POPULATION GROWTH

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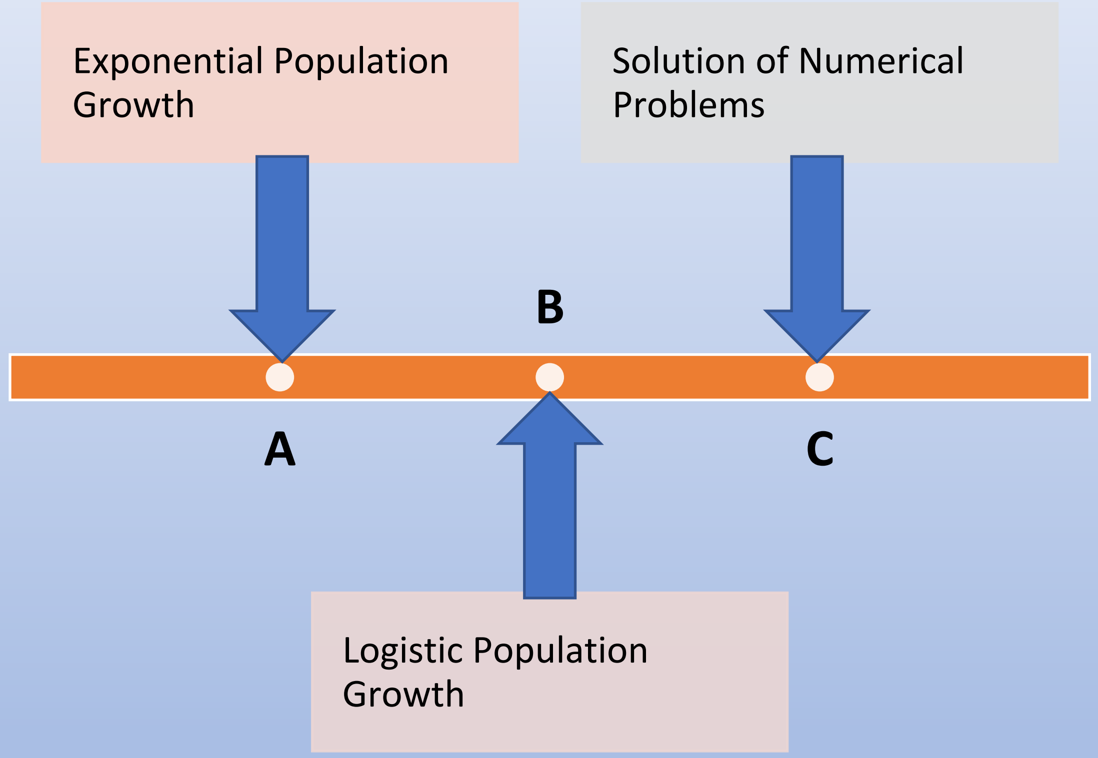

Exponential Growth of Population

The exponential growth of population is the **change of population with time and it is directly proportional to the size of the population at that time**.

If population size is denoted by 'N' and time by 't', then population growth i.e., rate of change of population with respect to time, can be expressed as –

$$\frac{dN}{dt} \propto N$$

$$\text{or, } \frac{dN}{dt} = RN \quad [R = \text{proportionality const.} = \text{rate const.}]$$

$$\text{or, } \frac{dN}{N} = Rdt$$

Integrating the above equation, we get

$$\int_{N_0}^{N_t} \frac{dN}{N} = R\int_0^t dt \qquad \begin{bmatrix} \text{at, time } 't' = 0,\ N = N_0 \\ \& \text{ time } 't' = t,\ N = N_t \end{bmatrix}$$

$$\text{or, } \left[\ln \frac{N_t}{N_0}\right] = R[t - 0]$$

$$\therefore N_t = N_0 e^{Rt}$$

If we plot the graph of N Vs t, it would be

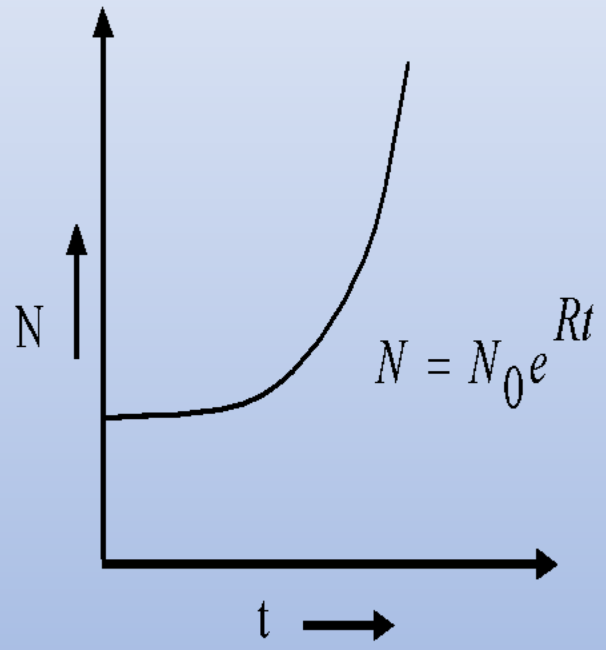

Fig: 1.2 Graph of Population Vs Time in exponential growth

Doubling Time of Population

The time required for the population becomes double of its present size (N) when it grows exponentially and growth rate is constant through out, is called 'Doubling time' (t_d).

We know that,

$$N_t = N_0 e^{Rt}$$

$$\therefore 2N_0 = N_0 e^{Rt_d} \quad \left[\begin{array}{l}\text{at doubling time } (t_d), \text{ the initial population} \\ (N_0) \text{ becomes double } 2N_0\end{array}\right]$$

or, $2 = e^{Rt_d}$

or, $\ln 2 = Rt_d$

$$\therefore t_d = \frac{\ln 2}{R} = \frac{0.693}{R}$$

If, growth rate is expressed as percentage, then

$$t_d = \frac{0.693}{R\%} \approx \frac{70}{R\%}$$

Logistic Growth of Population

Population growth is not always exponential. Growth of population usually depends on availability of natural resources. With the finite natural resources, a species cannot support any population beyond a certain size. There is an upper limit to the number of individual population which environment can support. This is called the '**carrying capacity**' of environment. Population growth of this kind in environment is known as **logistic growth**.

In such case, if we plot a graph of population (N) against the time (t), the curve so obtained shall be **'S' shaped**. This kind of curve is called '**sigmoid**' or '**logistic**' **curve**.

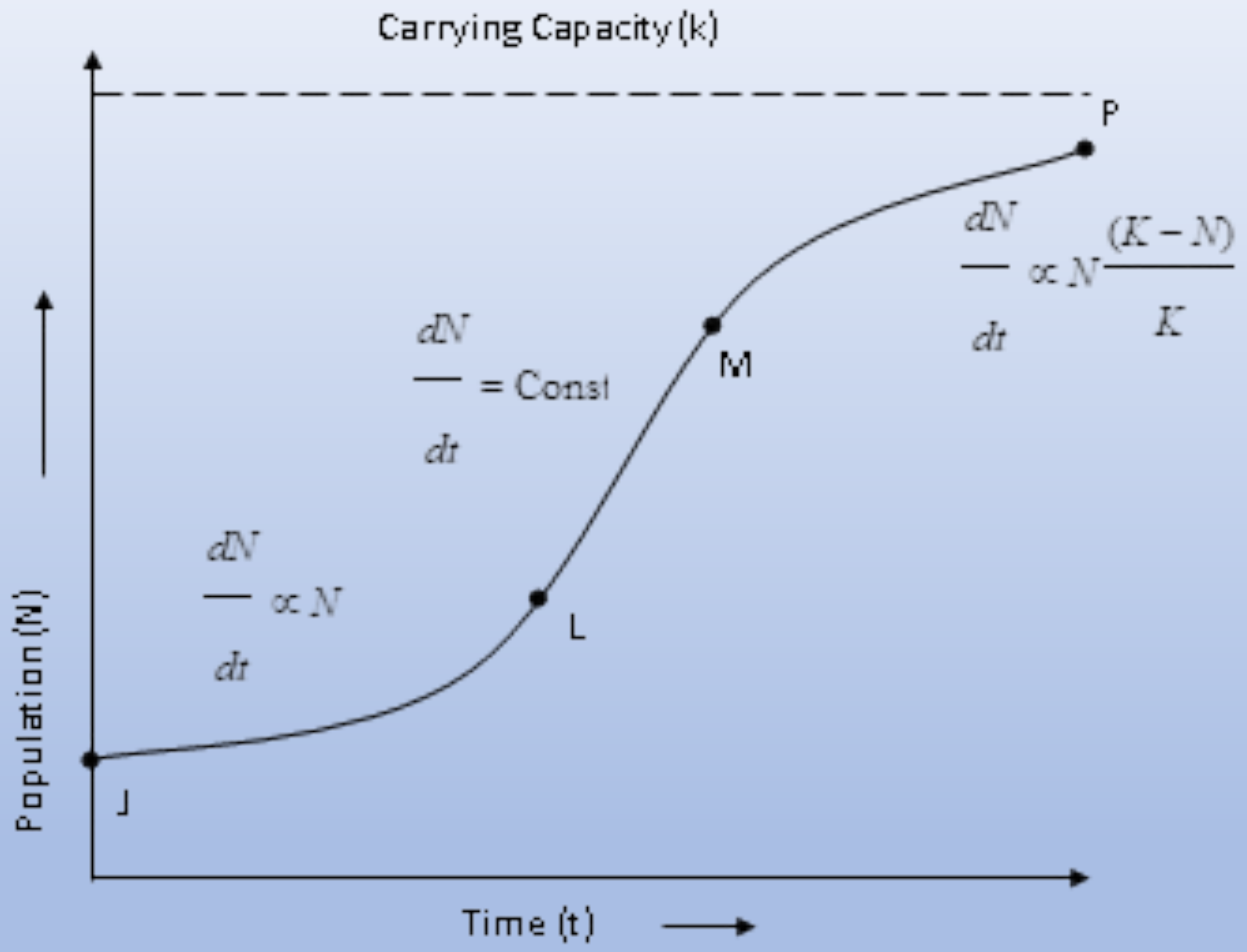

Fig: 1.3 Logistic Curve

The **early part** of the above mentioned **logistic curve** (JLMP) shows that growth rate is **directly proportional** to population, i.e., JL part of the curve is showing that population growth is increasing here **exponentially** $\left(\frac{dN}{dt} \propto N\right)$. Here, population grows exponentially because it is now **far below from the carrying capacity** i.e., **environment is suitable** for the growth of population. The growth between point L and M of the above mentioned curve shows that **population growth is constant** here $\left(\frac{dN}{dt} = \text{Const.}\right)$. This is called **transitional part of the logistic curve**. Later on the growth from M to P follows the **decreasing rate** i.e., $\frac{dN}{dt} \propto N\frac{(K-N)}{K}$.

As population approaches to the carrying capacity growth rate **should be reduced**. If population grows beyond the carrying capacity resource limitation shows its adverse effect on population by increasing death rates and reducing birth rates. This is called **'environmental resistance'** or **'growth realization factor'**. Thus the environmental resistance is responsible to change the 'exponential growth curve' to 'logistic growth curve'. The environmental resistance often denoted as $\left(1-\dfrac{N}{K}\right)$. The logistic equation represents the relationship between the biotic potential, the population growth curve and the environmental resistance.

When a population ***overshoots*** or surpasses the carrying capacity of its environment, death rates will begin to surpass the birth rates. The growth curve becomes negative, rather than positive, and the population may decrease as fast as, or faster, than it grew. We can call this as ***dieback*** a population crash.

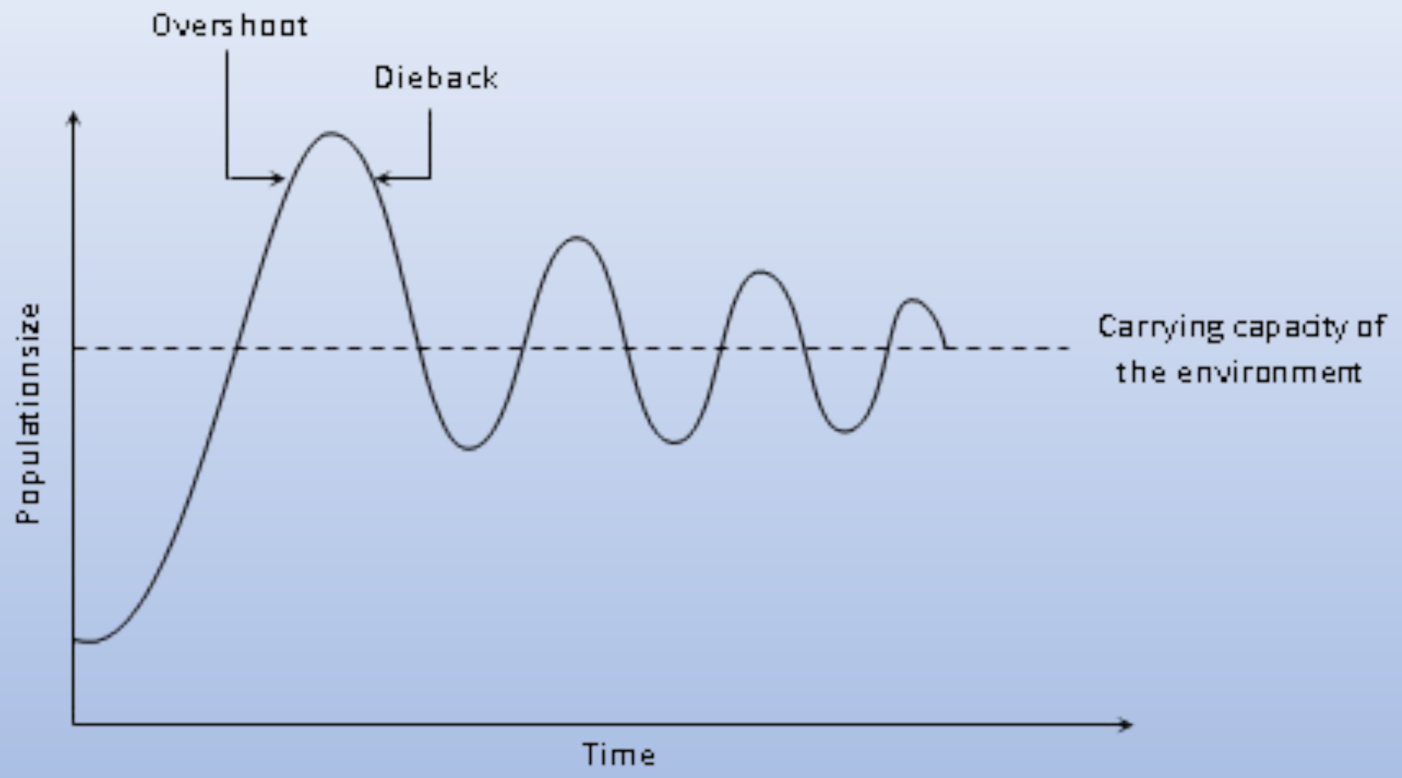

Fig: 1.4 Population Oscillations

Maximum Sustainable Yield

Maximum Sustainable Yield is the maximum rate that individuals can be removed without reducing the population size.

In the logistic growth curve of population, growth rate is **not equal at all the points**. The **growth rate** of population at any point can be determined with the help of the **slope at that point**. The growth rate would be **maximum** when the **slope is maximum**. The point can be determined when the derivative of the slope is equal to zero.

$$\text{i.e., } \frac{d}{dt}\left(\frac{dN}{dt}\right) = 0 \quad \begin{bmatrix} \text{Where, } N = \text{population size} \\ K = \text{carrying capacity} \\ r = \text{logistic growth rate const.} \end{bmatrix}$$

again,

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

$$\text{or,} \quad \frac{d}{dt}\left[rN\left(1 - \frac{N}{K}\right)\right] = 0$$

$$\text{or,} \quad r\frac{dN}{dt} - \frac{r}{K}\left(2N\frac{dN}{dt}\right) = 0$$

$$\text{or,} \quad r\frac{dN}{dt}\left(1 - \frac{2N}{K}\right) = 0$$

$$\text{or,} \quad 1 - \frac{2N}{K} = 0 \quad \left[\text{as, } r\frac{dN}{dt} \neq 0\right]$$

$$\therefore N = K/2 \qquad \ldots (1)$$

This indicates that growth rate would be maximum when population becomes half of its carrying capacity. At this point of the logistic curve, any removal of population will not reduce the population size. The growth at this particular point is called the Maximum Sustainable Yield.

As we know,

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

$\therefore$ Maximum Sustainable Yield

$$= \left(\frac{dN}{dt}\right)_{\max} = r\frac{K}{2}\left(1 - \frac{K/2}{K}\right)$$

or, $$\left(\frac{dN}{dt}\right)_{\max} = \frac{rK}{4}$$

Again, initially population grows exponentially, because, at that time $(t = 0)$ there is no '**environmental resistance**'. So, population growth

$$\frac{dN}{dt} = R_0 N_0 \qquad \ldots (3)$$

[Where, R_O = population growth rate const. at time $t = 0$

N_0 = initial population (at, $t = 0$)]

Here, $R_0 \neq r$.

Again, we know that, when population follows the logistic growth

$$\frac{dN}{dt} = rN_0\left(1 - \frac{N_0}{K}\right) \qquad \ldots (4)$$

After comparing these equations, we can write, at $t = 0$,

$$R_0 N_0 = rN_0\left(1 - \frac{N_0}{K}\right)$$

or, $r = \dfrac{R_0}{1 - \dfrac{N_0}{K}}$

Logistic Growth Rate Constant

Logistic growth can be derived from the following equation –

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$; [Here, r = logistic growth rate const.]

After, rearranging the above equation, we get

$$\frac{dN}{N\left(1 - \frac{N}{K}\right)} = rdt$$

If we consider that at time ‘t’, population is N and at time ‘t^*’, population is equal to half of the carrying capacity (K); i.e., $\left(K/2\right)$, then after integration we get –

$$\int_N^{K/2} \frac{dN}{N\left(1 - N/K\right)} = r\int_t^{t^*} dt$$

$$\text{or, } \int_N^{K/2} \frac{dN}{N} + \int_N^{K/2} \frac{\frac{1}{K}dN}{1 - \frac{N}{K}} = r\int_t^{t^*} dt$$

$$\text{or, } [\ln N]_N^{K/2} + \int_N^{K/2} \frac{dN}{K - N} = r[t]_t^{t^*}$$

$$\text{or, } \ln\frac{K/2}{N} - \int_N^{K/2} \frac{dN}{N - K} = r\left(t^* - t\right)$$

$$\text{or, } \ln\frac{K/2}{N} - \left[\ln(N-K)\right]_N^{K/2} = r\left(t^* - t\right)$$

$$\text{or, } \ln\frac{K/2}{N} - \ln\frac{\left(-K/2\right)}{N-K} = -r\left(t - t^*\right)$$

$$\ln\left[\frac{\frac{K/2}{N}}{\frac{\left(-K/2\right)}{N-K}}\right] = -r\left(t - t^*\right)$$

$$\ln\left[\frac{-(N-K)}{N}\right] = -r\left(t - t^*\right)$$

$$\ln\left[\frac{K-N}{N}\right] = -r\left(t - t^*\right)$$

$$\text{or, } \frac{(K-N)}{N} = e^{-r\left(t-t^*\right)}$$

$$\text{or, } \frac{K}{N} - 1 = e^{-r\left(t-t^*\right)}$$

$$\text{or, } N = \frac{K}{1 + e^{-r\left(t-t^*\right)}}$$

If we consider that, at time $\mathbf{t} = \mathbf{0}$, population $N = N_0$, then from the equation (2) we get –

$$N_0 = \frac{K}{1 + e^{-r(0-t^*)}}$$

or, $N_0 = \dfrac{K}{1 + e^{rt^*}}$

or, $1 + e^{rt^*} = \dfrac{K}{N_0}$

or, $e^{rt^*} = \dfrac{K}{N_0} - 1$

or, $rt^* = \ln\left(\dfrac{K}{N_0} - 1\right)$

$$\therefore\ t^* = \frac{1}{r}\ln\left(\frac{K}{N_0} - 1\right) \qquad \ldots (3)$$

or, $r = \dfrac{1}{t^*}\ln\left(\dfrac{K}{N_0} - 1\right)$

Problem 1

It took the world about 400 years to increase in population from 0.5 billion to 8.0 billion. If we assume exponential growth at constant rate over that period of time, what would be growth rate?

We know that,

$$N = N_0 e^{Rt}$$

$$\therefore 8.0 \times 10^9 = 0.5 \times 10^9 e^{R \times 400}$$

or, $$e^{R \times 400} = 16$$

$$\therefore R \times 400 = \ln 16$$

$$\therefore R = \frac{\ln 16}{400} = 0.00693$$

$$R = 0.693\% \quad \text{Ans.}$$

Problem 2

Suppose the human population follows a logistic curve until it stabilizes at 15.0 billion. In 1986 the world population was 5.0 billion and its growth rate was 1.7%. When would the population reach 7.5 billion and 14 billion?

We know that,

$$r = \frac{R_0}{1 - \dfrac{N_0}{K}}$$

$$\therefore r = \frac{0.017}{1 - \dfrac{(5.0 \times 10^9)}{(15.0 \times 10^9)}}$$

$$r = 0.0255$$

The logistic growth rate of population is 0.0255.

The time required to reach 7.5 billion, or half of the carrying capacity i.e., final population size should be

$$t^{*} = \frac{1}{r}\ln\left(\frac{K}{N_0} - 1\right)$$

$$= \frac{1}{0.0255}\ln\left(\frac{15\times10^{9}}{5\times10^{9}} - 1\right) \qquad = \frac{1}{0.0255}\ln 2$$

$$= \frac{0.693}{0.0255} = 27.18 \approx 27 \text{ years.} \qquad \text{Ans.}$$

To determine the number of years that it will take to reach 14.0 billion, we have to solve the following equation –

$$N = \frac{K}{1 + e^{-r(t-t^*)}}$$

$$\therefore t = t^* - \frac{1}{r}\ln\left(\frac{K}{N} - 1\right)$$

$$= 27 - \frac{1}{0.0255}\ln\left(\frac{15}{14} - 1\right) \qquad = 27 - \frac{1}{0.0255}\ln\frac{1}{14} \qquad = 130 \text{ years.}$$

$\therefore$ After 27 years population will be 7.5 billion while after 130 years it would be 14.0 billion.


Thank You
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