Part 3. Linear Algebraic Equations
Ch 10. LU Decomposition and Matrix Inversion

Lecture 11

Lower-Upper (LU) Decomposition

10.1

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Learning Outcomes

1. To decompose a nonsingular matrix into Lower , [L] and Upper , [U] triangular matrices

2. To find Inverse of a matrix using LU decomposition

LU Decomposition

- Provides an efficient way to compute matrix inverse by separating the time consuming elimination of the matrix [A] from manipulations of the right-hand side {B}.

- *Gauss elimination*, in which the forward elimination comprises the bulk of the computational effort, can be implemented as an LU decomposition.

LU Decomposition

The non-singular matrix **[*A*]**, can be written as two matrices of:

[L] - lower triangular matrix

[U] - upper triangular matrix

Where

$$\boxed{[A] = [L]\,[U]}$$

LU Decomposition

$$[A]=[L]\,[U]=\begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

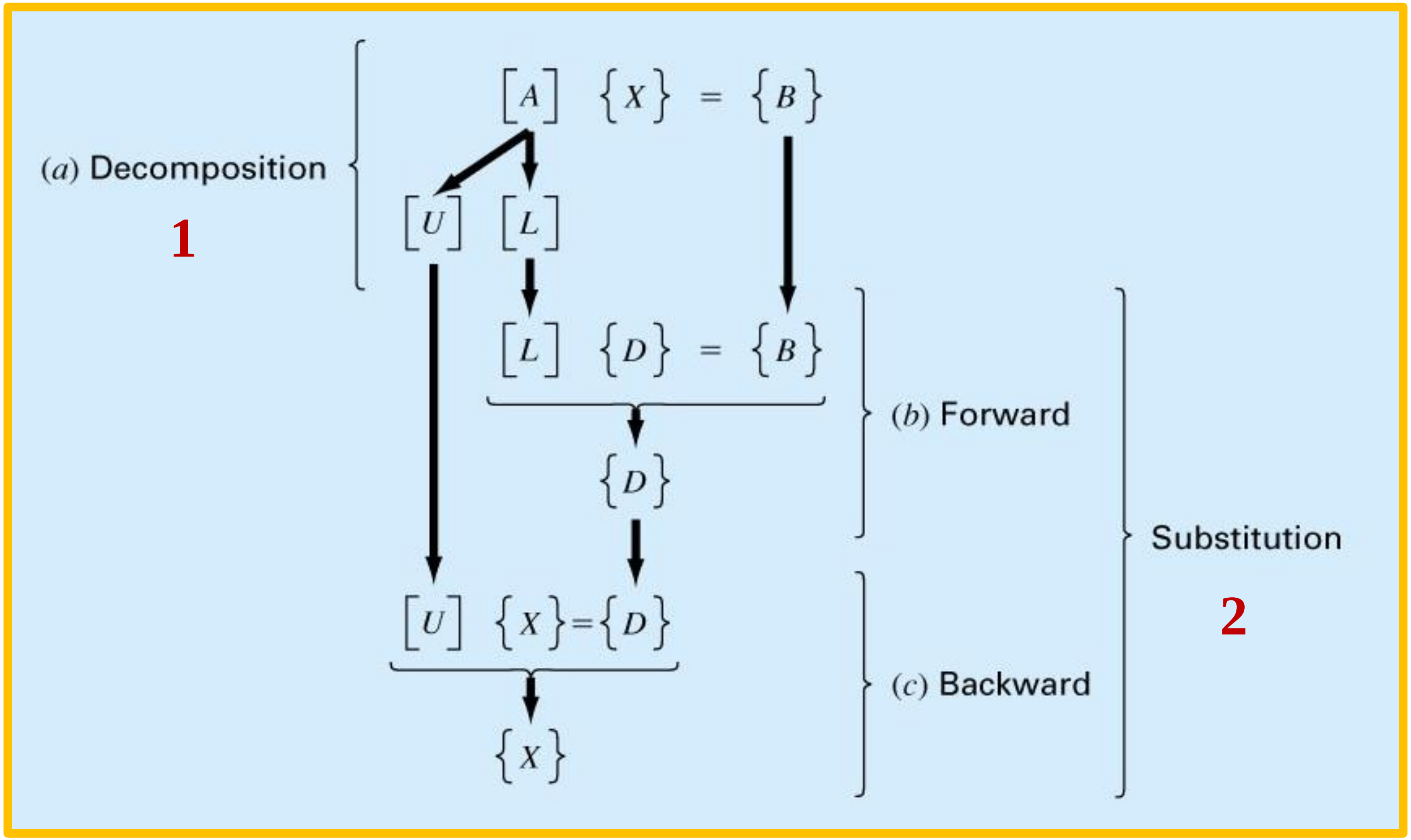
[*U*] is found at the end of the forward elimination step of [A]

[*L*] is obtained using the *multipliers* used in the forward elimination process

LU Decomposition Technique

• A set of linear equations:	$[A]\{X\} = \{B\}$
If $[A] = [L][U]$:	$[L][U]\{X\} = \{B\}$
Multiply by:	$[L]^{-1}$
Which gives:	$[L]^{-1}[L][U]\{X\} = [L]^{-1}\{B\}$
$[L]^{-1}[L] = [I]$:	$[I][U]\{X\} = [L]^{-1}\{B\}$
If $[I][U] = [U]$:	$[U]\{X\} = [L]^{-1}\{B\}$
Let:	$[L]^{-1}\{B\} = [D]$
Then:	$[L][D] = \{B\} \quad (1)$
	$[U]\{X\} = [D] \quad (2)$

LU Decomposition Technique



- Matrix [A] decomposed into [U]and [L]
- Solve for {D} as intermediate value
- Solve for {X} knowing {D}

LU Decomposition Technique

Given $[A][X] = [B]$

1. Decompose $[A]$ into $[L]$ and $[U]$

   - First find [U] by forward elimination steps

   - Then find elements of [L]

2. Solve $[L][D] = [B]$ for $[D]$

3. Solve $[U][X] = [D]$ for $[X]$

Finding Upper Triangular Matrix [U]

$$[A]=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Forward Elimination:

3 equations→ requires 2 steps of elimination

n equations→ requires n-1 steps of elimination

Step 1. Eliminate a_{21} and a_{31}

1.1. Make $a_{21} = 0$

Row 2= Row 2- (Row 1×($\mathbf{a_{21}}$ /a_{11}))

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \mathbf{0} & a'_{22} & a'_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

1.2. Make $a_{31} = 0$

Row 3= Row 3- (Row 1×($\mathbf{a_{31}}$ /a_{11}))

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \mathbf{0} & a'_{22} & a'_{23} \\ \mathbf{0} & a'_{32} & a'_{33} \end{bmatrix}$$

Step 2. Eliminate a_{32}

2.1. Make $a_{32} = 0$

Row 3= Row 3- (Row 2×($\mathbf{a'_{32}}$ /a'_{22}))

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \mathbf{0} & a'_{22} & a'_{23} \\ \mathbf{0} & \mathbf{0} & a''_{33} \end{bmatrix}$$

End of Eliminations

Equals to [U]

$$\begin{bmatrix} u_{11} & u_{12} & u \\ \mathbf{0} & u_{22} & u_{23} \\ \mathbf{0} & \mathbf{0} & u_{33} \end{bmatrix} = [U]$$

Finding Lower Triangular Matrix [L]

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

- Elements are obtained using the *multipliers* that were used in the forward elimination process
- Diagonal elements are always 1

Multipliers in Step 1
of forward elimination
for elements of 1st column of [L]

$$l_{21} = a_{21} / a_{11}$$

$$l_{31} = a_{31} / a_{11}$$

Multiplier in Step 2
of forward elimination
for element of 2nd column of [L]

$$l_{32} = a'_{32} / a'_{22}$$

Example 1. Solve the following system of 3 equations using LU decomposition

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Forward elimination to find [U]

Step 1. Eliminate a_{21} and a_{31}

1.1. Make $a_{21} = 0$

Row 2= Row 2- (Row 1×($\mathbf{a_{21}}$ /a_{11}))

$$\frac{64}{25} = 2.56; \quad Row2 - Row1(2.56) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 144 & 12 & 1 \end{bmatrix}$$

1.2. Make $a_{31} = 0$

Row 3= Row 3- (Row 1×($\mathbf{a_{31}}$ /a_{11}))

$$\frac{144}{25} = 5.76; \quad Row3 - Row1(5.76) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

Example 1. Continued. Forward elimination to find [U]- Step 2

At the end of Step 1 → $\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$

Step 2. Eliminate a_{32}
2.1. Make $a_{32} = 0$ Row 3= Row 3- (Row 2×(**a'$_{32}$** /a'$_{22}$))

$$\frac{-16.8}{-4.8} = 3.5; \quad Row3 - Row2(3.5) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Then $[U] = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$

Example 1. Continued. Find matrix [L]

Use multipliers used during the Forward Elimination

$$\begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}$$

From 1st step

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

$$\ell_{21} = \frac{a_{21}}{a_{11}} = \frac{64}{25} = 2.56$$

$$\ell_{31} = \frac{a_{31}}{a_{11}} = \frac{144}{25} = 5.76$$

From 2nd step

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

$$\ell_{32} = \frac{a'_{32}}{a'_{22}} = \frac{-16.8}{-4.8} = 3.5$$

Then

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix}$$

Example 1. Continued. Verification: [L] [U] = [A] ?

$$[L][U]=\begin{bmatrix}1 & 0 & 0\\ 2.56 & 1 & 0\\ 5.76 & 3.5 & 1\end{bmatrix}\begin{bmatrix}25 & 5 & 1\\ 0 & -4.8 & -1.56\\ 0 & 0 & 0.7\end{bmatrix}= \quad ?$$

Is it equal to? [A] = $\begin{bmatrix}25 & 5 & 1\\ 64 & 8 & 1\\ 144 & 12 & 1\end{bmatrix}$

Example 1. Continued. Setting [L] [D] = [B]

$$\begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Solve for $[D]$

$$d_1 = 106.8$$

$$2.56 d_1 + d_2 = 177.2$$

$$5.76 d_1 + 3.5 d_2 + d_3 = 279.2$$

$$[D] = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

Example 1. Continued. Setting [U] [X] = [D]

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

Solve for $[X]$

$$25x_1 + 5x_2 + x_3 = 106.8$$

$$-4.8x_2 - 1.56x_3 = -96.21$$

$$0.7x_3 = 0.735$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.2900 \\ 19.70 \\ 1.050 \end{bmatrix}$$

Example 2. Solve using LU decomposition

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

$$[\mathbf{U}] = \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.00333 & -0.93333 \\ 0 & 0 & 10.0120 \end{bmatrix}$$

Forward Elimination (Gaussian)

$$l_{21} = \frac{a_{21}}{a_{11}} = \frac{0.1}{3} \qquad l_{31} = \frac{a_{31}}{a_{11}} = \frac{0.3}{3} \qquad l_{32} = \frac{a'_{32}}{a'_{22}} = \frac{-0.19}{7.00333}$$

$$[\mathbf{L}] = \begin{bmatrix} 1 & 0 & 0 \\ 0.03333 & 1 & 0 \\ 0.1000 & -0.02713 & 1 \end{bmatrix}$$

One on diagonal,
zero above diagonal

Example 2. Continued. Verification

[L] **[U]**

$$\begin{bmatrix} 1 & \mathbf{0} & \mathbf{0} \\ 0.03333 & 1 & \mathbf{0} \\ 0.1000 & -0.02713 & 1 \end{bmatrix} \begin{bmatrix} 3 & -0.1 & -0.2 \\ \mathbf{0} & 7.00333 & -0.293333 \\ \mathbf{0} & \mathbf{0} & 10.0120 \end{bmatrix}$$

Check if [L] [U] = [A] ?

$$[A] = \begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix}$$

Forward substitution to find **[D]** using **[L] [D] = [B]**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.03333 & 1 & 0 \\ 0.1000 & -0.02713 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

$\boxed{\boldsymbol{d_1} = 7.85}$

$0.03333\,(7.85) + \boldsymbol{d_2} = -\,19.3 \rightarrow \boxed{\boldsymbol{d_2} = -19.5617}$

$0.1\,(7.85) - 0.02713\,(-19.5617) + \boldsymbol{d_3} = 71.4 \rightarrow \boxed{\boldsymbol{d_3} = 70.0843}$

Back substitution to find [X] using [U] [X] = [D]

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.00333 & -0.293333 \\ 0 & 0 & 10.0120 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.5617 \\ 70.0843 \end{Bmatrix}$$

$\boldsymbol{x_3}$ = 70.0843 / 10.0120 = 7.00003 => $\boxed{\boldsymbol{x_3} = 7}$

$7.00333\ \ \boldsymbol{x_2}$ - 0.293333 (7) = -19.5617 → $\boxed{\boldsymbol{x_2} = -2.5}$

3 $\boldsymbol{x_1}$ - 0.1 (- 2.5) – 0.2 (7.00003) – 7.85 → $\boxed{\boldsymbol{x_1} = 3}$

Notes

- Computational effort almost the same as Gaussian Elimination (G.E.)

- Storage similar, **[L]** and **[U]** can be stored in the same matrix

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Lecture 12

Matrix Inversion

10.2

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Inverse of a Matrix

The inverse [B] of a square matrix [A] is defined as

$$[A][B] = [I] = [B][A] \quad , \quad [B] = [A]^{-1}$$

$$[A]\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[A]\begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad [A]\begin{bmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \qquad [A]\begin{bmatrix} b_{1n} \\ b_{2n} \\ \vdots \\ b_{nn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

1^{st} Column of [B] **2^{nd} Column of [B]** **n^{th} Column of [B]**

Example 3. Matrix inverse. Find $[A]^{-1}$ using LU decomposition

$$[A]=\begin{bmatrix}25 & 5 & 1\\ 64 & 8 & 1\\ 144 & 12 & 1\end{bmatrix}$$

If $[A]^{-1}=[B]=\begin{bmatrix} b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{bmatrix}$

$$[A]=[L][U]=\begin{bmatrix}1 & 0 & 0\\ 2.56 & 1 & 0\\ 5.76 & 3.5 & 1\end{bmatrix}\begin{bmatrix}25 & 5 & 1\\ 0 & -4.8 & -1.56\\ 0 & 0 & 0.7\end{bmatrix}$$

Example 3. Matrix inverse. Continued. Overview

First Column

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Second Column

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}\begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Third Column

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}\begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Using LU decomposition

Using LU decomposition

$$\begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} -0.08333 \\ 1.417 \\ -5.000 \end{bmatrix}$$

$$\begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} -0.08333 \\ 1.417 \\ -5.000 \end{bmatrix}$$

$$\begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0.03571 \\ -0.4643 \\ 1.429 \end{bmatrix}$$

Example 3. Matrix inverse. Continued. Using [L] and [U]

Solving for the each column of $[B]$ requires two steps

1) Solve $[L]\ [Z] = [C]$ for $[Z]$, 2) Solve $[U]\ [X] = [Z]$ for $[X]$

Solve $[L]\ [Z] = [C]$ for $[Z]$

$$[L][Z]=[C]\rightarrow\begin{bmatrix}1 & 0 & 0\\ 2.56 & 1 & 0\\ 5.76 & 3.5 & 1\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix}=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$$

$$z_1 = 1$$
$$2.56z_1 + z_2 = 0$$
$$5.76z_1 + 3.5z_2 + z_3 = 0$$

$$[Z]=\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix}=\begin{bmatrix}1\\ -2.56\\ 3.2\end{bmatrix}$$

Example 3. Matrix inverse. Continued. Using [L] and [U]

Solving $[U][X] = [Z]$ for $[X]$ $$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix}$$

$$\begin{aligned} 25b_{11} + 5b_{21} + b_{31} &= 1 \\ -4.8b_{21} - 1.56b_{31} &= -2.56 \\ 0.7b_{31} &= 3.2 \end{aligned}$$

$$\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 0.04762 \\ -0.9524 \\ 4.571 \end{bmatrix}$$

1^{st} Column of [B]

Example 3. Matrix inverse. Continued. Using [L] and [U]

Repeat the process to find other columns

$$\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 0.04762 \\ -0.9524 \\ 4.571 \end{bmatrix} \qquad \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} -0.08333 \\ 1.417 \\ -5.000 \end{bmatrix} \qquad \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0.03571 \\ -0.4643 \\ 1.429 \end{bmatrix}$$

$$[A]^{-1} = \begin{bmatrix} 0.04762 & -0.08333 & 0.03571 \\ -0.9524 & 1.417 & -0.4643 \\ 4.571 & -5.000 & 1.429 \end{bmatrix}$$

Check:

$$[A][A]^{-1} = [I] = [A]^{-1}[A]$$


THE
END