Part 2. Roots of Equations
Chapter 6. Open Methods
Chapter 8. Case Studies

Lecture 7

The Secant Method &
Numerical Root Finding: An Engineering Application

6.3, 8.4

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The Secant Method

- A slight variation of Newton-Raphson's method for functions whose derivatives are difficult to evaluate.

- For these cases the derivative can be approximated by a backward finite divided difference.

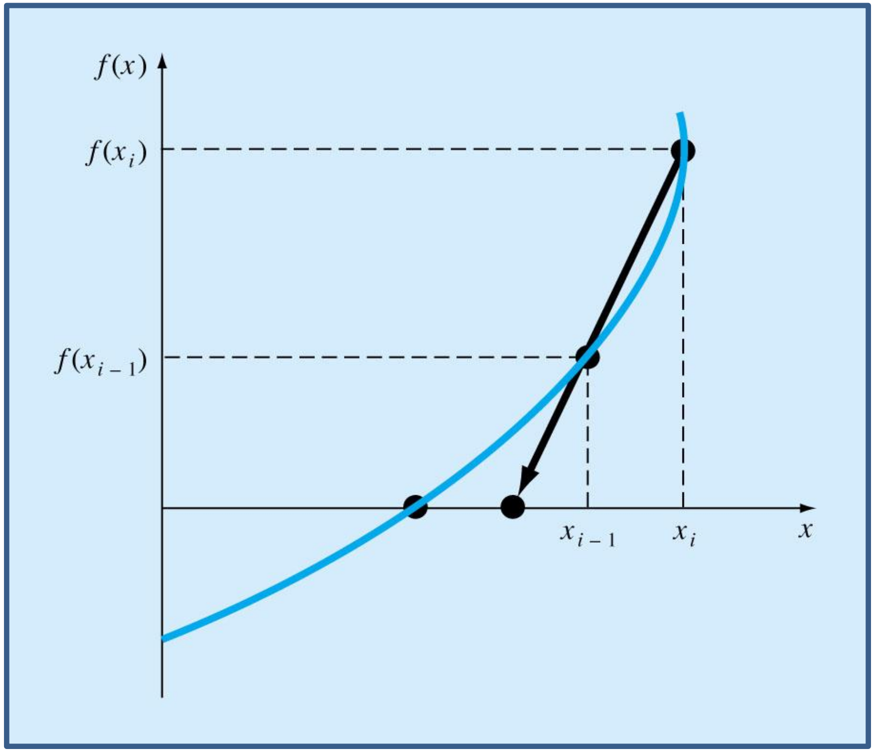
$$f'(x_i) \cong \frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})}$$

$$x_{i+1} = x_i - f(x_i)\frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})} \qquad i = 1,2,3,\ldots$$

The Secant Method

- Requires two initial estimates of x.

- Because f(x) is not required to change signs between estimates, it is not classified as a "bracketing" method.



- The scant method has the same properties as Newton's method. Convergence is not guaranteed for all x_o, f(x).

Example 1. Secant Method

Find roots of $f(x) = e^{-x} - x$ using the secant method and calculate the approximate relative error in each iteration for 3 iterations

Secant Method Formula

$$x_{i+1} = x_i - f(x_i)\frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})}$$

Requires previous (x_{i-1}) & current (x_i) value to estimate the next value (x_{i+1}); for iteration 1, needs: $\boldsymbol{x_0}$ (previous) , $\boldsymbol{x_1}$ (current)

Then, 2 initial guesses needed to start the computation
We choose: $x_0 = 0$, $x_1 = 1$

Check : $f(x_0) \neq f(x_1)$

Example 1. Secant Method

$$f(x) = e^{-x} - x$$

$$x_{i+1} = x_i - f(x_i)\frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})}$$

2 initial guesses:
$x_0 = 0$, $x_1 = 1$

Iteration 1 → i = 1

$f(x_0) = f(0) = e^{-0} - 0 = 1$

$f(x_1) = f(1) = e^{-1} - 1 = 0.36788 - 1 = -0.63212$

$x_2 = x_1 - f(x_1)\,[\,(x_1 - x_0) / (f(x_1) - f(x_0))\,]$

$x_2 = 1 - f(1)\,[\,(1 - 0) / (f(1) - f(0))\,]$

$x_2 = 1 - (-0.63212)\,[\,1 / (-0.63212 - 1)\,]$

$x_2 = 0.6127$

Example 1. Secant Method

$$f(x) = e^{-x} - x$$

$$x_{i+1} = x_i - f(x_i)\frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})}$$

From previous step:
$x_1 = 1$, $x_2 = 0.6127$

Iteration 2 → i = 2

$$f(x_1) = f(1) = e^{-1} - 1 = -0.63212$$

$$f(x_2) = f(0.6127) = e^{-0.6127} - (0.6127) = -0.7081$$

$$x_3 = x_2 - f(x_2)[(x_2 - x_1)/(f(x_2) - f(x_1))]$$

$$x_3 = 0.6127 - f(0.6127)[(0.6127 - 1)/(f(0.6127) - f(1))]$$

$$x_3 = 0.6127 + (-0.63212)[1/(-0.63212 - 1)]$$

$$x_3 = 0.56382$$

Example 1. Secant Method

Relative Percent Approximate Error: $E_a^{\,i} = |\,(x_{i+1} - x_I\,)\,/\,x_{i+1}\,| \times 100$

$E_a^{\,1} = |\,(x_2 - x_1\,)\,/\,x_2\,| \times 100 = |\,(0.6127 - 1)\;/\;0.6127\,| \times 100 = \;8.0\ \%$

$E_a^{\,2} = |\,(x_3 - x_2\,)\,/\,x_3\,| \times 100 = |\,(0.56382 - 0.6127)\;/\;0.56382\,] \times 100 = 0.58\ \%$

	Previous value	Current value			Next value	
i	x_{i-1}	x_i	$f(x_{i-1})$	$f(x_i)$	x_{i+1}	$E_a^{\,i}$
1	**0**	**1**	1.0	- 0.63212	0.6127	8.0%
2	1	0.6127	- 0.63212	- 0.7081	0.56382	0.58 %

0.56717 0.0048 %

Pros and Cons of The Secant Method

Converges fast, if it converges

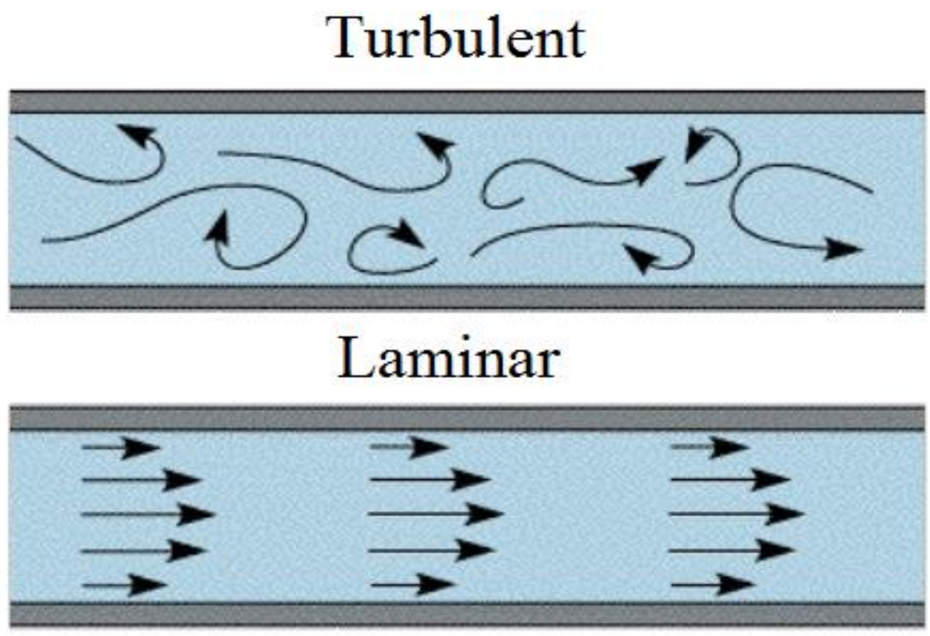
Requires 2 guesses that do not need to bracket the root

Division by zero

Root jumping

A Mech. Eng. Case Study Application: Fluid Flow In a Pipe

Comparing Different Numerical Methods



Application Example. Fluid Flow In a Pipe. Colebrook equation calculates the friction factor (f) for turbulent (high speed) flow:

$$g(f) = \frac{1}{\sqrt{f}} + 2.0 \log\left(\frac{\varepsilon}{3.7\,D} + \frac{2.51}{Re\,\sqrt{f}}\right)$$

Reynolds number:	$Re = \rho \mathrm{VD} / \mu$
Fluid's density:	$\rho = 1.23 \quad kg/m^3$
Dynamic viscosity:	$\mu = 1.79 \times 10^{-5}\, N \cdot s/m^2$
Diamater:	$D = 0.005 \quad m$
Velocity:	$V = 40 \quad m/s$
Roughness:	$\varepsilon = 0.0015 \quad mm$

Explore how different numerical methods are employed to determine ***f*** (friction factor) for air flow through a smooth, thin tube.
(Note that friction is between 0.008 – 0.08)

Solving for f Using Bisection Method

$$g(f) = \frac{1}{\sqrt{f}} + 2.0 \log\left(\frac{0.0015 \times 10^{-3}}{3.7\ (0.005)} + \frac{2.51}{13743\ \sqrt{f}}\right)$$

If + : $x_u = x_r$

Check sign of $f(x_l).\ f(x_r)$

If - : $x_l = x_r$

i Iteration #	x_l Lower boundary	x_u Upper boundary	x_r Interval Midpoint	$f(x_l)$	$f(x_u)$	$f(x_r)$	$E_a^{\ i}$
0	0.008	0.08	0.044	5.834	- 2.741	- 1.275	---
1	0.008	0.044	0.026	5.834	- 1.275	0.37	69.23
20	0.028968	0.028968	0.028968	7.86E - 06	- 5.8E - 08	3.9E-06	0.000119
21	0.028968	0.028968	0.028968	3.9E - 06	- 5.8E - 08	1.92E-06	5.93E - 05

Solving for f Using Newton-Raphson Method

$$g(f) = \frac{1}{\sqrt{f}} + 2.0 \log\left(\frac{0.0015 \times 10^{-3}}{3.7\ (0.005)} + \frac{2.51}{13743\sqrt{f}}\right)$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$g := \frac{1}{\mathrm{sqrt}(f)} + 2 \cdot \log 10\left(\frac{0.0015\mathrm{e}-3}{(3.7 \cdot .005)} + \frac{2.51}{(13743 \cdot \mathrm{sqrt}(f))}\right);$$

$$g := \frac{1}{\sqrt{f}} + \frac{2\ln\left(0.00008108108108 + \frac{0.0001826384341}{\sqrt{f}}\right)}{\ln(10)}$$

$$\mathit{diff}(g, f);$$

$$-\frac{1}{2 f^{3/2}} - \frac{0.0001826384341}{f^{3/2}\left(0.00008108108108 + \frac{0.0001826384341}{\sqrt{f}}\right)\ln(10)}$$

$$\mathit{simplify}\left(\frac{g}{\mathit{diff}(g, f)}\right);$$

$$-\frac{0.00008108108108\left(2.\ln\left(\frac{0.00008108108108\left(\sqrt{f} + 2.252540687\right)}{\sqrt{f}}\right)\sqrt{f} + \ln(2) + \ln(5)\right) f\left(\sqrt{f} + 2.252540687\right)}{0.0002759864784\sqrt{f} + 0.0002102702678}$$

Solving for *f* Using Newton-Raphson Method

iteration	f	g/g'	f_new	epsilon_a
1	0.008	-0.00777	0.015769	49.26693
2	0.015769	-0.00838	0.024154	34.71444
3	0.024154	-0.00422	0.02837	14.86287
4	0.02837	-0.00059	0.028959	2.032748
5	0.028959	-8.9E-06	0.028968	0.030841
6	0.028968	-2E-09	0.028968	6.87E-06

iteration	f	g/g'	f_new	epsilon_a
1	0.05	0.030812	0.019188	160.5862
2	0.019188	-0.00722	0.026403	27.32794
3	0.026403	-0.0024	0.028801	8.326348
4	0.028801	-0.00017	0.028967	0.573727
5	0.028967	-6.9E-07	0.028968	0.002399
6	0.028968	-1.2E-11	0.028968	4.15E-08

iteration	f	g/g'	f_new	epsilon_a
1	0.07	0.075141	-0.00514	1461.581
2	-5.14E-03	#NUM!	#NUM!	#NUM!
3	#NUM!	#NUM!	#NUM!	#NUM!
4	#NUM!	#NUM!	#NUM!	#NUM!
5	#NUM!	#NUM!	#NUM!	#NUM!
6	#NUM!	#NUM!	#NUM!	#NUM!

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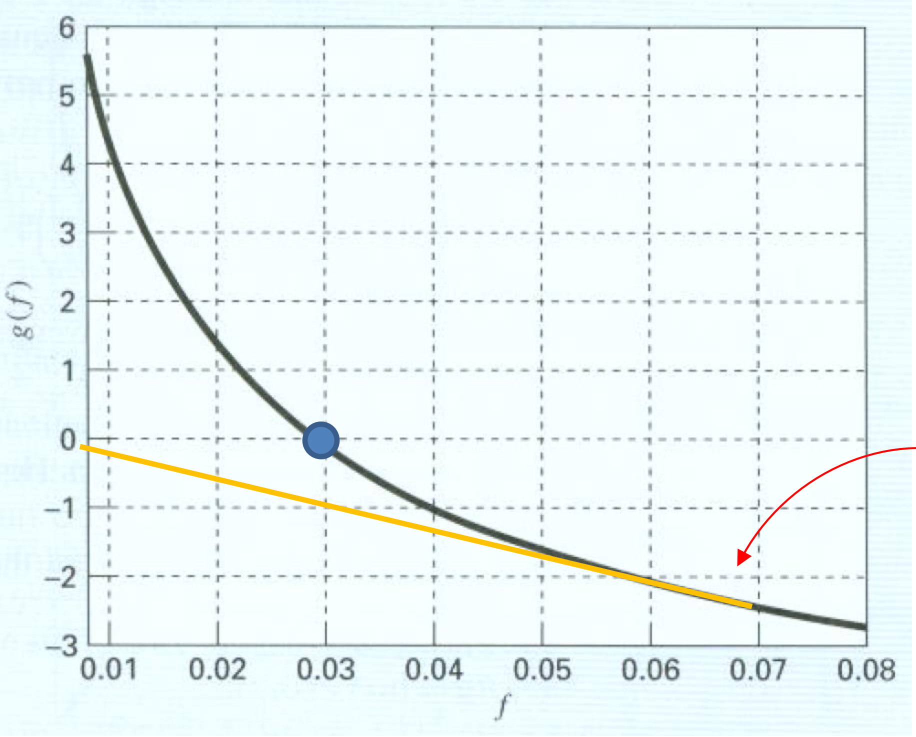


Plot of Colebrook function in MATLAB

For $f_1 < 0.066$ solution doesn't work

Solving for f Using Simple Fixed Point Iteration Method

$$g(f) = \frac{1}{\sqrt{f}} + 2.0 \log\left(\frac{0.0015 \times 10^{-3}}{3.7\,(0.005)} + \frac{2.51}{13743\,\sqrt{f}}\right)$$

$$f(x) = 0 \quad \Rightarrow \quad g(x) = x$$

$$x_k = g(x_{k-1}) \qquad x_o \text{ given}, k = 1, 2, \ldots$$

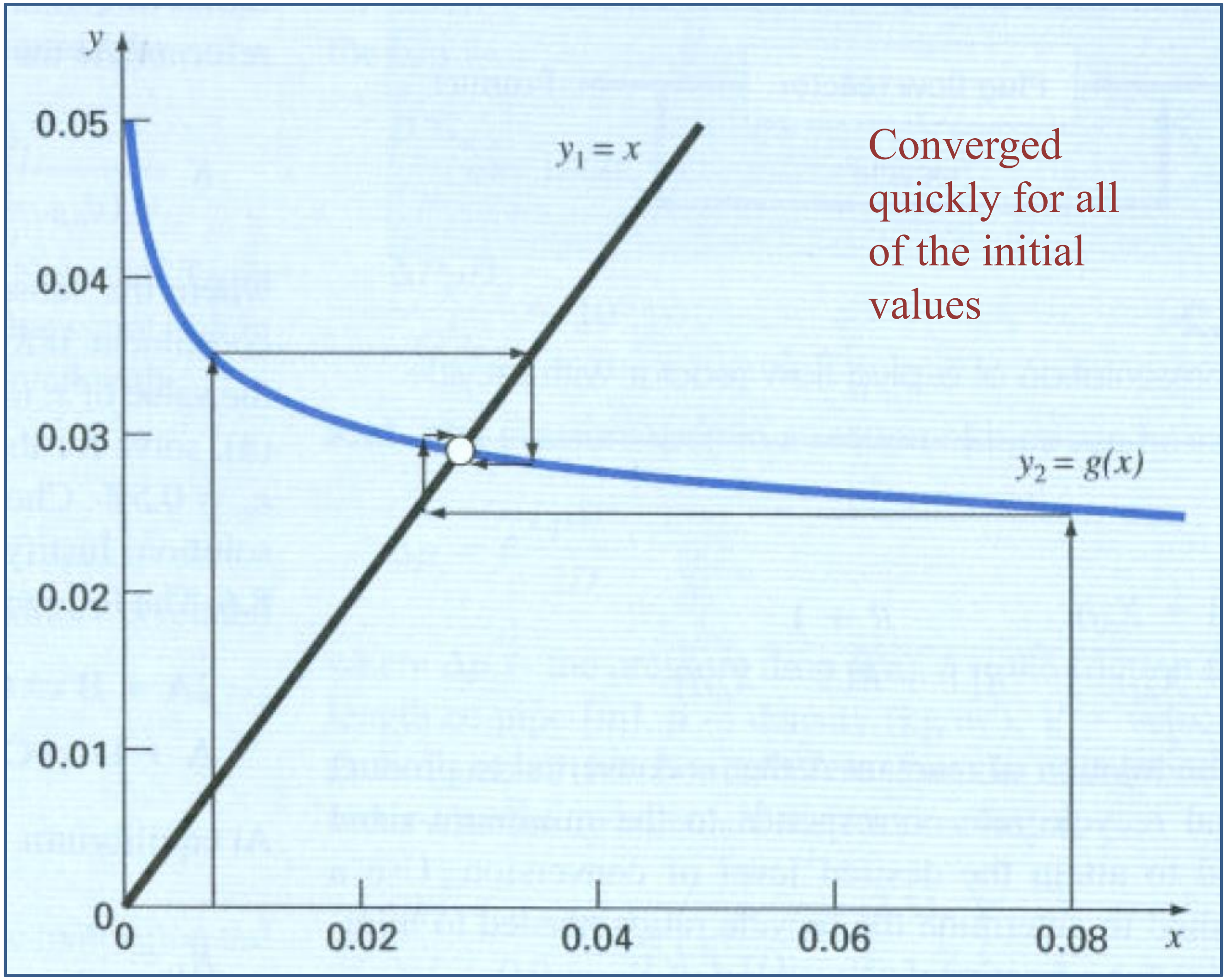
Intersection of 2 graph:

$y_1 = x$

$y_2 = g(x)$

$$f_{i+1} = \frac{0.25}{\left[\log\left(\frac{0.0015 \times 10^{-3}}{3.7\,(0.005)} + \frac{2.51}{13743\,\sqrt{f_i}}\right)\right]^2}$$

Solving for *f* Using Simple Fixed Point Iteration Method



Converged quickly for all of the initial values