MTE 203 – Advanced Calculus
Homework 6 (Solutions)

<u>Chain Rule for Partial Derivatives</u>

Problem 1: [12.6, Prob. 3]

Find the derivative below:

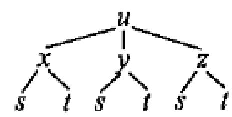
$$\left.\frac{\partial u}{\partial s}\right|_t$$

If, in general, $u = f(x,y,z),\ x = g(s,t),\ y = h(s,t),\ z = k(s,t)$
and, particulary, $u = \sqrt{(x^2+y^2+z^2)}, x = 2st, y = s^2+t^2, z = st$

Solution:

In general, $\dfrac{\partial u}{\partial s} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial s} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial s}$, and specifically,

$$\begin{aligned}\frac{\partial u}{\partial s} &= \frac{x}{\sqrt{x^2+y^2+z^2}}(2t) + \frac{y}{\sqrt{x^2+y^2+z^2}}(2s) + \frac{z}{\sqrt{x^2+y^2+z^2}}(t)\\ &= \frac{2xt+2ys+zt}{\sqrt{x^2+y^2+z^2}}.\end{aligned}$$



Problem 2: [12.6, Prob. 13]

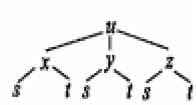
Find the derivative below:

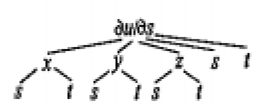
$$\left.\frac{\partial^2 u}{\partial s^2}\right|_t$$

If $u = x^2 + y^2 + z^2 + xyz$, $x = s^2 + t^2$, $y = s^2 - t^2$, and $z = st$

Solution:

$$\begin{aligned}\text{From } \frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}\\ &= (2x+yz)(2s) + (2y+xz)(2s) + (2z+xy)(t)\\ &= 2s(2x+2y+xz+yz) + t(2z+xy),\end{aligned}$$





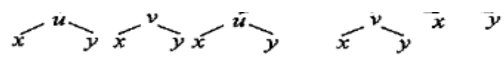
$$\begin{aligned}\frac{\partial^2 u}{\partial s^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial s}\right)\frac{\partial x}{\partial s} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial s}\right)\frac{\partial y}{\partial s} + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial s}\right)\frac{\partial z}{\partial s} + \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial s}\right)_{x,y,z,t}\\ &= (4s+2sz+ty)(2s) + (4s+2sz+tx)(2s) + (2sx+2sy+2t)(t) + 2(2x+2y+xz+yz)\\ &= 8s^2(z+2) + 2(x+y)(2st+z+2) + 2t^2.\end{aligned}$$

Problem 3: [12.6, Prob. 21]

If z=f(u, v), u=g(x, y), v=h(x, y) find the chain rule for the second derivative:

$$\frac{\partial^2 z}{\partial x^2}$$

Solution:



From $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$,

$$\begin{aligned}
\frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial x}\right)\frac{\partial v}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)_{u,v,y} \\
&= \left(\frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial u \partial v}\frac{\partial v}{\partial x}\right)\frac{\partial u}{\partial x} + \left(\frac{\partial^2 z}{\partial v \partial u}\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial x}\right)\frac{\partial v}{\partial x} \\
&\quad + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2} \\
&= \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + 2\frac{\partial^2 z}{\partial u \partial v}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2}.
\end{aligned}$$

Problem 4: [12.6, Prob. 27]

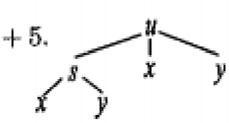
If $f(s)$ is a differentiable function, show that $u(x,y) = f(4x - 3y) + 5(y - x)$ satisfies the equation

$$3\frac{\partial u}{\partial x} + 4\frac{\partial u}{\partial y} = 5$$

Solution:

If we set $s = 4x - 3y$, then

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \left.\frac{\partial u}{\partial x}\right)_{s,y} = f'(s)(4) - 5, \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \left.\frac{\partial u}{\partial y}\right)_{s,x} = f'(s)(-3) + 5.$$



Hence, $3\frac{\partial u}{\partial x} + 4\frac{\partial u}{\partial y} = 3[4f'(s) - 5] + 4[-3f'(s) + 5] = 5.$

Problem 5: [12.6, Prob. 31]

If f(x, y) satisfies the first partial differential equation. Show that with change of independent variables, function F(u, v)=f[x(u, v), y(u, v)] must satisfy the second partial differential equation.

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = 0; u = \frac{x+y}{2}; v = \frac{x-y}{2}$$

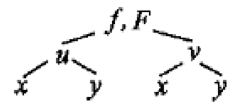
$$\left(\frac{\partial F}{\partial u}\right)^2 + \left(\frac{\partial F}{\partial v}\right)^2 = 0$$

Solution:

The schematic to the right describes the functional situation $f(x,y) = F[u(x,y), v(x,y)]$ where $u = u(x,y) = (x+y)/2$ and $v = v(x,y) = (x-y)/2$. It gives

$$\frac{\partial f}{\partial x} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} = \frac{1}{2}\frac{\partial F}{\partial u} + \frac{1}{2}\frac{\partial F}{\partial v},$$

$$\frac{\partial f}{\partial y} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} = \frac{1}{2}\frac{\partial F}{\partial u} - \frac{1}{2}\frac{\partial F}{\partial v}.$$



Hence,

$$0 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{1}{2}\frac{\partial F}{\partial u} + \frac{1}{2}\frac{\partial F}{\partial v}\right)^2 + \left(\frac{1}{2}\frac{\partial F}{\partial u} - \frac{1}{2}\frac{\partial F}{\partial v}\right)^2 = \frac{1}{2}\left[\left(\frac{\partial F}{\partial u}\right)^2 + \left(\frac{\partial F}{\partial v}\right)^2\right].$$

Problem 6: [12.6, Prob. 35] – Application Problem

An observer travels along the curve $x = t^2$, $y = 3t^3 + 1$, $z = 2t + 5$, where x, y, and z are in meters and $t \geq 0$ is in seconds. If the density ρ of a gas (in kg/m³) is given by

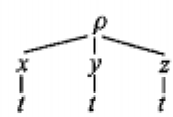
$$\rho = \frac{(3x^2 + y^2)}{z^2 + 5}$$

Find the time rate of change of the density of the gas as measured by the observer when $t = 2$ seconds.

Solution:

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial x}\frac{dx}{dt} + \frac{\partial \rho}{\partial y}\frac{dy}{dt} + \frac{\partial \rho}{\partial z}\frac{dz}{dt}$$

$$= \left(\frac{6x}{z^2+5}\right)(2t) + \left(\frac{2y}{z^2+5}\right)(9t^2) + \left[-\frac{2z(3x^2+y^2)}{(z^2+5)^2}\right](2).$$



When $t = 2$, we obtain $x = 4$, $y = 25$, $z = 9$, and

$$\frac{d\rho}{dt} = \left(\frac{24}{86}\right)(4) + \left(\frac{50}{86}\right)(36) + \left[-\frac{2(9)(673)}{86^2}\right](2) = 18.77 \text{ kg/m}^3/\text{s}.$$

Warm-Up Problems

Solutions to these problems can be found at the back of your textbook

1. S. 12.6, Probs. 2, 4, 6, 14

Extra Practice Problems

Solutions to these problems can be found at the back of your textbook

1. S. 12.6, Probs. 26, 30,

Extra Challenging Problems

Solutions to these problems can be found at the back of your textbook

1. S. 12.6, Probs. 34, 38, 44

Solutions to warm-up, extra practice, and extra challenging problems can be found at the back of your textbook.