Section 3 : Quantitative Theory & Math

Q5. Derive the optimal Kelly fraction for a strategy with Gaussian returns and autocorrelation. How does positive autocorrelation impact optimal leverage?

Proof:

To derive the **optimal Kelly fraction** for a strategy with **Gaussian returns and autocorrelation**, we must extend the classical Kelly Criterion to account for serial correlation in returns.

Step-1. Classical Kelly Criterion (No Autocorrelation)

For IID (independent and identically distributed) Gaussian returns:

Let:

- $r_t \sim \mathcal{N}(\mu, \sigma^2)$
- f: fraction of wealth invested (Kelly fraction)
- Then the log wealth grows as:

$$\mathbb{E}[\log(1 + f r_t)] \approx f\mu - \frac{1}{2} f^2 \sigma^2$$

Maximizing this over f, the optimal Kelly fraction is:

$$f^* = \frac{\mu}{\sigma^2}$$

Step-2. Extending to Autocorrelated Gaussian Returns

Now suppose returns follow a **Gaussian AR(1)** process:

$$r_t = \mu + \rho(r_{t-1} - \mu) + \epsilon_t, \quad \epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2)$$

Or equivalently, the return process has **mean** μ, **volatility** σ, and **autocorrelation coefficient** $\rho \in [-1, 1]$.

Step-3. Impact of Autocorrelation on Variance of Wealth

Over n periods, your compounded return depends not only on the mean and variance but also on the **autocovariance structure**.

The **variance of cumulative returns** becomes:

$$\mathrm{Var}\left(\sum_{t=1}^{n} r_t\right) = n\sigma^2 + 2\sum_{1\le i<j\le n} \mathrm{Cov}(r_i, r_j)$$

With **AR(1) autocorrelation**, the covariance between r_i and r_j is:

$$\mathrm{Cov}(r_i, r_j) = \sigma^2 \rho^{|i-j|}$$

So the total variance of returns over n periods is higher when $\rho > 0$, which **increases drawdown risk**.

Step-4. Adjusted Kelly Fraction with Autocorrelation

The **effective variance per unit time** increases due to autocorrelation:

$$\sigma_{\text{eff}}^2 = \sigma^2 \left(1 + 2\sum_{k=1}^{\infty} \rho^k\right) = \sigma^2 \left(\frac{1+\rho}{1-\rho}\right) \quad (\text{if } |\rho| < 1)$$

So the **adjusted Kelly fraction** becomes:

$$f^* = \frac{\mu}{\sigma_{\text{eff}}^2} = \frac{\mu(1-\rho)}{\sigma^2(1+\rho)}$$

Conclusion: Impact of Positive Autocorrelation

- **Positive autocorrelation** ($\rho > 0$) increases effective risk → **reduces** optimal Kelly leverage.
- **Negative autocorrelation** ($\rho < 0$) reduces risk → **increases** optimal leverage.
- When $\rho \to 1$, Kelly leverage goes to **zero**.
- When $\rho \to -1$, leverage increases sharply.

Final Formula:

$$\boxed{f^* = \frac{\mu(1-\rho)}{\sigma^2(1+\rho)}}$$

This formula reflects how autocorrelation skews risk-adjusted return, hence modifying optimal Kelly leverage.

Q6. Given a multivariate time series X_t \in \mathbb{R}^n, derive the Granger causality test in matrix form and describe how you would adapt it for non-stationary financial time series.

Proof:

Step-1. VAR Model (Multivariate Setup)

Let $X_t \in \mathbb{R}^n$ be an n-dimensional time series.

The **VAR(p)** model is:

$$X_t = A_1 X_{t-1} + A_2 X_{t-2} + \cdots + A_p X_{t-p} + \varepsilon_t$$

Where:

- $A_k \in \mathbb{R}^{n \times n}$ are coefficient matrices
- $\varepsilon_t \sim \mathcal{N}(0, \Sigma)$ is multivariate white noise

We can write this compactly using lagged regressors:

Let:

$$Z_t = \begin{bmatrix} X_{t-1} \\ X_{t-2} \\ \vdots \\ X_{t-p} \end{bmatrix} \in \mathbb{R}^{np}, \quad \Phi = \begin{bmatrix} A_1 & A_2 & \cdots & A_p \end{bmatrix} \in \mathbb{R}^{n \times np}$$

Then:

$$X_t = \Phi Z_t + \varepsilon_t$$

Step-2. Granger Causality Hypothesis in Matrix Form

We want to test if variable x_j Granger-causes variable x_i.

This means testing:

> Does the past of x_j help predict x_i beyond the past of other variables?

Formally, in the i-th row of Φ, the coefficients corresponding to all lags of x_j should be zero under the **null hypothesis**:

$$H_0 : R\Phi_i^\top = 0$$

Where:

- $\Phi_i^\top$ is the transpose of the i-th row of Φ (i.e., the coefficient vector predicting x_i)
- $R \in \mathbb{R}^{q \times np}$ is a **restriction matrix** that selects the coefficients on lagged values of x_j
- $q = p$ if testing all lags of x_j

Step-3. F-Test in Matrix Form

Estimate two models:

- **Unrestricted**: full model
- **Restricted**: set lag coefficients of xjx_jxj in the equation for xix_ixi to zero

Let:

- SSR_U: Sum of squared residuals from unrestricted model
- SSR_R: Sum of squared residuals from restricted model
- q: number of restrictions (lags of x_j)
- T: number of observations
- k: number of parameters in unrestricted model

Then the **F-statistic** is:

$$F = \frac{(SSR_R - SSR_U)/q}{SSR_U/(T-k)}$$

If F is greater than the critical value from the F-distribution with q and $T-k$ degrees of freedom, reject H_0: x_j **Granger-causes** x_i.

Adapting Granger Causality to Non-Stationary Time Series

Case 1: Non-stationary but not cointegrated

Use **first differences** to make the series stationary:

$$\Delta X_t = A_1 \Delta X_{t-1} + \cdots + A_p \Delta X_{t-p} + \varepsilon_t$$

Then run Granger causality on ΔX_t. This detects **short-term causality** only.

Case 2: Non-stationary and cointegrated

Use a **Vector Error Correction Model (VECM)**:

$$\Delta X_t = \Pi X_{t-1} + \sum_{i=1}^{p-1} \Gamma_i \Delta X_{t-i} + \varepsilon_t$$

Where:

- $\Pi = \alpha \beta^\top$
- β contains cointegration vectors (long-run relationships)
- α contains adjustment coefficients

In this case:

- **Short-term Granger causality** is tested via coefficients Γ_i
- **Long-term causality** is captured via the significance of α

So:

- If $\alpha_{ij} \neq 0$: x_j affects x_i in the long run
- If $\Gamma_{ij}^{(k)} \neq 0$: x_j Granger-causes x_i in the short run