SYDE252 - lecture notes

09/01/18

Presented by: John Zelek

Systems Design Engineering

note: some material (figures) borrowed from various sources

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4. Laplace

09/11/18

Presented by: John Zelek

Department of Engineering

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inspiration

- "You can certainly use the Laplace transform on this problem, but that would be akin to slicing your bread with a chainsaw: it's overkill, it's difficult to set up, and if you let your 5-year-old try it someone is going to end up calling 911. Well, maybe not that last one."   — Electrical engineering professor

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Laplace

The Laplace transform is a tool to map signal and system behaviours from the time domain into the frequency domain. We break $x(t)$ into exponential components of the form e^{st}, where s is the complex frequency $s = \alpha + j\omega$. The Laplace transform applies to general signals and not just sinusoids (e.g., the Fourier only applies to sinusoids and is a subset of the Laplace).

The Laplace transform is:

$$X(\alpha, \omega) = X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

and the inverse Laplace transform is

$$x(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(s) e^{st} ds$$

where c is a constant chosen to ensure convergence of the first integral. The path of integration along $c + i\omega$ with ω varying from $-\infty$ to ∞ must lie in the ROC (Region of Convergence).

Let us assume that all signals we will be dealing with are **causal**, meaning the they are on-sided (or unilateral) Laplace transform.
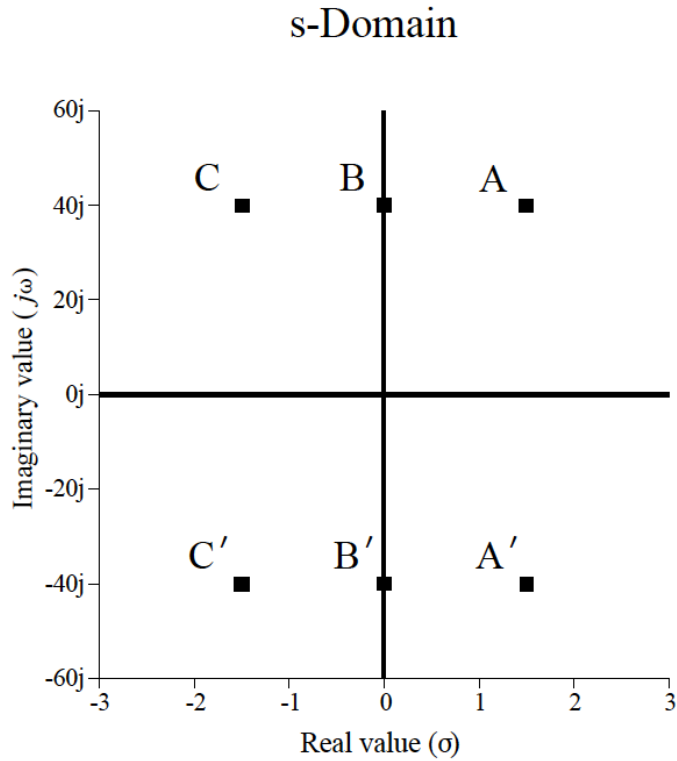
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s-Domain

Associated Waveforms

$\cos(40t)e^{-1.5t}$

A+A$'$

$\cos(40t)e^{0t}$

B+B$'$

$\cos(40t)e^{1.5t}$

C+C$'$

FIGURE 32-2
Waveforms associated with the s-domain. Each location in the s-domain is identified by two parameters: σ and ω. These parameters also define two waveforms associated with each location. If we only consider *pairs* of points (such as: A&A$'$, B&B$'$, and C&C$'$), the two waveforms associated with each location are sine and cosine waves of frequency ω, with an exponentially changing amplitude controlled by σ.

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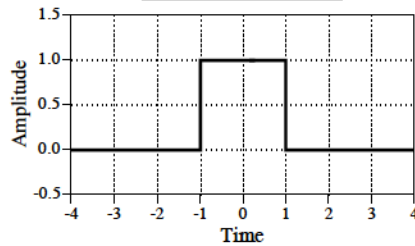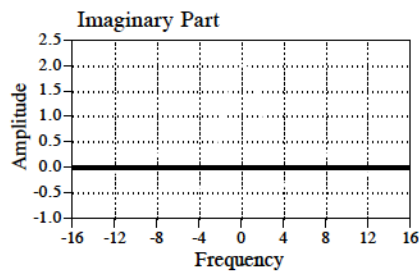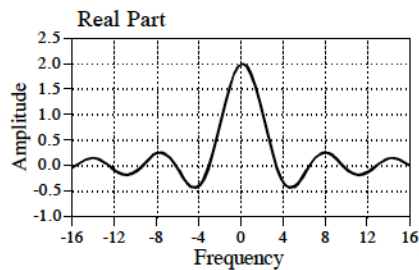**FACULTY OF ENGINEERING**
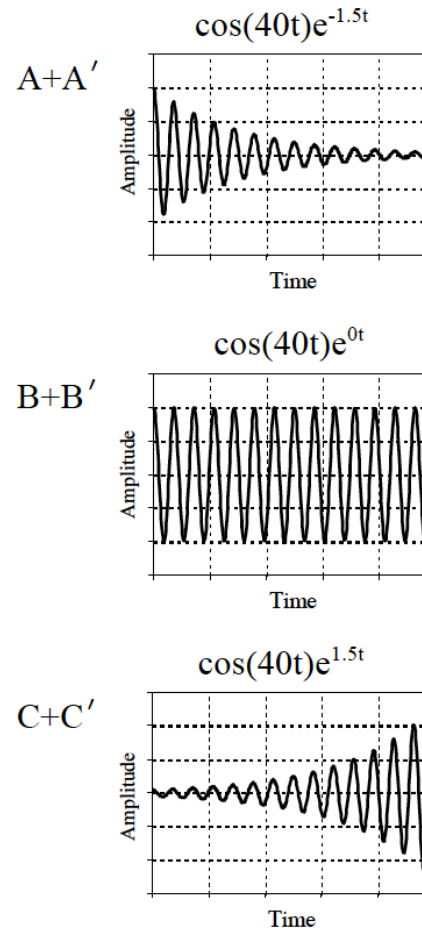
Time Domain

FIGURE 32-3
Time, frequency and s-domains. A time domain signal (the rectangular pulse) is transformed into the frequency domain using the Fourier transform, and into the s-domain using the Laplace transform.
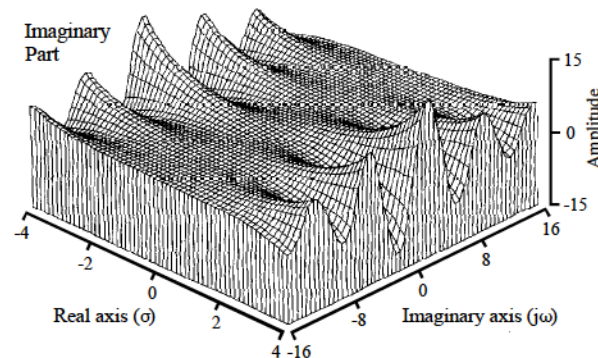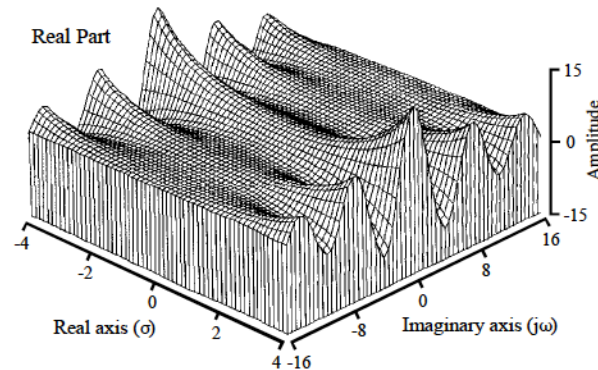
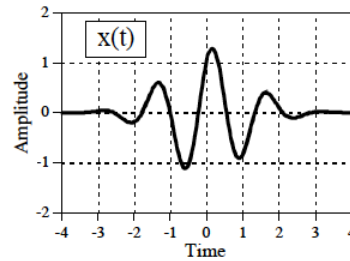*Fourier Transform*

Laplace Transform

Frequency Domain

Real Part

Imaginary Part

s-Domain

Real Part

Imaginary Part

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STEP 1
Start with the time domain signal
called $x(t)$

$x(t)$

Amplitude

Time

STEP 2
Multiply the time domain signal by
an infinite number of exponential
curves, each with a different decay
constant, σ. That is, calculate the
signal: $x(t)\, e^{-\sigma t}$ for each value of σ
from negative to positive infinity.

$\sigma = -3$ $\sigma = -2$ $\sigma = -1$ $\sigma = 0$ $\sigma = 1$ $\sigma = 2$ $\sigma = 3$
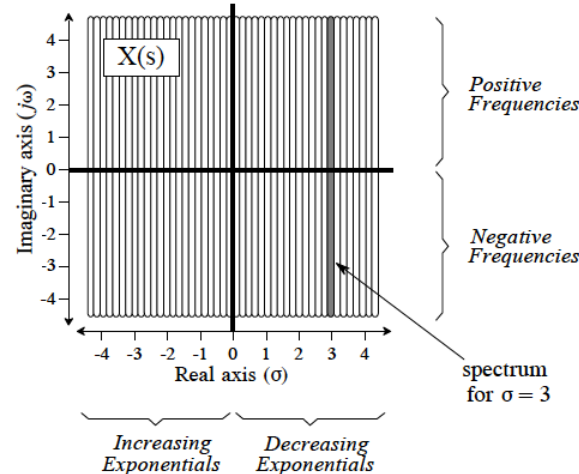
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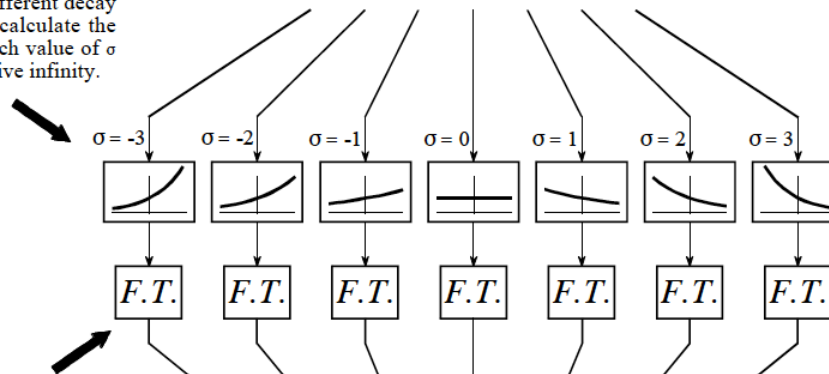
STEP 3
Take the complex Fourier Transform
of each exponentially weighted time
domain signal. That is, calculate:

$$\int_{-\infty}^{\infty} [\, x(t)\, e^{-\sigma t}\,]\, e^{-j\omega t}\, dt$$

for each value of σ from negative to
positive infinity.

STEP 4
Arrange each spectrum along a
vertical line in the s-plane. The
positive frequencies are in the
upper half of the s-plane while the
negative frequencies are in the
lower half.

$X(s)$

Imaginary axis ($j\omega$)

Positive
Frequencies

Negative
Frequencies

Real axis (σ)

spectrum
for $\sigma = 3$

Increasing   Decreasing
Exponentials   Exponentials

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Laplace transform

The Laplace transform is:

$$X(\alpha, \omega) = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

and the inverse Laplace transform is

$$x(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(s)e^{st}ds$$

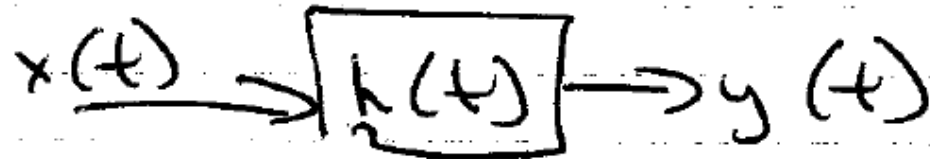where c is a constant chosen to ensure convergence of the first integral. The path of integration along $c + j\omega$ with

ω varying from $-\infty$ to ∞ must lie in the ROC (Region of Convergence).

Let us assume that all signals we will be dealing with are **causal**, meaning the they are on-sided (or unilateral) Laplace transform.

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Laplace transform

time

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$$

freq$^\omega$

$$x(t) \rightarrow \boxed{\int} \xrightarrow{X(s)} \boxed{H(s)} \xrightarrow{Y(s)} \boxed{\int^{-1}} \rightarrow y(t)$$

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Laplace transform examples

Determine the Laplace transforms of the signal waveforms displayed in Fig. 3-1.

Solution:

    (a) The step function in Fig. 3-1(a) is given by

$$x_1(t) = A\, u(t - T).$$
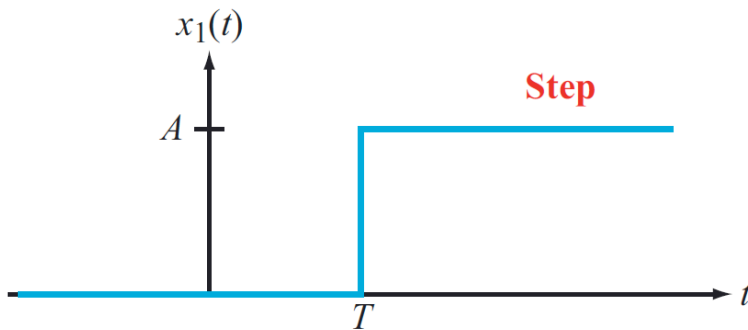
$$\mathbf{X}_1(\mathbf{s}) = \int_{0^-}^{\infty} x_1(t)\, e^{-st}\, dt$$

$$= \int_{0^-}^{\infty} A\, u(t - T)\, e^{-st}\, dt$$

$$= A \int_{T}^{\infty} e^{-st}\, dt$$

$$= -\frac{A}{s}\, e^{-st}\Big|_{T}^{\infty} = \frac{A}{s}\, e^{-sT}.$$

For the special case where $A = 1$ and $T = 0$ (i.e., the step occurs at $t = 0$), the transform pair becomes

$$u(t) \quad \longleftrightarrow \quad \frac{1}{s}. \qquad (3.6)$$

$x_1(t)$

Step

A

T

t

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Laplace transform examples

$x_2(t)$

Rectangle

T_1 T_2

(b)
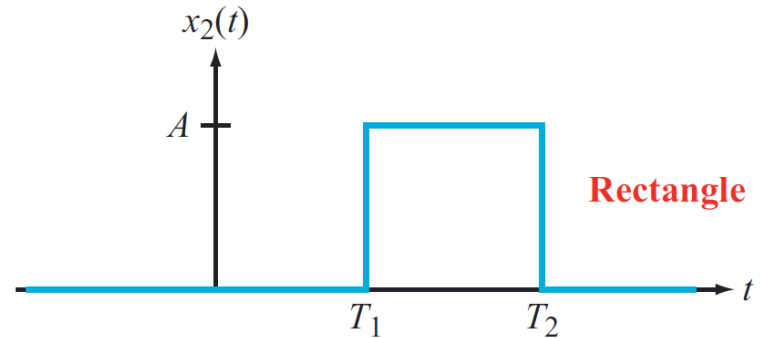
(b) The rectangle function in Fig. 3-1(b) can be constructed as the sum of two step functions:

$$x_2(t) = A[u(t - T_1) - u(t - T_2)],$$

and its Laplace transform is

$$\mathbf{X}_2(\mathbf{s}) = \int_{0^-}^{\infty} A[u(t - T_1) - u(t - T_2)]e^{-st}\, dt$$

$$= A \int_{0^-}^{\infty} u(t - T_1)\, e^{-st}\, dt - A \int_{0^-}^{\infty} u(t - T_2)\, e^{-st}\, dt$$

$$= \frac{A}{\mathbf{s}}[e^{-sT_1} - e^{-sT_2}].$$

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Laplace transform examples

(c)
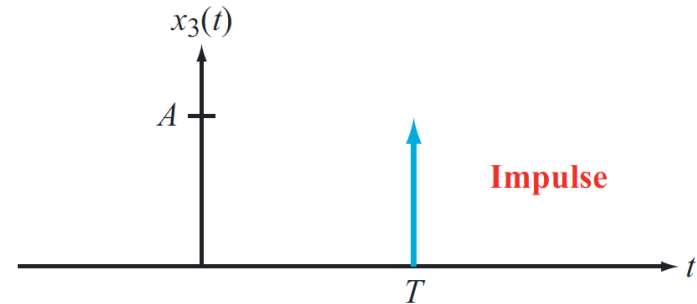
(c) The impulse function in Fig. 3-1(c) is given by

$$x_3(t) = A\, \delta(t - T),$$

and the corresponding Laplace transform is

$$\mathbf{X}_3(s) = \int_{0^-}^{\infty} A\, \delta(t - T)\, e^{-st}\, dt = Ae^{-sT},$$

defined by Eq. (1.29). For the special case where $A = 1$ and $T = 0$, the Laplace transform pair simplifies to

$$\delta(t) \longleftrightarrow 1. \qquad (3.7)$$

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Laplace transform examples

Example 3-2: Laplace Transform Pairs

Obtain the Laplace transforms of (a) $x_1(t) = e^{-at} u(t)$ and (b) $x_2(t) = [\cos(\omega_0 t)] u(t)$.

Solution:

(a) Application of Eq. (3.1) gives

$$X_1(s) = \int_{0^-}^{\infty} e^{-at} u(t) e^{-st} dt$$

$$= \left. \frac{e^{-(s+a)t}}{-(s+a)} \right|_0^{\infty} = \frac{1}{s+a}.$$

Hence,

$$e^{-at} u(t) \iff \frac{1}{s+a}. \qquad (3.8)$$

(b) We start by expressing $\cos(\omega_0 t)$ in the form

$$\cos(\omega_0 t) = \frac{1}{2}[e^{j\omega_0 t} + e^{-j\omega_0 t}].$$

Next, we take advantage of Eq. (3.8):

$$X_2(s) = \mathcal{L}[\cos(\omega_0 t) \, u(t)]$$

$$= \frac{1}{2} \mathcal{L}[e^{j\omega_0 t} \, u(t)] + \frac{1}{2} \mathcal{L}[e^{-j\omega_0 t} \, u(t)]$$

$$= \frac{1}{2} \frac{1}{s - j\omega_0} + \frac{1}{2} \frac{1}{s + j\omega_0}$$

$$= \frac{s}{s^2 + \omega_0^2}.$$

Hence,

$$[\cos(\omega_0 t)] \, u(t) \iff \frac{s}{s^2 + \omega_0^2}.$$

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Laplace transform examples (1)

$$x(t) = e^{-at}u(t) \quad , \text{a is real}$$

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Laplace transform examples (2)

$$x(t) = -e^{-at}u(-t) \qquad \text{, a is real}$$

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Laplace transform examples (3+)

$$x(t) = \delta(t)$$

$$x(t) = u(t)$$

$$x(t) = e^{at}u(t)$$

$$x(t) = \cos \omega_0 t u(t)$$

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Laplace transform pairs

Laplace transform pairs $x(t) \leftrightarrow X(s)$

$\delta(t) \Leftrightarrow 1$

$u(t) \Leftrightarrow \frac{1}{s}$

$tu(t) \Leftrightarrow \frac{1}{s^2}$

$t^n u(t) \Leftrightarrow \frac{n!}{s^{n+1}}$

$e^{\lambda t} u(t) \Leftrightarrow \frac{1}{s-\lambda}$

$t^n e^{\lambda t} u(t) \Leftrightarrow \frac{n!}{(s-\lambda)^{n+1}}$

Inverse Laplace is best done by using the tables available.

You will need to use partial fraction expansion before you can use the tables. See Appendix B for a review of partial fraction expansion. If the order of the numerator is equal to order of the denominator then you will need to add coefficient of the highest power in the numerator for partial fraction expansion.

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Table 3-2: Examples of Laplace transform pairs. Note that $x(t) = 0$ for $t < 0^-$ and $T \geq 0$.

	Laplace Transform Pairs	
	$x(t)$	$X(s) = \mathcal{L}[x(t)]$
1	$\delta(t)$ ⟷	1
1a	$\delta(t - T)$ ⟷	e^{-Ts}
2	$u(t)$ ⟷	$\dfrac{1}{s}$
2a	$u(t - T)$ ⟷	$\dfrac{e^{-Ts}}{s}$
3	$e^{-at} u(t)$ ⟷	$\dfrac{1}{s+a}$
3a	$e^{-a(t-T)} u(t - T)$ ⟷	$\dfrac{e^{-Ts}}{s+a}$
4	$t\, u(t)$ ⟷	$\dfrac{1}{s^2}$
4a	$(t - T)\, u(t - T)$ ⟷	$\dfrac{e^{-Ts}}{s^2}$
5	$t^2\, u(t)$ ⟷	$\dfrac{2}{s^3}$
6	$te^{-at}\, u(t)$ ⟷	$\dfrac{1}{(s+a)^2}$
7	$t^2 e^{-at}\, u(t)$ ⟷	$\dfrac{2}{(s+a)^3}$
8	$t^{n-1} e^{-at}\, u(t)$ ⟷	$\dfrac{(n-1)!}{(s+a)^n}$
9	$\sin(\omega_0 t)\, u(t)$ ⟷	$\dfrac{\omega_0}{s^2 + \omega_0^2}$
10	$\sin(\omega_0 t + \theta)\, u(t)$ ⟷	$\dfrac{s \sin\theta + \omega_0 \cos\theta}{s^2 + \omega_0^2}$
11	$\cos(\omega_0 t)\, u(t)$ ⟷	$\dfrac{s}{s^2 + \omega_0^2}$
12	$\cos(\omega_0 t + \theta)\, u(t)$ ⟷	$\dfrac{s \cos\theta - \omega_0 \sin\theta}{s^2 + \omega_0^2}$
13	$e^{-at} \sin(\omega_0 t)\, u(t)$ ⟷	$\dfrac{\omega_0}{(s+a)^2 + \omega_0^2}$
14	$e^{-at} \cos(\omega_0 t)\, u(t)$ ⟷	$\dfrac{s+a}{(s+a)^2 + \omega_0^2}$
15	$2e^{-at} \cos(bt - \theta)\, u(t)$ ⟷	$\dfrac{e^{j\theta}}{s+a+jb} + \dfrac{e^{-j\theta}}{s+a-jb}$
15a	$e^{-at} \cos(bt - \theta)\, u(t)$ ⟷	$\dfrac{(s+a)\cos\theta + b\sin\theta}{(s+a)^2 + b^2}$
16	$\dfrac{2t^{n-1}}{(n-1)!} e^{-at} \cos(bt - \theta)\, u(t)$ ⟷	$\dfrac{e^{j\theta}}{(s+a+jb)^n} + \dfrac{e^{-j\theta}}{(s+a-jb)^n}$

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inverse Laplace (1,2)

$$X(s) = \frac{7}{s^2}$$

$$X(s) = \frac{2s - 1}{s^2 + 4}$$

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inverse Laplace (3)

$$X(s) = \frac{7s - 6}{s^2 - s - 6}$$

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inverse Laplace (3) (ctn)

$$X(s) = \frac{7s - 6}{s^2 - s - 6}$$

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inverse Laplace (4)

$$X(s) = \frac{2s^2 + 5}{(s + 1)(s + 2)}$$

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Laplace properties

Time Shifting

if $x(t) \Leftrightarrow X(s)$
then $x(t - t_0) \Leftrightarrow X(s)e^{-st_0}$

Frequency Shifting

if $x(t) \Leftrightarrow X(s)$
then $x(t)e^{s_0 t} \Leftrightarrow X(s - s_0)$

Time Differentiation

if $x(t) \Leftrightarrow X(s)$
then $\frac{dx}{dt} \Leftrightarrow sX(s) - x(0^-)$
and $\frac{d^2 x}{dt^2} \Leftrightarrow s^2 X(s) - sx(0^-) - \dot{x}(0^-)$
and $\frac{d^n x}{dt^n} \Leftrightarrow s^n X(s) - \sum_{k=1}^{n} s^{n-k} x^{k-1}(0^-)$

Frequency Differentiation

if $x(t) \Leftrightarrow X(s)$
then $tx(t) \Leftrightarrow -\frac{d}{ds} X(s)$

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Laplace properties (2)

Time Integration

if $x(t) \Leftrightarrow X(s)$

then $\int_{0^-}^{t} x(\tau)d\tau \Leftrightarrow \frac{X(s)}{s}$

Scaling

if $x(t) \Leftrightarrow X(s)$

then $x(at) \Leftrightarrow \frac{1}{a}X(\frac{s}{a})$ for all a

Time Convolution

if $x_1(t) \Leftrightarrow X_1(s)$ and $x_2(t) \Leftrightarrow X_2(s)$

then $x_1(t) * x_2(t) \Leftrightarrow X_1(s)X_s(s)$

Frequency Convolution

if $x_1(t) \Leftrightarrow X_1(s)$ and $x_2(t) \Leftrightarrow X_2(s)$

then $x_1(t)x_2(t) \Leftrightarrow \frac{1}{2\pi j}X_1(s) * X_s(s)$

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Laplace properties (3)

Initial Value Theorem

if $x(t) \Leftrightarrow X(s)$

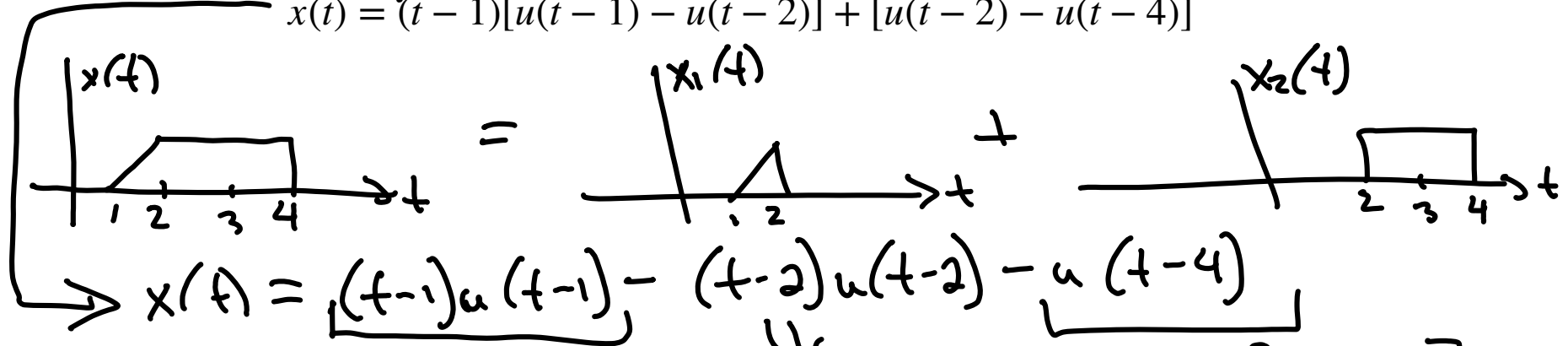then $\lim_{t \to 0} x(t) = x(0^+) = \lim_{s \to \infty} sX(s)$

Final Value Theorem

if $x(t) \Leftrightarrow X(s)$

then $\lim_{t \to \infty} x(t) = x(0^+) = \lim_{s \to 0} sX(s)$

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Laplace properties - examples (1) - time shift

$$x(t) = \overbrace{(t-1)[u(t-1) - u(t-2)]}^{x_1(t)} + \overbrace{[u(t-2) - u(t-4)]}^{x_2(t)}$$

$$\Rightarrow x(t) = \underline{(t-1)u(t-1)} - (t-2)u(t-2) - \underbrace{u(t-4)}$$

$$-(t-1)(u(t-2)) + u(t-2) = u(t-2)[-t+1+1]$$

$$u(t-2)(2-t)$$

$$\boxed{\begin{array}{l} u(t) \leftrightarrow \frac{1}{s} \\ t\, u(t) \leftrightarrow \frac{1}{s^2} \end{array}}$$

$$(t-1)u(t-1) \leftrightarrow \frac{1}{s^2} e^{-s}$$

$$(t-2)u(t-2) \leftrightarrow \frac{1}{s^2} e^{-2s}$$

$$u(t-4) \leftrightarrow \frac{1}{s} e^{-4s}$$

$$X(s) = \frac{1}{s^2} e^{-s} - \frac{1}{s^2} e^{-2s} - \frac{1}{s} e^{-4s}$$

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Laplace properties - examples (2) - frequency shift

$$x(t) = e^{-at}\cos(bt)u(t)$$

$$\cos bt\, u(t) \iff \frac{s}{s^2 + b^2}$$

$$e^{-at}\cos bt\, u(t) \iff \frac{s+a}{(s+a)^2 + b^2}$$

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Laplace properties - examples (3) - time differentiation
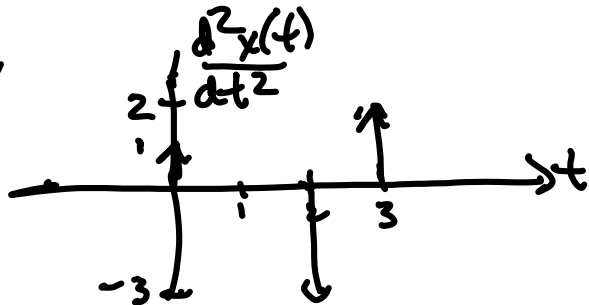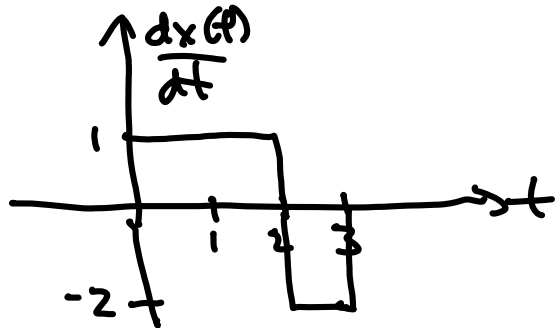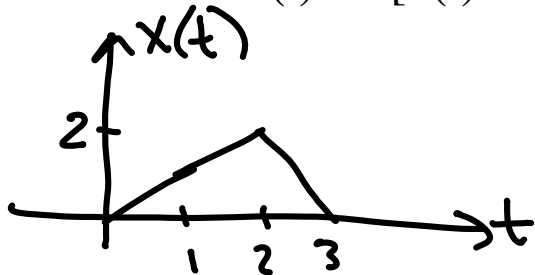
$$x(t) = t[u(t) - u(t-2)] + (3-t)[u(t-2) - u(t-3)]$$

$$\frac{d^2x}{dt^2} = \delta(t) - 3\delta(t-2) + 2\delta(t-3)$$

$$\mathcal{L}\left[\frac{d^2x}{dt^2}\right] = \mathcal{L}\left[\delta(t) - 3\delta(t-2) + 2\delta(t-3)\right]$$

$$\delta(t) \rightarrow 1$$

$$s^2 X(s) - sx(0^-) - \dot{x}(0^-)$$

$$s^2 X(s) = 1 - 3e^{-2s} + 2e^{-3s}$$

$$X(s) = \frac{1 - 3e^{-2s} + 2e^{-3s}}{s^2}$$

$$= \frac{1}{s^2}\left[1 - 3e^{-2s} + 2e^{-3s}\right]$$

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Laplace properties - examples (3) - time differentiation

$$x(t) = t[u(t) - u(t - 2)] + (3 - t)[u(t - 2) - u(t - 3)]$$

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Laplace properties - examples (4) - time convolution

$$c(t) = e^{at}u(t) * e^{bt}u(t)$$

from tables

$$e^{at}u(t) \Longleftrightarrow \frac{1}{s-a}$$

$$e^{bt}u(t) \Longleftrightarrow \frac{1}{s-b}$$

$$\therefore \ c(t) \Longleftrightarrow \frac{1}{(s-a)(s-b)}$$

$$C(s) = \frac{1}{(s-a)(s-b)}$$

$$C(s) = \frac{1}{(s-a)(s-b)} = \frac{1}{a-b}\left[\frac{1}{s-a} - \frac{1}{s-b}\right]$$

$$\therefore \ c(t) = \frac{1}{a-b}\left[e^{at} - e^{bt}\right]u(t)$$

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Laplace properties - examples (5) - initial, final value

$$Y(s) = \frac{5(3s + 1)}{s(s + 3)(s + 2)}$$

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Laplace - solving differential equations - ZI +ZS

$$\frac{d^2y}{dt^2} + 9y(y) = e^{4t}u(t)$$

I.C. $y(0) = 1$

         $\dot{y}(0) = 0$

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Laplace - solving differential equations - ZI +ZS (2)

$$\frac{d^2y}{dt^2} + 9y(y) = e^{4t}u(t)$$

I.C. $\quad y(0) = 1$

$\quad\quad\quad \dot{y}(0) = 0$

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Laplace - solving differential equations - ZI +ZS (3)

$$\frac{d^2y}{dt^2} + 9y(y) = e^{4t}u(t)$$

I.C. $y(0) = 1$

          $\dot{y}(0) = 0$

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Laplace - solving differential equations - ZI +ZS - example 2

$$(D^2 + 5D + 6)y(t) = (D + 1)x(t)$$

I.C. $y(0) = 2$ $\dot{y}(0) = 1$

$$x(t) = e^{-4t}u(t)$$

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System Description relationships