Contents

11 Session Eleven: Applications of Differentiation I

11.1 Session Objectives

By the end of this session, you should be able to:

(i) Use derivatives in getting the equations of tangents and normals to a curve.

(ii) Identify intervals at which the grap of a function is increasing or decreasing.

(iii) Sketch the graph of a function.

11.2 Introduction

Why learn differentiation? In this section and the next two sessions, we shall look at applications of differentiation. Here, we discuss applications of differentiation in tangents and normals to curves, Increasing and decreasing functions and relative extrema.

11.3 Tangents and Normals

Given the function $y = f(x)$, the derivative $\frac{dy}{dx}$ is the slope/gradient of the tangent to the curve at (x, y). Thus

$$m = \frac{dy}{dx}|_{x=a} = f'(a)$$

is the gradient of the tangent to the curve $y = f(x)$ at $(a, f(a))$. The equation of the tangent is

$$y - f(a) = f'(a)(x - a)$$

The normal to the curve $y = f(x)$ at $(a, f(a))$ is perpendicular to the tangent at $(a, f(a))$. The slope/gradient of the normal is $-\frac{1}{f'(a)}$ and its equation is

$$y - f(a) = -\frac{1}{f'(a)}(x - a)$$

Example **11.3.1.** *Find the equation of the tangent and normal to the curve $x + \sqrt{x}$ at the point $(1, 2)$.*

Solution: **Tangent Equation:**
$f(1) = 2, \quad f'(x) = 1 + \frac{1}{2\sqrt{x}} \Rightarrow f'(1) = \frac{3}{2}$.
Therefore, $y - 2 = \frac{3}{2}(x - 1) = \frac{3}{2}x + \frac{1}{2}$ or $2y = 3x + 1$.
Normal Equation:
$y - 2 = -\frac{2}{3}(x - 1) = -\frac{2}{3}x + \frac{8}{3}$

11.4 Increasing and decreasing functions

Definition 11.4.1 (Increasing and decreasing functions). *A function f is said to be increasing on an interval I, if for any two numbers x_1 and x_2 in I,*

$$f(x_1) < f(x_2) \text{ whenever } x_1 < x_2$$

A function f is said to be decreasing on an interval I, if for any two numbers x_1 and x_2 in I,

$$f(x_1) > f(x_2) \text{ whenever } x_1 > x_2$$

Test for intervals where $f(x)$ is increasing and decreasing
Suppose a function f has a derivative at each point in an open interval (a, b); then,

1. $f'(x) > 0 \Rightarrow$ for each $x \in (a, b)$, f is increasing on (a, b).
2. $f'(x) < 0 \Rightarrow$ for each $x \in (a, b)$, f is decreasing on (a, b).
3. $f'(x) = 0 \Rightarrow$ for each $x \in (a, b)$, f is a constant.

Definition 11.4.2 (Critical number). *If f is defined at c, then c is called a critical number of f if $f'(c) = 0$ or $f'(c)$ is not defined.*

Definition 11.4.3 (Critical point). *The point $(c, f(c))$ where c is a critical number is called a critical point.*

Example 11.4.1. *Find the critical numbers, critical points of $f(x) = \frac{9(x^2-3)}{x^3}$*

Solution: $f'(x) = \frac{9(9-x^2)}{x^4}$. So $x = \pm 3, 0$ are critical numbers since $f'(\pm 3) = 0$ and $f'(0)$ D.N.E. Therefore, $(-3, -2)$ and $(3, 2)$ are the critical points

Example 11.4.2. *Find the critical numbers, open intervals where the function $f(x) = x^3 + 3x^2 - 9x + 4$ is increasing and decreasing.*

Solution: $f'(x) = 3x^2 + 6x - 9 = 0 \Rightarrow (x+3)(x-1) = 0 \Rightarrow x = -3, 1$ are critical numbers. Note that there are no values of x for which $f'(x)$ fails to exist.

Table 1: Signs of $f'(x)$

x	-4	-3	0	1	2
Sign of f'	+		-		+
f is	incr.		decr.		incr.

From Table 1,
f is increasing in $(-\infty, -3)$ and $(1, \infty)$ and decreasing on $(-3, 1)$.

Example 11.4.3. *Find the critical numbers, open intervals where the function $f(x) = (x-1)^{\frac{2}{3}}$ is increasing and decreasing.*

Solution: $f'(x) = \frac{2}{3}(x-1)^{-\frac{1}{3}} = \frac{2}{3(x-1)^{\frac{1}{3}}}$.
$f'(x)$ is never zero and $f'(x)$ does not exist at $x = 1$, so $x = 1$ is a critical number

Table 2: Signs of $f'(x)$

x	0	1	2
Sign of f'	-		+
f is	decr.		incr.

From Table 2,
f is increasing in $(1, \infty)$ and decreasing on $(-\infty, 1)$.

Example 11.4.4. *$f(x) = \cos x$; $f'(x) = -\sin x$. $f'(0) = 0 \Rightarrow x = 0$ is a critical number and $(0, 1)$ is a critical point.*

11.5 Relative Extrema

Definition 11.5.1 (Relative extrema). *Let c be a number in the domain of a function f.*

1. *If there is an open interval (a, b) containing cin which $f(x) \leq f(c)$ for all x in (a, b), then $f(c)$ is called a **relative (or local)** maximum of f.*

2. *If there is an open interval (a, b) containing cin which $f(x) \geq f(c)$ for all x in (a, b), then $f(c)$ is called a **relative (or local)** minimum of f.*

Figure 1 shows both relative maximum and relative minimum.

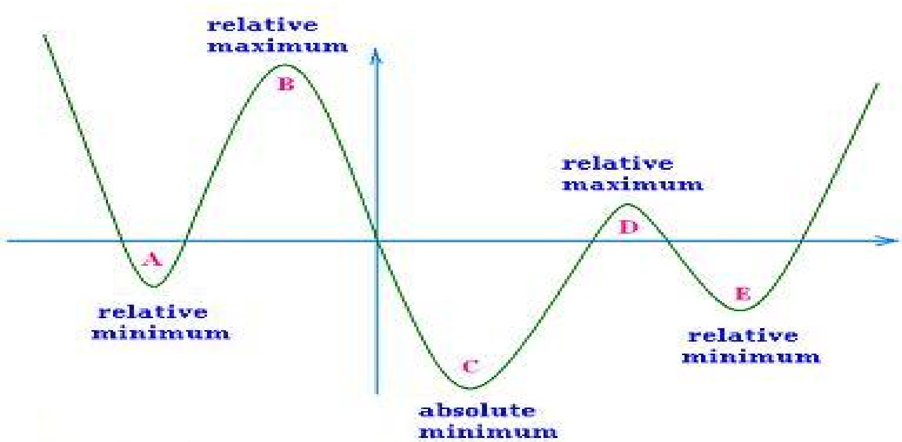

Figure 1: Relative extrema

First derivative test
Let c be a critical number of f. Then f is differentiable and

1. If f' changes from positive to negative at c, then $f(c)$ is a relative maximum of f.

2. If f' changes from negative to positive at c, then $f(c)$ is a relative minimum of f.

Example **11.5.1.** *Find the $x-$ value where $f(x) = x^4 - 18x^2 - 4$ has relative extrema and find the relative extreme values.*

Solution: $f'(x) = 4x^3 - 36x = 4x(x+3)(x-3) = 0 \Rightarrow x = -3, 0, 3$ are critical numbers.

Table 3: Signs of $f'(x)$

x	-4	-3	-1	0	1	3	4
Sign of f'	-		+		-		+
f is	decr.		incr.		decr.		incr.

Therefore, at $x = -4$ we have a relative maximum and at $x = -3, 3$ we have a relative minimum values -85 while at $x = 0$, we have a relative maximum value -4.

11.6 Session Summary

For more material on this section check out [1, 2] or visit Tangents and normals, increasing and decreasing functions/relative extrema. You can also watch the lecture videos:Tangents and normals, increasing and decreasing functions.

11.7 Student Activity

Exercise

Tangents and Normals

1. Find the equation of the tangent and normal to the curve $x = 3t^2 + 1$, $y = 2t^3 + 1$ at the point $(4, 3)$.
2. At what point of the curve $x = t^3 + 4t$, $y = 6t^2$ is the tangent parallel to the line with equation $x = -7t$, $y = 12t - 5$.
3. Find the equations of both lines through $(2, -3)$ that are tangents to the curve $y = x^2 + x$.
4. Find the points on the curve $y = x^3 - x^2 - x + 1$ where the tangent is horizontal.
5. Show that the curve $y = 6x^3 + 5x - 3$ has no tangent line with gradient 4.

Increasing and Decreasing Functions

In questions 1–6, find the critical numbers, open intervals where the function $f(x) = (x-1)^{\frac{2}{3}}$ is increasing and decreasing.

1. $f(x) = \frac{2}{3}x^3 - x^2 - 24x - 4$
2. $f(x) = \frac{x-1}{x+1}$
3. $f(x) = |x|$
4. $f(x) = 4x^3 - 15x^2 - 72x + 5$
5. $f(x) = 3x^4 + 8x^3 - 18x^2 + 15$
6. $f(x) = 3x^4 - 4x^3$

Relative Extrema

In questions 1–3, find the x-value where $f(x)$ has relative extrema and find the relative extreme values.

1. $f(x) = 6x^{\frac{2}{3}}$
2. $f(x) = xe^{2-x^2}$
3. $f(x) = x^2 - 10x + 33$

References

[1] E. Purcell D. Varberg and S. Rigdon. *Calculus*. Pearson Education, Inc., 9 edition, 2006. ISBN-13 : 978-0132306331.

[2] J. Stewart. *Calculus*. Cengage Learning 20 Channel Center Street, Boston, MA 02210, USA, 8 edition, 2016. ISBN-13: 978-1-305-27176-0.

