Contents

8 Session Eight: Derivatives of Functions III

8.1 Session Objectives

By the end of this session, you should be able to:

(i) Evaluate derivatives of logarithmic, exponential, hyperbolic and inverse hyperbolic functions

(ii) Deduce patterns in derivatives to get higher order derivatives

8.2 Introduction

In this session, we shall use the rules of logarithms in understanding thee derivative of logarithmic functions. Derivatives of exponential, hyperbolic and inverse hyperbolic functions are also discussed. We shall extend the concept of derivatives to higher orders.

8.3 Derivative of Logarithmic Functions

8.3.1 Rules of logarithms

1. $\log_b 1 = 0$
2. $\log_b b = 1$
3. $\log_b ac = \log_b a + \log_b c$
4. $\log_b \frac{a}{c} = \log_b a - \log_b c$
5. $\log_b a^r = r \log_b a$
6. $\log_b \frac{1}{c} = -\log_b c$

If $\log_c a = y$ then $c^y = a$. Taking logarithms on both sides we have:

$$\begin{aligned} \log_b c^y &= \log_b a \\ y \log_b c &= \log_b a \\ y &= \frac{\log_b a}{\log_b c} \\ \Rightarrow & \boxed{\log_c a = \frac{\log_b a}{\log_b c} \text{ where } b \text{ is any base}} \end{aligned}$$

The $\log_e b$ is called the natural logarithm abbreviated $\ln b$ where $e \approx 2.718281828\cdots$ is the natural number.

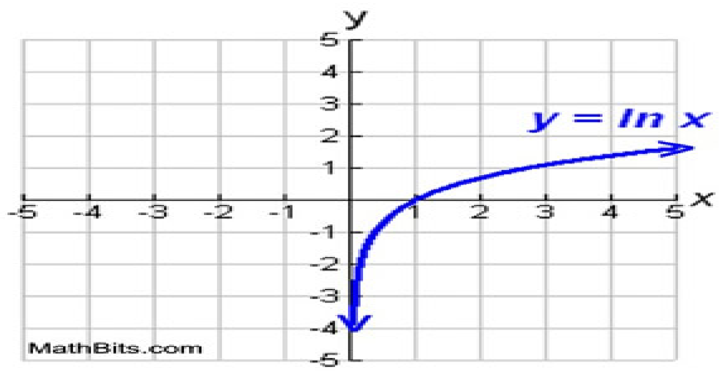

Figure 1: $y =\in x$

8.3.2 Properties of natural logarithms

1. $\ln ax = \ln a + \ln x$
2. $\ln x^n = n \ln x$
3. $\ln(\frac{x}{a}) = \ln x - \ln a$
4. $\ln e = \log_e e = 1$

We have that

$$\boxed{\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e}$$

Check:

x	$(1+x)^{\frac{1}{x}}$
3	$\sqrt[3]{4}$
2	$\sqrt{3}$
1	2
$\frac{1}{2}$	$\left(\frac{3}{2}\right)^2 = 2.25$
$\frac{1}{4}$	$\left(\frac{5}{4}\right)^2 = 2.4414063$
$\frac{1}{1000}$	$\left(\frac{1001}{1000}\right)^{1000} = 2.716923932236$
$\vdots$	$\vdots$
$\downarrow$	$\downarrow$
0	$e = 2.718281828\cdots$

$$\boxed{\lim_{x\to 0}(1+\frac{1}{x})^{x} = e}$$

Check:

x	$(1+\frac{1}{x})^x$
1	2
2	$\left(\frac{3}{2}\right)^2 = 2.25$
3	$\left(\frac{4}{3}\right)^3 = 2.3703702$
10	$\left(\frac{11}{10}\right)10 = 2.5937420$
$\vdots$	$\vdots$
$\downarrow$	$\downarrow$
0	$e = 2.718281828\cdots$

$$\begin{aligned}
\frac{d}{dx}(\log_b x) &= \lim_{h\to 0}\frac{\log_b(x+h)-\log_b x}{h} \\
&= \lim_{h\to 0}\frac{1}{h}\log_b\left(\frac{x+h}{h}\right) \\
&= \lim_{h\to 0}\frac{1}{h}\log_b\left(1+\frac{h}{x}\right) \\
\text{Let} \quad \underset{\text{as } h\to 0}{\frac{h}{x}=t} &\Rightarrow \underset{t=0}{h = tx} \\
\frac{d}{dx}(\log_b x) &= \lim_{t\to 0}\frac{1}{tx}\log_b(1+t) = \frac{1}{x}\lim_{t\to 0}\frac{1}{t}\log_b(1+t) \\
&= \frac{1}{x}\lim_{t\to 0}\log_b(1+t)^{\frac{1}{t}} = \frac{1}{x}\log_b\lim_{t\to 0}(1+t)^{\frac{1}{t}} \\
&= \frac{1}{x}\log_b e
\end{aligned}$$

Therefore,

$$\boxed{\frac{d}{dx}(\log_b x) = \frac{1}{x}\log_b e}$$

If $b = e$

$$\boxed{\frac{d}{dx}(\log_e x) = \frac{d}{dx}(\ln x) = \frac{1}{x}}$$

8.4 Exponential Functions

8.4.1 Derivative of $y = a^x$

Let $y = a^x$

$$\begin{aligned}
\ln y &= \ln a^x \\
\ln y &= a \ln a \\
\frac{1}{y}\frac{dy}{dx} &= \ln a \qquad (\text{Diff. both sides w.r.t } y) \\
\frac{dy}{dx} &= y \ln a \\
\frac{dy}{dx} &= a^x \ln a
\end{aligned}$$

Thus,

$$\boxed{\frac{d}{dx}(a^x) = a^x \ln a}$$

Example 8.4.1. *Find the derivative of* $y = 3^x$

Solution:

$$\begin{aligned}
\ln y &= x \ln 3 \\
\Rightarrow \frac{1}{y}\frac{dy}{dx} &= \ln 3 \\
\Rightarrow \frac{dy}{dx} &= y \ln 3 \\
&= 3^x \ln 3
\end{aligned}$$

Example 8.4.2. *Find* $\frac{dy}{dx}$ *in* $y = 3^{\sin x}$.

Solution:

$$\begin{aligned}
\ln y &= \sin x \ln 3 \\
\Rightarrow \frac{1}{y}\frac{dy}{dx} &= \cos x \ln 3 \\
\Rightarrow \frac{dy}{dx} &= y \cos x \ln 3 \\
&= 3^{\sin x} \cos x \ln 3
\end{aligned}$$

Example 8.4.3. *Find* $\frac{dy}{dx}$ *in* $y = x^x$.

Solution:

$$\begin{aligned}
\ln y &= x \ln x \\
\Rightarrow \frac{1}{y}\frac{dy}{dx} &= x\left(\frac{1}{x}\right) + \ln x \\
\Rightarrow \frac{dy}{dx} &= (\ln x + 1)y \\
&= x^x(\ln x + 1)
\end{aligned}$$

Example 8.4.4. *Find $\frac{dy}{dx}$ in $y = \ln(x^2+6)$.*

Solution: Let $u = x^2 + 1$; $y = \ln u \Rightarrow \frac{du}{dx} = 2x$ and $\frac{dy}{du} = \frac{1}{u}$. Therefore,

$$\frac{dy}{dx} = \frac{1}{u} \cdot 2x = \frac{2x}{x^2+6}$$

Example 8.4.5. *Find $\frac{dy}{dx}$ in $y = \ln\sqrt{x+1}$.*

Solution:

$$\begin{aligned} y &= \ln(x+1)^{\frac{1}{2}} = \frac{1}{2}\ln(x+1) \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{2} \cdot \frac{1}{x+1} \cdot 1 = \frac{1}{2(x+b1)} \end{aligned}$$

Example 8.4.6. *Find the derivative of $y = \ln\frac{x\sqrt{x+5}}{(x-1)^3}$.*

Solution:

$$\begin{aligned} \ln\frac{x\sqrt{x+5}}{(x-1)^3} &= \ln x\sqrt{x+5} - \ln(x-1)^3 \\ &= \ln x + \ln\sqrt{x+5} - \ln(x-1)^3 \\ &= \ln x + \frac{1}{2}\ln(x+5) - 3\ln(x-1) \end{aligned}$$

Example 8.4.7. *Find the derivative of $y = (x^2+1)^{10} + 10^{x^2+1}$.*

Solution: $\frac{dy}{dx} = 10(2x)(x^2+1)^9 + \frac{d}{dx}(10^{x^2+1}) = 20(x^2+1)^9 + 2x \cdot 10^{x^2+1}\ln 10$

8.4.2 Derivative of e^x

Since $\ln 2 < 1$ and $\ln 4 > 1$, then by intermediate value theorem $\exists$ a number between 2 and 4 such whose natural logarithm is equal to 1. Since $\ln x$ is 1-1, there is on;y one such number. We denote it by the letter e,named after the Swiss mathematician Euler who wrote about.

$$e = \ln^{-1} 1 \text{ and } \ln e = 1$$

Generally,

$$e^x = \ln^{-1} x$$

Since $y = e^x$ and $y = \ln x$ are inverses of one another, $\ln e^x = x$ and $e^{\ln x} = x$.
The function $y = e^x$ is differentiable since it is the inverse of a differential function whose derivative is never zero.

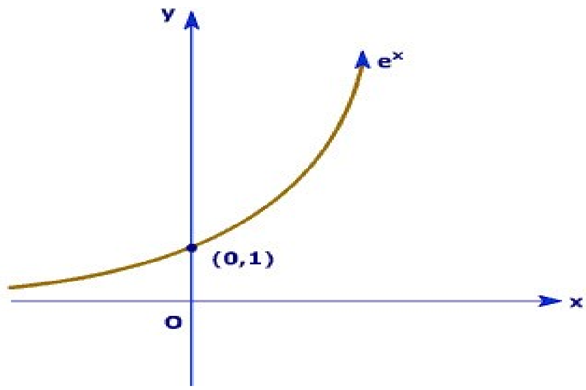

Figure 2: $y = e^x$

Let $y = e^x$, then taking natural logarithms on both sides we have

$$\begin{aligned} \ln y &= \ln e^x \\ \ln y &= x \\ \frac{1}{y}\frac{dy}{dx} &= 1 \\ \Rightarrow \frac{dy}{dx} &= y = e^x \end{aligned}$$

$$\boxed{\frac{d}{dx}(e^x) = e^x}$$

Example 8.4.8. *Find $\frac{dy}{dx}$ given $y = e^{ax}$.*

Solution: Let $u = ax$; $y = e^u \Rightarrow \frac{du}{dx} = a$ and $\frac{dy}{du} = e^u$. Therefore, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = ae^u = ae^{ax}$.

Example 8.4.9. *Find $\frac{dy}{dx}$ given $y = e^{\sin x}$.*

Solution: Let $u = \sin x$; $y = e^u \Rightarrow \frac{du}{dx} = \cos x$ and $\frac{dy}{du} = e^u$. Therefore, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \cos x e^u = e^{\sin x} \cos x$.

Example 8.4.10. *Find $\frac{dy}{dx}$ given $y = x^2 e^x$.*

Solution: $\frac{dy}{dx} = x^2 e^x + e^x(2x) = x^2 e^x + 2xe^x = xe^x(x + 2)$

8.5 Hyperbolic Functions

The hyperbolic functions are a combinations of the exponential e^x and e^{-x}. These are the hyperbolic sine, hyperbolic cosine, hyperbolic tangent, hyperbolic cosecant, hyperbolic secant and hyperbolic cotangent.

They are analogous to the trigonometric functions, and they have the same relationship to the hyperbola that the trigonometric functions have to the circle.

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad csch\ x = \frac{1}{\sinh x} = \frac{1}{e^x - e^{-x}}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad sech\ x = \frac{1}{\cosh x} = \frac{1}{e^x + e^{-x}}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

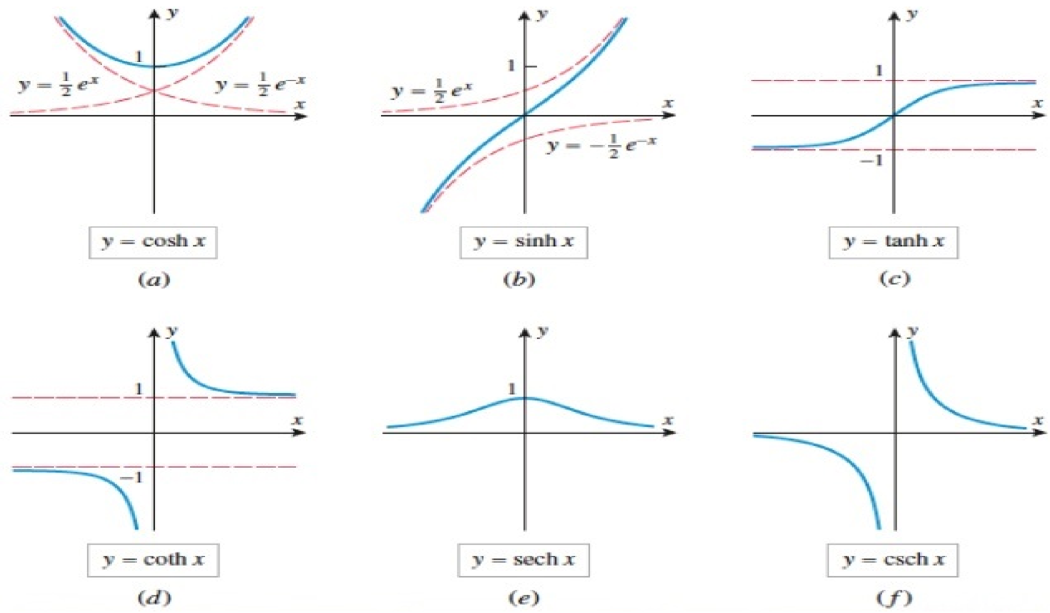

Figure 3: Graphs of Hyperbolic Functions

$$\sinh(-x) = -\sinh x \qquad 1 - \tanh^2 x = sech^2\ x$$

$$\cosh(-x) = \cosh x \qquad \coth^2 x - 1 = \cosh^2 x$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh(x + y) = \sinh x + \cosh y + \cosh x \sinh y$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x + \sinh y$$

Example 8.5.1. *Prove that* $\cosh^2 x - \sinh^2 x = 1$.

Solution:

$$\begin{aligned}
\cosh^2 x - \sinh^2 x = 1 &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\
&= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} \\
&= \frac{4}{4} \\
&= 1
\end{aligned}$$

Example 8.5.2. *Prove that* $\sinh(x+y) = \sinh x + \cosh y + \cosh x \sinh y$.

Solution:

$$\begin{aligned} R.H.S &= \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right) \\ &= \frac{e^{x+y}}{4} + \frac{e^{x-y}}{4} - \frac{e^{-x+y}}{4} - \frac{e^{-(x+y)}}{4} + \frac{e^{x+y}}{4} - \frac{e^{x-y}}{4} \\ &\quad + \frac{e^{-x+y}}{4} - \frac{e^{-(x+y)}}{4} \\ &= \frac{e^{x+y}}{4} - \frac{e^{-(x+y)}}{4} = \frac{e^{x+y} - e^{-(x+y)}}{2} \\ &= \sinh(x+y) \\ &= L.H.S \end{aligned}$$

Example 8.5.3. *Find the derivative of* $y = \sinh x$.

Solution:

$$\begin{aligned} \frac{d}{dx}(\sinh x) &= \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) \\ &= \frac{1}{2}\frac{d}{dx}(e^x - e^{-x}) \\ &= \frac{1}{2}(e^x + e^{-x}) \\ &= \frac{e^x + e^{-x}}{2} \\ &= \cosh x \end{aligned}$$

Example 8.5.4. *Find the derivative of* $y = \cosh x$.

Solution:

$$\begin{aligned} \frac{d}{dx}(\cosh x) &= \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) \\ &= \frac{1}{2}\frac{d}{dx}(e^x + e^{-x}) \\ &= \frac{1}{2}(e^x - e^{-x}) \\ &= \frac{e^x - e^{-x}}{2} \\ &= \sinh x \end{aligned}$$

Example 8.5.5. *Find the derivative of* $y = \tanh x$.

Solution:

$$\begin{aligned}
\frac{d}{dx}(\tanh x) &= \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) \\
&= \frac{\cosh x \cosh x - \sinh x \sinh x}{(\cosh x)^2} \\
&= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2} \\
&= \frac{1}{\cosh^2 x} \\
&= sech\ x
\end{aligned}$$

Exercise 8.5.1. *Find the derivative of* $\coth x$, $sech\ x$ *and* $cschx$.

Derivatives of Hyperbolic Functions Summary:

$\frac{d}{dx}(\sinh x) = \cosh x$	$\frac{d}{dx}(cschx) = -csch\ x \coth x$
$\frac{d}{dx}(\cosh x) = \sinh x$	$\frac{d}{dx}(sechx) = -sechx \tanh x$
$\frac{d}{dx}(\tanh x) = sech^2\ x$	$\frac{d}{dx}(\coth x) = -csch^2\ x$

Example 8.5.6. *Find the derivative of* $y = \cosh(x^2 + x)$

Solution: $u = x^2 + x$; $y = \cosh u \Rightarrow \frac{du}{dx} = 2x + 1$ and $\frac{dy}{du} = \sinh u$. Therefore, $\frac{dy}{dx} = (2x + 1)\sinh(x^2 + x)$

Example 8.5.7. *Find the derivative of* $y = \tanh 2x$.

Solution: $\frac{dy}{dx} = 2sech^2(2x)$

Example 8.5.8. *Find* $\frac{dy}{dx}$ *given* $\sinh y = \tan x$.

Solution: $\cosh y \frac{dy}{dx} = \sec^2 x \Rightarrow \frac{dy}{dx} = \frac{dy}{dx} = \frac{\sec^2 x}{\cosh y}$

8.6 Inverse Hyperbolic Functions

Since hyperbolic sine and hyperbolic tangent have positive derivatives, they are increasing functions and automatically have inverses. To obtain inverses for hyperbolic cosine and hyperbolic secant, we restrict their domains to $x \geq 0$. Thus

$$\begin{aligned}
x = \sinh^{-1} y &\iff y = \sinh x \\
x = \cosh^{-1} y &\iff y = \cosh x \text{ and } x \geq 0 \\
x = \tanh^{-1} y &\iff y = \tanh x \\
x = sech^{-1} y &\iff y = sech\ x \text{ and } x \geq 0
\end{aligned}$$

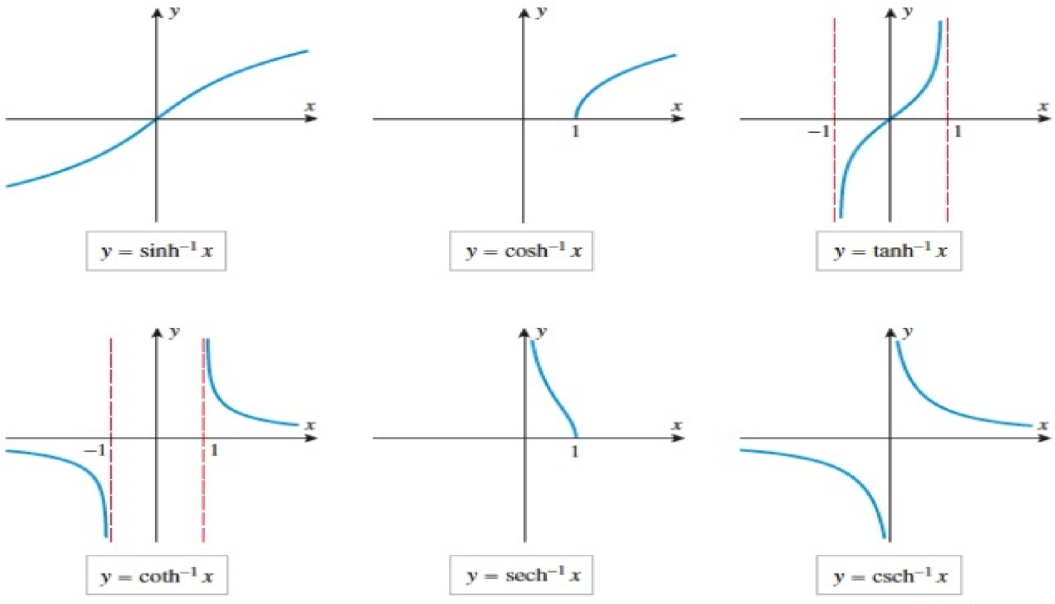

Figure 4: Inverse Hyperbolic Functions

Since the hyperbolic functions are defined in terms of e^x and e^{-x}, it is not suprisingly that the inverse hyperbolic functions can be expressed in terms of the natural logarithm. This is explained below.
Let $\sinh^{-1} x = y$, $\Rightarrow x = \sinh y = \frac{e^y - e^{-y}}{2}$. So $e^y - 2x - e^{-y}$. Multiplying by e^y we have

$$e^{2y} - 2xe^y - 1 = 0$$

This is a quadratic equation in e^y; $(e^y)^2 - 2x(e^y) - 1 = 0$. We solve the quadratic equation by quadratic formula.

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}$$

Note that e^y, but $x - \sqrt{x^2 + 1} < 0$ (because $x < \sqrt{x^2 + 1}$). Thus, the negative value is inadmissible and we have $e^y = x + \sqrt{x^2 + 1}$.
Therefore,

$$y = \ln(e^y) = \ln(x + \sqrt{x^2 + 1})$$

Similar arguments can apply to each of the inverse hyperbolic functions to get the following results.

$$\begin{aligned}
\sinh^{-1} x &= \ln(x + \sqrt{x^2+1}) \\
\cosh^{-1} x &= \ln(x + \sqrt{x^2-1}), \quad x \geq 1 \\
\tanh^{-1} &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \quad -1 < x < 1 \\
sech^{-1}x &= \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right), \quad 0 < x \leq 1 \\
csch^{-1}x &= \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^1}}{|x|}\right), \quad x \neq 0 \\
\coth^{-1} &= \frac{1}{2}\ln\frac{x+1}{x-1}, \quad |x| > 1
\end{aligned}$$

Example 8.6.1. *Prove that* $sech^{-1}x = \ln(x + \sqrt{x^2+1})$.

Solution: Let $sech^{-1}x = y$, $\Rightarrow x = sech\ y = \frac{1}{\cosh y} = \frac{2}{e^y+e^{-y}}$.
Thus

$$\begin{aligned}
x(e^y + e^{-y}) &= 2 \\
x\left(e^y + \frac{1}{e^y}\right) &= 2 \\
x\left(\frac{e^{2y}+1}{e^y}\right) &= 2 \\
x(e^{2y}+1) &= 2e^y \\
xe^{2y} + x - 2e^y &= 0
\end{aligned}$$

Let $e^y = m$, and we have $xm^2 - 2m + x$ which is a quadratic equation. Solving gives

$$\begin{aligned}
m &= \frac{2 \pm \sqrt{4-4x^2}}{2x} \\
&= \frac{2 \pm \sqrt{4(1-x^2)}}{2x} \\
&= \frac{1 \pm \sqrt{1-x^2}}{x} \\
&\Rightarrow e^y = \frac{1 \pm \sqrt{1-x^2}}{x} \\
&\Rightarrow e^y = \frac{1 + \sqrt{1-x^2}}{x}, \quad \text{Since } \frac{1-\sqrt{1-x^2}}{x} < 0 \\
&\Rightarrow y = sech^{-1}x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)
\end{aligned}$$

Example 8.6.2. *Find the derivative of* $y = \sinh^{-1} x$.

Solution: **Method 1**

$\sinh y = x \Rightarrow \cosh y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y}$. Since $\cosh^2 y - \sinh^2 y = 1$ and $\cosh y \geq 0$, we have $\cosh y = \sqrt{1+\sinh^2 y}$. Therefore

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\cosh y} \\ &= \frac{1}{\sqrt{1+\sinh^2 y}} \\ &= \frac{1}{\sqrt{1+x^2}} \end{aligned}$$

Method 2

$$\begin{aligned} \frac{d}{dx}(\sinh^{-1} x) &= \frac{d}{dx} \ln\left(x+\sqrt{x^2+1}\right) \\ &= \frac{1}{x+\sqrt{x^2+1}} \frac{d}{dx}\left(x+\sqrt{x^2+1}\right) \\ &= \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{x}{\sqrt{x^2+1}}\right) \\ &= \frac{1}{x+\sqrt{x^2+1}}\left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right) \\ &= \frac{1}{\sqrt{x^2+1}} \end{aligned}$$

Example 8.6.3. *Find the derivative of* $y = \cosh^{-1} x$.

Solution:

$$\begin{aligned} \cosh y &= x \\ \sinh y \frac{dy}{dx} &= 1 \\ \sinh y &= \sqrt{\cosh^2 y - 1} \\ \frac{dy}{dx} &= \frac{1}{\sinh y} = \frac{1}{\sqrt{x^2-1}} \end{aligned}$$

Example 8.6.4. *Find the derivative of* $y = \tanh^{-1} x$.

Solution:

$$\begin{aligned} \tanh y &= x \\ sech^2 y \frac{dy}{dx} &= 1 \\ sech^2 y &= 1 - \tanh^2 y \\ \frac{dy}{dx} &= \frac{1}{sech^2 y} = \frac{1}{1-x^2} \end{aligned}$$

The derivatives of the other ($\coth^{-1} x$, $sech^{-1}x$ and $csch^{-1}x$) can be obtained similarly. These derivatives are summarized as follows:
Derivatives of inverse hyperbolic functions Summary

$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$	$\frac{d}{dx}(csch^{-1}x) = -\frac{1}{\lvert x\rvert\sqrt{x^2+1}}$
$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$	$\frac{d}{dx}(sech^{-1}x) = -\frac{1}{x\sqrt{1-x^2}}$
$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$	$\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}$

Exercise

8.7 Higher Order Derivatives

If $y = f(x)$,

$$\begin{aligned}
f'(x) &= y' = \frac{dy}{dx} = D_x f(x) - 1^{st} \text{ derivative of } f(x)\\
f''(x) &= y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = D_x^2 f(x) - 2^{nd} \text{ derivative of } f(x)\\
f'''(x) &= y''' = \frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = D_x^3 f(x) - 3^{rd} \text{ derivative of } f(x)\\
\vdots & \quad \vdots\\
f^n(x) &= y^n = \frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = D_x^n f(x) - n^{th} \text{ derivative of } f(x)
\end{aligned}$$

Example 8.7.1. *Find* $\frac{d^7y}{dx^7}$ *if* $y = 3x^4 - 2x^3 + x^2 - 4x + 2$.

Solution:

$$\begin{aligned}
\frac{dy}{dx} &= 12x^3 - 6x^2 + 2x - 4\\
\frac{d^2y}{dx^2} &= 36x^2 - 12x + 2\\
\frac{d^3y}{dx^3} &= 72x - 12\\
\frac{d^4y}{dx^4} &= 72\\
\frac{d^ny}{dx^n} &= 0 \text{ for } n \geq 5\\
\text{Therefore,} &\quad \frac{d^7y}{dx^7} = 0
\end{aligned}$$

Example 8.7.2. *If* $f(x) = \frac{1}{x}$, *find* $f^n(x)$.

Solution: $f'(x) = \frac{1}{x} = x^{-1} = -\frac{1}{x^2}$ $f''(x) = 2 \cdot 1 \cdot x^{-3} = \frac{2}{x^3}$

$$\begin{aligned}
f'''(x) &= -3 \cdot 2 \cdot 1 \cdot x^{-4} = \frac{-3!}{x^4} \\
f^4(x) &= 4 \cdot 3 \cdot 2 \cdot 1 \cdot x^{-5} = \frac{4!}{x^5} \\
f^5(x) &= -5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot x^{-6} = \frac{-5!}{x^6} \\
\vdots \quad & \quad \vdots \\
f^n(x) &= (-1)^n (n-1)(n-2) \cdots 2 \cdot 1 \cdot x^{-(n+1)} \\
\text{Therefore,} \quad & \quad \frac{(-1)^n n!}{x^{n+1}}
\end{aligned}$$

Example 8.7.3. *Find $f^n(x)$ given $f(x) = \frac{1}{(1-x)^2}$.*

Solution:

$$\begin{aligned}
f(x) &= (1-x)^{-2} \\
f'(x) &= 2(1-x)^{-3} \\
f''(x) &= 6(1-x)^{-4} = 3!(1-x)^{-4} \\
f'''(x) &= 24(1-x)^{-5} = 4!(1-x)^{-5} \\
f^4(x) &= 120(1-x)^{-5} = 5!(1-x)^{-6} \\
\vdots \quad & \quad \vdots \\
f^n(x) &= \frac{(n+1)!}{(1-x)^{n+2}}
\end{aligned}$$

Example 8.7.4. *Given $f(x) = \cos x$, find $f^{24}(x)$, $f^{27}(x)$, $f^{92}(x)$.*

Solution:

$$\begin{aligned}
f'(x) &= -\sin x \\
f''(x) &= -\cos x \\
f'''(x) &= \sin x \\
f^4(x) &= \cos x \\
f^5(x) &= -\sin x
\end{aligned}$$

We observe that the successive derivatives occur in a cycle of length 4. In particular, $f^n(x) = \cos x$ whenever n is a multiple of 4.
Therefore,

$$f^{24}(x) = \cos x$$

Differentiating 3 more times we have

$$f^{27}(x) = \cos x$$

Clearly

$$f^{92}(x) = \cos x$$

8.8 Session Summary

For more material on this section check out [1, 2] or visit Logarithmic and exponential functions and hyperbolic and inverse hyperbolic functions. You can also watch logarithmic differentiation, exponential differentiation hyperbolic and inverse hyperbolic functions

8.9 Student Activity

Exercise

Logarithmic Functions

In the questions 1–10, find $\frac{d^2y}{dx}$.

1. $y = \ln(3x^2 + 2x)$
2. $y = (\ln x)^3$
3. $y = 7^x$
4. $y = \frac{x^3 \sqrt[3]{x-5}}{1+\sin^3 x}$
5. $y = \frac{\sqrt{\cos x}}{x^2 \sin x}$
6. $y = \ln \sqrt[3]{\cos x}$
7. $y = \ln(\ln x)$
8. $y = \frac{\ln x}{1+\ln x}$
9. $y = \frac{1}{2} \ln \frac{1+x}{1-x}$
10. $y = \sin x^{\tan x}$

Exponential Functions

In the questions 1–10, find $\frac{dy}{dx}$.

1. $y = e^{\sqrt{x^2+1}}$
2. $e^{2y} = x^2$
3. $y = e^{(x+1)}$
4. $e^{xy} - 3x + 3y^2 = 1$
5. $y = \frac{e^x - e^{-1}}{e^x + e^{-x}}$
6. $y = \tan^{-1}(e^x)$
7. $y = \ln \frac{e^x}{1+e^x}$
8. $y = \ln \sqrt{e^{2x} + e^[- 2x]}$
9. $y = \sin e^{-x}$
10. $xe^y + 2x - \ln(y+1) = 3$

Hyperbolic Functions

In the questions 1–12, find $\frac{dy}{dx}$.

1. $y = sech^3 x$
2. $y = \sinh x \cosh 4x$
3. $y = 4 \cosh \frac{x}{4}$
4. $y = \ln(\sinh x)$
5. $y = \cosh^2(5x)$
6. $y = \coth(\tan x)$
7. $y = x^4 \sinh x$
8. $x^2 \tanh y = \ln y$
9. $y = \cosh^2 x$
10. $y = \cosh 3x \sinh x$
11. If $\sinh x = -\frac{3}{4}$, find $\tanh x$
12. Evaluate $\cosh 0$

Inverse Hyperbolic Functions

1. Prove that

   (i) $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}), \quad x \geq 1$
   (ii) $\tanh^{-1} = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right), \quad -1 < x < 1$

2. In the questions (a)–(h), find $\frac{dy}{dx}$.

   (a) $y = \cosh^{-1} x$
   (b) $y = \coth^{-1} x$
   (c) $y = \sinh^{-1}(x^2)$
   (d) $y = \tanh^{-1}(2x - 3)$
   (e) $y = x \cosh^{-1}(3x)$
   (f) $y = \ln(\cosh^{-1} x)$
   (g) $y = \cosh^{-1}(\cos x)$
   (h) $y = x^2 \sinh^{-1}(x^5)$

Higher Order Derivatives

1. $y = x^{10-2x^8+13x}$, find y^4
2. $f(x) = (4x + 5)^8$, find $f'''(x)$
3. $y = \cos 2x$, find $f^{50} \cos 2x$
4. $f(x) = x \sin x$, find $D^{35} x \sin x$
5. Find y'' in (a) and (b)
   (a) $x^3 + y^3 = 1$
   (b) $x^2 + 6xy + y^2 = 8$
6. $f(x) = (2 - 3x)^{-\frac{1}{12}}$, find $f(0)$, $f'(0)$, $f''(0)$, $f'''(0)$,
7. $g(x) = \sec x$, find $g^4(\frac{\pi}{4})$

References

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