Part 8. Partial Differential Equations
Chapter 29 & 30. Elliptic and Parabolic Equations

Lecture 33

Boundary Conditions

29.3, 30.2.2

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Elliptic and Parabolic PDEs

Laplace (2D) $$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Diffusion (1D) $$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

Boundary Conditions (BCs)

2D Laplace Equation:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Requires four BCs for <u>particular solution:</u>

$$(x = 0, y) \qquad (x = L, y)$$

$$(x, y = 0) \qquad (x, y = H)$$

Types of boundary conditions

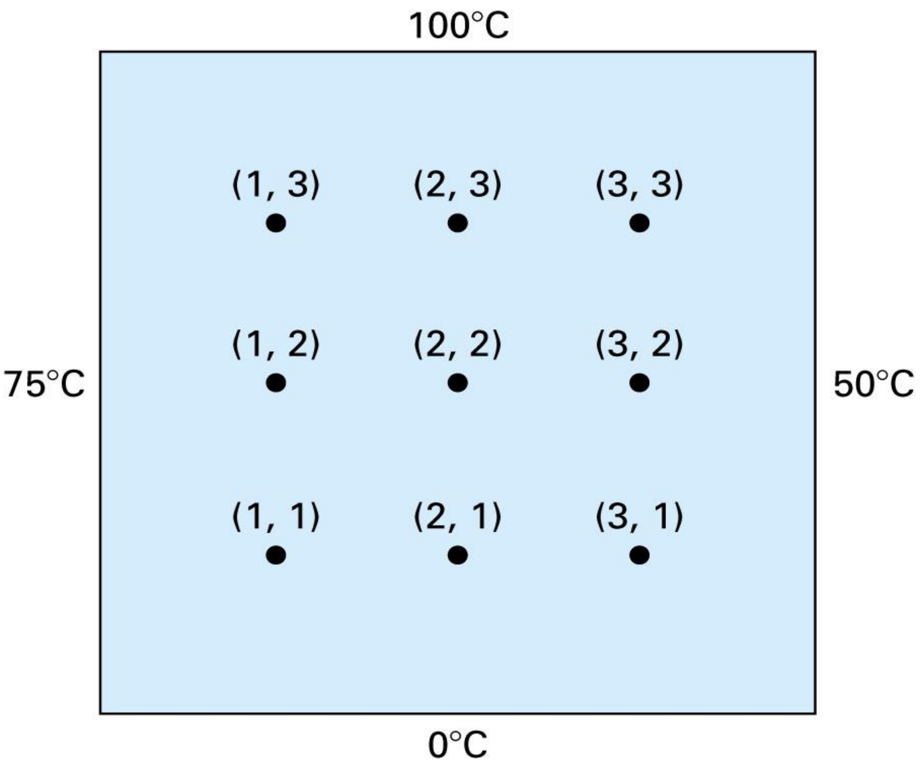
Dirichlet

Neumann

Dirichlet Condition

- Boundary conditions along the edges must be specified to obtain a unique solution.

- Dirichlet boundary condition is the simplest case is where the temperature at the boundary is set at a fixed value.

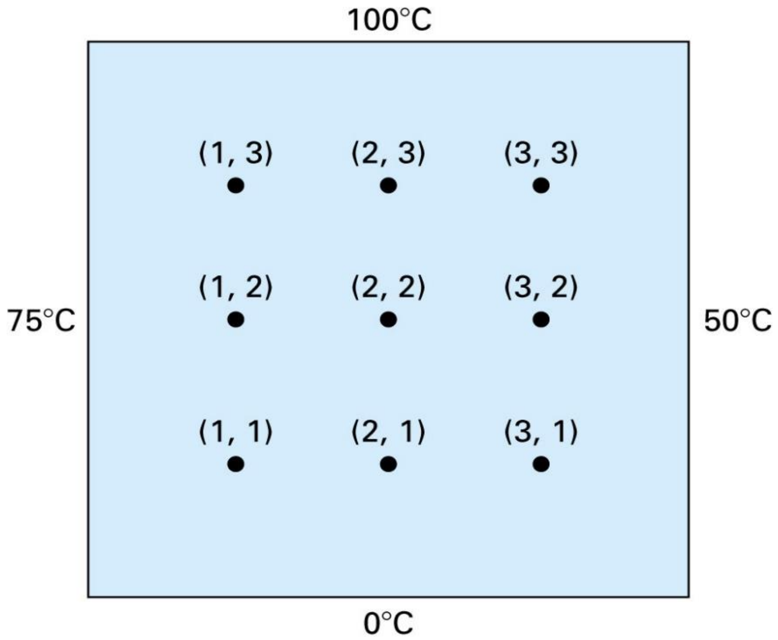


A heated plate where the boundary temperatures are held at constant level

Dirichlet Condition

- e.g. a balance for node (1,1) is:

$$T_{21} + T_{01} + T_{12} + T_{10} - 4T_{11} = 0$$
$$T_{01} = 75$$
$$T_{10} = 0$$
$$-4T_{11} + T_{12} + T_{21} + 75 = 0$$

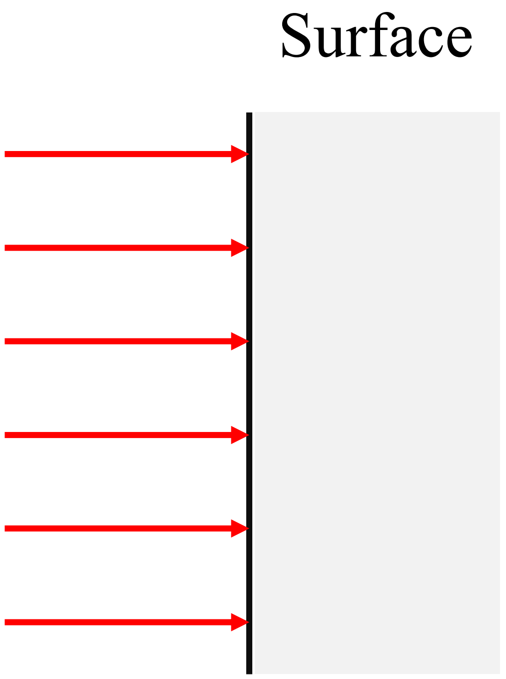


- Developing similar equations for other interior points result a set of simultaneous equations (can be solved using Gauss-Seidel)
- When Gauss-Seidel applied to PDEs, it is referred as Liebmann's method

What about Boundaries that are Irregularly Shaped or When BCs Are not Constant?

Neumann Condition
(Derivative Boundary Condition)

Fixed value of slope



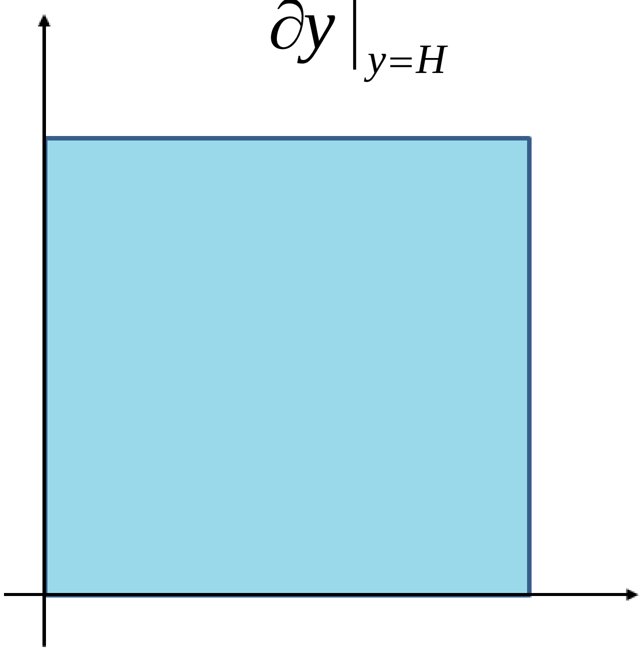
Q_0 Heat flux [W/m^2]

For the heated plate problem, heat flux is specified at the boundary, rather than the temperature.

If the edge is insulated, this derivative becomes zero.

Neumann Condition (Derivative Boundary Condition)

$$\left.\frac{\partial T}{\partial y}\right|_{y=H} = 0$$



$$\left.\frac{\partial T}{\partial y}\right|_{y=0} = 0$$

Special Case of Neumann BC is **Insulated Boundary**

Insulating a boundary means that heat flux and its gradients must be zero

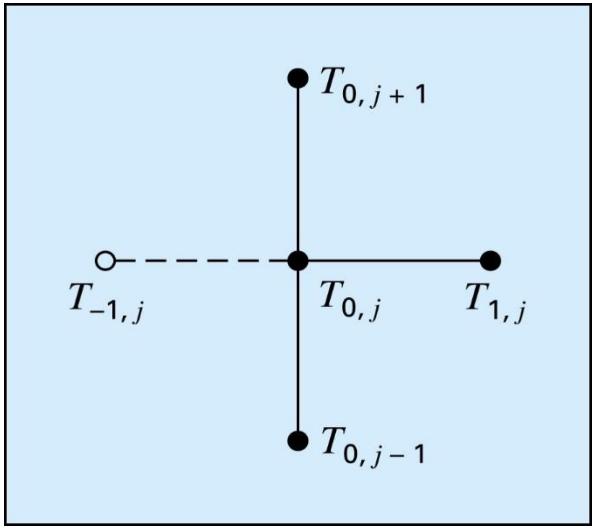
Neumann Condition (Derivative Boundary Condition)

$$T_{1,j} + T_{-1,j} + T_{0,j+1} + T_{0,j-1} - 4T_{0,j} = 0$$

$$\frac{\partial T}{\partial x} \cong \frac{T_{1,j} - T_{-1,j}}{2\Delta x}$$

$$T_{-1,j} = T_{1,j} - 2\Delta x \frac{\partial T}{\partial x}$$

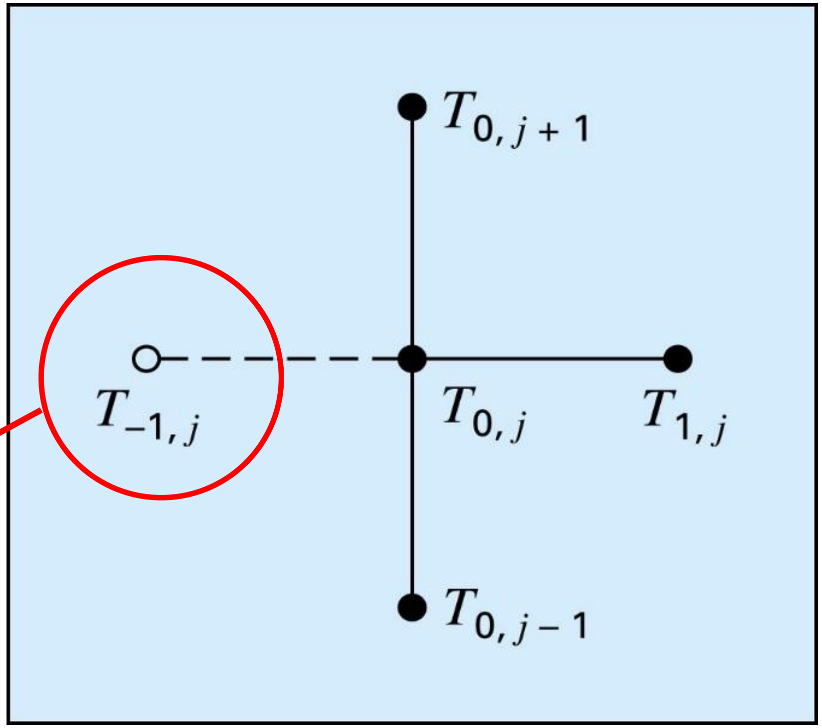
$$2T_{1,j} - 2\Delta x \frac{\partial T}{\partial x} + T_{0,j+1} + T_{0,j-1} - 4T_{0,j} = 0$$



- Thus, the derivative has been incorporated into the balance.
- Similar relationships can be developed for derivative boundary conditions at the other edges.

Neumann Condition
(Derivative Boundary Condition)

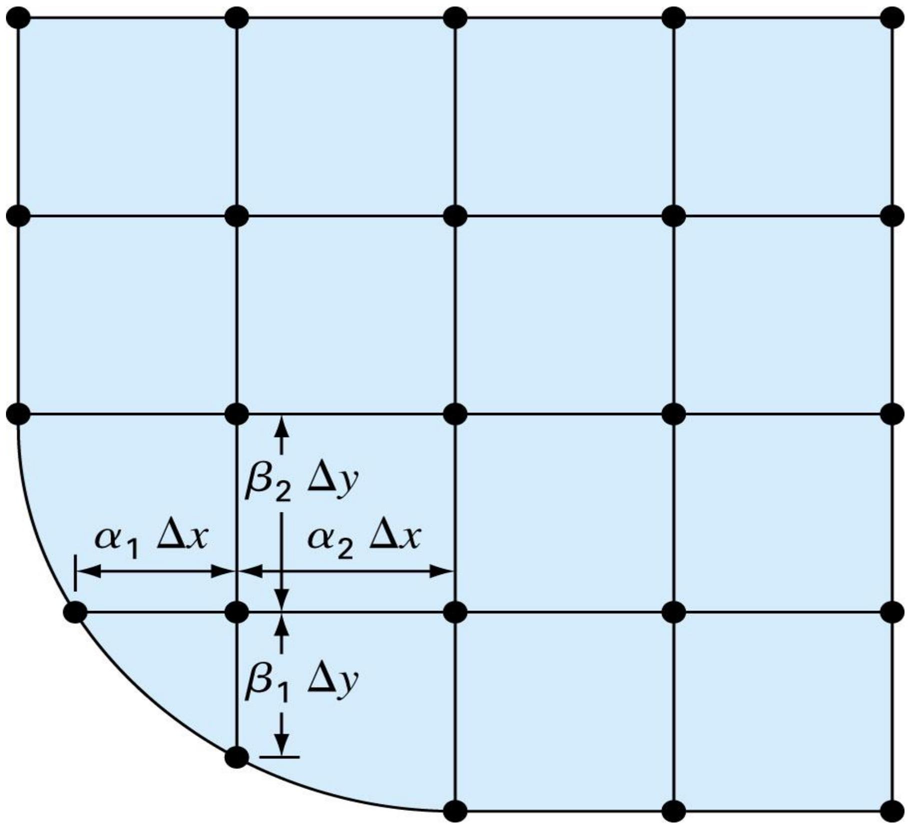
- Derivative boundary conditions can also be incorporated into parabolic equations.



Imaginary point i = -1 to characterize heat balance at the end nodes (with i=0)

$$T_i^{j+1} = T_i^j + \lambda \left(T_{i+1}^j - 2T_i^j + T_{i-1}^j \right)$$

Irregular Boundaries



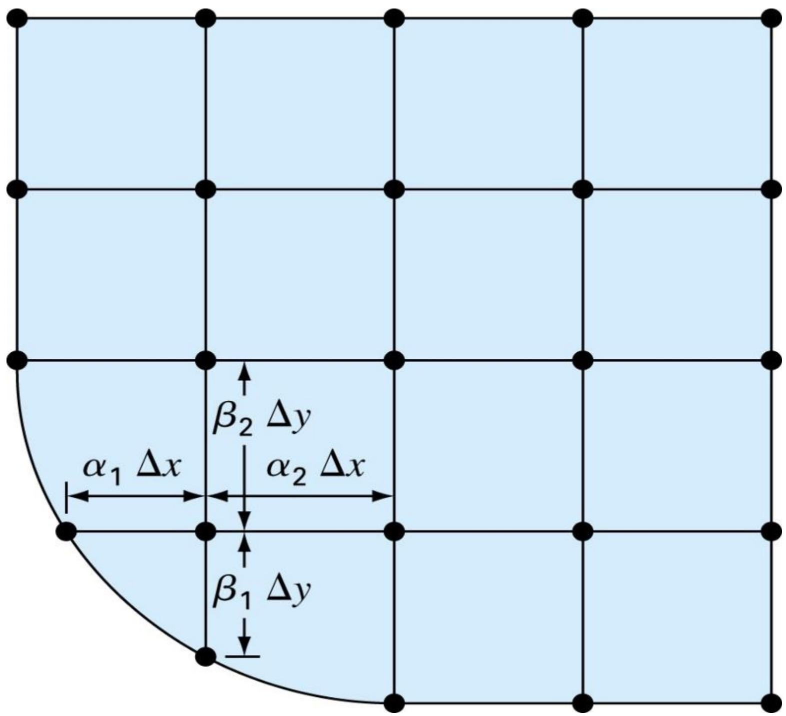
- Weighting coefficients are used to account for the non uniform spacing in the vicinity of the nonrectangular boundaries

Irregular Boundaries

First derivatives in the x direction can be approximated as:

$$\left(\frac{\partial T}{\partial x}\right)_{i-1,i} \cong \frac{T_{i,j} - T_{i-1,j}}{\alpha_1 \Delta x}$$

$$\left(\frac{\partial T}{\partial x}\right)_{i,i+1} \cong \frac{T_{i+1,j} - T_{i,j}}{\alpha_2 \Delta x}$$

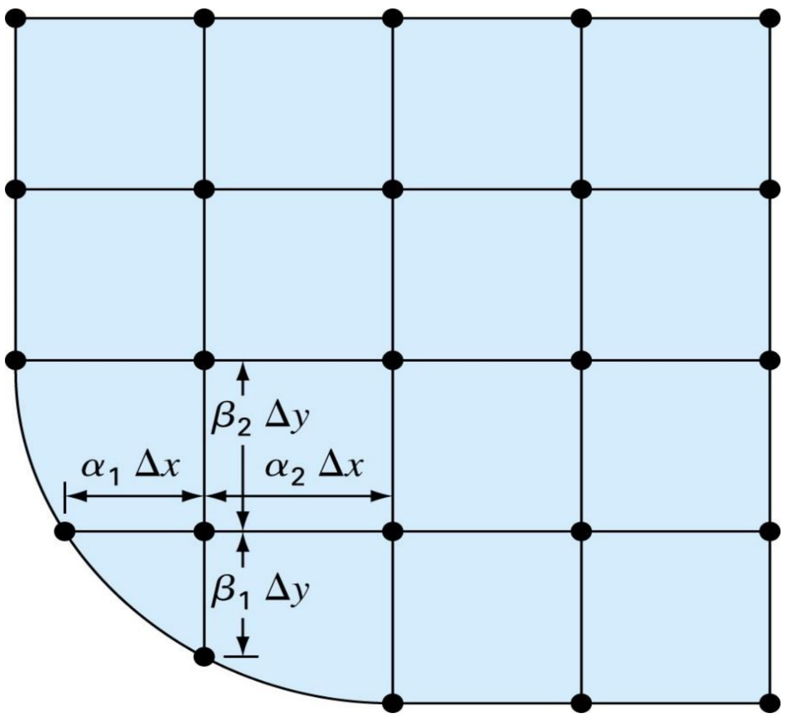


$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) = \frac{\left(\frac{\partial T}{\partial x}\right)_{i,i+1} - \left(\frac{\partial T}{\partial x}\right)_{i-1,i}}{\frac{\alpha_1 \Delta x + \alpha_2 \Delta x}{2}}$$

$$\frac{\partial^2 T}{\partial x^2} = 2\,\frac{\frac{T_{i,j} - T_{i-1,j}}{\alpha_1 \Delta x} - \frac{T_{i+1,j} - T_{i,j}}{\alpha_2 \Delta x}}{\frac{\alpha_1 \Delta x + \alpha_2 \Delta x}{2}}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{2}{\Delta x^2}\left[\frac{T_{i-1,j} - T_{i,j}}{\alpha_1(\alpha_1 + \alpha_2)} + \frac{T_{i+1,j} - T_{i,j}}{\alpha_2(\alpha_1 + \alpha_2)}\right]$$

Irregular Boundaries



- Similar equation can be developed in y-direction
- Final equation can be applied to any node that lies adjacent to an irregular, Dirichlet-type boundary.