2$

Now, solve for *y*.

$$x + 2 = 3y$$

$$\frac{1}{3}(x + 2) = y$$

$$y = \frac{x}{3} + \frac{2}{3}$$

Finally replace *y* with $f^{\ -1}$ (*x*).

$$f^{\ -1}\,(x) = \frac{x}{3} + \frac{2}{3}$$

Now, we need to verify the results. (Check that $(\,f\ \circ\ f^{\ -1}\,)(x) = x$ is true.)

$$(\,f\ \circ\ f^{\ -1}\,)(x) = f[f^{\ -1}(x)] = f\left[\frac{x}{3} + \frac{2}{3}\right] = 3\left(\frac{x}{3} + \frac{2}{3}\right) - 2 = x + 2 - 2 = x$$

Example 2 Given $g\ (x) = \sqrt{x - 3}$ find $g^{\ -1}(x)$.

Solution

$$y\ =\ \sqrt{x\ -\ 3}\ \Rightarrow\ x\ = \sqrt{\ y\ -\ 3}$$

$$x^2 = y - 3 \Rightarrow y = x^2 + 3$$

$$g^{\ -1}(x) = x^2 + 3$$

Verifying,

$$(\ g\ \circ\ g^{\ -1}\,)(x) = g[g^{\ -1}(x)] = g[x^2 + 3] = \sqrt{(x^2 + 3) - 3}\ = \sqrt{x^2}\ = x$$

Example 3 Given $h(x) = \frac{x+4}{2x-5}$ find $h^{\ -1}(x)$.

Solution

$y = \frac{x+4}{2x-5} \Rightarrow x = \frac{y+4}{2y-5}$

$$x(2y - 5) = y + 4$$

$$2xy - 5x = y + 4$$

$$2xy - y = 4 + 5x$$

$$y(2x - 1) = 4 + 5x$$

$$y = \frac{4 + 5x}{2x - 1}$$

$$h^{-1}(x) = \frac{4 + 5x}{2x - 1}$$

Verifying,

$$(h \circ h^{-1})(x) = h[h^{-1}(x)] = h\left[\frac{4 + 5x}{2x - 1}\right]$$

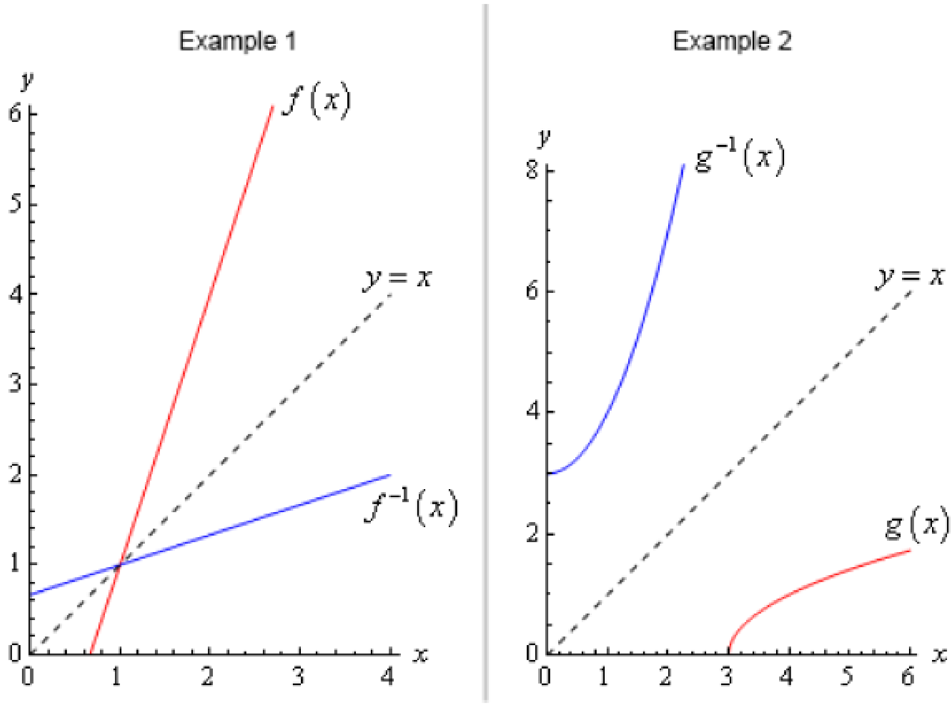
$$= \frac{\left(\frac{4 + 5x}{2x - 1}\right) + 4}{2\left(\frac{4 + 5x}{2x - 1}\right) - 5}$$

$$= \frac{\frac{(4 + 5x) + 4(2x - 1)}{2x - 1}}{\frac{2(4 + 5x) - 5(2x - 1)}{2x - 1}}$$

$$= \frac{4 + 5x + 8x - 4}{8 + 10x - 10x + 5} = \frac{13x}{13} = x$$

There is an interesting relationship between the graph of a function and the graph of its inverse.

Here is the graph of the function and inverse from the first two examples.



In both cases the graph of the inverse is a reflection of the actual function about the line $y = x$. This will always be the case with the graphs of a function and its inverse.