

# **SYDE252 - lecture notes**

09/01/18

Presented by: John Zelek  
Systems Design Engineering  
note: some material (figures) borrowed from various sources



**UNIVERSITY OF WATERLOO**  
**FACULTY OF ENGINEERING**

# 1. SIGNALS

09/01/18

Presented by: John Zelek  
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# inspiration

- You may delay, but time will not. \footnote{Confucius}.
- The most important question a human being has to face... What is it? The question, Why are we here?.\\--Elie Wiesel
- To study and not think is a waste. To think and not study is dangerous.  
\footnote{Confucius}.



# signals

- A signal can be represented in the time or frequency domain
- it can also be a discrete or continuous signal
- continuous signal
  - to go from a time representation to a frequency representation we use the Laplace or Fourier transform
- discrete signal
  - To go from a discrete signal time representation to a frequency representation we either use the Z transform or the discrete Fourier transform
- system is represented by a differential equation
- other concepts covered include convolution & filtering



# what is a signal

- any kind of physical variable subject to variations represents a signal
- it is a function
- it has independent and dependent variables
- examples
  - in time (t)       $y=f(t)$
  - in space (x,y,z)       $y = f(x_1, x_2)$    OR       $y = f(x_1, x_2, x_3)$



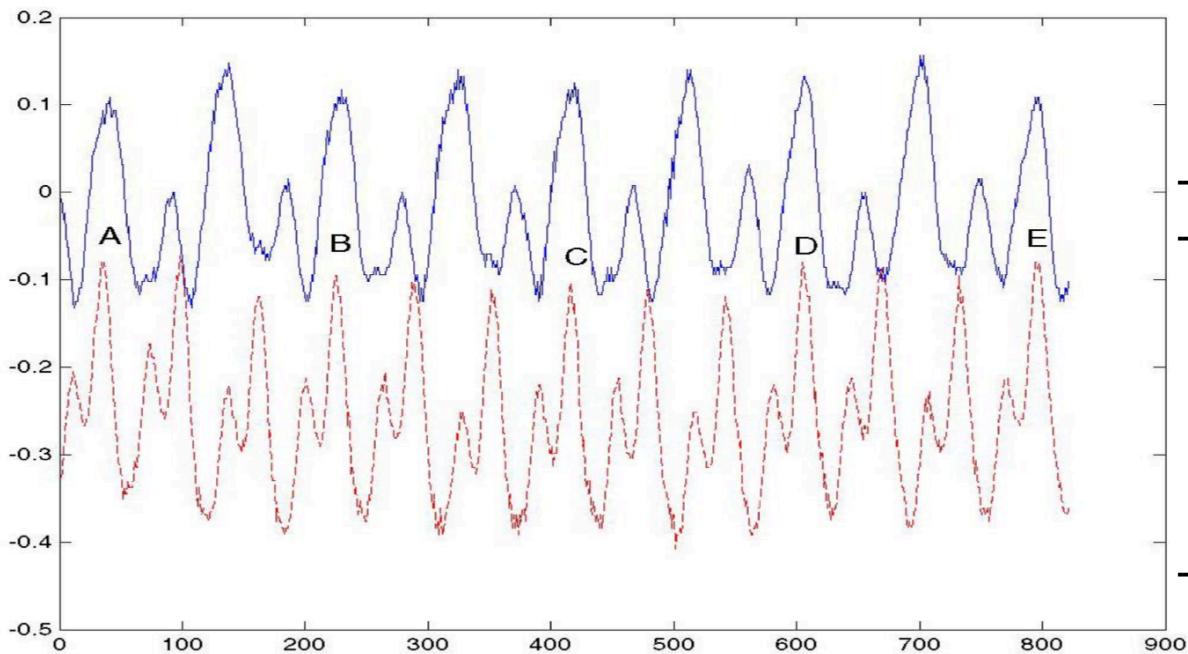
# what is a signal (2)

- 1D signal; EEG, voice, music, accelerometer
- 2D signal; images
- 3D signals; video sequences ( $x_1, x_2, t$ ) or volumetric data ( $x_1, x_2, x_3$ ), pixels, voxels
- 



# musical signals

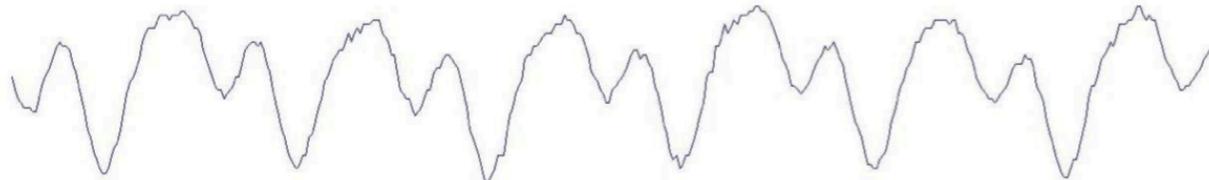
- female singer singing do (solid blue), and soh (dotted red shifted down)
  - fairly close to being periodic but not exactly periodic
  - blue curve for do shows 9 periods with major peaks interspersed with minor peaks
  - red curve shows 13 peaks
  - Look at the points marked A,B,C,D and E. At each letter both curves have peaks but between each pair, there is one extra peak for do and two for sol
  - In other words, two periods of do match three periods of sol
- (from Mumford notes (Brown)).



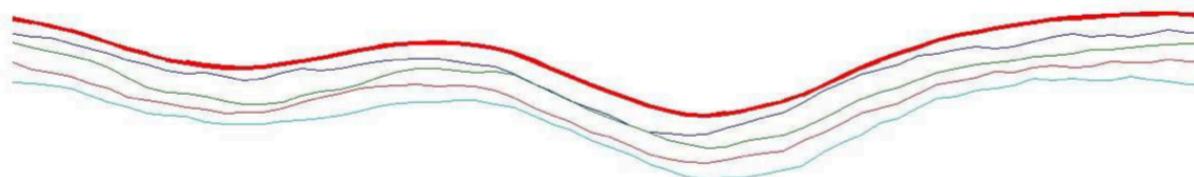
# musical signals

- female singing soh

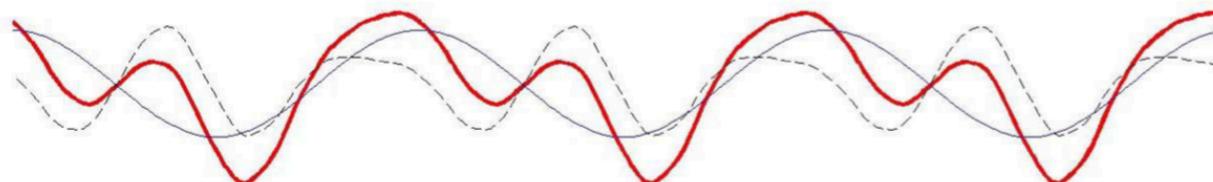
Six periods of a female voice singing the note sol



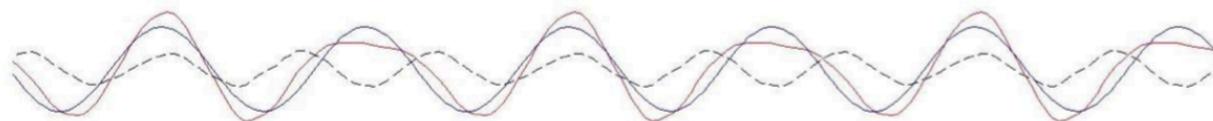
One period of the averaged detrended signal, compared to 4 samples



Three periods of a) the average signal (in red), b) its first harmonic (in blue) and c) the residual (dashed in black)



Three periods of a) the signal minus first harmonic (in red), b) its second harmonic (in blue) and c) the remaining residual (dashed in black)



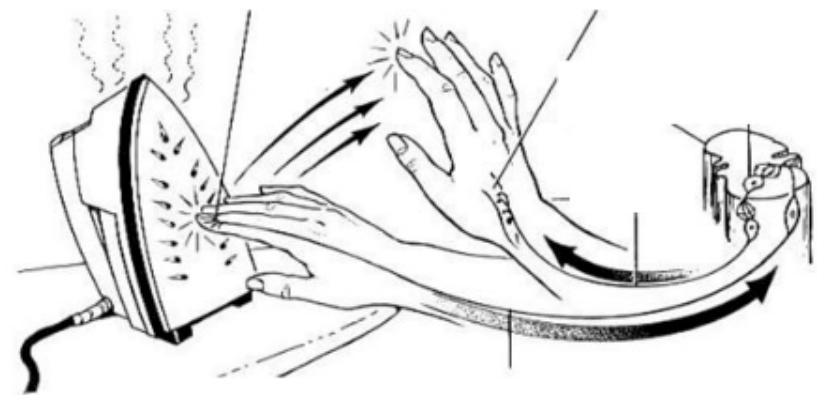
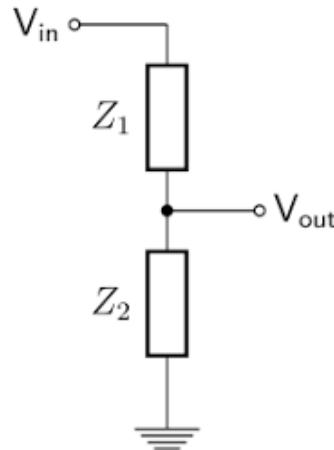
# musical signals

Let's put this in formulas. Let  $P(t)$  be the air pressure as a function of time. Then we model this by an exactly periodic function  $Q(t)$ , i.e. there is a period  $p$  such that  $Q(t + p) \equiv Q(t)$ , all  $t$ .  $P$  and  $Q$  will be very close to each other. We write  $Q$  as a sum of sinusoids like this:



# what is a system

- A process for transforming the input signal  $x(t)$  into an output signal  $y(t)$
- continuous functions of real independent variable, e.g., audio, ECG, images; 1D  $f = f(x)$ ; 2D  $f = f(x,y)$
- real valued functions of discrete variables; e.g., sampled signals or digital signals; 1D  $f = f[k]$ ; 2D  $f = f[i,j]$
- discrete functions of discrete variables (sampled and quantized); 1D  $f^d = f^d[k]$   
2D  $f^d = f^d[i, j]$



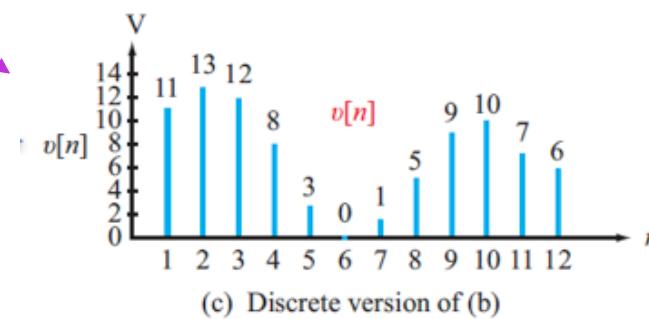
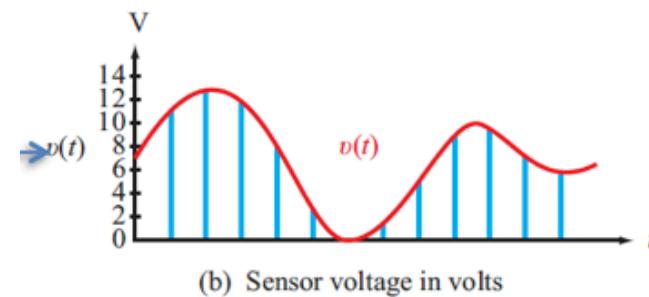
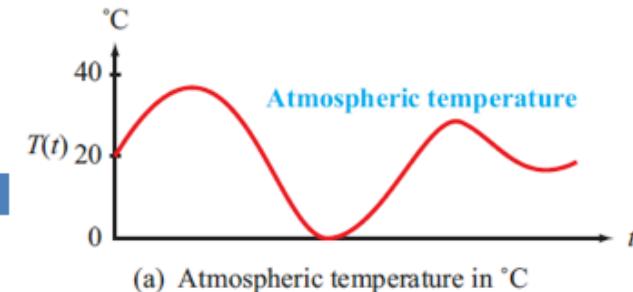
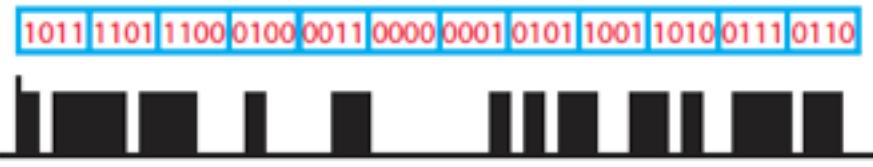
# signal classifications

- continuous time vs. discrete time
- analog vs. digital
- periodic vs. aperiodic
- real vs. complex
- causal vs. non causal signals vs. anti-causal
- even vs. odd signals
- deterministic vs. probabilistic signals
- finite length vs. infinite length signals



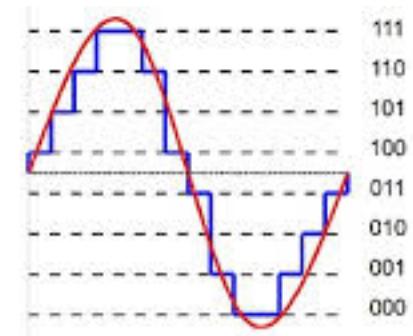
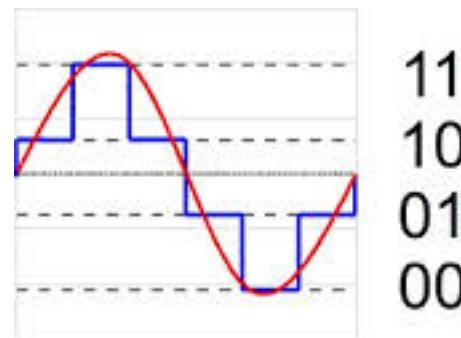
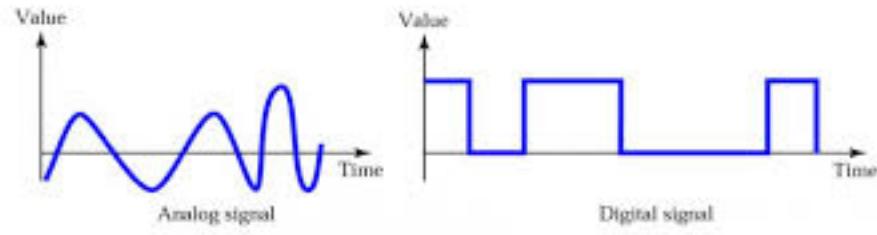
# continuous time vs. discrete time

- continuous-time - a signal is specified for every real value of the independent variable
- discrete-time - a signal specified for only discrete values of the independent variable
- horizontal axis is its or discrete
- digital signals



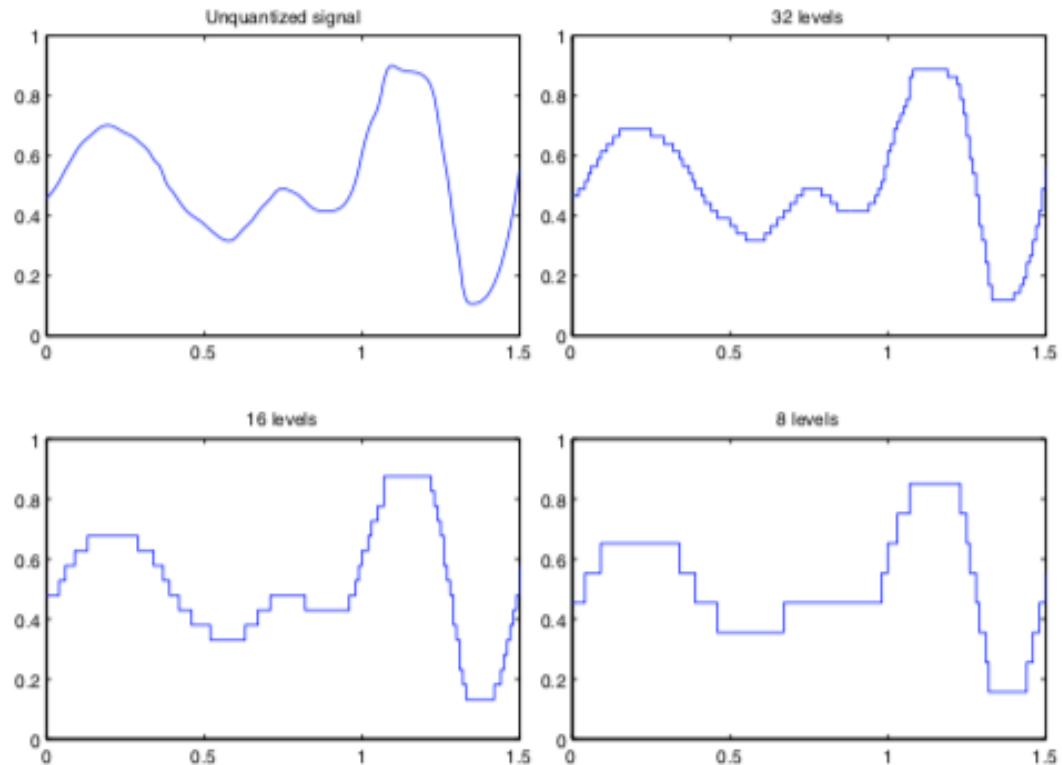
# analog vs. digital

- analog signal is where the amplitude (output, dependent variable) can take any value in a continuous range
- digital signal is where the amplitude can only take on a finite number of values (Quantized)



# quantization

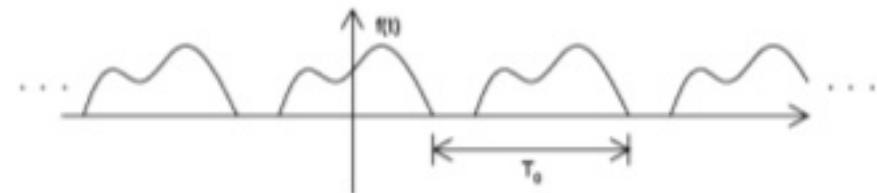
- different quantizations of the same signal
- 



# periodic vs. aperiodic

- a signal  $f(t)$  is periodic if there exists a positive constant  $T_0$  s.t. (such that)  $f(t + T_0) = f(t), \forall t$
- smallest  $T_0$  is called the period of the function  $f(t)$
- a periodic signal remains unchanged when time-shifted integer multiples of period
- an aperiodic signal is not periodic

Periodic signals repeat with some *period*  $T$ .



A signal is called aperiodic if it is not periodic.



# real vs. complex

- a signal can have a real and complex component

$$x(t) = x_r(t) + x_i(t)$$

$$x(t) = |x(t)| \exp^{j\angle x(t)}$$

$$|x(t)| = \sqrt{x_r^2(t) + x_i^2(t)}$$

$$\angle x(t) = \tan^{-1} \left[ \frac{x_i(t)}{x_r(t)} \right]$$

$$x_i(t) = |x(t)| \sin(\angle x(t))$$

$$x_r(t) = |x(t)| \cos(\angle x(t))$$

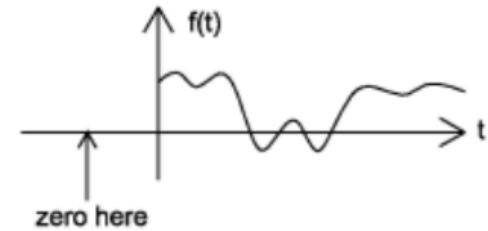


# causal vs. non-causal

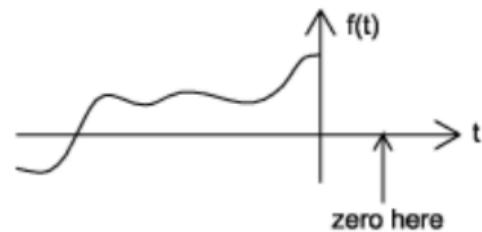
causal signals are zero for all negative time (or space)  $f(t) = 0, \forall t < 0$

anticausal signals are zero for all positive time (or space)  $f(t) = 0, \forall t \geq 0$

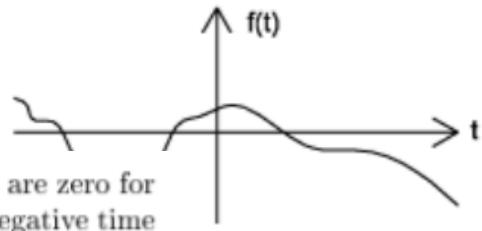
non causal signals have non zero values in both positive and negative time  $\exists t_1, f(t_1) \neq 0, \forall t_1 < 0$  and  $\exists t_2, f(t_2) \neq 0, \forall t_2 > 0$



(a)



(b)



(c)

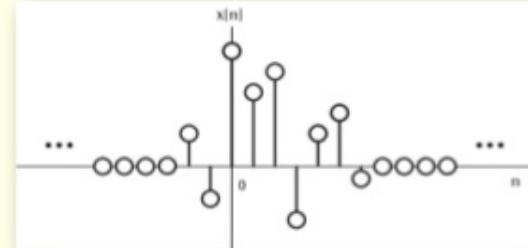
Figure 1.4: (a) A causal signal (b) An anticausal signal (c) A noncausal signal



# finite vs infinite length signals

- a finite signal is non zero over a finite set of values of the independent variable  $f = f(t), \forall t; t_1 \leq t \leq t_2; t_1 > -\infty; t_2 < +\infty$
- – an infinite signal is non zero over an infinite set of values of the independent variable ; e.g.,  $f(t) = \sin(wt)$
- 

Finite-length signal: nonzero over a finite interval  $t_{\min} < t < t_{\max}$

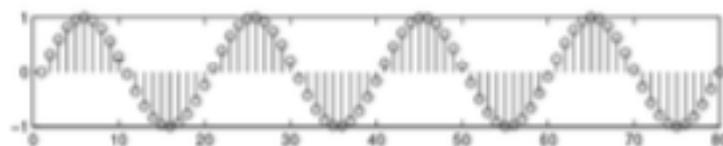


Infinite-length signal: nonzero over all real numbers

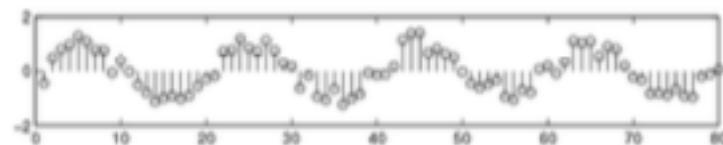


# deterministic vs. probabilistic signals

- deterministic signal - signal whose physical description is known completely
- probabilistic signal - signal has amplitude values that cannot be predicted precisely but only in terms of probabilistic descriptors



(a)



# even vs. odd signals

- an even signal is any signal  $f$  s.t.  $f(t) = f(-t)$
- an odd signal is any signal  $f$  s.t.  $f(t) = -(f(-t))$
- we can use even and odd signals to decompose any signal

- even signal



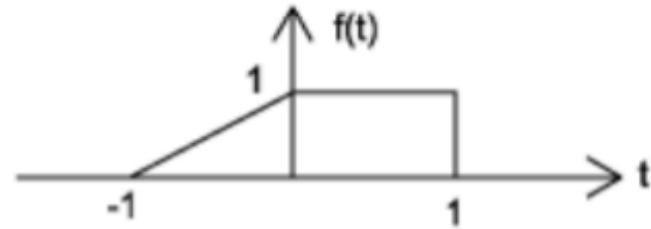
- odd signal



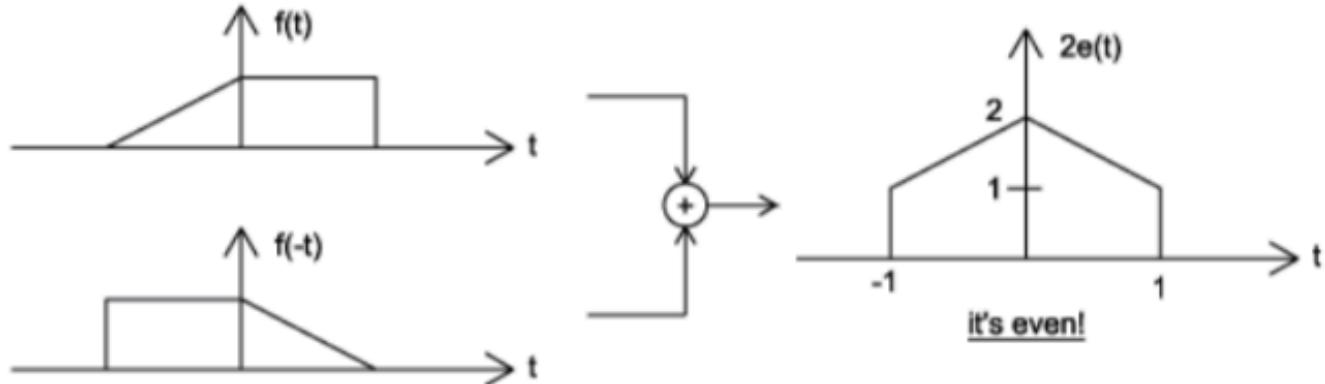
- even function multiplied by an odd function = odd function
- odd function multiplied by an odd function = even function
- even function multiplied by an even function = even function
- 



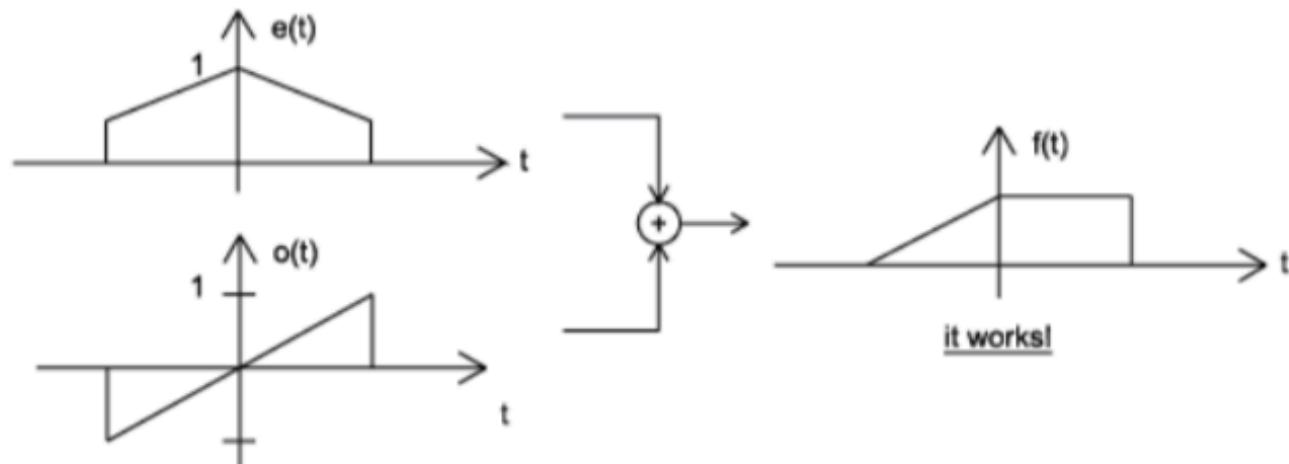
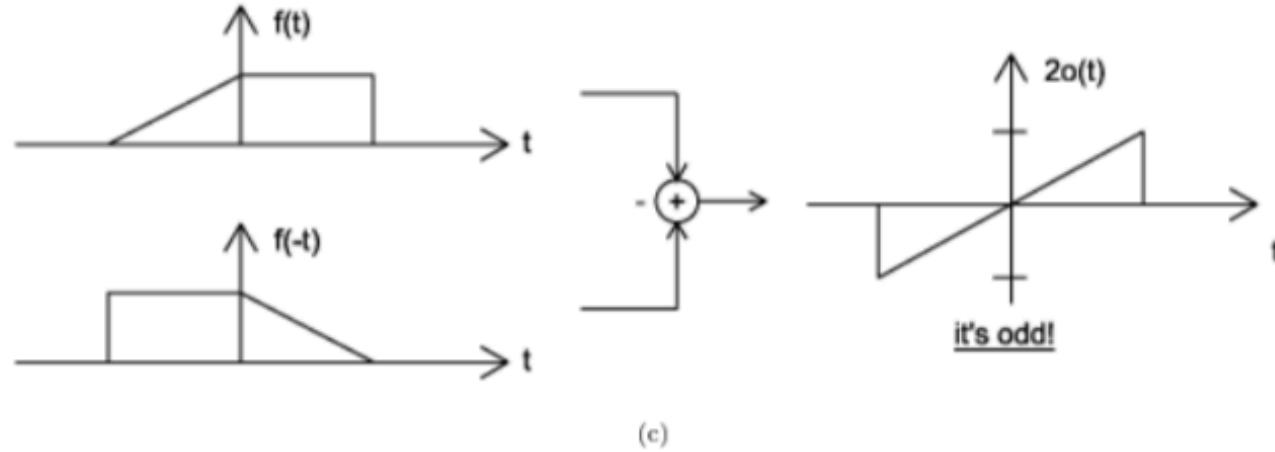
# even vs. odd signals



(a)



# even vs. odd signals



# energy & power of a signal

- Norms: size is the largeness or strength
- energy is represented by the area under the curve of the squared signal
- signal energy:



- signal power:

- signal to noise ratio:

$$SNR = 20\log_{10}\sqrt{\left(\frac{P_{signal}}{P_{noise}}\right)}$$

- a signal with finite energy is an energy signal; the amplitude must go to zero as independent variable goes to  $\infty$
- a signal with finite and different from zero power is called a power signal. The mean of an entity averaged over an infinite interval exists if either the entity is periodic or it has some statistical regularity
- all practical signals have finite energy and thus are energy signals. It is impossible to have infinite duration signals
- 



# energy & power of a signal

$$E = \lim_{T \rightarrow \infty} \int_{-T}^T |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt,$$

$$P_{\text{av}} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} p(t) dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt.$$

$$P_{\text{av}} = \lim_{T \rightarrow \infty} \frac{E}{T}$$

►  $P_{\text{av}}$  and  $E$  define three classes of signals:

(a) *Power signals*:

(b) *Energy signals*:

(c) *Non-physical sig*



# energy & power of a signal (example)

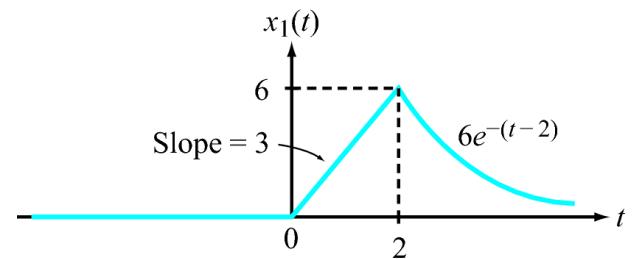
## Example [ ] Power and Energy

Evaluate  $P_{av}$  and  $E$  for each of the three signals displayed in Fig. [ ]

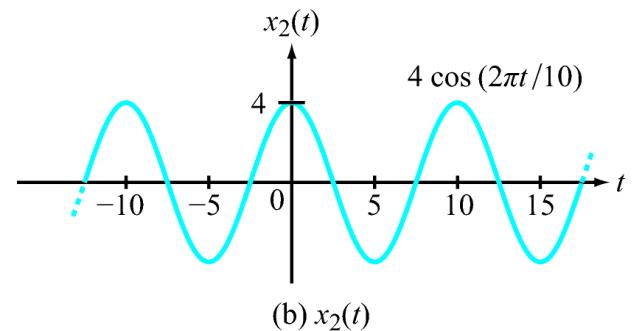
**Solution:**

(a) Signal  $x_1(t)$  is given by

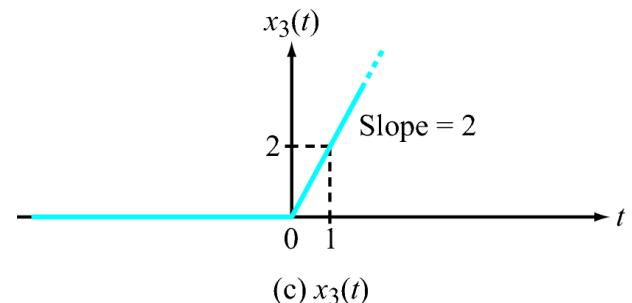
$$x_1(t) = \begin{cases} 0 & \text{for } t \leq 0, \\ 3t & \text{for } 0 \leq t \leq 2, \\ 6e^{-(t-2)} & \text{for } t \geq 2. \end{cases}$$



(a)  $x_1(t)$



(b)  $x_2(t)$



(c)  $x_3(t)$



# energy & power of a signal (example)



# energy & power of a signal (example)



# energy & power of a signal (example)



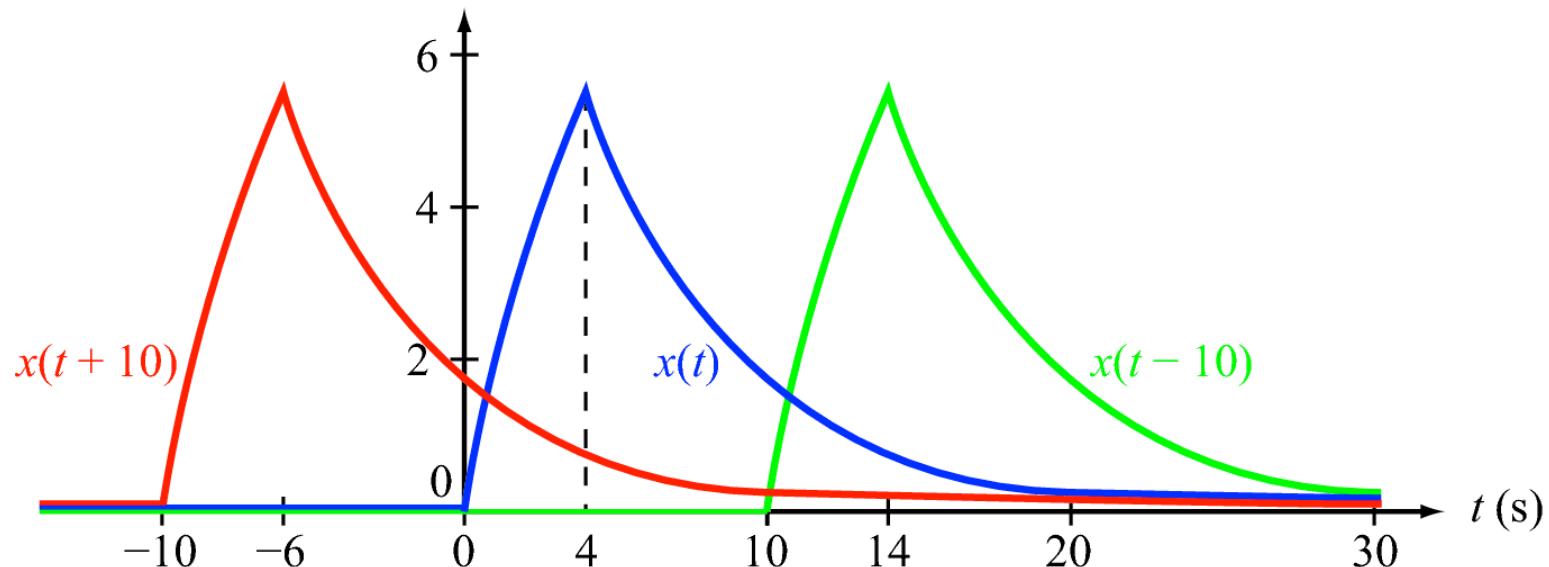
# signal operations

- time shifting
- time scaling
- time inversion
- amplitude shift
- addition
- multiplication
- combined operations



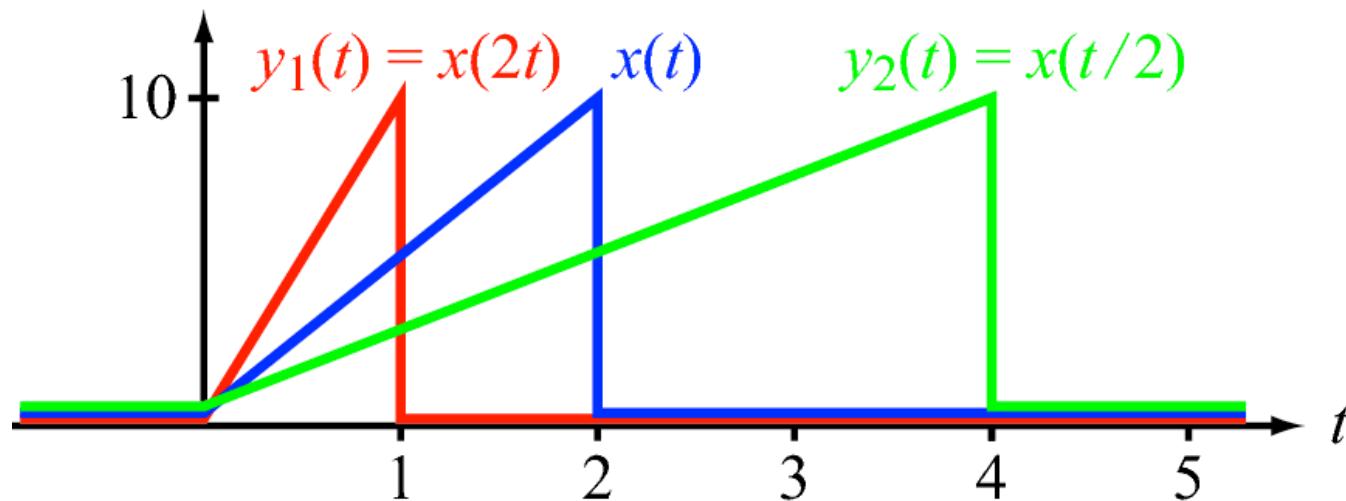
# signal operations - time shifting

- shifting:  $f(t)$  ; anticipated  $f(t + T)$  ; delayed  $f(t - T)$



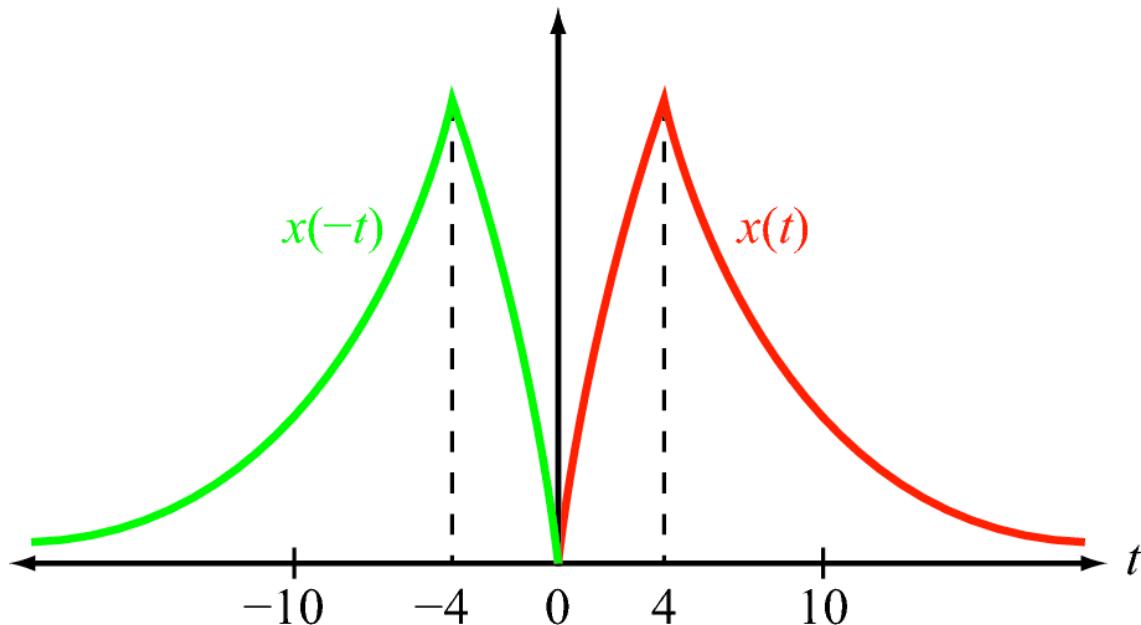
# signal operations - time scaling

- time scaling:  $f(t)$  ; compression  $f(at)$  ; expansion (dilated)  $f(t/a)$ ;  $a > 1$  ; note that  $a$  is a real constant,
- $0 < a < 1$  is time stretching;
- $a = 0$  is trivial ; however  $x(t)$  cannot be recovered from  $y(t)$  when  $a = 0$  ;
- For  $a > 1$  this is compression.



# signal operations - time inversion

- time inversion  $f(t)$  ;  $f(-t)$  ; inverted about  $f(t)$  axis



# signal operations - other operations

- amplitude shift  $y(t) = x(t) + b$
- addition  $y(t) = x(t) + z(t)$
- multiplication  $y(t) = x(t) z(t)$



# signal operations - combined operations

- transformations can be combined into a single transformation
- one way to handle this is to check by BRUTE force however a systematic approach is another way to handle this
- in the expression below, recognize that  $T=b/a$ .  $T$  is the time shift and  $a$  is the compression/expansion factor. The sign of  $a$  can also be an inversion factor.

$$y(t) = x(at - b) = x\left(a\left(t - \frac{b}{a}\right)\right) = x(a(t - T))$$



# signal operations - multiple transformations

example:

For signal  $x(t)$  profiled in Fig. 1-10(a), generate the corresponding profile of  $y(t) = x(-2t + 6)$ .

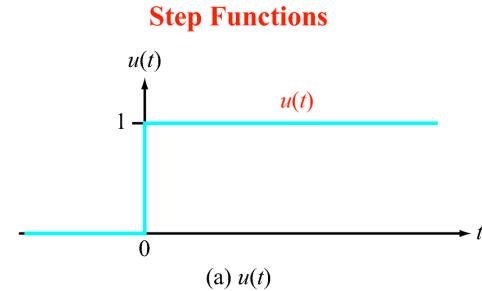
**Solution:**



# signals - useful functions - step

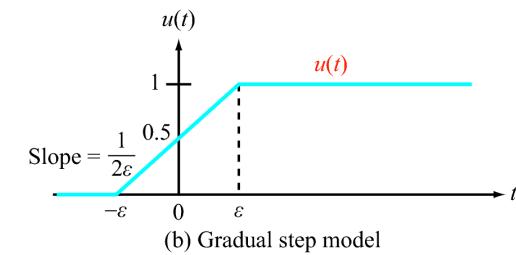
- unit step

$$\begin{aligned} u(t) &= 1, t \geq 0 \\ &= 0, t < 0 \end{aligned}$$



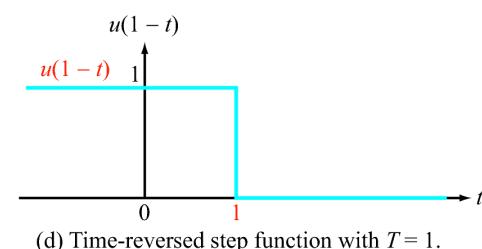
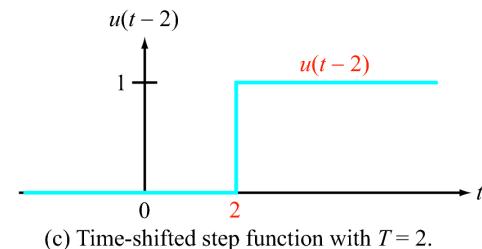
- realistic step

$$u(t) = \lim_{\epsilon \rightarrow 0} \begin{cases} 0 & \text{for } t \leq -\epsilon \\ \left[ \frac{1}{2} \left( \frac{t}{\epsilon} + 1 \right) \right] & \text{for } -\epsilon \leq t \leq \epsilon \\ 1 & \text{for } t \geq \epsilon, \end{cases}$$



- time shifted step

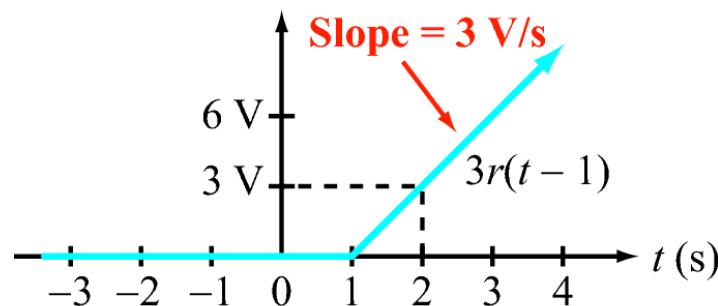
$$u(T-t) = \begin{cases} 1 & \text{for } t < T, \\ 0 & \text{for } t > T. \end{cases}$$



# signals - useful functions - ramp

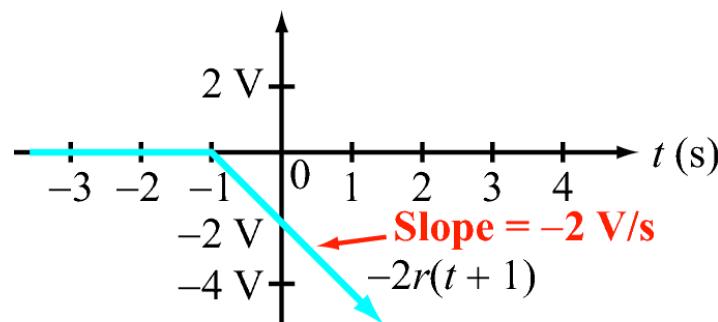
- ramp

## Ramp Functions



(a)

$$\begin{aligned} r(t, t_0) &= 0, t < 0 \\ &= \frac{t}{t_0}, 0 \leq t \leq t_0 \\ &= 1, t > t_0 \end{aligned}$$



(b)

$$\begin{aligned} r(t) &= t, t \geq 0 \\ &= 0, t < 0 \end{aligned}$$



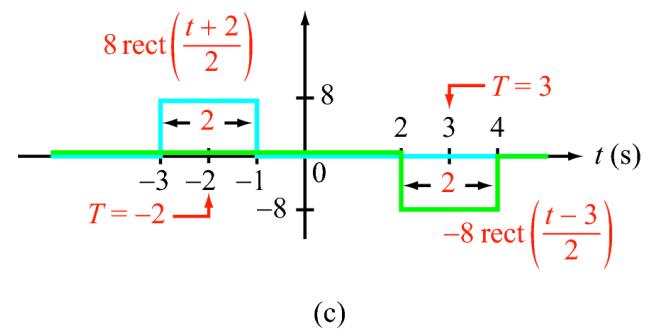
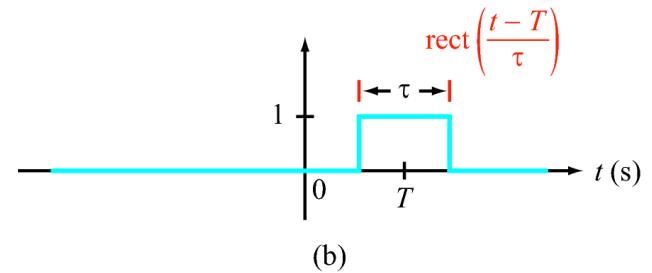
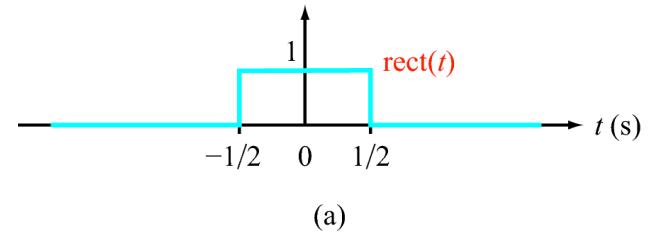
# signals - useful functions - unit pulse

- pulse

$$\begin{aligned}\Pi(t) &= 1, |t| < \frac{1}{2} \\ &= 0, t > \frac{1}{2}\end{aligned}$$

$$\text{rect}\left(\frac{t-T}{\tau}\right) = \begin{cases} 0 & \text{for } t < (T - \tau/2), \\ 1 & \text{for } (T - \tau/2) < t < (T + \tau/2), \\ 0 & \text{for } t > (T + \tau/2). \end{cases}$$

Rectangular Pulses



# signals - useful functions - triangle

- unit triangle function

$$\begin{aligned}\Lambda(t) &= t + 1, -1 \leq t < 0 \\ &= 1 - t, 0 \leq t < 1 \\ &= 0, \text{otherwise}\end{aligned}$$



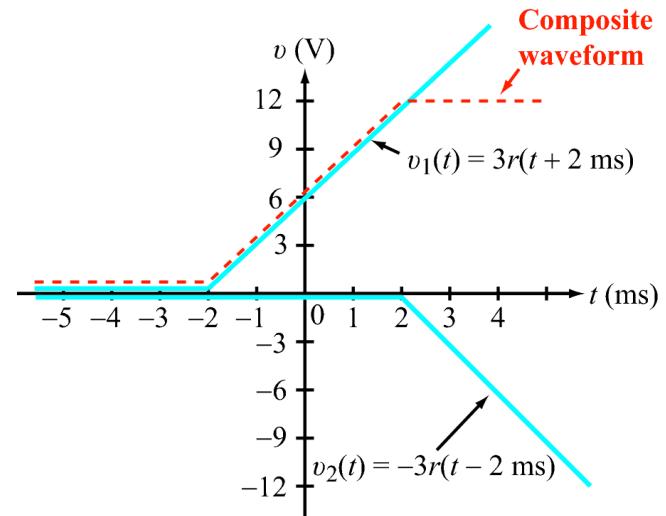
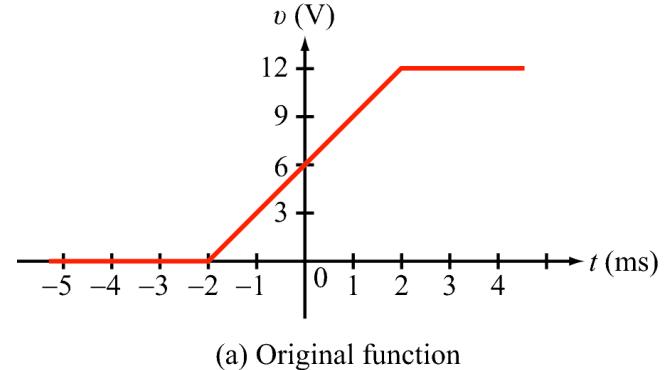
# signals - useful functions - waveform synthesis

- various combinations

$$\begin{aligned}v(t) &= v_1(t) + v_2(t) \\&= 3r(t + 2 \text{ ms}) - 3r(t - 2 \text{ ms})\end{aligned}$$

- equivalently

$$\begin{aligned}v(t) &= 3(t + 2 \text{ ms}) u(t + 2 \text{ ms}) \\&\quad - 3(t - 2 \text{ ms}) u(t - 2 \text{ ms})\end{aligned}$$

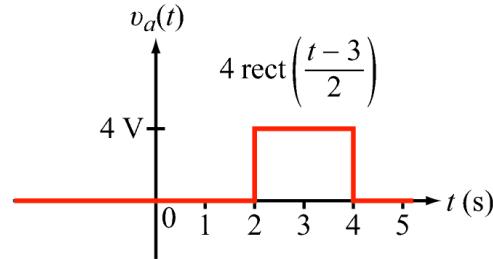


(b) As sum of two time-shifted ramp functions

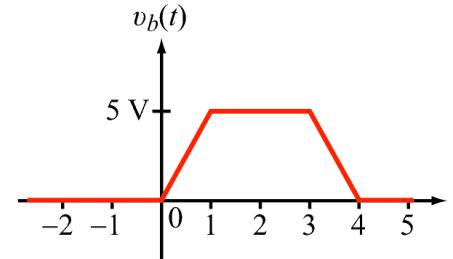


# signals - useful functions - waveform synthesis

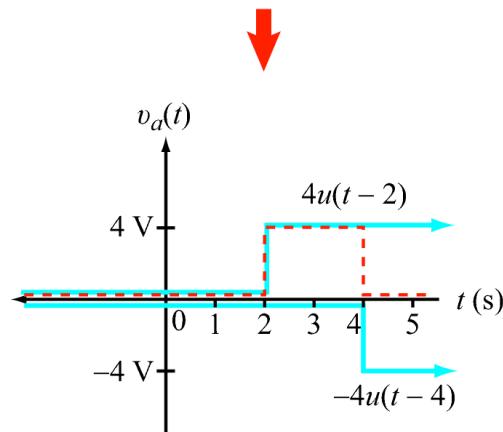
## Waveform Synthesis



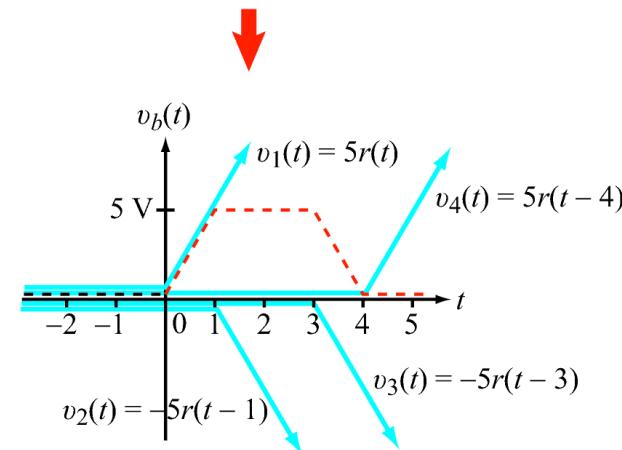
(a) Rectangular pulse



(b) Trapezoidal pulse



(c)  $v_a(t) = 4u(t-2) - 4u(t-4)$

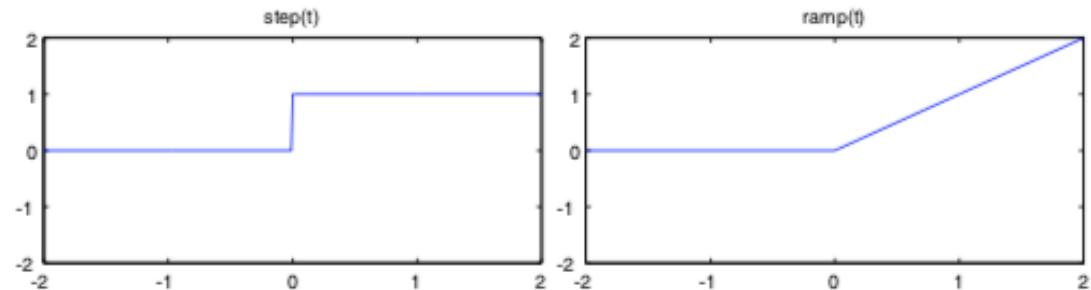


(d)  $v_b(t) = v_1(t) + v_2(t) + v_3(t) + v_4(t)$



# signals - some useful functions

$$\text{step}(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

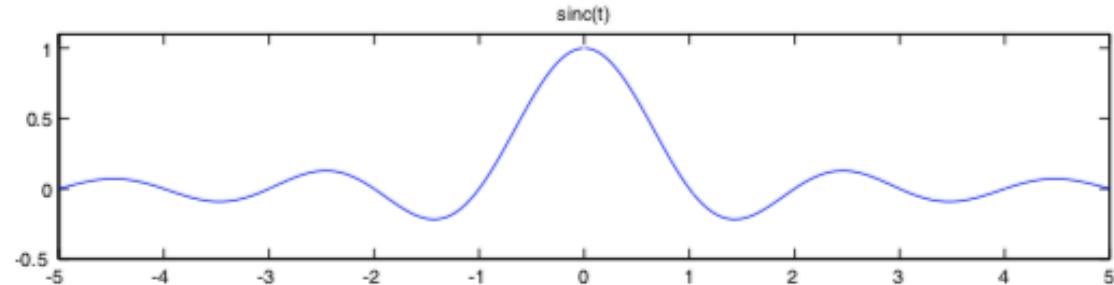
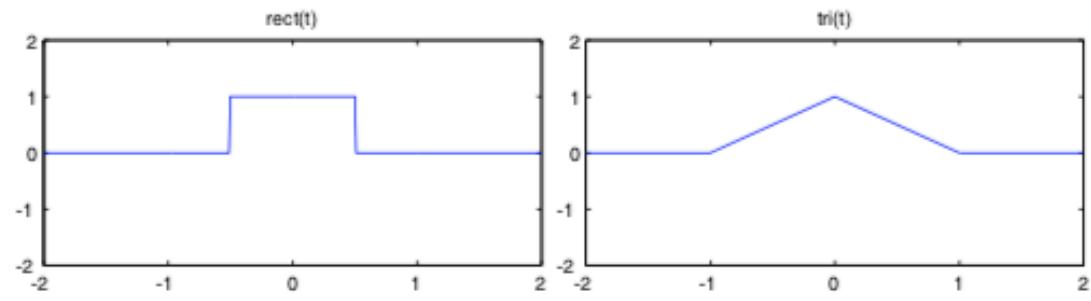


$$\text{ramp}(t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } t \geq 0 \end{cases}$$

$$\text{rect}(t) = \begin{cases} 1 & \text{if } -1/2 \leq t < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

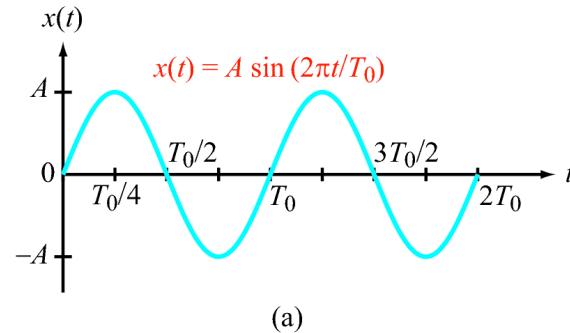
$$\text{tri}(t) = \begin{cases} 1 - |t| & \text{if } -1 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

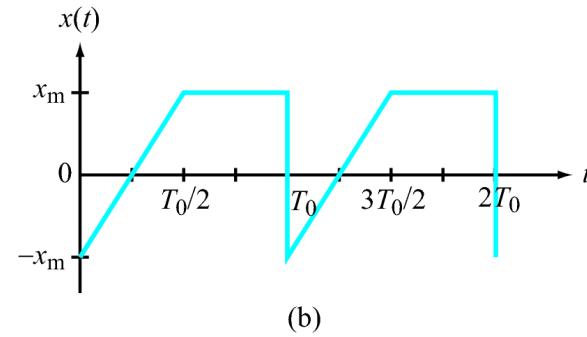


# signals - useful functions - periodic waveforms

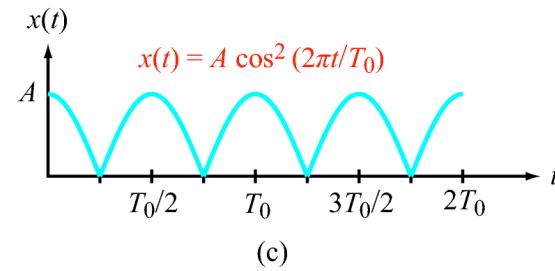
- a periodic signal  $x(t)$  of period  $T$  satisfies the periodicity property:
- $x(t) = x(t + nT)$
- for all integer values of  $n$  and all times  $t$



(a)



(b)



(c)



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# signals - useful functions - sinusoids

sinusoidal function:  $x(t) = A \cos(w_0 t + \theta)$ ;  $A$  is the amplitude of the signal,  $w_0$  is the radian frequency in radians per second units and  $\theta$  is the phase angle in radians; recall that  $w_0 = 2\pi f_0$

- if  $\theta = 0$ , the cosine waveform has a peak at the origin  $t_0 = 0$ , and the positive or negative peaks of the waveform would occur at time instants  $t_k$  solution to  $2\pi f_0 t_k = k\pi$  where  $k$  is an integer and  $t_k = \frac{k}{2f_0}$
- with a different value of  $\theta$ , first peak is shifted to  $t_0 = \frac{-\theta}{2\pi f_0}$  where  $t_k = \frac{k\pi - \theta}{2\pi f_0}$  which are solutions to  $2\pi f_0 t_k + \theta = k\pi$  where  $k$  is an integer



# signals - useful functions - impulse function

unit impulse function  $\delta(t) = 0, t \neq 0$  and  
 $\int_{-\infty}^{\infty} \delta(t)dt = 1$ ; some properties are: Figure 5

- multiplication of a function by an impulse

$$\phi(t)\delta(t) = \phi(0)\delta(t)$$

$$\phi(t)\delta(t-T) = \phi(T)\delta(t-T)$$

Time scaling property of Dirac

- sampling property

$$\int_{-\infty}^{\infty} \phi(t)\delta(t)dt = \int_{-\infty}^{\infty} \phi(0)\delta(t)dt =$$

$$\phi(0) \int_{-\infty}^{\infty} \delta(t)dt = \phi(0)$$

$$\delta(at) = \frac{1}{|a|}\delta(t).$$

- the area under the curve obtained by the product of the unit impulse function shifted by  $T$  and is the value of the function  $\phi(t)$  for  $t = T$

$$\int_{-\infty}^{\infty} \phi(t)\delta(t-T)dt = \phi(T)$$

- unit step is the integral of the unit impulse function

$$\frac{du(t)}{dt} = \delta(t); \int_{-\infty}^t \delta(t)dt = u(t)$$

- discrete time impulse function:  $\delta[n] = 1$  if  $n = 0$  or 0 otherwise



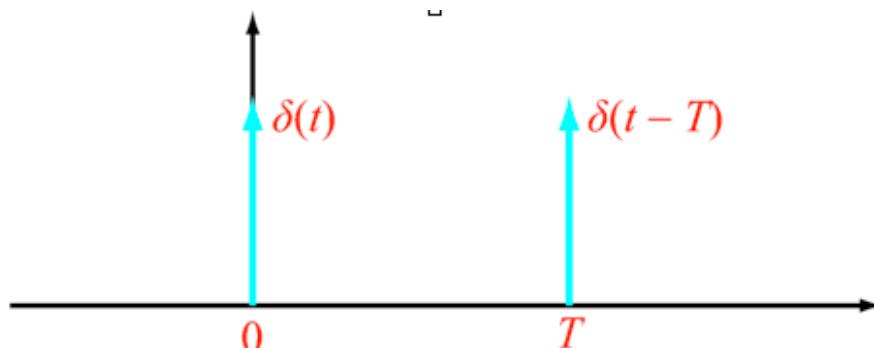
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# signals - useful functions - impulse function

$$\delta(t - T) = 0 \quad \text{for } t \neq T$$

and

$$\int_{-\infty}^{\infty} \delta(t - T) dt = 1.$$



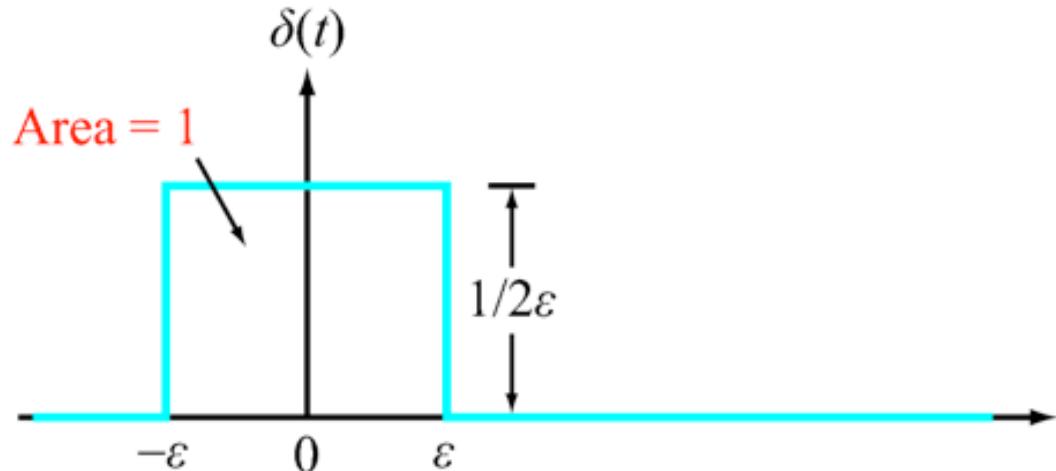
(a)  $\delta(t)$  and  $\delta(t - T)$



# signals - useful functions - impulse function

$$\frac{d}{dt} [u(t - T)] = \delta(t - T),$$

$$u(t - T) = \int_{-\infty}^t \delta(\tau - T) d\tau.$$



(b) Rectangle model for  $\delta(t)$



# signals - useful functions - sampling property of dirac (delta)

$$\int_{-\infty}^{\infty} x(t) \delta(t - T) dt = x(T).$$

(sampling property)



# signals - useful functions - example

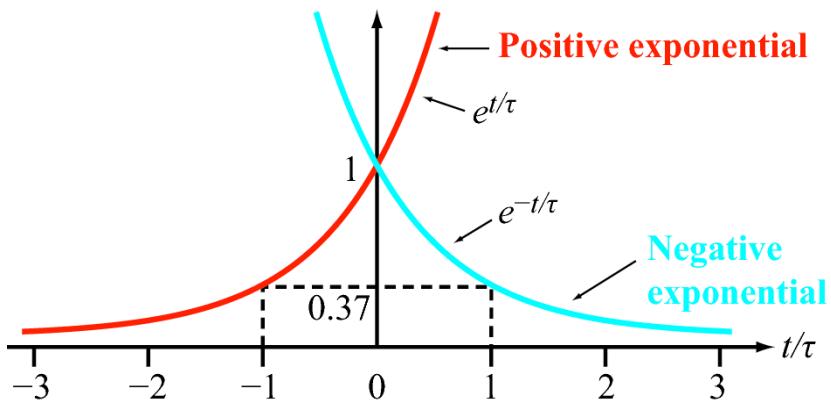
Evaluate  $\int_1^2 t^2 \delta(2t - 3) dt.$

**Solution:**

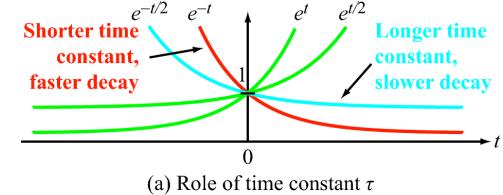


# signals - useful functions - exponential

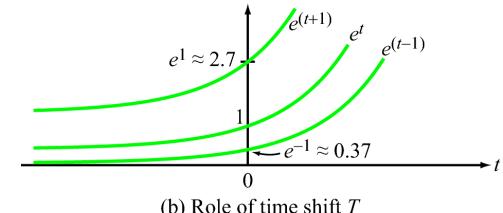
- exponential waveform



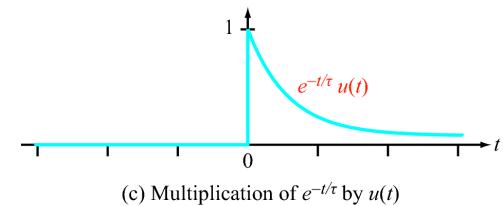
## Exponential Functions



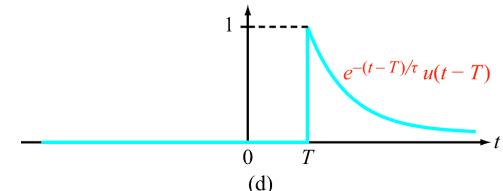
(a) Role of time constant  $\tau$



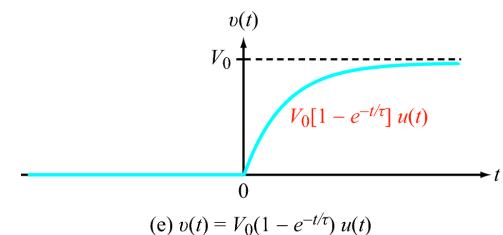
(b) Role of time shift  $T$



(c) Multiplication of  $e^{-t/\tau}$  by  $u(t)$



(d)



(e)  $v(t) = V_0(1 - e^{-t/\tau}) u(t)$



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# signals - useful functions - summary

Function	Expression	General Shape
Step	$u(t - T) = \begin{cases} 0 & \text{for } t < T \\ 1 & \text{for } t > T \end{cases}$	
Ramp	$r(t - T) = (t - T) u(t - T)$	
Rectangle	$\text{rect}\left(\frac{t - T}{\tau}\right) = u(t - T_1) - u(t - T_2)$ $T_1 = T - \frac{\tau}{2}; \quad T_2 = T + \frac{\tau}{2}$	
Impulse	$\delta(t - T)$	
Exponential	$\exp[-(t - T)/\tau] u(t - T)$	



# signals - useful functions - complex exponential

- complex exponential waveform
- review complex numbers

$$f(t) = Ae^{j\omega t}$$

- Eulers relations

$$Ae^{j\omega t} = A \cos(\omega t) + jA \sin(\omega t)$$

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$e^{j\omega t} = \cos(\omega t) + j \sin(\omega t)$$

- discrete time complex exponential

$$\begin{aligned} f[n] &= Be^{snT} \\ &= Be^{j\omega nT} \end{aligned}$$

where  $s = j\omega$  and  $k = nT$

- function generalization  $e^{st}$   
let  $s = \sigma + j\omega$   
therefore  $e^{st} = e^{(\sigma+j\omega)t} = e^{\sigma t}e^{j\omega t}$   
 $= e^{\sigma t}(\cos(\omega t) + j \sin(\omega t))$   
if  $s^* = \sigma - j\omega$  that being the conjugate of s, then  
 $e^{s^*t} = e^{(\sigma-j\omega)t} = e^{\sigma t}e^{-j\omega t}$   
 $= e^{\sigma t}(\cos(\omega t) - j \sin(\omega t))$
- $e^{st}$  can be used to model a large class of signals
  - \* if  $s = 0$  then the signal is a constant  $k$
  - \* if  $\omega = 0$  and  $s = \sigma$  then the signal is a monotonic exponential  $e^{\sigma t}$
  - \* if  $\sigma = 0$  and  $s = \pm j\omega$  then the signal is a sinusoid  $\cos(\omega t)$
  - \* if  $s = \sigma \pm j\omega$  then the signal is an exponentially varying sinusoid  $e^{\sigma t} \cos(\omega t)$



# signals - discrete time signals

- 3 ways to define
  - graphically
  - as a discrete function
  - as a list with a datum
- operations
  - arithmetic with constant
  - multiplication with constant
  - arithmetic 2 series
  - multiplication
  - time shift
  - time scale
  - time reversal

arithmetic with constant  $g[n] = x[n] + A$

multiplication with constant  $g[n] = Bx[n]$

arithmetic with 2 series  $g[n] = x[n] + y[n]$ ,  
note: make sure Indices line up

mutliplication with 2 series  $g[n] = x[n]y[n]$ ,  
same note, make sure indices line up

time shifting  $g[n] = x[n - k]$ ,  $k$  is an integer

time scaling  $g[n] = x[kn]$ ,  $k$  is an integer

time reversal  $g[n] = x[-n]$



# signals - discrete time signals

unit impulse

$$\begin{aligned}\delta[n] &= 1, n = 0 \\ &= 0, n \neq 0\end{aligned}$$

$$\begin{aligned}a\delta[n - n_1] &= a, n = n_1 \\ &= 0, n \neq n_1\end{aligned}$$

unit step function

$$\begin{aligned}u[n] &= 1, n \geq 0 \\ &= 0, n < 0\end{aligned}$$

unit ramp function

$$\begin{aligned}u[n] &= n, n \geq 0 \\ &= 0, n < 0\end{aligned}$$



# signals - discrete time signals

sinusoidal signals

$x[n] = A \cos(\Omega_0 n + \theta)$ ,  $A$  is amplitude,  $\Omega_0$  is the angular frequency in radians and  $\theta$  is the phase angle in radians

$$\begin{aligned}x_k[n] = x[k]\delta[n - k] &= x[k], n = k \\&= 0, n \neq k\end{aligned}$$

real vs. complex

$x[n] = x_r[n] + jx_i[n]$  or in polar form

$$x[n] = |x[n]|e^{j\angle x[n]}$$

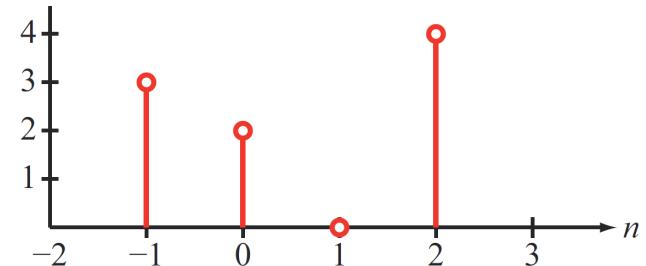


# signals - discrete time signals

- discrete-time signals  $x[n]$  are defined only at integer times  $n$
- continuous time signals  $x(t)$  are defined for all real numbers  $t$
- discrete time signals can be represented in 3 different ways:

1. Enclosing all non-zero values in **brackets**:  $x[n]=\{3,\underline{2},0,4\}$ .  
The underline indicates  $x[n]$  at time  $n=0$ . Here,  $x[0]=2$ .
2. Listing all of its non-zero values:
3. Using a **stem plot**:

$$x[n] = \begin{cases} 3 & \text{for } n = -1, \\ 2 & \text{for } n = 0, \\ 4 & \text{for } n = 2, \\ 0 & \text{for all other } n. \end{cases}$$

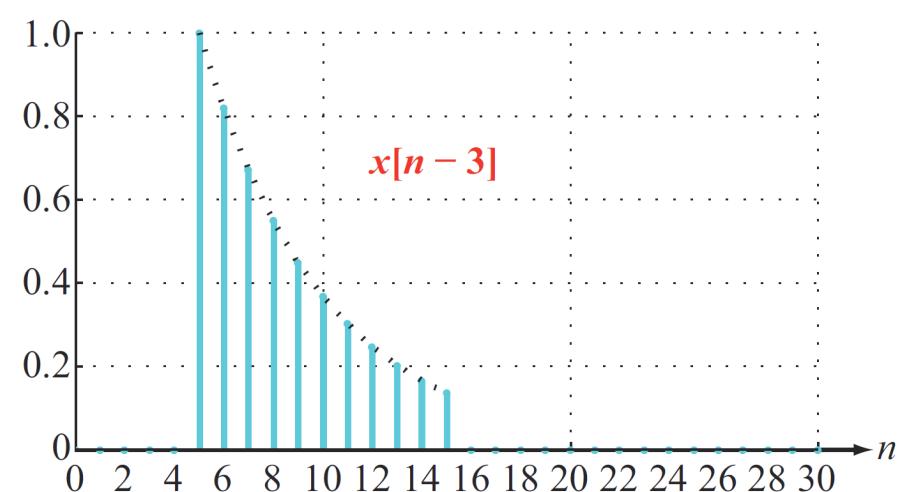
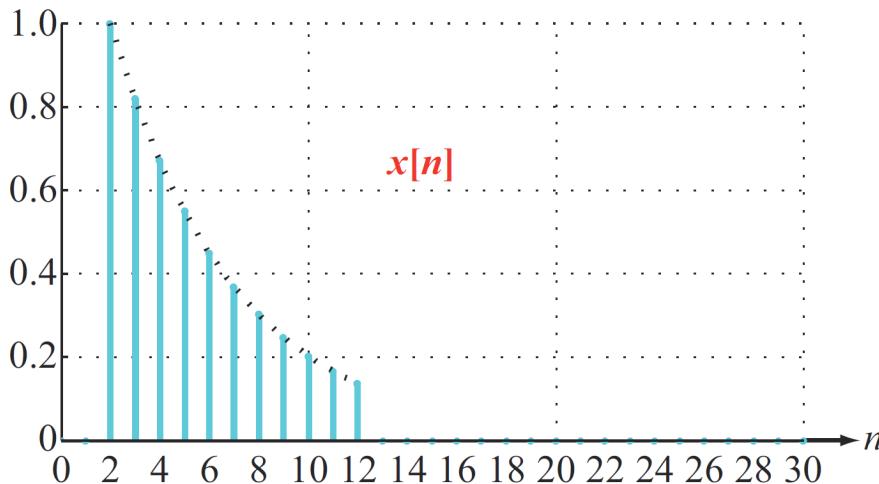


# signals - discrete time signals - finite length

Let  $x[n]=0$  for all  $n < a$  and all  $n > b$ . The **length** or **duration** of  $x[n]$  is  $b-a+1$ , **not**  $b-a$ .

**Example:** Let  $x[n]=\{3,2,0,4\}=0$  for  $n < -1$  and  $n > 2$ . The length of  $x[n]$  is  $2-(-1)+1=4$ .

**Time shifts** work the same way in discrete time as time delays work in continuous time.

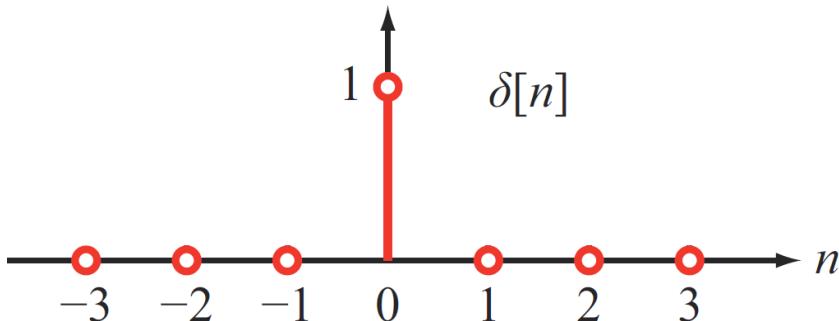


# signals - discrete time signals - functions

discrete-time impulse, discrete-time step

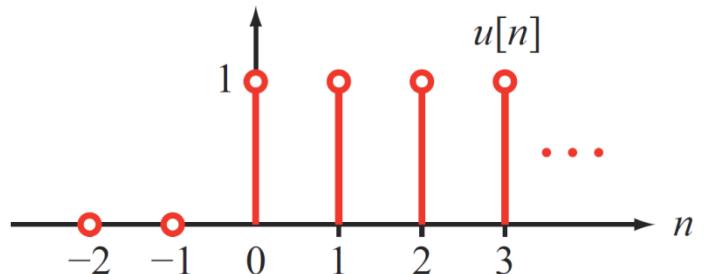
## Discrete-time Impulse:

$$\delta[n] = \{\underline{1}\} = \begin{cases} 1 & \text{for } n = 0, \\ 0 & \text{for } n \neq 0. \end{cases}$$



## Discrete-time Step:

$$u[n] = \{\underline{1}, 1, 1, \dots\} = \begin{cases} 1 & \text{for } n \geq 0, \\ 0 & \text{for } n < 0, \end{cases}$$



# signals - discrete time signals - impulses & steps

discrete vs. cts. time : sampling, steps, impulses

## Property:

### Sampling:

### Relation between step and impulse

### Value at time=0

## Discrete Time:

$$x[n] = \sum_{i=-\infty}^{\infty} x[i] \delta[i - n]$$

$$u[n] = \sum_{i=-\infty}^n \delta[i]$$

$$u[0]=1 \text{ and } u(0)=1.$$

## Continuous Time:

$$x(T) = \int_{-\infty}^{\infty} x(t) \delta(t - T) dt$$

$$u(t) = \int_{-\infty}^t \delta(\tau) d\tau$$

$$u(0) \text{ undefined}$$

$$u(0) \rightarrow \infty$$



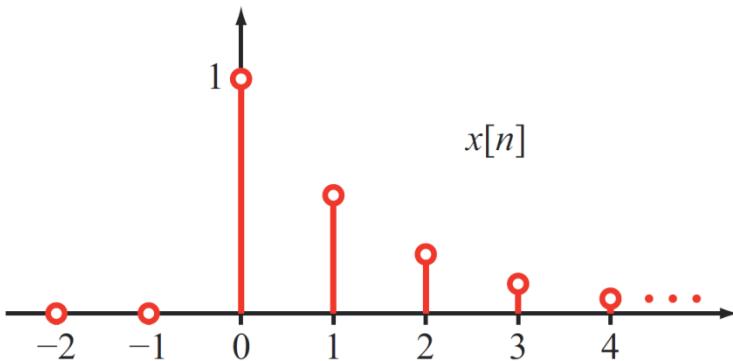
# signals - discrete time signals - exponentials

discrete vs. cts. time : geometric signals, exponential signals

## Discrete Time:

### Geometric Signals

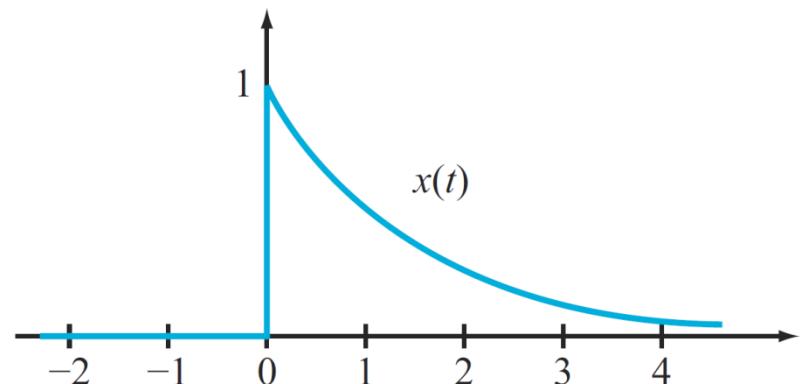
$$x[n] = p^n u[n]$$



## Continuous Time:

### Exponential Signals

$$x_1(t) = e^{-at} u(t)$$



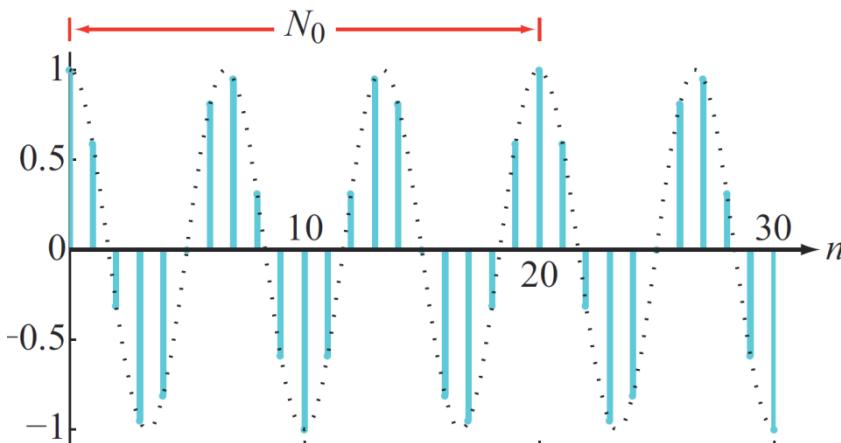
# signals - discrete time signals - sinusoids

discrete vs. cts. time : sinusoids

periodic vs. non-periodic, if signal satisfies  $x[n] = x[n + N]$  then the signal is periodic for an integer value  $n$  for  $N \neq 0$

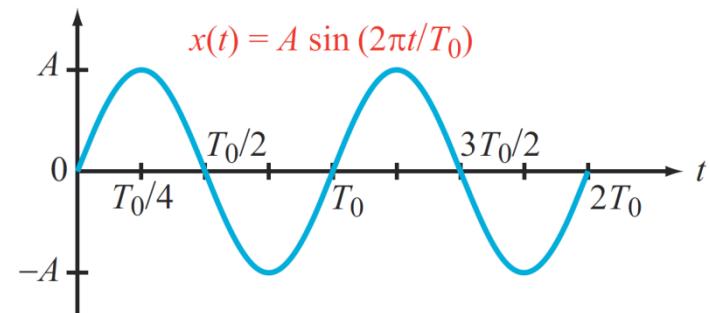
## Discrete Time:

$$= A \cos(\Omega n + \theta)$$



## Continuous Time:

$$x(t) = A \cos(\omega_0 t + \theta)$$



**Period:**  $N_0 = \frac{2\pi k}{\Omega}$  if  $k$  exists that makes this an integer



# signals - discrete time signals - example

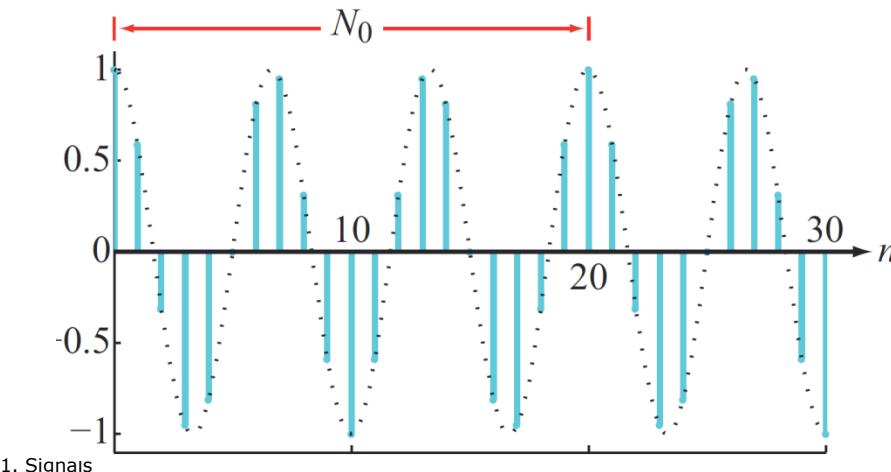
period of discrete time sinusoid

**Goal:** Determine the period of  $x[n] = \cos(0.3\pi n)$

**Solution:**  $N_0 = \frac{2\pi k}{\Omega} = \frac{2\pi k}{0.3\pi} = \frac{20k}{3}$   $k=3 \rightarrow \text{period}=20$ .

$$x[n+20] = \cos(0.3\pi(n+20)) = \cos(0.3\pi n + 6\pi) = \cos(0.3\pi n).$$

**Note:** Most discrete-time sinusoids are **not periodic!**



# signals - discrete time signals - example

To find the **integer** period  $N > 0$  (i.e., ( $N \in \mathbb{Z}^+$ ) of a general discrete-time sinusoid  $x[n] = A\cos(\Omega n + \phi)$ :

$$\begin{aligned}x[n] &= x[n + N] \\A\cos(\Omega n + \phi) &= A\cos(\Omega(n + N) + \phi) \\A\cos(\Omega n + \phi + 2\pi k) &= A\cos(\Omega n + \phi + \Omega N) \\\therefore 2\pi k &= \Omega N \\N &= \frac{2\pi k}{\Omega}\end{aligned}$$

where  $k \in \mathbb{Z}$ .

Note: there may not exist a  $k \in \mathbb{Z}$  such that  $\frac{2\pi k}{\Omega}$  is an integer.

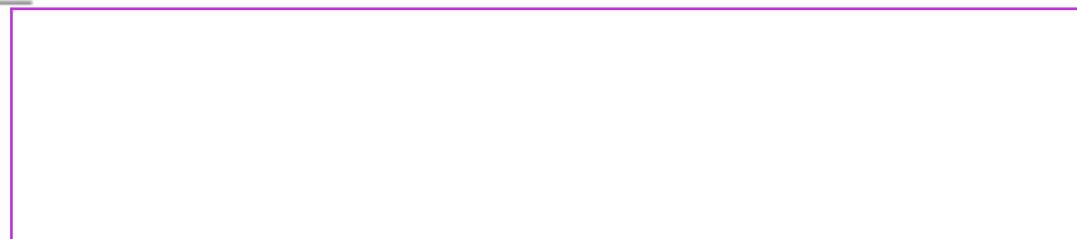


# signals - discrete time signals - example

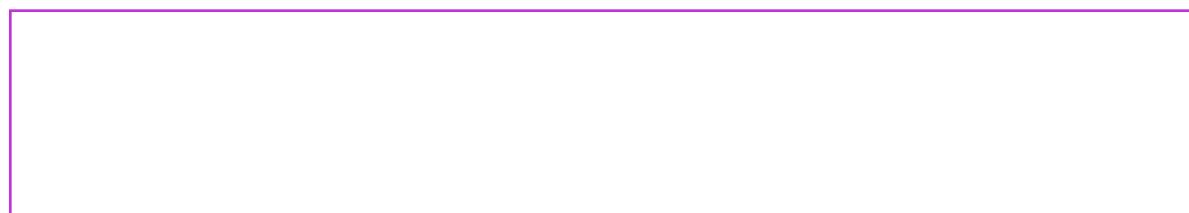
Example i:  $\Omega = \frac{37}{11}\pi$



Example ii:  $\Omega = 2$



Example iii:  $\Omega = \sqrt{2}\pi$



# signals - discrete time signals - example

$$N = \frac{2\pi k}{\Omega}$$

$$\Omega = \frac{2\pi k}{N} = 2\pi \frac{k}{N} = \pi \cdot$$



Therefore, a discrete-time sinusoid is periodic if its radian frequency  $\Omega$  is a rational multiple of  $\pi$ .

Otherwise, the discrete-time sinusoid is non-periodic.

# signals - discrete time signals - example

Example 1:  $\Omega = \pi/6 = \pi \cdot \frac{1}{6}$

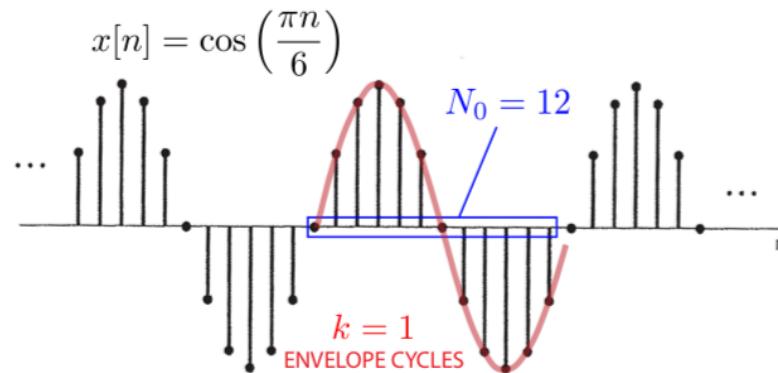
$$x[n] = \cos\left(\frac{\pi n}{6}\right)$$

$$N = \frac{2\pi k}{\pi/6} = 12k$$

$$k=1$$

periodic

$$N = \frac{2\pi k}{\Omega}$$



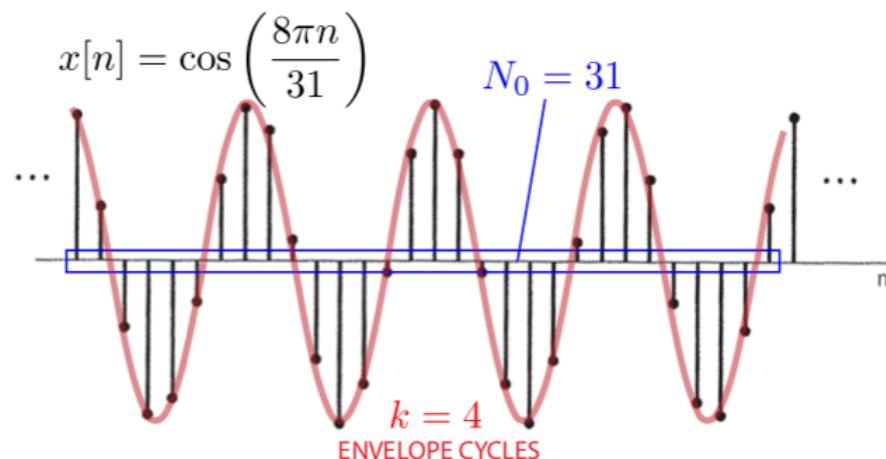
# signals - discrete time signals - example

Example 2:  $\Omega = 8\pi/31 = \pi \cdot \frac{8}{31}$

$$x[n] = \omega_s \left( \frac{8\pi n}{31} \right)$$

$$N_0 = \frac{2\pi k}{8\pi/31} = \frac{31}{4}k$$

$$N_0 = 31, k = 4$$



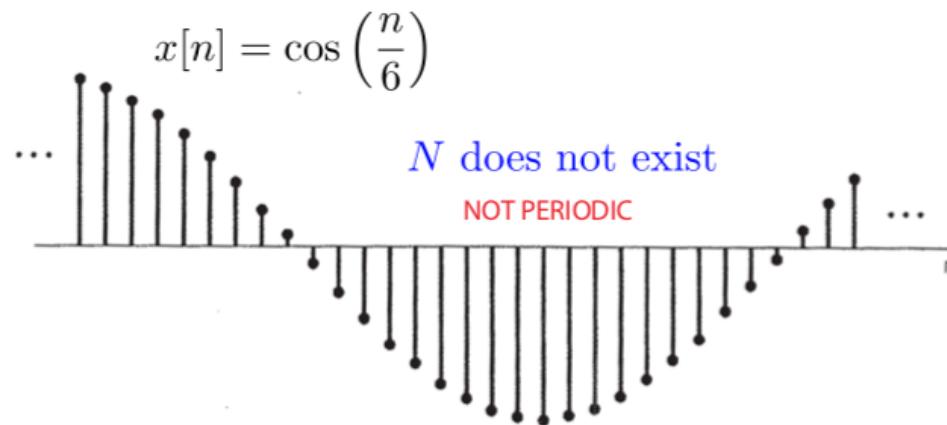
# signals - discrete time signals - example

Example 3:  $\Omega = 1/6 = \pi \cdot \frac{1}{6\pi}$

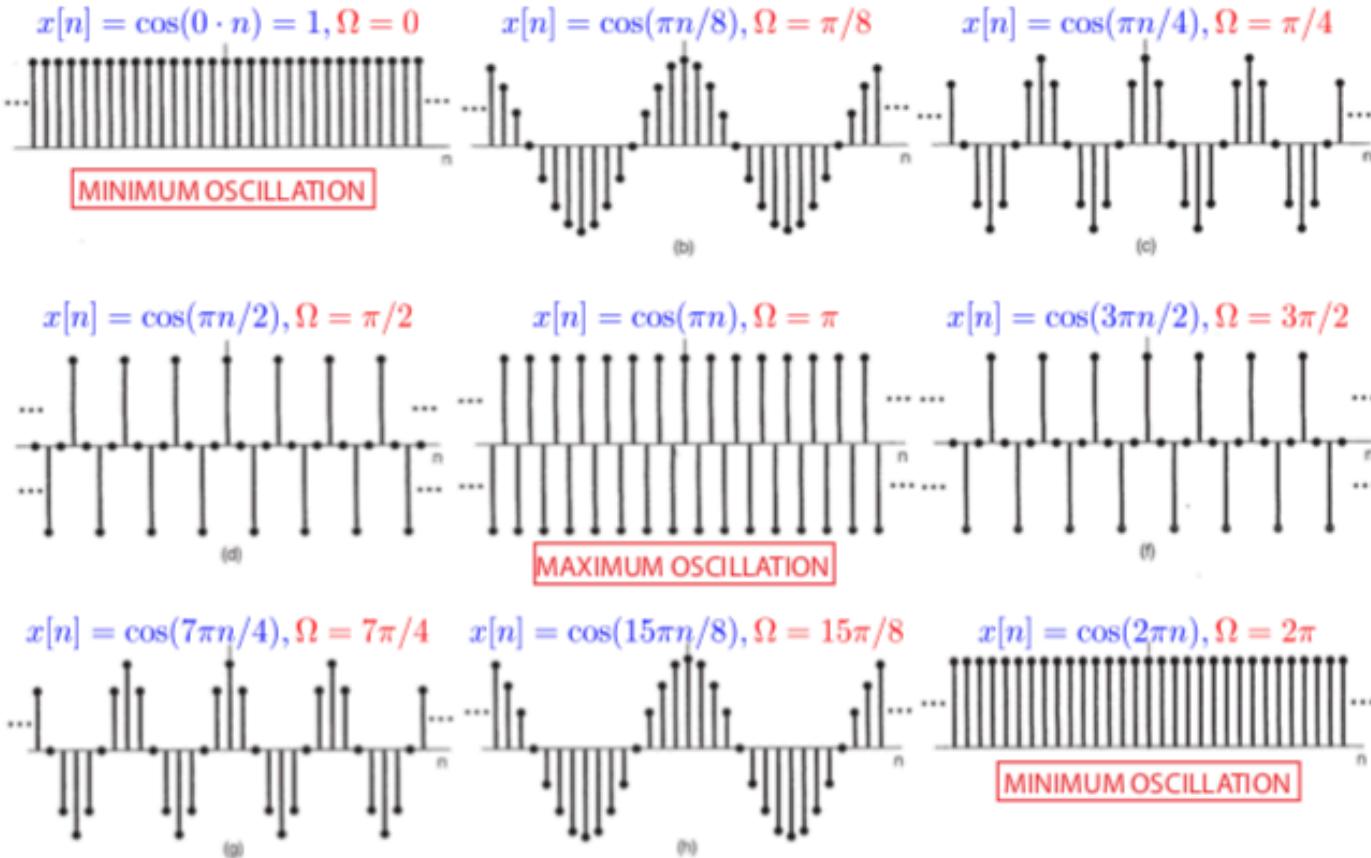
$$x[n] = \cos\left(\frac{n}{6}\right)$$

$$N = \frac{2\pi k}{1/6} = 12\pi k$$

$x[n]$  is non-periodic  
no  $k$  to make  $N$  int



# signals - discrete time signals - example



# signals - discrete time signals - properties

periodic vs. non-periodic, if signal satisfies  $x[n] = x[n + N]$  then the signal is periodic for an integer value  $n$  for  $N \neq 0$

energy of a signal

$$E_s = \sum_{n=-\infty}^{n=\infty} x^2[n] \text{ and}$$

$$E_s = \sum_{n=-\infty}^{n=\infty} |x[n]|^2 \text{ if complex}$$

time averaging  $\langle x[n] \rangle = \frac{1}{N} \sum_{n=0}^{n=N-1} x[n]$

power of a signal  $P_s = \langle x^2[n] \rangle$

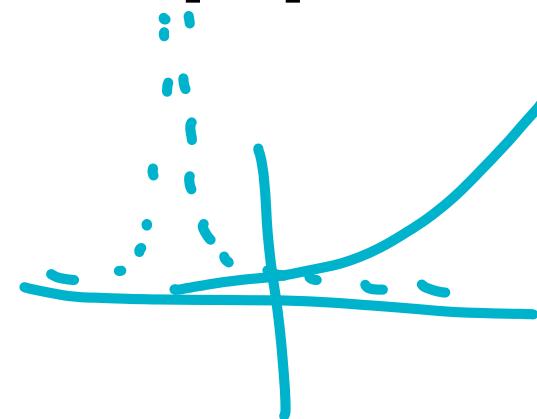


# signals - discrete time signals - qualitative properties

general

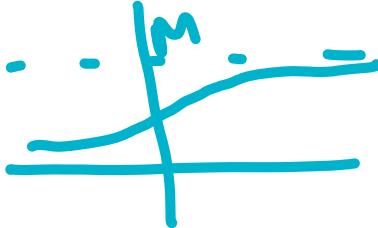
- $u$  decays if  $u(t) \rightarrow 0$  as  $t \rightarrow \infty$
- $u$  converges if  $u(t) \rightarrow a$  as  $t \rightarrow \infty$  where  $a$  is some constant
- $u$  is bounded if its peak is finite
- $u$  is unbounded or blows up if its peak is infinite
- $u$  is periodic if for some  $T > 0$ ,  $u(t+T) = u(t)$  holds for all  $t$

- in practice we are interested in more specific quantitative questions such as (i) how fast does  $u$  decay or converge or (ii) how large is the peak of  $u$



# signals - discrete time signals - qualitative properties

bounded



- a continuous signal bounded if there exists a  $M$  s.t. for all  $t \in R$ ,  $|x(t)| \leq M$
- a discrete signal is bounded if there exists an  $M$  s.t. for all  $n \in Z$ ,  $|x[n]| \leq M$
- a signal is unbounded if it is not bounded

cts

discrete

stable

- a continuous signal is absolutely integrable if  $\int_{-\infty}^{\infty} |x(t)| < \infty$
- a discrete time signal is absolutely summable if  $\sum_{-\infty}^{\infty} |x[n]| < \infty$
- a CT (continuous time) or DT (discrete time) signal is stable if it is absolutely integrable or summable

cts

discrete



# signals - discrete time signals - qualitative properties

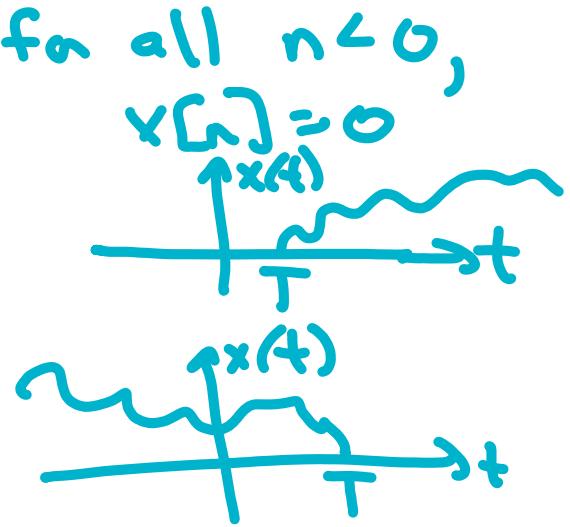
causal

- a CT signal is causal if for all  $t < 0$ ,  $x(t) = 0$

**- a DT signal is causal if for all  $n < 0$ ,  $x[n] = 0$**

side-ness

- a CT signal is right-sided if there exists a  $T$  s.t. for all  $t < T$ ,  $x(t) = 0$
- a DT signal is right-sided if there exists a  $T$  s.t. for all  $n < T$ ,  $x[n] = 0$
- a CT signal is left-sided if there exists a  $T$  s.t.  $t > T$ ,  $x(t) = 0$
- a DT signal is left-sided if there exists a  $T$  s.t. for all  $n > T$ ,  $x[n] = 0$
- a CT or DT signal is said to be two-sided if it is not left-sided or right-sided





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