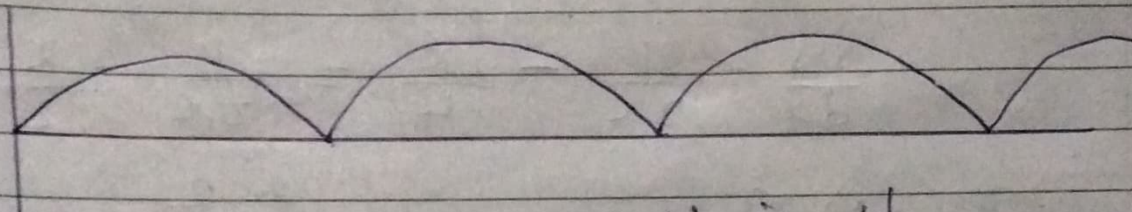
Example 4

Length of an arc of Cycloid

Cycloid is the curve traced by a point on a ~~cy~~ circle as it rolls along a straight line without slipping.



Alternatively, a cycloid is the curve traced out by a point on the circumference of a circle when the circle rolls along a straight line in its own plane.

The equations of a cycloid created by a circle of radius 1 are

$$x(t) = t - \sin t$$
$$y(t) = 1 - \cos t$$

The arc length s of the cycloid for $0 \le t \le 2\pi$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\frac{dx}{dt} = 1 - \cos t \quad ; \quad \frac{dy}{dt} = \sin t$$

$$ds = \sqrt{(1-\cos t)^2 + (\sin t)^2}\, dt = \sqrt{2 - 2\cos t}\, dt$$

$$= \sqrt{2 - 2\cos t}\, dt$$

$$S = \int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt$$

$$= \int_0^{2\pi} \sqrt{4\sin^2 \frac{t}{2}}\, dt$$

$$= \int_0^{2\pi} 2\sin \frac{t}{2}\, dt$$

$$= \left[-4\cos \frac{t}{2}\right]_0^{2\pi} = -4\left\{\cos\frac{2\pi}{2} - \cos 0\right\}$$

$$= -4[-1 - 1]$$

$$= \underline{\underline{8}} \text{ units}$$

From even powers of $\sin\theta$

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$2 - 2\cos t = 2(1-\cos t)$

$= 4\left(\frac{1}{2}\{1-\cos t\}\right)$

Let t be 2θ

$\Rightarrow \theta = \frac{t}{2}$

Thus

$\sin^2\theta = \sin^2\frac{t}{2} = \frac{1}{2}(1-\cos t)$

Thus

$2 - 2\cos t$

$= 4\left\{\sin^2\frac{t}{2}\right\}$

$= 4\sin^2\frac{t}{2}$

Question:

Find the Length of an arc of the cycloid

$x = r(\theta - \sin\theta)$

$y = r(1-\cos\theta)$

and $0 \le \theta \le 2\pi$

Soln

$$S = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$\frac{dx}{d\theta} = r(1-\cos\theta)\ ; \ \frac{dy}{d\theta} = r(-(-\sin\theta))$$

$$= r\sin\theta$$

$$S = \int_0^{2\pi} \sqrt{r^2(1-\cos\theta)^2 + r^2\sin^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{r^2 - 2r^2\cos\theta + r^2\cos^2\theta + r^2\sin^2\theta}\, d\theta$$

$$S = \int_0^{2\pi} \sqrt{r^2(1 - 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{1})}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{r^2(2 - 2\cos\theta)}\, d\theta$$

$$= r \int_0^{2\pi} \sqrt{2(1-\cos\theta)}\, d\theta$$

From $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\Rightarrow (1 - \cos 2x) = 2\sin^2 x$

Let $2x = \theta$

$\Rightarrow x = \frac{\theta}{2}$

Thus $1 - \cos\theta = 2\sin^2\frac{\theta}{2}$

$$\therefore S = r\int_0^{2\pi} \sqrt{2\left(2\sin^2\frac{\theta}{2}\right)}\, d\theta$$

$$= r\int_0^{2\pi} 2\sin\frac{\theta}{2}\, d\theta$$

$$= 4r\left[-\cos\frac{\theta}{2}\right]_0^{2\pi} = -4r\left[\cos\frac{\theta}{2}\right]_0^{2\pi}$$

$$= -4r\left[\cos\pi - \cos 0\right] = -4r[-1 - 1]$$

$$= -4r[-1-1] = \underline{\underline{8r}} \text{ units}$$

NOTE: The length of half an arc of cycloid is $4r$ and that of a complete arc is $8r$.

Exercise

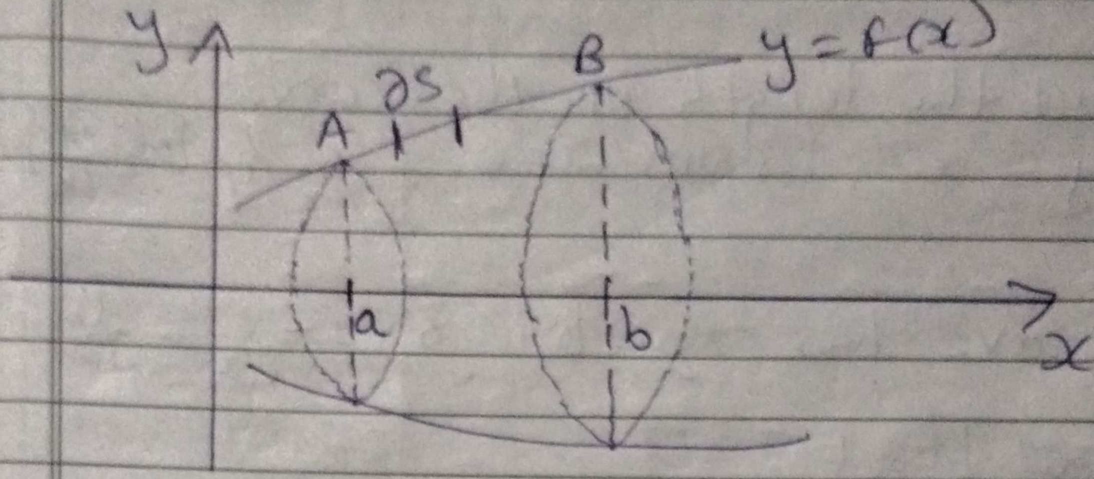
1. Find the length of the curve
$$y = \frac{e^{x} + e^{-x}}{2}$$
from $x = 0$ to $x = \ln 4$ [Ans : $1\frac{7}{8}$]

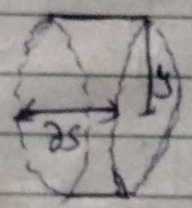
2. Find the length of the curve
$y = x^{\frac{3}{2}}$ from $x = 0$, $x = \frac{1}{4}$ [Ans: $\frac{61}{216}$]

3. A curve has equation $8x = \frac{-2}{y^2} - y^4$
show that
$$\sqrt{\left(\frac{dx}{dy}\right)^2 + 1} = \frac{1}{2}\left(\frac{1}{y^3} + y^3\right)$$
Hence find the length of the curve from $y = 1$ to $y = 4$. [Ans : $32\frac{7}{64}$]

AREAS OF SURFACES OF REVOLUTION

Suppose $y = f(x)$ between $x = a$ & $x = b$, is rotated about the x-axis.





$$\partial A = 2\pi r h$$
$$= 2\pi y \partial s$$

$$\partial s = \sqrt{(\partial x)^2 + (\partial y)^2}$$
$$= \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2}\,\partial x$$

$$\partial A = 2\pi y \partial s$$

$$A = \sum_{A}^{B} 2\pi y \partial s = \lim_{\partial x \to 0} \sum_{A}^{B} 2\pi y \partial s$$

$$= \int_{A}^{B} 2\pi y \, ds$$

where $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$

Similary, Suppose $x = h(y)$ between $y=c$ and $y=d$, is rotated about the y-axis, then

$$A = \int_c^d 2\pi x \, ds$$

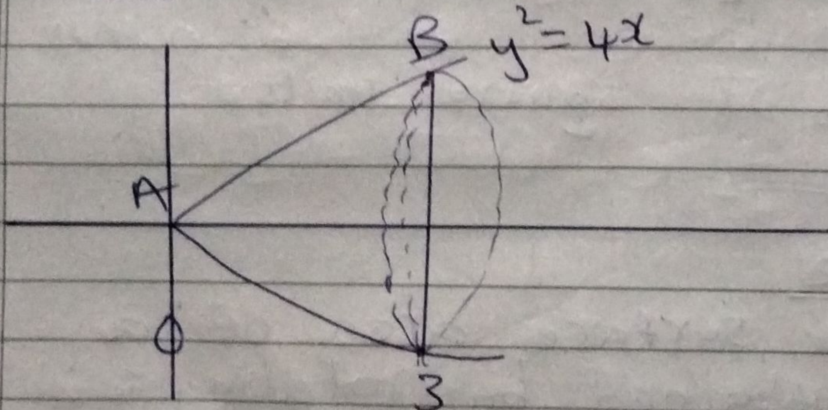
where

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$$

Example 1

Find the area of surface generated by $y^2 = 4x$ between $x=0$ and $x=3$ is rotated about x-axis.

Soln



$$A = \int_0^3 2\pi y \, ds$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

from $y^2 = 4x$

$$2y \frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{4}{2y} = \frac{2}{y}$$

$$ds = \sqrt{1 + \frac{4}{y^2}}\, dx = \frac{\sqrt{y^2+4}}{y}\, dx$$

but $y^2 = 4x$

$$= \frac{\sqrt{4x+4}}{y}\, dx$$

$$ds = \frac{2\sqrt{x+1}}{y}\, dx$$

$$A = \int_0^3 2\pi y \cdot \frac{2\sqrt{x+1}}{y}\, dx$$

$$= 4\pi\int_0^3 (x+1)^{\frac{1}{2}}\, dx = 4\pi \cdot \frac{2}{3}\left[(x+1)^{\frac{3}{2}}\right]_0^3$$

$$= \frac{8\pi}{3}[8-1]$$

$$= \frac{56}{3}\pi \text{ square units.}$$

Example 2

Find the surface area of a sphere obtained by rotating $x^2+y^2=a^2$ between $x=-a$ and $x=a$ about the x-axis.

Soln

$$A = \int_{-a}^{a} 2\pi y\, ds$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

from $x^2+y^2=a^2$

$$2x+2y\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{x}{y}$$

$$ds=\sqrt{1+\left(\frac{-x}{y}\right)^2}\,dx$$

$$=\sqrt{1+\frac{x^2}{y^2}}\,dx=\sqrt{\frac{y^2+x^2}{y^2}}\,dx$$

$$=\frac{\sqrt{x^2+y^2}}{y}\,dx=\frac{\sqrt{a^2}}{y}\,dx=\frac{a}{y}\,dx$$

$$ds=\frac{a}{y}\,dx$$

$$A=\int_{-a}^{a}2\pi y\cdot\frac{a}{y}\,dx=\int_{-a}^{a}2\pi a\,dx$$

$$=2\pi a[x]_{-a}^{a}=2\pi a[a--a]$$

$$=2\pi a(2a)$$

$$=\underline{4\pi a^2} \text{ square units.}$$

Exercise

1. Find the area of the surface generated by rotating the portion of the curve $y = \frac{1}{3}(x^2+2)^{3/2}$, between $x=0$ and $x=3$ about the y-axis. [Ans $\frac{99\pi}{2}$]

2. Find the area of the surface generated, when the arc of the parabola $y^2 = 4x$ between the origin and the point $(4,4)$ is rotated through an angle 2π about the x-axis. [Ans: $\frac{8\pi}{3}[5\sqrt{5}-1]$]

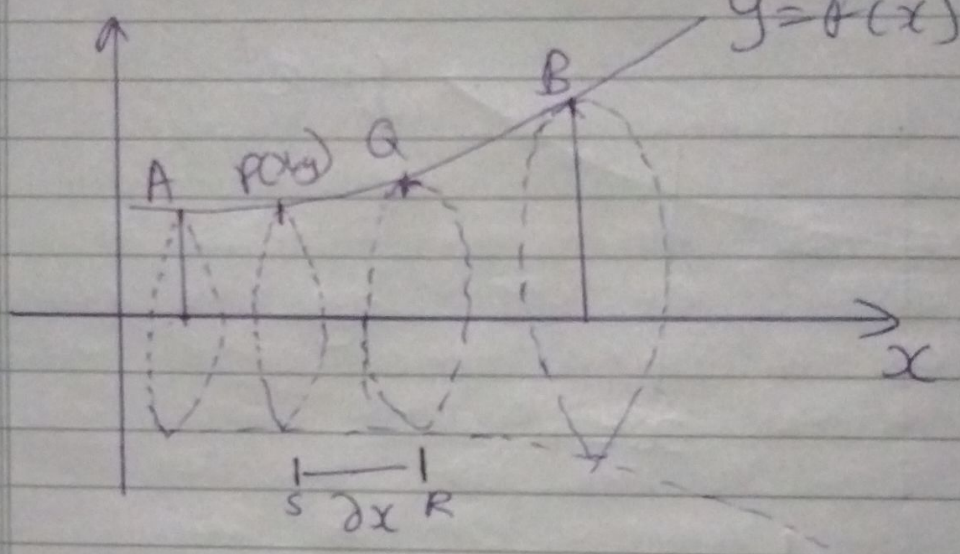
3. Find the area of the surface generated by rotating about the x-axis the arc of the curve $y = x^3$ between $x=0$ and $x=1$
Ans [$\frac{1}{27}\pi[10\sqrt{10}-1]$]

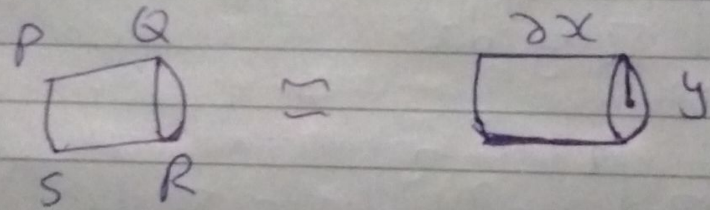
4. The curve described by the particle $P(x,y)$ $x = t+1$, $y = \frac{t^2}{2}+1$ from $t=0$ to $t=4$ is rotated about the y-axis. Find the surface area that is generated.
[Ans: $\frac{2}{3}\pi(26\sqrt{26}-2\sqrt{2})$]

5. Find the area of the surface swept out when the curve $12y^2 = x(x-4)^2$ rotates about x-axis.
[Ans: $\frac{16}{3}\pi$]

VOLUMES OF REVOLUTIONS

If part of a curve is rotated about a straight line, the solid formed is called solid of revolution. Such a solid is always symmetrical about the axis of rotation.





The section PQRS is approximately a cylinder whose radius is y and whose height is ∂x. Hence the volume PQRS is

$$\partial v \simeq \pi y^2 \partial x$$

$\therefore$ Volume V of the solid will be

$$V \simeq \sum_{A}^{B} \pi y^2 \partial x$$

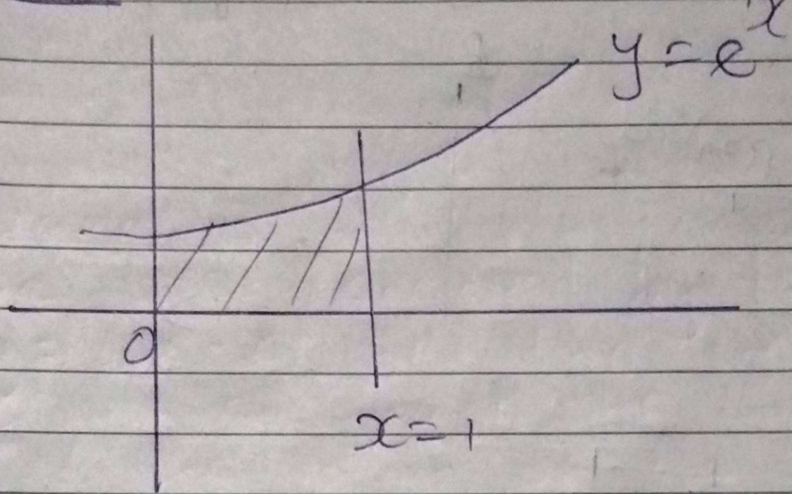
$$= \lim_{\partial x \to 0} \sum_{A}^{B} \pi y^2 \partial x$$

$$\boxed{V = \int_{A}^{B} \pi y^2 \, dx} \ *$$

Example 1

Find the volume generated when the area between $y = e^x$, the x-axis, the y-axis and $x = 1$ is rotated through one revolution about the x-axis

Soln



$$\text{Volume} = \int_0^1 \pi y^2 \, dx$$

$$= \pi \int_0^1 (e^x)^2 \, dx = \pi \int_0^1 e^{2x} \, dx$$

$$= \pi \left[\frac{e^{2x}}{2} \right]_0^1$$

$$= \frac{\pi}{2} \left[e^2 - 1 \right] \text{ cubic units}$$

Example 2

The area defined by the inequalities $y \geq x^2+1$, $x \geq 0$, $y \leq 2$ is rotated completely about the y-axis. Find the volume of the solid generated.

Soln

$y = x^2+1$; $x=0$; $y=2$

Turning point

$\frac{dy}{dx} = 0$

from $y = x^2+1$

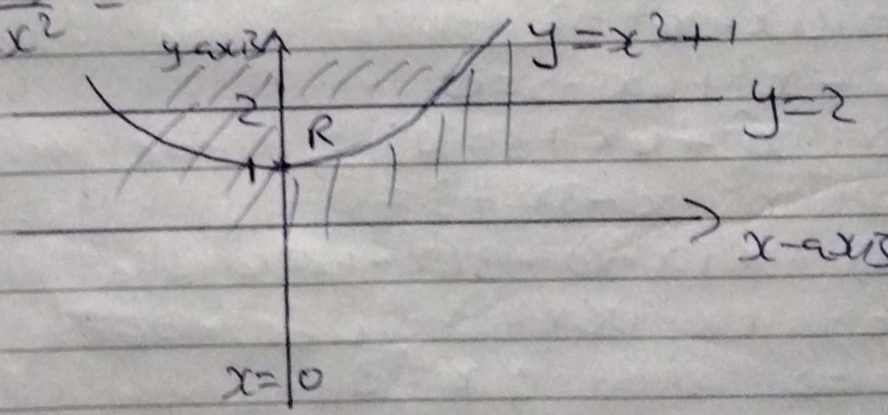
$\frac{dy}{dx} = 2x = 0$

$\Rightarrow x = 0$

Thus $y = 0^2+1 = 1$

Turning point $(x,y) = (0,1)$

$\frac{d^2y}{dx^2} = 2 > 0$ (minimum)



$$V = \int_{y=1}^{y=2} \pi x^2 \, dy \qquad \text{but } y = x^2+1$$

$$\Rightarrow x = \sqrt{y-1}$$

$$= \int_1^2 \pi (\sqrt{y-1})^2 \, dy$$

$$= \pi \int_1^2 (y-1) \, dy$$

$$V = \pi\left[\frac{y^2}{2} - y\right]_1^2$$

$$= \pi\left[(2-2) - \left(\frac{1}{2} - 1\right)\right]$$

$$= \frac{\pi}{2} \text{ cubic units}$$

Example 3

The area enclosed by the curve $y = 4x - x^2$ and the line $y = 3$ is rotated about the line $y = 3$. Find the volume of the solid generated.

Soln

$y = 4x - x^2$; $y = 3$

Point of intersection

$4x - x^2 = 3$

$\Rightarrow -x^2 + 4x - 3 = 0$

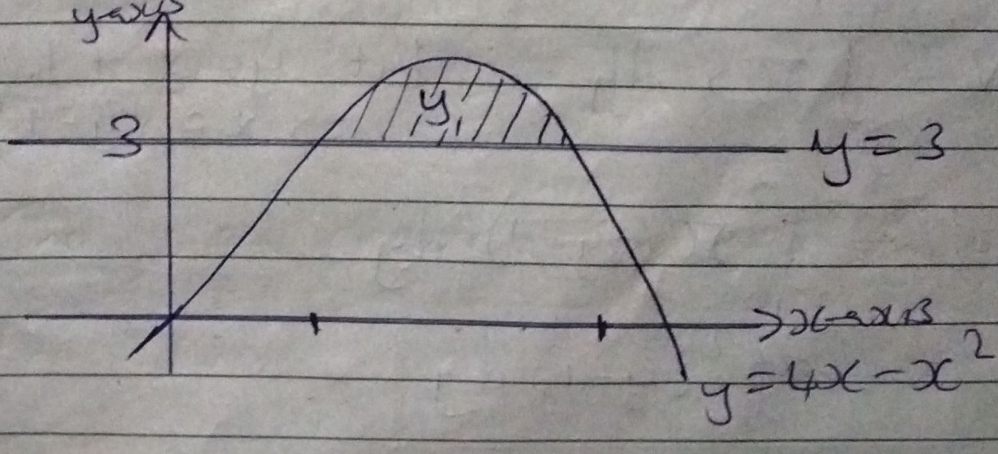
$\Rightarrow x^2 - 4x + 3 = 0$

$\Rightarrow x^2 - x - 3x + 3 = 0$

$\Rightarrow x(x-1) - 3(x-1) = 0$

$\Rightarrow (x-1)(x-3) = 0$

$\Rightarrow x = 1$; $x = 3$



where,

$$y_1 = (4x - x^2) - 3$$

$$V = \int_1^3 \pi y_1^2 \, dx$$

$$= \int_1^3 \pi (4x - x^2 - 3)^2 \, dx$$

$$= \pi \int_1^3 (x^4 - 8x^3 + 22x^2 - 24x + 9) \, dx$$

$$= \pi \left[\frac{x^5}{5} - \frac{8}{4}x^4 + \frac{22}{3}x^3 - \frac{24}{2}x^2 + 9x \right]_1^3$$

$$= \pi \left[\frac{x^5}{5} - 2x^4 + \frac{22}{3}x^3 - 12x^2 + 9x \right]_1^3$$

$$= \pi \left[108.6 - 2.5333 \right]$$

$$= 106.06\pi \text{ cubic units}$$

Example 4

Find the volume of the solid generated when the curve $y = \sqrt{x}$ over the interval $(1, 4)$ is revolved about the axis .

Soln

$$V = \int_1^4 \pi y^2 \, dx \quad ; y = \sqrt{x}$$

$$= \pi \int_1^4 (\sqrt{x})^2 \, dx = \pi \int_1^4 x \, dx$$

$$= \left. \frac{\pi x^2}{2} \right]_1^4 = \frac{\pi}{2}\left[4^2 - 1^2\right]$$

$$= \frac{15}{2}\pi \text{ cubic units.}$$

Exercise

1. Find the volume generated when the region bounded by the curves $y^2 = x$ and $y = x^3$ is rotated about the x-axis.

[Ans: $\frac{5}{14}\pi$]

2. ~~Find~~ If the area of the region defined by the inequalities $y^2 \leq x$, $y \geq x$, is rotated about the x-axis. Find the volume of the solid generated.

[Ans: $\frac{\pi}{6}$ cubic units]

3. Find the volume of the solid generated when the area enclosed by the curve $y = x + x^2$, the x-axis, the lines $x = 2$ & $x = 3$ is revolved through one revolution about the x-axis.

{Ans: $81\frac{1}{30}$ cubic units]

4. Find the volume generated when the area enclosed between $y^2 = x$ and $x = 1$ is rotated about the line $x = 1$.

[Ans: $\frac{16}{15}\pi$]

5. The area enclosed by $y = x^2$ and $y^2 = x$ is rotated about the x-axis. Find the volume generated.

[Ans: $\frac{3}{10}\pi$]

DISTANCE, VELOCITY, ACCELERATION

Recall

If $F(u)$ is an anti-derivative of $f(u)$ then

$$\int_a^b f(u)\,du = F(b) - F(a)$$

Suppose that, we want to let the upper limit of integration vary by replacing b by a variable x.
We think of a as a fixed starting value x_0.
Then

$$\int_{x_0}^{x} f(u)\,du = F(x) - F(x_0)$$

$$\Rightarrow \quad F(x) = F(x_0) + \int_{x_0}^{x} f(u)\,du$$

To obtain the position of an object at time t (say, on the x-axis) and we know its position at time t_0, let $s(t)$ denote the position of the object at time t. Then the net change in position between t_0 and t is $s(t) - s(t_0)$. Since $s(t)$ is an anti-derivative of the velocity function $v(t)$, we can write

$$s(t) = s(t_0) + \int_{t_0}^{t} v(u)\,du$$

Similarly since the velocity is an anti-derivative of the acceleration function $a(t)$, we have

$$v(t) = v(t_0) + \int_{t_0}^{t} a(u)\,du$$

Example 1

Suppose an object is acted upon by a constant force F. ~~and~~ Find $v(t)$ and $s(t)$. By Newton's law $F = ma$, so the acceleration is F/m, where m is the mass of the object.

i.e

$$a = \frac{F}{m}$$

Then

$$V(t) = v(t_0) + \int_{t_0}^{t} \frac{F}{m}\,du$$

$$= v(t_0) + \left[\frac{F}{m}u\right]_{t_0}^{t}$$

$$= v(t_0) + \frac{F}{m}(t - t_0)$$

$$v(t) = v_0 + \frac{F}{m}(t - t_0)$$

where $v_0 = v(t_0)$

$$s(t) = s(t_0) + \int_{t_0}^{t} \left(v_0 + \frac{F}{m}(\cancel{t} - t_0)\right) dt$$

let the difference $t - t_0$ be p

$p = t - t_0$

$dp = dt$

when $t = t$; $p = t - t_0$

when $t = t_0$; $p = t_0 - t_0 = 0$

$$s(t) = s(t_0) + \int_0^{t-t_0} \left(V_0 + \frac{F}{m} p\right) dp$$

$$= s(t_0) + \left[V_0 p + \frac{F}{m}\frac{p^2}{2}\right]_0^{t-t_0}$$

$$= s(t_0) + V_0(t - t_0) + \frac{F}{2m}(t - t_0)^2$$

$$s(t) = s_0 + V_0(t - t_0) + \frac{F}{2m}(t - t_0)^2$$

Example 2

The acceleration of an object is given by $a(t) = \cos \pi t$, and its velocity at time $t = 0$ is $1/2\pi$. Find both the net and the total distance traveled in the first 1.5 seconds.

Soln

$$v(t) = v(t_0) + \int_{t_0}^{t} a(u)\, du$$

$$= V(0) + \int_0^t \cos(\pi u)\, du = \frac{1}{2\pi} + \left[\frac{1}{\pi} \sin \pi u\right]_0^t$$

$$V(t) = \frac{1}{2\pi} + \frac{1}{\pi}\left[\sin(\pi t) - \sin 0\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{2} + \sin(\pi t)\right]$$

The net distance travelled is then

$$S(t) - S(t_0) = \int_{t_0}^{t} v(u)\,du$$

$$S\left(\tfrac{3}{2}\right) - S(0) = \int_0^{\frac{3}{2}} \frac{1}{\pi}\left(\frac{1}{2} + \sin(\pi t)\right) dt$$

$$= \left[\frac{1}{\pi}\left(\frac{1}{2}t - \frac{1}{\pi}\cos(\pi t)\right)\right]_0^{3/2}$$

$$= \frac{3}{4\pi} + \frac{1}{\pi^2}$$

$$\simeq \underline{\underline{0.340}} \text{ meters}$$

Example 3

Find the equation of motion for an object that moves along a straight line with constant acceleration a from an initial position x_0 with initial velocity V_0.

Soln.

$a(t) = a$ at all time t.

$$V(t) = \int a\,dt = at + c$$

$$V_0 = V(0) = a(0) + c = c$$

$$\Rightarrow V_0 = c$$

Thus

$v(t) = at + v_0$

$v(t) = v_0 + at$

Position function

$s(t) = \int v(t)\,dt = \int (v_0 + at)\,dt$

$= v_0 t + \frac{1}{2}at^2 + C$

For initial position s_0

$s_0 = s(0) = v_0(0) + \frac{1}{2}a(0)^2 + C = C$

$\Rightarrow \quad s_0 = C$

$s(t) = v_0(t) + \frac{1}{2}at^2 + s_0$

$s(t) = s_0 + v_0(t) + \frac{1}{2}at^2$

Example 4

A particle moves in a straightline with an acceleration a m/s^2 given by $a = t - \frac{5}{2}$ where t is the time in seconds from the start. The velocity v at the start is known to be 3m/s

(i) Find an expression for Velocity in terms of t.

(ii) Find the time t when the particle is at instantaneous rest.

(iii) If the particle started from a fixed point O on the line, how far is it from O after 2 seconds.

Soln

(i) $a = \frac{dv}{dt} = t - \frac{5}{2}$

$V = \int (t - \frac{5}{2})dt = \frac{t^2}{2} - \frac{5}{2}t + c$

when $t = 0$; $V(0) = 0^2 - 5$ $V(0) = 3$

$\Rightarrow 3 = \frac{0^2}{2} - \frac{5}{2}(0) + c$

$\Rightarrow c = 3$

Thus,

$V = \frac{t^2}{2} - \frac{5}{2}t + 3$

$= \frac{t^2 - 5t + 6}{2}$

$V = \frac{(t-3)(t-2)}{2}$

(ii) At instantaneous rest velocity is zero.

$V = 0 \quad \Rightarrow 0 = \frac{(t-3)(t-2)}{2}$

$\Rightarrow (t-3)(t-2) = 0$

Either $t - 3 = 0$ or $t - 2 = 0$

$\Rightarrow t = 3$ or 2

$\therefore$ At instantaneous rest, $t = 3$ or 2

(iii) $V = \frac{ds}{dt} = \frac{t^2}{2} - \frac{5}{2}t + 3$

$$\Rightarrow S = \int \left(\frac{t^2}{2} - \frac{5}{2}t + 3\right) dt$$

$$= \frac{t^3}{2 \times 3} - \frac{5}{2}\frac{t^2}{2} + 3t + c$$

$$= \frac{t^3}{6} - \frac{5}{4}t^2 + 3t + c$$

when $t = 0$; distance, $S = 0$

$$\Rightarrow 0 = \frac{0^3}{6} - \frac{5}{4}(0)^2 + 3(0) + c$$

$$\Rightarrow c = 0$$

$$\therefore S = \frac{t^3}{6} - \frac{5}{4}t^2 + 3t$$

After 2 seconds,

$$S = \frac{2^3}{6} - \frac{5}{4}(2^2) + 3(2)$$

$$= \frac{8}{6} - \frac{5}{4} \times 4 + 6 = \frac{8}{6} + 1$$

$$= \frac{14}{6} = \frac{7}{3}$$

If $t = 2$, $S = 2\frac{1}{3}$ m

Exercise

1. An object is shot upwards from ground level with an initial velocity of 2 meters per second; it is subject only to the force of gravity (no air resistance). Find its maximum altitude and the time at which it hits the ground.
   (Answer; $10/49$ m ; $20/49$ seconds)

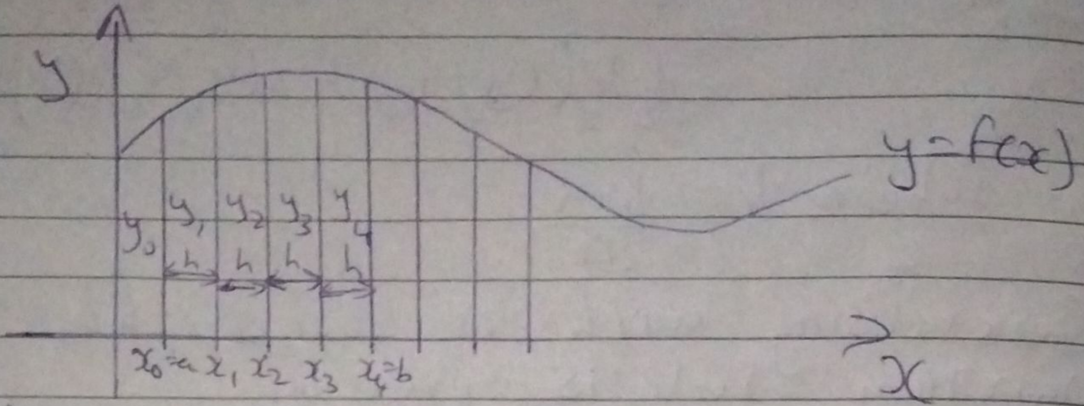
2. An object moves along a straight line with acceleration given by $a(t) = -\cos(t)$, and $s(0) = 1$ and $v(0) = 0$. Find the maximum distance the object travels from zero, and find its maximum speed. Describe the motion of the object.
   (Answer; $s(t) = \cos t$, $v(t) = -\sin t$, maximum distance is 1, maximum speed is 1.)

3. An object moves along a straight line with acceleration given by $a(t) = 1 + \sin(\pi t)$. Assume that when $t = 0$, $s(t) = v(t) = 0$. Find $s(t)$ and $v(t)$.
   (Answer: $s(t) = \frac{t^2}{2} - \frac{\sin(\pi t)}{\pi^2} + \frac{t}{\pi}$

   $v(t) = t - \frac{\cos(\pi t)}{\pi} + \frac{1}{\pi}$)

NUMERICAL INTEGRATION

1. Trapezium Rule



i.e $y_i = f(x_i)$

So that,

$$y_0 = f(x_0),\ y_1 = f(x_1), \ldots$$

If the area represented by $\int_a^b f(x)dx$ is divided into strips each of width h, then each such strip is approximately a trapezium.

Area of trapezium $= \frac{1}{2}$(sum of parallel sides) x height

Area of trapezium $1 = \frac{1}{2}(y_0 + y_1)h$

" " " $2 = \frac{1}{2}(y_1 + y_2)h$

" " " $3 = \frac{1}{2}(y_2 + y_3)h$

" " " $4 = \frac{1}{2}(y_3 + y_4)h$

$$\therefore \int_a^b f(x)dx = \frac{1}{2}(y_0 + y_1)h + \frac{1}{2}(y_1 + y_2)h + \frac{1}{2}(y_2 + y_3)h + \frac{1}{2}(y_3 + y_4)h$$

$$\therefore \int_a^b f(x)dx = \frac{1}{2}h(y_0+y_1+y_1+y_2+y_2+y_3+y_3+y_4)$$

$$= \frac{1}{2}h((y_0+y_4)+2(y_1+y_2+y_3))$$

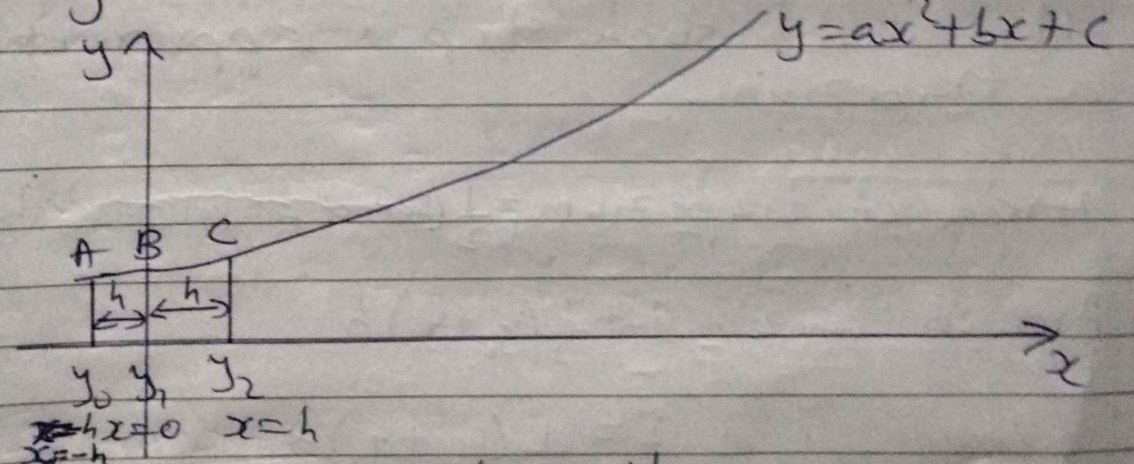
where $h = \frac{b-a}{n}$

Generalising with n strips, the trapezium rule is given by

$$\int_a^b f(x)dx \simeq \frac{h}{2}\{(y_0+y_n)+2(y_1+y_2+\cdots+y_{n-1})\}$$

2. Simpson's Rule

Suppose $y=f(x)$ is replaced by $y=ax^2+bx+c$



The area under the curve ABC

$$\int_{-h}^{h} f(x)dx \simeq \int_{-h}^{h}(ax^2+bx+c)dx$$

$$= \left[\frac{ax^3}{3}+\frac{bx^2}{2}+cx\right]_{-h}^{h}$$

$$= \frac{ah^3}{3}+\frac{bh^2}{2}+ch+\frac{ah^3}{3}-\frac{bh^2}{2}+ch$$

$$\int_{-h}^{h} f(x)\,dx = 2\frac{ah^3}{3} + 2ch \quad \cdots \text{(1)}$$

$$= \frac{2ah^2 h}{3} + 2ch$$

$$y = ax^2 + bx + c$$

$$y_0 = ah^2 - hb + c \quad \cdots (2)$$

$$y_1 = c \quad \cdots \text{(3)}$$

$$y_2 = ah^2 + bh + c \quad \cdots \text{(4)}$$

add eqns (2) \$ (4)

$$y_0 + y_2 = 2ah^2 + 2c$$

$$\Rightarrow 2ah^2 = y_0 + y_2 - 2c \qquad \text{but } c = y_1$$

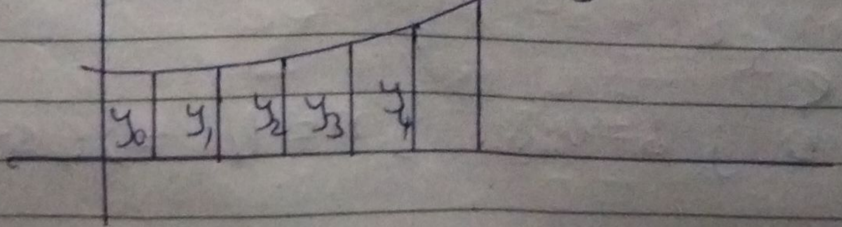
$$\Rightarrow 2ah^2 = y_0 + y_2 - 2y_1 \quad \cdots \text{(5)}$$

$$\Rightarrow \int_{-h}^{h} f(x)\,dx = \frac{(y_0 + y_2 - 2y_1)h}{3} + 2y_1 h$$

$$= \frac{h}{3}\{y_0 + y_2 - 2y_1 + 6y_1\}$$

$$= \frac{h}{3}\{y_0 + 4y_1 + y_2\}$$

This arrangement can be extended to another 2 strips i.e



The area through the parabolla y_0, y_1, y_2 is $\frac{h}{3}(y_0 + 4y_1 + y_2)$

The general area is :-

$$\frac{h}{3}\{(y_0 + y_4) + 2(y_2) + 4(y_1 + y_3)\}$$

hence, simpson's Rule with n strips is given by:

$$\int_a^b f(x)\,dx = \frac{h}{3}\{(y_0 + y_n) + 2(y_2 + y_4 + \dots + y_{n-2}) + 4(y_1 + y_3 + y_5 + \dots + y_{n-1})\}$$

__Examples__

1. Using trapezoidal rule and simpson's rule with $n = 4$. Evaluate $\int_0^{0.8} e^{x^2}\,dx$

__Soln__

$y = e^{x^2}$

$a = 0$; $b = 0.8$; $n = 4 \Rightarrow 4$ strips

$$h = \frac{b-a}{n} = \frac{0.8-0}{4} = 0.2$$

$x_0 = 0$

$x_1 = 0.2$

$x_2 = 0.4$

$x_3 = 0.6$

$x_4 = 0.8$

$y_0 = f(0) = e^0 = 1$

$y_1 = f(1) = e^{1^2} = 1.0408$

$y_2 = f(2) = e^{2^2} = 1.1735$

$y_3 = f(3) = e^{3^2} = 1.4333$

$y_4 = f(4) = e^{4^2} = 1.8965$

Trapezoidal rule

$$\int_0^{0.8} e^{x^2}\,dx \simeq \frac{h}{2}\left\{(y_0+y_4) + 2(y_1+y_2+y_3)\right\}$$

$$= \frac{0.2}{2}\left\{(1+1.8965) + 2(1.0408 + 1.1735 + 1.4333)\right\}$$

$$= \underline{\underline{1.01917}}$$

Simpson's rule

$$\int_0^{0.8} e^{x^2}\,dx \simeq \frac{h}{3}\left\{(y_0+y_4) + 2(y_2) + 4(y_1+y_3)\right\}$$

$$= \frac{0.2}{3}\left\{(1+1.8965) + 2(1.1735) + 4(1.0408 + 1.4333)\right\}$$

$$= \underline{\underline{1.00932667}} = \underline{\underline{1.00933}}$$

2. Use trapezoidal & simpson's rule with 5 ordinates to find approximate value of
$\int_0^1 \frac{4}{1+x^2}\,dx$. Compare your results with the exact value of the integrals.

<u>Soln</u>

Exact value of the integral.

$$\int_0^1 \frac{4}{1+x^2}\,dx$$

Using trigonometric substitution

from a^2+x^2 ; $x = a\tan\theta$

$\Rightarrow 1+x^2$; $x = 1\tan\theta = \tan\theta$

$dx = \sec^2\theta\, d\theta$

$$\int \frac{4}{1+x^2}\,dx = \int \frac{4}{\cancel{1+}\tan^2\theta} \cdot \sec^2\theta\, d\theta$$

$$= \int \frac{4}{\cancel{\sec^2\theta}} \cancel{\sec^2\theta}\, d\theta$$

$$= \int 4\, d\theta = 4\theta + c$$

but $x = \tan\theta$

$\Rightarrow \theta = \tan^{-1}x$

$$\therefore \int \frac{4}{1+x^2}\,dx = 4\tan^{-1}x + c$$

$$\int_0^1 \frac{4}{1+x^2}\,dx = \left[4\tan^{-1}x\right]_0^1$$

$$= 4\left[\tan^{-1}1 - \tan^{-1}0\right]$$

$$= 4\left[0.78539816339 - 0\right]$$

$$= 3.1415926556$$

$$\int_0^1 \frac{4}{1+x^2}\,dx \simeq 3.1416.$$

from $\int_0^1 \frac{4}{1+x^2}\,dx$

$y = \frac{4}{1+x^2}$; $a=0$; $b=1$; $n=5$ $n=4$

$h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$

$x_0 = 0$ $\qquad y_0 = 4$

$x_1 = 0.25$ $\qquad y_1 = \frac{4}{1+0.25^2} = 3.7642$

$x_2 = 0.5$ $\qquad y_2 = \frac{4}{1+0.5^2} = 3.2$

$x_3 = 0.75$ $\qquad y_3 = \frac{4}{1+0.75^2} = 2.56$

$x_4 = 1$ $\qquad y_4 = \frac{4}{1+1^2} = 2$

Trapezium rule

$$\int_0^1 \frac{4}{1+x^2}\,dx = \frac{h}{2}\left\{(y_0+y_4)+2(y_1+y_2+y_3)\right\}$$

$$= \frac{0.25}{2}\left\{(4+2)+2(3.7642+3.2+2.56)\right\}$$

$$= 3.1312$$

Simpson's rule

$$\int_0^1 \frac{4}{1+x^2}\,dx = \frac{h}{3}\left\{(y_0+y_4)+2(y_2)+4(y_1+y_3)\right\}$$

$$= \frac{0.25}{3}\left\{(4+2)+2(3.2)+4(3.7647+2.56)\right\}$$

$$\int_0^1 \frac{4}{1+x^2}\,dx \simeq \underline{\underline{3.1416}}$$

Comparing the results with the exact values of the integrals we notice that they are approximately the same.

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3. Use trapezoidal & simpson's rule to evaluate integral $\int_9^{10} y\,dx$ where y is given in terms of x by the following table

x	9.0	9.25	9.5	9.75	10.0
y	0.111	0.1081	0.1053	0.1026	0.1000

<u>soln</u>

$a = 9.0$; $b = 10.0$; $n = 4$

$y_0; y_1; y_2; y_3; y_4$

$$h = \frac{b-a}{4} = \frac{10.0 - 9.0}{4} = 0.25$$

~~x_0~~

<u>Trapezium rule</u>

$$\int_9^{10} y\,dx = \frac{h}{2}\{(y_0+y_4) + 2(y_1+y_2+y_3)\}$$

$$= \frac{0.25}{2}\{(0.111 + 0.1) + 2(0.1081 + 0.1053 + 0.1026)$$

$$= \underline{\underline{0.105375}}$$

Simpson's rule

$$\int_9^{10} y\,dx = \frac{h}{3}\left\{(y_0+y_4) + 2(y_2) + 4(y_1+y_3)\right\}$$

$$= \frac{0.25}{3}\left\{(0.111+0.1) + 2(0.1053) + 4(0.1081 + 0.1026)\right\}$$

$$= 0.1053666667$$

$$\approx \underline{\underline{0.1053667}}$$

Exercise

1. Use 11 ordinates and Simpson's rule to find an approximate value of $\int_1^2 \frac{dx}{x}$.
   [Ans 0.6931]

2. Find the approximate value of $\int_0 x^2 e^x\,dx$, using the trapezium rule with 5 strips.
   [Ans: 0.7456]

3. Find the estimate of $\int_0^{0.8} e^{x^2}\,dx$ using trapezium rule with six ordinates.
   [Ans: 1.0192]

4. Use Simpson's rule and 11 ordinates to obtaine approximate value of definite integral $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}$
   Compare your result with exact value of integral. [Ans: 0.5235]

5. Find the approximate value for $\int_0^{\pi} \sqrt{\sin\theta}\, d\theta$, using Simpson's rule with five ordinates.
[Ans: 2.2848]

6. Find an approximate arc length of an ellipse where

$$S = \int_0^{\frac{\pi}{2}} \sqrt{1 - 0.9\sin^2\theta}\, d\theta$$

use $n = 4$.
[Ans: 1.107]

7. Find the approximate value for $\int_0^1 \frac{1}{1+x^2}\, dx$ correct to 4 decimal places using five ordinates by applying
(i) the trapezium rule
(ii) the Simpson's rule
(iii) Determine the difference in the estimation of using the two methods.
[Ans: 0.7888 ; 0.7856 ; 0.0026]

8. For the following problems, evaluate using
(i) Trapezoidal rule
(ii) Simpson's rule
a) $\int_0^{0.6} x e^{x} dx$; $n=6$ [Ans: 0.2727 ; 0.27115]

b) $\int_0^2 \ln x\, dx$; $n=4$ [Ans: 0.99033, 0.9996]

c) $\int_0^1 \sqrt{1+x^2}\, dx$; $n=10$ [Ans: 1.14838 ; 1.14779]