Technical University of Mombasa

Session Three: Continuity

3.0 Session objectives

By the end of the session, you should be able to:

i. Define continuity of a function.
ii. State the types of discontinuity of a function
iii. State the intermediate value theorem and use it to prove the existence of roots of equations.

3.1 Continuity of a function

Definition

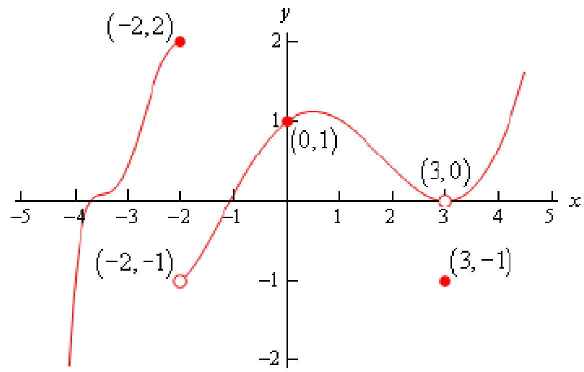
A function $f(x)$ is said to be **continuous** at $x = a$ if $\lim_{x\to a} f(x) = f(a)$

A function is said to be continuous on the interval $[a, b]$ if it is continuous at each point in the interval.

If $f(x)$ is continuous at $x = a$ then,

$\lim_{x\to a} f(x) = f(a)$ $\qquad$ $\lim_{x\to a^+} f(x) = f(a)$ $\qquad$ $\lim_{x\to a^-} f(x) = f(a)$

Example 1 Given the graph of $f(x)$, shown below, determine if $f(x)$ is continuous at $x = -2$, $x = 0$, and $x = 3$.



Solution

To answer the question for each point we'll need to get both the limit at that point and the function value at that point. If they are equal the function is continuous at that point and if they aren't equal the function isn't continuous at that point.

First $x = -2$.

$$f(-2) = 2 \qquad \lim_{x\to -2} f(x) \qquad \text{doesn't exist}$$

The function value and the limit aren't the same and so the function is not continuous at this point. This kind of discontinuity in a graph is called a **jump discontinuity**. Jump discontinuities occur where the graph has a break in it is as this graph does.

Now $x = 0$.

$$f(0) = 1 \qquad\qquad \lim_{x\to 0} f(x) = 1$$

The function is continuous at this point since the function and limit have the same value.

Finally $x = 3$.

$$f(3) = -1 \qquad\qquad \lim_{x\to 3} f(x) = 0$$

The function is not continuous at this point. This kind of discontinuity is called a **removable discontinuity**. Removable discontinuities are those where there is a hole in the graph as there is in this case.

A function is continuous if its graph has no holes or breaks in it.

For many functions it's easy to determine where it won't be continuous. Functions won't be continuous where we have things like division by zero or logarithms of zero.

Example 2 Determine where the function below is not continuous.

$$h(t) = \frac{4t + 10}{t^2 - 2t - 15}$$

Solution

Rational functions are continuous everywhere except where we have division by zero. So all that we need to is determine where the denominator is zero. That's easy enough to determine by setting the denominator equal to zero and solving.

$$t^2 - 2t - 15 = (t - 5)(t + 3) = 0$$

So, the function will not be continuous at *t=-3* and *t=5*.

If $f(x)$ is continuous at $x = b$ and $\lim_{x \to a} g(x) = b$ then, $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$

Example 3 Evaluate the following limit.

$$\lim_{x \to 0} e^{sinx}$$

Solution

Since we know that exponentials are continuous everywhere.

$$\lim_{x \to 0} e^{sinx} = e^{\lim_{x \to 0} sinx} = e^0 = 1$$

3.2 Intermediate Value Theorem

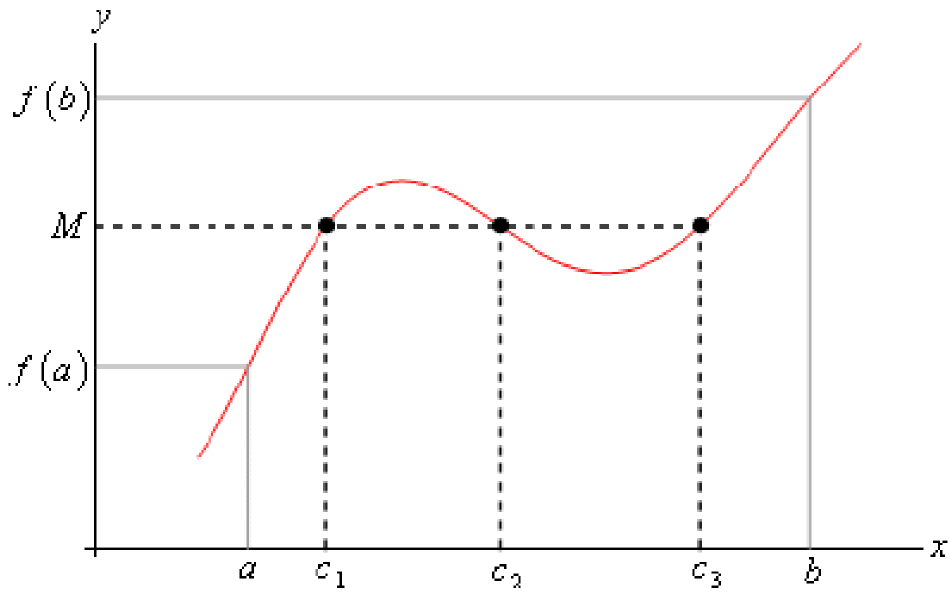
Suppose that $f(x)$ is continuous on $[a, b]$ and let M be any number between $f(a)$ and $f(b)$. Then there exists a number *c* such that,

i. $a < c < b$

ii. $f(c) = M$

That is a continuous function will take on all values between $f(a)$ and $f(b)$.

Below is a graph of a continuous function that illustrates the Intermediate Value Theorem.



From the graph if we pick any value, *M*, that is between the value of *f(a)* and the value of *f(b)* and draw a line straight out from this point the line will hit the graph in at least one point. In other words somewhere between *a* and *b* the function will take on the value of *M*. Also, as the figure shows the function may take on the value at more than one place.

It's also important to note that the Intermediate Value Theorem only says that the function will take on the value of *M* somewhere between *a* and *b*. It doesn't say just what that value will be. It only says that it exists.

The Intermediate Value Theorem may be used to prove the existence of roots of equations as the following example shows.

Example 4 Show that $p(x) = 2x^3 - 5x^2 - 10x + 5$ has a root somewhere in the interval $[-1,2]$.

Solution

What we're really asking here is whether or not the function will take on the value $p\,(x) = 0$ somewhere between -1 and 2. In other words, we want to show that there is a number c such that $-1 < c < 2$ and $p\,(c) = 0$. However if we define $M = 0$ and acknowledge that $a = -1$ and $b = 2$ we can see that these two condition on c are exactly the conclusions of the Intermediate Value Theorem.

So, this problem is set up to use the Intermediate Value Theorem and in fact, all we need to do is to show that the function is continuous and that $M = 0$ is between $p\,(-1)$ and $p\,(2)$ *i.e.* $p\,(-1) < 0 < p(2)\ or\ p\,(2) < 0 < p\,(-1)$ and we'll be done.

To do this all we need to do is compute,

$p\,(-1) = 8\ \ p\,(2) = -19$

So we have,

$$-19 = p\,(2) < 0 < p\,(-1) = 8$$

Therefore $M = 0$ is between $p\,(-1)$ and $p\,(2)$ and since $p\,(x)$ is a polynomial it's continuous everywhere and so in particular it's continuous on the interval [-1,2]. So by the Intermediate Value Theorem there must be a number $-1 < c < 2$ so that, $p\,(c) = 0$

Therefore the polynomial does have a root between -1 and 2.

Example 5 If possible, determine if $f\,(x) = 20\sin(x+3)\,cos\left(\frac{x^2}{2}\right)$ takes the following values in the interval [0,5].

(a) Does $f\,(x) = 10$?

(b) Does $f\,(x) = -10$?

Solution

Here we're being asked to determine, if possible, if the function takes on either of the two values above in the interval [0,5].

First, let's notice that this is a continuous function and so we know that we can use the Intermediate Value Theorem to do this problem.

Now, for each part we will let M be the given value for that part and then we'll need to show that M lives between $f(0)$ and $f(5)$. If it does then we can use the Intermediate Value Theorem to prove that the function will take the given value.

$f(0) = 2.8224$ $\qquad$ $f(5) = 19.7436$

a) In this case we'll define $M = 10$ and we can see that, $f(0) = 2.8224 < 10 < 19.7436 = f(5)$

   So, by the Intermediate Value Theorem there must be a number $0 \le c \le 5$ such that

   $f(c) = 10$

b) In this part we'll define $M = -10$. We now have a problem. In this part M does not live between $f(0)$ and $f(5)$. So, what does this mean for us? Does this mean that $f(x) \ne -10$ in [0,5]?

   Unfortunately for us, this doesn't mean anything. It is possible that $f(x) \ne -10$ in [0,5], but is it also possible that $f(x) = -10$ in [0,5]. The Intermediate Value Theorem will only tell us that c's will exist. The theorem will NOT tell us that c's don't exist.

   In this case it is not possible to determine if $f(x) = -10$ in [0,5] using the Intermediate Value Theorem.

Note: The Intermediate Value Theorem will not always be able to tell us what we want to know. Sometimes we can use it to verify that a function will take some value in a given interval and in other cases we won't be able to use it.

3.3 Student's Activity

i. Given the function $f(x) = \begin{cases} c - x \text{ } if \text{ } x \leq \pi \\ csinx \text{ } if \text{ } x > \pi \end{cases}$

a) Find the values of the constant c so that the function $f(x)$ is continuous.

b) For the value of c found above verify whether the 3 conditions for continuity are satisfied.

ii. a) Use the Intermediate Value Theorem to show that $2^x = \frac{10}{x}$ for some $x > 0$.

b) Show that the equation $2^x = \frac{10}{x}$ has no solution for $x < 0$.

iii. Assume that $f(x) = \begin{cases} 2 + \sqrt{x} \text{ } if \text{ } x \geq 1 \\ \frac{x}{2} + \frac{5}{2} \text{ } if \text{ } x < 1 \end{cases}$

Determine whether or not f is continuous at $x = 1$. Justify your answer.

iv. Give one example of a function $f(x)$ that is continuous for all values of x except $x = 3$, where it has a removable discontinuity. Explain how you know that f is discontinuous at $x = 3$, and how you know that the discontinuity is removable.