MTE 203 – Advanced Calculus
Homework 5 (Solution)

Drawing and setting up multivariable functions

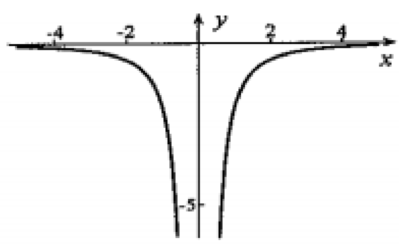
Problem 1: [12.1, Prob. 5]

Find and illustrate geometrically the largest possible domain for the function:

$$f\left(x, y\right) = \sin^{-1}\left(x^2 y + 1\right)$$

Solution:

For $-1 \le x^2 y + 1 \le 1$, we require $-2 \le x^2 y \le 0$ or $-2/x^2 \le y \le 0$. Points are below the x-axis and above the curve $y = -2/x^2$. Points on the boundary are also included.

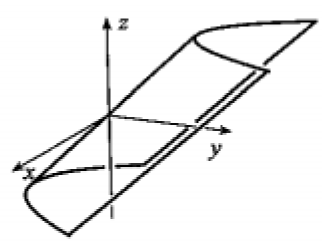


Problem 2: [12.1, Prob.17, 19, 21]

Draw the surface defined by the following functions:

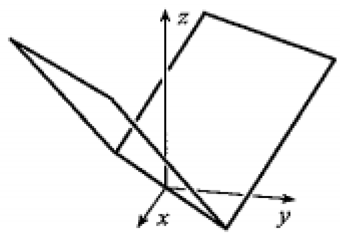
a. $f\left(x, y\right) = y - x^2$

Solution:



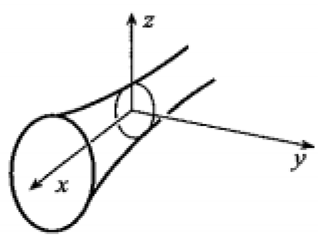
b. $f(x,y) = |x - y|$

Solution:



c. $f(x,y) = \sqrt{1 + x^2 - y^2}$

Solution:



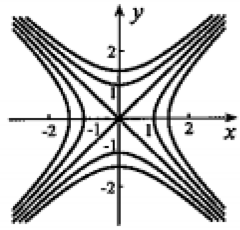
Problem 3: [12.1, Prob. 25]

Draw the level curves $f(x,y) = C$ corresponding to the values $C = -2, -1, 0, 1, 2$ for the curve below:

Solution:

$$f(x,y) = x^2 - y^2$$

Level curves are defined by $x^2 - y^2 = C$. They are hyperbolas, except when $C = 0$ when they are the lines $y = \pm x$.



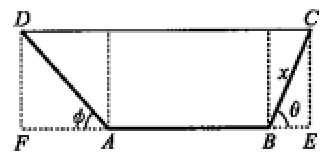
Problem 4: [S.12.1, Prob. 31] – Application Problem

A long piece of metal 1 m wide is bent in two places A and B (figure below) to form a channel with three straight sides. Find a formula for the cross-sectional area of the channel in terms of x, θ, and φ.

Solution:

Since

$$\begin{aligned} \|DF\| &= \|CE\| = x\sin\theta, \\ \|AF\| &= \|DF\|\cot\phi = x\sin\theta\cot\phi, \\ \|AB\| &= 1 - \|BC\| - \|AD\| \\ &= 1 - x - x\sin\theta\csc\phi, \end{aligned}$$



the cross-sectional area is

$$\begin{aligned} \text{Area} &= \|AB\|\|CE\| + \frac{1}{2}\|BE\|\|CE\| + \frac{1}{2}\|FA\|\|DF\| \\ &= \|CE\|\left(\|AB\| + \frac{1}{2}\|BE\| + \frac{1}{2}\|FA\|\right) \\ &= x\sin\theta\left[(1 - x - x\sin\theta\csc\phi) + \frac{1}{2}(x\cos\theta) + \frac{1}{2}(x\sin\theta\cot\phi)\right]. \end{aligned}$$

Partial Derivatives

Problem 5: [S.12.3, Probs. 21, 23]

Evaluate the partial derivatives as indicated

1. $\frac{\partial f}{\partial x}$ $if\ f(x,y,z) = xyze^{x^2+y^2}$
2. $\frac{\partial f}{\partial y}$ $at\ (1,1,0)\ if\ f(x,y,z) = xy(x^2+y^2+z^2)^{\frac{1}{3}}$

Solution:

21. $$\frac{\partial f}{\partial x} = yze^{x^2+y^2} + xyze^{x^2+y^2}(2x) = yz(1+2x^2)e^{x^2+y^2}$$

23. Since $\dfrac{\partial f}{\partial y} = x(x^2+y^2+z^2)^{1/3} + (xy/3)(x^2+y^2+z^2)^{-2/3}(2y)$, the partial derivative at $(1,1,0)$ is $2^{1/3} + (1/3)2^{-2/3}(2) = 2^{7/3}/3$.

Problem 6: [12.3, Prob. 39] – Application Problem

The equation of continuity for three-dimensional unsteady flow of a compressible fluid is

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

Where $\rho(x, y, z, t)$ is the density of the fluid, and

$$u\hat{\imath} + v\hat{\jmath} + w\hat{k}$$

is the velocity of the fluid at position $\left(x, y, z\right)$ and time t. Determine whether the continuity equation is satisfied if,

a. $\rho = constant, u = (2x^2 - xy + z^2)t, v = (x^2 - 4xy + y^2)t, w = (-2xy - yz + y^2)t$
b. $\rho = xy + zt, u = x^2y + t, v = y^2z - 2t^2, w = 5x + 2z$

Solution:

(a) $\dfrac{\partial \rho}{\partial t} + \dfrac{\partial}{\partial x}(\rho u) + \dfrac{\partial}{\partial y}(\rho v) + \dfrac{\partial}{\partial z}(\rho w) = 0 + \rho(4xt - yt) + \rho(-4xt + 2yt) + \rho(-yt) = 0$

(b) $$\begin{aligned}\frac{\partial \rho}{\partial t} &+ \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = z + \frac{\partial}{\partial x}(x^3y^2 + xyt + x^2yzt + zt^2) \\ &+ \frac{\partial}{\partial y}(xy^3z - 2xyt^2 + y^2z^2t - 2zt^3) + \frac{\partial}{\partial z}(5x^2y + 2xyz + 5xzt + 2z^2t) \\ &= z + (3x^2y^2 + yt + 2xyzt) + (3xy^2z - 2xt^2 + 2yz^2t) + (2xy + 5xt + 4zt) \neq 0\end{aligned}$$

<u>**Higher Order Partial Derivatives**</u>

Problem 7: [12.5, Prob. 21]

If $z = x^2 + xy + y^2 \sin\left(\frac{x}{y}\right)$, show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z = x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\,\partial y} + y^2\frac{\partial^2 z}{\partial y^2}$$

Solution:

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=x\left[2x+y+y\cos\left(\frac{x}{y}\right)\right]+y\left[x+2y\sin\left(\frac{x}{y}\right)-x\cos\left(\frac{x}{y}\right)\right]$$
$$=2\left[x^2+xy+y^2\sin\left(\frac{x}{y}\right)\right]=2f(x,y)$$

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=x^2\left[2-\sin\left(\frac{x}{y}\right)\right]+2xy\left[1+\cos\left(\frac{x}{y}\right)+\frac{x}{y}\sin\left(\frac{x}{y}\right)\right]$$
$$+y^2\left[2\sin\left(\frac{x}{y}\right)-\frac{2x}{y}\cos\left(\frac{x}{y}\right)-\frac{x^2}{y^2}\sin\left(\frac{x}{y}\right)\right]$$
$$=2x^2+2xy+(-x^2+2x^2+2y^2-x^2)\sin\left(\frac{x}{y}\right)+(2xy-2xy)\cos\left(\frac{x}{y}\right)$$
$$=2\left[x^2+xy+y^2\sin\left(\frac{x}{y}\right)\right]=2f(x,y)$$

Problem 8: [12.5, Prob. 27] – Application Problem

A function is said to be a harmonic function in a region R if it satisfies the Laplace's equation in R and has continuous second partial derivatives in R. The Laplace's equation for a function $f\left(x,y,z\right)$ of three variables is

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}=0$$

Find a region (if possible) in which the function,

$$f\left(x,y,z\right)=\frac{1}{\sqrt{x^2+y^2+z^2}}$$

is harmonic.

Solution:

From $\dfrac{\partial f}{\partial x}=\dfrac{-x}{(x^2+y^2+z^2)^{3/2}}$, $\quad \dfrac{\partial^2 f}{\partial x^2}=\dfrac{-1}{(x^2+y^2+z^2)^{3/2}}+\dfrac{3x^2}{(x^2+y^2+z^2)^{5/2}}=\dfrac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$.
With similar results for second derivatives with respect to y and z,

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}=\frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}+\frac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{5/2}}+\frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}=0.$$

Since second partial derivatives are continuous except at $(0,0,0)$, the function is harmonic in any region not containing $(0,0,0)$.

Problem 9: [12.5, Prob. 31] – Challenging Application Problem

The figure below shows a plate bounded by the lines $x = 0$, $y = 0$, $x = 1$, and $y = 1$. Temperature along the first three sides is kept at 0°C, while that along $y = 1$ varies according to

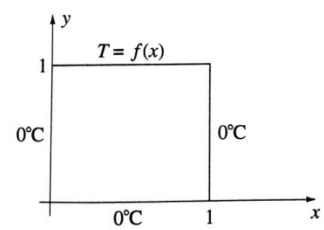
$$f(x) = \sin(3\pi x) - 2\sin(4\pi x),\ 0 \le x \le 1.$$

The temperature at any point interior to the plate is then

$$T(x, y) = C(e^{3\pi y} - e^{-3\pi y})\sin(3\pi x) + D(e^{4\pi y} - e^{-4\pi y})\sin(4\pi x)$$

Where $C = (e^{3\pi} - e^{-3\pi})^{-1}$ and $D = (e^{4\pi} - e^{-4\pi})^{-1}$.

Show that $T(x, y)$ is harmonic in the region $0 < x < 1,\ \ 0 < y < 1$, and that it also satisfies the boundary conditions $T(0, y) = 0$, $T(1, y) = 0$, $T(x, 0) = 0$, and $T(x, 1) = f(x)$.



Solution:

As long as C and D are constants,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = [-9\pi^2 C(e^{3\pi y} - e^{-3\pi y})\sin(3\pi x) - 16\pi^2 D(e^{4\pi y} - e^{-4\pi y})\sin(4\pi x)]$$
$$+ [9\pi^2 C(e^{3\pi y} - e^{-3\pi y})\sin(3\pi x) + 16\pi^2 D(e^{4\pi y} - e^{-4\pi y})\sin(4\pi x)] = 0.$$

Since second partial derivatives are continuous, $T(x, y)$ is harmonic in the plate. It is obvious that $T(x, y)$ satisfies $T(0, y) = T(1, y) = T(x, 0) = 0$. Finally

$$T(x, 1) = C(e^{3\pi} - e^{-3\pi})\sin(3\pi x) + D(e^{4\pi} - e^{-4\pi})\sin(4\pi x)$$
$$= \sin(3\pi x) - 2\sin(4\pi x).$$