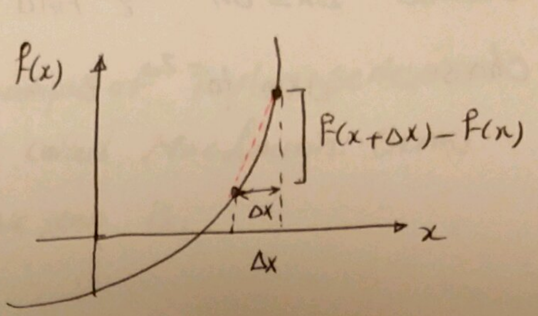
# Truncation Error – Example of Differentiation

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$




$$f'(x) \cong \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

↳ can be approximated as this.

$f(x) = 6x^2$

Find $f'(3) = ?$ First use the approximate formula & then use exact value:

Approximation

$$f'(x) \cong \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$\begin{cases} x = 3 \\ \Delta x = 0.2 \end{cases}$ → choose this & see what will be the value of $f'(3)$

$$f'(3) \cong \frac{f(3+0.2) - f(3)}{0.2} = \frac{f(3.2) - f(3)}{0.2} = \frac{6(3.2)^2 - 6(3)^2}{0.2} = 37.2$$

Then $f'(3)$, by choosing $\Delta x = 0.3$ & approximating the mathematical procedure of differentiation was 37.2.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{6(x+\Delta x)^2 - 6x^2}{\Delta x} = \frac{6(x^2 + 2(\Delta x)x + \Delta x^2) - 6x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{12 x \Delta x + 6\Delta x^2}{\Delta x} = 12x + 6\Delta x = 12x$$

$f'(x) = 12x$ → $f'(3) = 12(3) = 36$

You could drive this, based on what you learned about differentials:

$$\frac{d}{dx}(x^n) = n x^{n-1} \quad (n \neq 0) \rightarrow \frac{d}{dx}(6x^2) = 6(2x^{2-1}) = 12x$$

$\Rightarrow \begin{cases} f'(3) = 12 \times 3 = 36 & \text{(Exact Value)} \\ f'(3) = 37.2 & (\Delta x = 0.2) \text{ Approximate} \end{cases}$

Trunction Error $= \underset{\text{True Value}}{36} - \underset{\text{Approximated Value}}{37.2} = -1.2$

Choose $\Delta X = 0.1$ & find the T.E. ②

Choose $\Delta X = 10^{-20}$ & " " "

(very small $\Delta X$ is also not a good idea when doing numerical Differentiation)

# Taylor Series (Revisiting)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

one example of Taylor series which is called MacLaurin Series

Some other MacLaurin series that you might have seen is:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

## What is the Taylor Theorem?

Higher Degree approximation.

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \cdots + \frac{f^{(n)}(x)}{n!}h^n$$

- This series is an infinite series, where you have all the functions & their der
- It says that give me the value of $f(x)$, $f'(x)$, $f''(x)$, ---- and all other derivatives, then if you find $f(x+h)$ → you can find this. on the point that is $h$ away from $x$.
- All derivatives have to be contineous & exist in $[x, x+h]$

  if this is not met, Taylor series can not be the case.

$h$ → does not to be small. can be million, ---- but if the $h$ value is small the contribution of the following terms after $\frac{f''(x)}{2!}h^2$ are becoming smaller & smaller. (e.g. Smaller than 1)

## Taylor Series – Example

$f(4) = 125$ is given & $f'(4) = 74$, $f''(4) = 30$, $f'''(4) = 6$

& all other derivatives are zero.

Find $f(6) = ?$

$$f(6) = f(\overset{x}{4} + \overset{h}{2}) = f(4) + \frac{f'(4)(2)}{1!} + \frac{f''(4)(2)^2}{2!} + \frac{f'''(4)(2)^3}{3!} + 0$$

$$f(6) = 125 + \frac{(74)(2)}{1!} + \frac{(30)(4)}{2!} + \frac{(6)(8)}{3!}$$

$$f(6) = 125 + 148 + \frac{120}{2} + \frac{48}{6} =$$

$$= 125 + 148 + 60 + 8 = 341$$

$$\begin{array}{r} 125 \\ 148 \\ 68 \\ \hline 341 \end{array}$$

## Taylor series – Example of exp(x) or $e^x$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \cdots$$

$f(x) = e^x$ $\quad$ $x = 0 \to e^0 = 1$ $\quad$ $f(0) = 1$

$f'(x) = e^x$ $\quad$ $f'(0) = 1$

$f''(x) = e^x$ $\quad$ $f''(0) = 1$

$f'''(x) = e^x$ $\quad$ $f'''(0) = 1$

$\vdots$

$$f(0+h) = f(0) + f'(0)h + \frac{f''(0)}{2!}h^2 + \frac{f'''(0)}{3!}h^3 + \cdots$$

$$f(h) = 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots$$

if $h = x \to$

$$\boxed{f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x}$$

## Relative Approximate Error

Numerical Methods Guy

$f(x) = 7e^{0.5x}$, $f'(2) = ?$, $\Delta x = 0.3$

Use this formula: $f'(x) \simeq \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

give this function instead $\to$ $f(x) = 7 \times 10^{-6} e^{0.5x}$, $f'(2) = ?$, $\Delta x = 0.3$

| | $f'(2)$, $\Delta x = 0.3$ (using $\Delta x = 0.3$) | $f'(2)$, $\Delta x = 0.15$ (using $\Delta x = 0.15$) |
|---|---|---|
| $f(x) = 7e^{0.5x}$ | 10.265 | 9.8799 |
| $f(x) = 7 \times 10^{-6} e^{0.5x}$ | $10.265 \times 10^{-6}$ | $9.8799 \times 10^{-6}$ |

The approximate error differ by $10^{-6}$

we need to define the relative approximate error.

$$\epsilon_a = \frac{\text{Approximate Error}}{\text{current approximation}}$$

$f(x) = 7e^{0.5x}$

$f'(2) = ?$

$\Delta x = 0.3$, $\Delta x = 0.15$

$E_a = 9.8799 - 10.265 = -0.38474$

$$\epsilon_a = \frac{-0.38474}{9.8799} = -0.0389 = -3.89\%$$

$|\epsilon_a| = 0.0389$ or $3.89\%$

if $f(x) = 7\times10^{-6}\, e^{0.5x}$ & $\Delta x = 0.3$ & $\Delta x = 0.15$

$$E_a = 9.8799\times10^{-6} - 10.265\times10^{-6} = -0.38474\times10^{-6}$$

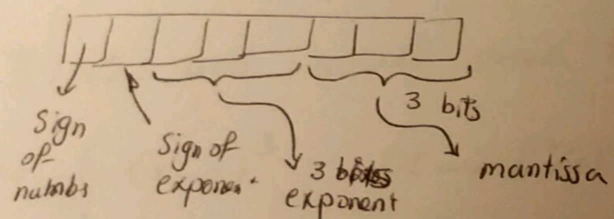
But the relative error is!

$$\epsilon_a = \frac{-0.38474\times10^{-6}}{9.8799\times10^{-6}} = -0.038942 \text{ or } -3.8942\%$$

Almos thE Same as the previous one.

Example of Floating Representation!

8 bit word




$(-13.9)_{10} \longrightarrow$ Repesent in floating-point

$(13)_{10} = (1101)_2$ (Homework)

$(0.9)_{10} \cong (0.11100)_2$ "

$$(13.9)_{10} \cong -(1101.11100)_2 = -(1.101111)_2$$

$$= (1.10111100)_2 \times 2^{3}$$

$$\simeq -(1.101)_2 \times 2^{3}$$

mantissa has only space for 3

$(3) = (11)_2 = (011)_2$ (HW)

$(-13.9)_{10} \simeq -(1.011)_2 \times 2^{(011)_2}$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 1 | 1 | 0 | 1 | 1 |

It is

positive

mantissa

# Trunction Error & Taylor series example.

→ Let $f(x) = x^3$ → Find $P_2(x)$ with $x_0 = 0$

a

We know that → $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$

In this case → $a = 0$, we find $f'(x), f''(x)$ & evaluate at $x_0 = 0$.

$f(x_0) = 0^3 = 0$ $\qquad x_0 = 0$

$f'(x_0) = 3(0)^2 = 0$

$f''(x_0) = 6(0) = 0$

$P_2(x) = f(0) + f'(0)(x-0) + f''(0)(x-0)/2!$

b Find $P_2(0.5)$ and actual error to approximate.

$R_2(0.5) = \frac{6}{3!}(x-0)^3$ $\qquad$ 1st → $f'''(0) = 6$

$R_2(0.5) = (0.5)^3 = 0.125$

actual error

$f(0.5) = (0.5)^3 = 0.125$

$P_2(0.5) = (0)^3 + 0(0.5) + \frac{0(0.5)^2}{2!} + R_2(0.5)$

$f(x) = P(x) + R(x)$

$f(0.5) = 0 + 0.125$

c Let $x_0 = 1$ → $P_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2}$

@ $x_0 = 1$ → $R_2(x) = \frac{6}{3!}(\xi)(x-1)^3$

again $f(0.5) = 0.125$ actual $\qquad P_2(0.5) = 1 + 3(-0.5)$ @ $x_0 = 1$

$\begin{cases} P(0.5) = 1 + (-1.5) + 3(2.5) \\ P(0.5) = 1.75 + (-1.5) = 0.25 \end{cases}$ $\qquad$ Then the Error:

$f(x) = \frac{f'''(\xi(x))}{3!}(x-1)^3$ $\qquad f(0.5) = \frac{6}{3!}(0.5-1)^3 = -0.125$

$f(0.5) = 0.125 = 0.25 - 0.125 = 0.125$ ✓