**Problem Set #4 Solutions**

**11.9** The first iteration can be implemented as

$$c_1 = \frac{3800 + 3c_2 + c_3}{15} = \frac{3800 + 3(0) + 0}{15} = 253.3333$$

$$c_2 = \frac{1200 + 3c_1 + 6c_3}{18} = \frac{1200 + 3(253.3333) + 6(0)}{18} = 108.8889$$

$$c_3 = \frac{2400 + 4c_1 + c_2}{12} = \frac{2350 + 4(253.3333) + 108.8889}{12} = 289.3519$$

Second iteration:

$$c_1 = \frac{3300 + 3c_2 + c_3}{15} = \frac{3800 + 3(108.8889) + 289.3519}{15} = 294.4012$$

$$c_2 = \frac{1200 + 3c_1 + 6c_3}{18} = \frac{1200 + 3(294.4012) + 6(289.3519)}{18} = 212.1842$$

$$c_3 = \frac{2400 + 4c_1 + c_2}{12} = \frac{2350 + 4(294.4012) + 212.1842}{12} = 311.6491$$

The error estimates can be computed as

$$\varepsilon_{a,1} = \left|\frac{294.4012 - 253.3333}{294.4012}\right| \times 100\% = 13.95\%$$

$$\varepsilon_{a,2} = \left|\frac{212.1842 - 108.8889}{212.1842}\right| \times 100\% = 48.68\%$$

$$\varepsilon_{a,3} = \left|\frac{311.6491 - 289.3519}{311.6491}\right| \times 100\% = 7.15\%$$

The remainder of the calculation proceeds until all the errors fall below the stopping criterion of 5%. The entire computation can be summarized as

| iteration | unknown | value | $\varepsilon_a$ | maximum $\varepsilon_a$ |
|---|---|---|---|---|
| 1 | $c_1$ | 253.3333 | 100.00% | |
| | $c_2$ | 108.8889 | 100.00% | |
| | $c_3$ | 289.3519 | 100.00% | 100.00% |
| 2 | $c_1$ | 294.4012 | 13.95% | |
| | $c_2$ | 212.1842 | 48.68% | |
| | $c_3$ | 311.6491 | 7.15% | 48.68% |
| 3 | $c_1$ | 316.5468 | 7.00% | |
| | $c_2$ | 223.3075 | 4.98% | |
| | $c_3$ | 319.9579 | 2.60% | 7.00% |
| 4 | $c_1$ | 319.3254 | 0.87% | |
| | $c_2$ | 226.5402 | 1.43% | |
| | $c_3$ | 321.1535 | 0.37% | 1.43% |

Thus, after 4 iterations, the maximum error is 1.43% and we arrive at the result: $c_1 = 319.3254$, $c_2 = 226.5402$ and $c_3 = 321.1535$. Note that after 11 iterations, the process would converge on $c_1 = 320.2073$, $c_2 = 227.2021$ and $c_3 = 321.5026$.

**11.13** The equations must first be rearranged so that they are diagonally dominant

$$
\begin{aligned}
-8x_1 + x_2 - 2x_3 &= -20 \\
2x_1 - 6x_2 - x_3 &= -38 \\
-3x_1 - x_2 + 7x_3 &= -34
\end{aligned}
$$

**(a)** The first iteration can be implemented as

$$x_1 = \frac{-20 - x_2 + 2x_3}{-8} = \frac{-20 - 0 + 2(0)}{-8} = 2.5$$

$$x_2 = \frac{-38 - 2x_1 + x_3}{-6} = \frac{-38 - 2(2.5) + 0}{-6} = 7.166667$$

$$x_3 = \frac{-34 + 3x_1 + x_2}{7} = \frac{-34 + 3(2.5) + 7.166667}{7} = -2.761905$$

Second iteration:

$$x_1 = \frac{-20 - 7.166667 + 2(-2.761905)}{-8} = 4.08631$$

$$x_2 = \frac{-38 - 2x_1 + x_3}{-6} = \frac{-38 - 2(4.08631) + (-2.761905)}{-6} = 8.155754$$

$$x_3 = \frac{-34 + 3x_1 + x_2}{7} = \frac{-34 + 3(4.08631) + 8.155754}{7} = -1.94076$$

The error estimates can be computed as

$$\varepsilon_{a,1} = \left|\frac{4.08631 - 2.5}{4.08631}\right| \times 100\% = 38.82\%$$

$$\varepsilon_{a,2} = \left|\frac{8.155754 - 7.166667}{8.155754}\right| \times 100\% = 12.13\%$$

$$\varepsilon_{a,3} = \left|\frac{-1.94076 - (-2.761905)}{-1.94076}\right| \times 100\% = 42.31\%$$

The remainder of the calculation proceeds until all the errors fall below the stopping criterion of 5%. The entire computation can be summarized as

| iteration | unknown | value | $\varepsilon_a$ | maximum $\varepsilon_a$ |
|---|---|---|---|---|
| 0 | x1 | 0 | | |
| | x2 | 0 | | |
| | x3 | 0 | | |
| 1 | x1 | 2.5 | 100.00% | |
| | x2 | 7.166667 | 100.00% | |
| | x3 | -2.7619 | 100.00% | 100.00% |
| 2 | x1 | 4.08631 | 38.82% | |
| | x2 | 8.155754 | 12.13% | |
| | x3 | -1.94076 | 42.31% | 42.31% |
| 3 | x1 | 4.004659 | 2.04% | |
| | x2 | 7.99168 | 2.05% | |
| | x3 | -1.99919 | 2.92% | 2.92% |

Thus, after 3 iterations, the maximum error is 2.92% and we arrive at the result: $x_1 = 4.004659$, $x_2 = 7.99168$ and $x_3 = -1.99919$.

**(b)** The same computation can be developed with relaxation where $\lambda = 1.2$.

<u>First iteration:</u>

$$x_1 = \frac{-20 - x_2 + 2x_3}{-8} = \frac{-20 - 0 + 2(0)}{-8} = 2.5$$

Relaxation yields: $x_1 = 1.2(2.5) - 0.2(0) = 3$

$$x_2 = \frac{-38 - 2x_1 + x_3}{-6} = \frac{-38 - 2(3) + 0}{-6} = 7.333333$$

Relaxation yields: $x_2 = 1.2(7.333333) - 0.2(0) = 8.8$

$$x_3 = \frac{-34 + 3x_1 + x_2}{7} = \frac{-34 + 3(3) + 8.8}{7} = -2.3142857$$

Relaxation yields: $x_3 = 1.2(-2.3142857) - 0.2(0) = -2.7771429$

Second iteration:

$$x_1 = \frac{-20 - x_2 + 2x_3}{-8} = \frac{-20 - 8.8 + 2(-2.7771429)}{-8} = 4.2942857$$

Relaxation yields: $x_1 = 1.2(4.2942857) - 0.2(3) = 4.5531429$

$$x_2 = \frac{-38 - 2x_1 + x_3}{-6} = \frac{-38 - 2(4.5531429) - 2.7771429}{-6} = 8.3139048$$

Relaxation yields: $x_2 = 1.2(8.3139048) - 0.2(8.8) = 8.2166857$

$$x_3 = \frac{-34 + 3x_1 + x_2}{7} = \frac{-34 + 3(4.5531429) + 8.2166857}{7} = -1.7319837$$

Relaxation yields: $x_3 = 1.2(-1.7319837) - 0.2(-2.7771429) = -1.5229518$

The error estimates can be computed as

$$\varepsilon_{a,1} = \left|\frac{4.5531429 - 3}{4.5531429}\right| \times 100\% = 34.11\%$$

$$\varepsilon_{a,2} = \left|\frac{8.2166857 - 8.8}{8.2166857}\right| \times 100\% = 7.1\%$$

$$\varepsilon_{a,3} = \left|\frac{-1.5229518 - (-2.7771429)}{-1.5229518}\right| \times 100\% = 82.35\%$$

The remainder of the calculation proceeds until all the errors fall below the stopping criterion of 5%. The entire computation can be summarized as

| iteration | unknown | value | relaxation | $\varepsilon_a$ | maximum $\varepsilon_a$ |
|---|---|---|---|---|---|
| 1 | *x*1 | 2.5 | 3 | 100.00% | |
| | *x*2 | 7.3333333 | 8.8 | 100.00% | |
| | *x*3 | -2.314286 | -2.777143 | 100.00% | 100.000% |
| 2 | *x*1 | 4.2942857 | 4.5531429 | 34.11% | |
| | *x*2 | 8.3139048 | 8.2166857 | 7.10% | |
| | *x*3 | -1.731984 | -1.522952 | 82.35% | 82.353% |
| 3 | *x*1 | 3.9078237 | 3.7787598 | 20.49% | |
| | *x*2 | 7.8467453 | 7.7727572 | 5.71% | |
| | *x*3 | -2.12728 | -2.248146 | 32.26% | 32.257% |
| 4 | *x*1 | 4.0336312 | 4.0846055 | 7.49% | |
| | *x*2 | 8.0695595 | 8.12892 | 4.38% | |
| | *x*3 | -1.945323 | -1.884759 | 19.28% | 19.280% |
| 5 | *x*1 | 3.9873047 | 3.9678445 | 2.94% | |
| | *x*2 | 7.9700747 | 7.9383056 | 2.40% | |
| | *x*3 | -2.022594 | -2.050162 | 8.07% | 8.068% |
| 6 | *x*1 | 4.0048286 | 4.0122254 | 1.11% | |
| | *x*2 | 8.0124354 | 8.0272613 | 1.11% | |
| | *x*3 | -1.990866 | -1.979007 | 3.60% | 3.595% |

Thus, relaxation actually seems to retard convergence. After 6 iterations, the maximum error is 3.595% and we arrive at the result: $x_1 = 4.0122254$, $x_2 = 8.0272613$ and $x_3 = -1.979007$.

**11.18** Define the quantity of transistors, resistors, and computer chips as $x_1$, $x_2$ and $x_3$. The system equations can then be defined as

$$4x_1 + 3x_2 + 2x_3 = 960$$
$$x_1 + 3x_2 + x_3 = 510$$
$$2x_1 + x_2 + 3x_3 = 610$$

The solution can be implemented in Excel as shown below:

| | A | B | C | D |
|---|---|---|---|---|
| 1 | A: | | | B: |
| 2 | 4 | 3 | 2 | 960 |
| 3 | 1 | 3 | 1 | 510 |
| 4 | 2 | 1 | 3 | 610 |
| 5 | | | | |
| 6 | AI: | | | X: |
| 7 | 0.421053 | -0.36842 | -0.15789 | 120 |
| 8 | -0.05263 | 0.421053 | -0.10526 | 100 |
| 9 | -0.26316 | 0.105263 | 0.473684 | 90 |