Contents

7 Session Seven: Derivatives of Functions II

In this session, we look at the application of trigonometric identities in evaluating the derivatives of trigonometric and inverse trigonometric functions. For functions that the dependent variable is not explicitly expressed in terms of the independent variable, we perform implicit differentiation. We also examine parametric equations and their graphs..

7.1 Session Objectives

By the end of this session, you should be able to:

(i) Apply trigonometric identities in getting the derivatives of trigonometric and inverse trigonometric functions.

(ii) Plot a curve described by parametric equations.

(iii) Convert the parametric equations of a curve into the form $y = f(x)$.

(iv) Recognize the parametric equations of basic curves, such as a line and a circle.

(v) Perform parametric differentiation

7.2 Introduction

7.3 Derivatives of Trigonometric Functions

Recall:
$\lim\limits_{h\to 0} \sin x = 0$, $\lim\limits_{h\to 0} \cos x = 1$, $\lim\limits_{h\to 0} \frac{\sin h}{h} = 1$,

$$\begin{aligned}
\lim_{h\to 0} \frac{1-\cos x}{h} &= \lim_{h\to 0} \frac{(1-\cos x)(1+\cos x)}{h(1+\cos x)} \\
&= \lim_{h\to 0} \frac{1-\cos^2 x}{h(1+\cos x)} \\
&= \lim_{h\to 0} \frac{\sin^2 x}{h(1+\cos x)} \\
&= \lim_{h\to 0} \frac{\sin h}{h} \cdot \lim_{h\to 0} \frac{\sin h}{1+\cos h} \\
&= 1 \cdot \frac{0}{2} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\sin(A \pm B) &= \sin A \cos B \pm \sin B \cos A \\
\cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\
\tan(A + B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B} \\
\sin^2 x + \cos^2 x &= 1
\end{aligned}$$

$f(x) = \sin x$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}\\
&= \lim_{h\to 0}\frac{(\sin x\cos h+\sin h\cos x)-\sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin x(\cos h-1)+\sin h\cos x}{h}\\
&= \lim_{h\to 0}\sin x\frac{\cos h-1}{h}+\lim_{h\to 0}\cos x\frac{\sin h}{h}\\
&= \lim_{h\to 0}-\sin x\frac{1-\cos h}{h}+\lim_{h\to 0}\cos x\frac{\sin h}{h}\\
&= \lim_{h\to 0}\sin x(0)+\lim_{h\to 0}\cos x(1)\\
&= \cos x
\end{aligned}$$

Example **7.3.1.** $f(x) = \tan x$

Solution:

$$\begin{aligned}
f'(x) &= \tan x = \frac{\sin x}{\cos x}\\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right)\\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{\sin x\cos h+\cos x\sin h}{\cos x\cos h-\sin x\sin h}-\frac{\sin x}{\cos x}\right)\\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{\sin x\cos x\cos h+\cos^2\sin h-\sin x\cos x\cos h+\sin^2 x\sin h}{\cos x(\cos x\cos h-\sin x\sin h)}\right)\\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{\cos x^2\sin h+\sin^2 x\sin h}{\cos 62x\cos h-\cos x\sin x\sin h}\right)\\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{\sin h}{\cos^2 x\cos h-\cos x\sin x\sin h}\right)\\
&= \lim_{h\to 0}\frac{\sin h}{h}\lim_{h\to 0}\frac{1}{\cos^2 x\cos h-\cos x\sin x\sin h}\\
&= \frac{1}{\cos^2 x}\\
&= \sec^2 x
\end{aligned}$$

***Alternatively:* Use Quotient rule**

$$\begin{aligned}\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)\\ &= \frac{\cos x\cos x-\sin x(-\sin x)}{\cos^2 x}\\ &= \frac{\cos^2 x+\sin^2 x}{\cos^2 x}\\ &= \sec^2 x\end{aligned}$$

Example **7.3.2.** $f(x)=x^2\tan x$
Solution: $f'(x)=x^2\sec^2 x+\tan x(2x)=x^2\sec^2 x+2x\tan x$

Example **7.3.3.** $f(x)=\frac{\sin x}{1+\cos x}$
Solution:

$$\begin{aligned}f'(x) &= \frac{(1+\cos x)\cos x-\sin x(-\sin x)}{(1+\cos x)^2}\\ &= \frac{\cos x+\cos^2 x+\sin^2 x}{(1+\cos x)^2}\\ &= \frac{1+\cos x}{(1+\cos x)^2}\\ &= \frac{1}{1+\cos x}\end{aligned}$$

Example **7.3.4.** $f(x)=\sin 2x$
Solution: *Let* $u=2x$; $y=\sin u\Rightarrow \frac{du}{dx}=2$ *and* $\frac{dy}{du}=\cos u$. *Therefore,* $\frac{dy}{dx}=2\cos 2x$

Example **7.3.5.** $f(x)=4\cos x^3$
Solution: *Let* $u=x^3$; $y=4\cos u\Rightarrow \frac{du}{dx}=3x^2$ *and* $\frac{dy}{du}=\sin u$. *Therefore,* $\frac{dy}{dx}=-4\sin x^3(3x^2)=-12x^2\sin x^3$

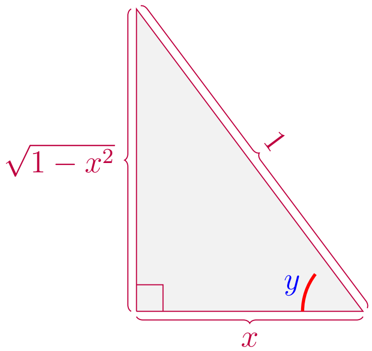
Example **7.3.6.** $f(x)=(\sin x)^2$
Solution: *Let* $u=\sin x$; $y=u^2\Rightarrow \frac{du}{dx}=\cos x$ *and* $\frac{dy}{du}=2u$. *Therefore,* $\frac{dy}{dx}=2\sin x\cos x$

7.4 Derivatives of inverse trigonometric functions

These are $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\csc^{-1}x$, $\sec^{-1}x$ and $\cot^{-1}x$.

Solution:

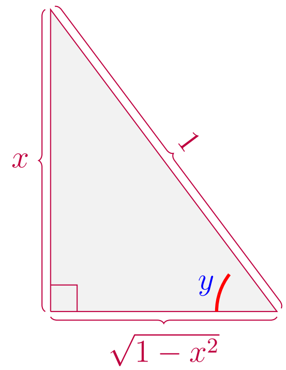
Example 7.4.1. *Evaluate* $\tan\left(\cos^{-1} x\right)$



Let $y = \cos^{-1} x$*. W*
$\tan y$*. From the tria*
have $\cos y = x \Rightarrow \tan$

Solution:

Example 7.4.2. *Find the derivative of* $y = \sin^{-1} x$



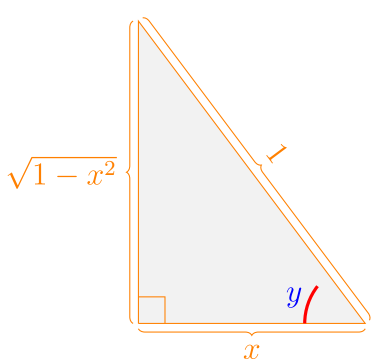
$y = \sin^{-1} x \Rightarrow \sin y = x$. *Differentiating w.r.t we have* $\cos y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$

Alternatively:

$\sin^2 y = x^2 \Rightarrow 1 - \cos^2 y = x^2 \Rightarrow \cos^2 y = x^2 \Rightarrow \cos y = \sqrt{1-x^2}$ *and* $-\sin y \frac{dy}{dx} = \frac{-2x}{2\sqrt{1-x^2}}$. *Therefore,* $\frac{dy}{dx} = \frac{-2x}{-2x\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$.

Solution:

Example 7.4.3. *Find the derivative of* $y = \cos^{-1} x$



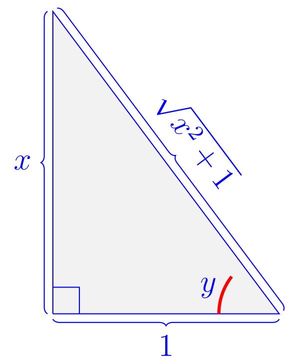
$y = \cos^{-1} x \Rightarrow \cos y = x$. *Differentiating w.r.t we have* $-\sin y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{-\sin y} = \frac{1}{-\sqrt{1-x^2}}$

Alternatively:

$\cos^2 y = x^2 \Rightarrow 1 - \sin^2 y = x^2 \Rightarrow \sin^2 y = x^2 \Rightarrow \sin y = \sqrt{1-x^2}$ *and* $\cos y \frac{dy}{dx} = \frac{-2x}{2\sqrt{1-x^2}}$. *Therefore,* $\frac{dy}{dx} = \frac{-2x}{2x\sqrt{1-x^2}} = \frac{1}{-\sqrt{1-x^2}}$.

Example 7.4.4. *Find the derivative of* $y = \tan^{-1} x$

Solution:

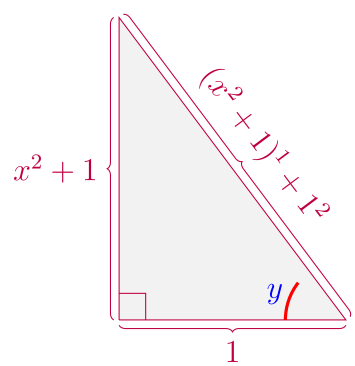


$y = \tan^{-1} x \Rightarrow \tan y = x$. *Differentiating w.r.t* x *we have* $\sec^2 y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} =$ *Similarly, we have* $\frac{1}{x^2+1}$

$\boxed{\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}}$, $\boxed{\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}}$, $\boxed{\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{x^2+1}}$

Example 7.4.5. *Find the derivative of* $y = \tan^{-1}(x^2+1)$

Solution:



$y = \tan^{-1} x \Rightarrow \tan y = x^2 + 1$. *Differentiating w.r.t we have* $\sec^2 y \frac{dy}{dx} = 2x \Rightarrow \frac{dy}{dx} = \frac{2x}{\sec^2 y} = \frac{2x}{x^4+2x+2}$.

Example 7.4.6. *Find the derivative of* $\sin^{-1} x + \cos^{-1} x$.

Solution: $\frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-x^2}} = 0$. *Why?*. *This is because* $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

Example 7.4.7. *Find the derivative of* $x^3 + x\tan^{-1} y = e^y$

Solution:

$$\begin{aligned} 3x^2+\tan^{-1}y+\frac{x}{1+y^2}\frac{dy}{dx} &= e^y\frac{dy}{dx} \\ \frac{x}{1+y^2}\frac{dy}{dx}-e^y\frac{dy}{dx} &= -3x-\tan^{-1}y \\ \frac{dy}{dx} &= \frac{-3x-\tan^{-1}y}{\frac{x}{1+y^2}\frac{dy}{dx}-e^y} \end{aligned}$$

Table 1 gives a summary of the derivatives of inverse trigonometric functions.

Table 1: Summary of derivatives of inverse trigonometric functions

Basic Rule	**Generalized Rule**
$\frac{d}{dx}\left[\sin^{-1}x\right]=\frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\left[\sin^{-1}u\right]=\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
$\frac{d}{dx}\left[\cos^{-1}x\right]=-\frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\left[\cos^{-1}u\right]=-\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
$\frac{d}{dx}\left[\tan^{-1}x\right]=-\frac{1}{x^2+1}$	$\frac{d}{dx}\left[\tan^{-1}u\right]=-\frac{1}{u^2+1}\frac{du}{dx}$
$\frac{d}{dx}\left[\sec^{-1}x\right]=\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\frac{d}{dx}\left[\sec^{-1}u\right]=\frac{1}{\lvert u\rvert\sqrt{u^2-1}}\frac{du}{dx}$
$\frac{d}{dx}\left[\csc^{-1}x\right]=-\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\frac{d}{dx}\left[\csc^{-1}u\right]=-\frac{1}{\lvert u\rvert\sqrt{u^2-1}}$
$\frac{d}{dx}\left[\cot^{-1}x\right]=-\frac{1}{x^2+1}$	$\frac{d}{dx}\left[\cot^{-1}u\right]=-\frac{1}{u^2+1}5$

7.5 Implicit differentiation

Most of the functions that we have discussed so far are equations that express y explicitly in terms of x. Quite often however, we encounter equations like $y^3+7y=x^3$ and $x^2+y^2-4=0$ in which y cannot explicitly be expressed in terms of x. Nevertheless, each of these equations define a relation between x and y.

Example **7.5.1.** *Find* $\frac{dy}{dx}$ *in* $xy=1$

Solution: *We differentiate both sides w.r.t* x*. On the L.H.S, we use product rule.* $x\frac{dy}{dx}+y=0\Rightarrow\ \frac{dy}{dx}=\frac{-y}{x}=-\frac{1}{x^2}$ *(since* $y=\frac{1}{x}$*).*
This method is called implicit differentiation.
A check by explicit differentiation: $y=\frac{1}{x}=x^{-1}\ \frac{dy}{dx}=-\frac{1}{x^2}$.

Example **7.5.2.** *Find* $\frac{dy}{dx}$ *if* $4x^2y-3y=x^3-1$
Solution:

$$\begin{aligned} 4x^2\frac{dy}{dx}+y\cdot 8x-3\frac{dy}{dx} &= 3x^2 \\ \frac{dy}{dx}(4x^2-3) &= 3x^2-8xy \\ \frac{dy}{dx} &= \frac{3x^2-8xy}{4x^2-3} \end{aligned}$$

Example **7.5.3.** *Find* $\frac{dy}{dx}$ *if* $4x^2-2y^2=9$
Solution: $8x-4y\frac{dy}{dx}=0\Rightarrow\frac{dy}{dx}=\frac{-8x}{-4y}=\frac{2x}{y}$

Example 7.5.4. *Find* $\frac{dy}{dx}$ *if* $x^2 + y^2 - 6xy + 3x - 2y + 5 = 0$
Solution:

$$\begin{aligned}
2x + 2y\frac{dy}{dx} + 3 - 2\frac{dy}{dx} - 6(y + x\frac{dy}{dx}) &= 0 \\
\frac{dy}{dx}(2y - 6x - 2) &= 6y - 3 - 2x \\
\frac{dy}{dx} &= \frac{6y - 3 - 2x}{2y - 6x - 2}
\end{aligned}$$

Example 7.5.5. *Find* $\frac{dy}{dx}$ *if* $x = \cos y$
Solution: $-\sin y\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{-\sin y} = -\frac{1}{\sqrt{1-x^2}}$

Example 7.5.6. *Find the derivative of* $x^2 + 2xy + y^2 = 3$
Solution:

$$\begin{aligned}
2x + 2(y + x\frac{dy}{dx}) + 2y\frac{dy}{dx} &= 0 \\
2x + 2y + 2x\frac{dy}{dx} + 2y\frac{dy}{dx} &= 0 \\
\frac{dy}{dx}(2x + 2y) &= -2x - 2y \\
\frac{dy}{dx} &= \frac{-2x - 2y}{2x + 2y} \\
&= \frac{-2(x + y)}{2(x + y)} \\
&= -1
\end{aligned}$$

7.6 Derivatives of Parametric Functions

7.6.1 Parametric Equations

In the two-dimensional coordinate system, parametric equations are useful for describing curves that are not necessarily functions. The parameter is an independent variable that both x and y depend on, and as the parameter increases, the values of x and y trace out a path along a plane curve. For example, if the parameter is t (a common choice), then$t$ might represent time. Then x and y are defined as functions of time, and $(x(t), y(t))$ can describe the position in the plane of a given object as it moves along a curved path.

Consider the orbit of Earth around the Sun. Our year lasts approximately 365 days. On January 1 of each year, the physical location of Earth with respect to the Sun is nearly the same, except for leap years, when the lag introduced by the extra $\frac{1}{4}$ day of orbiting time is built into the calendar.
The number of the day in a year can be considered a variable that determines Earth's position in its orbit. As Earth revolves around the Sun, its physical location changes relative to the Sun. After one full year, we are back where we started, and a new year begins. According

to Kepler's laws of planetary motion, the shape of the orbit is elliptical, with the Sun at one focus of the ellipse.
Figure 1 depicts Earth's orbit around the Sun during one year with coordinate axes superimposed over the graph . The point labeled F_2 is one of the foci of the ellipse; the other focus is occupied by the Sun. We can assign ordered pairs to each point on the ellipse. Each x value on the graph is a value of position as a function of time, and each y value is also a value of position as a function of time. Therefore, each point on the graph corresponds to a value of Earth's position as a function of time.

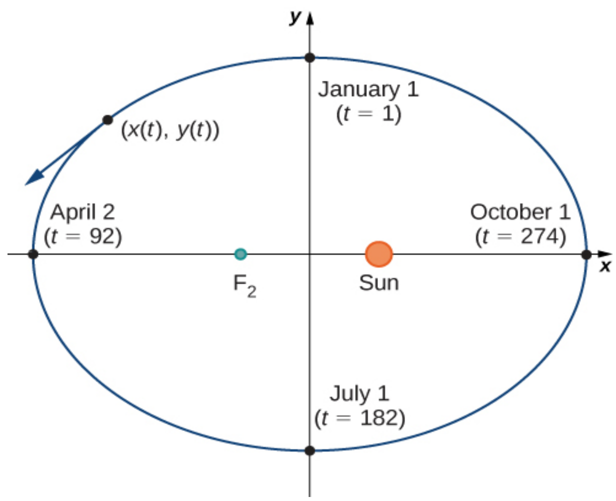


Figure 1: Coordinate axes superimposed on the orbit of Earth

We can determine the functions for $x(t)$ and $y(t)$, thereby parameterizing the orbit of Earth around the Sun. The variable t is called an independent parameter and, in this context, represents time relative to the beginning of each year.

Definition 7.6.1. *If x and y are continuous functions of t on an interval I, then the equations*

$$
\begin{aligned}
x &= x(t) \\
y &= y(t)
\end{aligned}
$$

are called ***parametric equations*** *and t is called the parameter. The set of points (x,y) obtained as t varies over the interval I is called the graph of the* ***parametric equations****. The graph of parametric equations is called a* ***parametric curve or plane curve***

Example 7.6.1. *Sketch the curves described by the following parametric equations:*

(a) $x(t) = t - 1,\ y(t) = 2t + 4,\ -3 \leq t \leq 2$

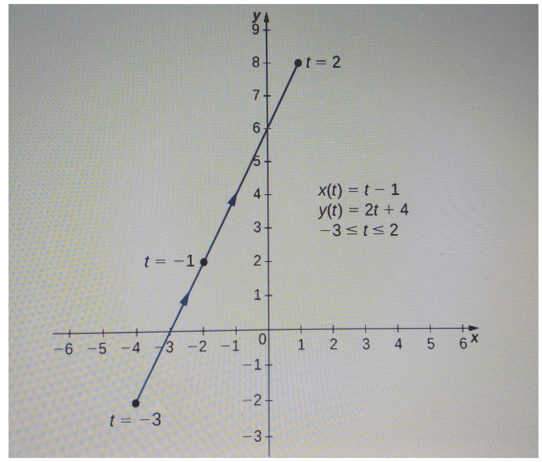
(b) $x(t) = t^2 - 3,\ y(t) = 2t + 1,\ -2 \leq t \leq 3$

(c) $x(t) = 4\cos t,\ y(t) = 4\sin t,\ 0 \leq t \leq 2\pi$

Solution:

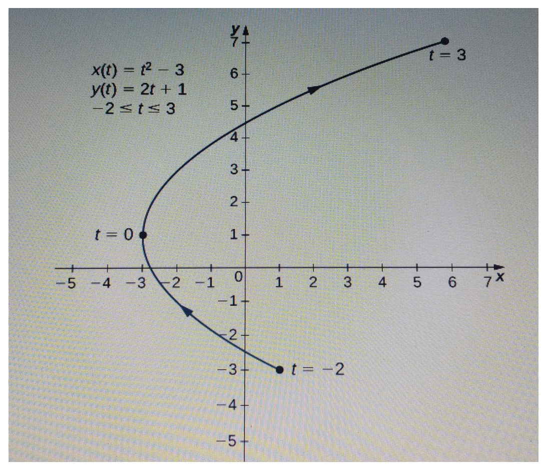
(a) *To create a graph of this curve, first set up a table of value.*

t	-3	-2	-1	0	1	2
x=x(t)	-4	-3	-2	-1	0	1
y=y(t)	-2	0	2	4	6	8



(b) *Similarly, we set up a table of value.*

t	-2	-1	0	1	2	3
x=x(t)	1	-2	-3	-2	1	6
y=y(t)	-3	-1	1	3	5	7



(c) *Try out this as an exercise.*

7.6.2 Eliminating the Parameter

To better understand the graph of a curve represented parametrically, it is useful to rewrite the two equations as a single equation relating the variables x and y

Example **7.6.2.** *Eliminate the parameter for each of the plane curve described by the following parametric equations:*

$$x(t) = \sqrt{2t+4}, \; y(t) = 2t+1, \; -2 \leq t \leq 6$$

Solution: $x = \sqrt{2t+4} \Rightarrow x^2 = 2t+4 \Rightarrow 2t = x^2 - 4.$
Therefore, $y = 2t+1 = (x^2-4)+1 = x^2-3$

Exercise 7.6.1. *Eliminate the parameter for each of the plane curve described by the following parametric equations:*

$$x(t) = 4\cos t,\ y(t) = 3\sin t,\ 0 \le t \le 2\pi$$

7.6.3 Parameterizing a Curve

Suppose we are given the equation of a curve expressed in terms of x and y. How can we determine a pair of parametric equations for that curve? This is certainly possible, and in fact it is possible to do so in many different ways for a given curve. The process is known as **parameterization of a curve.**

Example 7.6.3. *Find two different pairs of parametric equations to represent $y = 2x^2 - 3$.*
Solution: *First, it is always possible to parameterize a curve by defining $x(t) = t$, then replacing x with t in the equation for $y(t)$. This gives the parameterization*

$$x(t) = t,\ y(t) = 2t^2 - 3$$

Since there is no restriction on the domain in the original graph, there is no restriction on the values of t.
For the second choice, we can have $x(t) = 3t-2$. We need to check if there is any restrictions imposed on x. In our case there is none, that is, the range of $x(t)$ is all real numbers. Substituting $x(t) = 3t-2$ for x gives $y(t) = 2(3t-2)^2-2 = 2(9t^2-12t+4)-3 = 18t^2-24t+5$. Therefore, a second parameterization of the curve can be written as

$$x(t) = 3t - 2 \text{ and } y(t) = 18t^2 - 24t + 5$$

Exercise 7.6.2. *Find two different sets of parametric equations to represent the curve $y = x^2 + 2x$.*

7.6.4 Parametric Differentiation

Suppose that $x = f(t)$ and $yg(t)$ are differentiable functions of t and that $\frac{dx}{dt}$ is never zero. Then

$$\frac{dy}{dt} = \frac{dy}{dt} \times \frac{dt}{dx}$$

Therefore

$$\boxed{\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt}, \quad \frac{dx}{dt} \neq 0}$$

Example 7.6.4. *Sketch the graph of $x = 2t$, $y = t^2 - 1$ for $-1 \le t \le 2$ and find $\frac{dy}{dx}$.*
Solution:

t	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{3}{2}$	2
x	-2	-1	0	1	$\frac{3}{2}$	2	3	4
y	0	$-\frac{3}{4}$	-1	$-\frac{3}{4}$	$-\frac{7}{16}$	0	$\frac{5}{4}$	3

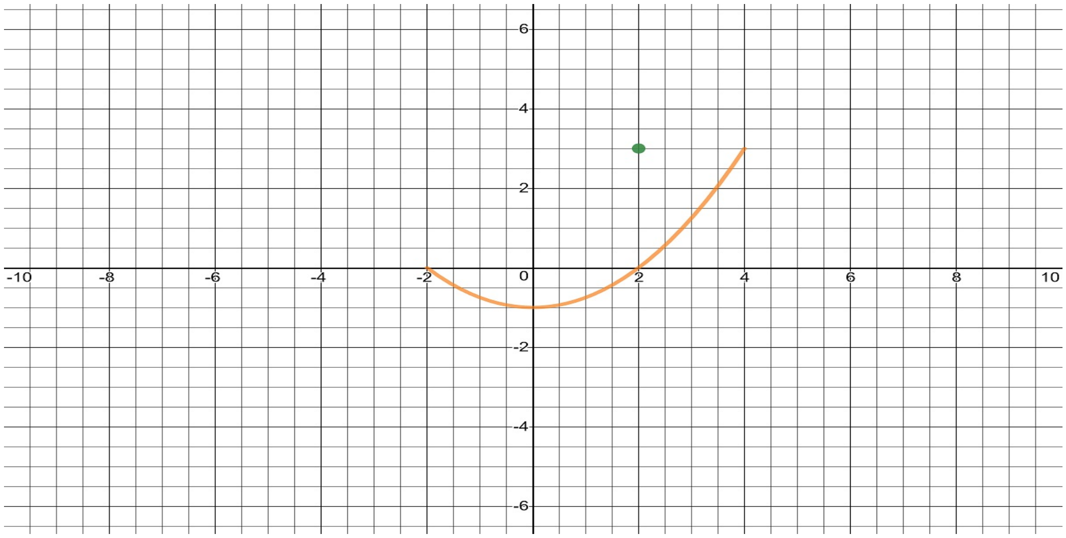


Figure 2: $x = 2t; y = t^2 - 1$

$\frac{dx}{dt} = 2,\ \frac{dy}{dt} = 2t \Rightarrow \frac{dy}{dx} = \frac{dy}{dt}/\frac{dx}{dt} = \frac{2t}{2} = t = \frac{1}{2}x$

Example 7.6.5. *Given that* $x = t^3 + t^2$, $y = t^2 + t$, *show that* $\frac{dy}{dx} = \frac{2t+1}{t(3t+2)}$, $y^3 = x^2 + xy$.

Solution: $\frac{dx}{dt} = 3t^2 + 2t,\ \frac{dy}{dt} = 2t + 1 \Rightarrow \frac{dy}{dx} = \frac{2t+1}{3t^2+2t} = \frac{2t+1}{t(3t+2)}$

$y^3 = t^6 + 3t^5 + 3t^4 + t^3$

$$\begin{aligned} x^3 + xy &= (t^3 + t^2) + (t^3 + t^2)(t^2 + t) \\ &= t^6 + 3t^5 + 3t^4 + t^3 \end{aligned}$$

Therefore,

$$y^3 = x^2 + xy$$

Example 7.6.6. *Find* $\frac{dy}{dx}$ *in* $x = \sin t$, $y = \cos t$.

Solution: $\frac{dx}{dt} = -\sin t,\ \frac{dy}{dt} = \cos t \Rightarrow \frac{dy}{dx} = \frac{dy}{dt}/\frac{dx}{dt} = \frac{\cos t}{-\sin t} = -\frac{x}{y}$

Alternatively:

$x^2 + y^2 = 1 \Rightarrow 2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-x}{y}$.

Example 7.6.7. *If* $x = \frac{-2t}{t+2}$, $y = \frac{3t}{t+3}$, *find* $\frac{dy}{dx}$ *in terms of* t.

Solution: $\frac{dx}{dt} = \frac{2(t+2)-2t}{(t+2)^2} = \frac{4}{(t+2)^2},\ \frac{dy}{dt} = \frac{3(t+2)-3t}{(t+3)^2} = \frac{9}{(t+3)^2}$.

Therefore, $\frac{dy}{dx} = \frac{9}{(t+2)^2} \cdot \frac{(t+3)^2}{4} = \frac{9}{4}$.

7.6.5 Parametric Differentiation for $\frac{d^2y}{dx^2}$

The second derivative of y w.r.t x is obtained by differentiating y w.r.t x twice.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) == \frac{d}{dx}(y')$$

Therefore,

$$\boxed{\frac{d^2y}{dx^2} = \frac{dy'}{dt}/\frac{dx}{dt}, \quad \frac{dx}{dt} \neq 0}$$

Example 7.6.8. *Find* $\frac{d^2y}{dx^2}$ *if* $x = t - t^2$ *and* $y = t - t^3$.
Solution: $\frac{dx}{dt} = 1 - 2t, \quad \frac{dy}{dt} = 1 - 3t^2 \Rightarrow \frac{dy}{dx} = y' = \frac{dy}{dt} / \frac{dx}{dt} = \frac{1-3t^2}{1-2t}$
$\Rightarrow \frac{dy'}{dt} = \frac{(1-2t)(-6t)-(1-3t^2)(-2)}{(1-2t)^2} = \frac{2-6t+6t^2}{(1-2t)^2}$. *Therefore,*

$$\frac{d^2y}{dx^2} = \frac{dy'}{dt} / \frac{dx}{dt} = \frac{2 - 6t + 6t^2}{(1-2t)^3}$$

Example 7.6.9. *Given that* $x = \cos^2 t$, $y = \sin^2 t \quad 0 \leq t \leq 2\pi$, *find* $\frac{d^2y}{dx^2}$.
Solution:

$$\begin{aligned}
\frac{dx}{dt} &= -2\sin t \cos t \\
\frac{dy}{dt} &= 2\cos t \sin t \\
\frac{dy}{dx} &= y' \frac{2\cos t \sin t}{-2\sin t \cos t} \\
&= -1 \\
\frac{dy'}{dt} &= 0 \\
\frac{d^2y}{dx^2} &= \frac{0}{-\sin t \cos t} \\
&= 0
\end{aligned}$$

7.7 Session Summary

In implicit differentiation, whenever we differentiate a dependent variable, say y, we should always remember to introduce $\frac{dy}{dx}$.

In parameterization of the equation of a curve, we have complete freedom in the choice for $x(t)$ from which we deduce $y(t)$.

For more material on this section check out [1, 2] or visit trigonometric differentiation, inverse trigonometric functions, implicit differentiation, parametric equations. You can also watch the lecture videos implicit differentiation, derivatives of trigonometric functions, inverse trigonometric functions, parametric differentiation.

7.8 Student Activity

Exercise

1. In questions 1-12, find $\dfrac{dy}{dx}$

(a) $y = \tan^2 5x$ (b) $y = \sqrt{x}\tan^3(x^{\frac{1}{2}})$ (c) $y = \sqrt{x}\tan^3 x^{\frac{1}{2}}$

(d) $\cos^2(3\sqrt{x})$ (e) $y = \frac{\sin x}{\sec(x+1)}$ (f) $y = 2\sin x + 3\cos x$

(g) $y = \tan(4x^2)$ (h) $y = \cos^3(\sin 2x)$ (i) $y = \frac{\sin x + \cos x}{\tan x}$

(j) $y = \sin\left(\frac{1}{x^2}\right)$ (k) $y = \sin^2 x^2$ (l) $y = \sin x \cos x$

2. Find the derivatives of the following
   (a) $y = \cos^{-1}(x^2 + 1)$ (b) $y = x^3 \tan^{-1}(e^x)$

   (c) $y = \sec^{-1}(2\sqrt{x})$ (d) $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

   (e) $y = (1 + \sin^{-1} x)^3$ (f) $y = \tan^{-1}(\ln x^2)$

   (g) $y = \sin^{-1}\left(\frac{x-1}{x+1}\right)$ (h) $y = (\sec^{-1} x)^3$

3. In questions (a)–(d), prove the identities given.
   (a) $\tan(\sin^{-1}) = \frac{x}{\sqrt{1-x^2}}$ (b) $\tan(2\tan^{-1}) = \frac{2x}{\sqrt{1-x^2}}$

   (c) $\sin(\tan^{-1}) = \frac{x}{\sqrt{1+x^2}}$ (d) $\cos(2\sin^{-1}) = 1 - x^2$

4. Find $\dfrac{dy}{dx}$ by implicit differentiation.
   (a) $x^2 - 3xy + y^2 - 2y + 4x = 0$ (b) $x^2 = \frac{x+y}{x-y}$

   (c) $x^2 + xy + y^2 = 7$ (d) $\tan^3(xy^2 + y) = x$

   (e) $x^2 + 2xy - 2y^2 + x - 2 = 0$ (f) $3xy = (x^3 + y^2)^{\frac{3}{2}}$

   (g) $x^3y^2 - 5x^2y + x = 1$ (h) $\sqrt{x} + \sqrt{y} = 8$

   (i) $\frac{xy^3}{1+\sec y} = 1 + y^4$ (j) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$

   (k) $\sqrt{5xy} + 2y = y^2 + xy^3$ (l) $\cos(xy^2) = y^2 + x$

5. In the questions (a)–(h), find $\dfrac{dy}{dx}$.
   (a) $x = at^2$, $y = 2at$ (b) $x = t$, $y = \sqrt{1-t^2}$, $-1 \le t \le 1$

   (c) $x = (t+1)^2$, $y = (t^2 - 1)$ (d) $x = t^2 - \frac{\pi}{2}$, $y = \sin(t^2)$

   (e) $x = \sqrt{t}$, $y = \sqrt{t}$, $t \ge 0$ (f) $x = \cos t$, $y = 1 - \sin^2 t$

   (g) $x = \cos t$, $y = \sin(t^2)$ (h) $x = \frac{1}{t} + t$, $y = t^2 - t + 1$

6. In the questions (a)–(h), find $\dfrac{d^2y}{dx}$.

(a) $x = 2t,\ y = 4t - 7$

(b) $x = \cos t,\ y = 1 + \sin t$

(c) $x = 3t,\ y = 9t^2$

(d) $x = 80,\ y = 64t - 16t^2$

(e) $x = t,\ y = \sqrt{t},\ t \geq 0$

(f) $x = t^2 - \frac{\pi}{2},\ y = \sin t^2$

(g) $x = t,\ y = \frac{1}{t}$

(h) $x = \cos t,\ y = 1 - \sin^2 t$

References

[1] E. Purcell D. Varberg and S. Rigdon. *Calculus*. Pearson Education, Inc., 9 edition, 2006. ISBN-13 : 978-0132306331.

[2] J. Stewart. *Calculus*. Cengage Learning 20 Channel Center Street, Boston, MA 02210, USA, 8 edition, 2016. ISBN-13: 978-1-305-27176-0.

