Problem Set #1 Solutions

3.1 (a)

$$(101101)_2 = (1\times 2^5) + (0\times 2^4) + (1\times 2^3) + (1\times 2^2) + (0\times 2^1) + (1\times 2^0)$$
$$= 32 + 8 + 4 + 1 = 45$$

(b)

$$(101.011)_2 = (1\times 2^2) + (0\times 2^1) + (0\times 2^0) + (0\times 2^{-1}) + (1\times 2^{-2}) + (1\times 2^{-3})$$
$$= 4 + 1 + 0.25 + 0.125 = 5.375$$

(c)

$$(0.01101)_2 = (0\times 2^{-1}) + (1\times 2^{-2}) + (1\times 2^{-3}) + (0\times 2^{-4}) + (1\times 2^{-5})$$
$$= 0.25 + 0.125 + 0.03125 = 0.40625$$

3.7 The true value can be computed as

$$f'(0.577) = \frac{6(0.577)}{(1-3\times 0.577^2)^2} = 2{,}352{,}911$$

Using 3-digits with chopping

$$6x = 6(0.577) = 3.462 \xrightarrow{\text{chopping}} 3.46$$
$$x = 0.577$$
$$x^2 = 0.332929 \xrightarrow{\text{chopping}} 0.332$$
$$3x^2 = 0.996$$
$$1-3x^2 = 0.004$$

$$f'(0.577) = \frac{3.46}{(1-0.996)^2} = \frac{3.46}{0.004^2} = 216{,}250$$

This represents a percent relative error of

$$\varepsilon_t = \left|\frac{2{,}352{,}911 - 216{,}250}{2{,}352{,}911}\right| = 90.8\%$$

Using 4-digits with chopping

$$6x = 6(0.577) = 3.462 \xrightarrow{\text{chopping}} 3.462$$
$$x = 0.577$$
$$x^2 = 0.332929 \xrightarrow{\text{chopping}} 0.3329$$
$$3x^2 = 0.9987$$
$$1-3x^2 = 0.0013$$

$$f'(0.577)=\frac{3.462}{(1-0.9987)^2}=\frac{3.462}{0.0013^2}=2{,}048{,}521$$

This represents a percent relative error of

$$\varepsilon_t=\left|\frac{2{,}352{,}911-2{,}048{,}521}{2{,}352{,}911}\right|=12.9\%$$

Although using more significant digits improves the estimate, the error is still considerable. The problem stems primarily from the fact that we are subtracting two nearly equal numbers in the denominator. Such subtractive cancellation is worsened by the fact that the denominator is squared.

3.8 First, the correct result can be calculated as

$$y=1.37^3-5(1.37)^2+6(1.37)+0.55=1.956853$$

(*a*) Using 3-digits with chopping

1.37^3	$\rightarrow$	2.571353	$\rightarrow$	2.57
$-5(1.37)^2$	$\rightarrow$	$-5(1.87)$	$\rightarrow$	-9.35
$6(1.37)$	$\rightarrow$	10.96	$\rightarrow$	8.22
				+ 0.55
				1.99

This represents an error of

$$\varepsilon_t=\left|\frac{1.956853-1.99}{1.956853}\right|=1.694\%$$

(*b*) Using 3-digits with chopping

$$y=((1.37-5)1.37+6)1.37+0.55$$
$$y=(-3.63\times1.37+8)1.37+0.55$$
$$y=(-4.97+8)1.37+0.55$$
$$y=1.03\times1.37+0.55$$
$$y=1.41+0.55$$
$$y=1.96$$

This represents an error of

$$\varepsilon_t=\left|\frac{1.956853-1.96}{1.956853}\right|=0.161\%$$

Hence, the second form is superior because it tends to minimize round-off error.

4.5 True value: $f(3) = 554$.

zero order:

$$f(3) = f(1) = -62 \qquad \varepsilon_t = \left|\frac{554-(-62)}{554}\right| \times 100\% = 111.191\%$$

first order:

$$f(3) = -62 + f'(1)(3-1) = -62 + 70(2) = 78 \qquad \varepsilon_t = 85.921\%$$

second order:

$$f(3) = 78 + \frac{f''(1)}{2}(3-1)^2 = 78 + \frac{138}{2}4 = 354 \qquad \varepsilon_t = 36.101\%$$

third order:

$$f(3) = 354 + \frac{f^{(3)}(1)}{6}(3-1)^3 = 354 + \frac{150}{6}8 = 554 \qquad \varepsilon_t = 0\%$$

Thus, the third-order result is perfect because the original function is a third-order polynomial.

4.11 $$\frac{\partial v}{\partial c} = \frac{cgte^{-(c/m)t} - gm\left(1 - e^{-(c/m)t}\right)}{c^2} = -1.38807$$

$$\Delta v(\tilde{c}) = \left|\frac{\partial v}{\partial c}\right| \Delta\tilde{c} = 1.38807(1.5) = 2.082112$$

$$v(12.5) = \frac{9.81(50)}{12.5}\left(1 - e^{-12.5(6)/50}\right) = 30.48437$$

$$v = 30.48437 \pm 2.082112$$

Thus, the bounds computed with the first-order analysis range from 28.40226 to 32.5648. This result can be verified by computing the exact values as

$$v(c - \Delta c) = \frac{9.81(50)}{11}\left(1 - e^{-(11/50)6}\right) = 32.6791$$

$$v(c + \Delta c) = \frac{9.81(50)}{14}\left(1 - e^{-(14/50)6}\right) = 28.50597$$

Thus, the range of ± 2.086567 is close to the first-order estimate.

4.12 $v(12.5) = \dfrac{9.81(50)}{12.5}\left(1 - e^{-12.5(6)/50}\right) = 30.48437$

$$\Delta v(\tilde{c}, \tilde{m}) = \left|\frac{\partial v}{\partial c}\right| \Delta\tilde{c} + \left|\frac{\partial v}{\partial m}\right| \Delta\tilde{m}$$

$$\frac{\partial v}{\partial c} = \frac{cgte^{-(c/m)t} - gm\left(1 - e^{-(c/m)t}\right)}{c^2} = -1.38807$$

$$\frac{\partial v}{\partial m} = -\frac{gt}{m} e^{-(c/m)t} + \frac{g}{c}\left(1 - e^{-(c/m)t}\right) = 0.347019$$

$$\Delta v(\tilde{c}, \tilde{m}) = \left|-1.38807\right|(1.5) + \left|0.347019\right|(2) = 2.082112 + 0.694037 = 2.776149$$

$$v = 30.48437 \pm 2.776149$$