MTE 203 – Advanced Calculus
Homework 11 (Solutions)

Line Integrals Involving Vector Functions

Problem 1: [S. 14.3, Prob. 13]

Evaluate the line integral $\int_C xy\,dx + x\,dy$ from $(-5, 3, 0)$ to $(4, 0, 0)$ along each of the following curves:

a. The straight line joining the points $(-5, 3, 0)$ and $(4, 0, 0)$
b. $x = 4 - y^2, z = 0$
c. $3y = x^2 - 16, z = 0$

Solution:

(a) Along the straight line with parametric equations $C_1 :\ x = -5 + 9t,\ y = 3 - 3t,\ 0 \le t \le 1,$

$$\int_{C_1} xy\,dx + x\,dy = \int_0^1 (-5 + 9t)(3 - 3t)(9\,dt) + (-5 + 9t)(-3\,dt)$$
$$= 3\int_0^1 (-40 + 117t - 81t^2)\,dt = 3\left\{-40t + \frac{117t^2}{2} - 27t^3\right\}_0^1 = -\frac{51}{2}.$$

(b) Along the parabola with parametric equations $C_2 :\ x = 4 - t^2,\ y = -t,\ -3 \le t \le 0,$

$$\int_{C_2} xy\,dx + x\,dy = \int_{-3}^0 (4 - t^2)(-t)(-2t\,dt) + (4 - t^2)(-dt) = \int_{-3}^0 (-4 + 9t^2 - 2t^4)\,dt$$
$$= \left\{-4t + 3t^3 - \frac{2t^5}{5}\right\}_{-3}^0 = -\frac{141}{5}.$$

(c) Along the parabola with equation $C_3 :\ y = (x^2 - 16)/3,\ -5 \le x \le 4,$

$$\int_{C_3} xy\,dx + x\,dy = \int_{-5}^4 x\left(\frac{x^2 - 16}{3}\right)dx + x\left(\frac{2x\,dx}{3}\right) = \frac{1}{3}\int_{-5}^4 (x^3 + 2x^2 - 16x)\,dx$$
$$= \frac{1}{3}\left\{\frac{x^4}{4} + \frac{2x^3}{3} - 8x^2\right\}_{-5}^4 = \frac{141}{4}.$$

Problem 2: [S. 14.3, Prob. 35]

Suppose a gas flows through a region D of space. At each $P\left(x,y,z\right)$ in D and time t, the gas has velocity $\vec{v}(x,y,z,t)$. If C is a closed curve in D, the line integral:

$$\Gamma = \oint_C \vec{v}.\vec{r}$$

is called *the circulation of the flow* for the curve C. If C is the circle $x^2+y^2=r^2, \quad z=1$ (directed clockwise as viewed from the origin), calculate Γ for the following flow vectors:

a. $\vec{v} = \frac{x\hat{\imath}+y\hat{\jmath}+z\hat{k}}{(x^2+y^2+z^2)^{3/2}}$
b. $\vec{v} = -y\hat{\imath}+x\hat{\jmath}$

Solution:

With parametric equations $x = r\cos t$, $y = r\sin t$, $z = 1$, $-\pi < t \leq \pi$,

$$\text{(a) } \Gamma = \int_C \frac{x\,dx + y\,dy + z\,dz}{(x^2+y^2+z^2)^{3/2}} = \int_{-\pi}^{\pi} \frac{r\cos t(-r\sin t\,dt) + r\sin t(r\cos t\,dt)}{(r^2+1)^{3/2}} = 0$$

$$\text{(b) } \Gamma = \int_C -y\,dx + x\,dy = \int_{-\pi}^{\pi} -r\sin t(-r\sin t\,dt) + r\cos t(r\cos t\,dt) = r^2\int_{-\pi}^{\pi} dt = 2\pi r^2$$

Path Independence

Problem 3: [S. 14.4, Prob.5]

Show that the line integral is independent of the path and evaluate it.

$$\int_C -\frac{y}{z}\sin x\ dx + \frac{1}{z}\cos x\,dy - \frac{y}{z^2}\cos x\,dz$$

where C is the helix $x = \ 2\,cos\,t, y = 2\,sin\,t, z = t\,from\ (2,0,2\pi)\ to\ (2,0,4\pi)$.

Solution:

According to Theorem 14.3, the line integral $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ is independent of the path in D if and only if there exists a function $\varphi(x,y,z)$ in D such that $\nabla\varphi = \boldsymbol{F}$.

Since $\nabla\left(\frac{y}{z}\cos x\right) = \left(-\frac{y}{z}\sin x\right)\hat{\mathbf{i}} + \left(\frac{1}{z}\cos x\right)\hat{\mathbf{j}} - \left(\frac{y}{z^2}\cos x\right)\hat{\mathbf{k}}$, the line integral is independent of path in any domain not containing points in the xy-plane. Since C does not pass through the xy-plane,

$$\int_C -\frac{y}{z}\sin x\,dx + \frac{1}{z}\cos x\,dy - \frac{y}{z^2}\cos x\,dz = \left\{\frac{y}{z}\cos x\right\}_{(2,0,2\pi)}^{(2,0,4\pi)} = 0.$$

Problem 4: [S. 14.4, Prob.11]

Show that if $f(x)$, $g(y)$ and $h(z)$ have continuous first derivatives, then the line integral $\int_C f(x)dx + g(y)dy + h(z)dz$ is independent of path.

Solution:

To answer this problem we can use theorem 14.4

Since $\nabla\times[f(x)\hat{\mathbf{i}} + g(y)\hat{\mathbf{j}} + h(z)\hat{\mathbf{k}}] = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f(x) & g(y) & h(z) \end{vmatrix} = \mathbf{0}$, the line integral is independent of path.

Problem 5: [S. 14.4, Prob.17]

Evaluate $\int_C -\frac{1}{x}\tan^{-1}y\ dx + \frac{1}{x+xy^2}\ dy$, where C is the curve $x = y^2 + 1$ from (2,-1) to (10,3).

Solution

Since $\nabla\left(\frac{1}{x}\mathrm{Tan}^{-1}y\right) = \left(-\frac{1}{x^2}\mathrm{Tan}^{-1}y\right)\hat{\mathbf{i}} + \frac{1}{x(1+y^2)}\hat{\mathbf{j}}$, the line integral is independent of path in any domain not containing points on the y-axis. Since C does not pass through this axis,

$$\int_C -\frac{1}{x^2}\mathrm{Tan}^{-1}y\,dx + \frac{1}{x+xy^2}dy = \left\{\frac{1}{x}\mathrm{Tan}^{-1}y\right\}_{(2,-1)}^{(10,3)} = \frac{1}{10}\mathrm{Tan}^{-1}3 + \frac{\pi}{8}.$$

Conservative Fields

Problem 6: [S. 14.5, Prob.5]

Determine whether the force field is conservative. Identify conservative force field and find a potential energy function.

$$\boldsymbol{F}(x, y, z) = GMm\frac{x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}}{(x^2+y^2+z^2)^{3/2}} \text{ where } G, M \text{ and } m \text{ are constant.}$$

Solution

According to definition 14.5, in order to have a conservative field, $\int \boldsymbol{F} \cdot d\boldsymbol{r}$ should be independent of the path. Theorem 14.3 says that the line integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ is independent of a path in D if and only if there exists a function $\varphi(x, y, z)$ in D such that $\nabla\varphi = \boldsymbol{F}$.

$\nabla\left(\frac{-GMm}{\sqrt{x^2+y^2+z^2}}\right) = \frac{GMm}{(x^2+y^2+z^2)^{3/2}}(x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}})$, the force field is conservative in any domain not containing the origin. It is the gravitational force between masses M and m. A potential energy function is $V = \frac{GMm}{\sqrt{x^2+y^2+z^2}}$.

Problem 7: [S. 14.5, Prob.7]

How do the equipotential surfaces of the forces in exercise 5 from section 14.5 [Problem 4] look like?

Solution

The potential formula is $V = \frac{GMm}{(x^2+y^2+z^2)^{1/2}}$. $x^2 + y^2 + z^2 = a^2$ defines a sphere centered at origin with a radius of a, thus for all the points on the sphere surface $V = \frac{GMm}{(a^2)^{1/2}}$. Therefore equipotential surfaces are spheres centered at origin.

Green's Theorem

Problem 8: [S. 14.6, Prob.7]

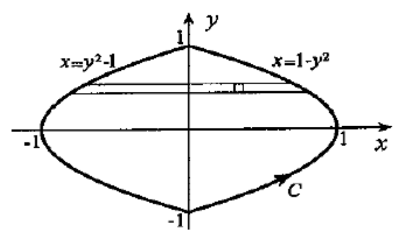
Use Green's theorem to evaluate the line integral:

$$\oint_C (x^3 + y^3)dx + (x^3 - y^3)dy,$$

where C is the curve enclosing the region bounded by $x = y^2 - 1$ and $x = 1 - y^2$

Solution

By Green's theorem,



$$\oint_C (x^3 + y^3)\,dx + (x^3 - y^3)\,dy = \iint_R (3x^2 - 3y^2)\,dA$$

$$= 6\int_0^1 \int_{y^2-1}^{1-y^2} (x^2 - y^2)\,dx\,dy$$

$$= 6\int_0^1 \left\{\frac{x^3}{3} - xy^2\right\}_{y^2-1}^{1-y^2} dy$$

$$= 2\int_0^1 [(1-y^2)^3 - 3y^2(1-y^2) - (y^2-1)^3 + 3y^2(y^2-1)]\,dy$$

$$= 2\int_0^1 \{(2 - 12y^2 + 12y^4 - 2y^6)\,dy = 2\left\{2y - 4y^3 + \frac{12y^5}{5} - \frac{2y^7}{7}\right\}_0^1 = \frac{8}{35}$$

Problem 9: [S. 14.6, Prob.25]

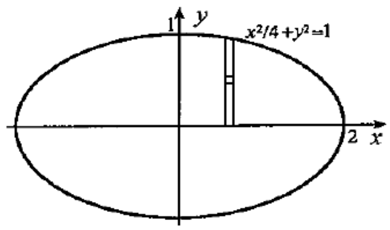
Use Green's theorem to evaluate the line integral:

$$\oint_C \left(2xye^{x^2y} + 3x^2y\right)dx + \left(x^2e^{x^2y}\right)dy, \text{ where C is the ellipse } x^2 + 4y^2 = 4.$$

Solution

By Green's theorem,

$$\oint_C (2xye^{x^2y} + 3x^2y)\,dx + x^2e^{x^2y}\,dy$$
$$= \iint_R (2xe^{x^2y} + 2x^3ye^{x^2y} - 2xe^{x^2y} - 2x^3ye^{x^2y} - 3x^2)\,dA$$
$$= -12\int_0^2\int_0^{(1/2)\sqrt{4-x^2}} x^2\,dy\,dx = -12\int_0^2 \left\{x^2y\right\}_0^{(1/2)\sqrt{4-x^2}} dx$$
$$= -6\int_0^2 x^2\sqrt{4-x^2}\,dx$$



If we set $x = 2\sin\theta$ and $dx = 2\cos\theta\,d\theta$,

$$\oint_C (2xye^{x^2y} + 3x^2y)\,dx + x^2e^{x^2y}\,dy = -6\int_0^{\pi/2} 4\sin^2\theta(2\cos\theta)(2\cos\theta\,d\theta) = -96\int_0^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)^2 d\theta$$
$$= -24\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2}\right)d\theta = -12\left\{\theta - \frac{1}{8}\sin 4\theta\right\}_0^{\pi/2} = -6\pi.$$

Surface Integrals

Problem 10: [S. 14.7, Prob.9]

Set up double iterated integrals for the surface integral of a function $f(x, y, z)$ over the surface defined by $z = 4 - x^2 - 4y^2$, $(x, y, z) \geq 0$, if the surface is projected onto the xy-, the xz-, and yz-planes.

Solution

For projection in the xy-plane, $dS = \sqrt{1 + (-2x)^2 + (-8y)^2}\, dA = \sqrt{1 + 4x^2 + 64y^2}\, dA$. Thus,

$$\iint_S f(x, y, z)\, dS = \int_0^1 \int_0^{\sqrt{4-4y^2}} f(x, y, 4 - x^2 - 4y^2)\sqrt{1 + 4x^2 + 64y^2}\, dx\, dy$$

For projection in the xz-plane,

$$dS = \sqrt{1 + \left(\frac{-x}{2\sqrt{4 - x^2 - z}}\right)^2 + \left(\frac{-1}{4\sqrt{4 - x^2 - z}}\right)^2}\, dA = \frac{1}{4}\sqrt{\frac{65 - 12x^2 - 16z}{4 - x^2 - z}}\, dA.$$

Thus,

$$\iint_S f(x, y, z)\, dS = \frac{1}{4}\int_0^2 \int_0^{4-x^2} f(x, \sqrt{4 - x^2 - z}/2, z)\sqrt{\frac{65 - 12x^2 - 16z}{4 - x^2 - z}} dz\, dx.$$

For projection in the yz-plane,

$$dS = \sqrt{1 + \left(\frac{-4y}{\sqrt{4 - 4y^2 - z}}\right)^2 + \left(\frac{-1}{2\sqrt{4 - 4y^2 - z}}\right)^2}\, dA = \frac{1}{2}\sqrt{\frac{17 + 48y^2 - 4z}{4 - 4y^2 - z}}\, dA.$$

Thus,

$$\iint_S f(x, y, z)\, dS = \frac{1}{2}\int_0^1 \int_0^{4-4y^2} f(\sqrt{4 - 4y^2 - z}, y, z)\sqrt{\frac{17 + 48y^2 - 4z}{4 - 4y^2 - z}} dz\, dy.$$

Problem 11: [S. 14.7, Prob.19]

Evaluate the surface integral by projecting the surface into one of the coordinate planes and also by using spherical coordinate area element ($dS = \rho^2 sin\,\varphi\; d\varphi\; d\theta$) given in equation 14.56.

$$\oiint_S x^2z^2\, dS, \text{ where S is the sphere } x^2+y^2+z^2=R^2$$

Solution

Since $f = x^2z^2$ has the same values for (x, z) and $(-x, -z)$, we only need to evaluate the integral for the first octant and then multiply the result by 8.

If S_{xy} is the projection of the first octant part of the sphere in the xy-plane,

$$dS = \sqrt{1+\left(\frac{-x}{\sqrt{1-x^2-y^2}}\right)^2+\left(\frac{-y}{\sqrt{1-x^2-y^2}}\right)^2}\,dA = \frac{1}{\sqrt{1-x^2-y^2}}dA.$$

Thus,

$$\iint_S x^2z^2\,dS = 8\iint_{S_{xz}} x^2(1-x^2-y^2)\frac{1}{\sqrt{1-x^2-y^2}}dA = 8\int_0^{\pi/2}\int_0^1 r^2\cos^2\theta\sqrt{1-r^2}\,r\,dr\,d\theta.$$

If we set $u = 1-r^2$ and $du = -2r\,dr$,

$$\begin{aligned}\iint_S x^2z^2\,dS &= 8\int_0^{\pi/2}\int_1^0 (1-u)\sqrt{u}\cos^2\theta\left(\frac{du}{-2}\right)d\theta = 4\int_0^{\pi/2}\int_0^1 (\sqrt{u}-u^{3/2})\cos^2\theta\,du\,d\theta \\ &= 4\int_0^{\pi/2}\left\{\left(\frac{2u^{3/2}}{3}-\frac{2u^{5/2}}{5}\right)\cos^2\theta\right\}_0^1 d\theta = \frac{16}{15}\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta \\ &= \frac{8}{15}\left\{\theta+\frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} = \frac{4\pi}{15}.\end{aligned}$$

Alternatively, using area element 14.56 with $R = 1$,

$$\begin{aligned}\iint_S x^2z^2\,dS &= 8\int_0^{\pi/2}\int_0^{\pi/2}(\sin^2\phi\cos^2\theta)\cos^2\phi\sin\phi\,d\phi\,d\theta \\ &= 8\int_0^{\pi/2}\int_0^{\pi/2}\cos^2\theta(1-\cos^2\phi)\cos^2\phi\sin\phi\,d\phi\,d\theta \\ &= 8\int_0^{\pi/2}\left\{\cos^2\theta\left(-\frac{1}{3}\cos^3\phi+\frac{1}{5}\cos^5\phi\right)\right\}_0^{\pi/2}d\theta = \frac{16}{15}\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta \\ &= \frac{8}{15}\left\{\theta+\frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} = \frac{4\pi}{15}.\end{aligned}$$