SYDE252 - lecture notes

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note: some material (figures) borrowed from various sources

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6. ZT Z Transform

09/11/18

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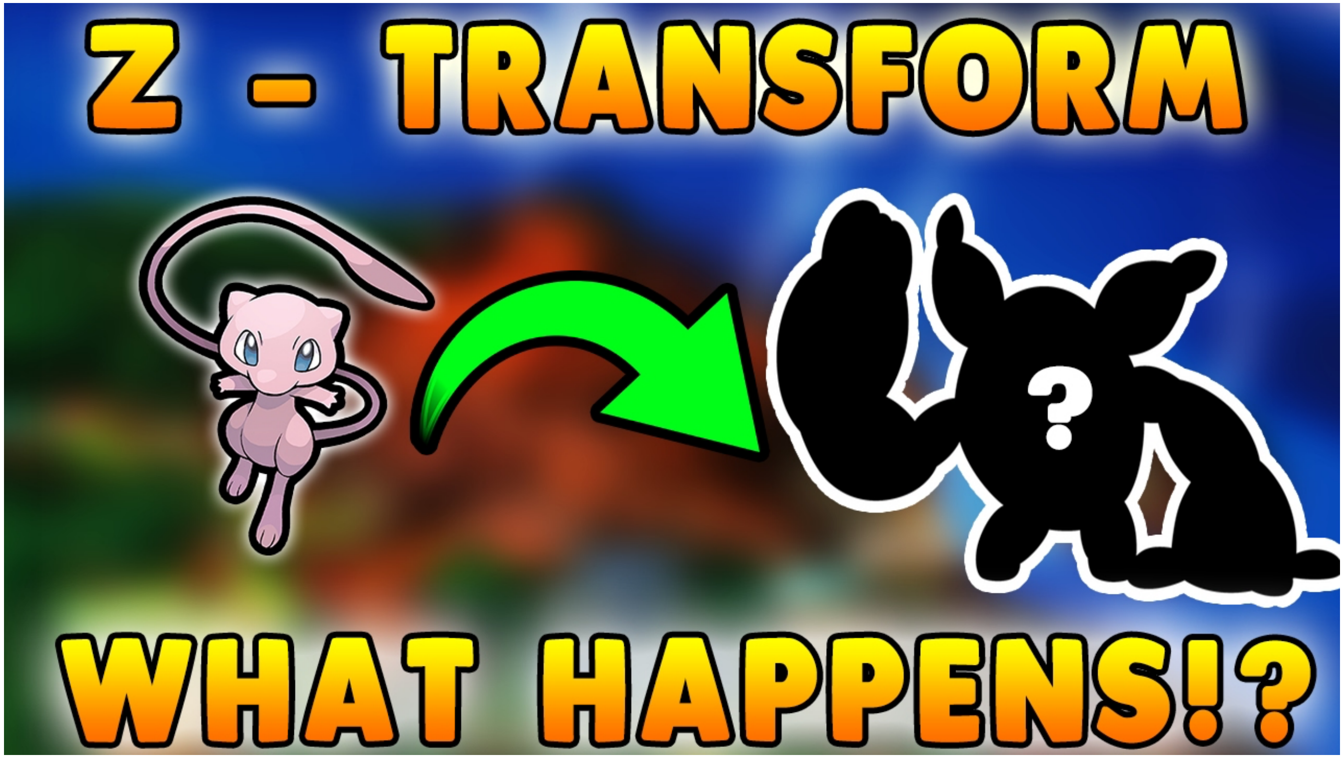
inspiration



totally irrelevant however i could not find something for this - https://www.youtube.com/watch?v=4Bv2NNkcdTA

discrete vs. cts. transforms

Discrete Time:

$$\mathbb{Z}[x[n]] = \mathbf{X}(\mathbf{z}) = \sum_{n=0}^{\infty} x[n]\,\mathbf{z}^{-n}$$

$$\boxed{\delta[n] \quad \longleftrightarrow \quad 1}$$

$$\mathbf{a}^{n-1}\, u[n-1] \quad \longleftrightarrow \quad \frac{1}{\mathbf{z}-\mathbf{a}}$$

$$\boxed{x[0] = \lim_{\mathbf{z}\to\infty} \mathbf{X}(\mathbf{z})}$$

$$x_1[n] * x_2[n] \quad \longleftrightarrow \quad \mathbf{X}_1(\mathbf{z})\,\mathbf{X}_2(\mathbf{z})$$

Continuous Time:

$$\mathcal{L}[x(t)] = \mathbf{X}(\mathbf{s}) = \int_0^{\infty} x(t)\,(e^{\mathbf{s}})^{-t}\,dt$$

$$\boxed{\delta(t) \quad \longleftrightarrow \quad 1}$$

$$e^{\mathbf{a}t}\, u(t) \quad \longleftrightarrow \quad \frac{1}{\mathbf{s}-\mathbf{a}}$$

$$\boxed{x(0^+) = \lim_{\mathbf{s}\to\infty} \mathbf{s}\,\mathbf{X}(\mathbf{s})}$$

$$x_1(t) * x_2(t) \quad \longleftrightarrow \quad \mathbf{X}_1(\mathbf{s})\,\mathbf{X}_2(\mathbf{s})$$

z transform

- **Laplace Transform**: $X(s) = \int_0^\infty x(t)e^{-st}dt$. The purpose of the Laplace transform is that it converts integral-differential equations to algebraic equations. It deals with continuous time systems and is used for signal analysis including stability analysis.

- **Z Transform**: $X[z] = \sum_{n=0}^{\infty} x[n]z^{-n}$. The ZT converts difference equations into algebraic equations. It deals with discrete time systems and signal analysis and including stability analysis. It is the discrete equivalent of the Laplace transform.

ZT intro

Consider a discrete signal $x(t)$ sampled every T seconds.

$x(t) = x_0\delta(t) + x_1\delta(t-T) + x_2\delta(t-2T) + x_3\delta(t-3T) + \ldots$

The Laplace transform of $x(t)$ is
$X(s) = x_0 + x_1e^{-sT} + x_2e^{-s2T} + x_3e^{-s3T} + \ldots$

Now lets define $z = E^{sT} = e^{(\sigma+j\omega)T} = e^{\sigma T}cos(\omega T) + je^{\sigma T}sin(\omega T)$

Therefore:
$X[z] = x_0 + x_1z^{-1} + x_2z^{-2} + x_3z^{-3} + \ldots$

Z^{-1} is the sample period delay operator.
From Laplace, we know $z = e^{sT}$ is the time advance by T seconds (T is the sampling period). Therefore $z^{-1} = e^{sT}$ corresponds to THE UNIT SAMPLE PERIOD DELAY. Therefore all the sampled data (and discrete time system) can be expressed in terms of z.

ZT intro (2)

The unilateral z-transform of a causal sampled sequence:
$x[n] = x[0] + x[1] + x[2] + x[3] + \dots$
is
$= \sum_{n=0}^{\infty} x[n]z^{-n}$

Therefore the bilateral z-transform for a general sampled sequence is:
$X[z] = \sum_{-\infty}^{\infty} x[n]z^{-n}$

We will only use the unilateral z-transform:
$X[z] = \sum_{n=0}^{\infty} x[n]z^{-n}$

ZT and difference equations

Recall that Laplace transform is applied to differential equations. In a similar way, the Z transform is applied to difference equations. Recall that IC (Initial Conditions) lead to the zero-input solution and that the input leads to the zero-state response.

If a difference equation is in the advance form, this will require a knowledge of ancillary conditions $y[0], y[1], \ldots y[N-1]$ rather than IC $y[-1], y[-2], \ldots$. This can be overcome by writing the difference equation in the delay form.

Note that we need some time reference, i.e., $n = 0$. It is $y[n-k]u[n]$ and not $y[n-k]u[n-k]$.

ZT and transfer fn

$$TF = H(z) = ZT\{h[n]\}$$
$$x[n] \to LTI \to y[n]$$
$$= h[n] * x[n]$$
is the same in the frequency domain as $H(z)X(z)$

therefore $H(z) = \frac{Y(z)}{X(z)}$

$z^n \;\to\; h[n] \;\to\; z^n H(z)$ is z^n in $\to$ scaled z^n out.

$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n}$

$h[n] * z^n = \sum h[i] z^{n-i} = z^n \sum h[i] z^{-i} = z^n H(z).$

$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}}$

Note that z^n plays the same role that e^{st} in continuous time for the Laplace.

$H(z)$ is the transfer function. $H(e^{j\omega}) = H(z)|_{z=e^{j\omega}}$

ZT and frequency response

$$x[n] = e^{j\omega n} \rightarrow LTI \rightarrow y[n] = h[n] * e^{j\omega n}$$
$$= \sum_{i=0}^{\infty} h[i] e^{j\omega(n-i)}$$
$$= \sum_{i=0}^{\infty} h[i] e^{-j\omega i} e^{j\omega n}$$
$$= H(e^{j\omega}) e^{j\omega n}$$

as an example, using $H(z) = 2 - 3z^{-1} + z^{-2}$
compute the response of the system to $\cos\left(\frac{\pi}{2}n\right)$
$$H(e^{j\frac{\pi}{2}}) = H(j) = 2 - 3\frac{1}{j} + 1\frac{1}{j^2} = 1 + 3j = \sqrt{10}e^{j1.25}$$
so the response is
$$x[n] = cost\frac{\pi}{2} \rightarrow LTI \rightarrow y[n] = \sqrt{10}cos(\frac{\pi}{2}n + 1.25)$$

ZT example 1

$$\{\underline{a_0}, a_1, a_2, \ \ldots\ , a_m\} \longleftrightarrow a_0 + \frac{a_1}{\mathbf{z}} + \frac{a_2}{\mathbf{z}^2} + \cdots + \frac{a_m}{\mathbf{z}^m}$$

Compute the $\mathbf{z}$-transform of $x[n] = \{\underline{4}, 2, 0, 5\}$.

$$= 4 + 2\mathbf{z}^{-1} + 5\mathbf{z}^{-3}$$

ZT example 2

Often convenient to use this formula: $\mathbf{a}^n\ u[n] \longleftrightarrow \dfrac{\mathbf{z}}{\mathbf{z}-\mathbf{a}}$

Setting **a**=1 gives the special case: $u[n] \longleftrightarrow \dfrac{\mathbf{z}}{\mathbf{z}-1}$

Compute $\mathbb{Z}[\{\underline{1}, 3\} + 4(2^n)\ u[n]]$. Do this term-by-term:

$$\begin{aligned}\mathbb{Z}[\{\underline{1}, 3\} + 4(2^n)\ u[n]] &= \mathbb{Z}[\{\underline{1}, 3\}] + 4\mathbb{Z}[2^n\ u[n]] \\ &= (1 + 3\mathbf{z}^{-1}) + 4\,\frac{\mathbf{z}}{\mathbf{z}-2} = \frac{5\mathbf{z}^2 + \mathbf{z} - 6}{\mathbf{z}^2 - 2\mathbf{z}}\end{aligned}$$

ZT properties

Properties of the **z**-transform for causal signals for $m > 0$.

Property	$x[n]$		$\mathbf{X}(\mathbf{z})$
1. Linearity	$C_1\ x_1[n] + C_2\ x_2[n]$	$\longleftrightarrow$	$C_1\ \mathbf{X}_1(\mathbf{z}) + C_2\ \mathbf{X}_2(\mathbf{z})$
2. Time delay by 1	$x[n-1]\ u[n]$	$\longleftrightarrow$	$\frac{1}{\mathbf{z}}\ \mathbf{X}(\mathbf{z}) + x[-1]$
2a. Time delay by m	$x[n-m]\ u[n]$	$\longleftrightarrow$	$\frac{1}{\mathbf{z}^m}\ \mathbf{X}(\mathbf{z}) + \frac{1}{\mathbf{z}^m} \sum_{i=1}^{m} x[-i]\ \mathbf{z}^i$
3. Right shift by m	$x[n-m]\ u[n-m]$	$\longleftrightarrow$	$\frac{1}{\mathbf{z}^m}\ \mathbf{X}(\mathbf{z})$
4. Time advance by 1	$x[n+1]\ u[n]$	$\longleftrightarrow$	$\mathbf{z}\ \mathbf{X}(\mathbf{z}) - \mathbf{z}\ x[0]$
4a. Time advance by m	$x[n+m]\ u[n]$	$\longleftrightarrow$	$\mathbf{z}^m\ \mathbf{X}(\mathbf{z}) + \mathbf{z}^m \sum_{i=0}^{m-1} x[i]\ \mathbf{z}^{-i}$
5. Multiplication by a^n	$\mathbf{a}^n\ x[n]\ u[n]$	$\longleftrightarrow$	$\mathbf{X}\left(\frac{\mathbf{z}}{\mathbf{a}}\right)$
6. Multiplication by n	$n\ x[n]\ u[n]$	$\longleftrightarrow$	$-\mathbf{z}\ \frac{d\mathbf{X}(\mathbf{z})}{d\mathbf{z}}$

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ZT table

No.	$x[n]$	$X[z]$
1	$\delta[n-n]$	z^{-k}
2	$u[n]$	$\dfrac{z}{z-1}$
3	$nu[n]$	$\dfrac{z}{(z-1)^2}$
4	$n^2u[n]$	$\dfrac{z(z+1)}{(z-1)^3}$
5	$n^3u[n]$	$\dfrac{z(z^2+4z+1)}{(z-1)^4}$
6	$\gamma^n u[n]$	$\dfrac{z}{z-\gamma}$
7	$\gamma^{n-1}u[n-1]$	$\dfrac{1}{z-\gamma}$
8	$n\gamma^n u[n]$	$\dfrac{\gamma z}{(z-\gamma)^2}$
9	$n^2\gamma^n u[n]$	$\dfrac{\gamma z(z+\gamma)}{(z-\gamma)^3}$
10	$\dfrac{n(n-1)(n-2)\cdots(n-m+1)}{\gamma^m m!}\gamma^n u[n]$	$\dfrac{z}{(z-\gamma)^{m+1}}$
11a	$\lvert\gamma\rvert^n \cos \beta n\, u[n]$	$\dfrac{z(z-\lvert\gamma\rvert\cos\beta)}{z^2-(2\lvert\gamma\rvert\cos\beta)z+\lvert\gamma\rvert^2}$
11b	$\lvert\gamma\rvert^n \sin \beta n\, u[n]$	$\dfrac{z\lvert\gamma\rvert\sin\beta}{z^2-(2\lvert\gamma\rvert\cos\beta)z+\lvert\gamma\rvert^2}$
12a	$r\lvert\gamma\rvert^n\cos(\beta n+\theta)u[n]$	$\dfrac{rz[z\cos\theta-\lvert\gamma\rvert\cos(\beta-\theta)]}{z^2-(2\lvert\gamma\rvert\cos\beta)z+\lvert\gamma\rvert^2}$
12b	$r\lvert\gamma\rvert^n\cos(\beta n+\theta)u[n]$ $\quad \gamma=\lvert\gamma\rvert e^{j\beta}$	$\dfrac{(0.5re^{j\theta})z}{z-\gamma}+\dfrac{(0.5re^{-j\theta})z}{z-\gamma^*}$
12c	$r\lvert\gamma\rvert^n\cos(\beta n+\theta)u[n]$	$\dfrac{z(Az+B)}{z^2+2az+\lvert\gamma\rvert^2}$

$$r=\sqrt{\frac{A^2\lvert\gamma\rvert^2+B^2-2AaB}{\lvert\gamma\rvert^2-a^2}}$$

$$\beta=\cos^{-1}\frac{-a}{\lvert\gamma\rvert}$$

$$\theta=\tan^{-1}\frac{Aa-B}{A\sqrt{\lvert\gamma\rvert^2-a^2}}$$

discrete vs. cts. inverse transforms

	Discrete Time:	**Continuous Time:**
Rational Function:	$\mathbf{X}(\mathbf{z}) = \dfrac{\mathbf{N}(\mathbf{z})}{D_0(\mathbf{z}-\mathbf{p}_1)(\mathbf{z}-\mathbf{p}_2)\cdots(\mathbf{z}-\mathbf{p}_N)}$	$\mathbf{X}(\mathbf{s}) = \dfrac{\mathbf{N}(\mathbf{s})}{\mathbf{D}(\mathbf{s})} = \dfrac{\mathbf{N}(\mathbf{s})}{(\mathbf{s}-p_1)(\mathbf{s}-p_2)\ldots(\mathbf{s}-p_n)}$
Partial Fraction:	$\mathbf{X}(\mathbf{z}) = A_0 + \sum_{i=1}^{N} \dfrac{\mathbf{A}_i}{\mathbf{z}-\mathbf{p}_i}$	$\mathbf{X}(\mathbf{s}) = \dfrac{A_1}{\mathbf{s}-p_1} + \dfrac{A_2}{\mathbf{s}-p_2} + \cdots + \dfrac{A_n}{\mathbf{s}-p_n}$
Using:	$\mathbf{a}^{n-1}\, u[n-1] \leftrightarrow \dfrac{1}{\mathbf{z}-\mathbf{a}}$	$e^{\mathbf{a}t}\, u(t) \leftrightarrow \dfrac{1}{\mathbf{s}-\mathbf{a}}$
Inverse Transform:	$x[n] = A_0\,\delta[n] + \sum_{i=1}^{N} \mathbf{A}_i \mathbf{p}_i^{n-1}\, u[n-1].$	$= [A_1 e^{p_1 t} + A_2 e^{p_2 t} + \cdots + A_n e^{p_n t}]\, u(t).$

Complex Poles:

$$2|\mathbf{a}|^n \cos(\Omega n + \theta)\, u[n] \leftrightarrow \frac{\mathbf{z}e^{j\theta}}{\mathbf{z}-\mathbf{a}} + \frac{\mathbf{z}e^{-j\theta}}{\mathbf{z}-\mathbf{a}^*}, \qquad \mathbf{a} = |\mathbf{a}|e^{j\Omega}$$

z transform examples 1

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$\delta[n] = x[n]$

$x[0] = 1,\ x[2] = x[3] = \ldots = 0$

$\therefore X[z] = \sum_{n=0}^{\infty} x[n] z^{-n} = x[0] + \frac{x[1]}{z} + \frac{x[2]}{z^2} + \ldots$

$= 1$

$\therefore\ \delta[n] \Leftrightarrow 1$ for all z

z transform examples 2

$$u[n] = x[n] \Rightarrow x[0] = x[1] = x[2] = \dots = 1$$

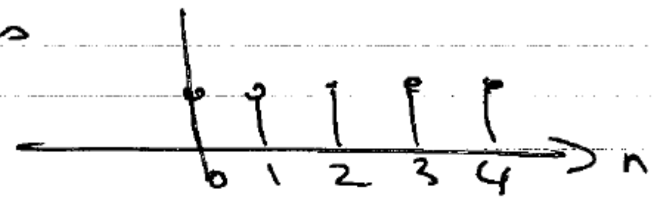
$$\therefore X[z] = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots$$

$$= \frac{1}{1 - \frac{1}{z}} \qquad \left|\frac{1}{z}\right| < 1 \Rightarrow \frac{z}{z-1} \qquad |z| > 1$$

$$\therefore u[n] \longleftrightarrow \frac{z}{z-1} \qquad |z| > 1$$

z transform examples 3

ZT of 5 impulses



by def^{n}

$$X[z] = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4}$$

recall for a power series

$$\sum_{k=0}^{n} r^k = \frac{r^{n+1} - 1}{r - 1}$$

let $r = z^{-1}$ & $n = 4$

$$X[z] = \frac{z^{-5} - 1}{z^{-1} - 1} = \frac{z}{z-1}\left(1 - z^{-5}\right)$$

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inverse z transform examples 1

eg ① $X[z] = \dfrac{8z-19}{(z-2)(z-3)}$

1st divide both sides by z

$$\frac{X[z]}{z} = \frac{8z-19}{z(z-2)(z-3)}$$

need to do this because if we don't, we have $u[n-1]$ terms & not $u[n]$

partial fraction expansion

$$\frac{X[z]}{z} = \frac{-19/6}{z} + \frac{3/2}{z-2} + \frac{5/3}{z-3}$$

multiply both sides by z

$$X[z] = -\frac{19}{6} + \frac{3}{2}\frac{z}{z-2} + \frac{5}{3}\frac{z}{z-3}$$

use tables

$$x[n] = -\frac{19}{6}\delta[n] + \left[\frac{3}{2}(2)^n + \left(\frac{5}{3}\right)3^n\right]u[n]$$

inverse z transform examples 2

e.g. 2 Inverse ZT ⑥ (repeated real poles)

$$X[z] = \frac{z(2z^2-11z+12)}{(z-1)(z-2)^3}$$

$$\frac{X[z]}{z} = \frac{2z^2-11z+12}{(z-1)(z-2)^3} = \frac{k}{z-1} + \frac{a_0}{(z-2)^3} + \frac{a_1}{(z-2)^2} + \frac{a_2}{(z-2)}$$

– use covering to get k & a_0

$$k = \left.\frac{2z^2-11z+12}{\cancel{(z-1)}(z-2)^3}\right|_{z=1} = -3 \qquad a_0 = \left.\frac{2z^2-11z+12}{(z-1)\cancel{(z-2)^3}}\right|_{z=2} = -2$$

$$\frac{X[z]}{z} = \frac{-3}{z-1} - \frac{2}{(z-2)^3} + \frac{a_1}{(z-2)^2} + \frac{a_2}{(z-2)} = \frac{2z^2-11z+12}{(z-1)(z-2)^3}$$

– to get a_2 multiply both sides by z & let $z \to \infty$

$$0 = -3 - 0 + 0 + a_2 \Rightarrow a_2 = 3$$

inverse z transform examples 2b

– to find a_1, let $z=0$

$$\frac{12}{8} = 3 + \frac{1}{4} + \frac{a_1}{4} + \frac{3}{2} \Rightarrow a_1 = -1$$

$$\therefore \frac{X[z]}{z} = \frac{-3}{z-1} - \frac{2}{(z-2)^3} - \frac{1}{(z-2)^2} + \frac{3}{z-2}$$

$$X[z] = \frac{-3z}{z-1} - \frac{2z}{(z-2)^3} - \frac{z}{(z-2)^2} + \frac{3z}{z-2}$$

using tables

$$x[n] = \left[-3 - 2n\frac{(n-1)}{8}(2)^n - \frac{n}{2}(2)^n + 3(2)^n\right]u[n]$$

$$= -\left\{3 + \frac{1}{4}(n^2+n-12)2^n\right]u[n]$$

inverse z transform examples 3

e.g., 3) inverse ZT (complex poles)

$$X[z] = \frac{2z\,(3z+17)}{(z-1)(z^2-6z+25)}$$

$$= \frac{2z\,(3z+17)}{(z-1)(z-3-j4)(z-3+j4)}$$

$$\frac{X[z]}{z} = \frac{2(3z+17)}{(z-1)(z^2-6z+25)} = \frac{2}{z-1} + \frac{Az+B}{6z^2-6z+25}$$

multiply both sides by z & let $z \to \infty$

$$0 = 2 + A \Rightarrow A = -2$$

- to find B, set $z=0$ $\therefore -\frac{34}{25} = -2 + \frac{B}{25} \Rightarrow B = 16$

$$\therefore \frac{X[z]}{z} = \frac{2}{z-1} + \frac{-2z+16}{(z^2-6z+25)}$$

$$X[z] = \frac{2z}{z-1} + \frac{z(-2z+16)}{(z^2-6z+25)}$$

use $r|\gamma|^n \cos(\beta n+\theta)\,u[n] \Leftrightarrow \frac{z(Az+B)}{z^2+2az+|\gamma|^2}$

$\left\{ \begin{array}{l} A=-2 \\ B=16 \\ |\gamma|=5 \\ a=-3 \\ \Downarrow \\ r=3.2 \\ \beta=.927\text{rad} \\ \theta=-2.246\text{rad} \end{array} \right.$

$$\boxed{\therefore x[n] = [2 + 3.2(5)^n \cos(.927n - 2.246)]\,u[n]}$$

inverse z transform

Divide X(z) by **z** first:

$$\mathbf{X}'(\mathbf{z}) = \frac{\mathbf{X}(\mathbf{z})}{\mathbf{z}}$$

Compute partial fraction expansion:

$$\mathbf{X}'(\mathbf{z}) = \sum_{i=1}^{N} \frac{\mathbf{A}_i}{\mathbf{z} - \mathbf{p}_i}$$

Multiply through by **z**:

$$\mathbf{X}(\mathbf{z}) = \sum_{i=1}^{N} \frac{\mathbf{A}_i \mathbf{z}}{\mathbf{z} - \mathbf{p}_i}$$

Use the z-transform pair:

$$\mathbf{A}\mathbf{p}^n \, u[n] \longleftrightarrow \frac{\mathbf{A}\mathbf{z}}{\mathbf{z} - \mathbf{p}}$$

To get the inverse z-transform:

$$x[n] = A_0 \, \delta[n] + \sum_{i=1}^{N} \mathbf{A}_i \mathbf{p}_i^n$$

ZT & difference equations solving

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ex $y[n+2] - 5y[n+1] + 6y[n] = 3x[n+1] + 5x[n]$

given i/p $x[n] = 2^{-n} u[n]$

IC $y[-1] = \frac{11}{6}$, $y[-2] = \frac{37}{36}$

re-arrange difference eq^n to be in backward form

$$y[n] - 5y[n-1] + 6y[n-2] = 3x[n-1] + 5x[n-2]$$

use ZT

$$y[n]u[n] \Leftrightarrow Y(z)$$

$$y[n-1]u[n] \Leftrightarrow \frac{1}{z} Y[z] + y(-1)$$

$$= \frac{1}{z} Y[z] + \frac{11}{6}$$

$$y[n-2]u[n] \Leftrightarrow \frac{1}{z^2} Y[z] + \frac{1}{z} y[-1] + y[-2]$$

$$= \frac{1}{z^2} Y[z] + \frac{11}{6z} + \frac{37}{36}$$

$$x[n]u[n] = 0.5^n u[n] \Leftrightarrow X[z] = \frac{z}{z-0.5}$$

$$x[n-1]u[n] \Leftrightarrow \frac{1}{z} X[z] + x[-1]$$

$$= \frac{1}{z} \frac{z}{z-0.5} + 0$$

$$= \frac{1}{z-0.5}$$

$$x[n-2]u[n] \Leftrightarrow \frac{1}{z^2}X[z] + \frac{1}{z}x[-1] + x[-2]$$

$$= \frac{1}{z^2}X[z] + 0 + 0 = \frac{1}{z(z-0.5)}$$

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lets apply to difference eq^{n}

$$Y[z] - 5\left(\frac{1}{z}Y[z] + \frac{11}{6}\right) + 6\left(\frac{1}{z^2}Y[z] + \frac{11}{6z} + \frac{37}{36}\right) = \frac{3}{z-0.5} + \frac{5}{z(z-0.5)}$$

$$\left(1 - \frac{5}{z} + \frac{6}{z^2}\right)Y[z] - \left(3 - \frac{11}{z}\right) = \frac{3}{z-0.5} + \frac{5}{z(z-0.5)}$$

$$\left(1 - \frac{5}{z} + \frac{6}{z^2}\right)Y[z] = \frac{3z+5}{z(z-0.5)} + \left(3 - \frac{11}{z}\right)$$

$$\left(\frac{z^2-5z+6}{z^2}\right)Y(z) = \frac{3z^2-9.5z+10.5}{z(z-0.5)}$$

$$\therefore Y(z) = \cancel{\frac{z(3z^2-9.5z+10.5)}{z(z-0.5)}} = \frac{z(3z^2-9.5z+10.5)}{(z-0.5)(z-2)(z-3)}$$

$$= \underbrace{\frac{26}{15}\left(\frac{z}{z-0.5}\right) - \frac{7}{3}\left(\frac{z}{z-2}\right) + \frac{18}{5}\left(\frac{z}{z-3}\right)}_{\text{from PFE}}$$

$$\therefore y[n] = \left(\frac{26}{15}(0.5)^n - \frac{7}{3}(2)^n + \frac{18}{5}(3)^n\right)u[n]$$

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- we can separate into separate terms

$$(z^2-5z+6)Y(z) = \underbrace{\frac{z(3z+5)}{(z-0.5)}}_{\text{i/p terms}} + \underbrace{z(3z-11)}_{\text{IC}}$$

$$\therefore Y(z) = \underbrace{\frac{z(3z+5)}{(z-0.5)(z^2-5z+6)}}_{\text{zero-state}} + \underbrace{\frac{z(3z-11)}{(z^2-5z+6)}}_{\text{zero input}}$$

$$\therefore Y(z) = \underbrace{\frac{26}{15}\left(\frac{z}{z-0.5}\right) - \frac{22}{3}\left(\frac{z}{z-2}\right) + \frac{28}{5}\left(\frac{z}{z-3}\right)}_{\text{zero-state response}}$$

$$+ \underbrace{5\left(\frac{z}{z-2}\right) - 2\left(\frac{z}{z-3}\right)}_{\text{zero-input response}}$$

$$\therefore y[n] = \Big\{ \underbrace{\frac{26}{15}(0.5)^n - \frac{22}{3}(2)^n + \frac{28}{5}(3)^n}_{\text{zero state response}} + \underbrace{5(2)^n - 2(3)^n}_{\text{zero-input response}} \Big\} u[n]$$

- recall natural response is all the characteristic modes terms in ZI sol^n in the response; remaining terms are forced response

$$\therefore y[n] = \underbrace{\left\{ -\frac{7}{3}(2)^n + \frac{18}{5}(3)^n \right\} u[n]}_{\text{natural response}} + \underbrace{\frac{26}{15}(0.5)^n u[n]}_{\text{forced response}}$$

example:

e.g., ① $x[n] = \left(\frac{1}{2}\right)^{n+1} u[n]$

$y[n] = \delta[n] - \delta[n-1]$

compute step response?

Sol^{n}

$$H(z) = \frac{Y(z)}{X(z)} = (1 - z^{-1}) \Big/ \left(\frac{1}{2} \frac{z}{z-(\frac{1}{2})}\right)$$

$$= \underline{(1-z^{-1})} \cdot (2 - z^{-1})$$

$$= 2 - 3z^{-1} + z^{-2} \Rightarrow \&$$

$$\Downarrow$$

$$h[n] = 2\delta[n] - 3\delta[n-1] + \delta[n-2]$$

OR $[\underset{\uparrow}{2}, -3, 1]$

② for above, what is step response

$$Y(z) = H(z)X(z) = (2 - 3z^{-1} + z^{-2}) \frac{z}{z-1}$$

$$= \frac{(2z-1)(z-1)}{z^2} \frac{z}{z-1} = \frac{2z-1}{z}$$

$$= 2 - z^{-1}$$

$$\Downarrow$$

$y[n] = \{2, -1\}$ OR $= 2\delta[n] - \delta[n-1]$

NOTE

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example:

NOTE

$$x[n] = e^{j\omega n} \rightarrow \boxed{\text{LTI}} \rightarrow y[n] = h[n] * e^{j\omega n}$$

$$= \sum_{i=-\infty}^{\infty} h[i]\, e^{j\omega(n-i)}$$

$$= \sum_{i=-\infty}^{\infty} h[i]\, e^{-j\omega i}\, e^{j\omega n}$$

$$= H(e^{j\omega})\, e^{j\omega n}$$

example:

e.g., 3) using previous example

$$H(z) = 2 - 3z^{-1} + z^{-2}$$

compute response of system to $\cos\left(\left(\frac{\pi}{2}\right)n\right)$

$$\therefore H(e^{j\frac{\pi}{2}}) = H(j) = 2 - 3\frac{1}{j} + 1\frac{1}{j^2} = 1 + 3j$$

$$= \sqrt{10}\, e^{j1.25}$$

so response is

$$x[n] = \cos\left(\frac{\pi}{2}n\right) \Rightarrow \boxed{\text{LTI}} \rightarrow y[n]$$

$$y[n] = \sqrt{10}\cos\left(\frac{\pi}{2}n + 1.25\right)$$


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