MTE 203 – Advanced Calculus
Homework 10 (Solutions)

Triple Iterated Integrals in Cylindrical Coordinates

Problem 1: [13.11, Prob. 23]

Evaluate the triple integral $\int_0^4 \int_0^{\sqrt{4y-y^2}} \int_0^{x^2+y} dz\ dx\ dy$

Solution:

The limits define the first octant volume under $z = y + x^2$ and inside the cylinder $x^2 + y^2 = 4y$. The value of the triple iterated integral is therefore given by

$$\int_0^{\pi/2} \int_0^{4\sin\theta} \int_0^{r\sin\theta + r^2\cos^2\theta} r\,dz\,dr\,d\theta = \int_0^{\pi/2} \int_0^{4\sin\theta} r(r\sin\theta + r^2\cos^2\theta)\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left\{ \frac{r^3}{3}\sin\theta + \frac{r^4}{4}\cos^2\theta \right\}_0^{4\sin\theta} d\theta = \frac{1}{12}\int_0^{\pi/2} [4\sin\theta(4\sin\theta)^3 + 3\cos^2\theta(4\sin\theta)^4]\,d\theta$$

$$= \frac{64}{3}\int_0^{\pi/2} (\sin^4\theta + 3\cos^2\theta\sin^4\theta)\,d\theta = \frac{64}{3}\int_0^{\pi/2} \left[\left(\frac{1-\cos 2\theta}{2}\right)^2 + 3\left(\frac{1+\cos 2\theta}{2}\right)\left(\frac{1-\cos 2\theta}{2}\right)^2 \right] d\theta$$

$$= \frac{8}{3}\int_0^{\pi/2} [2(1 - 2\cos 2\theta + \cos^2 2\theta) + 3(1 - 2\cos 2\theta + \cos^2 2\theta) + 3\cos 2\theta(1 - 2\cos 2\theta + \cos^2 2\theta)]\,d\theta$$

$$= \frac{8}{3}\int_0^{\pi/2} \left[5 - 7\cos 2\theta - \left(\frac{1+\cos 4\theta}{2}\right) + 3\cos 2\theta(1 - \sin^2 2\theta) \right] d\theta$$

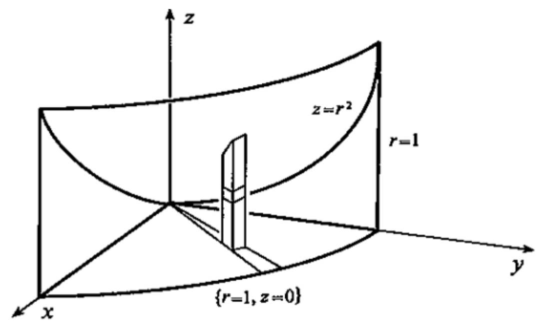
$$= \frac{8}{3}\left\{ \frac{9\theta}{2} - 2\sin 2\theta - \frac{1}{8}\sin 4\theta - \sin^3 2\theta \right\}_0^{\pi/2} = 6\pi$$

Problem 2: [13.11, Prob. 25]

Evaluate the triple integral $\int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^{x^2+y^2} y^2 dz\ dx\ dy$

Solution:

The limits determine the volume in the first octant bounded by the paraboloid $z = x^2 + y^2$ and the right circular cylinder $x^2 + y^2 = 1$. If we use a triple iterated integral with respect to z, r, and θ, then



$$I = \int_0^{\pi/2}\int_0^1\int_0^{r^2} r^2 \sin^2\theta\ r\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^1 r^5 \sin^2\theta\,dr\,d\theta = \int_0^{\pi/2} \frac{1}{6}\sin^2\theta\,d\theta$$
$$= \frac{1}{6}\int_0^{\pi/2}\left(\frac{1-\cos 2\theta}{2}\right)d\theta = \frac{1}{12}\left\{\theta - \frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} = \frac{\pi}{24}.$$

Problem 3: [13.11, Prob. 20] Application problem for Moment of Inertia - Cartesian Coordinates

Find the moment of inertia of a uniform sphere of radius *R* about any line through its center.

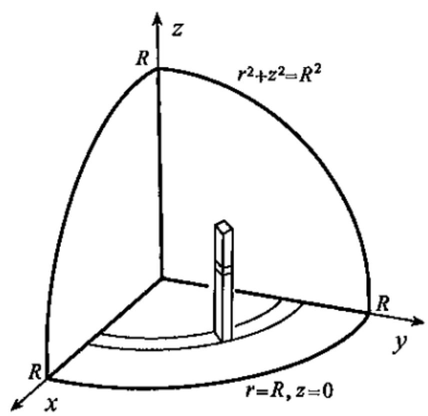
Solution:

We multiply the moment of inertia about the z-axis of that part in the first octant by eight.

$$I_z = 8\int_0^R\int_0^{\pi/2}\int_0^{\sqrt{R^2-r^2}} r^2\rho\,r\,dz\,d\theta\,dr$$
$$= 8\rho\int_0^R\int_0^{\pi/2} r^3\sqrt{R^2-r^2}\,d\theta\,dr$$
$$= 4\pi\rho\int_0^R r^3\sqrt{R^2-r^2}\,dr$$

If we set $u = R^2 - r^2$, then $du = -2r\,dr$, and

$$I_z = 4\pi\rho\int_{R^2}^0 (R^2-u)\sqrt{u}\left(-\frac{du}{2}\right)$$
$$= 2\pi\rho\left\{\frac{2}{3}R^2u^{3/2} - \frac{2}{5}u^{5/2}\right\}_0^{R^2} = \frac{8\pi\rho R^5}{15}.$$



Triple Iterated Integrals in Spherical Coordinates

Problem 4: [13.12, Prob. 13] Application problem for Centre of Mass- Spherical Coordinates

Find the center of mass of a uniform hemispherical solid.

Hint: $M = \left(\frac{2}{3}\right)\pi\rho R^3$

Solution:

For the hemisphere bounded by $z = \sqrt{R^2 - x^2 - y^2}$ and $z = 0$, $\overline{x} = \overline{y} = 0$. Since $M = (2/3)\pi\rho R^3$, and

$$M\overline{z} = 4\int_0^{\pi/2}\int_0^{\pi/2}\int_0^R (\Re\cos\phi)\rho\Re^2\sin\phi\, d\Re\, d\phi\, d\theta = 4\rho\int_0^{\pi/2}\int_0^{\pi/2}\left\{\frac{\Re^4}{4}\cos\phi\sin\phi\right\}_0^R d\phi\, d\theta$$

$$= \rho R^4\int_0^{\pi/2}\left\{\frac{1}{2}\sin^2\phi\right\}_0^{\pi/2} d\theta = \frac{\rho R^4}{2}\left\{\theta\right\}_0^{\pi/2} = \frac{\pi\rho R^4}{4},$$

it follows that $\overline{z} = \frac{\pi\rho R^4}{4}\frac{3}{2\pi\rho R^3} = \frac{3R}{8}$.

Problem 5: [13.12, Prob. 23]

Find the volume bounded by the surface $(x^2 + y^2 + z^2)^2 = 2z(x^2 + y^2)$.

Solution:

The equation of the surface in spherical coordinates is $\Re^4 = 2\Re\cos\phi(\Re^2\sin^2\phi) \implies \Re = 2\sin^2\phi\cos\phi$.

$$V = 4\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{2\sin^2\phi\cos\phi}\Re^2\sin\phi\, d\Re\, d\phi\, d\theta = 4\int_0^{\pi/2}\int_0^{\pi/2}\left\{\frac{\Re^3}{3}\sin\phi\right\}_0^{2\sin^2\phi\cos\phi} d\phi\, d\theta$$

$$= \frac{4}{3}\int_0^{\pi/2}\int_0^{\pi/2} 8\sin^7\phi\cos^3\phi\, d\phi\, d\theta = \frac{32}{3}\int_0^{\pi/2}\int_0^{\pi/2}\sin^7\phi(1-\sin^2\phi)\cos\phi\, d\phi\, d\theta$$

$$= \frac{32}{3}\int_0^{\pi/2}\left\{\frac{1}{8}\sin^8\phi - \frac{1}{10}\sin^{10}\phi\right\}_0^{\pi/2} d\theta = \frac{4}{15}\left\{\theta\right\}_0^{\pi/2} = \frac{2\pi}{15}$$

Problem 6: [13.12, Prob. 25]

A sphere of constant density ρ and radius R is located at the origin (figure below). If a mass m is situated at a point P on the z-axis (distance $d > R$ from the center of the sphere) and dV is small element of the volume of the sphere, then according to Newton's universal law of gravitation, the z-component of the force on m due to the mass in dV is given by

$$-\frac{Gm\rho dV cos\psi}{s^2}$$

Where G is a constant and s is distance between P and dV.

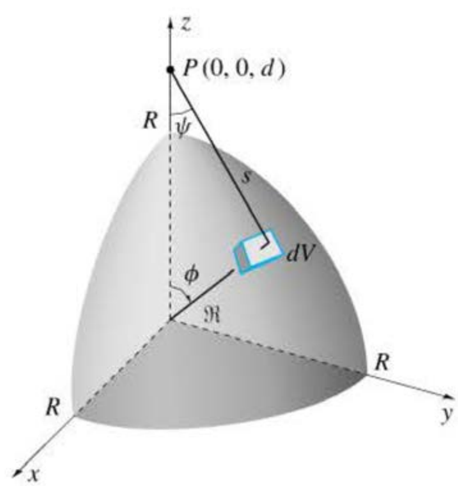
(a) Show that in spherical coordinates total force on m due to the entire sphere has z-component

$$F_z = -\frac{Gm\rho}{2d}\int_{-\pi}^{\pi}\int_0^{\pi}\int_0^R \left(\frac{s^2 + d^2 - \mathfrak{R}^2}{s^3}\right) \mathfrak{R}^2 \sin\varphi \, d\mathfrak{R} d\varphi d\theta$$

(b) Use the transformation in Exercise 24(b) to write F_z in the form

$$F_z = -\frac{Gm\rho}{2d^2}\int_{-\pi}^{\pi}\int_0^R \int_{d-\mathfrak{R}}^{d+\mathfrak{R}} \mathfrak{R}\left(\frac{s^2 + d^2 - \mathfrak{R}^2}{s^2}\right) ds d\mathfrak{R} d\theta$$

And show that $F_z = -GmM/d^2$, where M is the total mass of the sphere.



Solution:

(a) The cosine law for the triangle joining O, P, and dV gives $\Re^2 = s^2 + d^2 - 2sd\cos\psi$, and therefore

$$F_z = \iiint_V -\frac{Gm\rho}{s^2}\left(\frac{s^2+d^2-\Re^2}{2sd}\right)dV = -\frac{Gm\rho}{2d}\int_{-\pi}^{\pi}\int_0^{\pi}\int_0^R\left(\frac{s^2+d^2-\Re^2}{s^3}\right)\Re^2\sin\phi\,d\Re\,d\phi\,d\theta.$$

(b) In order to replace ϕ with s we first write

$$F_z = -\frac{Gm\rho}{2d}\int_{-\pi}^{\pi}\int_0^{R}\int_0^{\pi}\left(\frac{s^2+d^2-\Re^2}{s^3}\right)\Re^2\sin\phi\,d\phi\,d\Re\,d\theta.$$

If we set $s = \sqrt{\Re^2+d^2-2d\Re\cos\phi} \implies s^2 = \Re^2+d^2-2d\Re\cos\phi$, from which $2s\,ds = 2d\Re\sin\phi\,d\phi$, then

$$\begin{aligned}
F_z &= -\frac{Gm\rho}{2d}\int_{-\pi}^{\pi}\int_0^{R}\int_{d-\Re}^{d+\Re}\left(\frac{s^2+d^2-\Re^2}{s^3}\right)\Re^2\left(\frac{s\,ds}{d\Re}\right)d\Re\,d\theta \\
&= -\frac{Gm\rho}{2d^2}\int_{-\pi}^{\pi}\int_0^{R}\int_{d-\Re}^{d+\Re}\Re\left(\frac{s^2+d^2-\Re^2}{s^2}\right)ds\,d\Re\,d\theta \\
&= -\frac{Gm\rho}{2d^2}\int_{-\pi}^{\pi}\int_0^{R}\left\{\Re\left(s-\frac{d^2-\Re^2}{s}\right)\right\}_{d-\Re}^{d+\Re}d\Re\,d\theta \\
&= -\frac{Gm\rho}{2d^2}\int_{-\pi}^{\pi}\int_0^{R}\Re\left(d+\Re-\frac{d^2-\Re^2}{d+\Re}-d+\Re+\frac{d^2-\Re^2}{d-\Re}\right)d\Re\,d\theta \\
&= -\frac{2Gm\rho}{d^2}\int_{-\pi}^{\pi}\int_0^{R}\Re^2\,d\Re\,d\theta = -\frac{2Gm\rho}{d^2}\int_{-\pi}^{\pi}\left\{\frac{\Re^3}{3}\right\}_0^R d\theta \\
&= -\frac{2Gm\rho R^3}{3d^2}\left\{\theta\right\}_{-\pi}^{\pi} = -\frac{4\pi Gm\rho R^3}{3d^2} = -\frac{GmM}{d^2},
\end{aligned}$$

where M is the mass of the sphere.

Vector Fields:

Problem 7: [S. 14.1, Prob. 45]

Find all functions $f(x, y)$ such that $\vec{\nabla} f = \vec{F}$ and $\vec{F} = 2xy\hat{\imath} + x^2\hat{\jmath}$.

Solution:

If $\nabla f = 2xy\hat{\mathbf{i}} + x^2\hat{\mathbf{j}}$, then $\dfrac{\partial f}{\partial x} = 2xy$, $\dfrac{\partial f}{\partial y} = x^2$. Integrating the first gives $f(x, y) = x^2 y + v(y)$. Substitution into the second requires $x^2 + v'(y) = x^2 \implies v(y) = C = \text{constant}$. Thus, $f(x, y) = x^2 y + C$.

Plotting Vector Fields:

Problem 8:

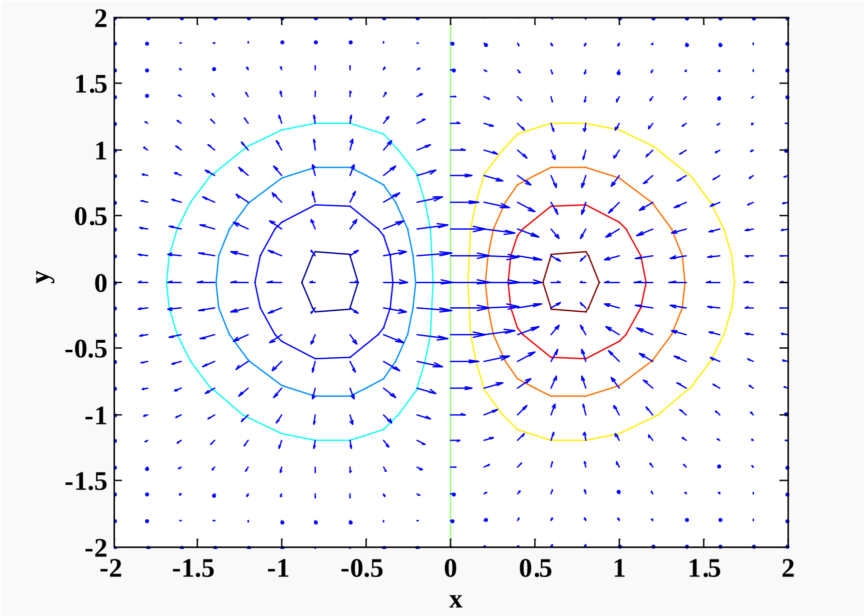
Use Matlab to plot the gradient of the function $z = xe^{-x^2-y^2}$. Then on the same plot, draw the contours of $z(x, y)$.

Solution:

Table below lists the gradient of z for different points

x	y	$\frac{\partial z}{\partial x}$	$\frac{\partial z}{\partial y}$
0	0.6	0.67	0
0.6	-1.2	0.048	0.238
-1.2	0.2	-0.41	0.105
-0.4	-0.4	0.482	-0.224

The vector field given by the gradient and the respective contour plots are shown below:

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%%% MATLAB m-file code:

[X,Y] = meshgrid(-2:.2:2);

Z = X.*exp(-X.^2 - Y.^2);

[DX,DY] = gradient(Z,.2,.2);

figure

contour(X,Y,Z)

hold on
quiver(X,Y,DX,DY)
hold off
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Line Integrals:

Problem 9: [S. 14.2, Prob. 13]

Evaluate the line integral $\int_C \; xyds$, where C is the curve $x = 1 - y^2, z = 0$ from (1,0,0) to (0,1,0)

Solution:

Using y as the parameter along the curve,

$$\int_C xy\,ds = \int_0^1 (1-y^2)y\sqrt{1+(-2y)^2}\,dy.$$

If we set $u = 1+4y^2$ and $du = 8y\,dy$, then

$$\int_C xy\,ds = \int_1^5 \left(1 - \frac{u-1}{4}\right)\sqrt{u}\left(\frac{du}{8}\right) = \frac{1}{32}\int_1^5 (5\sqrt{u} - u^{3/2})\,du$$

$$= \frac{1}{32}\left\{\frac{10u^{3/2}}{3} - \frac{2u^{5/2}}{5}\right\}_1^5 = \frac{25\sqrt{5}-11}{120}$$

Problem 10: [S. 14.2, Prob. 25]

The average value of function $f(x, y, z)$ along a curve C is defined as the value of the line integral of the function along the curve divided by the length of the curve.

Find the average value of the function along the curve

$f(x, y, z) = x^2 + y^2 + z^2$ along $x = cos\ t, y = sin\ t. z = t, 0 \leq t \leq \pi$

Solution:

Since $\quad L = \int_0^\pi \sqrt{(-\sin t)^2 + (\cos t)^2 + 1}\,dt = \int_0^\pi \sqrt{2}\,dt = \sqrt{2}\pi,$

$$\overline{f} = \frac{1}{\sqrt{2}\pi}\int_C (x^2+y^2+z^2)\,ds = \frac{1}{\sqrt{2}\pi}\int_0^\pi (\cos^2 t + \sin^2 t + t^2)\sqrt{2}\,dt$$

$$= \frac{1}{\pi}\int_0^\pi (1+t^2)\,dt = \frac{1}{\pi}\left\{t + \frac{t^3}{3}\right\}_0^\pi = \frac{3+\pi^2}{3}.$$

Problem 11: [S. 14.2, Prob. 31]

In polar coordinates, small lengths along a curve can be expressed in the form $ds = \sqrt{r^2 + (dr/d\theta)^2 d\theta}$. Use this fact to evaluate the following line integral $\oint_C (x^2 + y^2) ds$ where C is the cardioid $r = 1 + \cos\theta$.

Solution:

$$\oint_C (x^2 + y^2)\, ds = \int_{-\pi}^{\pi} r^2 \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_{-\pi}^{\pi} (1 + \cos\theta)^2 \sqrt{(1 + \cos\theta)^2 + (-\sin\theta)^2}\, d\theta$$

$$= \int_{-\pi}^{\pi} (1 + \cos\theta)^2 \sqrt{2 + 2\cos\theta}\, d\theta = \sqrt{2} \int_{-\pi}^{\pi} (1 + \cos\theta)^{5/2}\, d\theta$$

$$= \sqrt{2} \int_{-\pi}^{\pi} [1 + 2\sin^2(\theta/2) - 1]^{5/2}\, d\theta = 8 \int_{-\pi}^{\pi} \sin^5(\theta/2)\, d\theta$$

$$= 8 \int_{-\pi}^{\pi} \sin(\theta/2)[1 - \cos^2(\theta/2)]^2\, d\theta = 8 \int_{-\pi}^{\pi} \sin(\theta/2)[1 - 2\cos^2(\theta/2) + \cos^4(\theta/2)]\, d\theta$$

$$= 8 \left\{ -2\cos(\theta/2) + \frac{4}{3}\cos^3(\theta/2) - \frac{2}{5}\cos^5(\theta/2) \right\}_{-\pi}^{\pi} = \frac{256}{15}$$