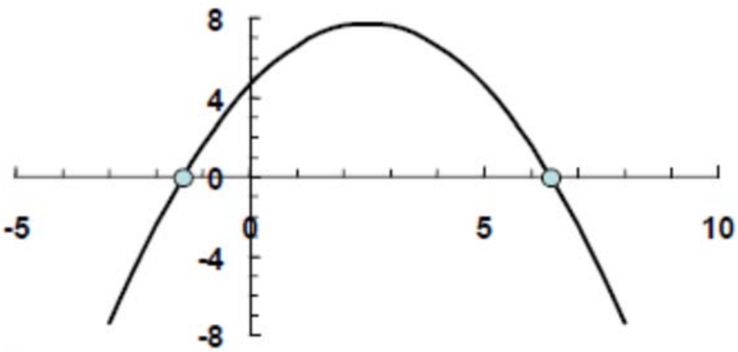
Problem Set #2 Solutions

5.1 (a) A plot indicates that roots occur at about $x = -1.4$ and 6.4.



(b)

$$x = \frac{-2.5 \pm \sqrt{(2.5)^2 - 4(-0.5)(4.5)}}{2(-0.5)} = \begin{matrix} -1.40512 \\ \\ 6.40512 \end{matrix}$$

(c) First iteration:

$$x_r = \frac{5+10}{2} = 7.5$$

$$\varepsilon_t = \left|\frac{6.40512 - 7.5}{6.40512}\right| \times 100\% = 17.09\% \qquad \varepsilon_a = \left|\frac{10-5}{10+5}\right| \times 100\% = 33.33\%$$

$$f(5)f(7.5) = 4.5(-4.875) = -21.9375$$

Therefore, the bracket is $x_l = 5$ and $x_u = 7.5$.

Second iteration:

$$x_r = \frac{5+7.5}{2} = 6.25$$

$$\varepsilon_t = \left|\frac{6.40512 - 6.25}{6.40512}\right| \times 100\% = 2.42\% \qquad \varepsilon_a = \left|\frac{7.5-5}{7.5+5}\right| \times 100\% = 20.00\%$$

$$f(5)f(6.25) = 4.5(0.59375) = 2.672$$

Consequently, the new bracket is $x_l = 6.25$ and $x_u = 7.5$.

Third iteration:

$$x_r = \frac{6.25+7.5}{2} = 6.875$$

$$\varepsilon_t = \left|\frac{6.40512 - 6.875}{6.40512}\right| \times 100\% = 7.34\% \qquad \varepsilon_a = \left|\frac{7.5-6.25}{7.5+6.25}\right| \times 100\% = 9.09\%$$

5.5 A graph of the function can be generated with MATLAB

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>> x=[-0.5:0.1:1.5];
>> f=sin(x)-x.^2;
>> plot(x,f)
>> grid
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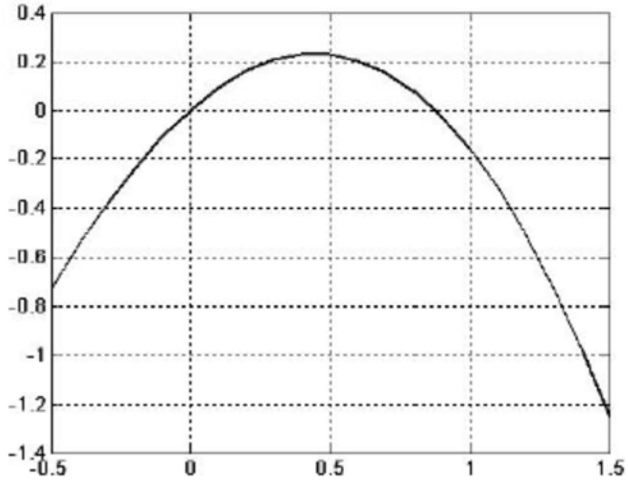


This plot indicates that a nontrivial root (i.e., nonzero) is located at about 0.85.

Using bisection, the first iteration is

$$x_r = \frac{0.5+1}{2} = 0.75$$
$$f(0.5)f(0.75) = 0.229426(0.1191388) = 0.027333$$

Therefore, the root is in the second interval and the lower guess is redefined as $x_l = 0.75$. The second iteration is

$$x_r = \frac{0.75+1}{2} = 0.875$$
$$\varepsilon_a = \left|\frac{0.875-0.75}{0.875}\right| 100\% = 14.29\%$$
$$f(0.75)f(0.875) = 0.119139(0.0019185) = 0.000229$$

Because the product is positive, the root is in the second interval and the lower guess is redefined as x_l = 0.875. The remainder of the iterations are displayed in the following table:

i	x_l	$f(x_l)$	x_u	$f(x_u)$	x_r	$f(x_r)$	$\|\varepsilon_a\|$
1	0.5	0.229426	1	−0.158529	0.75	0.1191388	
2	0.75	0.119139	1	−0.158529	0.875	0.0019185	14.29%
3	0.875	0.001919	1	−0.158529	0.9375	−0.0728251	6.67%
4	0.875	0.001919	0.9375	−0.0728251	0.90625	−0.0340924	3.45%
5	0.875	0.001919	0.90625	−0.0340924	0.890625	−0.0157479	1.75%

Therefore, after five iterations we obtain a root estimate of **0.890625** with an approximate error of 1.75%, which is below the stopping criterion of 2%. As in the above table, the function value at the root estimate is −0.0157479.

5.14 The function to evaluate is

$$f(c)=\frac{9.81(82)}{c}\left(1-e^{-(c/82)4}\right)-36=0$$

The first iteration is

$$x_r=\frac{3+5}{2}=4$$
$$f(3)f(4)=0.50386(-0.35099)=-0.17685$$

Therefore, the root is in the first interval and the upper guess is redefined as $x_u=4$. The second iteration is

$$x_r=\frac{3+4}{2}=3.5$$
$$\varepsilon_a=\left|\frac{3.5-4}{3.5}\right|100\%=14.29\%$$
$$f(3)f(3.5)=0.50386(0.07301)=0.03679$$

Therefore, the root is in the upper interval and the lower bound is redefined as $x_l=3.5$. The remaining iterations are displayed in the following table:

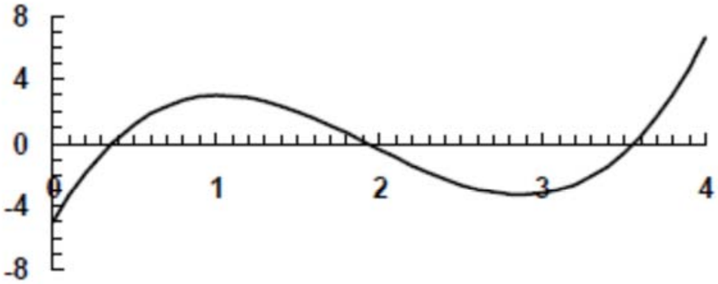
i	x_l	$f(x_l)$	x_u	$f(x_u)$	x_r	$f(x_r)$	$\lvert\varepsilon_a\rvert$	$f(x_l)\times f(x_r)$
1	3	0.50386	5	-1.17892	4	-0.35099		-0.17685
2	3	0.50386	4	-0.35099	3.5	0.07301	14.29%	0.03679
3	3.5	0.07301	4	-0.35099	3.75	-0.13983	6.67%	-0.01021
4	3.5	0.07301	3.75	-0.13983	3.625	-0.03362	3.45%	-0.00245
5	3.5	0.07301	3.625	-0.03362	3.5625	0.01964	1.75%	0.00143

Thus, after five iterations, we obtain a root estimate of **3.5625** with an approximate error of 1.75%. This result can be checked by substituting your final answer into the original equation to yield

$$v=\frac{9.81(82)}{3.5625}\left(1-e^{-(3.5625/82)4}\right)=36.01964$$

. Note that the iterations could be continued to yield the exact result of 3.58553 in 18 iterations which would yield the exact result of 36 m/s.

6.2 (a) Graphical



Root ≈ 3.58

(b) Fixed point

The equation can be solved in numerous ways. A simple way that converges is to solve for the x that is not raised to a power to yield

$$x = \frac{5 - 2x^3 + 11.7x^2}{17.7}$$

The resulting iterations are

i	x_i	ε_a
0	3	
1	3.180791	5.68%
2	3.333959	4.59%
3	3.442543	3.15%

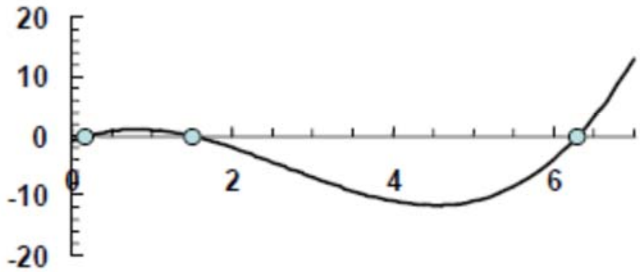
(c) Newton-Raphson

i	x_i	$f(x)$	$f'(x)$	ε_a
0	3	-3.2	1.5	
1	5.133333	48.09007	55.68667	41.56%
2	4.26975	12.95624	27.17244	20.23%
3	3.792934	2.947603	15.26344	12.57%

(d) Secant

i	x_{i-1}	$f(x_{i-1})$	x_i	$f(x_i)$	ε_a
0	3	-3.2	4	6.6	
1	4	6.6	3.326531	-1.9688531	20.25%
2	3.326531	-1.96885	3.481273	-0.7959153	4.44%
3	3.481273	-0.79592	3.586275	0.2478695	2.93%

6.4 (a) A graph of the function indicates that there are 3 real roots at approximately 0.2, 1.5, and 6.3.



(b) The Newton-Raphson method can be set up as

$$x_{i+1} = x_i - \frac{-1 + 5.5x_i - 4x_i^2 + 0.5x_i^3}{5.5 - 8x_i + 1.5x_i^2}$$

This formula can be solved iteratively to determine the three roots as summarized in the following tables:

i	x_i	$f(x)$	$f'(x)$	ε_a
0	0	-1	5.5	
1	0.181818	-0.12923	4.095041	100.000000%
2	0.213375	-0.0037	3.861294	14.789338%
3	0.214332	-3.4E-06	3.85425	0.446594%
4	0.214333	-2.8E-12	3.854244	0.000408%

i	x_i	$f(x)$	$f'(x)$	ε_a
0	2	-2	-4.5	
1	1.555556	-0.24143	-3.31481	28.571429%
2	1.482723	-0.00903	-3.06408	4.912085%
3	1.479775	-1.5E-05	-3.0536	0.199247%
4	1.479769	-4.6E-11	-3.05358	0.000342%

i	x_i	$f(x)$	$f'(x)$	ε_a
0	6	-4	11.5	
1	6.347826	0.625955	15.15974	5.479452%
2	6.306535	0.009379	14.7063	0.654728%
3	6.305898	2.22E-06	14.69934	0.010114%

Therefore, the roots are 0.214333, 1.479769, and 6.305898.

6.11 The Newton-Raphson method can be set up as

$$x_{i+1} = x_i - \frac{e^{-0.5x_i}(4 - x_i) - 2}{-e^{-0.5x_i}(3 - 0.5x_i)}$$

(a)

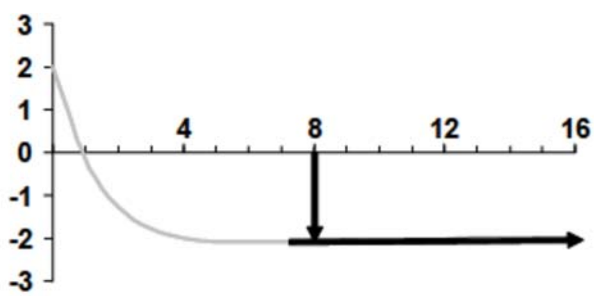
i	x	$f(x)$	$f'(x)$
0	2	-1.26424	-0.73576
1	0.281718	1.229743	-2.48348
2	0.776887	0.18563	-1.77093
3	0.881708	0.006579	-1.64678
4	0.885703	9.13E-06	-1.64221
5	0.885709	1.77E-11	-1.6422
6	0.885709	0	-1.6422

(b) The case does not work because the derivative is zero at $x_0 = 6$.

(c)

i	x	$f(x)$	$f'(x)$
0	8	-2.07326	0.018316
1	121.1963	-2	2.77E-25
2	7.21E+24	-2	0

This guess breaks down because, as depicted in the following plot, the near zero, positive slope sends the method away from the root.



6.19 The equation to be solved is

$$f(h) = \pi R h^2 - \left(\frac{\pi}{3}\right)h^3 - V$$

Because this equation is easy to differentiate, the Newton-Raphson is the best choice to achieve results efficiently. It can be formulated as

$$x_{i+1} = x_i - \frac{\pi R x_i^2 - \left(\frac{\pi}{3}\right)x_i^3 - V}{2\pi R x_i - \pi x_i^2}$$

or substituting the parameter values,

$$x_{i+1} = x_i - \frac{\pi(3)x_i^2 - \left(\frac{\pi}{3}\right)x_i^3 - 30}{2\pi(3)x_i - \pi x_i^2}$$

The iterations can be summarized as

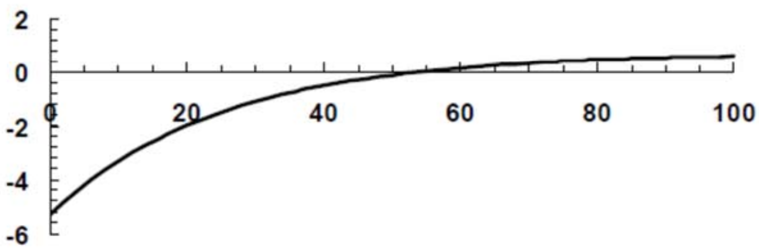
iteration	x_i	$f(x_i)$	$f'(x_i)$	$\lvert\varepsilon_a\rvert$
0	3	26.54867	28.27433	
1	2.061033	0.866921	25.50452	45.558%
2	2.027042	0.003449	25.30035	1.677%
3	2.026906	5.68E-08	25.29952	0.007%

Thus, after only three iterations, the root is determined to be 2.026906 with an approximate relative error of 0.007%.

8.4 The function to be solved is

$$f(t) = 10\left(1 - e^{-0.04t}\right) + 4e^{-0.04t} - 9.3 = 0$$

A plot of the function indicates a root at about $t = 55$.



Bisection with initial guesses of 0 and 60 can be used to determine a root of 53.711 after 16 iterations with $\varepsilon_a = 0.002\%$.

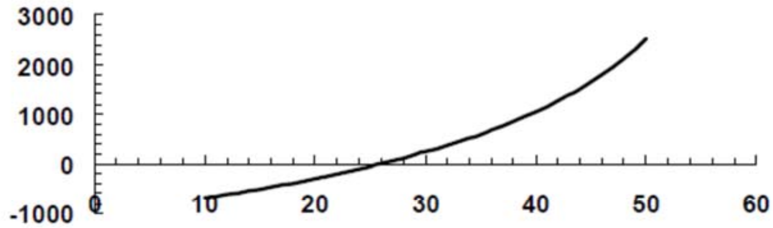
8.39 The solution can be formulated as

$$f(t) = u \ln \frac{m_0}{m_0 - qt} - gt - v$$

Substituting the parameter values gives

$$f(t) = 2{,}200 \ln \frac{160{,}000}{160{,}000 - 2{,}680t} - 9.81t - 1{,}000$$

A plot of this function indicates a root at about $t = 26$.



Because two initial guesses are given, a bracketing method like bisection can be used to determine the root,

i	t_l	t_u	t_r	$f(t_l)$	$f(t_r)$	$f(t_l)\times f(t_r)$	ε_a
1	10	50	30	-694.791	241.6514	-167897	
2	10	30	20	-694.791	-298.67	207513.3	50.00%
3	20	30	25	-298.67	-51.5865	15407.33	20.00%
4	25	30	27.5	-51.5865	88.38228	-4559.33	9.09%
5	25	27.5	26.25	-51.5865	16.86085	-869.792	4.76%
6	25	26.25	25.625	-51.5865	-17.7329	914.7789	2.44%
7	25.625	26.25	25.9375	-17.7329	-0.53026	9.403139	1.20%
8	25.9375	26.25	26.09375	-0.53026	8.141517	-4.31716	0.60%

Thus, after 8 iterations, the approximate error falls below 1% with a result of $t = 26.09375$. Note that if the computation is continued, the root can be determined as 25.94708.