Part 7. Ordinary Differential Equations
Chapter 27. Boundary-Value & Eigenvalue Problems

Lecture 27

General Methods for Boundary Value Problems: Finite-Difference Method

27.1 (27.1.2)

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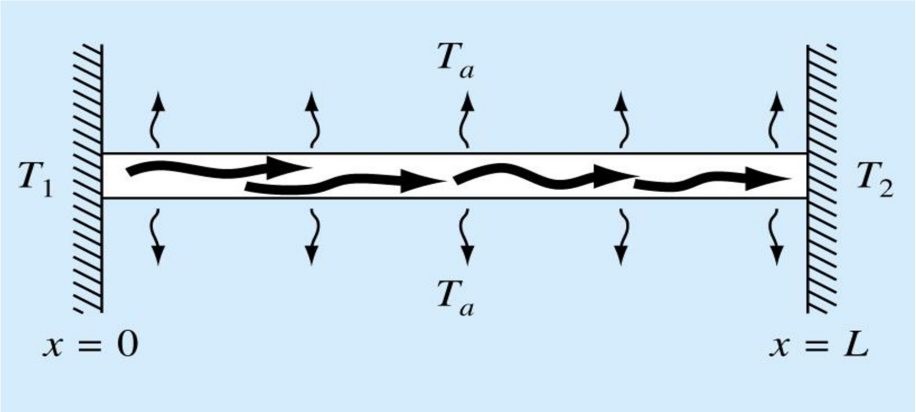
Learning Outcomes

- Understand Finite-Difference method
- Apply finite difference method to solve boundary-value problems (2^{nd} –order differential equations)

Finite Differences Methods

- The most common alternatives to the shooting method.
- Finite differences are substituted for the derivatives in the original equation.



$$\frac{d^2T}{dx^2} = \frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2}$$

$$\frac{d^2T}{dx^2} + h'(T_a - T) = 0$$

$$\frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2} - h'(T_i - T_a) = 0$$

$$-T_{i-1} + (2 + h'\Delta x^2)T_i - T_{i+1} = h'\Delta x^2 T_a$$

Finite Differences Methods

- Finite differences equation applies for each of the interior nodes.
- The first and last interior nodes, T_{i-1} and T_{i+1}, respectively, are specified by the boundary conditions.
- Thus, a linear equation transformed into a set of simultaneous algebraic equations can be solved efficiently.

$$-T_{i-1} + (2 + h'\Delta x^2)\ T_i - T_{i+1} = h'\Delta x^2\ T_a$$

Example 1. Solve the following 2^{nd} order ODE, with the provided boundary values at x =0 and x=1.

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = e^{-x} \qquad y(0) = 2 \qquad y(1) = 1$$

Step 1. Subdivide region into 5 subsections (or 4 nodes) ($\Delta x=0.2$)

Step 2. Apply second order approximation (central difference) at x_i

Step 3. Form finite difference equations at each non-boundary x_i value

Step 4. Continue the process for all nodes (subsections) till x=1.

Step 5. Solve system of equations using the method of your choice

e.g. Gauss Elimination or Gauss Seidel

Notes

- "Shooting" and "Finite Difference" are two methods are for solving boundary-value problems
- Other methods for solving BVPs are:

  ❖ Steady-state solution of 2D –BVPs

  ❖ Transient solution of 2D-BVPs

  ❖ Steady state solution of 1D problems with finite-element approach

Part 8. Partial Differential Equations

Lecture 28

Introduction to Partial Differential Equations (PDEs)

PT 8.1, PT 8.2

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Learning Outcomes

- Understand difference between ODE and PDE

- Know the general form of linear, 2^{nd} order PDE Recognize difference between elliptic, parabolic and hyperbolic PDEs.

- See development of 1D diffusion PDE equation for a heated rod

as well as 2D and 3D equations.

Introduction to PDEs

Partial Differential Equation (PDE): equation containing partial derivatives

Partial Derivatives: derivatives of multivariable functions

Multivariable Function: function of more than one independent variable

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Introduction to PDEs

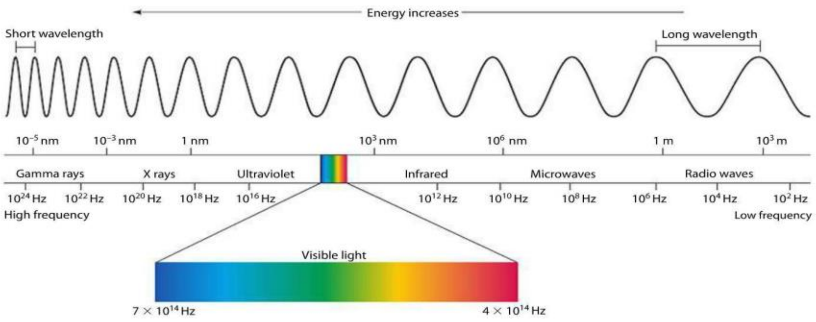
- In solution of ODEs integration yielded constant values, C_1 , C_2 , etc.
- In solution of PDEs integration yields functions, $f(x)$, $g(x)$, etc.
- Particular solution includes boundary and/or initial conditions to find $f(x)$, $g(x)$, etc.
- Order of PDEs: The highest order partial derivative appearing in the equation
- Linear PDE: If it is linear in the unknown function and all its derivatives with coefficients depending only on the independent variables

Linear 2nd Order PDEs

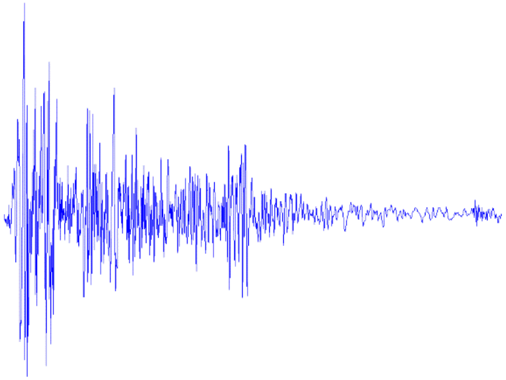
$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0$$

$B^2 - 4AC$	Category	Example
< 0	Elliptic	Laplace equation (steady state with two spatial dimensions) $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$
$= 0$	Parabolic	Heat conduction equation (time variable with one spatial dimension) $\frac{\partial T}{\partial t} = k' \frac{\partial^2 T}{\partial x^2}$
> 0	Hyperbolic	Wave equation (time variable with one spatial dimension) $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$

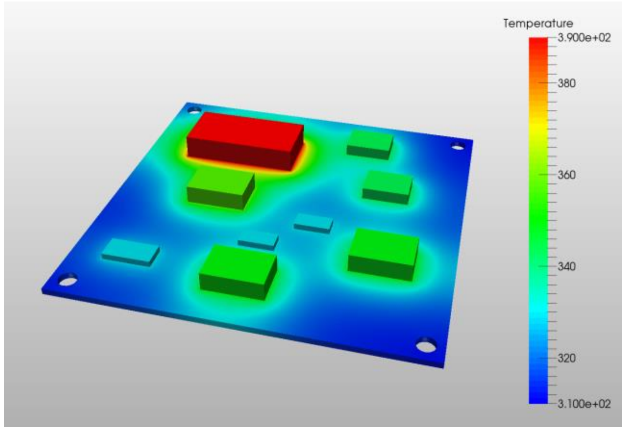
Engineering Applications of PDEs



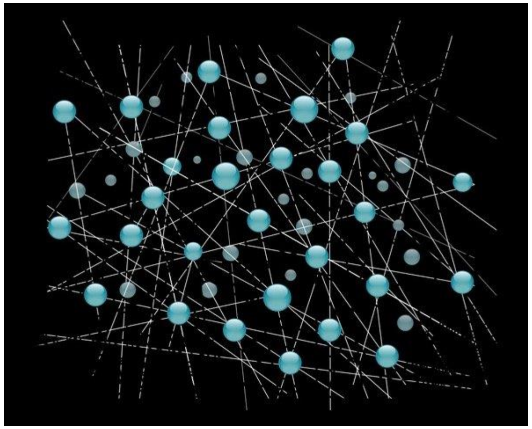
Electro-magnetics



Vibrations and acoustics



Heat transfer and fluid mechanics



Quantum mechanics

Example. PDE. 1D Diffusion Equations

1D Diffusion Equation: heat conduction in uniform rod, length = L, area = A

Assumptions

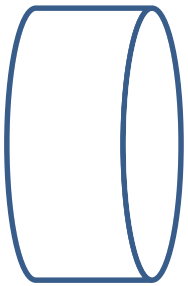
– Boundary conditions

– Initial condition

– Insulated sides

– Constant properties

Example. PDE. 1D Diffusion Equations



Step 1: Sketch system element

Step 2: Apply conservation of energy

Step 3: Evaluate with Taylor series expansion

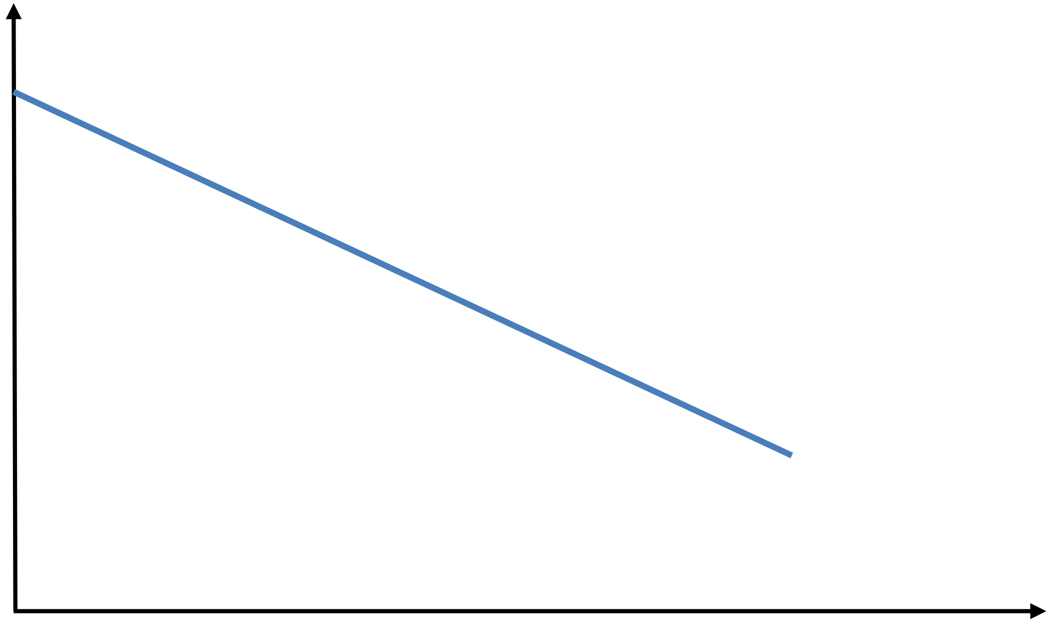
Step 4: Relate Q and T with physical laws

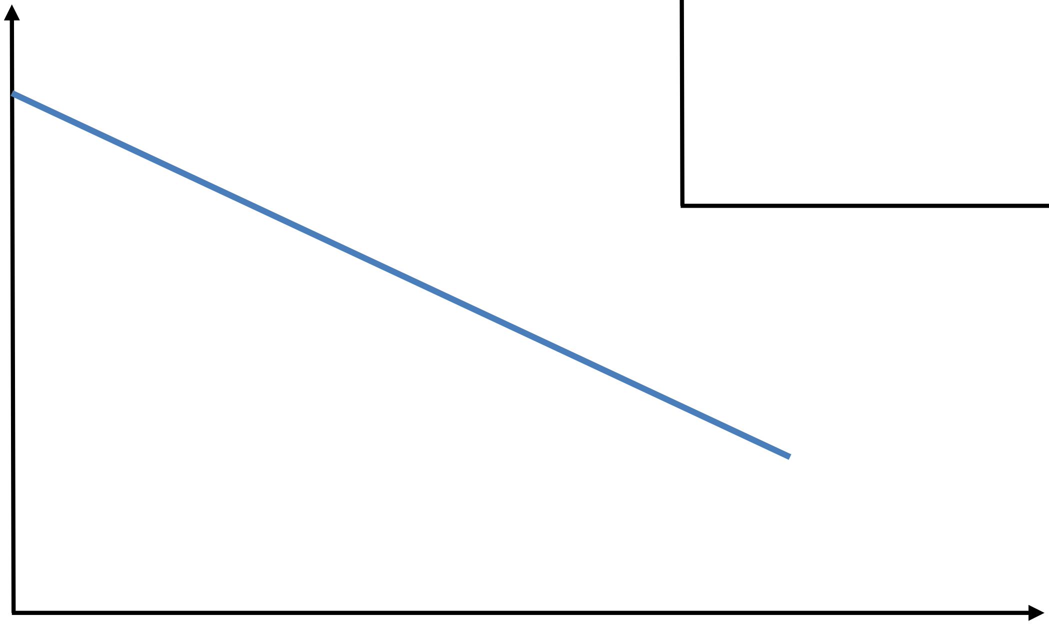
Step 5: Solve PDE with boundary conditions - what are limiting cases for the diffusion equation if $T_h > T_c = T_0$?

Example. PDE. 1D Diffusion Equations

$$t \to \infty$$

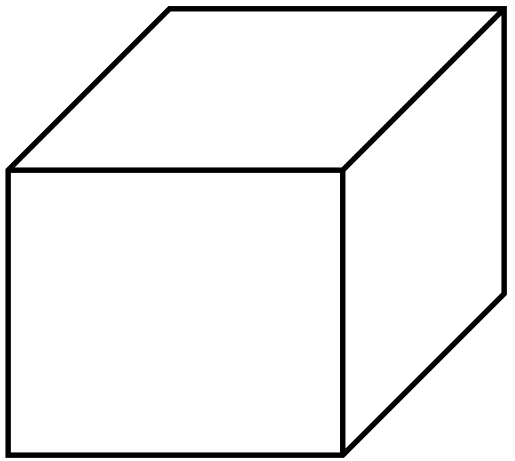
$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$





Example. PDE. 2D and 3D Diffusion Equations

Step 1: Sketch system element



Step 2: Apply conservation of energy

Steps 3, 4: Evaluate with Taylor series expansion and Fourier's law

$$\frac{\partial T(x,y,t)}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

$$\frac{\partial T(x,y,z,t)}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

2D and 3D Diffusion Equations

Step 2: Apply conservation of energy

Steps 3, 4: Evaluate with Taylor series expansion and Fourier's law

$$\frac{\partial T(x,y,t)}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

$$\frac{\partial T(x,y,z,t)}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$