SYDE252 - lecture notes

09/01/18

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note: some material (figures) borrowed from various sources

UNIVERSITY OF WATERLOO
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3. Convolution

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inspiration

- " Chance does not speak essentially through words nor can it be seen in their convolution. It is the eruption of language, its sudden appearance. It's not a night twinkle with stars, an illuminated sleep, nor a drowsy vigil. It is the very edge of consciousness." Michel Foucault
- "life is convoluted when there are several doors yet to open, but keys in meander of thoughts due to no prioritization" Harshitha Lakshmi
- "i am somehow less interested in the weight and convolutions of Einstein's brain than in the near certainty that people of equal talent have lived and died in cotton fields and sweatshops" Steven Jay Gould

convolution

- the impulse response of a system is a way of characterizing the system; meaning that if we know the impulse response, we can find the response of the system to any other input

LTI System with Zero Initial Conditions

$$\delta(t) \rightarrow \boxed{\text{LTI}} \rightarrow y(t) = h(t)$$

$$\delta(t-\tau) \rightarrow \boxed{\text{LTI}} \rightarrow y(t) = h(t-\tau)$$

$$x(\tau)\,\delta(t-\tau) \rightarrow \boxed{\text{LTI}} \rightarrow y(t) = x(\tau)\,h(t-\tau)$$

$$\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau \rightarrow \boxed{\text{LTI}} \rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

$$x(t) \rightarrow \boxed{\text{LTI}} \rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

convolution

The following integral

$$y(t) = \int_{=\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

is called the CONVOLUTION INTEGRAL

in general,

$$x_1(t) * x_2(t) == \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau$$

discrete vs. continuous convolution

Discrete Time:

$$y[n] = x[n] * h[n] = \sum_{i=-\infty}^{\infty} x[i]\, h[n-i]$$

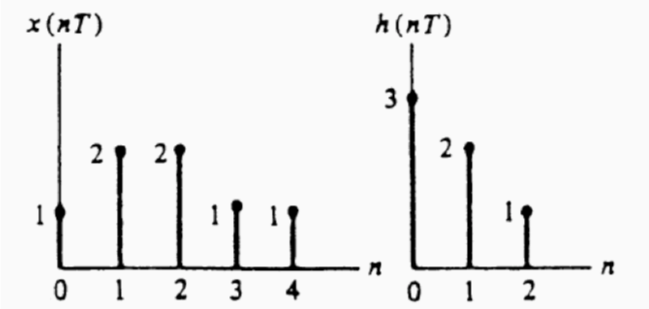
$$x[n] \rightarrow \boxed{\text{LTI}} \rightarrow \sum_{i=-\infty}^{\infty} x[i]\, h[n-i]$$

Continuous Time:

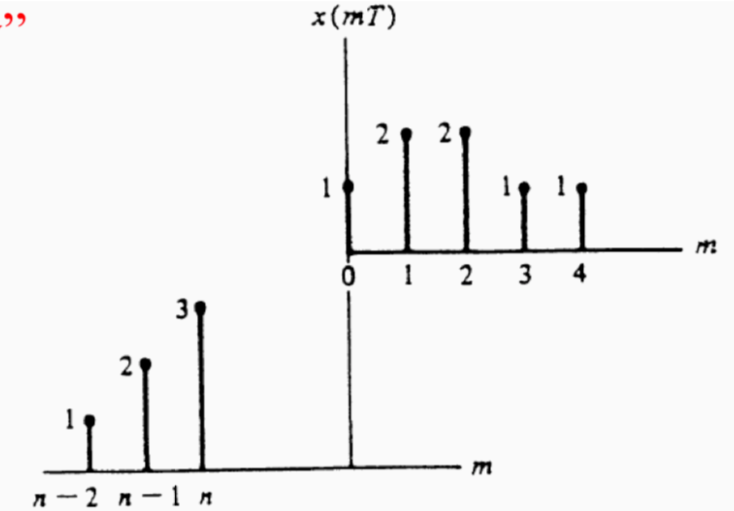
$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$x(t) \rightarrow \boxed{\text{LTI}} \rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Convolution Example "Table view"



(a) Input and unit pulse response



(b) Functions for computing convolution sum

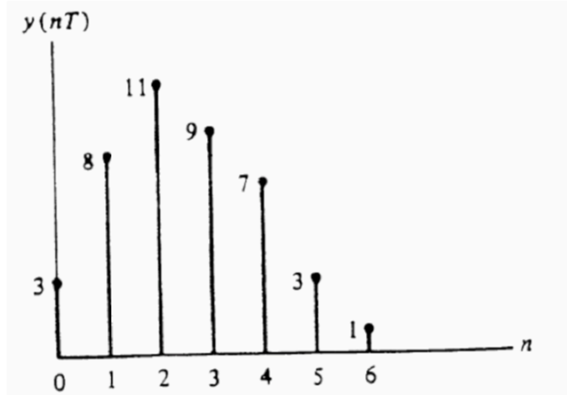
Samples $x(mT)$

	0	0	1	2	2	1	1	0
$n = 0$	1	2	3	0	0	0	0	0
$n = 1$	0	1	2	3	0	0	0	0
$n = 2$	0	0	1	2	3	0	0	0
$n = 3$	0	0	0	1	2	3	0	0
$n = 4$	0	0	0	0	1	2	3	0
$n = 5$	0	0	0	0	0	1	2	3
$n = 6$	0	0	0	0	0	0	1	2

$h(-m)$

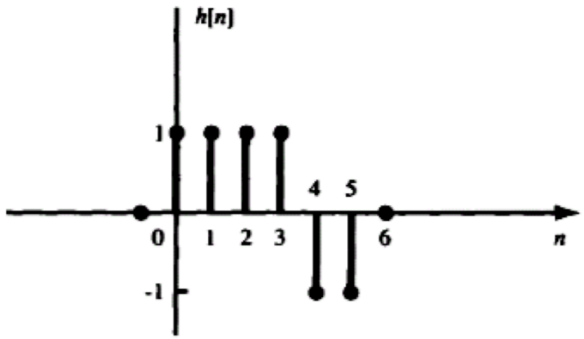
$h(1-m)$

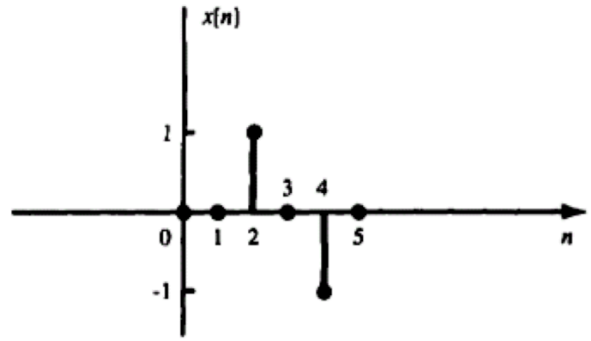
(c) Table for evaluating summation



(d) Output

The impulse response h[n] of a discrete-time LTI system. (a). Determine and sketch the output y[n] of this system to the input x[n]. (b) without using the convolution technique.





$$h[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] - \delta[n-4] - \delta[n-5],$$

$$x[n] = \delta[n-2] - \delta[n-4]$$

$$x[n] * h[n] = x[n] * \{\delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] - \delta[n-4] - \delta[n-5]\}$$

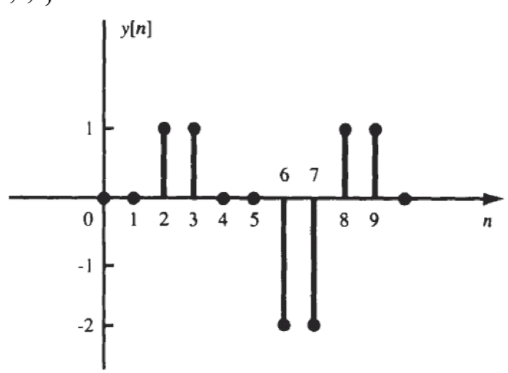
$$= x[n] + x[n-1] + x[n-2] + x[n-3] - x[n-4] - x[n-5]$$

$$y[n] = \delta[n-2] - \delta[n-4] + \delta[n-3] - \delta[n-5] + \delta[n-4] - \delta[n-6] + \delta[n-5] - \delta[n-7]$$

$$- \delta[n-6] + \delta[n-8] - \delta[n-7] + \delta[n-9]$$

$$= \delta[n-2] + \delta[n-3] - 2\delta[n-6] - 2\delta[n-7] + \delta[n-8] + \delta[n-9]$$

$$y[n] = \{0,0,1,1,0,0,-2,-2,1,1\}$$



convolution - properties (1)

1. Commutative Property
   $x_1(t) * x_2(t) = \int_{=\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau$
   $= \int_{=\infty}^{\infty} x_2(\tau)x_1(t-\tau)d\tau$
   $= x_2(t) * x_1(t)$

2. Associative Property
   $x_1(t) * [x_2(t) * x_3(t)] = [x_1(t) * x_2(t)] * x_3(t)$

3. Distributive Property
   $x_1(t)*[x_2(t)+x_3(t)] = x_1(t)*x_2(t)+x_1(t)*x_3(t)$

4. Shift Property
   if $x_1(t) * x_2(t) = c(t)$ then
   $x_1(t) * x_2(t-T) = x_1(t-T) * x_2(t) = c(t-T)$,
   also
   $x_1(t-T_1) * x_2(t-T_2) = c(t-T_1-T_2)$

convolution - properties (2)

5. Impulse Property
$x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - i\tau)d\tau = x(t)$

6. Width Property
if $x_1(t)$ duration is T_1 and $x_2(t)$ duration is T_2 then the duration of $x_1(t) * x_2(t)$ is $T_1 + T_2$

7. Causality Property
if both system's impulse $h(t)$ and input $x(t)$ are causal, then so is the output
$y(t) = x(t) * h(t) = \int_{0^-}^{t} x(\tau)h(t - \tau)d\tau, t \geq 0$
$= \int_{0^-}^{t} h(\tau)x(t - \tau)d\tau$
$= 0, t < 0$

convolution - properties - summary

Commutative: $f_1(t) * f_2(t) = f_2(t) * f_1(t)$

Associative: $f_1(t) * \left[f_2(t) * f_3(t)\right] = \left[f_1(t) * f_2(t)\right] * f_3(t)$

Distributive: $f_1(t) * \left[f_2(t) + f_3(t)\right] = f_1(t) * f_2(t) + f_1(t) * f_3(t)$

Shift Property: If $f_1(t) * f_2(t) = c_1(t)$ then $f_1(t) * f_2(t-T) = c_1(t-T)$

$f_1(t-T) * f_2(t) = c_1(t-T)$

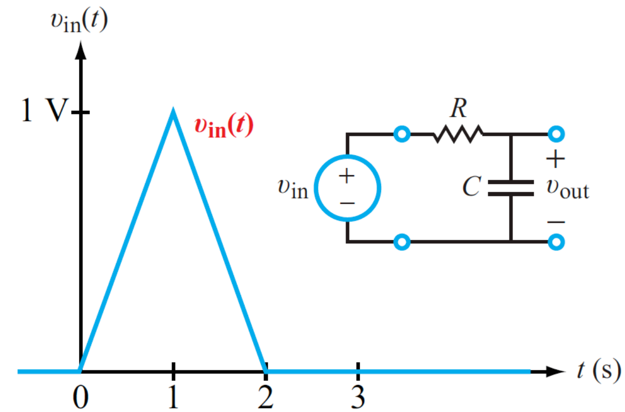
Convolution with an impulse: $f_1(t) * \delta(t) = f(t)$

Width Property: If $f_1(t)$ and $f_2(t)$ have durations of T_1 and T_2 respectively, then the duration of $f_1(t) * f_2(t)$ is $T_1 + T_2$.

convolution - example

Solution:
The input signal, measured in volts, is given by

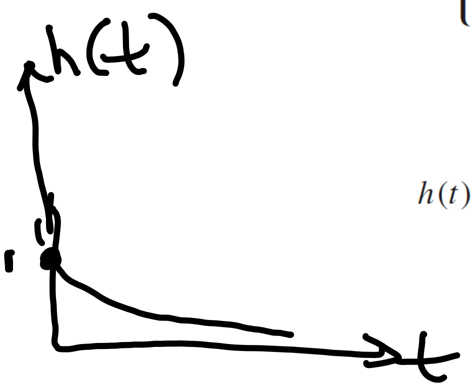
$$v_{\text{in}}(t) = \begin{cases} 0 & \text{for } t \leq 0, \\ t & \text{for } 0 \leq t \leq 1 \text{ s}, \\ 2 - t & \text{for } 1 \leq t \leq 2 \text{ s}, \\ 0 & \text{for } t \geq 2 \text{ s}, \end{cases}$$



(a) Triangular pulse

the impulse response for $\tau_c = 1$ is

$$h(t) = \frac{1}{\tau_c} e^{-t/\tau_c} u(t)$$
$$= e^{-t} u(t).$$



$$v_{\text{out}}(t) = v_{\text{in}}(t) * h(t)$$
$$= \int_0^t v_{\text{in}}(\tau)\, h(t - \tau)\, d\tau$$

response of circuit with RC = 1 s to triangular pulse

with

$$h(t - \tau) = e^{-(t-\tau)}\, u(t - \tau) = \begin{cases} 0 & \text{for } t < \tau, \\ e^{-(t-\tau)} & \text{for } t > \tau. \end{cases}$$

convolution - example

(1) $t < 0$:

$t<0$

The lowest value that the integration variable τ can assume is zero. Therefore, when $t < 0$, $t < \tau$ and $h(t-\tau) = 0$. Consequently,

$$\upsilon_{\text{out}}(t) = 0 \qquad \text{for } t < 0.$$

(2) $0 \leq t \leq 1$ s:

$0 \leq t \leq 1$

$0 \leq t < 1$

$$h(t-\tau) = e^{-(t-\tau)}, \qquad \upsilon_{\text{in}}(\tau) = \tau,$$

and

$$\upsilon_{\text{out}}(t) = \int_0^t \tau e^{-(t-\tau)}\, d\tau$$

$$= e^{-t} + t - 1, \qquad \text{for } 0 \leq t \leq 1 \text{ s}.$$

$$\int_0^t \tau e^{-t} e^{\tau}\, d\tau = e^{-t} \int_0^t \tau e^{\tau}\, d\tau$$

(3) $1 \text{ s} \leq t \leq 2$ s:

$$\upsilon_{\text{in}}(\tau) = \begin{cases} \tau & \text{for } 0 \leq \tau \leq 1 \text{ s}, \\ 2-\tau & \text{for } 1 \text{ s} \leq \tau \leq 2 \text{ s}, \end{cases}$$

and

$$\upsilon_{\text{out}}(t) = \int_0^1 \tau e^{-(t-\tau)}\, d\tau + \int_1^t (2-\tau) e^{-(t-\tau)}\, d\tau$$

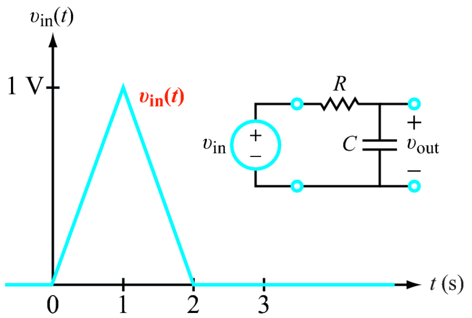
$$= (1-2e)e^{-t} - t + 3, \qquad \text{for } 1 \text{ s} \leq t \leq 2 \text{ s}.$$

(4) $t \geq 2$ s:

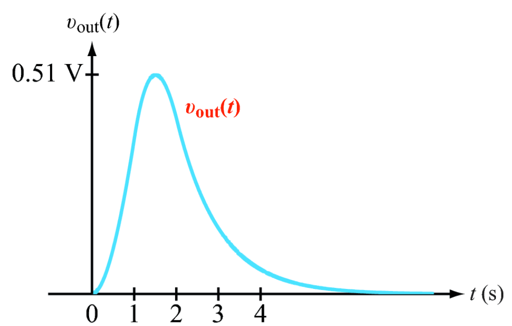
$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau$$

$$\upsilon_{\text{out}}(t) = \int_0^1 \tau e^{-(t-\tau)}\, d\tau + \int_1^2 (2-\tau) e^{-(t-\tau)}\, d\tau$$

$$= (1 - 2e + e^2)e^{-t} \qquad \text{for } t \geq 2 \text{ s}.$$



(a) Triangular pulse



(b) Output response

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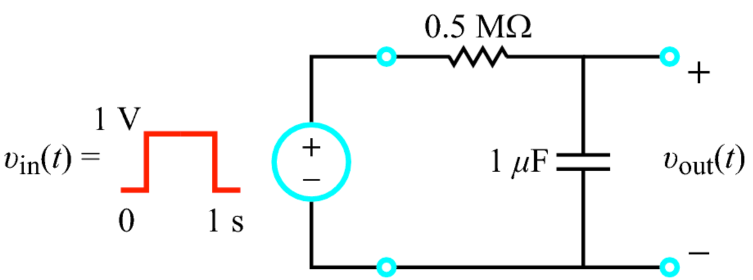
convolution - example (2)



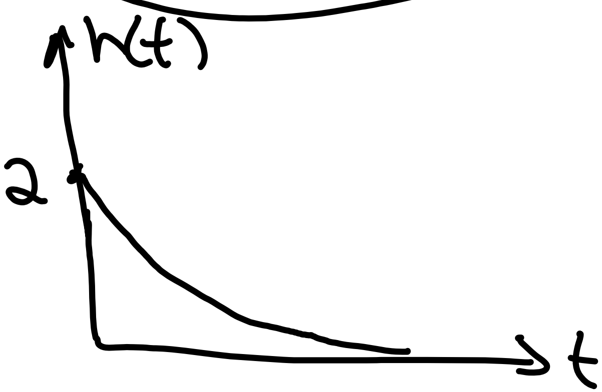
(a) RC lowpass filter

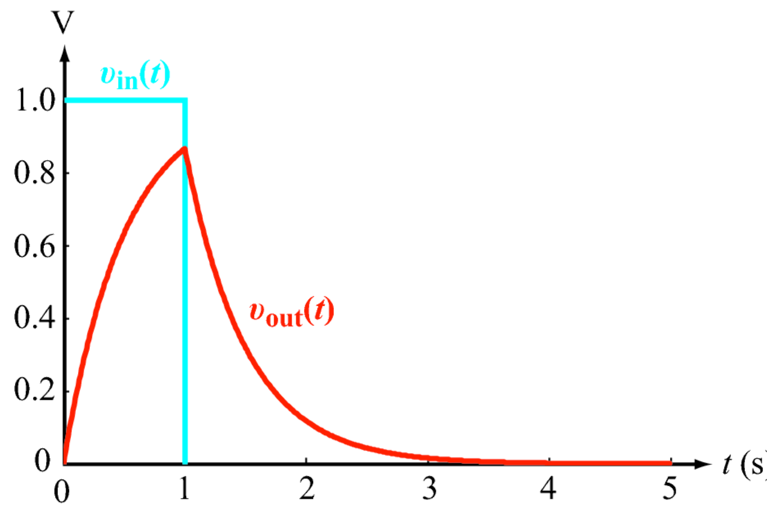
Solution:
The time constant of the RC circuit is $\tau_c = RC = (0.5 \times 10^6) \times 10^{-6} = 0.5$ s. In view of Eq. (2.17), the impulse response of the circuit is

$$h(t) = \frac{1}{\tau_c} e^{-t/\tau_c}\, u(t) = 2e^{-2t}\, u(t). \tag{2.53}$$

The input voltage is

$$v_{in}(t) = [u(t) - u(t-1)] \text{ V}. \tag{2.54}$$





(b) Output response

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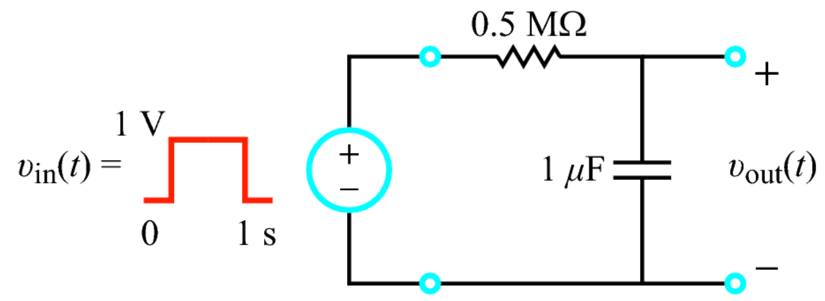
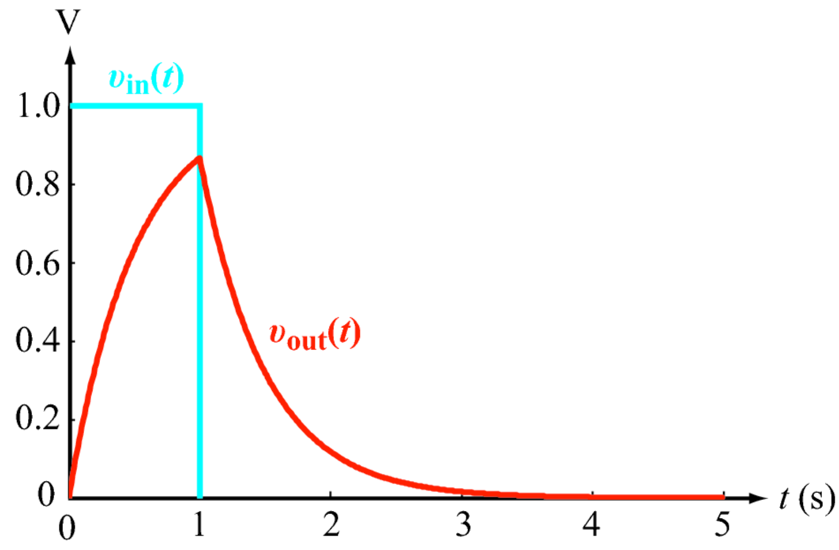
convolution - example (2)

$$
\begin{aligned}
v_{\text{out}}(t) &= v_{\text{in}}(t) * h(t) \\
&= u(t) \int_0^t v_{\text{in}}(\tau)\, h(t-\tau)\, d\tau \\
&= u(t) \int_0^t [u(\tau) - u(\tau-1)] \\
&\qquad \times 2e^{-2(t-\tau)}\, u(t-\tau)\, d\tau \\
&= u(t) \int_0^t 2e^{-2(t-\tau)}\, u(\tau)\, u(t-\tau)\, d\tau \\
&\quad - u(t) \int_0^t 2e^{-2(t-\tau)}\, u(\tau-1)\, u(t-\tau)\, d\tau.
\end{aligned}
$$

$$
\begin{aligned}
v_{\text{out}}(t) &= \left[\int_0^t 2e^{-2(t-\tau)}\, d\tau\right] u(t) \\
&\quad - \left[\int_1^t 2e^{-2(t-\tau)}\, d\tau\right] u(t-1) \\
&= \frac{2}{2}\, e^{-2(t-\tau)}\Big|_0^t\, u(t) - \frac{2}{2}\, e^{-2(t-\tau)}\Big|_1^t\, u(t-1) \\
&= [1 - e^{-2t}]\, u(t) - [1 - e^{-2(t-1)}]\, u(t-1)\ \text{V},
\end{aligned}
$$



(a) RC lowpass filter



(b) Output response

convolution - zero state response

CONTINUOUS-TIME SYSTEMS

The **Zero-state Response** can be written as the **convolution integral** of the Input and the Unit Impulse Response. If $f(t)$ and $h(t)$ are causal, the limits of integration are 0 to t.

$$\overbrace{y(t)}^{\text{Zero-state Response}} = \underbrace{f(t)}_{\text{Input}} * \overbrace{h(t)}^{\text{Unit Impulse Response}} = \overbrace{\int_0^t f(\tau)\, h(t-\tau)\, d\tau}^{\text{Convolution Integral}}, \qquad t \geq 0$$

graphical convolution

Essentially the steps are FLIP, SLIDE, MULTIPLY and INTEGRATE.
The graphical steps are:

1. Plot $x(\tau) * h(\tau)$
2. select which signal to flip, say $h(\tau)$. In general, the signal of shorter duration is the one to flip, or the easier one.
3. Flip $h(\tau)$ about the vertical axis giving $h(-\tau)$.
4. Plot $h(-\tau + t)$ on τ axis, This is just $h(-\tau)$ shifted RIGHT by t.
5. Identify regions of overlap as t varies from $-\infty$ to ∞ where there are breaks in either function.
6. Multiply and integrate for each of the identified regions.

graphical convolution

Graphical understanding of convolution

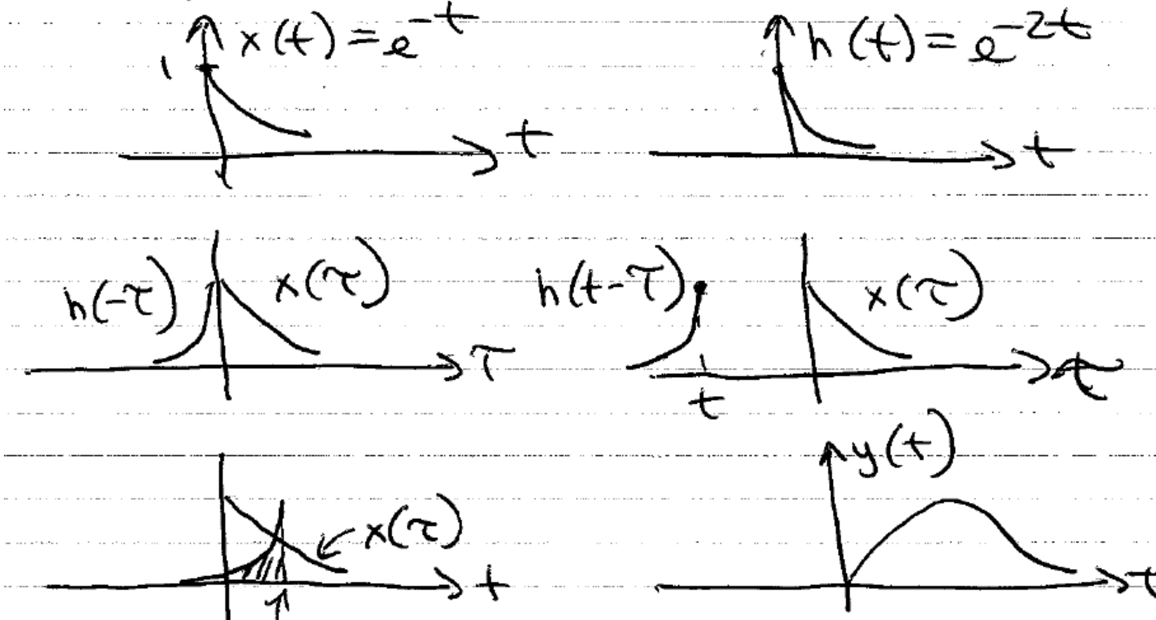
$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$h(\tau) \rightarrow_{FLIP} h(-\tau) \rightarrow_{SLIDE} h(t-\tau) \rightarrow_{MULTIPLY}$$

$$x(\tau)h(t-\tau) \rightarrow_{INTEGRATE} \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

graphical convolution

Examples Graphical cts.



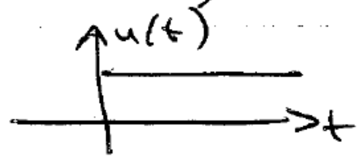
sum all
areas under
curve that resulted
to sliding here

graphical convolution

Example 6 Graphical Continuous (cts.)

$$y(t) = u(t) * [u(t) - u(t-1)]$$

=



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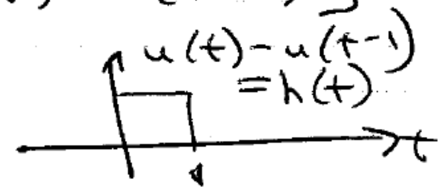
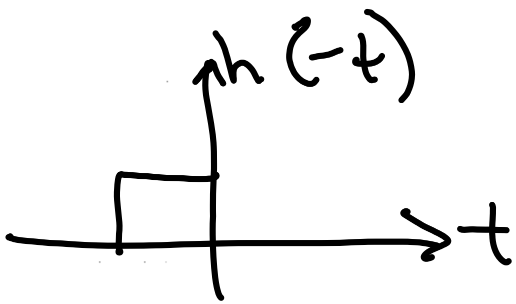
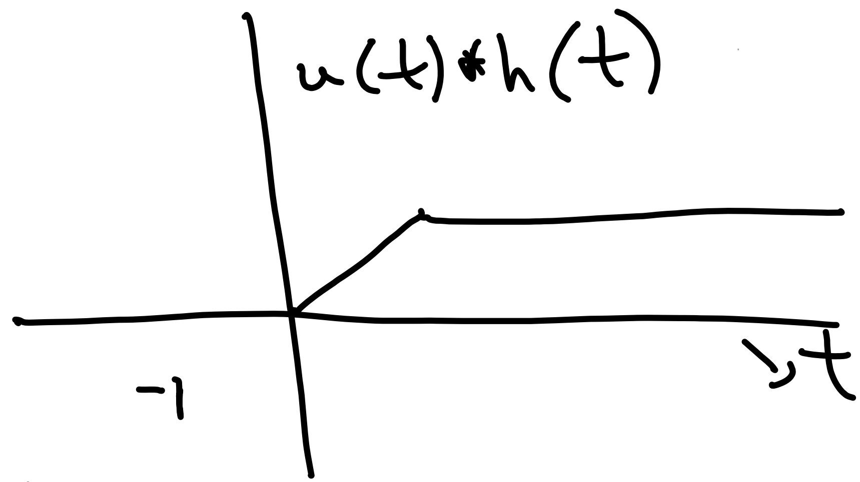
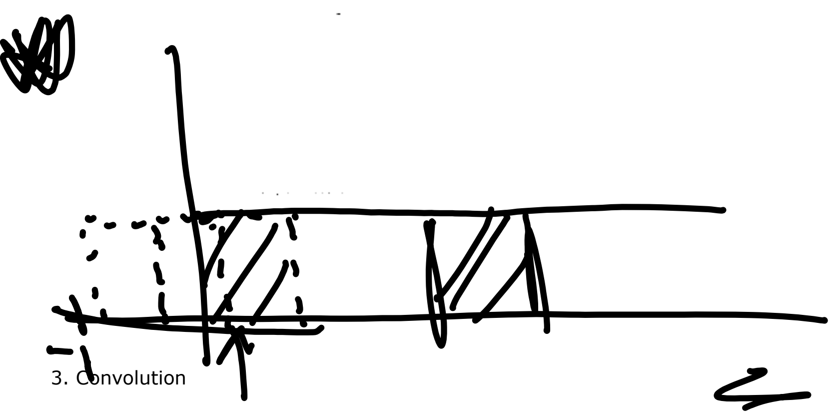








convolution by integration

Example 7 convolution by integration

$$h(t) = e^{-2t}u(t)$$

$$x(t) = e^{-t}u(t)$$

$$y(t) = \int_0^t x(\tau)h(t-\tau)\,d\tau$$

$$= \int_0^t e^{-\tau}e^{-2(t-\tau)}\,d\tau = e^{-2t}\int_0^t e^{\tau}\,d\tau$$

$$= e^{-2t}(e^{t}-1)$$

$$= e^{-t} - e^{-2t}, \quad t \geq 0$$

$$\therefore y_{zs}(t) = (e^{-t} - e^{-2t})u(t)$$

convolution tables

Table 3.1: *some models*

No.	$x_1(t)$	$x_2(t))$	result
1.	$x(t)$	$\delta(t-T)$	$x(t-T)$
2.	$e^{\lambda t}u(t)$	$u(t)$	$\frac{1-e^{\lambda t}}{-\lambda}u(t)$
3.	$u(t)$	$u(t)$	$tu(t)$
4.	$e^{\lambda_1 t}u(t)$	$e^{\lambda_2 t}u(t)$	$\frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}u(t),\ \lambda_1 \neq \lambda_2$
5.	$e^{\lambda t}u(t)$	$e^{\lambda t}u(t)$	$te^{\lambda t}u(t)$
6.	$te^{\lambda t}u(t)$	$e^{\lambda t}u(t)$	$\frac{1}{2}t^2e^{\lambda t}u(t)$

	$f_1(t)$	$f_2(t)$	$f_1(t)*f_2(t)=f_2(t)*f_1(t)$
1	$f(t)$	$\delta(t-T)$	$f(t-T)$
2	$e^{\lambda t}u(t)$	$u(t)$	$\frac{-1}{\lambda}\left(1-e^{\lambda t}\right)u(t)$
3	$u(t)$	$u(t)$	$tu(t)$
4	$e^{\lambda_1 t}u(t)$	$e^{\lambda_2 t}u(t)$	$\frac{-1}{\lambda_1-\lambda_2}\left(e^{\lambda_1 t}-e^{\lambda_2 t}\right)u(t), \quad \lambda_1\neq\lambda_2$
5	$e^{\lambda t}u(t)$	$e^{\lambda t}u(t)$	$te^{\lambda t}u(t)$
6	$te^{\lambda t}u(t)$	$e^{\lambda t}u(t)$	$\frac{1}{2}t^2e^{\lambda t}u(t)$
7	$t^n u(t)$	$e^{\lambda t}u(t)$	$\frac{n!}{\lambda^{n+1}}e^{\lambda t}u(t)-\sum_{j=0}^{n}\frac{n!}{\lambda^{j+1}(n-j)!}t^{n-j}u(t)$
8	$t^m u(t)$	$t^n u(t)$	$\frac{m!n!}{(n+m+1)!}t^{m+n+1}u(t)$
9	$te^{\lambda_1 t}u(t)$	$e^{\lambda_2 t}u(t)$	$\frac{1}{(\lambda_1-\lambda_2)^2}\left[e^{\lambda_2 t}-e^{\lambda_1 t}+(\lambda_1-\lambda_2)te^{\lambda_1 t}\right]u(t)$
10	$t^m e^{\lambda t}u(t)$	$t^n e^{\lambda t}u(t)$	$\frac{m!n!}{(n+m+1)!}t^{m+n+1}e^{\lambda t}u(t)$
11	$t^m e^{\lambda_1 t}u(t)$	$t^n e^{\lambda_2 t}u(t)$	$\sum_{j=0}^{m}\frac{(-1)^j m!(n+j)!}{j!(m-j)!(\lambda_1-\lambda_2)^{n+j+1}}t^{m-j}e^{\lambda_1 t}u(t)$ $+\sum_{k=0}^{n}\frac{(-1)^k n!(m+k)!}{k!(n-k)!(\lambda_2-\lambda_1)^{m+k+1}}t^{n-k}e^{\lambda_2 t}u(t),$ $\lambda_1\neq\lambda_2$
12	$e^{-\alpha t}\cos(\beta t+\theta)u(t)$	$e^{\lambda t}u(t)$	$\frac{\cos(\theta-\phi)e^{\lambda t}-e^{-\alpha t}\cos(\beta t+\theta-\phi)}{\sqrt{(\alpha+\lambda)^2+\beta^2}}u(t)$ $\phi=\tan^{-1}\left[-\beta/(\alpha+\lambda)\right]$
13	$e^{\lambda_1 t}u(t)$	$e^{\lambda_2 t}u(-t)$	$\frac{1}{\lambda_2-\lambda_1}\left[e^{\lambda_1 t}u(t)+e^{\lambda_2 t}u(-t)\right],$ $\text{Re}\,\lambda_2>\text{Re}\,\lambda_1$
14	$e^{\lambda_1 t}u(-t)$	$e^{\lambda_2 t}u(-t)$	$\frac{1}{\lambda_2-\lambda_1}\left(e^{\lambda_1 t}-e^{\lambda_2 t}\right)u(-t)$

Example 8 convolution by tables

$$x(t) = 10e^{-3t}u(t) \qquad h(t) = (2e^{-2t} - e^{-t})u(t)$$

response is $y(t) = x(t) * h(t)$

$$= 10e^{-3t}u(t) * [2e^{-2t} - e^{-t}]u(t)$$

$$e^{\lambda_1 t}u(t) * e^{\lambda_2 t}u(t) = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}u(t) \qquad \lambda_1 \neq \lambda_2$$

$$10e^{-3t}u(t) * 2e^{-2t}u(t) - 10e^{-3t}u(t) * e^{-t}u(t)$$

$$= 20\underbrace{[e^{-3t}u(t) * e^{-2t}u(t)]}_{\frac{e^{-3t} - e^{-2t}}{-1}u(t)} - 10[e^{-3t}u(t) * e^{-t}u(t)]$$

$\therefore y(t)$

$=$

$=$

impulse properties revisited

► An LTI system is causal ***if and only if*** its impulse response is a causal function: $h(t) = 0$ for $t < 0$. ◄

► An LTI system is BIBO stable *if and only if* its impulse response $h(t)$ is ***absolutely integrable*** (i.e., if $\int_{-\infty}^{\infty} |h(t)|\, dt$ is finite). ◄

convolution properties revisited

Convolution Integral	$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$
• **Causal Systems:**	Replace lower limit with 0

Property	Description
1. Commutative	$x(t) * h(t) = h(t) * x(t)$
2. Associative	$[g(t) * h(t)] * x(t) = g(t) * [h(t) * x(t)]$
3. Distributive	$x(t) * [h_1(t) + \cdots + h_N(t)] = x(t) * h_1(t) + \cdots + x(t) * h_N(t)$
4. Causal * Causal = Causal	$y(t) = u(t) \int_{0}^{t} h(\tau)\, x(t-\tau)\, d\tau$
5. Time-shift	$h(t-T_1) * x(t-T_2) = y(t-T_1-T_2)$
6. Convolution with Impulse	$x(t) * \delta(t-T) = x(t-T)$
7. Width	Width of $y(t)$ = width of $x(t)$ + width of $h(t)$
8. Area	Area of $y(t)$ = area of $x(t)$ $\times$ area of $h(t)$
9. Convolution with $u(t)$	$y(t) = x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$ (Ideal integrator)
10a. Differentiation	$\left(\frac{d^m x}{dt^m}\right) * \left(\frac{d^n h}{dt^n}\right) = \frac{d^{m+n} y}{dt^{m+n}}$
10b. Integration	$\int_{-\infty}^{t} y(\tau)\, d\tau = x(t) * \left[\int_{-\infty}^{t} h(\tau)\, d\tau\right] = \left[\int_{-\infty}^{t} x(\tau)\, d\tau\right] * h(t)$

discrete vs. continuous convolution

Discrete Time:

$$y[n] = x[n] * h[n] = \sum_{i=-\infty}^{\infty} x[i]\, h[n-i]$$

$$x[n] \rightarrow \boxed{\text{LTI}} \rightarrow \sum_{i=-\infty}^{\infty} x[i]\, h[n-i]$$

Continuous Time:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$x(t) \rightarrow \boxed{\text{LTI}} \rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

discrete vs. continuous convolution

Property	Continuous Time	Discrete Time
Definition	$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$	$y[n] = h[n] * x[n] = \sum_{i=-\infty}^{\infty} h[i]\, x[n-i]$
1. Commutative	$x(t) * h(t) = h(t) * x(t)$	$x[n] * h[n] = h[n] * x[n]$
2. Associative	$[g(t) * h(t)] * x(t) = g(t) * [h(t) * x(t)]$	$[g[n] * h[n]] * x[n] = g[n] * [h[n] * x[n]]$
3. Distributive	$x(t) * [h_1(t) + \cdots + h_N(t)] =$ $x(t) * h_1(t) + \cdots + x(t) * h_N(t)$	$x[n] * [h_1[n] + \cdots + h_N[n]] =$ $x[n] * h_1[n] + \cdots + x[n] * h_N[n]$
4. Causal $*$ Causal $=$ Causal	$y(t) = u(t) \int_0^t h(\tau)\, x(t-\tau)\, d\tau$	$y[n] = u[n] \sum_{i=0}^{n} h[i]\, x[n-i]$
5. Time-shift	$h(t-T_1) * x(t-T_2) = y(t-T_1-T_2)$	$h[n-a] * x[n-b] = y[n-a-b]$
6. Convolution with Impulse	$x(t) * \delta(t-T) = x(t-T)$	$x[n] * \delta[n-a] = x[n-a]$
7. Width	width $y(t) =$ width $x(t) +$ width $h(t)$	width $y[n] =$ width $x[n] +$ width $h[n] - 1$
8. Area	area of $y(t) =$ area of $x(t) \times$ area of $h(t)$	$\sum_{n=-\infty}^{\infty} y[n] = \left(\sum_{n=-\infty}^{\infty} h[n]\right)\left(\sum_{n=-\infty}^{\infty} x[n]\right)$
9. Convolution with step	$y(t) = x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$	$x[n] * u[n] = \sum_{i=-\infty}^{n} x[i]$

discrete convolution - example

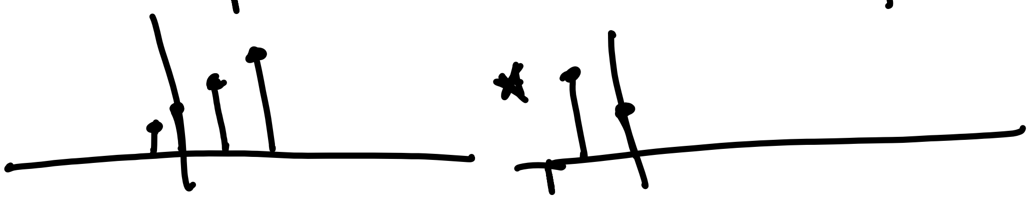
Example 1

$x[n] = \delta[n+1] + 2\delta[n] + 3\delta[n-1] + 4\delta[n-2]$

$h[n] = -\delta[n+2] + 5\delta[n+1] + 3\delta[n]$

$x[n] * h[n]$

$$[1, 2, 3, 4] * [-1, 5, 3]$$



$y[-3] = (-1)(1) = -1$

$y[-2] = (5)(1) + (-1)(2) = 3$

$y[-1] = (3)(1) + (5)(2) + (-1)(3) = 10$

$y[0] = 3(2) + 5(3) + (-1)(4) = 17$

$y[1] = 3(3) + 5(4) + (-1)0 = 29$

$y[2] = 3(4) + 5(0) + (-1)0 = 12$

rest are 0

discrete convolution - example - how did we do this?

$h[n] =$

$x[n] =$

$h[-n] =$

for $y[-3] \Rightarrow$

for $y[-2] \Rightarrow$

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discrete convolution - example - how did we do this? (2)

for $y[-1] \Rightarrow$

for $y[0] \Rightarrow$

for $y[1] \Rightarrow$

for $y[2] \Rightarrow$

discrete convolution - example 2

in general $x[n] = \{x_0, x_1, x_2, x_3, x_4\}$

$h[n] = \{h_0, h_1, h_2, h_3\}$

flip, shift, multiply

$y[0] =$ $\boxed{h_3}\boxed{h_2}\boxed{h_1}\boxed{h_0}$

$\qquad\qquad\qquad\quad \boxed{x_0}\boxed{x_1}\boxed{x_2}\boxed{x_3}\boxed{x_4}$

$= x_0 h_0$

$y[1] =$ $\boxed{h_3}\boxed{h_2}\boxed{h_1}\boxed{h_0}$

$\qquad\qquad\qquad\quad \boxed{x_0}\boxed{x_1}\boxed{x_2}\boxed{x_3}\boxed{x_4}$

$= x_0 h_1 + x_1 h_0$

$y[2] =$ $\boxed{h_3}\boxed{h_2}\boxed{h_1}\boxed{h_0}$

$\qquad\qquad\qquad\quad \boxed{x_0}\boxed{x_1}\boxed{x_2}\boxed{x_3}\boxed{x_4}$

$= x_0 h_2 + x_1 h_1 + x_2 h_0$

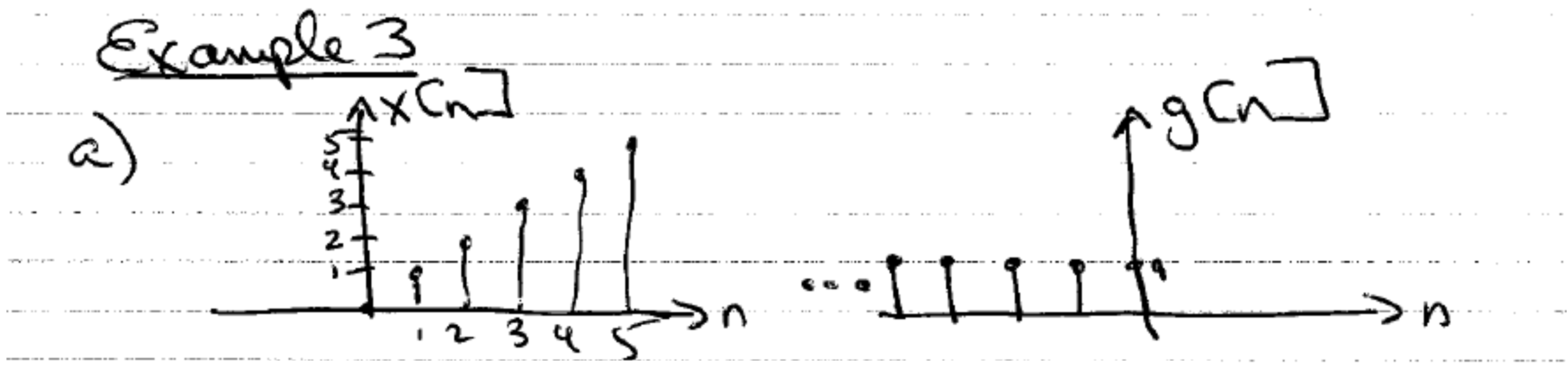
$y[3] =$ $\boxed{h_3}\boxed{h_2}\boxed{h_1}\boxed{h_0}$

$\qquad\qquad\qquad\quad \boxed{x_0}\boxed{x_1}\boxed{x_2}\boxed{x_3}\boxed{x_4}$

$= x_0 h_3 + x_1 h_2 + x_2 h_1 + x_3 h_0$

$\vdots$ etc

discrete convolution - example 3



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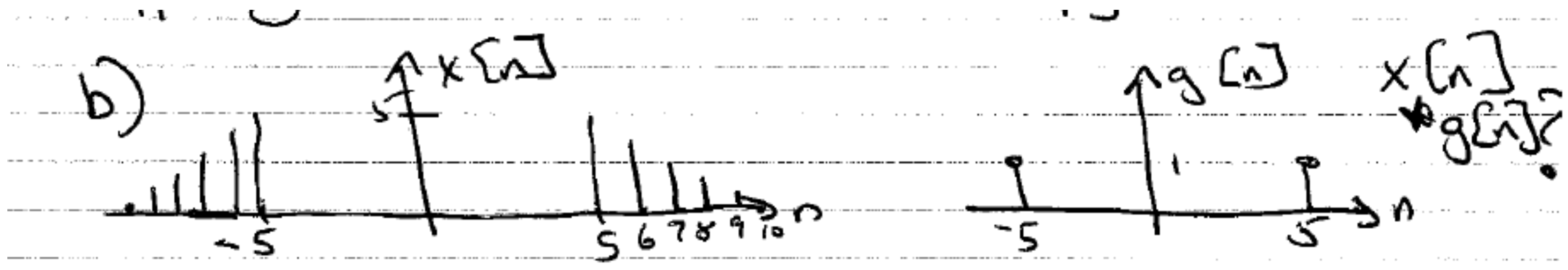
discrete convolution - example 4



computing discrete convolution 5

Compute the convolution {$\underline{2}$,3,4}*{$\underline{5}$,6,7} using $y[n] = u[n] \sum_{i=0}^{n} h[i]\, x[n-i]$

Solution: The length of the convolution is 3+3-1=5: {y[0]…y[4]}. Compute each of y[0],y[1],y[2],y[3],y[4] in succession as follows:

,

computing discrete convolution 6

Compute the convolution $\{\underline{2},3,4\}*\{\underline{5},6,7\}$ graphically.

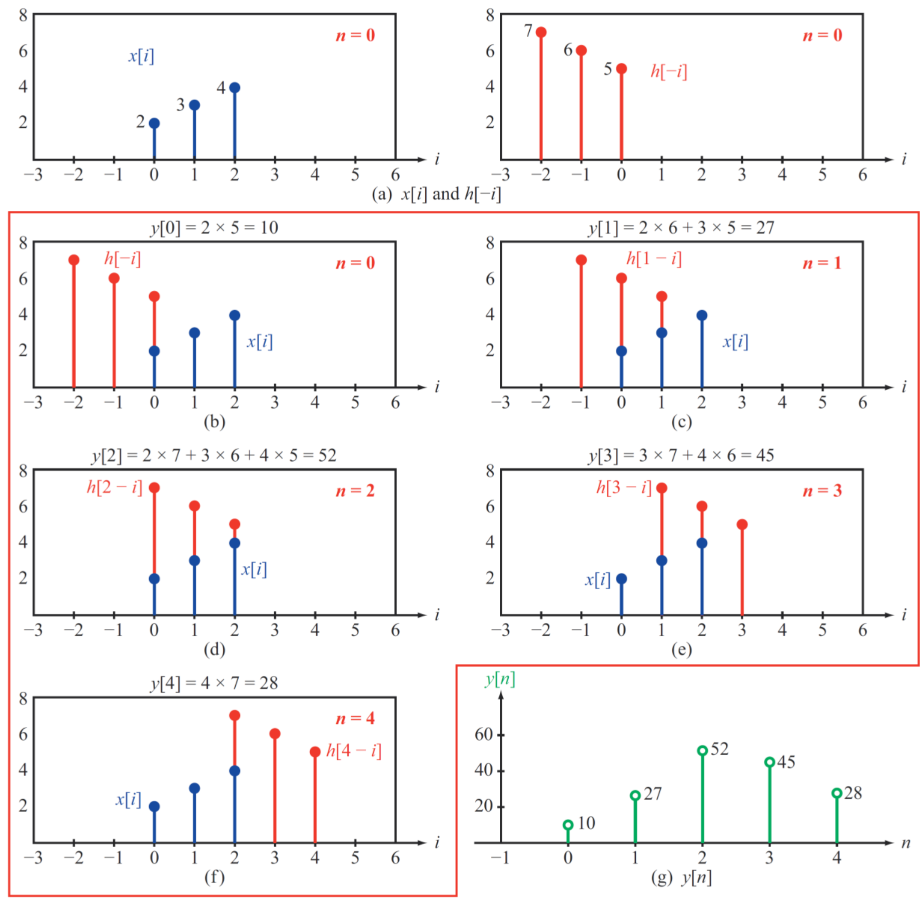


Figure 7-12: Graphical computation of convolution sum.

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computing discrete convolution 7

Example 4

find $y[k] = (.8)^{k+1} u[k] * u[k]$

$$= \sum_{n=-\infty}^{\infty} (.8)^{n+1} u[n] u[k-n]$$

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CONVOLUTION

- discrete $\quad y[k] = x_1[k] * x_2[k] = \sum_{n=-\infty}^{\infty} x_1[n]\, x_2[k-n]$

- continuous $\quad y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau = x(t) * h(t)$

① integration
② tables
③ graphically

$\left.\right\}$ ways of solving

$\delta(t) \rightarrow \boxed{H} \rightarrow h(t)$

$\delta[n] \rightarrow \boxed{H} \rightarrow h[n]$

impulse response function

- if we know $h(t)$ / $h[n]$ we can find response of system to any other input $x(t)$ / $x[n]$

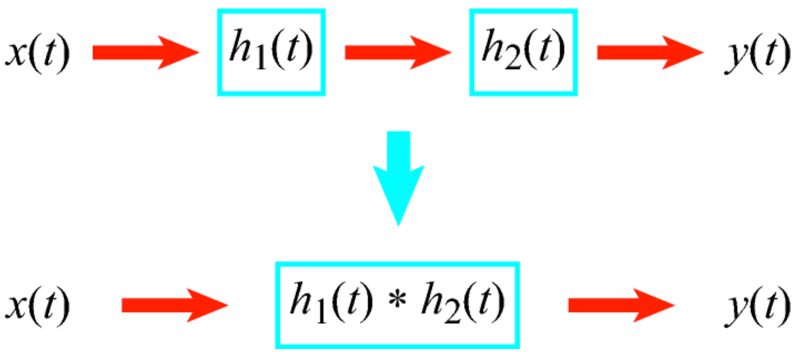
$$y(t) = x(t) * h(t)$$

$$y[n] = x[n] * h[n]$$

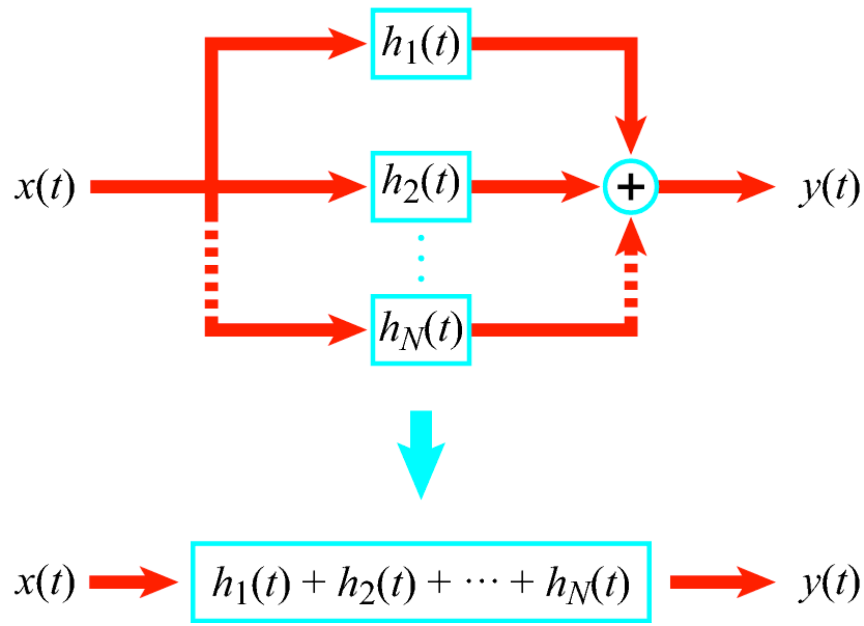
- graphical method of convolution

FLIP, SLIDE, MULTIPLY, & INTEGRATE

Systems Connected In-Series



Systems Connected In-Parallel

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