Part 7. Ordinary Differential Equations
Chapter 27. Boundary-Value & Eigenvalue Problems

Lecture 26

General Methods for Boundary Value Problems: Shooting Method

27.1

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Learning Outcomes

- Understand the difference between initial-value problems (IVPs) and boundary-value problems (BVPs).

- Learn and apply shooting method to solve BVPs

Introduction: ODE Conditions

An ODE is accompanied by supplementary conditions.

Conditions are used to evaluate the integral that result during the solution of the equation.

An n^{th} order equation requires n conditions.

Initial-Value (IVP) Problems vs Boundary-Value (BVP) Problems

IVP

- When all conditions are specified at the same value of the independent variable.

BVP

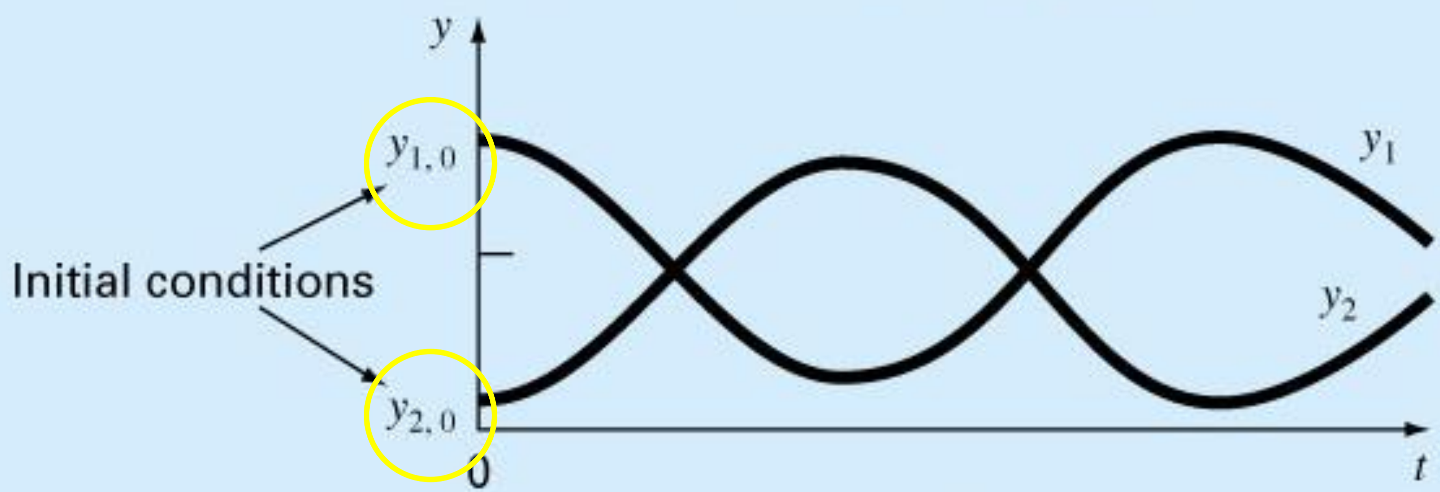
- When the conditions are specified at different values of the independent variable, usually at extreme points or boundaries of a system.

Initial-Value (IVP) Problems

$$\frac{dy_1}{dt} = f_1(t, y_1, y_2)$$

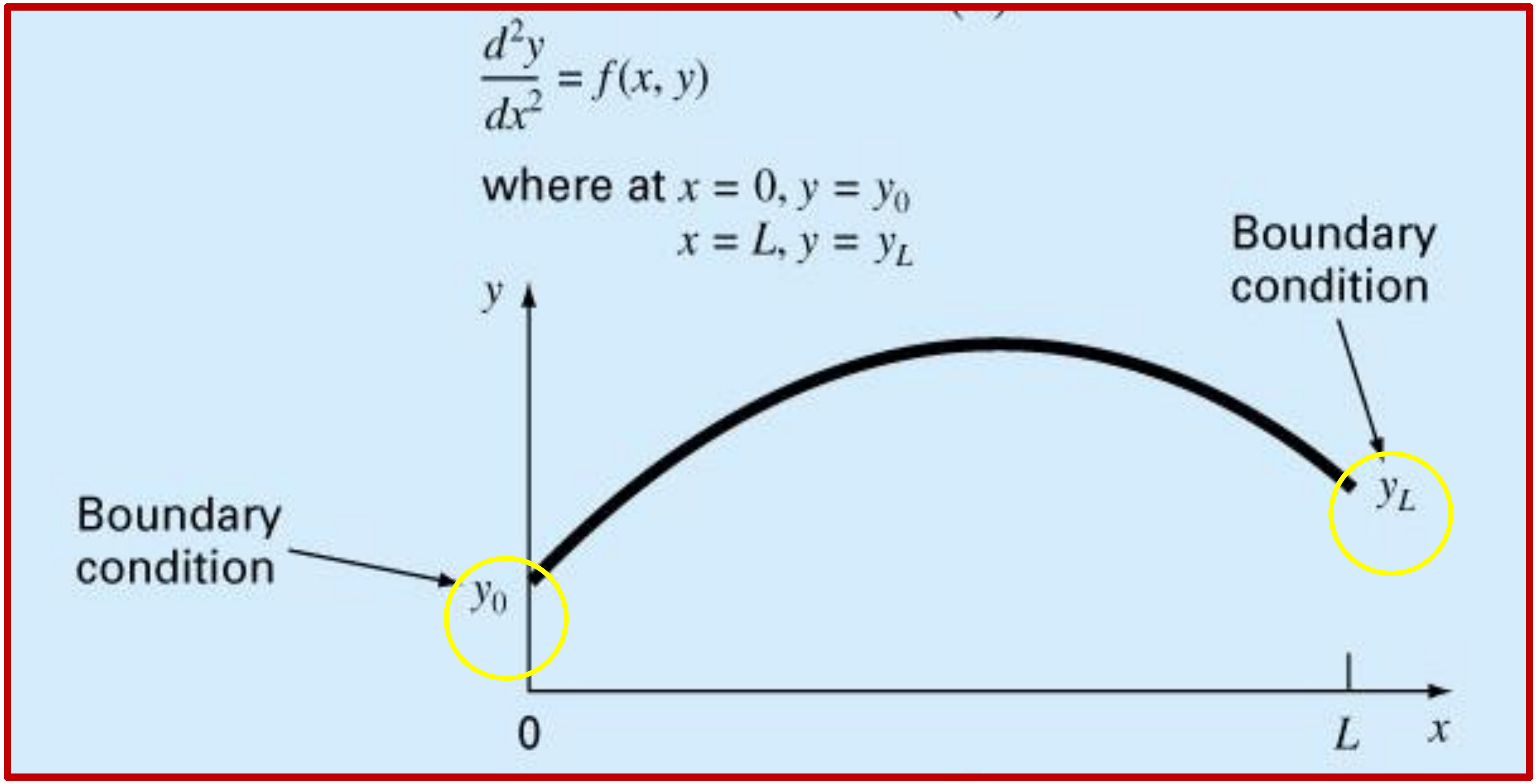
$$\frac{dy_2}{dt} = f_2(t, y_1, y_2)$$

where at $t = 0$, $y_1 = y_{1,0}$ and $y_2 = y_{2,0}$



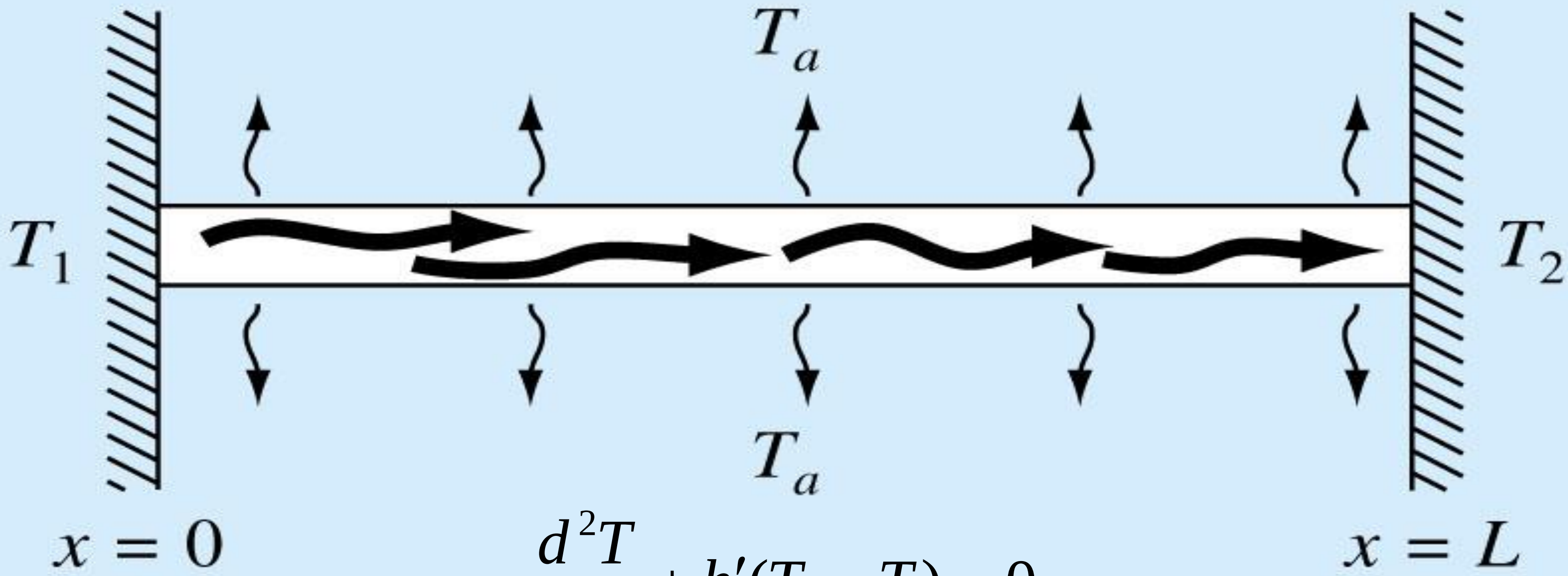
Conditions for y_1 and y_2 are specified at the same value of the independent variable ($t = 0$).

Boundary-Value (IVP) Problems



Conditions of y_0 and y_L are specified at different values of the independent variable, x (at x= 0 and x = L), at extreme points.

Example of a Boundary-Value Problem



$$\frac{d^2T}{dx^2} + h'(T_a - T) = 0$$

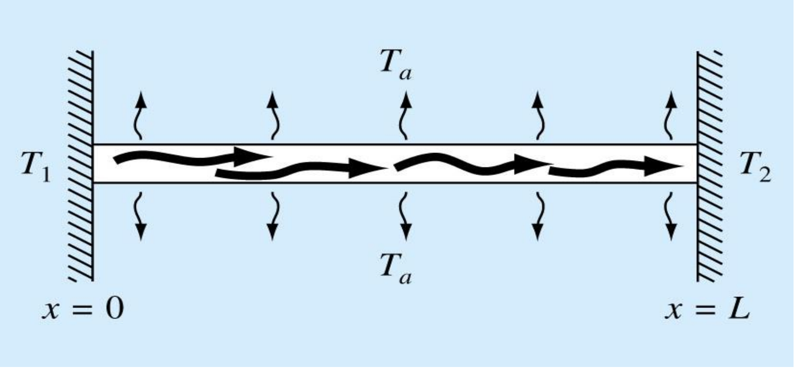
Noninsulated uniform rod positioned between two bodies of constant but different temperature

Example of a Boundary-Value Problem

h' = Heat transfer coefficient

T_a = Air temperature

$T_1 > T_2$, $T_2 > T_a$



$$\boxed{\begin{aligned} T(0) &= T_1 = 40 \\ T(L) &= T_2 = 200 \end{aligned}}$$

Boundary Conditions

$$\frac{d^2T}{dx^2} + h'(T_a - T) = 0$$

$$T_a = 20$$

$$L = 10\,m$$

$$h' = 0.01 m^{-2}$$

Analytical Solution:

$$T = 73.4523\ e^{0.1x} - 53.4523\ e^{-0.1x} + 20$$

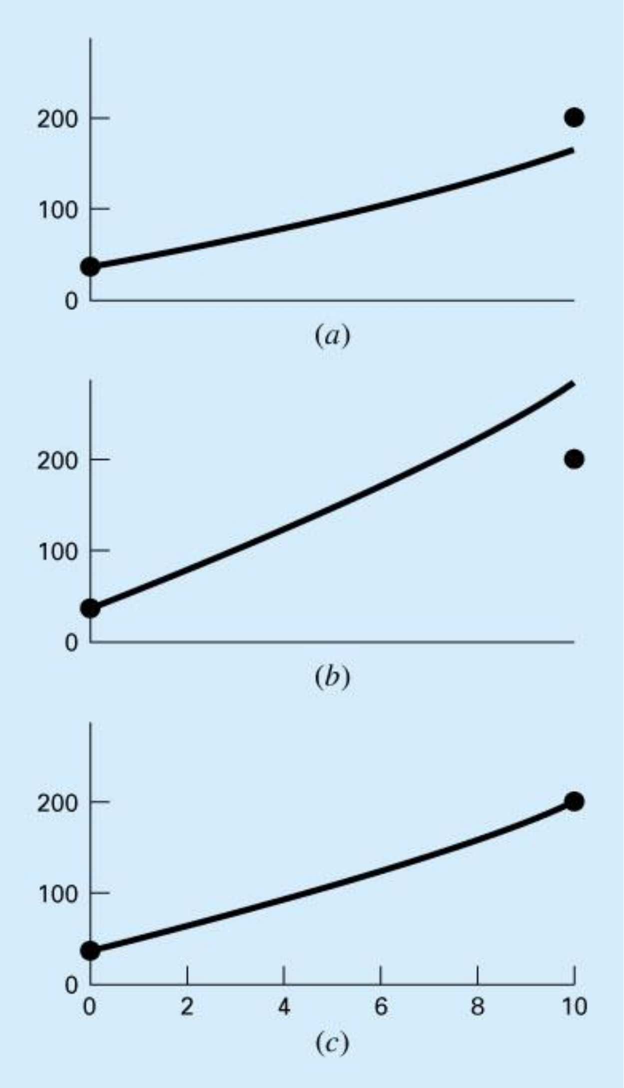
The Shooting Method

- Converts BVP to IVP.
- Trial-and-error approach used for solving IVP.
- For example, the 2nd-order equation can be expressed as two 1st-order ODEs:

$$\frac{dT}{dx} = z$$

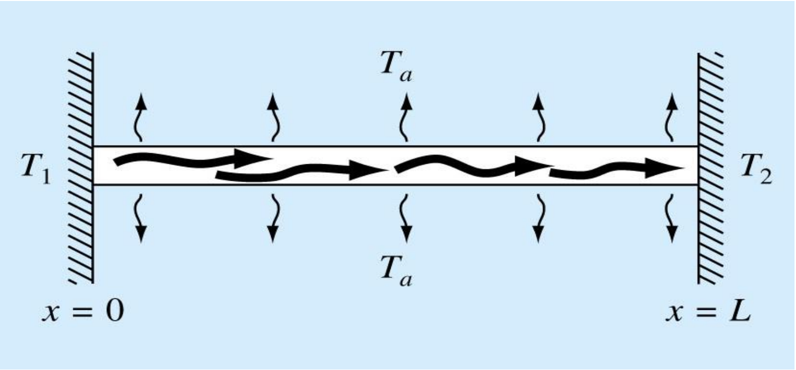
$$\frac{dz}{dx} = h'(T - T_a)$$

- An initial value is guessed, say z(0)=10.
- The solution is then obtained by integrating the two 1st order ODEs simultaneously.



The Shooting Method

- Using a 4th order RK method with a step size of *2*:



1) z(0)=10 → T(10)=168.3797

- This differs from T(10)=200 → Make a new guess : z(0)=20
  Perform computation again.

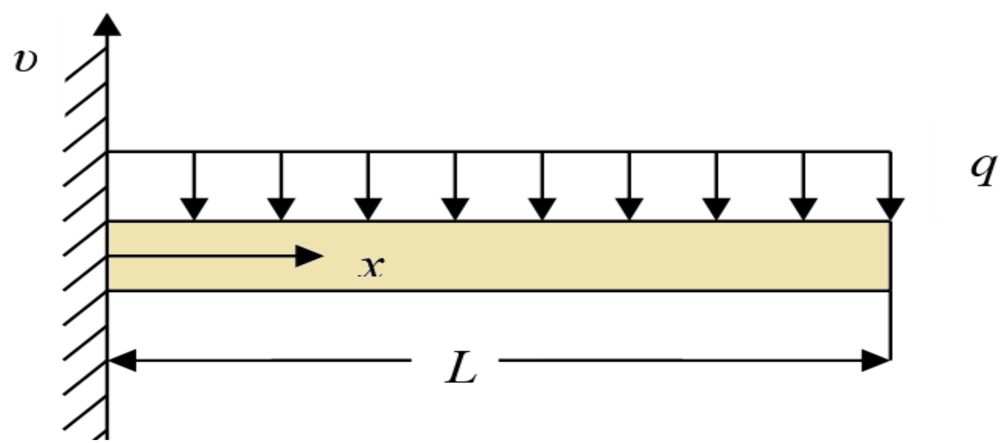
2) z(0) = 20 → T(10)=285.8980

- Two points, $(z, T)_1$ and $(z, T)_2$, are linearly related→
  Use linear interpolation to compute the value of z(0) as 12.6907 to determine the correct solution.

Example 1. In order to calculate deflection along the length of a cantilevered uniformly loaded beam, the following 2nd order ODE needs to be solved:

$$\frac{d^2 v}{dx^2} = \frac{q}{2EI}(L-x)^2$$

where **x** is the location from the fixed wall, **q** is uniform load, **L** is total length of beam, **E** is Young's modulus of the beam, **I** is the second moment of area of the cross-section of the beam. Specify if this is IVP or BVP? How many initial conditions are required? What would be the procedure if you choose shooting method for numerical solution of ODE?

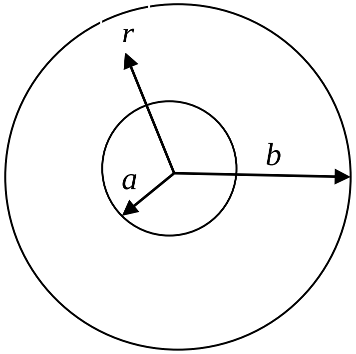


Example 2. Pressure Vessel Radial Displacement

For a thick pressure vessel with dimensions shown in figure, the radial displacement in u of a point along the thickness is given by the 2nd order ODE as shown. Assume a=5 inch and b=8 inch, and the material of the pressure vessel is ASTM36 steel, with yield strength of 36 ksi (kilo pound per square inch). Two strain gages that are bonded tangentially at the inner and the outer radius measure the normal tangential strain in the pressure vessel as

$$\epsilon_{t/r=a} = 0.00077462$$
$$\epsilon_{t/r=b} = 0.00038462$$

at maximum needed pressure. Setup the procedure of numerical analysis for this ODE, using the shooting method.



$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0$$

Notes

- Solution requires two initial guesses
- Can use accurate methods like RK for 1^{st} order ODE
- Solution requires iterations