CHAPTER

2 Basic Structures: Sets

Lesson Objectives

Much of discrete mathematics is devoted to the study of discrete structures, used to represent discrete objects. Many important discrete structures are built using sets, which are collections of objects. Among the discrete structures built from sets are combinations, unordered collections of objects used extensively in counting; relations, sets of ordered pairs that represent relationships between objects; graphs, sets of vertices and edges that connect vertices; and finite state machines, used to model computing machines. These are some of the topics we will study in later chapters.

The concept of a function is extremely important in discrete mathematics. A function assigns to each element of a first set exactly one element of a second set, where the two sets are not necessarily distinct. Functions play important roles throughout discrete mathematics. They are used to represent the computational complexity of algorithms, to study the size of sets, to count objects, and in a myriad of other ways. Useful structures such as sequences and strings are special types of functions. In this chapter, we will introduce the notion of sequences, which represent ordered lists of elements. Furthermore, we will introduce some important types of sequences and we will show how to define the terms of a sequence using earlier terms. We will also address the problem of identifying a sequence from its first few terms.

In our study of discrete mathematics, we will often add consecutive terms of a sequence of numbers. Because adding terms from a sequence, as well as other indexed sets of numbers, is such a common occurrence, a special notation has been developed for adding such terms. In this chapter, we will introduce the notation used to express summations. We will develop formulae for certain types of summations that appear throughout the study of discrete mathematics. For instance, we will encounter such summations in the analysis of the number of steps used by an algorithm to sort a list of numbers so that its terms are in increasing order.

The relative sizes of infinite sets can be studied by introducing the notion of the size, or cardinality, of a set. We say that a set is countable when it is finite or has the same size as the set of positive integers. In this chapter we will establish the surprising result that the set of rational numbers is countable, while the set of real numbers is not. We will also show how the concepts we discuss can be used to show that there are functions that cannot be computed using a computer program in any programming language.

Matrices are used in discrete mathematics to represent a variety of discrete structures. We will review the basic material about matrices and matrix arithmetic needed to represent relations and graphs. The matrix arithmetic we study will be used to solve a variety of problems involving these structures.

2.1 Sets

2.1.1 Introduction

In this section, we study the fundamental discrete structure on which all other discrete structures are built, namely, the set. Sets are used to group objects together. Often, but not always, the objects in a set have similar properties. For instance, all the students who are currently enrolled in your school make up a set. Likewise, all the students currently taking a course in discrete mathematics at any school make up a set. In addition, those students enrolled in your school who are taking a course in discrete mathematics form a set that can be obtained by taking the elements common to the first two collections. The language of sets is a means to study such collections in an organized fashion. We now provide a definition of a set. This definition is an intuitive definition, which is not part of a formal theory of sets.

Definition 1

A *set* is an unordered collection of distinct objects, called *elements* or *members* of the set. A set is said to *contain* its elements. We write $a \in A$ to denote that a is an element of the set A. The notation $a \notin A$ denotes that a is not an element of the set A.

It is common for sets to be denoted using uppercase letters. Lowercase letters are usually used to denote elements of sets.

There are several ways to describe a set. One way is to list all the members of a set, when this is possible. We use a notation where all members of the set are listed between braces. For example, the notation $\{a, b, c, d\}$ represents the set with the four elements a, b, c, and d. This way of describing a set is known as the **roster method**.

EXAMPLE 1 The set V of all vowels in the English alphabet can be written as $V = \{a, e, i, o, u\}$. ◀

EXAMPLE 2 The set O of odd positive integers less than 10 can be expressed by $O = \{1, 3, 5, 7, 9\}$. ◀

EXAMPLE 3 Although sets are usually used to group together elements with common properties, there is nothing that prevents a set from having seemingly unrelated elements. For instance, $\{a$, 2, Fred, New Jersey$\}$ is the set containing the four elements a, 2, Fred, and New Jersey. ◀

Sometimes the roster method is used to describe a set without listing all its members. Some members of the set are listed, and then *ellipses* (…) are used when the general pattern of the elements is obvious.

EXAMPLE 4 The set of positive integers less than 100 can be denoted by $\{1, 2, 3, \dots, 99\}$. ◀

Extra Examples

Another way to describe a set is to use **set builder** notation. We characterize all those elements in the set by stating the property or properties they must have to be members. The general form of this notation is $\{x \mid x \text{ has property } P\}$ and is read "the set of all x such that x has property P." For instance, the set O of all odd positive integers less than 10 can be written as

$$O = \{x \mid x \text{ is an odd positive integer less than } 10\},$$

or, specifying the universe as the set of positive integers, as

$$O = \{x \in \mathbf{Z}^+ \mid x \text{ is odd and } x < 10\}.$$

We often use this type of notation to describe sets when it is impossible to list all the elements of the set. For instance, the set $\mathbf{Q}^+$ of all positive rational numbers can be written as

$$\mathbf{Q}^+ = \{x \in \mathbf{R} \mid x = \tfrac{p}{q}, \text{ for some positive integers } p \text{ and } q\}.$$

Beware that mathematicians disagree whether 0 is a natural number. We consider it quite natural.

These sets, each denoted using a boldface letter, play an important role in discrete mathematics:

$\mathbf{N} = \{0, 1, 2, 3, \dots\}$, the set of all **natural numbers**
$\mathbf{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$, the set of all **integers**
$\mathbf{Z}^+ = \{1, 2, 3, \dots\}$, the set of all **positive integers**
$\mathbf{Q} = \{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z}, \text{ and } q \neq 0\}$, the set of all **rational numbers**
$\mathbf{R}$, the set of all **real numbers**

R$^+$, the set of all **positive real numbers**
C, the set of all **complex numbers**.

(Note that some people do not consider 0 a natural number, so be careful to check how the term *natural numbers* is used when you read other books.)

Among the sets studied in calculus and other subjects are **intervals**, sets of all the real numbers between two numbers a and b, with or without a and b. If a and b are real numbers with $a \leq b$, we denote these intervals by

$$[a, b] = \{x \mid a \leq x \leq b\}$$

$$[a, b) = \{x \mid a \leq x < b\}$$

$$(a, b] = \{x \mid a < x \leq b\}$$

$$(a, b) = \{x \mid a < x < b\}.$$

Note that $[a, b]$ is called the **closed interval** from a to b and (a, b) is called the **open interval** from a to b. Each of the intervals $[a, b]$, $[a, b)$, $(a, b]$, and (a, b) contains all the real numbers strictly between a and b. The first two of these contain a and the first and third contain b.

Remark: Some books use the notations $[a, b[$, $]a, b]$, and $]a, b[$ for $[a, b)$, $(a, b]$, and (a, b), respectively.

Sets can have other sets as members, as Example 5 illustrates.

EXAMPLE 5 The set $\{\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}\}$ is a set containing four elements, each of which is a set. The four elements of this set are **N**, the set of natural numbers; **Z**, the set of integers; **Q**, the set of rational numbers; and **R**, the set of real numbers. ◀

Remark: Note that the concept of a datatype, or type, in computer science is built upon the concept of a set. In particular, a **datatype** or **type** is the name of a set, together with a set of operations that can be performed on objects from that set. For example, *boolean* is the name of the set $\{0, 1\}$, together with operators on one or more elements of this set, such as AND, OR, and NOT.

Because many mathematical statements assert that two differently specified collections of objects are really the same set, we need to understand what it means for two sets to be equal.

Definition 2 Two sets are *equal* if and only if they have the same elements. Therefore, if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$. We write $A = B$ if A and B are equal sets.

EXAMPLE 6 The sets $\{1, 3, 5\}$ and $\{3, 5, 1\}$ are equal, because they have the same elements. Note that the order in which the elements of a set are listed does not matter. Note also that it does not matter if an element of a set is listed more than once, so $\{1, 3, 3, 3, 5, 5, 5, 5\}$ is the same as the set $\{1, 3, 5\}$ because they have the same elements. ◀

THE EMPTY SET There is a special set that has no elements. This set is called the **empty set**, or **null set**, and is denoted by ∅. The empty set can also be denoted by $\{\ \}$ (that is, we represent the empty set with a pair of braces that encloses all the elements in this set). Often, a set of elements with certain properties turns out to be the null set. For instance, the set of all positive integers that are greater than their squares is the null set.

{∅} has one more element than ∅.

A set with one element is called a **singleton set**. A common error is to confuse the empty set ∅ with the set {∅}, which is a singleton set. The single element of the set {∅} is the empty set itself! A useful analogy for remembering this difference is to think of folders in a computer file system. The empty set can be thought of as an empty folder and the set consisting of just the empty set can be thought of as a folder with exactly one folder inside, namely, the empty folder.

Links

NAIVE SET THEORY Note that the term *object* has been used in the definition of a set, Definition 1, without specifying what an object is. This description of a set as a collection of objects, based on the intuitive notion of an object, was first stated in 1895 by the German mathematician Georg Cantor. The theory that results from this intuitive definition of a set, and the use of the intuitive notion that for any property whatever, there is a set consisting of exactly the objects with this property, leads to **paradoxes**, or logical inconsistencies. This was shown by the English philosopher Bertrand Russell in 1902 (see Exercise 50 for a description of one of these paradoxes). These logical inconsistencies can be avoided by building set theory beginning with axioms. However, we will use Cantor's original version of set theory, known as **naive set theory**, in this book because all sets considered in this book can be treated consistently using Cantor's original theory. Students will find familiarity with naive set theory helpful if they go on to learn about axiomatic set theory. They will also find the development of axiomatic set theory much more abstract than the material in this text. We refer the interested reader to [Su72] to learn more about axiomatic set theory.

2.1.2 Venn Diagrams

Sets can be represented graphically using Venn diagrams, named after the English mathematician John Venn, who introduced their use in 1881. In Venn diagrams the **universal set** U, which contains all the objects under consideration, is represented by a rectangle. (Note that the universal set varies depending on which objects are of interest.) Inside this rectangle, circles or other geometrical figures are used to represent sets. Sometimes points are used to represent the particular elements of the set. Venn diagrams are often used to indicate the relationships between sets. We show how a Venn diagram can be used in Example 7.

Assessment

EXAMPLE 7 Draw a Venn diagram that represents V, the set of vowels in the English alphabet.

Solution: We draw a rectangle to indicate the universal set U, which is the set of the 26 letters of the English alphabet. Inside this rectangle we draw a circle to represent V. Inside this circle we indicate the elements of V with points (see Figure 1). ◀

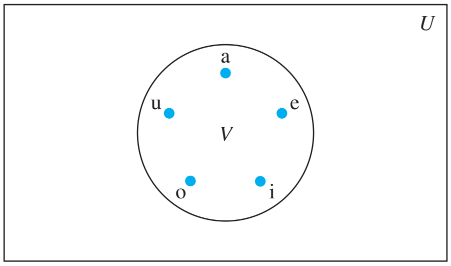


FIGURE 1 **Venn diagram for the set of vowels.**

2.1.3 Subsets

It is common to encounter situations where the elements of one set are also the elements of a second set. We now introduce some terminology and notation to express such relationships between sets.

Definition 3

The set A is a *subset* of B, and B is a *superset* of A, if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B. If, instead, we want to stress that B is a superset of A, we use the equivalent notation $B \supseteq A$. (So, $A \subseteq B$ and $B \supseteq A$ are equivalent statements.)

We see that $A \subseteq B$ if and only if the quantification

$$\forall x(x \in A \to x \in B)$$

is true. Note that to show that A is not a subset of B we need only find one element $x \in A$ with $x \notin B$. Such an x is a counterexample to the claim that $x \in A$ implies $x \in B$.

We have these useful rules for determining whether one set is a subset of another:

Showing that A is a Subset of B To show that $A \subseteq B$, show that if x belongs to A then x also belongs to B.

Showing that A is Not a Subset of B To show that $A \nsubseteq B$, find a single $x \in A$ such that $x \notin B$.

EXAMPLE 8 The set of all odd positive integers less than 10 is a subset of the set of all positive integers less than 10, the set of rational numbers is a subset of the set of real numbers, the set of all computer science majors at your school is a subset of the set of all students at your school, and the set of all people in China is a subset of the set of all people in China (that is, it is a subset of itself). Each of these facts follows immediately by noting that an element that belongs to the first set in each pair of sets also belongs to the second set in that pair. ◀

EXAMPLE 9 The set of integers with squares less than 100 is not a subset of the set of nonnegative integers because -1 is in the former set [as $(-1)^2 < 100$], but not the latter set. The set of people who have taken discrete mathematics at your school is not a subset of the set of all computer science majors at your school if there is at least one student who has taken discrete mathematics who is not a computer science major. ◀

Theorem 1 shows that every nonempty set S is guaranteed to have at least two subsets, the empty set and the set S itself, that is, $\emptyset \subseteq S$ and $S \subseteq S$.

THEOREM 1 For every set S, $(i)\ \emptyset \subseteq S$ and $(ii)\ S \subseteq S$.

Proof: We will prove (i) and leave the proof of (ii) as an exercise.

Let S be a set. To show that $\emptyset \subseteq S$, we must show that $\forall x(x \in \emptyset \to x \in S)$ is true. Because the empty set contains no elements, it follows that $x \in \emptyset$ is always false. It follows that the conditional statement $x \in \emptyset \to x \in S$ is always true, because its hypothesis is always false and a conditional statement with a false hypothesis is true. Therefore, $\forall x(x \in \emptyset \to x \in S)$ is true. This completes the proof of (i). Note that this is an example of a vacuous proof. ◁

When we wish to emphasize that a set A is a subset of a set B but that $A \neq B$, we write $A \subset B$ and say that A is a **proper subset** of B. For $A \subset B$ to be true, it must be the case that $A \subseteq B$ and there must exist an element x of B that is not an element of A. That is, A is a proper subset of B if and only if

$$\forall x(x \in A \to x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

is true. Venn diagrams can be used to illustrate that a set A is a subset of a set B. We draw the universal set U as a rectangle. Within this rectangle we draw a circle for B. Because A is a subset of B, we draw the circle for A within the circle for B. This relationship is shown in Figure 2.

Recall from Definition 2 that sets are equal if they have the same elements. A useful way to show that two sets have the same elements is to show that each set is a subset of the other. In other words, we can show that if A and B are sets with $A \subseteq B$ and $B \subseteq A$, then $A = B$. That is, $A = B$ if and only if $\forall x(x \in A \to x \in B)$ and $\forall x(x \in B \to x \in A)$ or equivalently if and only if $\forall x(x \in A \leftrightarrow x \in B)$, which is what it means for the A and B to be equal. Because this method of showing two sets are equal is so useful, we highlight it here.

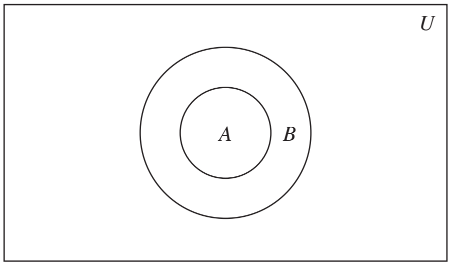


FIGURE 2 **Venn diagram showing that A is a subset of B.**

Showing Two Sets are Equal To show that two sets A and B are equal, show that $A \subseteq B$ and $B \subseteq A$.

Sets may have other sets as members. For instance, we have the sets

$$A = \{\emptyset, \{a\}, \{b\}, \{a, b\}\} \qquad \text{and} \qquad B = \{x \mid x \text{ is a subset of the set } \{a, b\}\}.$$

Note that these two sets are equal, that is, $A = B$. Also note that $\{a\} \in A$, but $a \notin A$.

2.1.4 The Size of a Set

Sets are used extensively in counting problems, and for such applications we need to discuss the sizes of sets.

Definition 4 Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a *finite set* and that n is the *cardinality* of S. The cardinality of S is denoted by $|S|$.

Remark: The term *cardinality* comes from the common usage of the term *cardinal number* as the size of a finite set.

EXAMPLE 10 Let A be the set of odd positive integers less than 10. Then $|A| = 5$. ◀

EXAMPLE 11 Let S be the set of letters in the English alphabet. Then $|S| = 26$. ◀

EXAMPLE 12 Because the null set has no elements, it follows that $|\emptyset| = 0$. ◀

We will also be interested in sets that are not finite.

Definition 5 A set is said to be *infinite* if it is not finite.

EXAMPLE 13 The set of positive integers is infinite. ◀

2.1.5 Power Sets

Many problems involve testing all combinations of elements of a set to see if they satisfy some property. To consider all such combinations of elements of a set S, we build a new set that has as its members all the subsets of S.

Definition 6 Given a set S, the *power set* of S is the set of all subsets of the set S. The power set of S is denoted by $\mathcal{P}(S)$.

EXAMPLE 14 What is the power set of the set $\{0, 1, 2\}$?

Extra Examples

Solution: The power set $\mathcal{P}(\{0, 1, 2\})$ is the set of all subsets of $\{0, 1, 2\}$. Hence,

$$\mathcal{P}(\{0, 1, 2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

Note that the empty set and the set itself are members of this set of subsets. ◀

EXAMPLE 15 What is the power set of the empty set? What is the power set of the set $\{\emptyset\}$?

Solution: The empty set has exactly one subset, namely, itself. Consequently,

$$\mathcal{P}(\emptyset) = \{\emptyset\}.$$

The set $\{\emptyset\}$ has exactly two subsets, namely, $\emptyset$ and the set $\{\emptyset\}$ itself. Therefore,

$$\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}.$$ ◀

If a set has n elements, then its power set has 2^n elements. We will demonstrate this fact in several ways in subsequent sections of the text.

2.1.6 Cartesian Products

The order of elements in a collection is often important. Because sets are unordered, a different structure is needed to represent ordered collections. This is provided by **ordered *n*-tuples**.

Definition 7 The *ordered n-tuple* $(a_1, a_2, \dots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, …, and a_n as its nth element.

We say that two ordered n-tuples are equal if and only if each corresponding pair of their elements is equal. In other words, $(a_1, a_2, \dots, a_n) = (b_1, b_2, \dots, b_n)$ if and only if $a_i = b_i$, for $i = 1, 2, \dots, n$. In particular, ordered 2-tuples are called **ordered pairs**. The ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$. Note that (a, b) and (b, a) are not equal unless $a = b$.

Many of the discrete structures we will study in later chapters are based on the notion of the *Cartesian product* of sets (named after René Descartes). We first define the Cartesian product of two sets.

Definition 8

Let A and B be sets. The *Cartesian product* of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}.$$

EXAMPLE 16

Extra Examples

Let A represent the set of all students at a university, and let B represent the set of all courses offered at the university. What is the Cartesian product $A \times B$ and how can it be used?

Solution: The Cartesian product $A \times B$ consists of all the ordered pairs of the form (a, b), where a is a student at the university and b is a course offered at the university. One way to use the set $A \times B$ is to represent all possible enrollments of students in courses at the university. Furthermore, observe that each subset of $A \times B$ represents one possible total enrollment configuration, and $\mathcal{P}(A \times B)$ represents all possible enrollment configurations. ◀

EXAMPLE 17

What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$?

Solution: The Cartesian product $A \times B$ is

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}.$$ ◀

Note that the Cartesian products $A \times B$ and $B \times A$ are not equal unless $A = \emptyset$ or $B = \emptyset$ (so that $A \times B = \emptyset$) or $A = B$ (see Exercises 33 and 40). This is illustrated in Example 18.

EXAMPLE 18

Show that the Cartesian product $B \times A$ is not equal to the Cartesian product $A \times B$, where A and B are as in Example 17.

Solution: The Cartesian product $B \times A$ is

$$B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}.$$

This is not equal to $A \times B$, which was found in Example 17. ◀

The Cartesian product of more than two sets can also be defined.

Definition 9 The *Cartesian product* of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where a_i belongs to A_i for $i = 1, 2, \ldots, n$. In other words,

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}.$$

EXAMPLE 19 What is the Cartesian product $A \times B \times C$, where $A = \{0, 1\}$, $B = \{1, 2\}$, and $C = \{0, 1, 2\}$?

Solution: The Cartesian product $A \times B \times C$ consists of all ordered triples (a, b, c), where $a \in A$, $b \in B$, and $c \in C$. Hence,

$$\begin{aligned} A \times B \times C = \{&(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2),\\ &(1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2)\}. \end{aligned}$$ ◀

Remark: Note that when A, B, and C are sets, $(A \times B) \times C$ is not the same as $A \times B \times C$ (see Exercise 41).

We use the notation A^2 to denote $A \times A$, the Cartesian product of the set A with itself. Similarly, $A^3 = A \times A \times A$, $A^4 = A \times A \times A \times A$, and so on. More generally,

$$A^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A \text{ for } i = 1, 2, \ldots, n\}.$$

EXAMPLE 20 Suppose that $A = \{1, 2\}$. It follows that $A^2 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$ and $A^3 = \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$. ◀

A subset R of the Cartesian product $A \times B$ is called a **relation** from the set A to the set B. The elements of R are ordered pairs, where the first element belongs to A and the second to B. For example, $R = \{(a, 0), (a, 1), (a, 3), (b, 1), (b, 2), (c, 0), (c, 3)\}$ is a relation from the set $\{a, b, c\}$ to the set $\{0, 1, 2, 3\}$, and it is also a relation from the set $\{a, b, c, d, e\}$ to the set $\{0, 1, 3, 4)$. (This illustrates that a relation need not contain a pair (x, y) for every element x of A.) A relation from a set A to itself is called a relation on A.

EXAMPLE 21 What are the ordered pairs in the less than or equal to relation, which contains (a, b) if $a \leq b$, on the set $\{0, 1, 2, 3\}$?

Solution: The ordered pair (a, b) belongs to R if and only if both a and b belong to $\{0, 1, 2, 3\}$ and $a \leq b$. Consequently, $R = \{(0, 0), (0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$. ◀

We will study relations and their properties at length in Chapter 9.

2.1.7 Using Set Notation with Quantifiers

Sometimes we restrict the domain of a quantified statement explicitly by making use of a particular notation. For example, $\forall x \in S(P(x))$ denotes the universal quantification of $P(x)$ over all elements in the set S. In other words, $\forall x \in S(P(x))$ is shorthand for $\forall x(x \in S \to P(x))$. Similarly, $\exists x \in S(P(x))$ denotes the existential quantification of $P(x)$ over all elements in S. That is, $\exists x \in S(P(x))$ is shorthand for $\exists x(x \in S \wedge P(x))$.

EXAMPLE 22 What do the statements $\forall x \in \mathbf{R}\ (x^2 \geq 0)$ and $\exists x \in \mathbf{Z}\ (x^2 = 1)$ mean?

Solution: The statement $\forall x \in \mathbf{R}(x^2 \geq 0)$ states that for every real number x, $x^2 \geq 0$. This statement can be expressed as "The square of every real number is nonnegative." This is a true statement.

The statement $\exists x \in \mathbf{Z}(x^2 = 1)$ states that there exists an integer x such that $x^2 = 1$. This statement can be expressed as "There is an integer whose square is 1." This is also a true statement because $x = 1$ is such an integer (as is -1). ◀

2.1.8 Truth Sets and Quantifiers

We will now tie together concepts from set theory and from predicate logic. Given a predicate P, and a domain D, we define the **truth set** of P to be the set of elements x in D for which $P(x)$ is true. The truth set of $P(x)$ is denoted by $\{x \in D \mid P(x)\}$.

EXAMPLE 23 What are the truth sets of the predicates $P(x)$, $Q(x)$, and $R(x)$, where the domain is the set of integers and $P(x)$ is "$|x| = 1$," $Q(x)$ is "$x^2 = 2$," and $R(x)$ is "$|x| = x$."

Solution: The truth set of P, $\{x \in \mathbf{Z} \mid |x| = 1\}$, is the set of integers for which $|x| = 1$. Because $|x| = 1$ when $x = 1$ or $x = -1$, and for no other integers x, we see that the truth set of P is the set $\{-1, 1\}$.

The truth set of Q, $\{x \in \mathbf{Z} \mid x^2 = 2\}$, is the set of integers for which $x^2 = 2$. This is the empty set because there are no integers x for which $x^2 = 2$.

The truth set of R, $\{x \in \mathbf{Z} \mid |x| = x\}$, is the set of integers for which $|x| = x$. Because $|x| = x$ if and only if $x \geq 0$, it follows that the truth set of R is $\mathbf{N}$, the set of nonnegative integers. ◀

Note that $\forall x P(x)$ is true over the domain U if and only if the truth set of P is the set U. Likewise, $\exists x P(x)$ is true over the domain U if and only if the truth set of P is nonempty.

Exercises

1. List the members of these sets.
   a) $\{x \mid x \text{ is a real number such that } x^2 = 1\}$
   b) $\{x \mid x \text{ is a positive integer less than } 12\}$
   c) $\{x \mid x \text{ is the square of an integer and } x < 100\}$
   d) $\{x \mid x \text{ is an integer such that } x^2 = 2\}$

2. Use set builder notation to give a description of each of these sets.
   a) $\{0, 3, 6, 9, 12\}$
   b) $\{-3, -2, -1, 0, 1, 2, 3\}$
   c) $\{m, n, o, p\}$

3. Which of the intervals $(0, 5)$, $(0, 5]$, $[0, 5)$, $[0, 5]$, $(1, 4]$, $[2, 3]$, $(2, 3)$ contains
   a) 0? **b)** 1?
   c) 2? **d)** 3?
   e) 4? **f)** 5?

4. For each of these intervals, list all its elements or explain why it is empty.
   a) $[a, a]$ **b)** $[a, a)$
   c) $(a, a]$ **d)** (a, a)
   e) (a, b), where $a > b$ **f)** $[a, b]$, where $a > b$

5. For each of these pairs of sets, determine whether the first is a subset of the second, the second is a subset of the first, or neither is a subset of the other.

a) the set of airline flights from New York to New Delhi, the set of nonstop airline flights from New York to New Delhi
b) the set of people who speak English, the set of people who speak Chinese
c) the set of flying squirrels, the set of living creatures that can fly

6. For each of these pairs of sets, determine whether the first is a subset of the second, the second is a subset of the first, or neither is a subset of the other.

a) the set of people who speak English, the set of people who speak English with an Australian accent
b) the set of fruits, the set of citrus fruits
c) the set of students studying discrete mathematics, the set of students studying data structures

7. Determine whether each of these pairs of sets are equal.

a) $\{1, 3, 3, 3, 5, 5, 5, 5, 5\}$, $\{5, 3, 1\}$
b) $\{\{1\}\}$, $\{1, \{1\}\}$ **c)** $\emptyset$, $\{\emptyset\}$

8. Suppose that $A = \{2, 4, 6\}$, $B = \{2, 6\}$, $C = \{4, 6\}$, and $D = \{4, 6, 8\}$. Determine which of these sets are subsets of which other of these sets.

9. For each of the following sets, determine whether 2 is an element of that set.

a) $\{x \in \mathbf{R} \mid x \text{ is an integer greater than } 1\}$
b) $\{x \in \mathbf{R} \mid x \text{ is the square of an integer}\}$
c) $\{2, \{2\}\}$ **d)** $\{\{2\}, \{\{2\}\}\}$
e) $\{\{2\}, \{2, \{2\}\}\}$ **f)** $\{\{\{2\}\}\}$

10. For each of the sets in Exercise 9, determine whether $\{2\}$ is an element of that set.

11. Determine whether each of these statements is true or false.

a) $0 \in \emptyset$ **b)** $\emptyset \in \{0\}$
c) $\{0\} \subset \emptyset$ **d)** $\emptyset \subset \{0\}$
e) $\{0\} \in \{0\}$ **f)** $\{0\} \subset \{0\}$
g) $\{\emptyset\} \subseteq \{\emptyset\}$

12. Determine whether these statements are true or false.

a) $\emptyset \in \{\emptyset\}$ **b)** $\emptyset \in \{\emptyset, \{\emptyset\}\}$
c) $\{\emptyset\} \in \{\emptyset\}$ **d)** $\{\emptyset\} \in \{\{\emptyset\}\}$
e) $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$ **f)** $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$
g) $\{\{\emptyset\}\} \subset \{\{\emptyset\}, \{\emptyset\}\}$

13. Determine whether each of these statements is true or false.

a) $x \in \{x\}$ **b)** $\{x\} \subseteq \{x\}$ **c)** $\{x\} \in \{x\}$
d) $\{x\} \in \{\{x\}\}$ **e)** $\emptyset \subseteq \{x\}$ **f)** $\emptyset \in \{x\}$

14. Use a Venn diagram to illustrate the subset of odd integers in the set of all positive integers not exceeding 10.

15. Use a Venn diagram to illustrate the set of all months of the year whose names do not contain the letter R in the set of all months of the year.

16. Use a Venn diagram to illustrate the relationship $A \subseteq B$ and $B \subseteq C$.

17. Use a Venn diagram to illustrate the relationships $A \subset B$ and $B \subset C$.

18. Use a Venn diagram to illustrate the relationships $A \subset B$ and $A \subset C$.

19. Suppose that A, B, and C are sets such that $A \subseteq B$ and $B \subseteq C$. Show that $A \subseteq C$.

20. Find two sets A and B such that $A \in B$ and $A \subseteq B$.

21. What is the cardinality of each of these sets?

a) $\{a\}$ **b)** $\{\{a\}\}$
c) $\{a, \{a\}\}$ **d)** $\{a, \{a\}, \{a, \{a\}\}\}$

22. What is the cardinality of each of these sets?

a) $\emptyset$ **b)** $\{\emptyset\}$
c) $\{\emptyset, \{\emptyset\}\}$ **d)** $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

23. Find the power set of each of these sets, where a and b are distinct elements.

a) $\{a\}$ **b)** $\{a, b\}$ **c)** $\{\emptyset, \{\emptyset\}\}$

24. Can you conclude that $A = B$ if A and B are two sets with the same power set?

25. How many elements does each of these sets have where a and b are distinct elements?

a) $\mathcal{P}(\{a, b, \{a, b\}\})$
b) $\mathcal{P}(\{\emptyset, a, \{a\}, \{\{a\}\}\})$
c) $\mathcal{P}(\mathcal{P}(\emptyset))$

26. Determine whether each of these sets is the power set of a set, where a and b are distinct elements.

a) $\emptyset$ **b)** $\{\emptyset, \{a\}\}$
c) $\{\emptyset, \{a\}, \{\emptyset, a\}\}$ **d)** $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

27. Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ if and only if $A \subseteq B$.

28. Show that if $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$

29. Let $A = \{a, b, c, d\}$ and $B = \{y, z\}$. Find

a) $A \times B$. **b)** $B \times A$.

30. What is the Cartesian product $A \times B$, where A is the set of courses offered by the mathematics department at a university and B is the set of mathematics professors at this university? Give an example of how this Cartesian product can be used.

31. What is the Cartesian product $A \times B \times C$, where A is the set of all airlines and B and C are both the set of all cities in the United States? Give an example of how this Cartesian product can be used.

32. Suppose that $A \times B = \emptyset$, where A and B are sets. What can you conclude?

33. Let A be a set. Show that $\emptyset \times A = A \times \emptyset = \emptyset$.

34. Let $A = \{a, b, c\}$, $B = \{x, y\}$, and $C = \{0, 1\}$. Find

a) $A \times B \times C$. **b)** $C \times B \times A$.
c) $C \times A \times B$. **d)** $B \times B \times B$.

35. Find A^2 if

a) $A = \{0, 1, 3\}$. **b)** $A = \{1, 2, a, b\}$.

36. Find A^3 if

a) $A = \{a\}$. **b)** $A = \{0, a\}$.

37. How many different elements does $A \times B$ have if A has m elements and B has n elements?

38. How many different elements does $A \times B \times C$ have if A has m elements, B has n elements, and C has p elements?

39. How many different elements does A^n have when A has m elements and n is a positive integer?

40. Show that $A \times B \neq B \times A$, when A and B are nonempty, unless $A = B$.

41. Explain why $A \times B \times C$ and $(A \times B) \times C$ are not the same.

42. Explain why $(A \times B) \times (C \times D)$ and $A \times (B \times C) \times D$ are not the same.

43. Prove or disprove that if A and B are sets, then $\mathcal{P}(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)$.

44. Prove or disprove that if A, B, and C are nonempty sets and $A \times B = A \times C$, then $B = C$.

45. Translate each of these quantifications into English and determine its truth value.

a) $\forall x \in \mathbf{R}\ (x^2 \neq -1)$ **b)** $\exists x \in \mathbf{Z}\ (x^2 = 2)$
c) $\forall x \in \mathbf{Z}\ (x^2 > 0)$ **d)** $\exists x \in \mathbf{R}\ (x^2 = x)$

46. Translate each of these quantifications into English and determine its truth value.

a) $\exists x \in \mathbf{R}\ (x^3 = -1)$ **b)** $\exists x \in \mathbf{Z}\ (x + 1 > x)$
c) $\forall x \in \mathbf{Z}\ (x - 1 \in \mathbf{Z})$ **d)** $\forall x \in \mathbf{Z}\ (x^2 \in \mathbf{Z})$

47. Find the truth set of each of these predicates where the domain is the set of integers.

a) $P(x)$: $x^2 < 3$ **b)** $Q(x)$: $x^2 > x$
c) $R(x)$: $2x + 1 = 0$

48. Find the truth set of each of these predicates where the domain is the set of integers.

a) $P(x)$: $x^3 \geq 1$ **b)** $Q(x)$: $x^2 = 2$
c) $R(x)$: $x < x^2$

***49.** The defining property of an ordered pair is that two ordered pairs are equal if and only if their first elements are equal and their second elements are equal. Surprisingly, instead of taking the ordered pair as a primitive concept, we can construct ordered pairs using basic notions from set theory. Show that if we define the ordered pair (a, b) to be $\{\{a\}, \{a, b\}\}$, then $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$. [*Hint:* First show that $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$ if and only if $a = c$ and $b = d$.]

***50.** This exercise presents **Russell's paradox**. Let S be the set that contains a set x if the set x does not belong to itself, so that $S = \{x \mid x \notin x\}$.

Links

a) Show the assumption that S is a member of S leads to a contradiction.

b) Show the assumption that S is not a member of S leads to a contradiction.

By parts (a) and (b) it follows that the set S cannot be defined as it was. This paradox can be avoided by restricting the types of elements that sets can have.

***51.** Describe a procedure for listing all the subsets of a finite set.

2.2 Set Operations

2.2.1 Introduction

Links

Two, or more, sets can be combined in many different ways. For instance, starting with the set of mathematics majors at your school and the set of computer science majors at your school, we can form the set of students who are mathematics majors or computer science majors, the set of students who are joint majors in mathematics and computer science, the set of all students not majoring in mathematics, and so on.

Definition 1 Let A and B be sets. The *union* of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.

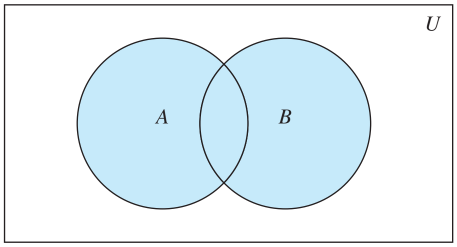
An element x belongs to the union of the sets A and B if and only if x belongs to A or x belongs to B. This tells us that

$$A \cup B = \{x \mid x \in A \vee x \in B\}.$$

The Venn diagram shown in Figure 1 represents the union of two sets A and B. The area that represents $A \cup B$ is the shaded area within either the circle representing A or the circle representing B.

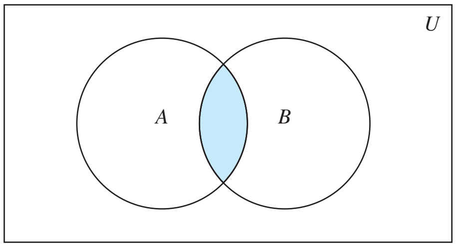
We will give some examples of the union of sets.

EXAMPLE 1 The union of the sets $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{1, 2, 3, 5\}$; that is, $\{1, 3, 5\} \cup \{1, 2, 3\} = \{1, 2, 3, 5\}$. ◀



$A \cup B$ is shaded.

FIGURE 1 Venn diagram of the union of A and B.



$A \cap B$ is shaded.

FIGURE 2 Venn diagram of the intersection of A and B.

EXAMPLE 2 The union of the set of all computer science majors at your school and the set of all mathematics majors at your school is the set of students at your school who are majoring either in mathematics or in computer science (or in both). ◀

Definition 2 Let A and B be sets. The *intersection* of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.

An element x belongs to the intersection of the sets A and B if and only if x belongs to A and x belongs to B. This tells us that

$$A \cap B = \{x \mid x \in A \land x \in B\}.$$

The Venn diagram shown in Figure 2 represents the intersection of two sets A and B. The shaded area that is within both the circles representing the sets A and B is the area that represents the intersection of A and B.

We give some examples of the intersection of sets.

EXAMPLE 3 The intersection of the sets $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{1, 3\}$; that is, $\{1, 3, 5\} \cap \{1, 2, 3\} = \{1, 3\}$. ◀

EXAMPLE 4 The intersection of the set of all computer science majors at your school and the set of all mathematics majors is the set of all students who are joint majors in mathematics and computer science. ◀

Definition 3 Two sets are called *disjoint* if their intersection is the empty set.

EXAMPLE 5 Let $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 4, 6, 8, 10\}$. Because $A \cap B = \emptyset$, A and B are disjoint. ◀

Be careful not to overcount!

We are often interested in finding the cardinality of a union of two finite sets A and B. Note that $|A| + |B|$ counts each element that is in A but not in B or in B but not in A exactly once, and each element that is in both A and B exactly twice. Thus, if the number of elements that are in both A and B is subtracted from $|A| + |B|$, elements in $A \cap B$ will be counted only once. Hence,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

The generalization of this result to unions of an arbitrary number of sets is called the **principle of inclusion–exclusion**. The principle of inclusion–exclusion is an important technique used in enumeration. We will discuss this principle and other counting techniques in detail in Chapters 6 and 8.

There are other important ways to combine sets.

Definition 4 Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set containing those elements that are in A but not in B. The difference of A and B is also called the *complement of B with respect to A*.

Remark: The difference of sets A and B is sometimes denoted by $A \backslash B$.

An element x belongs to the difference of A and B if and only if $x \in A$ and $x \notin B$. This tells us that

$$A - B = \{x \mid x \in A \wedge x \notin B\}.$$

The Venn diagram shown in Figure 3 represents the difference of the sets A and B. The shaded area inside the circle that represents A and outside the circle that represents B is the area that represents $A - B$.

We give some examples of differences of sets.

EXAMPLE 6 The difference of $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{5\}$; that is, $\{1, 3, 5\} - \{1, 2, 3\} = \{5\}$. This is different from the difference of $\{1, 2, 3\}$ and $\{1, 3, 5\}$, which is the set $\{2\}$. ◀

EXAMPLE 7 The difference of the set of computer science majors at your school and the set of mathematics majors at your school is the set of all computer science majors at your school who are not also mathematics majors. ◀

Once the universal set U has been specified, the **complement** of a set can be defined.

Definition 5 Let U be the universal set. The *complement* of the set A, denoted by $\overline{A}$, is the complement of A with respect to U. Therefore, the complement of the set A is $U - A$.

Remark: The definition of the complement of A depends on a particular universal set U. This definition makes sense for any superset U of A. If we want to identify the universal set U, we would write "the complement of A with respect to the set U."

An element belongs to $\overline{A}$ if and only if $x \notin A$. This tells us that

$$\overline{A} = \{x \in U \mid x \notin A\}.$$

In Figure 4 the shaded area outside the circle representing A is the area representing $\overline{A}$.

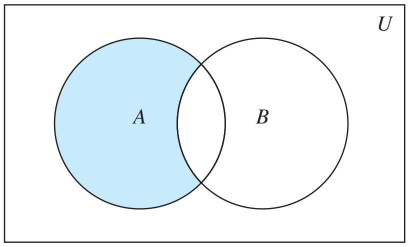
We give some examples of the complement of a set.

EXAMPLE 8 Let $A = \{a, e, i, o, u\}$ (where the universal set is the set of letters of the English alphabet). Then $\overline{A} = \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z\}$. ◀

EXAMPLE 9 Let A be the set of positive integers greater than 10 (with universal set the set of all positive integers). Then $\overline{A} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. ◀

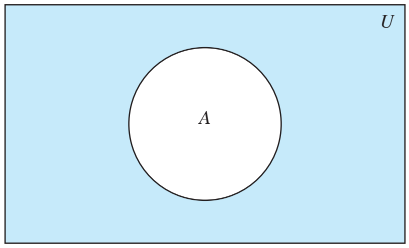
It is left to the reader (Exercise 21) to show that we can express the difference of A and B as the intersection of A and the complement of B. That is,

$$A - B = A \cap \overline{B}.$$



$A - B$ is shaded.

FIGURE 3 Venn diagram for the difference of A and B.



$\overline{A}$ is shaded.

FIGURE 4 Venn diagram for the complement of the set A.

2.2.2 Set Identities

Set identities and propositional equivalences are just special cases of identities for Boolean algebra.

Table 1 lists the most important identities of unions, intersections, and complements of sets. We will prove several of these identities here, using three different methods. These methods are presented to illustrate that there are often many different approaches to the solution of a problem. The proofs of the remaining identities will be left as exercises. The reader should note the similarity between these set identities and the logical equivalences discussed in Section 1.3. (Compare Table 6 of Section 1.6 and Table 1.) In fact, the set identities given can be proved directly from the corresponding logical equivalences. Furthermore, both are special cases of identities that hold for Boolean algebra (discussed in Chapter 12).

TABLE 1 Set Identities.

Identity	***Name***
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Before we discuss different approaches for proving set identities, we briefly discuss the role of Venn diagrams. Although these diagrams can help us understand sets constructed using two or three **atomic sets** (the sets used to construct more complicated combinations of these sets), they provide far less insight when four or more atomic sets are involved. Venn diagrams for four or more sets are quite complex because it is necessary to use ellipses rather than circles to represent the sets. This is necessary to ensure that every possible combination of the sets is represented by a nonempty region. Although Venn diagrams can provide an informal proof for some identities, such proofs should be formalized using one of the three methods we will now describe.

This identity says that the complement of the intersection of two sets is the union of their complements.

One way to show that two sets are equal is to show that each is a subset of the other. Recall that to show that one set is a subset of a second set, we can show that if an element belongs to the first set, then it must also belong to the second set. We generally use a direct proof to do this. We illustrate this type of proof by establishing the first of De Morgan's laws.

EXAMPLE 10 Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Extra Examples

Solution: We will prove that the two sets $\overline{A \cap B}$ and $\overline{A} \cup \overline{B}$ are equal by showing that each set is a subset of the other.

First, we will show that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$. We do this by showing that if x is in $\overline{A \cap B}$, then it must also be in $\overline{A} \cup \overline{B}$. Now suppose that $x \in \overline{A \cap B}$. By the definition of complement, $x \notin A \cap B$. Using the definition of intersection, we see that the proposition $\neg((x \in A) \wedge (x \in B))$ is true.

By applying De Morgan's law for propositions, we see that $\neg(x \in A)$ or $\neg(x \in B)$. Using the definition of negation of propositions, we have $x \notin A$ or $x \notin B$. Using the definition of the complement of a set, we see that this implies that $x \in \overline{A}$ or $x \in \overline{B}$. Consequently, by the definition of union, we see that $x \in \overline{A} \cup \overline{B}$. We have now shown that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$.

Next, we will show that $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$. We do this by showing that if x is in $\overline{A} \cup \overline{B}$, then it must also be in $\overline{A \cap B}$. Now suppose that $x \in \overline{A} \cup \overline{B}$. By the definition of union, we know that $x \in \overline{A}$ or $x \in \overline{B}$. Using the definition of complement, we see that $x \notin A$ or $x \notin B$. Consequently, the proposition $\neg(x \in A) \vee \neg(x \in B)$ is true.

By De Morgan's law for propositions, we conclude that $\neg((x \in A) \wedge (x \in B))$ is true. By the definition of intersection, it follows that $\neg(x \in A \cap B)$. We now use the definition of complement to conclude that $x \in \overline{A \cap B}$. This shows that $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$.

Because we have shown that each set is a subset of the other, the two sets are equal, and the identity is proved. ◀

We can more succinctly express the reasoning used in Example 10 using set builder notation, as Example 11 illustrates.

EXAMPLE 11 Use set builder notation and logical equivalences to establish the first De Morgan law $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Solution: We can prove this identity with the following steps.

$$
\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} && \text{by definition of complement}\\
&= \{x \mid \neg(x \in (A \cap B))\} && \text{by definition of does not belong symbol}\\
&= \{x \mid \neg(x \in A \wedge x \in B)\} && \text{by definition of intersection}\\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} && \text{by the first De Morgan law for logical equivalences}\\
&= \{x \mid x \notin A \vee x \notin B\} && \text{by definition of does not belong symbol}\\
&= \{x \mid x \in \overline{A} \vee x \in \overline{B}\} && \text{by definition of complement}\\
&= \{x \mid x \in \overline{A} \cup \overline{B}\} && \text{by definition of union}\\
&= \overline{A} \cup \overline{B} && \text{by meaning of set builder notation}
\end{aligned}
$$

Note that besides the definitions of complement, union, set membership, and set builder notation, this proof uses the second De Morgan law for logical equivalences. ◀

Proving a set identity involving more than two sets by showing each side of the identity is a subset of the other often requires that we keep track of different cases, as illustrated by the proof in Example 12 of one of the distributive laws for sets.

EXAMPLE 12 Prove the second distributive law from Table 1, which states that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ for all sets A, B, and C.

Solution: We will prove this identity by showing that each side is a subset of the other side.

Suppose that $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. By the definition of union, it follows that $x \in A$, and $x \in B$ or $x \in C$ (or both). In other words, we know that the compound proposition $(x \in A) \wedge ((x \in B) \vee (x \in C))$ is true. By the distributive law for conjunction over disjunction, it follows that $((x \in A) \wedge (x \in B)) \vee ((x \in A) \wedge (x \in C))$. We conclude that either $x \in A$ and $x \in B$, or $x \in A$ and $x \in C$. By the definition of intersection, it follows that $x \in A \cap B$ or $x \in A \cap C$. Using the definition of union, we conclude that $x \in (A \cap B) \cup (A \cap C)$. We conclude that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Now suppose that $x \in (A \cap B) \cup (A \cap C)$. Then, by the definition of union, $x \in A \cap B$ or $x \in A \cap C$. By the definition of intersection, it follows that $x \in A$ and $x \in B$ or that $x \in A$ and $x \in C$. From this we see that $x \in A$, and $x \in B$ or $x \in C$. Consequently, by the definition of union we see that $x \in A$ and $x \in B \cup C$. Furthermore, by the definition of intersection, it follows that $x \in A \cap (B \cup C)$. We conclude that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. This completes the proof of the identity. ◀

Set identities can also be proved using **membership tables**. We consider each combination of the atomic sets (that is, the original sets used to produce the sets on each side) that an element can belong to and verify that elements in the same combinations of sets belong to both the sets in the identity. To indicate that an element is in a set, a 1 is used; to indicate that an element is not in a set, a 0 is used. (The reader should note the similarity between membership tables and truth tables.)

EXAMPLE 13 Use a membership table to show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution: The membership table for these combinations of sets is shown in Table 2. This table has eight rows. Because the columns for $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$ are the same, the identity is valid. ◀

TABLE 2 A Membership Table for the Distributive Property.

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

Once we have proved set identities, we can use them to prove new identities. In particular, we can apply a string of identities, one in each step, to take us from one side of a desired identity to the other. It is helpful to explicitly state the identity that is used in each step, as we do in Example 14.

EXAMPLE 14 Let A, B, and C be sets. Show that

$$\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}.$$

Solution: We have

$$\begin{aligned}
\overline{A \cup (B \cap C)} &= \overline{A} \cap (\overline{B \cap C}) && \text{by the first De Morgan law} \\
&= \overline{A} \cap (\overline{B} \cup \overline{C}) && \text{by the second De Morgan law} \\
&= (\overline{B} \cup \overline{C}) \cap \overline{A} && \text{by the commutative law for intersections} \\
&= (\overline{C} \cup \overline{B}) \cap \overline{A} && \text{by the commutative law for unions.}
\end{aligned}$$

◀

We summarize the three different ways for proving set identities in Table 3.

TABLE 3 Methods of Proving Set Identities.

Description	***Method***
Subset method	Show that each side of the identity is a subset of the other side.
Membership table	For each possible combination of the atomic sets, show that an element in exactly these atomic sets must either belong to both sides or belong to neither side
Apply existing identities	Start with one side, transform it into the other side using a sequence of steps by applying an established identity.

2.2.3 Generalized Unions and Intersections

Because unions and intersections of sets satisfy associative laws, the sets $A \cup B \cup C$ and $A \cap B \cap C$ are well defined; that is, the meaning of this notation is unambiguous when A, B, and C are sets. That is, we do not have to use parentheses to indicate which operation comes first because $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$. Note that $A \cup B \cup C$ contains those elements that are in at least one of the sets A, B, and C, and that $A \cap B \cap C$ contains those elements that are in all of A, B, and C. These combinations of the three sets, A, B, and C, are shown in Figure 5.

EXAMPLE 15 Let $A = \{0, 2, 4, 6, 8\}$, $B = \{0, 1, 2, 3, 4\}$, and $C = \{0, 3, 6, 9\}$. What are $A \cup B \cup C$ and $A \cap B \cap C$?

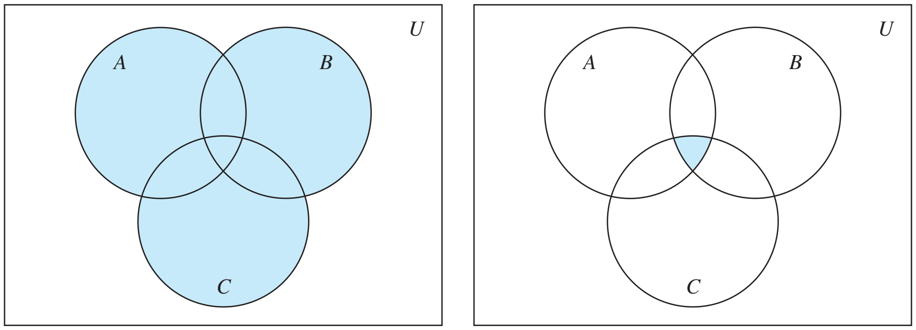
Solution: The set $A \cup B \cup C$ contains those elements in at least one of A, B, and C. Hence,

$$A \cup B \cup C = \{0, 1, 2, 3, 4, 6, 8, 9\}.$$

The set $A \cap B \cap C$ contains those elements in all three of A, B, and C. Thus,

$$A \cap B \cap C = \{0\}.$$

◀



(a) $A \cup B \cup C$ is shaded. (b) $A \cap B \cap C$ is shaded.

FIGURE 5 The union and intersection of A, B, and C.

We can also consider unions and intersections of an arbitrary number of sets. We introduce these definitions.

Definition 6 The *union* of a collection of sets is the set that contains those elements that are members of at least one set in the collection.

We use the notation

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$$

to denote the union of the sets $A_1, A_2, \ldots, A_n$.

Definition 7 The *intersection* of a collection of sets is the set that contains those elements that are members of all the sets in the collection.

We use the notation

$$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$$

to denote the intersection of the sets $A_1, A_2, \ldots, A_n$. We illustrate generalized unions and intersections with Example 16.

EXAMPLE 16 For $i = 1, 2, \ldots$, let $A_i = \{i, i+1, i+2, \ldots\}$. Then,

Extra Examples

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{1, 2, 3, \ldots\},$$

and

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{n, n+1, n+2, \ldots\} = A_n.$$

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We can extend the notation we have introduced for unions and intersections to other families of sets. In particular, to denote the union of the infinite family of sets $A_1, A_2, \ldots, A_n, \ldots$, we use the notation

$$A_1 \cup A_2 \cup \cdots \cup A_n \cup \cdots = \bigcup_{i=1}^{\infty} A_i.$$

Similarly, the intersection of these sets is denoted by

$$A_1 \cap A_2 \cap \cdots \cap A_n \cap \cdots = \bigcap_{i=1}^{\infty} A_i.$$

More generally, when I is a set, the notations $\bigcap_{i \in I} A_i$ and $\bigcup_{i \in I} A_i$ are used to denote the intersection and union of the sets A_i for $i \in I$, respectively. Note that we have $\bigcap_{i \in I} A_i = \{x \mid \forall i \in I\,(x \in A_i)\}$ and $\bigcup_{i \in I} A_i = \{x \mid \exists i \in I\,(x \in A_i)\}$.

EXAMPLE 17 Suppose that $A_i = \{1, 2, 3, \ldots, i\}$ for $i = 1, 2, 3, \ldots$. Then,

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \{1, 2, 3, \ldots, i\} = \{1, 2, 3, \ldots\} = \mathbf{Z}^+$$

and

$$\bigcap_{i=1}^{\infty} A_i = \bigcap_{i=1}^{\infty} \{1, 2, 3, \ldots, i\} = \{1\}.$$

To see that the union of these sets is the set of positive integers, note that every positive integer n is in at least one of the sets, because it belongs to $A_n = \{1, 2, \ldots, n\}$, and every element of the sets in the union is a positive integer. To see that the intersection of these sets is the set $\{1\}$, note that the only element that belongs to all the sets $A_1, A_2, \ldots$ is 1. To see this note that $A_1 = \{1\}$ and $1 \in A_i$ for $i = 1, 2, \ldots$. ◀

2.2.4 Computer Representation of Sets

There are various ways to represent sets using a computer. One method is to store the elements of the set in an unordered fashion. However, if this is done, the operations of computing the union, intersection, or difference of two sets would be time consuming, because each of these operations would require a large amount of searching for elements. We will present a method for storing elements using an arbitrary ordering of the elements of the universal set. This method of representing sets makes computing combinations of sets easy.

Assume that the universal set U is finite (and of reasonable size so that the number of elements of U is not larger than the memory size of the computer being used). First, specify an arbitrary ordering of the elements of U, for instance $a_1, a_2, \ldots, a_n$. Represent a subset A of U with the bit string of length n, where the ith bit in this string is 1 if a_i belongs to A and is 0 if a_i does not belong to A. Example 18 illustrates this technique.

EXAMPLE 18 Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and the ordering of elements of U has the elements in increasing order; that is, $a_i = i$. What bit strings represent the subset of all odd integers in U, the subset of all even integers in U, and the subset of integers not exceeding 5 in U?

Solution: The bit string that represents the set of odd integers in U, namely, $\{1, 3, 5, 7, 9\}$, has a one bit in the first, third, fifth, seventh, and ninth positions, and a zero elsewhere. It is

10 1010 1010.

(We have split this bit string of length ten into blocks of length four for easy reading.) Similarly, we represent the subset of all even integers in U, namely, $\{2, 4, 6, 8, 10\}$, by the string

01 0101 0101.

The set of all integers in U that do not exceed 5, namely, $\{1, 2, 3, 4, 5\}$, is represented by the string

11 1110 0000. ◀

Using bit strings to represent sets, it is easy to find complements of sets and unions, intersections, and differences of sets. To find the bit string for the complement of a set from the bit string for that set, we simply change each 1 to a 0 and each 0 to 1, because $x \in A$ if and only if $x \notin \overline{A}$. Note that this operation corresponds to taking the negation of each bit when we associate a bit with a truth value—with 1 representing true and 0 representing false.

EXAMPLE 19 We have seen that the bit string for the set $\{1, 3, 5, 7, 9\}$ (with universal set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$) is

10 1010 1010.

What is the bit string for the complement of this set?

Solution: The bit string for the complement of this set is obtained by replacing 0s with 1s and vice versa. This yields the string

01 0101 0101,

which corresponds to the set $\{2, 4, 6, 8, 10\}$. ◀

To obtain the bit string for the union and intersection of two sets we perform bitwise Boolean operations on the bit strings representing the two sets. The bit in the ith position of the bit string of the union is 1 if either of the bits in the ith position in the two strings is 1 (or both are 1), and is 0 when both bits are 0. Hence, the bit string for the union is the bitwise *OR* of the bit strings for the two sets. The bit in the ith position of the bit string of the intersection is 1 when the bits in the corresponding position in the two strings are both 1, and is 0 when either of the two bits is 0 (or both are). Hence, the bit string for the intersection is the bitwise *AND* of the bit strings for the two sets.

EXAMPLE 20 The bit strings for the sets $\{1, 2, 3, 4, 5\}$ and $\{1, 3, 5, 7, 9\}$ are 11 1110 0000 and 10 1010 1010, respectively. Use bit strings to find the union and intersection of these sets.

Solution: The bit string for the union of these sets is

$$11\ 1110\ 0000 \vee 10\ 1010\ 1010 = 11\ 1110\ 1010,$$

which corresponds to the set $\{1, 2, 3, 4, 5, 7, 9\}$. The bit string for the intersection of these sets is

$$11\ 1110\ 0000 \wedge 10\ 1010\ 1010 = 10\ 1010\ 0000,$$

which corresponds to the set $\{1, 3, 5\}$. ◀

2.2.5 Multisets

Sometimes the number of times that an element occurs in an unordered collection matters. A **multiset** (short for multiple-membership set) is an unordered collection of elements where an element can occur as a member more than once. We can use the same notation for a multiset as we do for a set, but each element is listed the number of times it occurs. (Recall that in a set, an element either belongs to a set or it does not. Listing it more than once does not affect the membership of this element in the set.) So, the multiset denoted by $\{a, a, a, b, b\}$ is the multiset that contains the element a thrice and the element b twice. When we use this notation, it must be clear that we are working with multisets and not ordinary sets. We can avoid this ambiguity by using an alternate notation for multisets. The notation $\{m_1 \cdot a_1, m_2 \cdot a_2, \ldots, m_r \cdot a_r\}$ denotes the multiset with element a_1 occurring m_1 times, element a_2 occurring m_2 times, and so on. The numbers m_i, $i = 1, 2, \ldots, r$, are called the **multiplicities** of the elements a_i, $i = 1, 2, \ldots, r$. (Elements not in a multiset are assigned 0 as their multiplicity in this set.) The cardinality of a multiset is defined to be the sum of the multiplicities of its elements. The word *multiset* was introduced by Nicolaas Govert de Bruijn in the 1970s, but the concept dates back to the 12th century work of the Indian mathematician Bhaskaracharya.

Let P and Q be multisets. The **union** of the multisets P and Q is the multiset in which the multiplicity of an element is the maximum of its multiplicities in P and Q. The **intersection** of P and Q is the multiset in which the multiplicity of an element is the minimum of its multiplicities in P and Q. The **difference** of P and Q is the multiset in which the multiplicity of an element is the multiplicity of the element in P less its multiplicity in Q unless this difference is negative, in which case the multiplicity is 0. The **sum** of P and Q is the multiset in which the multiplicity of an element is the sum of multiplicities in P and Q. The union, intersection, and difference of P and Q are denoted by $P \cup Q$, $P \cap Q$, and $P - Q$, respectively (where these operations should not be confused with the analogous operations for sets). The sum of P and Q is denoted by $P + Q$.

EXAMPLE 21 Suppose that P and Q are the multisets $\{4 \cdot a, 1 \cdot b, 3 \cdot c\}$ and $\{3 \cdot a, 4 \cdot b, 2 \cdot d\}$, respectively. Find $P \cup Q$, $P \cap Q$, $P - Q$, and $P + Q$.

Solution: We have

$$\begin{aligned} P \cup Q &= \{\max(4, 3) \cdot a, \max(1, 4) \cdot b, \max(3, 0) \cdot c, \max(0, 2) \cdot d\} \\ &= \{4 \cdot a, 4 \cdot b, 3 \cdot c, 2 \cdot d\}, \end{aligned}$$

$$\begin{aligned} P \cap Q &= \{\min(4, 3) \cdot a, \min(1, 4) \cdot b, \min(3, 0) \cdot c, \min(0, 2) \cdot d\} \\ &= \{3 \cdot a, 1 \cdot b, 0 \cdot c, 0 \cdot d\} = \{3 \cdot a, 1 \cdot b\}, \end{aligned}$$

$$P - Q = \{\max(4-3, 0) \cdot a, \max(1-4, 0) \cdot b, \max(3-0, 0) \cdot c, \max(0-2, 0) \cdot d\}$$
$$= \{1 \cdot a, 0 \cdot b, 3 \cdot c, 0 \cdot d\} = \{1 \cdot a, 3 \cdot c\}, \text{ and}$$

$$P + Q = \{(4+3) \cdot a, (1+4) \cdot b, (3+0) \cdot c, (0+2) \cdot d\}$$
$$= \{7 \cdot a, 5 \cdot b, 3 \cdot c, 2 \cdot d\}.$$

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Exercises

1. Let A be the set of students who live within one mile of school and let B be the set of students who walk to classes. Describe the students in each of these sets.

a) $A \cap B$ **b)** $A \cup B$
c) $A - B$ **d)** $B - A$

2. Suppose that A is the set of sophomores at your school and B is the set of students in discrete mathematics at your school. Express each of these sets in terms of A and B.

a) the set of sophomores taking discrete mathematics in your school
b) the set of sophomores at your school who are not taking discrete mathematics
c) the set of students at your school who either are sophomores or are taking discrete mathematics
d) the set of students at your school who either are not sophomores or are not taking discrete mathematics

3. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$. Find

a) $A \cup B$. **b)** $A \cap B$.
c) $A - B$. **d)** $B - A$.

4. Let $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d, e, f, g, h\}$. Find

a) $A \cup B$. **b)** $A \cap B$.
c) $A - B$. **d)** $B - A$.

In Exercises 5–10 assume that A is a subset of some underlying universal set U.

5. Prove the complementation law in Table 1 by showing that $\overline{\overline{A}} = A$.

6. Prove the identity laws in Table 1 by showing that

a) $A \cup \emptyset = A$. **b)** $A \cap U = A$.

7. Prove the domination laws in Table 1 by showing that

a) $A \cup U = U$. **b)** $A \cap \emptyset = \emptyset$.

8. Prove the idempotent laws in Table 1 by showing that

a) $A \cup A = A$. **b)** $A \cap A = A$.

9. Prove the complement laws in Table 1 by showing that

a) $A \cup \overline{A} = U$. **b)** $A \cap \overline{A} = \emptyset$.

10. Show that

a) $A - \emptyset = A$. **b)** $\emptyset - A = \emptyset$.

11. Let A and B be sets. Prove the commutative laws from Table 1 by showing that

a) $A \cup B = B \cup A$.
b) $A \cap B = B \cap A$.

12. Prove the first absorption law from Table 1 by showing that if A and B are sets, then $A \cup (A \cap B) = A$.

13. Prove the second absorption law from Table 1 by showing that if A and B are sets, then $A \cap (A \cup B) = A$.

14. Find the sets A and B if $A - B = \{1, 5, 7, 8\}$, $B - A = \{2, 10\}$, and $A \cap B = \{3, 6, 9\}$.

15. Prove the second De Morgan law in Table 1 by showing that if A and B are sets, then $\overline{A \cup B} = \overline{A} \cap \overline{B}$

a) by showing each side is a subset of the other side.
b) using a membership table.

16. Let A and B be sets. Show that

a) $(A \cap B) \subseteq A$. **b)** $A \subseteq (A \cup B)$.
c) $A - B \subseteq A$. **d)** $A \cap (B - A) = \emptyset$.
e) $A \cup (B - A) = A \cup B$.

17. Show that if A and B are sets in a universe U then $A \subseteq B$ if and only if $\overline{A} \cup B = U$.

18. Given sets A and B in a universe U, draw the Venn diagrams of each of these sets.

a) $A \to B = \{x \in U \mid x \in A \to x \in B\}$
b) $A \leftrightarrow B = \{x \in U \mid x \in A \leftrightarrow x \in B\}$

19. Show that if A, B, and C are sets, then $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$

a) by showing each side is a subset of the other side.
b) using a membership table.

20. Let A, B, and C be sets. Show that

a) $(A \cup B) \subseteq (A \cup B \cup C)$.
b) $(A \cap B \cap C) \subseteq (A \cap B)$.
c) $(A - B) - C \subseteq A - C$.
d) $(A - C) \cap (C - B) = \emptyset$.
e) $(B - A) \cup (C - A) = (B \cup C) - A$.

21. Show that if A and B are sets, then

a) $A - B = A \cap \overline{B}$.
b) $(A \cap B) \cup (A \cap \overline{B}) = A$.

22. Show that if A and B are sets with $A \subseteq B$, then

a) $A \cup B = B$.
b) $A \cap B = A$.

23. Prove the first associative law from Table 1 by showing that if A, B, and C are sets, then $A \cup (B \cup C) = (A \cup B) \cup C$.

24. Prove the second associative law from Table 1 by showing that if A, B, and C are sets, then $A \cap (B \cap C) = (A \cap B) \cap C$.

25. Prove the first distributive law from Table 1 by showing that if A, B, and C are sets, then $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

26. Let A, B, and C be sets. Show that $(A - B) - C = (A - C) - (B - C)$.

27. Let $A = \{0, 2, 4, 6, 8, 10\}$, $B = \{0, 1, 2, 3, 4, 5, 6\}$, and $C = \{4, 5, 6, 7, 8, 9, 10\}$. Find

a) $A \cap B \cap C$. **b)** $A \cup B \cup C$.
c) $(A \cup B) \cap C$. **d)** $(A \cap B) \cup C$.

28. Draw the Venn diagrams for each of these combinations of the sets A, B, and C.

a) $A \cap (B \cup C)$ **b)** $\overline{A} \cap \overline{B} \cap \overline{C}$
c) $(A - B) \cup (A - C) \cup (B - C)$

29. Draw the Venn diagrams for each of these combinations of the sets A, B, and C.

a) $A \cap (B - C)$ **b)** $(A \cap B) \cup (A \cap C)$
c) $(A \cap \overline{B}) \cup (A \cap \overline{C})$

30. Draw the Venn diagrams for each of these combinations of the sets A, B, C, and D.

a) $(A \cap B) \cup (C \cap D)$ **b)** $\overline{A} \cup \overline{B} \cup \overline{C} \cup \overline{D}$
c) $A - (B \cap C \cap D)$

31. What can you say about the sets A and B if we know that

a) $A \cup B = A$? **b)** $A \cap B = A$?
c) $A - B = A$? **d)** $A \cap B = B \cap A$?
e) $A - B = B - A$?

32. Can you conclude that $A = B$ if A, B, and C are sets such that

a) $A \cup C = B \cup C$? **b)** $A \cap C = B \cap C$?
c) $A \cup C = B \cup C$ and $A \cap C = B \cap C$?

33. Let A and B be subsets of a universal set U. Show that $A \subseteq B$ if and only if $\overline{B} \subseteq \overline{A}$.

34. Let A, B, and C be sets. Use the the identity $A - B = A \cap \overline{B}$, which holds for any sets A and B, and the identities from Table 1 to show that $(A - B) \cap (B - C) \cap (A - C) = \emptyset$.

35. Let A, B, and C be sets. Use the identities in Table 1 to show that $\overline{(A \cup B)} \cap \overline{(B \cup C)} \cap \overline{(A \cup C)} = \overline{A} \cap \overline{B} \cap \overline{C}$.

36. Prove or disprove that for all sets A, B, and C, we have

a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
b) $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

37. Prove or disprove that for all sets A, B, and C, we have

a) $A \times (B - C) = (A \times B) - (A \times C)$.
b) $\overline{A} \times \overline{(B \cup C)} = \overline{A \times (B \cup C)}$.

The **symmetric difference** of A and B, denoted by $A \oplus B$, is the set containing those elements in either A or B, but not in both A and B.

38. Find the symmetric difference of $\{1, 3, 5\}$ and $\{1, 2, 3\}$.

39. Find the symmetric difference of the set of computer science majors at a school and the set of mathematics majors at this school.

40. Draw a Venn diagram for the symmetric difference of the sets A and B.

41. Show that $A \oplus B = (A \cup B) - (A \cap B)$.

42. Show that $A \oplus B = (A - B) \cup (B - A)$.

43. Show that if A is a subset of a universal set U, then

a) $A \oplus A = \emptyset$. **b)** $A \oplus \emptyset = A$.
c) $A \oplus U = \overline{A}$. **d)** $A \oplus \overline{A} = U$.

44. Show that if A and B are sets, then

a) $A \oplus B = B \oplus A$. **b)** $(A \oplus B) \oplus B = A$.

45. What can you say about the sets A and B if $A \oplus B = A$?

***46.** Determine whether the symmetric difference is associative; that is, if A, B, and C are sets, does it follow that $A \oplus (B \oplus C) = (A \oplus B) \oplus C$?

***47.** Suppose that A, B, and C are sets such that $A \oplus C = B \oplus C$. Must it be the case that $A = B$?

48. If A, B, C, and D are sets, does it follow that $(A \oplus B) \oplus (C \oplus D) = (A \oplus C) \oplus (B \oplus D)$?

49. If A, B, C, and D are sets, does it follow that $(A \oplus B) \oplus (C \oplus D) = (A \oplus D) \oplus (B \oplus C)$?

50. Show that if A and B are finite sets, then $A \cup B$ is a finite set.

51. Show that if A is an infinite set, then whenever B is a set, $A \cup B$ is also an infinite set.

***52.** Show that if A, B, and C are sets, then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

(This is a special case of the inclusion–exclusion principle, which will be studied in Chapter 8.)

53. Let $A_i = \{1, 2, 3, \ldots, i\}$ for $i = 1, 2, 3, \ldots$. Find

a) $\bigcup_{i=1}^{n} A_i$. **b)** $\bigcap_{i=1}^{n} A_i$.

54. Let $A_i = \{\ldots, -2, -1, 0, 1, \ldots, i\}$. Find

a) $\bigcup_{i=1}^{n} A_i$. **b)** $\bigcap_{i=1}^{n} A_i$.

55. Let A_i be the set of all nonempty bit strings (that is, bit strings of length at least one) of length not exceeding i. Find

a) $\bigcup_{i=1}^{n} A_i$. **b)** $\bigcap_{i=1}^{n} A_i$.

56. Find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$ if for every positive integer i,

a) $A_i = \{i, i+1, i+2, \ldots\}$.
b) $A_i = \{0, i\}$.
c) $A_i = (0, i)$, that is, the set of real numbers x with $0 < x < i$.
d) $A_i = (i, \infty)$, that is, the set of real numbers x with $x > i$.

57. Find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$ if for every positive integer i,

a) $A_i = \{-i, -i+1, \ldots, -1, 0, 1, \ldots, i-1, i\}$.
b) $A_i = \{-i, i\}$.
c) $A_i = [-i, i]$, that is, the set of real numbers x with $-i \leq x \leq i$.
d) $A_i = [i, \infty)$, that is, the set of real numbers x with $x \geq i$.

58. Suppose that the universal set is $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Express each of these sets with bit strings where the ith bit in the string is 1 if i is in the set and 0 otherwise.

a) $\{3, 4, 5\}$
b) $\{1, 3, 6, 10\}$
c) $\{2, 3, 4, 7, 8, 9\}$

59. Using the same universal set as in the last exercise, find the set specified by each of these bit strings.

a) 11 1100 1111
b) 01 0111 1000
c) 10 0000 0001

60. What subsets of a finite universal set do these bit strings represent?

a) the string with all zeros
b) the string with all ones

61. What is the bit string corresponding to the difference of two sets?

62. What is the bit string corresponding to the symmetric difference of two sets?

63. Show how bitwise operations on bit strings can be used to find these combinations of $A = \{a, b, c, d, e\}$, $B = \{b, c, d, g, p, t, v\}$, $C = \{c, e, i, o, u, x, y, z\}$, and $D = \{d, e, h, i, n, o, t, u, x, y\}$.

a) $A \cup B$ **b)** $A \cap B$
c) $(A \cup D) \cap (B \cup C)$ **d)** $A \cup B \cup C \cup D$

64. How can the union and intersection of n sets that all are subsets of the universal set U be found using bit strings?

The **successor** of the set A is the set $A \cup \{A\}$.

65. Find the successors of the following sets.

a) $\{1, 2, 3\}$ **b)** $\emptyset$
c) $\{\emptyset\}$ **d)** $\{\emptyset, \{\emptyset\}\}$

66. How many elements does the successor of a set with n elements have?

67. Let A and B be the multisets $\{3 \cdot a, 2 \cdot b, 1 \cdot c\}$ and $\{2 \cdot a, 3 \cdot b, 4 \cdot d\}$, respectively. Find

a) $A \cup B$. **b)** $A \cap B$. **c)** $A - B$.
d) $B - A$. **e)** $A + B$.

68. Assume that $a \in A$, where A is a set. Which of these statements are true and which are false, where all sets shown are ordinary sets, and not multisets. Explain each answer.

a) $\{a, a\} \cup \{a, a, a\} = \{a, a, a, a, a\}$
b) $\{a, a\} \cup \{a, a, a\} = \{a\}$
c) $\{a, a\} \cap \{a, a, a\} = \{a, a\}$
d) $\{a, a\} \cap \{a, a, a\} = \{a\}$
e) $\{a, a, a\} - \{a, a\} = \{a\}$

69. Answer the same questions as posed in Exercise 68 where all sets are multisets, and not ordinary sets.

70. Suppose that A is the multiset that has as its elements the types of computer equipment needed by one department of a university and the multiplicities are the number of pieces of each type needed, and B is the analogous multiset for a second department of the university. For instance, A could be the multiset $\{107 \cdot \text{personal computers}, 44 \cdot \text{routers}, 6 \cdot \text{servers}\}$ and B could be the multiset $\{14 \cdot \text{personal computers}, 6 \cdot \text{routers}, 2 \cdot \text{mainframes}\}$.

a) What combination of A and B represents the equipment the university should buy assuming both departments use the same equipment?
b) What combination of A and B represents the equipment that will be used by both departments if both departments use the same equipment?
c) What combination of A and B represents the equipment that the second department uses, but the first department does not, if both departments use the same equipment?
d) What combination of A and B represents the equipment that the university should purchase if the departments do not share equipment?

The **Jaccard similarity** $J(A, B)$ of the finite sets A and B is $J(A, B) = |A \cap B|/|A \cup B|$, with $J(\emptyset, \emptyset) = 1$. The **Jaccard distance** $d_J(A, B)$ between A and B equals $d_J(A, B) = 1 - J(A, B)$.

71. Find $J(A, B)$ and $d_J(A, B)$ for these pairs of sets.

a) $A = \{1, 3, 5\}$, $B = \{2, 4, 6\}$
b) $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$
c) $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{1, 2, 3, 4, 5, 6\}$
d) $A = \{1\}$, $B = \{1, 2, 3, 4, 5, 6\}$

72. Prove that each of the properties in parts (a)–(d) holds whenever A and B are finite sets.

a) $J(A, A) = 1$ and $d_J(A, A) = 0$
b) $J(A, B) = J(B, A)$ and $d_J(A, B) = d_J(B, A)$
c) $J(A, B) = 1$ and $d_J(A, B) = 0$ if and only if $A = B$
d) $0 \leq J(A, B) \leq 1$ and $0 \leq d_J(A, B) \leq 1$
****e)** Show that if A, B, and C are sets, then $d_J(A, C) \leq d_J(A, B) + d_J(B, C)$. (This inequality is known as the **triangle inequality** and together with parts (a), (b), and (c) implies that d_J is a **metric**.)

Fuzzy sets are used in artificial intelligence. Each element in the universal set U has a **degree of membership**, which is a real number between 0 and 1 (including 0 and 1), in a fuzzy set S. The fuzzy set S is denoted by listing the elements Links with their degrees of membership (elements with 0 degree of membership are not listed). For instance, we write {0.6 Alice, 0.9 Brian, 0.4 Fred, 0.1 Oscar, 0.5 Rita} for the set F (of famous people) to indicate that Alice has a 0.6 degree of membership in F, Brian has a 0.9 degree of membership in F, Fred has a 0.4 degree of membership in F, Oscar has a 0.1 degree of membership in F, and Rita has a 0.5 degree of membership in F (so that Brian is the most famous and Oscar is the least famous of these people). Also suppose that R is the set of rich people with $R =$ {0.4 Alice, 0.8 Brian, 0.2 Fred, 0.9 Oscar, 0.7 Rita}.

73. The **complement** of a fuzzy set S is the set $\overline{S}$, with the degree of the membership of an element in $\overline{S}$ equal to 1 minus the degree of membership of this element in S. Find $\overline{F}$ (the fuzzy set of people who are not famous) and $\overline{R}$ (the fuzzy set of people who are not rich).

74. The **union** of two fuzzy sets S and T is the fuzzy set $S \cup T$, where the degree of membership of an element in $S \cup T$ is the maximum of the degrees of membership of this element in S and in T. Find the fuzzy set $F \cup R$ of rich or famous people.

75. The **intersection** of two fuzzy sets S and T is the fuzzy set $S \cap T$, where the degree of membership of an element in $S \cap T$ is the minimum of the degrees of membership of this element in S and in T. Find the fuzzy set $F \cap R$ of rich and famous people.