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Lecture - Tuesday Sept. 18, 2018 (2 hr session)

Slide 1,2 Recall: Roots of Equations → Bracketing & open methods (overview provided)

we started with graphical method & showed number of ways roots can occur in an interval between x_L (lower boundary) & x_u (upper boundary) range.

Slide 4
- if $f(x_L) \cdot f(x_u) > 0$ (same sign) → no roots or even # of roots
- if $f(x_L) \cdot f(x_u) < 0$ (opposite signs) → at least 1 root or odd # of roots.

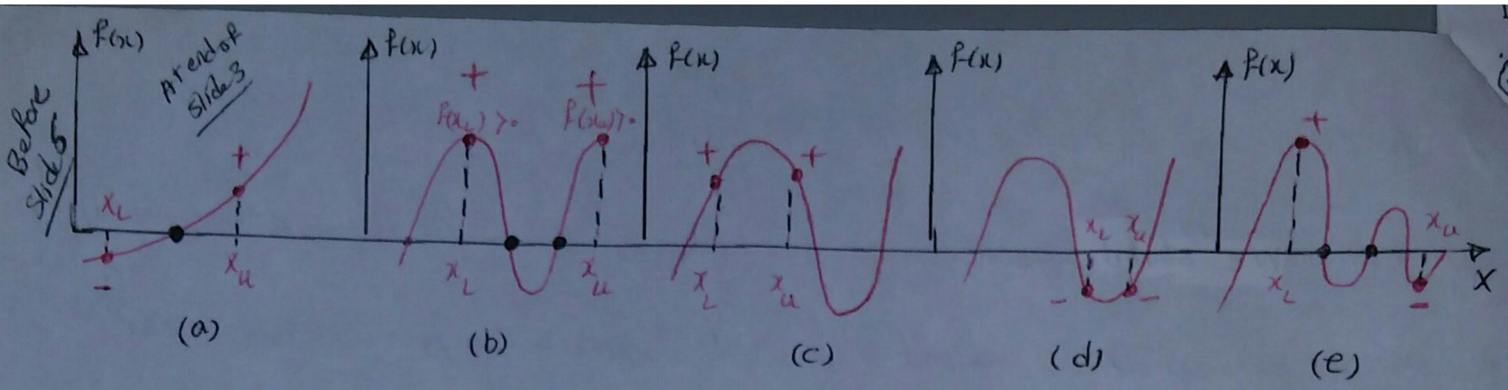
Exceptions → $f(x)$ tangent to x-axis as minimum or maximum or discontineous.

5.2. The Bisection method or Binary Chopping or Interval Halving or Bolzano's method.

Slide 3 Incremental search → locating an interval where the function changes sign. Identifies the location of sign change by dividing interval into number of subintervals. Each of these subintervals is searched to locate sign change. The subintervals are divided into finer increments → process repeated → root estimate is refined.

The Bisection method → Interval is always divided in half. If a function changes sign over an interval (x_L, x_u), the function value at the midpoint (x_m, x_r) is evaluated. The location of the root is then determined as lying at midpoint of the subinterval within which the sign change occurs. → process repeated → root refined.

* since method is based on finding root between two points, it falls under the category of bracketing. → This midpoint creat two interval $[x_L, x_m]$, $[x_m, x_u]$ (x_m: x_r)



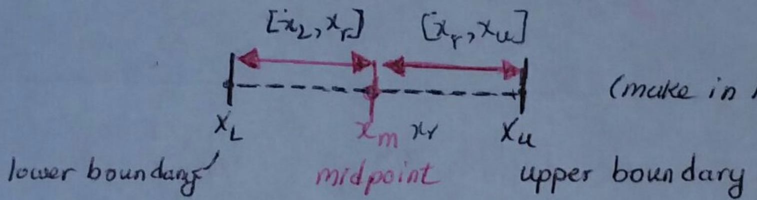
a) at least one root between x_L & x_u if $f(x)$ is real, continuous & changes sign.

b) $f(x_L)$ & $f(x_u) > 0$ → yet roots may still exist between x_L & x_u

c) & d) $f(x_L) \cdot f(x_u) > 0$ (no sign change) → may not be any root between x_L & x_u

d) Sign change $f(x_L) f(x_u) < 0$ → more than one root may exist between x_L & x_u.

Note → In Bisection method
Before slide 5



(make in half)

$[x_L, x_u]$ → Divided in 2 → Two brackets $[x_L, x_m]$, $[x_m, x_u]$

Then we can evaluate if in each section there is a sign change → first $[x_L, x_m]$ range then in the range of $[x_m, x_u]$. → These intervals can also be halved.

Algorithm for Bisection method (slide 5)

1. Start with two initial guesses for x_L and x_u; choose in such a way that $f(x_L) \cdot f(x_u) < 0$, or $f(x)$ changes sign between $[x_L, x_u]$

2. Estimate root as x_m (midpoint) of equation $f(x) = 0$ $\quad x_r \; x_m = \frac{x_L + x_u}{2}$

3. Check: if $f(x_L) \cdot f(x_m) < 0$ → root between x_L and x_m → $x_l = x_l$ and $x_u = x_m$

   if $f(x_L) \cdot f(x_m) > 0$ → root between x_m and x_u → $x_l = x_m$ & $x_u = x_u$.

   if $f(x_L) \cdot f(x_m) = 0$ → root is x_m (stop algorithm if this is true)

(4) Find new estimate of root $x_m = \frac{x_L + x_u}{2}$ (slide 8)

Find absolute relative approximate error: $|\epsilon_a| = \left| \frac{x_m^{new} - x_m^{old}}{x_m^{new}} \right| \times 100\%$

x_m^{new} = estimated root from present iteration

x_m^{old} = " " " previous "

(5) Compare the absolute relative approximate error $|\epsilon_a|$ with the pre-specified relative error tolerance ϵ_s (stopping criterion). If $|\epsilon_a| > \epsilon_s$, then go to step (3), else stop the algorithm. the number of iteration needs to be checked against the maximum number of iterations allowed.

Advantages of Bisection method: → always convergent as it brackets the root. Robust algorithm – always works

→ Intervals gets halved in iterations → one can garantee the error in the solution.

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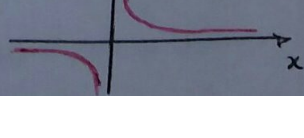
Drawbacks of bisection method: (1) Slow convergence (based on interval halving) cannot handle multiple roots

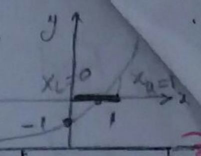
(2) If one of the initial guesses is closer to the root, it will take larger number of iterations to reach the root. (3) If $f(x)$ only touches x-axis (e.g. $x^2=0$) → it will be unable to find the lower guess x_L & upper guess x_u such that $f(x_L).f(x_u) < 0$ (4) For $f(x)$ where there is sigularity and it reverses sign at singularity, the bisection method may converge on the singularity. [Singularity in a function is defined as a point where the function becomes infinite (e.g. for $\frac{1}{x}$, $x=0$ is point of singularity, as it is infinite)

$f(x) = \frac{1}{x}$ $x_L = -2$, $x_u = 3$ are valid initial guesses that $f(x_L).f(x_u) < 0$ → but function is not contineous & then the theorem that root exist is not applicable.



Slide 9

Example 1. Bisection method. Find the roots of $f(x) = x^2 - e^{-x}$



① 2 initial guess $\to x_L = 0$, $\to x_u = 1$

in such a way that $f(x_L) \cdot f(x_u) < 0$

$f(0) = -1$, $f(1) = 0.632$

$f(0) \cdot f(1) < 0$

Check $f(x_L) \cdot f(m)$

now: $(-1)(-0.356) > 0$

$\Rightarrow x_l = x_m$ (x_r) & $x_u = x_u$ (x_u stays the same) (x_m value will be the value of lower bound)

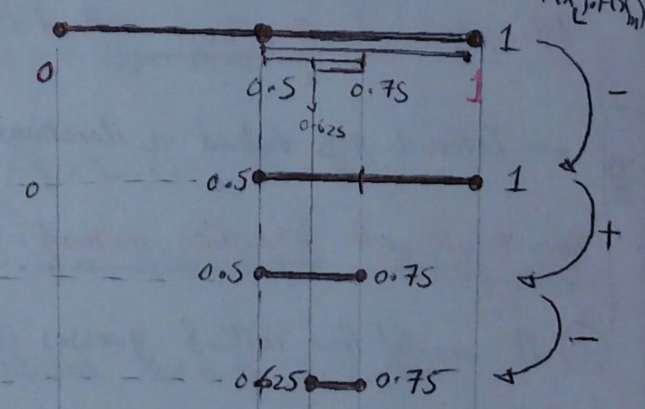
iteration	x_l	x_u	x_m or x_r	$f(x_l)$	$f(x_u)$	$f(x_m)$	$f(x_l)f(x_m)$
1	0	1	0.5	-1	0.632	-0.356	$-$
2	0.5	1	0.75	-0.356	0.632	0.0901	$+$
3	0.5	0.75	0.625	-0.356	0.0901	-0.145	$-$
4	0.625	0.75	0.6875 etc.				

② $x_l = 0.5$, $x_u = 1$ $\quad x_m = \dfrac{1+0.5}{2} = 0.75$

$f(0.5) = -0.356$ $\quad f(1) = 0.632$ $\quad f(0.75) = 0.901$

$f(0.5) \cdot f(0.75) = (-0.356)(+0.0901) < 0$

$\Rightarrow$ root between x_L and $x_m \Rightarrow x_l = x_l$, $x_u = x_m$ (x_r)



③ $x_l = 0.5$, $x_u = 0.75$ $\quad x_m$ (x_r) $= \dfrac{0.5+0.75}{2} = 0.625$

$f(0.5) = -0.356$, $f(0.75) = (0.0901)$ $f(0.625) = -0.145$

$f(x_l) \cdot f(x_m) = f(0.5) \cdot f(0.625) > 0 \to x_l = x_m$ & $x_u = x_u$

Continued until terminated based on specified criteria, approximate error.

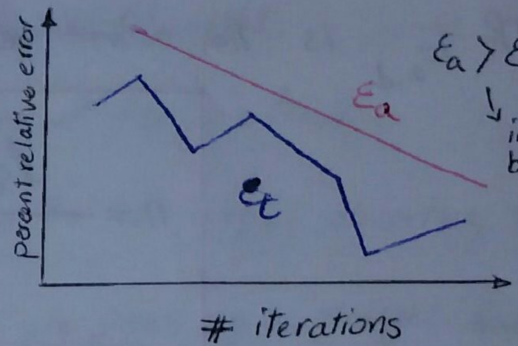
$$\varepsilon_a = \left[\frac{x_m^{new} - x_m^{old}}{x_m^{new}} \right] \times 100\%$$

	ε_a (Relative approximate error)
2	$\dfrac{0.75 - 0.5}{0.75} \times 100 = 33.33\%$
3	$\dfrac{0.625 - 0.75}{0.625} \times 100 = 20\%$
4	$\dfrac{0.6875 - 0.625}{0.6875} \times 100 = 9.09\%$

10th iteration

$\simeq 0.11\%$

you can also claculate the ε_t (Relative percent true erro) & compare the results with the "Relative percent approximate error" → You will see that the true error has "ragged" nature. This is due to the fact that, for bisection, the true root can lie anywhere within the bracketing interval. The true & approximate errors are far apart when the interval happens to be centered on the true root. They are close when the true root falls at either end of the interval.



why? $\varepsilon_a > \varepsilon_t$ in bisection

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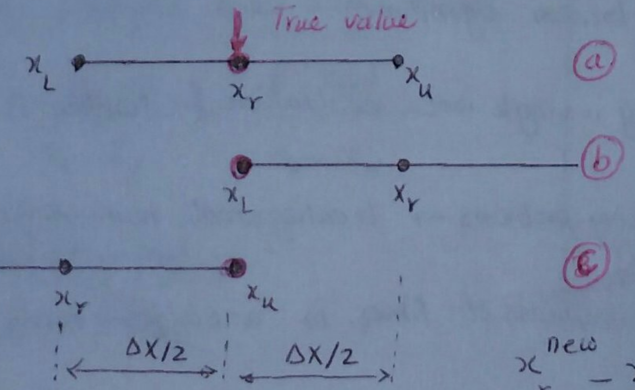
Best case:

Worst case:



(a) 3 ways that interval may bracket the root. (a) True value at the center of interval (c), (b) true value near the extreme

(b)

(c)

* discrepency between true value & midpoint never exceeds half the interval length.

$$x_r^{new} - x_r^{old} = \frac{\Delta X}{2} = \text{Upper bound on } \varepsilon_t$$

Calculating number of iterations to achieve desired error level: Slide 11

one of the benefits of bisection method is that the number of iterations required to attain an absolute error can be computed "a priori" - before starting the iterations.

$$E_a^0 = \Delta X^0 = X_u^0 - X_L^0 \qquad E_a^0 = \text{starting absolute error}$$

$$E_a^1 = \Delta X^0/2 \qquad \text{1st iteration}$$

$$E_a^2 = \frac{\Delta X^1}{2} = \frac{\Delta X^0}{2\times 2} = \frac{\Delta X^0}{2^2} \qquad \text{2nd "}$$

$$E_a^n = \frac{\Delta X^0}{2^n} \qquad \text{n th iteration}$$

$$n = \frac{\log_{10}\left(\frac{\Delta X^0}{E_{a,d}}\right)}{\log_{10} 2}$$

$E_{a,d}$ → desired error

False-position Method (Linear Interpolation Method)

Bisection method → Valid → But has "brute-force" approach → inefficient.

False-position → alternative method based on a graphical insight.

section method, no account is taken of the magnitudes of $f(x_l)$ & $f(x_u)$.

f $f(x_l)$ is much closer to zero than $f(x_u)$, then the root may be closer to x_l than

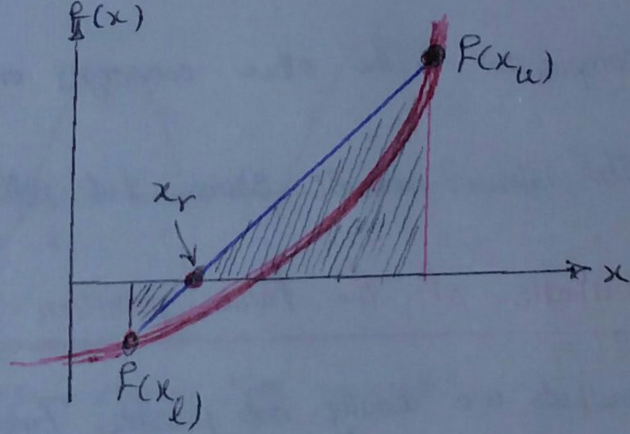
In false-position method, $f(x_l)$ & $f(x_u)$ points are joined by a straight line.

instead of curve). Replacement of the curve by a straight line gives "false-position"

f the root. → This method is also called "linear interpolation method".

he intersection of straight line with x-axis

an be estimated: $\dfrac{f(x_l)}{x_r - x_l} = \dfrac{f(x_u)}{x_r - x_u}$



$$x_r = x_u - \frac{f(x_u)(x - x_u)}{f(x_l) - f(x_u)}$$

$$(x_l)(x_r - x_u) = f(x_u)(x_r - x_l)$$

$$\boxed{x_r = \frac{x_u f(x_l) - x_l f(x_u)}{f(x_l) - f(x_u)}}$$ ← False-position method (one of the forms)

lso possible: $x_r = \dfrac{x_u f(x_l)}{f(x_l) - f(x_u)} - \dfrac{x_l f(x_u)}{f(x_l) - f(x_u)}$

Another form

$$x_r = x_u + \frac{x_u f(x_l)}{f(x_l) - f(x_u)} - x_u - \frac{x_l f(x_u)}{f(x_l) - f(x_u)} \Rightarrow \boxed{x_r = x_u + \frac{x_u f(x_u)}{f(x_l) - f(x_u)} - \frac{x_l f(x_u)}{f(x_l) - f(x_u)}}$$

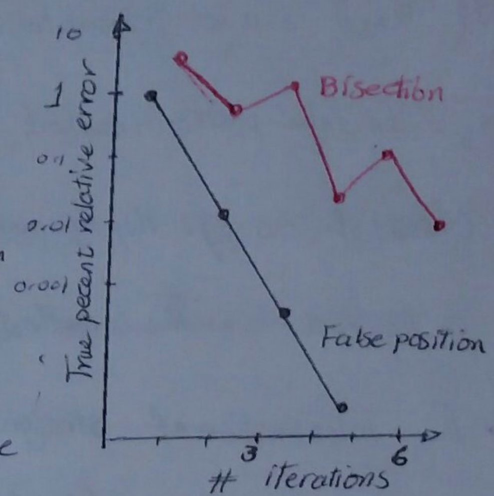
r $x_r = x_u - \dfrac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$

Comparison of Bisection & False-position

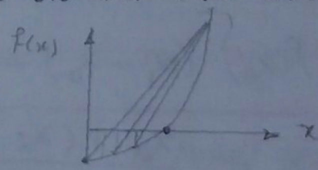
- If the true percent relative error is plotted against iteration # , it is noticeable how the error for false position decreases much faster than bisection. Why?

Because of the more efficient scheme for root location in the false-position method.

- In Bisection interval of (x_L, x_u) becomes smaller & smaller during computation. 1st iteration $\rightarrow \frac{\Delta x}{2} = \frac{|x_u - x_L|}{2}$ interval provided a measure of the error. However in the False-position one of the initial guesses may stay fixed throughout the computation as the other converges on the root. In this case the interval doesn't shrink but rather approaches a constant value.



- **pitfalls of the false-position method:** Generalization regarding root-locating methods are usually not possible. False-position usually is superior to bisection $\rightarrow$ Yet some cases violate this general conclusion:



- "one-Sideness" $\rightarrow$ as iterations are proceeding, one of the bracketing points will tend to stay fixed $\rightarrow$ lead to poor convergence $\rightarrow$ esp. for functions with significant curvature.

- **Modified version:** mitigate "one-sided" nature $\rightarrow$ detecting when one of the bounds is struck $\rightarrow$ if happens, the function value at stagnent bound can be divided in half.

5.4. Incremental searches and Determining Initial Guesses: (optional)

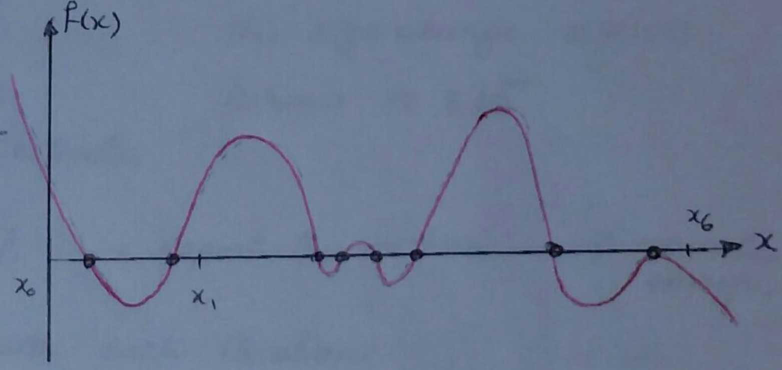
→ check each root located → check if all roots located → plotting is useful

→ Incremental search at the begining of the computer program, is an option.

→ Starting at one end of the region of intrest and then making function evaluations at small increments across the region. → Sign changes → means root falls within increment.

- Choice of increment length → if too small → time-consuming search
  ↓
  (more difficult for multiple roots) → if too large → closely spaced roots might be missed.

Cases where roots may be missed ←

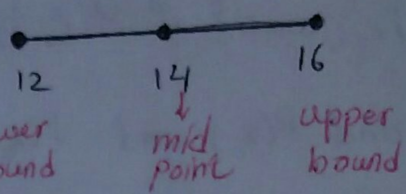


Increment length of search procedure is too large

Remedy: compute $f'(x)$ at the begining & end of each interval. → if sign changes → Min or Max occured → interval should be examined more closely for the existence of a possible root.

The estimate creates true percent relative error → $\varepsilon_t = 5.3\%$

Now lower bound Midpoint product is found:



$f(12)\,f(14) = 6.114\,(1.611) = 9.850$

lower bound — midpoint

$f(12) \cdot f(14) > 0$ ⇒ No sign change occurs. between lower bound & mid point.

Between 14 & 16 →

$x_r = \frac{14+16}{2} = 15$ → $\varepsilon_t = 1.3\%$

$f(14) \cdot f(15) = 1.611\,(-0.384) = -0.619 < 0$ →

the sign change occurs between 14 & 15

$x_r = \frac{14+15}{2} = 14.5$ new root estimate

→ $\varepsilon_t = 2.0\%$ → Repeat the method → until accurate enough

Note: true error doesn't decrease with each iteration.

→ The interval that contains the root is halved

Termination criteria & Error estimates → Decide when to terminate process

→ When true error falls below some prespecified level.

→ We Require an error estimate that is not contingent on foreknowledge of root.

$$\varepsilon_a = \left| \frac{x_r^{new} - x_r^{old}}{x_r^{new}} \right| \times 100\%$$

x_r^{new} = root for present iteration

x_r^{old} = " " previous "

→ approximate percent relative error

when $\varepsilon_a < \varepsilon_s$ → The computation can be terminated.

ε_s → stopping criterion

5.2. The Bisection Method or binary chopping or interval halving or Bolzano method

In general if $f(x)$ is real and continuous in the interval from x_l to x_u and $f(x_l)$ and $f(x_u)$ have ~~positive~~ opposite signs $\rightarrow f(x_l).f(x_u) < 0$ there is at least one real root between x_l and x_u.

Incremental Search methods $\rightarrow$ locating an interval where the function changes sign. Identifies the location of sign change by dividing the interval into ~~many~~ number of subintervals. Each of these subintervals is searched to locate the sign change. The subintervals are divided into finer increments $\rightarrow$ process repeated $\rightarrow$ root estimate refined

The Bisection method $\rightarrow$ Interval is always divided into half. If a function changes sign over an interval, the function value at the midpoint is evaluated. The location of the root is then determined as lying at the midpoint of the subinterval within which the sign change occurs. $\rightarrow$ process repead $\rightarrow$ root refined

Example. $f(c) = \frac{(9.81)(68.1)}{c}\left(1 - e^{-(c/68.1)(10)}\right) - 40$

$m = 68.1\,kg$, $v = 40\ m/s$, $t = 10s$, $g = 9.81\ m/s^2$ $\qquad f(c) = \frac{668.06}{c}\left(1 - e^{-0.146843c}\right) - 40$

Numerical solution for finding c, using bisection method:

① Guess two values of unknown C that gives value for $f(c)$ with two different signs. $\rightarrow$ The function changes sign between 12 & 16 $\rightarrow x_r = \frac{12+16}{2} = 14$

initial estimate of the root.

Example – Bisection method : $f(x) = x^2 - e^{-x}$

Procedure : 1) choose x_l and x_u such that $f(x)$ changes sign

2) Estimate root by $x_r = \frac{x_l + x_u}{2}$

3) Identify position of x_r, if $f(x_l)$ & $f(x_r)$ have same sign then $x_l = x_r$

if $f(x_u)$ and $f(x_r)$ have same sign, then $x_u = x_r$

4) Repeat the process.

until the specified criteria is met.

lower bound upper bound Root estimate

it#	x_l	x_u	x_r	$f(x_l)$	$f(x_u)$	$f(x_r)$
1	0	1	0.5	−1	0.632	−0.356
2	0.5	1	0.75	−0.356	0.632	0.0901
3	0.5	0.75	0.625	−0.356	0.0901	−0.145
4	0.625	0.75	⋮			