Part 7. Ordinary Differential Equations
Chapter 25. Runge-Kutta Methods

Lecture 24

Euler's Method and Its Improvements

25.1, 25.2

Homeyra Pourmohammadali

Learning Outcomes

- Understand the visual representation of Euler's method and its improved versions: (Heun's & the midpoint) for solving ODEs

- Know relationship of Euler's method to Tylor series expansion and its insight into the error

- Know the general form of Runge-Kutta (RK) method, understand derivation of the 2^{nd}-order RK method and how it related to Taylor series expansion

Ordinary Differential Equations

- Are composed of an unknown function and its derivatives.

- ODEs play a fundamental role in engineering

- Many physical phenomena are formulated mathematically in terms of their rate of change.

$$\frac{dv}{dt} = g - \frac{c}{m} v$$

v- dependent variable

t- independent variable

Types of Ordinary Differential Equations

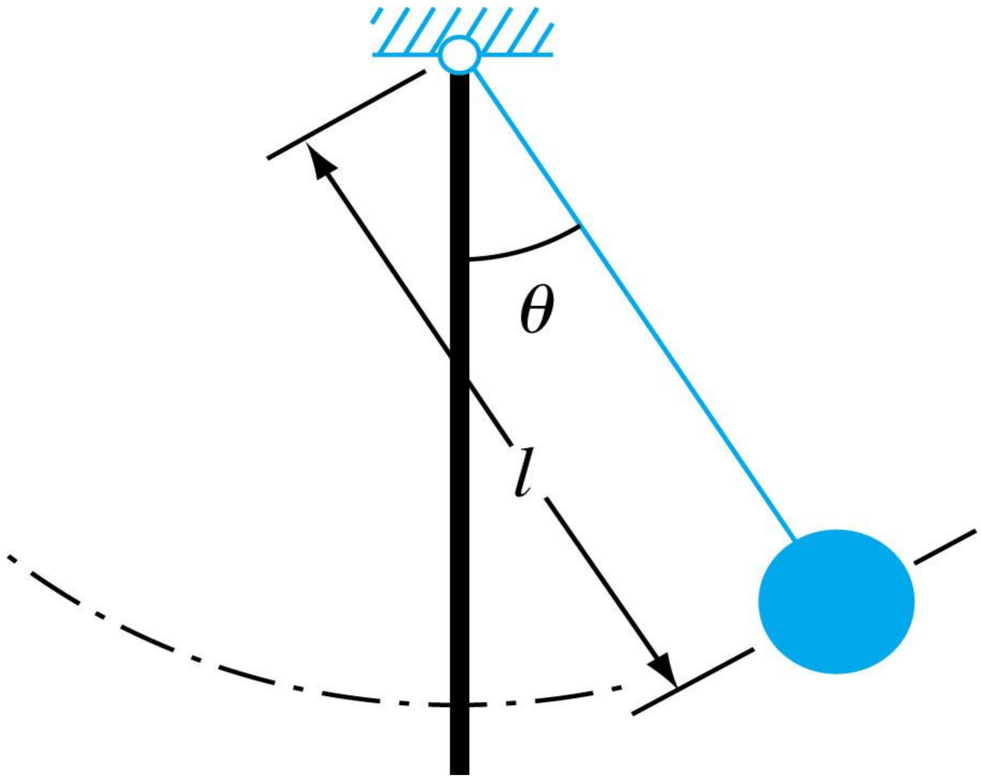
ODE → involves 1 dependent variable

1st order ODE includes a 1st derivative as its highest derivative.

2nd order ODE includes a 2nd derivative.

PDE → involves 2 or more independent variables

ODEs and Engineering Practice



Swinging Pendulum

One-Step Methods

- Solving ODEs of the form

$$\frac{dy}{dx} = f(x, y)$$

We focus on:

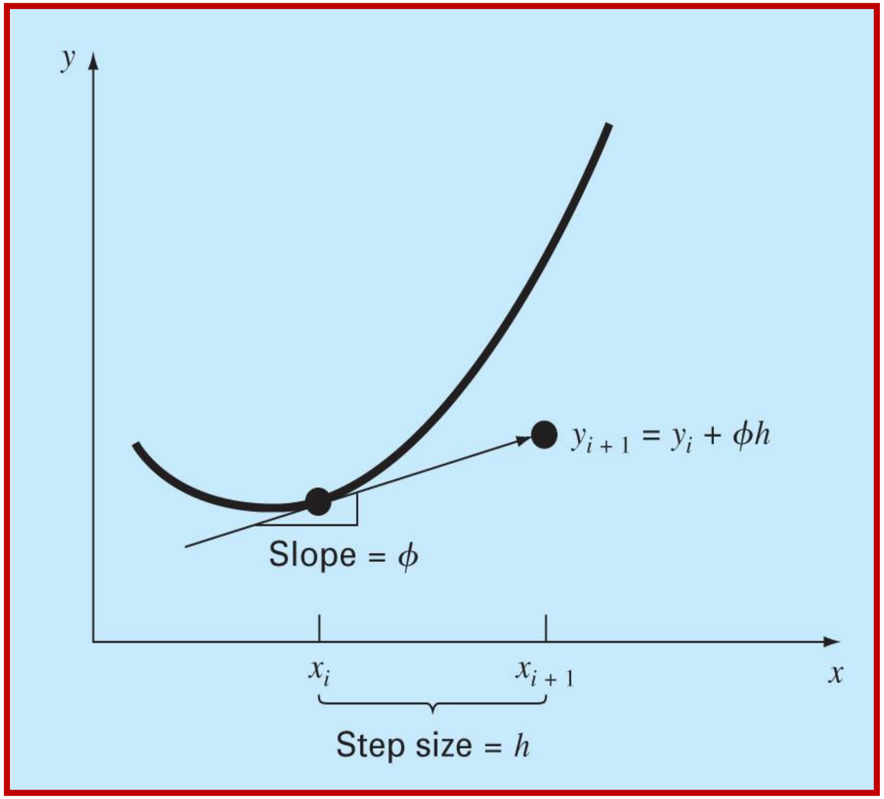
1) Euler's method

2) Improved Euler's methods

 - Heun's method

 - The midpoint

3) Runge-Kutta (RK) methods



Slope at x_i is taken as approximation of average slope over the whole interval

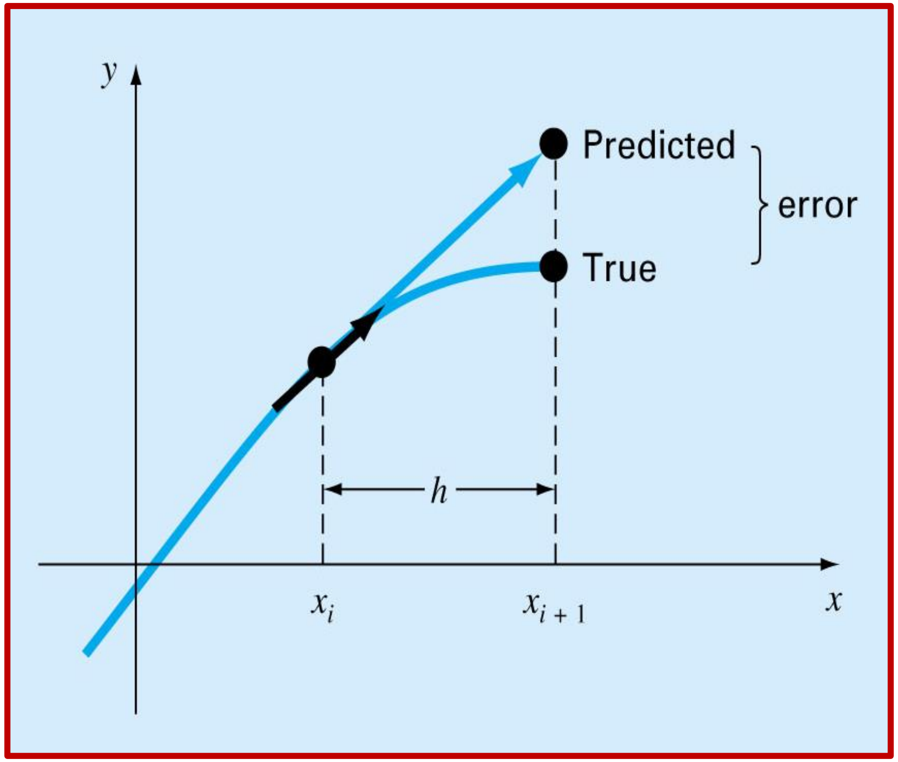
Euler's Method (Point-Slope Method)

- Solving ODEs of the form $dy/dx = f(x, y)$
- Explicit Euler's method, estimates the slope at x_i *as:*

$$\phi = f(x_i, y_i)$$

$$\boxed{y_{i+1} = y_i + f(x_i, y_i)\ h}$$

- New ***y*** is predicted using the slope to extrapolate linearly over the step size ***h***.



Error Analysis for Euler's Method

- Numerical solutions of ODEs involves two types of error:

Round-Off Errors

Total (Global) Truncation Error

Local truncation error + **Propagated truncation error**

the error created during a single step of method

produced during previous steps of method

Euler's Method & Taylor Series Relationship

- Local truncation error in Euler's method can be quantified by Taylor series expansion:

$$(y_{i+1}) = (y_i) + (y_i{}')h + (y_i{}'')\frac{h^2}{2!} + (y_i{}''')\frac{h^3}{3!} + \ldots \quad + O(h^{n+1})$$

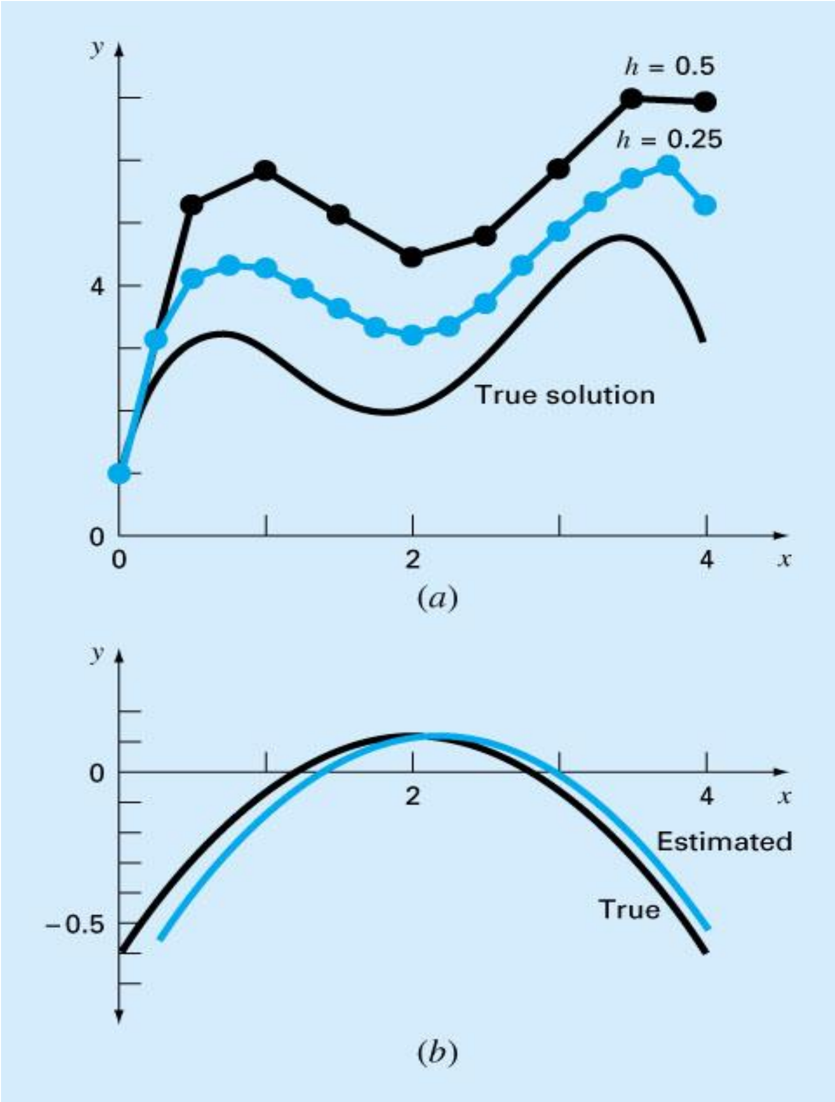
$h = x_{i+1} - x_i$ replace → $y_i' = f(x_i, y_i)$

$$E_a = \frac{f'(x_i, y_i)}{2!}h^2 \quad , \quad E_a = O(h^2)$$ Local Error order

- Complicated functions are not in form of simple polynomials
- Their derivatives not easily obtained to be used in Taylor series

Effect of Step Size in Error of Euler's Method

- Less error by choosing smaller step size
- If the solution to ODE is linear, the method will provide error free predictions as for a straight line the 2nd derivative would be zero.



Example 1. Euler's Method.

$$\frac{dy}{dx} = \frac{x^3 + 1}{y} \qquad y(0) = 2 \qquad 0 \le x \le 10, \qquad h = 0.5$$

$$y_{i+1} = y_i + f(x_i, y_i)\ h \qquad\qquad f(x_i, y_i) = \frac{x^3 + 1}{y}$$

i	x_i	y_i	$\frac{x_i^3 + 1}{y_i}$	y_{i+1}
0	x_0			
1	x_0+h			
2	x_1+h			

Example. Euler's Method. Continued (with h = 0.5)

i	*x*	*y*	*f(x,y)*	*y_new*	*y_exact*	*error*
0	0	2.00	0.50	2.25	2.00	0.00
1	0.5	2.25	0.50	2.50	2.24	0.01
2	1	2.50	0.80	2.90	2.55	0.05
3	1.5	2.90	1.51	3.65	3.09	0.19
4	2	3.65	2.46	4.89	4.00	0.35
17	8.5	50.26	12.24	56.38	51.29	1.03
18	9	56.38	12.95	62.85	57.47	1.09
19	9.5	62.85	13.66	69.68	64.00	1.14
20	10	69.68	14.37	76.86	70.88	1.20

Example. Euler's Method. Continued (with h = 0.25)

i	*x*	*y*	*f(x,y)*	*y_new*	*y_exact*	*error*
0	0	2.00	0.50	2.25	2.00	0.00
1	0.25	2.25	0.45	2.36	2.12	0.13
2	0.5	2.36	0.48	2.48	2.24	0.12
3	0.75	2.48	0.57	2.63	2.38	0.10
4	1	2.63	0.76	2.82	2.55	0.08
37	9.25	60.13	13.18	63.42	60.69	0.56
38	9.5	63.42	13.53	66.81	64.00	0.57
39	9.75	66.81	13.89	70.28	67.39	0.59
40	10	70.28	14.24	73.84	70.88	0.60

Euler's Method (Implicit) (Iterative)

- Apply ODE at x_{i+1}
- dy/dx evaluated at x_{i+1}, *is* equal to $f(x_{i+1}, y_{i+1})$
- Approximate using backward difference

$$\boldsymbol{y_{i+1} = y_i + h\, f(x_{i+1}, y_{i+1})}$$

Use iterative solution to find y_i

<u>Implicit form</u>: y_{i+1} on both sides of equation

Example 2. Euler's Method (Iterative)

$$\frac{dy}{dx} = \frac{x^3 + 1}{y} \qquad y(0) = 2 \qquad 0 \leq x \leq 10, \qquad h = 0.5$$

$$\boldsymbol{y_{i+1} = y_i + h\, f\,(x_{i+1}, y_{i+1})} \qquad f\,(x_i, y_i) = \frac{x^3 + 1}{y}$$

Example 2. Method (Iterative). Continued

x	y_{exact}	y_{Euler}	$y_{Implicit\ Euler}$
0	2	2	2
1	2.55	2.5	2.63
2	4.00	3.65	4.33
3	7.106	6.58	6.97
4	11.83	11.23	12.02

Notes

- Both implicit and explicit Euler's methods are first order accurate ($\boldsymbol{E} \propto \boldsymbol{h}$)

- Given $\boldsymbol{h}$, implicit Euler's method is more accurate

- But implicit Euler's method requires iteration at each step → time and resource consuming

Example 3. Heat Transfer Application. A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right), \quad \theta(0) = 1200K$$

Find the temperature at t = 480 seconds using Euler's method. Assume a step size of h = 240 seconds.

Euler’s Method Improvements

Heun’s Method

Midpoint Method

Why Improve Euler's Method?

- A fundamental source of error in Euler's method is that the derivative at the beginning of the interval is assumed to apply across the entire interval.
- Depending on the nature of function can either under- or over-estimate the change.
- Method is less successful for more complicated situations.
- Satisfactory results may achieve with smaller step sizes: slow the process time of algorithm

Strategies for Improvement

1) Averaging the derivatives in the initial and final points of interval and use "predictor-corrector" approach

   → called "Heun's method"

2) Predict value at the midpoint of interval and use it for slope calculation

   → called the "Midpoint method"

Both belong to larger class of solution technique

→ called the "Runge-Kutta method"

Heun's Method (Improved Euler's)

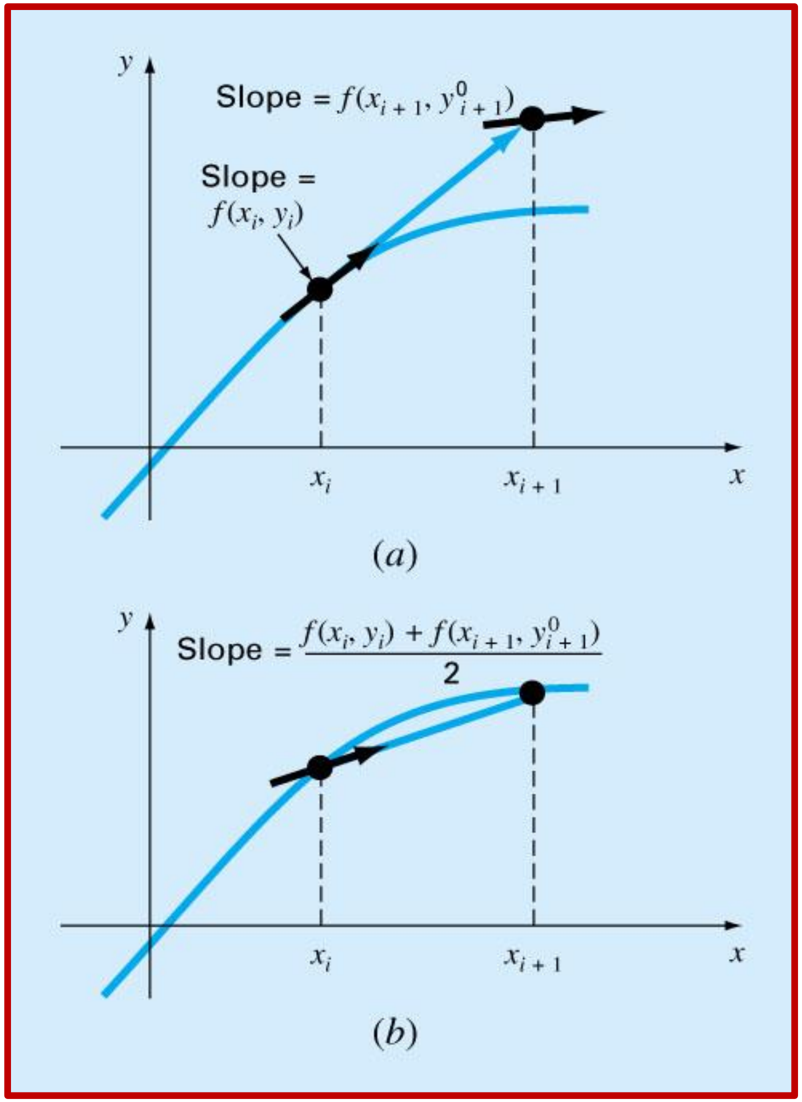
- This method improves the estimate of the slope by determining two derivatives for the interval at the 1) initial point, 2) end point.

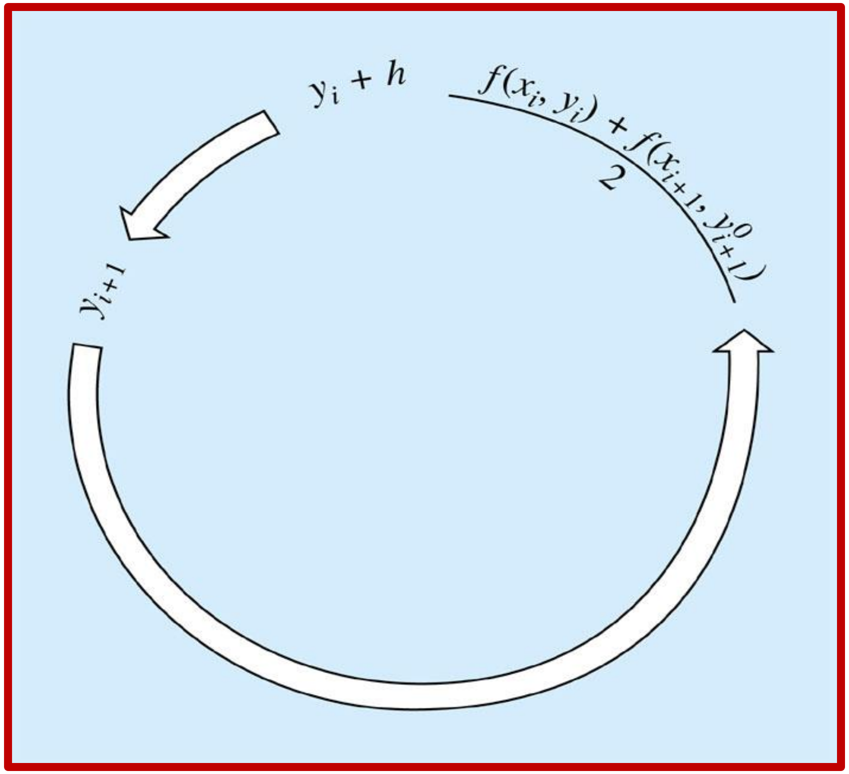
- The two derivatives are then averaged to obtain an improved estimate of the slope for the entire interval.

$$\text{Predictor}: \quad y_{i+1}^{0} = y_i + f(x_i, y_i)h$$

$$\text{Corrector}: \quad y_{i+1} = y_i + \frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{0})}{2}h$$

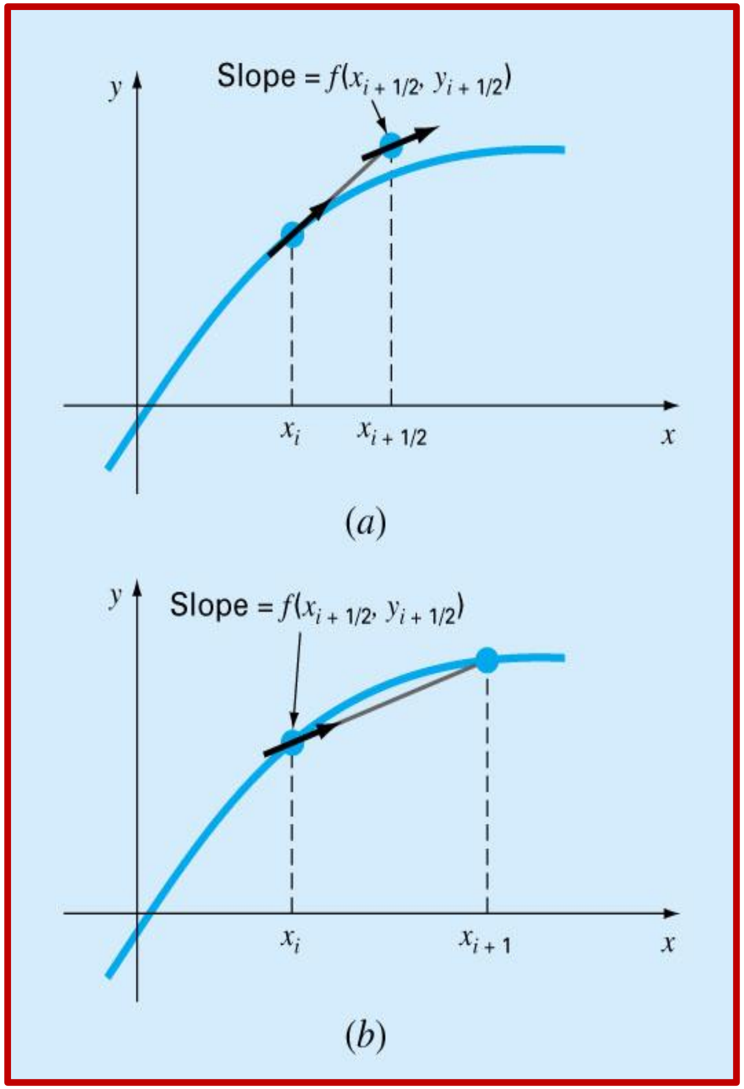
Heun's Method (Improved Euler's)





Iterating Corrector of Heun's method

The Midpoint (Improved Euler's) Method



- Uses Euler's method to predict a value of *y* at the midpoint of the interval:

$$y_{i+1} = y_i + f(x_{i+1/2}, y_{i+1/2})\ h$$

Example 4. Heun's Method (Improved Euler's)

$$\frac{dy}{dx} = \frac{x^3 + 1}{y} \quad y(0) = 2 \quad 0 \le x \le 10, \quad h = 0.5$$

$$f(x_i, y_i) = \frac{x^3 + 1}{y}$$

$$\text{Predictor}: \quad y_{i+1}^{0} = y_i + f(x_i, y_i)h$$

$$\text{Corrector}: \quad y_{i+1} = y_i + \frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{0})}{2}h$$

Example 4. Heun's Method (Improved Euler's)

x	y_{exact}	$y_{Heun's}$	$y_{Imp\ Euler}$
0	2	2	2
1	2.55	2.575	2.63
2	4.00	4.073	4.33
3	7.106	7.20	6.97
4	11.83	11.929	12.02

Part 7. Ordinary Differential Equations
Chapter 25. Runge-Kutta Methods

Lecture 25

Runge-Kutta Methods

25.3

Homeyra Pourmohammadali

Runge-Kutta Methods (RK)

- Runge-Kutta methods achieve the accuracy of a Taylor series approach without requiring the calculation of higher derivatives.

Runge-Kutta Methods (RK)

$$y_{i+1} = y_i + \phi(x_i, y_i, h)h$$

Increment function

$$\phi = a_1 k_1 + a_2 k_2 + \cdots + a_n k_n$$

$$a's = \text{constants}$$

k's are recurrence functions

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h)$$

$$k_3 = f(x_i + p_3 h, y_i + q_{21} k_1 h + q_{22} k_2 h)$$

$$\vdots$$

$$k_n = f(x_i + p_{n-1} h, y_i + q_{n-1} k_1 h + q_{n-1,2} k_2 h + \cdots + q_{n-1,n-1} k_{n-1} h)$$

p's and q's are constants

Runge-Kutta Methods (RK)

- Because each ***k*** is a functional evaluation, this recurrence makes RK methods efficient for computer calculations.

- Various types of RK methods can be devised by employing different number of terms in the increment function as specified by ***n***.

- First order RK method with $n=1$ is in fact Euler's method.

- Once n is chosen, values of a's, p's, and q's are evaluated by setting general equation equal to terms in a Taylor series expansion.

2^{nd} Order
Runge-Kutta Methods

2^{nd} Order Runge-Kutta Methods

$$y_{i+1} = y_i + (a_1\ k_1 + a_2\ k_2)\ h$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + p_1\ h, y_i + q_{11}\ k_1\ h)$$

- Values of a_1, a_2, p_1, and q_{11} are evaluated by setting the second order equation to Taylor series expansion to the second order term. (Check the notes for derivation)

2^{nd} Order Runge-Kutta Methods

- Three equations to evaluate four unknowns constants are derived.

$$\left.\begin{aligned} a_1 + a_2 &= 1 \\ a_2 p_1 &= \frac{1}{2} \\ a_2 q_{11} &= \frac{1}{2} \end{aligned}\right\}$$

A value is assumed for one of the unknowns to solve for the other three.

We can choose an infinite number of values for $a_2 \rightarrow$ there are an infinite number of 2^{nd}-order RK methods.

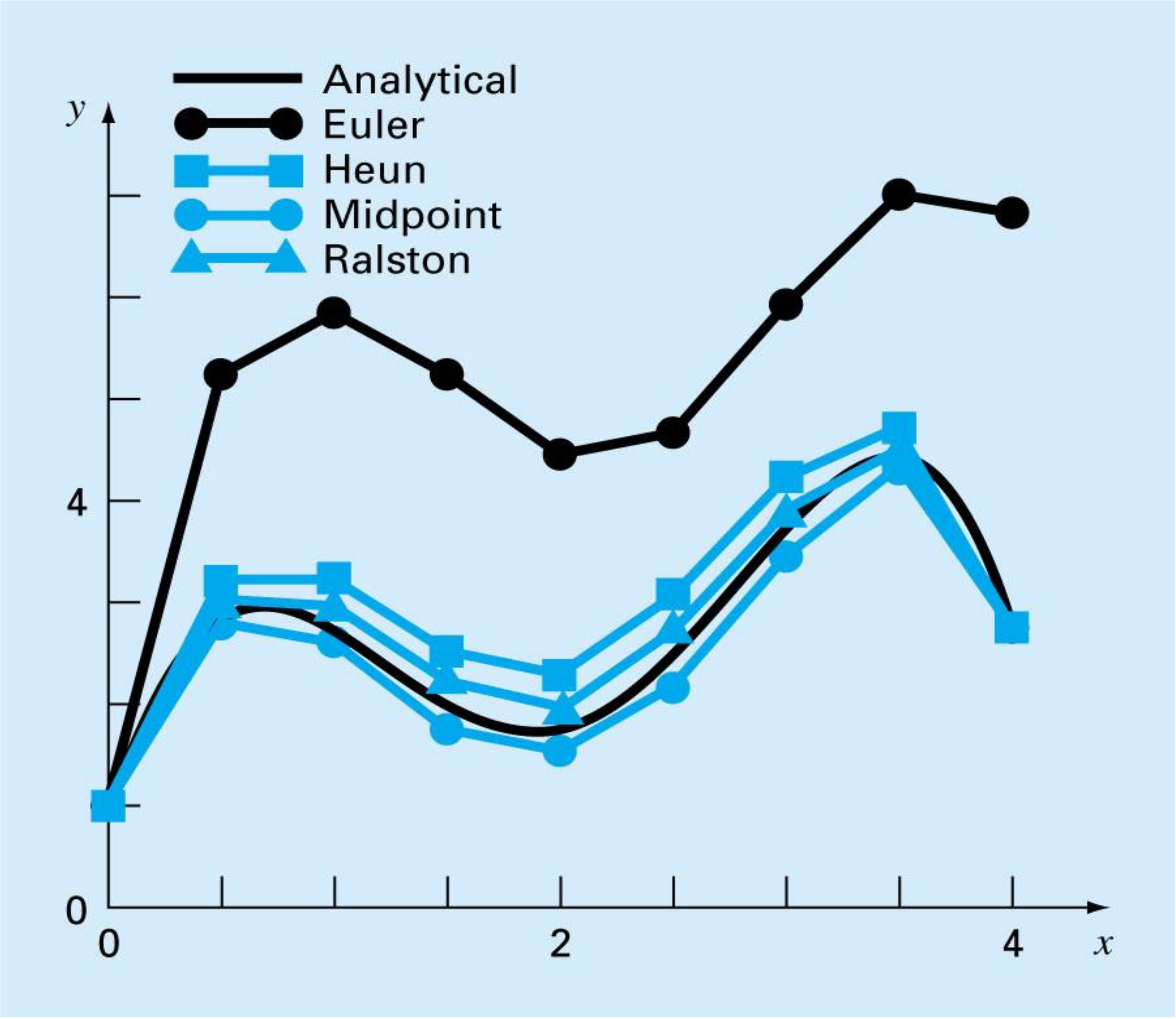
2^{nd} Order Runge-Kutta Methods

- Every version would yield exactly the same results if the solution to ODE were quadratic, linear, or a constant.

- They yield different results if the solution is more complicated (typically the case).

- Three of the most commonly used methods are:

$a_2 = 1/2$ → Huen Method with a Single Corrector
$a_2 = 1$ → The Midpoint Method
$a_2 = 2/3$ → Raltson's Method



Analytical
Euler
Heun
Midpoint
Ralston
y
4
0
0
2
4
x

4^{th} Order
Runge-Kutta Methods

4^{th} Order Runge-Kutta Method

- The most common form of the RK method in most books
- Computed using slope at 3 points

$$y_{i+1} = y_i + \frac{1}{6} \ (k_1 + 2k_2 + 2k_3 + k_4) \ h$$

$$k_1 = f(x_i, y_i)$$
$$k_2 = f(x_i + h/2, y_i + k_1 h/2)$$
$$k_3 = f(x_i + h/2, y_i + k_2 h/2)$$
$$k_4 = f(x_i + h, y_i + k_3 h)$$

Example 5. The 4th Order Runge-Kutta Method.

$$\frac{dy}{dx} = \frac{x^3+1}{y} \qquad y(0) = 2 \qquad 0 \le x \le 10, \qquad h = 0.5$$

$$f(x_i, y_i) = \frac{x^3+1}{y}$$

$$y_{i+1} = y_i + \frac{1}{6}\,(k_1 + 2k_2 + 2k_3 + k_4)\,h$$

$$k_1 = f(x_i, y_i)$$
$$k_2 = f(x_i + h/2, y_i + k_1 h/2)$$
$$k_3 = f(x_i + h/2, y_i + k_2 h/2)$$
$$k_4 = f(x_i + h, y_i + k_3 h)$$