Part 8. Partial Differential Equations
Chapter 29. Finite Difference: Parabolic Equations

Lecture 31 & 32

The Heat Conduction Equations (30.1)
Explicit & Implicit Methods (30.2, 30.3)
The Crank-Nicolson Method (30.4)

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Finite Difference: Parabolic Equations

- Parabolic equations are employed to characterize time-variable (*unsteady-state*) problems.

- Conservation of energy can be used to develop an *unsteady-state* energy balance for the differential element in a long, thin insulated rod.

Introduction: Parabolic PDEs

- General form for 2nd order linear PDE with 2 independent and 1 dependent variables:

$$\boxed{A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0}$$

- Criteria for parabolic Eq: $B^2 - 4AC = 0$

- e.g. in heat-conduction equation: $\alpha\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$

$A = \alpha, \;\; B = 0$

$C = 0, \;\; D = -1$

$B^2 - 4AC = 0 - 4(\alpha)(0) = 0$ **Parabolic**

Example. Heat Conduction Equation for Metal Rod

- Energy balance together with Fourier's law of heat conduction yields heat-conduction equation:

$$k\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

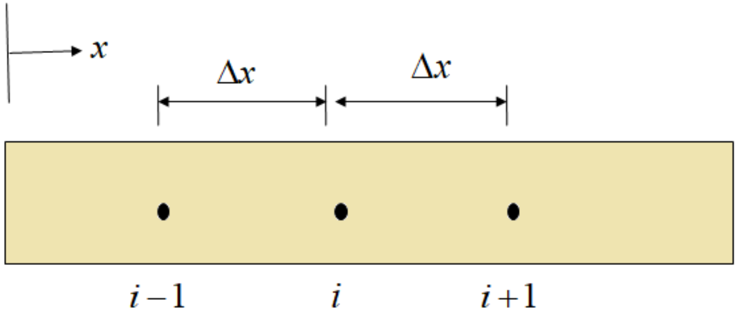
- Just as elliptic PDEs, parabolic equations can be solved by substituting finite divided differences for the partial derivatives.

- In contrast to elliptic PDEs, we must now consider changes in time as well as in space.

Example of Parabolic PDE: Heat Conduction Equation for Metal Rod

The internal temperature of a metal rod exposed to two different temperatures at each end can be found using the heat conduction equation.

$$\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$



Discretizing Parabolic PDE

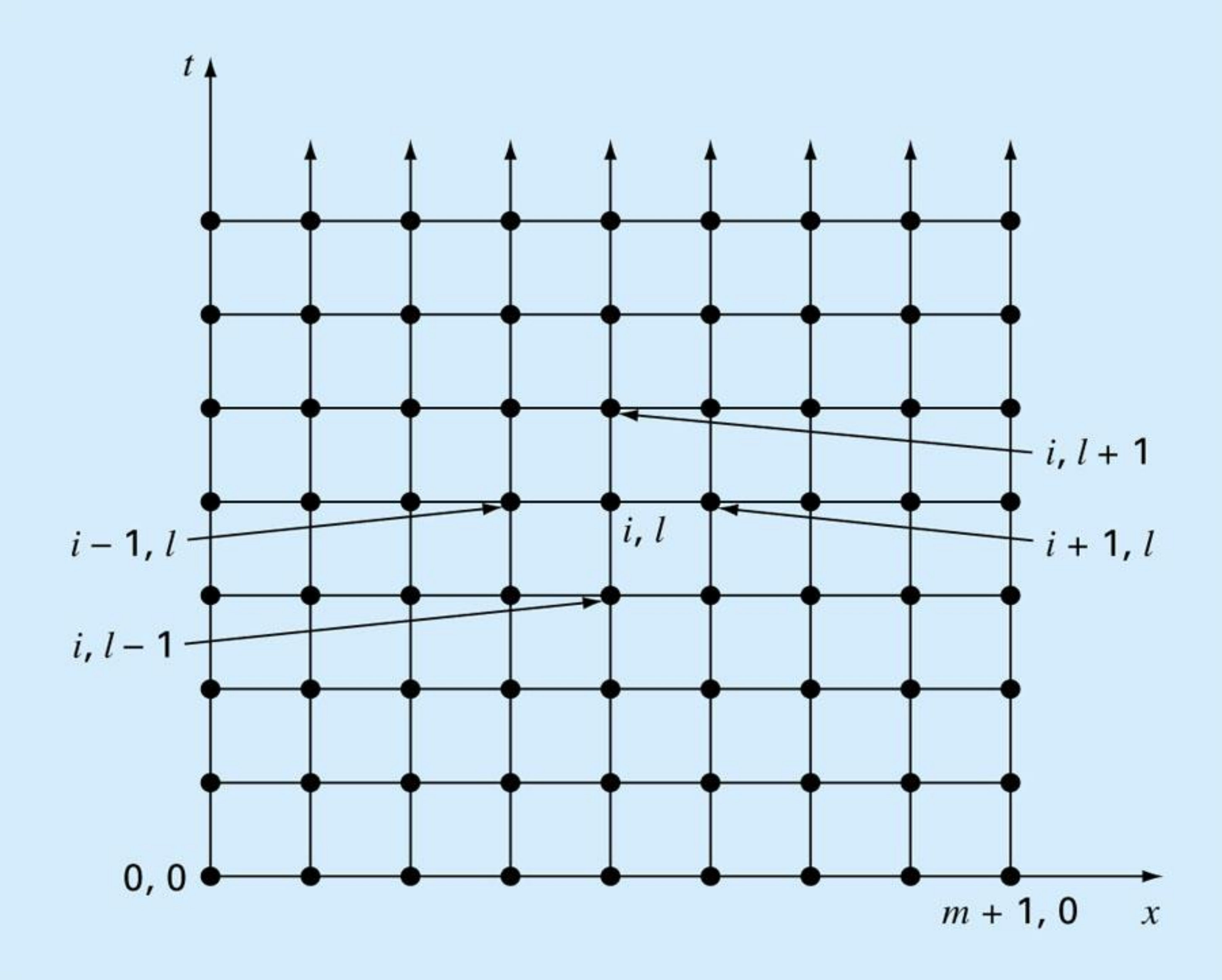
For a rod of length L divided into n+1 nodes: $\Delta x = L / n$

The time is similarly broken into time steps of Δt

Then:

$$x = (i)(\Delta x)$$

$$t = (j)(\Delta t)$$



t
i, l + 1
i − 1, l
i, l
i + 1, l
i, l − 1
0, 0
m + 1, 0
x

Methods for Solution of Parabolic PDE

- Explicit Method
- Implicit Method
- Crank-Nocolson Method (implicit)

Explicit Method

Explicit Method

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{i,j} \cong \frac{T_{i+1}^{j} - 2T_i^{j} + T_{i-1}^{j}}{(\Delta x)^2}$$

$$\left.\frac{\partial T}{\partial t}\right|_{i,j} \cong \frac{T_i^{j+1} - T_i^{j}}{\Delta t}$$

$$\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \quad \Rightarrow \quad \alpha \frac{T_{i+1}^{j} - 2T_i^{j} + T_{i-1}^{j}}{(\Delta x)^2} = \frac{T_i^{j+1} - T_i^{j}}{\Delta t}$$

$$T_i^{j+1} = T_i^{j} + \alpha \frac{\Delta t}{(\Delta x)^2}\left(T_{i+1}^{j} - 2T_i^{j} + T_{i-1}^{j}\right)$$

$$\lambda = \alpha \frac{\Delta t}{(\Delta x)^2}$$

$$\boxed{T_i^{j+1} = T_i^{j} + \lambda\left(T_{i+1}^{j} - 2T_i^{j} + T_{i-1}^{j}\right)}$$

Explicit Method

$$T_i^{j+1} = T_i^j + \lambda\left(T_{i+1}^j - 2T_i^j + T_{i-1}^j\right)$$

- Eq. can be solved explicitly → can be written for each internal location node of the rod for time node j in terms of the temperature at time node j+1 .

- If the temperature at node j=0 , and the boundary temperatures are known, the temperature at the next time step can be found.

- The process is continued by first finding the temperature at all nodes of j=1, and using these to find the temperature at the next time node, j=2. This process continues until the time at which we are interested in finding the temperature is reached.

Implicit Method

Implicit Method

Why ?

•Using the explicit method, we were able to find the temperature at each node, one equation at a time.

•However, the temperature at a specific node was only dependent on the temperature of the neighboring nodes from the previous time step. This is contrary to what we expect from the physical problem.

•The implicit method allows us to solve this and other problems by developing a system of simultaneous linear equations for the temperature at all interior nodes at a particular time.

Implicit Method

$$\left.\frac{\partial T}{\partial t}\right|_{i,j+1} \approx \frac{T_i^{j+1} - T_i^{j}}{\Delta t} \qquad \left.\frac{\partial^2 T}{\partial x^2}\right|_{i,j+1} \approx \frac{T_{i+1}^{j+1} - 2T_i^{j+1} + T_{i-1}^{j+1}}{(\Delta x)^2}$$

The second derivative on the left hand side of the equation is approximated by the Central divided difference scheme at time level j+1 at node (i) as

$$\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \qquad \alpha \frac{T_{i+1}^{j+1} - 2T_i^{j+1} + T_{i-1}^{j+1}}{(\Delta x)^2} = \frac{T_i^{j+1} - T_i^{j}}{\Delta t}$$

$$\lambda = \alpha \frac{\Delta t}{(\Delta x)^2}$$

$$\boxed{-\lambda\, T_{i-1}^{j+1} + (1 + 2\lambda)\; T_i^{j+1} - \lambda\, T_{i+1}^{j+1} = T_i^{j}}$$

Example. Solve the 1D heat conduction in rod equation using **explicit** and **implicit** method, if L=10 cm and one side is 100 C and one side of the rod is 0 C . T (x, t=0)=20 C.

$$\Delta x = 2cm \quad \Delta t = 1s \quad \lambda = 0.243$$

Solve using new time step, j=1

1.486	-0.243	0	0
-0.243	1.486	-0.243	0
0	-0.243	1.486	-0.243
0	0	-0.243	1.486

Solve using new time step, j=2

1.486	-0.243	0	0
-0.243	1.486	-0.243	0
0	-0.243	1.486	-0.243
0	0	-0.243	1.486

Convergence and Stability

- Convergence means that as Δx and Δt approach zero, the results of the finite difference method approach the true solution.

- Stability means that errors at any stage of the computation are not amplified but are attenuated as the computation progresses.
- The explicit method is both convergent and stable if

$$\lambda \leq 1/2$$

$$or$$

$$\Delta t \leq \frac{1}{2}\frac{\Delta x^2}{k}$$

Pros and Cons of Implicit Method

Advantages

- Unconditionally stable
- Can use larger time step values
- More accurate than explicit method

Challenges

- Computationally intense
- First order accurate in time

Crank-Nicolson Method (implicit)

Crank-Nicolson Method (implicit)

WHY:

Using the implicit method:

approximation of $\dfrac{\partial^2 T}{\partial x^2}$ is of $O(\Delta x)^2$ accuracy,

approximation of $\dfrac{\partial T}{\partial t}$ is of $O(\Delta t)$ accuracy.

Crank-Nicolson Method (implicit)

approximating the second derivative at the midpoint of the time step.

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{i,j} \approx \frac{\alpha}{2}\left[\frac{T_{i+1}^{j} - 2T_i^{j} + T_{i-1}^{j}}{(\Delta x)^2} + \frac{T_{i+1}^{j+1} - 2T_i^{j+1} + T_{i-1}^{j+1}}{(\Delta x)^2}\right] \qquad \left.\frac{\partial T}{\partial t}\right|_{i,j} \approx \frac{T_i^{j+1} - T_i^{j}}{\Delta t}$$

$$\frac{\alpha}{2}\left[\frac{T_{i+1}^{j} - 2T_i^{j} + T_{i-1}^{j}}{(\Delta x)^2} + \frac{T_{i+1}^{j+1} - 2T_i^{j+1} + T_{i-1}^{j+1}}{(\Delta x)^2}\right] = \frac{T_i^{j+1} - T_i^{j}}{\Delta t} \qquad \lambda = \alpha\frac{\Delta t}{(\Delta x)^2}$$

$$-\lambda T_{i-1}^{j+1} + 2(1+\lambda)T_i^{j+1} - \lambda T_{i+1}^{j+1} = \lambda T_{i-1}^{j} + 2(1-\lambda)T_i^{j} + \lambda T_{i+1}^{j}$$

Example. Solve the 1D heat conduction in rod equation using **Crank-Nicolson** method, if L=10 cm and one side is 100 C and one side of the rod is 0 C . T (x, t=0)=20 C.

$$\Delta x = 2cm \quad \Delta t = 1s \quad \lambda = 0.243$$

Comparison of 3 methods

	Explicit	**Implicit**	**Crank Nicolson**
T_1^1	39.4	33.4	27.89
T_2^1	20	22.2	20.8
T_3^1	20	19.8	19.9
T_4^1	15.1	16.7	16.1

Group Problem Solving

Consider a steel rod that is subjected to a temperature of 100 C on the left end and 25 C on the right end. If the rod is of length 0.05 m, find the temperature distribution in the rod from t = 0 and t = 9 seconds using three methods blow. Use Δx = 0.01 m and Δt = 3s. The initial temperature of the rod is 20C. By knowing the following constants:

$$\begin{cases} k = 54 \dfrac{W}{m-K} \\ \rho = 7800 \dfrac{kg}{m^3} \\ C = 490 \dfrac{J}{kg-K} \end{cases}$$

Using:

- Explicit method (Group 1)
- Implicit Method (Group 2)
- Crank-Nicolson Method (Group 3)