MTE 203 – Advanced Calculus
Homework 11

Line Integrals Involving Vector Functions

Problem 1: [S. 14.3, Prob. 13]

Evaluate the line integral $\int_C xy\,dx + x\,dy$ from $(-5, 3, 0)$ to $(4, 0, 0)$ along each of the following curves:

a. The straight line joining the points $(-5, 3, 0)$ and $(4, 0, 0)$
b. $x = 4 - y^2, z = 0$
c. $3y = x^2 - 16, z = 0$

Problem 2: [S. 14.3, Prob. 35]

Suppose a gas flows through a region D of space. At each $P(x, y, z)$ in D and time t, the gas has velocity $\vec{v}(x, y, z, t)$. If C is a closed curve in D, the line integral:

$$\Gamma = \oint_C \vec{v}.\vec{r}$$

is called *the circulation of the flow* for the curve C. If C is the circle $x^2 + y^2 = r^2, \quad z = 1$ (directed clockwise as viewed from the origin), calculate Γ for the following flow vectors:

a. $\vec{v} = \frac{x\hat{\imath}+y\hat{\jmath}+z\hat{k}}{(x^2+y^2+z^2)^{3/2}}$
b. $\vec{v} = -y\hat{\imath} + x\hat{\jmath}$

Path Independence

Problem 3: [S. 14.4, Prob.5]

Show that the line integral is independent of the path and evaluate it.

$$\int_C -\frac{y}{z}\sin x\ dx + \frac{1}{z}\cos x\, dy - \frac{y}{z^2}\cos x\, dz$$

where C is the helix $x = \ 2\, cos\, t, y = 2\, sin\, t, z = t\ from\ (2{,}0{,}2\pi)\ to\ (2{,}0{,}4\pi)$.

Problem 4: [S. 14.4, Prob.11]

Show that if $f(x)$, $g(y)$ and $h(z)$ have continuous first derivatives, then the line integral $\int_C\ f(x)dx + g(y)dy + h(z)dz$ is independent of path.

Problem 5: [S. 14.4, Prob.17]

Evaluate $\int_C\ -\frac{1}{x}\tan^{-1} y\ dx + \frac{1}{x+xy^2}\ dy$, where C is the curve $x = y^2 + 1$ from (2,-1) to (10,3).

Conservative Fields

Problem 6: [S. 14.5, Prob.5]

Determine whether the force field is conservative. Identify conservative force field and find a potential energy function.

$$\boldsymbol{F}(x, y, z) = GMm\frac{x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}}{(x^2+y^2+z^2)^{3/2}} \text{ where } G, M \text{ and } m \text{ are constant.}$$

Problem 7: [S. 14.5, Prob.7]

How do the equipotential surfaces of the forces in exercise 5 from section 14.5 (previous homework problem) look like?

Green's Theorem

Problem 8: [S. 14.6, Prob.7]

Use Green's theorem to evaluate the line integral:

$$\oint_C (x^3 + y^3)dx + (x^3 - y^3)dy,$$

where C is the curve enclosing the region bounded by $x = y^2 - 1$ and $x = 1 - y^2$

Problem 9: [S. 14.6, Prob.25]

Use Green's theorem to evaluate the line integral:

$\oint_C \left(2xye^{x^2y} + 3x^2y\right)dx + \left(x^2e^{x^2y}\right)dy$, where C is the ellipse $x^2 + 4y^2 = 4$.

Surface Integrals

Problem 10: [S. 14.7, Prob.9]

Set up double iterated integrals for the surface integral of a function $f(x, y, z)$ over the surface defined by $z = 4 - x^2 - 4y^2, (x, y, z) \geq 0$, if the surface is projected onto the xy-, the xz-, and yz-planes.

Problem 11: [S. 14.7, Prob.19]

Evaluate the surface integral by projecting the surface into one of the coordinate planes and also by using spherical coordinate area element ($dS = \rho^2 sin\, \varphi\, d\varphi\, d\theta$) given in equation 14.56.

$\oiint_S x^2z^2\, dS$, where S is the sphere $x^2 + y^2 + z^2 = R^2$

Warm-Up Problems

Solutions to these problems can be found at the back of your textbook

1. S. 14.3, Probs. 2, 6, 12
2. S. 14.4, Probs. 2, 6, 8
3. S. 14.5, Probs. 2, 4
4. S. 14.6, Probs. 2, 6, 12
5. S. 14.7, Probs. 2, 8, 12

Extra Practice Problems

Solutions to these problems can be found at the back of your textbook

1. S. 14.3, Probs. 16, 22, 34
2. S. 14.4, Prob. 14, 18, 22, 24
3. S. 14.5, Probs. 6, 8
4. S. 14.6, Probs. 18, 26, 28
5. S. 14.7, Probs. 18, 22

Extra Challenging Problems

Solutions to these problems can be found at the back of your textbook

1. S. 14.3, Probs. 36
2. S. 14.5, Probs. 10
3. S. 14.6, Probs. 30, 32