Session Two: Limits

2.0 Session objectives

By the end of the session, you should be able to:

i. Define the limit of a function.

ii. Evaluate one-sided limits of a function, unbounded limits, Two-sided limits, Limits of trigonometric functions, Limits at infinity.

2.1 Informal definition of limits

If f is some function then $\lim_{x\to a} f(x) = L$ is read "the limit of $f(x)$ as x approaches a is L". It means that if you choose values of x which are close but not equal to a, then $f(x)$ will be close to the value L; moreover, $f(x)$ gets closer and closer to L as x gets closer and closer to a.

The following alternative notation is sometimes used

$f(x) \to L$ as $x \to a$; (read "$f(x)$ approaches L as x approaches a" or "$f(x)$ goes to L is x goes to a".)

Example: If $f(x) = x + 3$ then

$\lim_{x\to 4} f(x) = 7,$

is true, because if you substitute numbers x close to 4 in $f(x) = x + 3$ the result will be close to 7.

Example: **Substituting numbers to guess a limit**. What (if anything) is

$\lim_{x \to 2} \frac{x^2-2x}{x^2-4}$?

Here $f(x) = \frac{x^2-2x}{x^2-4}$ and $a = 2$.

We first try to substitute $x = 2$, but this leads to

$$f(2) = \frac{2^2 - 2.2}{2^2 - 4} = \frac{0}{0}$$

which does not exist. Next we try to substitute values of x close but not equal to 2. The table below suggests that $f(x)$ approaches 0.5.

x	***f(x)***
3.000000	0.600000
2.500000	0.555556
2.100000	0.512195
2.010000	0.501247
2.001000	0.500125

Example: Substituting numbers can suggest the wrong answer. The previous example shows that our first definition of “limit" is not very precise, because it says “x close to a," but how close is close enough? Suppose we had taken the function

$$g(x) == \frac{101000x}{100000x + 1}$$

and we had asked for the limit $\lim_{x \to 0} g(x)$.

Then substitution of some “small values of x" could lead us to believe that the limit is 1.00…

x	$g(x)$
1.000000	1.009990
0.500000	1.009980
0.100000	1.009899
0.010000	1.008991
0.001000	1.000000

Only when you substitute even smaller values do you find that the limit is 0 (zero)!

2.2 **The formal definition of limit**

L is the limit of $f(x)$ as $x \to a$, if

i. $f(x)$ need not be defined at $x = a$, but it must be defined for all other x in some interval which contains a.

ii. for every $\varepsilon > 0$ one can find a $\delta > 0$ such that for all x in the domain of f

$|x - a| < \delta$ implies $|f(x) - L| < \varepsilon$.

Example: Show that $\lim_{x \to 5} 2x + 1 = 11$.

Solution

We have $f(x) = 2x + 1$, $a = 5$ and $L = 11$, and the question we must answer is "how close should x be to 5 if want to be sure that $f(x) = 2x + 1$ differs less than ε from $L = 11$?"

To figure this out we try to get an idea of how big $|f(x) - L|$is:

$$|f(x) - L| = |(2x + 1) - 11| = |2x - 10| = 2.|x - 5| = 2.|x - a|$$

So, if $2.|x - a| < \varepsilon$ then we have$|f(x) - L| < \varepsilon$, i.e.

if $|x - a| < \frac{1}{2}\varepsilon$ then $|f(x) - L| < \varepsilon$

We can therefore choose $\delta = \frac{1}{2}\varepsilon$. No matter what $\varepsilon > 0$ we are given our δ will also be positive, and if $|x - 5| < \delta$ then we can guarantee $|(2x + 1) - 11| < \varepsilon$.

That shows that $\lim_{x \to 5} 2x + 1 = 11$.

Example: Show that $\lim_{x \to 1} x^2 = 1$

Solution

We have $f(x) = x^2$, $a = 1$, $L = 1$, and again the question is, "how small should $|x - 1|$be to guarantee $|x^2 - 1| < \varepsilon$?"

We begin by estimating the difference $|x^2 - 1|$

$$|x^2 - 1| = |(x - 1)(x + 1)| = |x - 1|.|x + 1|$$

As x approaches 1 the factor $|x - 1|$ becomes small, and if the other factor $|x + 1|$were a constant then we could find δ as before, by dividing ε by that constant.

Here is a trick that allows you to replace the factor $|x + 1|$with a constant. We hereby agree *that we always choose our δ so that $\delta \le 1$*. If we do that, then we will always have

$|x - 1| < \delta \le 1$, i.e. $|x - 1| < 1$,

and x will always be beween 0 and 2. Therefore

$$|x^2 - 1| = |x - 1|.|x + 1| < 3|x - 1|$$

If we now want to be sure that $|x^2 - 1| < \varepsilon$, then this calculation shows that we should require $3|x - 1| < \varepsilon$, i.e. $|x - 1| < \frac{1}{3}\varepsilon$. So we should choose $\delta \le \frac{1}{3}\varepsilon$. We must also live up to our promise never to choose $\delta > 1$, so if we are handed an ε for which $\frac{1}{3}\varepsilon > 1$, then we choose $\delta = 1$ instead of $\delta = \frac{1}{3}\varepsilon$. To summarize, we are going to choose δ the smaller of 1 and $\frac{1}{3}\varepsilon$.

We have shown that if you choose δ this way, then $|x - 1| < \delta$ implies $|x^2 - 1| < \varepsilon$, no matter what $\varepsilon > 0$ is.

The expression "the smaller of a and b" shows up often, and is abbreviated to $min(a, b)$. We could therefore say that in this problem we will choose δ to be

$\delta = \min\left(1, \frac{1}{3}\varepsilon\right)$.

Example: Show that $\lim_{x \to 4} \frac{1}{x} = \frac{1}{4}$

Solution:

We apply the definition with $a = 4$, $L = \frac{1}{4}$ and $f(x) = \frac{1}{x}$.

Thus, for any $\varepsilon > 0$ we try to show that if $|x - 4|$is small enough then one has $\left|f(x) - \frac{1}{4}\right| < \varepsilon$.

We begin by estimating $\left|f(x) - \frac{1}{4}\right|$ in terms of $|x - 4|$:

$$\left|f(x) - \frac{1}{4}\right| = \left|\frac{1}{x} - \frac{1}{4}\right| = \left|\frac{4 - x}{4x}\right| = \frac{|x - 4|}{|4x|} = \frac{1}{|4x|}|x - 4|$$

As before, things would be easier if $\frac{1}{|4x|}$ were a constant. To achieve that we again agree not to take $\delta > 1$.

If we always have $\delta \leq 1$, then we will always have $|x - 4| < 1$, and hence $3 < x < 5$. How large can $\frac{1}{|4x|}$ be in this situation? Answer: the quantity $\frac{1}{|4x|}$ increases as you decrease x, so if $3 < x < 5$ then it will never be larger than $\frac{1}{|4.3|} = \frac{1}{12}$

We see that if we never choose $\delta > 1$, we will always have

$\left|f(x) - \frac{1}{4}\right| \leq \frac{1}{12}|x - 4|$ for $|x - 4| < \delta$.

To guarantee that $\left|f(x) - \frac{1}{4}\right| < \varepsilon$ we could therefore require

$\frac{1}{12}|x - 4| < \varepsilon$, i.e. $|x - 4| < 12\varepsilon$

Hence if we choose $\delta = 12\varepsilon$ or any smaller number, then $|x-4| < \delta$ implies $\left|f(x) - \frac{1}{4}\right| < \varepsilon$. Of course we have to honor our agreement never to choose $\delta > 1$, so our choice of δ is

δ = the smaller of 1 and $12\varepsilon = \min(1, 12\varepsilon\ )$

2.3 Left and Right limits

When we let "x approach a" we allow x to be both larger or smaller than a, as long as x gets close to a. If we explicitly want to study the behaviour of $f(x)$ as x approaches a through values larger than a, then we write

$\lim_{x \searrow a} f(x)$ or $\lim_{x \to a+} f(x)$ or $\lim_{x \to a+0} f(x)$ or $\lim_{x \to a, x > a} f(x)$.

All four notations are in use. Similarly, to designate the value which $f(x)$ approaches as x approaches a through values below a one writes

$\lim_{x \nearrow a} f(x)$ or $\lim_{x \to a-} f(x)$ or $\lim_{x \to a-0} f(x)$ or $\lim_{x \to a, x < a} f(x)$.

2.3.1 Definition of right-limit.

Let f be a function. Then

$\lim_{x \searrow a} f(x) = L$

means that for every $\varepsilon > 0$ one can find a $\delta > 0$ such that

$$a < x < a + \delta \Rightarrow |f(x) - L| < \varepsilon$$

holds for all x in the domain of f.

2.3.2 Definition of left-limit.

The left-limit, i.e. the one-sided limit in which x approaches a through values less than a is defined

As

Let f be a function. Then

$$\lim_{x \nearrow a} f(x) = L$$

means that for every $\varepsilon > 0$ one can find a $\delta > 0$ such that

$$a + \delta < x < a \Rightarrow |f(x) - L| < \varepsilon$$

holds for all x in the domain of f.

The following theorem tells you how to use one-sided limits to decide if a function $f(x)$ has a limit at $x = a$.

2.3.3 Theorem. If both one-sided limits

$\lim_{x \searrow a} f(x) = L_+$, and $\lim_{x \nearrow a} f(x) = L_-$

exist, then

$\lim_{x \to a} f(x)$ exists $\Leftrightarrow L_+ = L_-$

In other words, if a function has both left- and right-limits at some $x = a$, then that function has a limit at $x = a$ if the left- and right-limits are equal.

2.4.0 Limits at infinity

Instead of letting x approach some finite number, one can let x become "larger and larger" and ask what happens to $f(x)$. If there is a number L such that $f(x)$ gets arbitrarily close to L if one chooses x sufficiently large, then we write

$\lim_{x \to \infty} f(x) = L$, or $\lim_{x \uparrow \infty} f(x) = L$, or $\lim_{x \nearrow \infty} f(x) = L$.

("The limit for x going to infinity is L.")

Example: Limit of $\frac{1}{x}$.

The larger you choose x, the smaller its reciprocal $\frac{1}{x}$ becomes.

Therefore, it seems reasonable to say

$\lim_{x \to \infty} \frac{1}{x} = 0.$

2.4.1 Definition of limit at ∞.

Let f be some function which is defined on some interval $x_0 < x < \infty$.

If there is a number L such that for every $\varepsilon > 0$ one can find an A such that

$$x > A \Rightarrow |f(x) - L| < \varepsilon$$

for all x, then we say that the limit of $f(x)$ for $x \to \infty$ is L.

The definition is very similar to the original definition of the limit. Instead of δ which specifies how close x should be to a, we now have a number A which says how large x should be, which is a way of saying “how close x should be to infinity."

Example Prove that $\lim_{x\to\infty} \frac{1}{x} = 0$

Solution

We apply the definition to $f(x) = \frac{1}{x}, L = 0.$

For given $\varepsilon > 0$ we need to show that

$\left|\frac{1}{x} - \mathrm{L}\right| < \varepsilon$ for all $x > A$

provided we choose the right A.

How do we choose A? A is not allowed to depend on x, but it may depend on ε.

If we assume for now that we will only consider positive values of x, we have

$\frac{1}{x} < \varepsilon$ which is equivalent to $x > \frac{1}{\varepsilon}$

This tells us how to choose A. Given any positive ε, we will simply choose

$A = \frac{1}{\varepsilon}$.

Then one has $\left|\frac{1}{x} - 0\right| = \frac{1}{x} < \varepsilon$ for all $x > A$. Hence we have proved that $\lim_{x\to\infty} \frac{1}{x} = 0$

2.5. Properties of the Limit

The precise definition of the limit is not easy to use, and fortunately we won't use it very often in this class. Instead, there are a number of properties that limits have which allow you to compute them without having to resort to “epsiloncy."

a) **Limits of constants and of x**.

If a and c are constants, then $\lim_{x\to a} c = c$ and $\lim_{x\to a} x = c$

b) **Limits of sums, products and quotients**.

Let F_1 and F_2 be two given functions whose limits for $x \to a$ we know,

$\lim_{x\to a} F_1(x) = L_1$ and $\lim_{x\to a} F_2(x) = L_2$

Then

$$\lim_{x\to a}(F_1(x) + F_2(x)) = L_1 + L_2$$

$$\lim_{x\to a}(F_1(x) - F_2(x)) = L_1 - L_2$$

$$\lim_{x\to a}(F_1(x).F_2(x)) = L_1.L_2$$

Finally, if $\lim_{x\to a} F_2(x) \neq 0$,

$$\lim_{x\to a}\frac{F_1(x)}{F_2(x)} = \frac{L_1}{L_2}$$

2.6 Examples of limit computations

1. Find $\lim_{x\to 2} x^2$

Solution

$\lim_{x\to 2} x^2$ =$\lim_{x\to 2} x.x = \left(\lim_{x\to 2} x\right)\left(\lim_{x\to 2} x\right) = 2.2 = 4$

2. Find $\lim_{x \to 2} x^3$

Solution

$\lim_{x \to 2} x^3 = \lim_{x \to 2} x.x^2 = \left(\lim_{x \to 2} x\right)\left(\lim_{x \to 2} x^2\right) = 2.4 = 8$

3. $\lim_{x \to 2} x^2 - 1$

Solution

$\lim_{x \to 2} x^2 - 1 = \lim_{x \to 2} x^2 - \lim_{x \to 2} 1 = 4 - 1 = 3$

4. $\lim_{x \to 2} x^3 - 1$

Solution

$\lim_{x \to 2} x^3 - 1 = \lim_{x \to 2} x^3 - \lim_{x \to 2} 1 = 8 - 1 = 7$

5. $\lim_{x \to 2} \frac{x^3-1}{x^2-1}$

Solution

$$\lim_{x \to 2} \frac{x^3 - 1}{x^2 - 1} = \frac{\lim_{x \to 2} x^3 - 1}{\lim_{x \to 2} x^2 - 1} = \frac{7}{3}$$

Note: *Always check that the denominator ("L2") is not zero.*

6. Compute $\lim_{x \to 2} \frac{x^2-2x}{x^2-4}$

Solution

$$\lim_{x \to 2} x^2 - 2x = \lim_{x \to 2} x^2 - \lim_{x \to 2} (2x) = 4 - 4 = 0$$

$$\lim_{x \to 2} x^2 - 4 = \lim_{x \to 2} x^2 - \lim_{x \to 2} 4 = 4 - 4 = 0$$

Thus, $\lim_{x \to 2} \frac{x^2-2x}{x^2-4} = \frac{\lim_{x \to 2} x^2-2x}{\lim_{x \to 2} x^2-4} = \frac{0}{0}$

The denominator is zero and the result doesn't mean anything anyway.

We have to do something else.

The function we are dealing with is a ***rational function***, which means that it is the quotient of two polynomials. For such functions there is an algebra trick which always allows you to compute the limit even if you first get $\frac{0}{0}$. The thing to do is to simplify the fraction.

In our case we have

$x^2 - 2x = x(x-2)$, $x^2 - 4 = (x+2)(x-2)$

so that

$$\lim_{x\to 2}\frac{x^2-2x}{x^2-4} = \lim_{x\to 2}\frac{x(x-2)}{(x+2)(x-2)} = \lim_{x\to 2}\frac{x}{x+2} = \frac{\lim\limits_{x\to 2} x}{\lim\limits_{x\to 2} x+2} = \frac{2}{4} = \frac{1}{2}$$

7. Find $\lim\limits_{x\to 2}\sqrt{x}$

   Solution

Suppose that there is a number L with

$$\lim_{x\to 2}\sqrt{x} = L$$

Then

$$L^2 = \left(\lim_{x\to 2}\sqrt{x}\right)\left(\lim_{x\to 2}\sqrt{x}\right) = \lim_{x\to 2}\sqrt{x}\,.\sqrt{x} = \lim_{x\to 2} x = 2$$

In other words, $L^2 = 2$, and hence L must be either $\sqrt{2}$ or $-\sqrt{2}$. We can reject the latter because whatever x does, its square root is always a positive number, and hence it can never “get close to" a negative number like $-\sqrt{2}$

Our conclusion: if the limit exists, then

$\lim\limits_{x\to 2}\sqrt{x} = \sqrt{2}$

8. Find $\lim_{x \to 2} \frac{\sqrt{x}-\sqrt{2}}{x-2}$

Solution

The function is a fraction whose numerator and denominator vanish when $x = 2$, i.e. the limit is of the form $\frac{0}{0}$

We factor numerator and denominator:

$$\frac{\sqrt{x}-\sqrt{2}}{x-2} = \frac{\sqrt{x}-\sqrt{2}}{\left(\sqrt{x}-\sqrt{2}\right)\left(\sqrt{x}+\sqrt{2}\right)} = \frac{1}{\sqrt{x}+\sqrt{2}}$$

Now one can use the limit properties to compute

$$\lim_{x \to 2} \frac{\sqrt{x}-\sqrt{2}}{x-2} = \lim_{x \to 2} \frac{1}{\sqrt{x}+\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

2.7 Limit as $x \to \infty$ of rational functions.

A rational function is the quotient of two polynomials,

$$R(x) = \frac{a_n x^n + \cdots + a_1 x + a_0}{b_n x^m + \cdots + b_1 x + b_0}$$

To find $\lim_{x \to \infty} R(x)$ divide numerator and denominator by x^m (the highest power of x occurring in the denominator).

Example: Compute $\lim_{x \to \infty} \frac{3x^2+3}{5x^2+7x-39}$

Solution

Divide the numerator and the denominator by x^2, and you get

$$\lim_{x \to \infty} \frac{3x^2+3}{5x^2+7x-39} = \lim_{x \to \infty} \frac{3+\frac{3}{x^2}}{5+\frac{7}{x}-\frac{39}{x^2}}$$

$$= \frac{\lim_{x\to\infty} 3 + \frac{3}{x^2}}{\lim_{x\to\infty} 5 + \frac{7}{x} - \frac{39}{x^2}} = \frac{3}{5}$$

Note: $\lim_{x\to\infty} \frac{3}{x^2} = \lim_{x\to\infty} 3\left(\frac{1}{x}\right)^2 = \left(\lim_{x\to\infty} 3\right)\left(\lim_{x\to\infty} \frac{1}{x}\right)^2 = 3.0^2 = 0$

$\lim_{x\to\infty} \frac{7}{x} = \lim_{x\to\infty} 3\left(\frac{1}{x}\right) = \left(\lim_{x\to\infty} 7\right)\left(\lim_{x\to\infty} \frac{1}{x}\right) = 7.0 = 0$

$\lim_{x\to\infty} \frac{39}{x^2} = \lim_{x\to\infty} 39\left(\frac{1}{x}\right)^2 = \left(\lim_{x\to\infty} 39\right)\left(\lim_{x\to\infty} \frac{1}{x}\right)^2 = 39.0^2 = 0$

Example: Compute $\lim_{x\to\infty} \frac{x}{x^3+5}$

Solution

Divide numerator and denominator by x^3.

$$\lim_{x\to\infty} \frac{x}{x^3+5} = \lim_{x\to\infty} \frac{\frac{1}{x^2}}{1+\frac{5}{x^3}} = \frac{\lim_{x\to\infty} \frac{1}{x^2}}{\lim_{x\to\infty} 1 + \frac{5}{x^3}} = \frac{0}{1} = 0.$$

2.8 When limits fail to exist

2.8.1 Definition. If there is no number L such that $\lim_{x\to a} f(x) = L$, then we say that the limit $\lim_{x\to a} f(x)$ does not exist.

2.8.2 The sign function near $= 0$.

The “sign function" is defined by

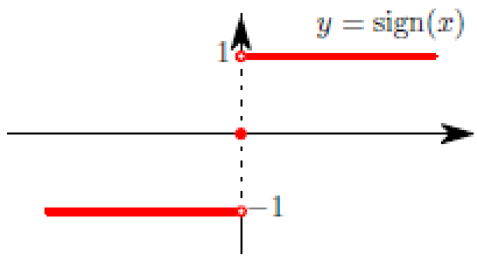
$$sign(x) = \begin{cases} -1 \text{ for } x < 0 \\ 0 \quad \text{for } x = 0 \\ 1 \text{ for } x > 0 \end{cases}$$

Note that “the sign of zero" is defined to be zero. But does the sign function have a limit at $x = 0$, i.e. does $\lim_{x\to 0} sign(x)$ exist? And is it also zero? The answers are no and no, and here is why: suppose that for some number L one had

$$\lim_{x\to 0} sign(x) = L\,,$$

then since for arbitrary small positive values of x one has $sign(x) = +1$ one would think that $L = +1$. But for arbitrarily small negative values of x one has $sign(x) = -1$, so one would conclude that $L = -1$. But one number L can't be both +1 and -1 at the same time, so there is no such L, i.e. there is no limit.

$\lim_{x \to 0} sign(x)$ does not exist.



The Sign function

In this example the one-sided limits do exist, namely,

$\lim_{x \searrow 0} sign(x) = 1$ and $\lim_{x \nearrow 0} sign(x) = -1$

All this says is that when x approaches 0 through positive values, its sign approaches +1, while if x goes to 0 through negative values, then its sign approaches -1.

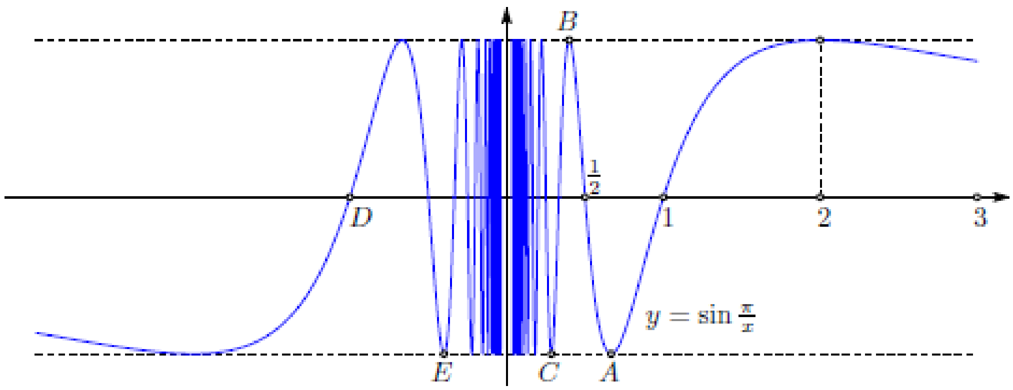
2.8.3 The example of the backward sine.

Contemplate the limit as $x \to 0$ of the "backward sine," i.e.

$$\lim_{x \to 0} sin\left(\frac{\pi}{x}\right)$$

When $x = 0$ the function $f(x) = sin\left(\frac{\pi}{x}\right)$ is not defined, because its definition involves division by x. What happens to $f(x)$ as $x \to 0$? First, $\frac{\pi}{x}$ becomes larger and larger ("goes to infinity") as $x \to 0$. Then, taking the sine, we see that $sin\left(\frac{\pi}{x}\right)$ oscillates between +1 and -1 infinitely often as

$x \to 0$. This means that $f(x)$ gets close to any number between -1 and +1 as $x \to 0$, but that the function $f(x)$ never stays close to any particular value because it keeps oscillating up and down.



Graph of $y = sin\left(\frac{\pi}{x}\right)$ for $-3 < x < 3, x \neq 0$

Here again, the limit $\lim_{x \to 0} f(x)$ does not exist. We have arrived at this conclusion by only considering what $f(x)$ does for small positive values of x. So the limit fails to exist in a stronger way than in the example of the sign-function. There, even though the limit didn't exist, the one-sided limits existed. In the present example we see that even the one-sided limit

$$\lim_{x \searrow 0} sin\left(\frac{\pi}{x}\right)$$

does not exist.

2.8.4 Trying to divide by zero using a limit.

The expression $\frac{1}{0}$ is not defined, but what about

$\lim_{x \to 0} \frac{1}{x}$?

This limit also does not exist. Here are two reasons:

Note that if you divide by a small number you get a large number, so as $x \searrow 0$ the quotient $\frac{1}{x}$ will not be able to stay close to any particular finite number, and the limit can't exist.

The limit can't exist, because, suppose that there were an number L such that

$\lim\limits_{x\to 0}\frac{1}{x} = L.$

Then

$$\lim_{x\to 0}\left(\frac{1}{x}.x\right) = \left(\lim_{x\to 0}\frac{1}{x}\right).\left(\lim_{x\to 0}x\right) = L.0 = 0$$

On the other hand $\frac{1}{x}.x = 1$ so the above limit should be 1! A number can't be both 0 and 1 at the same time, so we have a contradiction. The assumption that $\lim\limits_{x\to 0}\frac{1}{x}$ exists is to blame, so it must go.

2.8.5 Using limit properties to show a limit does not exist.

If $\lim\limits_{x\to a}g(x) + h(x)$ does not exist then either $\lim\limits_{x\to a}g(x)$ or $\lim\limits_{x\to a}h(x)$ does not exist (or both limits fail to exist).

For instance, the limit

$\lim\limits_{x\to 0}\frac{1}{x} - x$ can't exist, for if it did, then the limit

$\lim\limits_{x\to 0}\frac{1}{x} = \lim\limits_{x\to 0}\left(\frac{1}{x} - x + x\right) = \lim\limits_{x\to 0}\left(\frac{1}{x} - x\right) + \lim\limits_{x\to 0}x$

would also have to exist, and we know $\lim\limits_{x\to 0}\frac{1}{x}$ doesn't exist.

2.8.6 Limits at ∞ which don't exist.

If you let x go to ∞, then x will not get “closer and closer" to any particular number L, so it seems reasonable to guess that

$\lim\limits_{x\to\infty}x$ does not exist.

To prove this, let's consider

$L = \lim\limits_{x\to\infty}\frac{x^2+2x-1}{x+2}$

we divide numerator and denominator by the highest power in the denominator (i.e. x)

$L = \lim\limits_{x\to\infty}\frac{x+2-\frac{1}{x}}{1+\frac{2}{x}}$

Here the denominator has a limit (1), but the numerator does not, for if $\lim_{x\to\infty} x + 2 - \frac{1}{x}$ existed then,

Since $\lim_{x\to\infty}\left(2 - \frac{1}{x}\right) = 2$ exists,

$\lim_{x\to\infty} x = \lim_{x\to\infty}\left[\left(x + 2 - \frac{1}{x}\right) - \left(2 - \frac{1}{x}\right)\right]$ would also have to exist, and $\lim_{x\to\infty} x$ doesn't exist.

Note that L is the limit of a fraction in which the denominator has a limit, but the numerator does not. In this situation the limit L itself can never exist. If it did, then

$\lim_{x\to\infty}\left(x + 2 - \frac{1}{x}\right) = \lim_{x\to\infty}\frac{x+2-\frac{1}{x}}{1+\frac{2}{x}}.\left(1 + \frac{2}{x}\right)$

would also have to have a limit.

2.9 Limits and Inequalities

This section has two theorems which let you compare limits of different functions. The properties in these theorems are not formulas that allow you to compute limits. Instead, they allow you to reason about limits, i.e. they let you say that this or that limit is positive, or that it must be the same as some other limit which you find easier to think about.

2.9.1 Theorem: Let f and g be functions whose limits for $x \to a$ exist, and assume that $f(x) \le g(x)$ holds for all x. Then

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$$

A useful special case arises when you set $f(x) = 0$. The theorem then says that if a function g never has negative values, then its limit will also never be negative.

2.9.2 The Sandwich Theorem: Suppose that

$$f(x) \le g(x) \le h(x)$$

$(for\ all\ x)$ and that

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x)$$

Then

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \lim_{x \to a} h(x)$$

The theorem is useful when you want to know the limit of g, and when you can sandwich it between two functions f and h whose limits are easier to compute.

Example: **A Backward Cosine Sandwich**. The Sandwich Theorem says that if the function $g(x)$ is sandwiched between two functions $f(x)$ and $h(x)$ and the limits of the outside functions f and h exist and are equal, then the limit of the inside function g exists and equals this common value. For example

$$-|x| \leq x cos \frac{1}{x} \leq |x|$$

since the cosine is always between -1 and 1. Since

$$\lim_{x \to 0} -|x| = \lim_{x \to 0} |x| = 0$$

the sandwich theorem tells us that

$$\lim_{x \to 0} x cos \frac{1}{x} = 0$$

Note that the limit $\lim_{x \to 0} cos\left(\frac{1}{x}\right)$ does not exist, multiplying with x changed that.

2.10 Two Limits in Trigonometry

For small angles θ one has

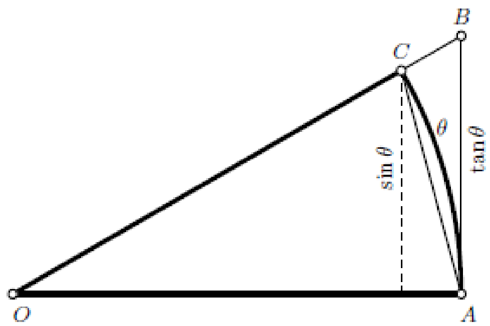
$sin\,\theta \approx \theta$ and $cos\,\theta \approx 1 - \frac{1}{2}\theta^2$

We will use these limits when we compute the derivatives of Sine, Cosine and Tangent.

Theorem: $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

Proof

Consider the circular wedge OAC below that contains triangle OAC and is contained in the right triangle OAB.



We begin by only considering positive angles, $0 < \theta < \frac{\pi}{2}$.

Since the wedge OAC contains the triangle OAC its area must be larger. The area of the wedge is $\frac{1}{2}\theta$ and the area of the triangle is $\frac{1}{2}\sin\theta$, we find that

$0 < \sin\theta < \theta$ for $0 < \theta < \frac{\pi}{2}$

The Sandwich Theorem implies that

$$\lim_{\theta \searrow 0} \sin\theta = 0.$$

Moreover, we also have

$$\lim_{\theta \searrow 0} \cos\theta = \lim_{\theta \searrow 0} \sqrt{1 - \sin^2\theta} = 1.$$

.Next we compare the areas of the wedge OAC and the larger triangle OAB. Since OAB has area $\frac{1}{2}\tan\theta$

we find that

$\theta < \tan\theta$ for $0 < \theta < \frac{\pi}{2}$

Since $tan\theta = \frac{sin\theta}{cos\theta}$

we can multiply with $cos\theta$ and divide by θ to get

$cos\theta < \frac{sin\theta}{\theta}$ for $0 < \theta < \frac{\pi}{2}$

$$cos\theta < \frac{sin\theta}{\theta} < 1$$

The Sandwich Theorem gives

$$\lim_{\theta \searrow 0} \frac{sin\theta}{\theta} = 1$$

This is a one-sided limit.

Example: Show that

$$\lim_{\theta \to 0} \frac{1 - cos\theta}{\theta^2} = \frac{1}{2}$$

Solution

This follows from$sin^2\theta + cos^2\theta = 1$. Namely,

$$\frac{1 - cos\theta}{\theta^2} = \frac{1}{1 + cos\theta} \frac{1 - cos^2\theta}{\theta^2}$$

$$= \frac{1}{1 + cos\theta} \frac{sin^2\theta}{\theta^2}$$

$$= \frac{1}{1 + cos\theta} \left\{\frac{sin\theta}{\theta}\right\}^2$$

Since $cos\theta \to 1$ and $\frac{sin\theta}{\theta} \to 1$ as $\theta \to 0$, it follows that

$$\frac{1}{1 + cos\theta} \left\{\frac{sin\theta}{\theta}\right\}^2 = \frac{1}{1 + 1} \{1\}^2 = \frac{1}{2}$$

2.11 Student's Activity

1. Use the $\varepsilon - \delta$ definition to prove the following limits:

   a) $\lim_{x \to 1} 2x - 4 = 6$

   b) $\lim_{x \to 3} x^3 = 27$

   c) $\lim_{x \to 3} \sqrt{x + 6} = 3$

   d) $\lim_{x \to 2} \frac{1+x}{4+x} = \frac{1}{2}$

   e) $\lim_{x \to 3} \frac{x}{6-x} = 1$

2.