Contents

4 Session Four: Derivatives of Functions I

4.1 Session Objectives

By the end of this session, you should be able to:

(i) Define the derivative of a function.

(ii) Find the derivative of a function by first principles.

(iii) Apply diferent rules of differentiation in evaluating derivatives

4.2 Introduction

Having looked at limits, we now use the the knowledge gained to introduce the concept of derivative of a function. We define the derivative of a function and evaluate derivatives by first principles. We shall also discuss the properties and rules of derivatives.

Suppose as an engineer you are to make a tool box out of a rectangular plane metallic sheet. The main interest would be to have a box with as much space (volume) as possible. A business person's main objective is always to minimize on cost so as to get maximum profits. The rate of growth of bacteria is a differential equation that has a great importance in the medical world in making human insulin as it is in yogurt, bread, beer and wine making. These and many other areas require differentiation in achieving accurate results.

we use limits to introduce the concept of derivative of a function.

4.3 Definition of Derivative

Let $P(x, f(x))$ be any point on the graph of $f(x)$. Let $Q(x+h, f(x+h))$ be another point on the same graph (as shown in Figure 1)

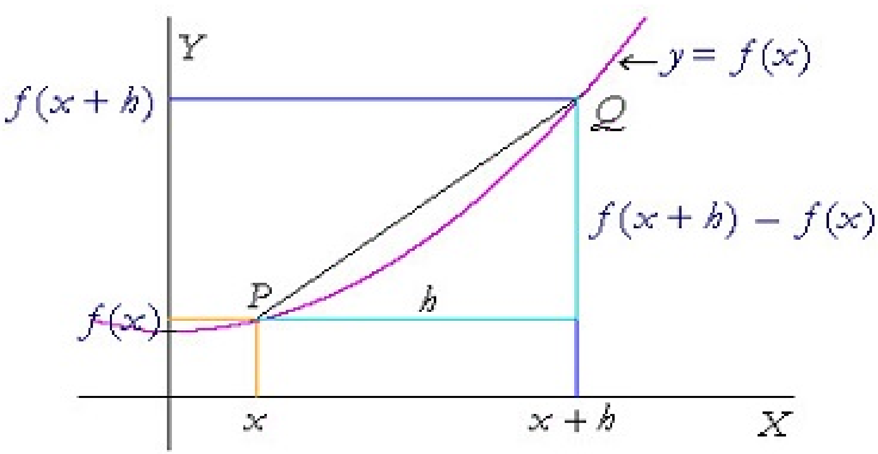

Figure 1: The Derivative

The slope of the secant line PQ is given by

$$m_{PQ} = \frac{f(x+h) - f(x)}{(x+h) - h} = \frac{f(x+h) - f(x)}{h}$$

Let m be the slope of the tangent line at point P.

$$m = \lim_{h \to 0} m_{PQ} = \lim_{x \to h} \frac{f(x+h) - f(x)}{h}$$

This means $m_{PQ} \to m$ as $h \to 0$.

Remark 4.3.1. *If the tangent line is vertical, the slope is undefined since the limit does not exist.*

Definition 4.3.1. *The derivative of a function f is another function f' (read "f prime") whose value at any number x is*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

If this limit exists, we say that f is differentiable at x. Finding a derivative is called differentiation; the part of calculus associated with the derivative is called differential calculus.

Evaluating the derivative of a function using limits is called **differentiation by first principles.**

Example 4.3.1. *If $f(x) = x^2$, find $f'(x)$.*
Solution:

$$\begin{aligned}
f'(x4) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{h(2x+h)}{h} \\
&= \lim_{h \to 0} 2x + h \\
&= \lim_{h \to 0} 2x
\end{aligned}$$

Example 4.3.2. *Find $f'(x)$ given $f(x) = \sqrt{x}$.*

Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \\ &= \lim_{h\to 0} \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

Example **4.3.3.** *Find the derivative of* $f(x) = \frac{1}{x}$.
Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h} \\ &= \lim_{h\to 0} \frac{\frac{x-x+h}{x(x+h)}}{h} \\ &= \lim_{h\to 0} \frac{h}{h(x+h)} \\ &= \lim_{h\to 0} \frac{1}{x(x+h)} \\ &= \frac{1}{x^2} \\ &= x^{-1} \end{aligned}$$

4.4 Equivalent Forms of Derivative

The derivative of $f(x)$ at $x = c$ can be given by

$$f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \text{or } f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$$

Example **4.4.1.** *Let* $f(x) = 13x - 6$. *Find* $f'(4)$.
Solution:

$$\begin{aligned} f'(4) &= \lim_{h\to 0} \frac{f(4+h)-f(4)}{h} \\ &= \lim_{h\to 0} \frac{[13(4+h)-6]-[13(4)-6]}{h} \\ &= \lim_{h\to 0} \frac{13h}{h} \\ &= \lim_{h\to 0} 13 \\ &= 13 \end{aligned}$$

Remark 4.4.1 (Differentiability Implies Continuity). *If a curve has a tangent line at a point, then that curve cannot take a jump or wiggle too badly at the point.*

Theorem 4.4.1. *(Differentiability Implies Continuity) If $f'(c)$ exists, then f is continuous at c.*

Proof. See [1, Pg. 102] □

The converse of Theorem 4.4.1 is false, i.e., if f is continuous at c, it is not necessarily differentiable at c (See [1, Pg. 103]).

4.5 Rules for Finding Derivatives

1. **Constant Funtion Rule:**
   If $f(x) = k$, where k is a constant, then for any x, $f'(x) = \frac{d}{dx}(k) = 0$.

2. **Identity Function Rule:**
   If $f(x) = x$, then $f'(x) = \frac{d}{dx}(x) = 1$.

3. **Power Rule:**
   If $f(x) = x^n$, where n is a positive integer, then $f'(x) = \frac{d}{dx}(x^n) = nx^{n-1}$.

4. **Constant Multiple:**
   If k is a constant and f is a differentiable function, then $(kf)'(x) = k \cdot f'(x)$.

5. **Sum Rule:**
   If f and g are differentiable functions, then $(f + g)'(x) = f'(x) + g'(x)$.

6. **Difference Rule:**
   If f and g are differentiable functions, then $(f - g)'(x) = f'(x) - g'(x)$.

7. **Product Rule:**
   If f and g are differentiable functions, then $(f \cdot g)'(x) = f(x)g'(x) + g(x)f'(x)$ or $\frac{d}{dx}(uv) = uv' + vu'$.

8. **Quotient Rule:**
   If f and g are differentiable functions, then $\left(\frac{f}{g}\right)(x) = \frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}$ or $\frac{d}{dx}(\frac{u}{v}) = \frac{vu'-uv'}{v^2}$.

Example 4.5.1. *Find* $\dfrac{dy}{dx}$ *given* $f(x) = 3x^4 - 2x^3 - 5x^2 + \pi x + \pi^2$.
Solution: $f'(x) = 12x^3 - 6x^2 - 10x + \pi$

Example 4.5.2. *Find* $f'(x)$ *given* $f(x) = (x^2 + 2)(x^3 + 1)$.
Solution: *Let* $u = (x^2 + 2)$ *and* $v = (x^3 + 1)$. *Then*

$$\begin{aligned} f'(x) &= uv' + vu' \\ &= (x^2 + 2)(3x^2) + (x^3 + 1)(2x) \\ &= = 3x^4 + 6x^2 + 3x^4 + 2x. \end{aligned}$$

Example 4.5.3. *Find $f'(x)$ given $f(x) = \dfrac{x+1}{x^2-2}$.*

Solution: *$u = x+1$ and $v = x^2-2$. Then*
$f'(x) = \dfrac{(x^2-2)(1)-(x+1)(2x)}{(x^2-2)^2} = \dfrac{-x^2-2x-2}{(x^2-2)^2}$

Example 4.5.4. *Find $\dfrac{dy}{dx}$ given $y = \dfrac{5x^2+2x-6}{3x-1}$*

Solution: *$u = 5x^2+2x-6$ and $v = 3x-1$. Then*

$$\begin{aligned}\frac{dy}{dx} &= \frac{(3x-1)(10x+2)-(5x^2+2x-6)(3)}{(3x-1)^2} \\ &= \frac{-x^2-2x-2}{(x^2-2)^2} \\ &= \frac{(3x-1)(10x+2)-(5x^2+2x-6)(3)}{(3x-1)^2} \\ &= \frac{15x^2-10x-16}{(3x-1)^2}\end{aligned}$$

4.6 Chain Rule Differentiation

Let $y = f(u)$ and $u = g(x)$. If g is differentiable at x and f is differentiable at $u = g(x)$, then the composite function $f \circ g$, defined by $(f \circ g)(x) = f(g(x))$, is differentiable at x and

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

or

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example 4.6.1. *If $y = (2x^2-4x+1)^{60}$, find $f'(x)$*

Solution: *Let $u = 2x^2-4x+1$ and $y = u^{60}$. Then $\dfrac{du}{dx} = 4x-4$ and $\dfrac{dy}{du} = 60u^{59}$ hence* $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = 60(2x^2-4x+1)^{59}(4x-4)$

Example 4.6.2. *Find $D_x(x^2+3x)^7$.*

Solution: *$u = (x^2+3x)$ and $y = u^7 \Rightarrow \dfrac{du}{dx} = 2x+3$ and $\dfrac{dy}{du} = 7u^6$. Therefore, $\dfrac{dy}{dx} =$* $7(2x+3)(x^2+3x)^6$

Example 4.6.3. *Find $f'(x)$ given $f(x) = \frac{1}{(2x-1)^3}$.*

Solution: *Writing $f(x) = (2x-1)^{-3}$, we have $u = (2x-1)$ and $y = u^{-3} \Rightarrow \dfrac{du}{dx} = 2$ and $\dfrac{dy}{du} = -3u^{-4}$. Therefore, $\dfrac{dy}{dx} = (-3)(2)(2x-1)^{-4}$*

Example 4.6.4. *Find* $\frac{dy}{dt}$ *in* $y = \left(\frac{t^3-2t+1}{t^4+3}\right)^{13}$

Solution: $u = \frac{t^3-2t+1}{t^4+3}$ *and* $y = u^{13} \Rightarrow \frac{du}{dt} = \frac{(t^4+3)(3t^2-2)-(t^3-2t+1)4t^3}{(t^4+3)^2} = \frac{-t^6+6t^4-4t^3+9t^2-6}{(t^4+3)^2}$ *and* $\frac{dy}{du} = 13u^{12}$. *Therefore,* $\frac{dy}{dt} = 13\left(\frac{t^3-2t+1}{t^4+3}\right)^{12}\frac{-t^6+6t^4-4t^3+9t^2-6}{(t^4+3)^2}$

4.7 Session Summary

Whenever one is required to find derivative by first principles, this means using the limit definition of a derivative.

It is important to know which rule is applicable in a given derivative problem. More than one rule may be required in a single derivative problem.

For more material on differentiation by first principles and rules of differentiation check out [1, 2] or visit differentiation. You can also watch the lecture video on differentiation by first principles, rules of differentiation.

4.8 Student Activity

Exercise

1. Use $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ to find the following derivatives:
   (a) $h(x) = \frac{2}{x}$
   (b) $f(x) = x^4$
   (c) $f(x) = \sqrt{3x}$
   (d) $f(x) = \frac{x-1}{x+1}$
2. Use the definition
$$f'(c) = \lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$
   to find the following derivatives:
   (a) $f'(1)$ if $f(x) = x^3$
   (b) $f'(3)$ if $f(t) = t^2 - t$
   (c) $f'(2)$ if $f(t) = 2(t^2)$
   (d) $f'(4)$ if $f(s) = \frac{1}{s-1}$
3. Find the derivatives of the following:
   (a) $y = \frac{3x-x^4}{x^2+1}$
   (b) $y = \sqrt{\frac{(1+x)^3}{x+2}}$
   (c) $y = \frac{x^2}{\sqrt{1+x^2}}$
   (d) $y = \frac{1}{3x^2+1}$
   (e) $y = \sqrt{\frac{1+x}{2+x}}$
   (f) $y\frac{100}{x^5}$
4. Find the derivatives of the following functions

(a) $y = (1+x)^{15}$

(b) $y = \cos^3\left(\frac{x^2}{1-x}\right)$

(c) $y = (x^2 - x + 1)^7$

(d) $y = (3x-2)^2(3-x^2)^2$

(e) $y = \frac{1}{(x+3)^5}$

(f) $y = \frac{1}{(3x^2-2x+1)^3}$

(g) $y = \left(\frac{x+1}{x-1}\right)^3$

(h) $y = (6x^3 - 4x)^{-2}$

(i) $y = \cos\left(\frac{3x^2}{x+2}\right)$

(j) $y = (2-3x^2)^4(x^7+3)^3$

References

[1] E. Purcell D. Varberg and S. Rigdon. *Calculus*. Pearson Education, Inc., 9 edition, 2006. ISBN-13 : 978-0132306331.

[2] J. Stewart. *Calculus*. Cengage Learning 20 Channel Center Street, Boston, MA 02210, USA, 8 edition, 2016. ISBN-13: 978-1-305-27176-0.