SYDE252 - lecture notes

09/01/18

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Systems Design Engineering
note: some material (figures) borrowed from various sources

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6. DFT (Discrete Fourier Transform)

09/11/18

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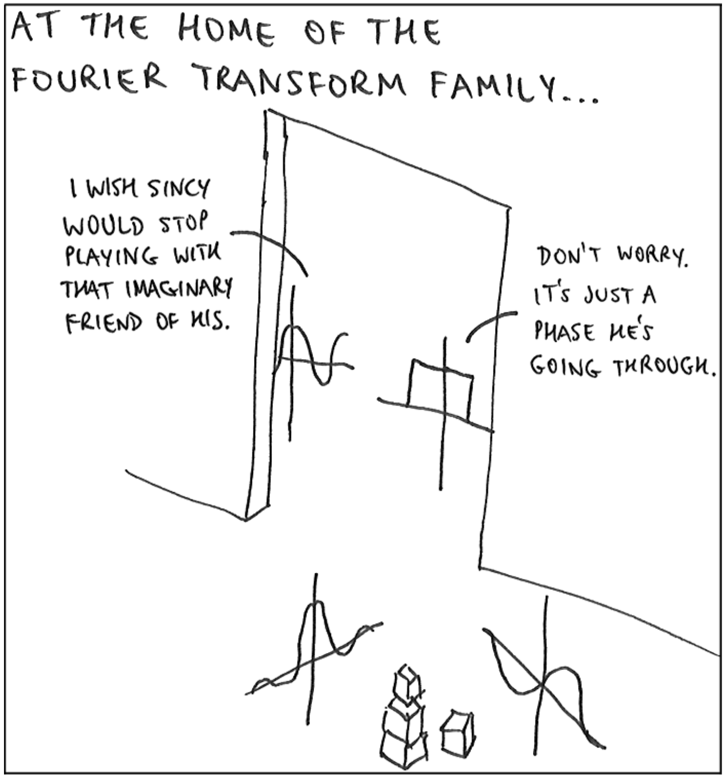
inspiration



https://dsp.stackexchange.com/questions/37119/dsp-or-signal-image-data-processing-jokes

discrete vs. cts. fourier transform

	Discrete Time:	Continuous Time:
Periodic:	$x[n] = x[n + N_0]$	$x(t) = x(t + T_0)$
Fourier Series:	$x[n] = \sum_{k=0}^{N_0-1} \mathbf{x}_k e^{jk\Omega_0 n}$	$x(t) = \sum_{n=-\infty}^{\infty} \mathbf{x}_n e^{jn\omega_0 t}$
Computing Coefficient:	$\mathbf{x}_k = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n]\, e^{-jk\Omega_0 n}$	$\mathbf{x}_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)\, e^{-jn\omega_0 t}\, dt.$
# Terms:	N_0	Infinity, in general

discrete vs. cts. fourier series

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# Terms:	N_0	Infinity, in general

discrete fourier series (DTFS) - example

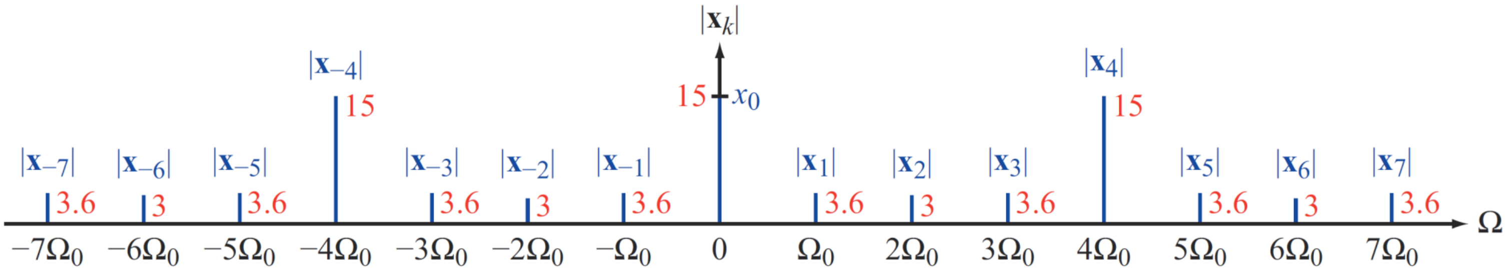
Compute the DTFS of the periodic signal

$$x[n] = \{\ldots, \underline{24}, 8, 12, 16, 24, 8, 12, 16, \ldots\}.$$

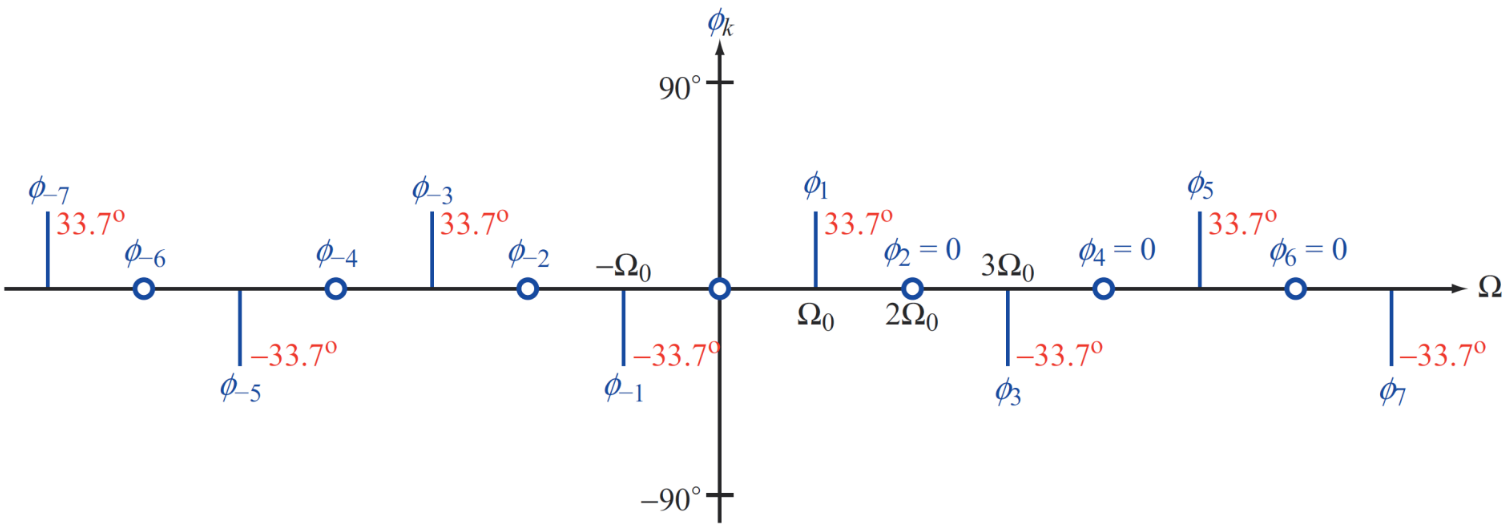
discrete fourier series (DTFS) - example

Magnitude and phase line spectra of x[n] on next slide.

discrete fourier series (DTFS) - example



(b) Magnitude line spectrum



(c) Phase line spectrum

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discrete vs. Cts. Fourier transform

Discrete-Time

$$\mathbf{X}(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{\Omega_1}^{\Omega_1+2\pi} \mathbf{X}(e^{j\Omega})\, e^{j\Omega n}\, d\Omega,$$

Continuous-Time

$$\hat{\mathbf{X}}(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\mathbf{X}}(\omega)\, e^{j\omega t}\, d\omega$$

If x[n] and x(t) are causal signals, then:

$$\mathbf{X}(e^{j\Omega}) = \mathbf{X}(\mathbf{z})\Big|_{\mathbf{z}=e^{j\Omega}}$$

$$\hat{\mathbf{X}}(\omega) = \mathbf{X}(\mathbf{s})\Big|_{\mathbf{s}=j\omega}$$

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discrete time fourier transform (DTFT)

Define x(t) from x[n] using $x(t) = \sum^{\infty} x[n]\,\delta(t-n).$

Fourier transform of x(t): $\hat{\mathbf{X}}(\omega) = \sum_{n=-\infty}^{\infty} x[n]\,\mathcal{F}[\delta(t-n)] = \mathbf{X}(e^{j\Omega})$

1. DTFT is continuous Fourier transform of chain-of-impulses.
2. DTFT is a continuous- Fourier series expansion of
3. DTFT is periodic in with period 2π. CRUCIAL property!
4. Other DTFT properties follow from continuous-time Fourier transform properties (see the next slide).

DTFT pairs

Discrete-time Fourier transform (DTFT) pairs.

	$x[n]$		$\mathbf{X}(e^{j\Omega})$	Condition
1.	$\delta[n]$	$\leftrightarrow$	1	
1a.	$\delta[n-m]$	$\leftrightarrow$	$e^{-jm\Omega}$	$m =$ integer
2.	1	$\leftrightarrow$	$2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)$	
3.	$u[n]$	$\leftrightarrow$	$\dfrac{e^{j\Omega}}{e^{j\Omega}-1} + \sum_{k=-\infty}^{\infty} \pi\delta(\Omega - 2\pi k)$	
3a.	$\mathbf{a}^n\, u[n]$	$\leftrightarrow$	$\dfrac{e^{j\Omega}}{e^{j\Omega}-\mathbf{a}}$	$\lvert\mathbf{a}\rvert < 1$
3b.	$n\mathbf{a}^n\, u[n]$	$\leftrightarrow$	$\dfrac{\mathbf{a}e^{j\Omega}}{(e^{j\Omega}-\mathbf{a})^2}$	$\lvert\mathbf{a}\rvert < 1$
4.	$e^{j\Omega_0 n}$	$\leftrightarrow$	$2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi k)$	
5.	$\mathbf{a}^{-n}\, u[-n-1]$	$\leftrightarrow$	$\dfrac{\mathbf{a}e^{j\Omega}}{1-\mathbf{a}e^{j\Omega}}$	$\lvert\mathbf{a}\rvert < 1$
6.	$\cos(\Omega_0 n)$	$\leftrightarrow$	$\pi \sum_{k=-\infty}^{\infty} [\delta(\Omega - \Omega_0 - 2\pi k) + \delta(\Omega + \Omega_0 - 2\pi k)]$	
7.	$\sin(\Omega_0 n)$	$\leftrightarrow$	$\dfrac{\pi}{j} \sum_{k=-\infty}^{\infty} [\delta(\Omega - \Omega_0 - 2\pi k) - \delta(\Omega + \Omega_0 - 2\pi k)]$	
8.	$\mathbf{a}^n \cos(\Omega_0 n + \theta)\, u[n]$	$\leftrightarrow$	$\dfrac{e^{j2\Omega}\cos\theta - \mathbf{a}e^{j\Omega}\cos(\Omega_0 - \theta)}{e^{j2\Omega} - 2\mathbf{a}e^{j\Omega}\cos\Omega_0 + \mathbf{a}^2}$	$\lvert\mathbf{a}\rvert < 1$
9.	$\mathrm{rect}\left[\dfrac{n}{N}\right] = u[n+N] - u[n-1-N]$	$\leftrightarrow$	$\dfrac{\sin\left[\Omega\left(N+\frac{1}{2}\right)\right]}{\sin\left(\frac{\Omega}{2}\right)}$	
10.	$\dfrac{\sin[\Omega_0 n]}{\pi n}$	$\leftrightarrow$	$\sum_{k=-\infty}^{\infty} [u(\Omega + \Omega_0 - 2\pi k) - u(\Omega - \Omega_0 - 2\pi k)]$	

DTFT properties

Properties of the DTFT.

Property	$x[n]$		$\mathbf{X}(e^{j\Omega})$
1. Linearity	$k_1\ x_1[n] + k_2\ x_2[n]$	$\longleftrightarrow$	$k_1\ \mathbf{X}_1(e^{j\Omega}) + k_2\ \mathbf{X}_2(e^{j\Omega})$
2. Time shift	$x[n - n_0]$	$\longleftrightarrow$	$\mathbf{X}(e^{j\Omega})\ e^{-jn_0\Omega}$
3. Frequency shift	$x[n]\ e^{j\Omega_0 n}$	$\longleftrightarrow$	$\mathbf{X}(e^{j(\Omega-\Omega_0)})$
4. Multiplication by n (frequency differentiation)	$n\ x[n]$	$\longleftrightarrow$	$j\dfrac{d\mathbf{X}(e^{j\Omega})}{d\Omega}$
5. Time Reversal	$x[-n]$	$\longleftrightarrow$	$\mathbf{X}(e^{-j\Omega})$
6. Time convolution	$x_1[n] * x_2[n]$	$\longleftrightarrow$	$\mathbf{X}_1(e^{j\Omega})\ \mathbf{X}_2(e^{j\Omega})$
7. Frequency convolution	$x_1[n]\ x_2[n]$	$\longleftrightarrow$	$\dfrac{1}{2\pi}\ \mathbf{X}_1(e^{j\Omega}) * \mathbf{X}_2(e^{j\Omega})$
8. Conjugation	$x^*[n]$	$\longleftrightarrow$	$\mathbf{X}^*(e^{-j\Omega})$
9. Parseval's theorem	$\displaystyle\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2$	$=$	$\dfrac{1}{2\pi}\displaystyle\int_{\Omega_1}^{\Omega_1+2\pi} \lvert\mathbf{X}(e^{j\Omega})\rvert^2\ d\Omega$
10. Conjugate symmetry	$\mathbf{X}^*(e^{j\Omega}) = \mathbf{X}(e^{-j\Omega})$		

example - computing DTFT

Compute the DTFT of

(a) $x_1[n] = \{3, 1, \underline{4}, 2, 5\}$,

(b) $x_2[n] = \left(\frac{1}{2}\right)^n \; u[n]$,

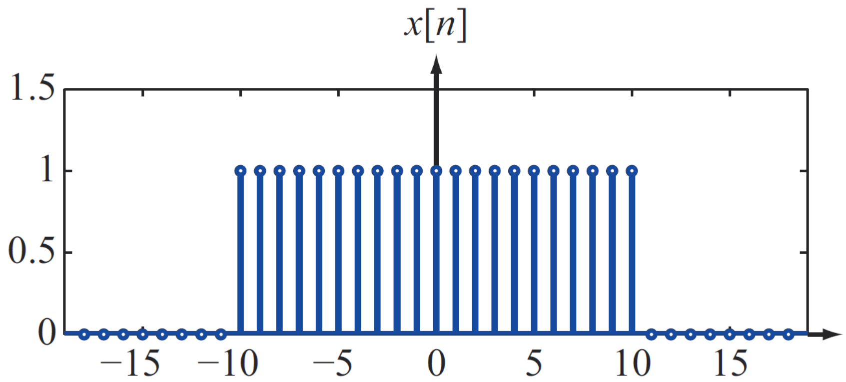
(c) $x_3[n] = 4\sin(0.3n)$,

Easiest way to do (a) and (b) is to use
Note that all three DTFTs are periodic with period 2π.

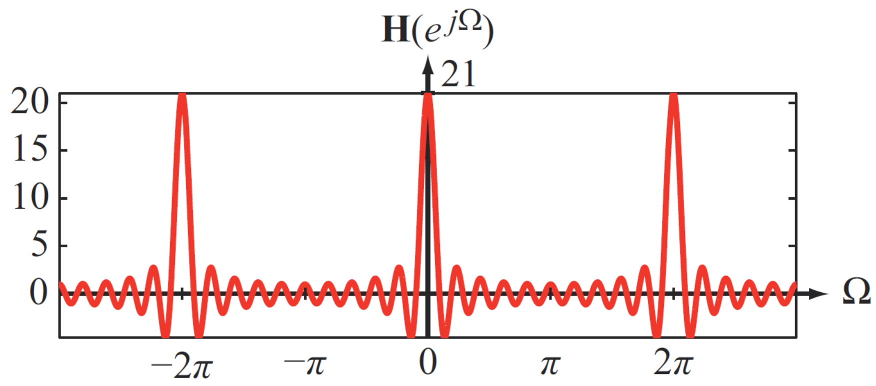
example (discrete sinc) - computing DTFT [pulse]

$$x[n] = \begin{cases} 1 & \text{for } |n| \le N, \\ 0 & \text{for } |n| > N. \end{cases}$$

$$\mathbf{X}(e^{j\Omega}) = \frac{\sin\left[\Omega\left(N+\frac{1}{2}\right)\right]}{\sin\left(\frac{\Omega}{2}\right)}$$



(a) $x[n] = \text{rect}(n/10)$



(b) $\mathbf{H}(e^{j\Omega})$

discrete fourier transform (DFT)

DFT:

$$\mathbf{X}_k = \sum_{n=0}^{N_0-1} x[n]\, e^{-jk\Omega_0 n}$$

$$x[n] = \frac{1}{N_0} \sum_{k=0}^{N_0-1} \mathbf{X}_k e^{jk\Omega_0 n},$$

$$\Omega_0 = \frac{2\pi}{N_0} \qquad \mathbf{X}_k = N_0 \mathbf{x}_k.$$

DTFS:

$$\mathbf{x}_k = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n]\, e^{-jk\Omega_0 n},$$

$$x[n] = \sum_{n=0}^{N_0-1} \mathbf{x}_k e^{jk\Omega_0 n},$$

$$\mathbf{X}_k = \mathbf{X}(e^{j\Omega})\Big|_{\Omega=k\Omega_0} = \mathbf{X}(\mathbf{z})|_{\mathbf{z}=e^{jk\Omega_0}}$$

DFT **DTFT** **z-transform**

discrete fourier series (DTFS) - example ctn. Parsevals

Average power is the same
In time or frequency domain

$$\frac{1}{N_0}\sum_{n=0}^{N_0-1}|x[n]|^2 = \sum_{k=0}^{N_0-1}|\mathbf{x}_k|^2.$$

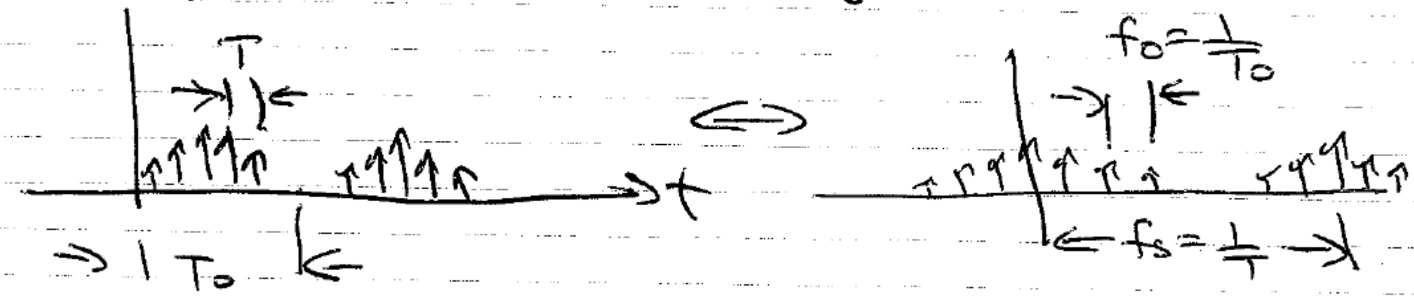
For the previous example, we have:

Time Domain:

Frequency Domain:

discrete fourier transform

– now sample the spectrum which is done by repeating copies of $x(t)$



\# of time samples in T_0

$$N_0 = \frac{T_0}{T}$$

\# of freq^n samples in f_s $\quad N_0' = \frac{f_s}{f_0}$

$$N_0 = \frac{T_0}{T} = \frac{1/f_0}{1/f_s} = \frac{f_s}{f_0} = N_0' \quad \left(\text{note } N_0 = N_0'\right)$$

picket fence effect: i/f between samples in frequency is missing

– improve resolution by increasing samples N_0 (or N_0')

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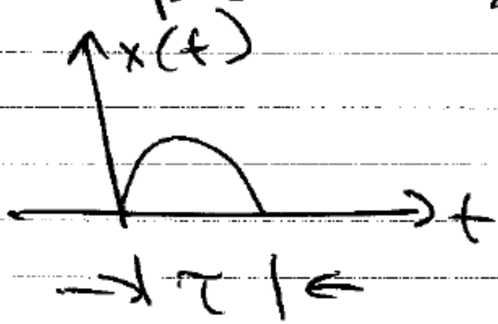
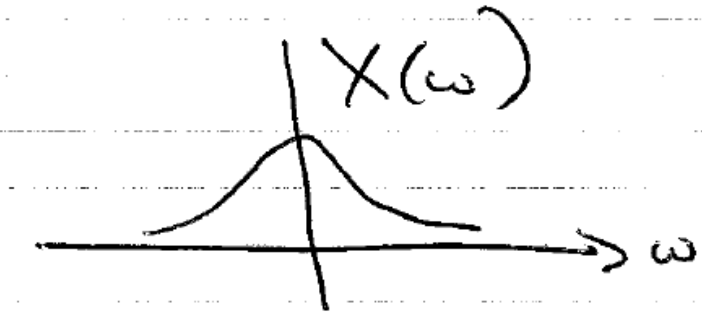
discrete fourier transform

- consider a time-limited signal $x(t)$ with spectrum $X(\omega)$



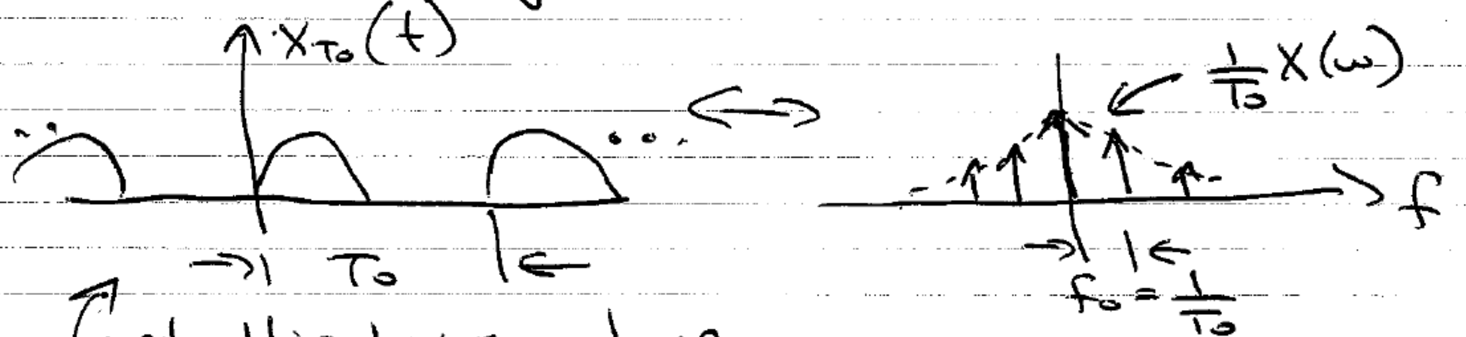
$\overset{FT}{\longleftrightarrow}$



by def^n $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt = \int_{0}^{\tau} x(t) e^{-j\omega t}\, dt$

discrete fourier transform

- make a periodic signal $x_{T_0}(t)$, the spectrum of this will be discrete (i.e. Fourier series)

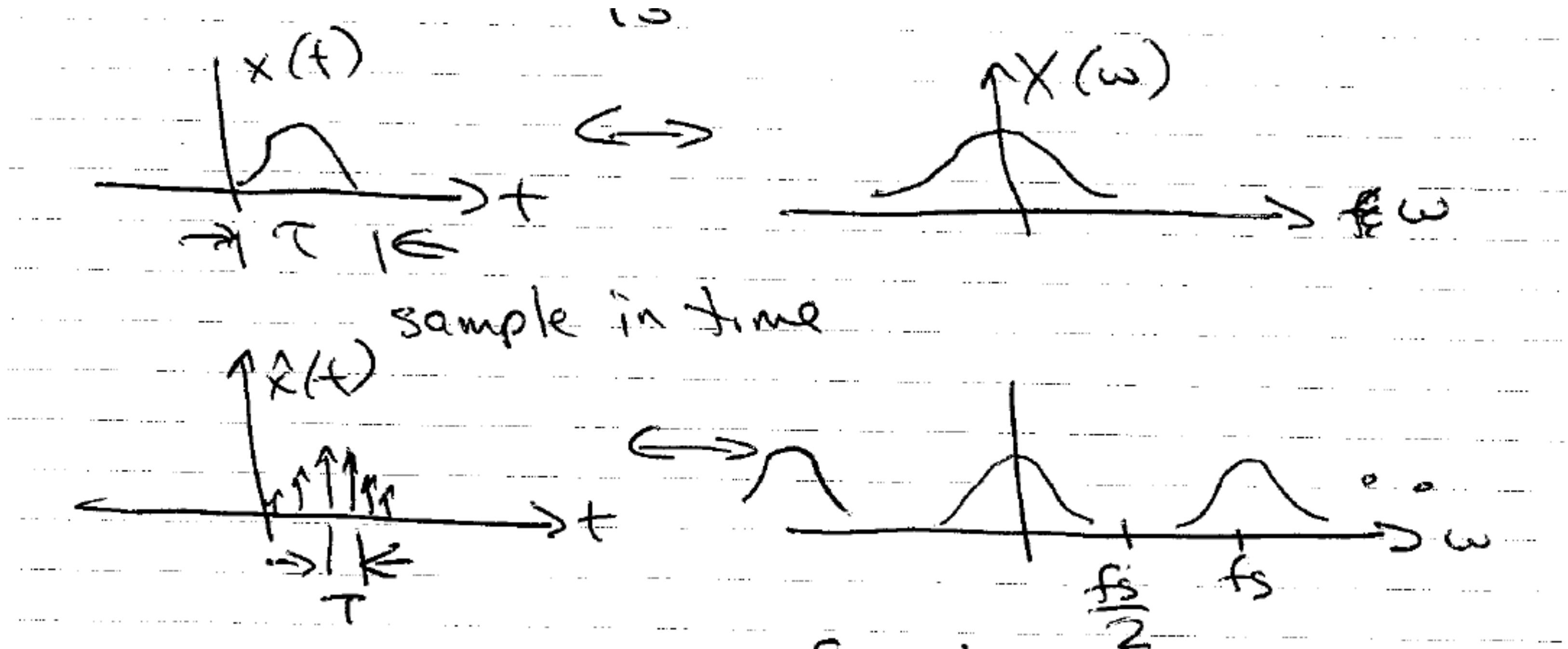


get this by convolving with impulse train in time

spectral sampling - which is multiplying by frequency impulse train

$$\omega_0 = \frac{2\pi}{T_0}$$

discrete fourier transform

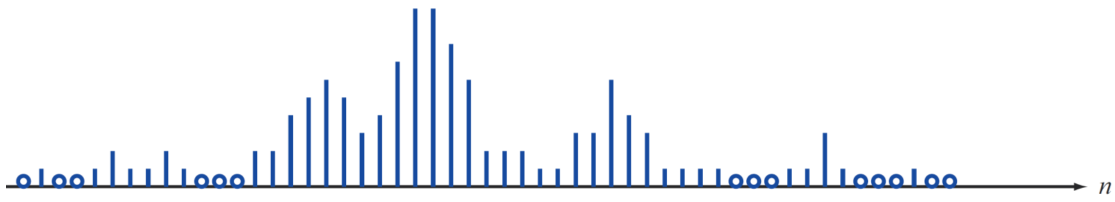


$$f_s = \frac{1}{T}$$

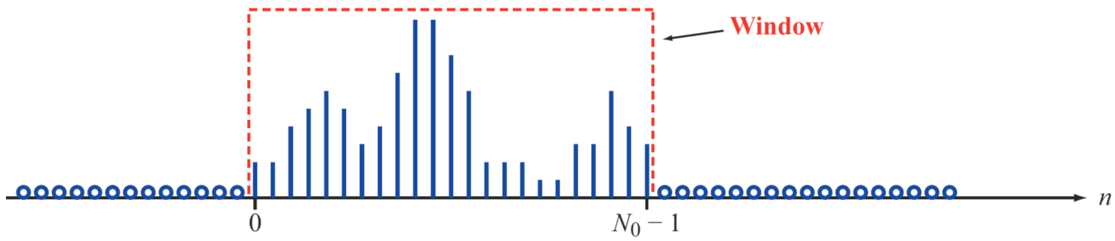
$$\omega = 2\pi f = \frac{2\pi}{T}$$

computing spectra of non periodic signals (dft & windowing)

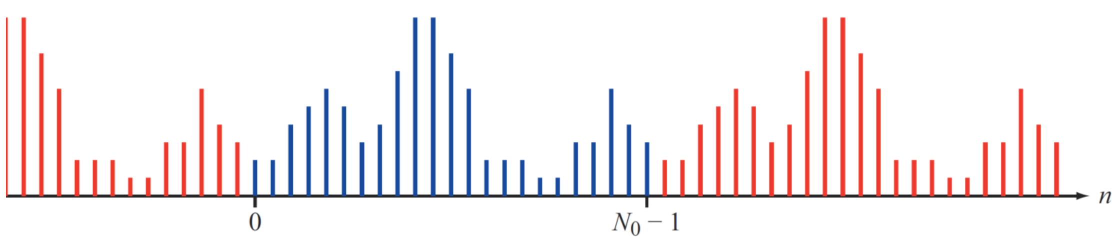
Multiply x[n] by a rectangular window that sets x[n] to 0 outside window, take the periodic extension of the result, and find its DTFS spectrum using the DFT.



(a) Original $x[n]$



(b) Windowed $x[n]$



(c) Periodic version

using DFT to compute convolutions

Goal: To compute the convolution of x[n] and y[n].
Lengths: x[n]:L, y[n]:M, x[n]*y[n]: N=L+M-1.

Procedure: Zero-pad x[n] and y[n] to lengths N by appending N-L zeros to x[n] and N-M to y[n].

DFT: x[n]*y[n] = IDFT{DFT{x[n]}DFT{y[n]}} where all DFTs and inverse DFTs (IDFTs) have lengths N.

using DFT to compute convolutions - example

Example: Compute convolution {4,5}*{1,2,3} using DFTs.

Solution: The convolution has length 2+3-1=4. Zero-pad the two signals to be convolved to {4,5,0,0} and {1,2,3,0}.
DFT{{4,5,0,0}}={9, 4-j5, -1, 4+j5}.
DFT{{1,2,3,0}}={6, -2-j2, 2, -2+j2}.

Multiply these: {54, -18+j2, -2, -18-j2}.
IDFT{{54, -18+j2, -2, -18-j2}}={4,13,22,15}.

So {4,5}*{1,2,3}={4,13,22,15}.

Fast Fourier Transform (FFT)

FFT: A fast algorithm for computing the
DFT: A transform for computing spectra.

Idea: Divide-and-conquer approach:
Break up a large DFT into smaller ones.
If N_0 is a power of 2, reduce computation from N_0^2 to $(N_0/2)\log_2 N_0$. If $N_0 = 512$, this reduces from 262,144 to 2304 multiplications!

Goal: Compute $\mathbf{X}_k = \sum_{n=0}^{N_0-1} x[n]\, W_{N_0}^{nk}$ where $W_{N_0} = e^{-j2\pi/N_0}$

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Fast Fourier Transform (FFT) formulae

Split x[n] into values at even and odd times:

$$x_e[n] = x[2n],$$

$$x_o[n] = x[2n+1]$$

Then use these formulae to break up N-point DFT into two N/2-point DFTs and some multiplications:

$$\mathbf{X}_k = \underbrace{\sum_{n=0}^{N_0/2-1} x_e[n]\, W_{N_0/2}^{nk}}_{N_0/2\text{-point DFT}} + W_{N_0}^{k} \underbrace{\sum_{n=0}^{N_0/2-1} x_o[n]\, W_{N_0/2}^{nk}}_{N_0/2\text{-point DFT}}$$

$$\mathbf{X}_{k+N_0/2} = \underbrace{\sum_{n=0}^{N_0/2-1} x_e[n]\, W_{N_0/2}^{nk}}_{N_0/2\text{-point DFT}} - W_{N_0}^{k} \underbrace{\sum_{n=0}^{N_0/2-1} x_o[n]\, W_{N_0/2}^{nk}}_{N_0/2\text{-point DFT}}$$

compute 4 pt DFT using FFT

Goal: Use the FFT to break down the 4-point DFT of {a,b,c,d} into two 2-point DFTs and multiplications by j.

Solution: 2-point DFT: $\mathbf{X}_0 = x[0] + x[1]$ and $\mathbf{X}_1 = x[0] - x[1]$

$x[n]$ ={a,b,c,d}. ={a,c}. ={b,d}. =-j.

2-point DFT of {a,c} is {a+c, a-c}.
2-point DFT of {b,d} is {b+d, b-d}.
4-point DFT of {a,b,c,d} is then computed from these as:
{(a+c)+(b+d), (a-c)-j(b-d), (a+c)-(b+d), (a-c)+j(b-d)}= result of direct computation of the 4-point DFT, which is {a+b+c+d, a-jb-c+jd, a-b+c-d, a+jb-c-jd}.

Ft, DFT

- **Fourier Transform**: $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$. The purpose of the Foruier transform is to convert an aperiodic time signal to the frequency domain. It deals with continuous time. We also explored Fourier series which deals with periodic continuous signals.

- **Discrete Fourier Transform**: $X[n\omega_0] = \sum_{n=0}^{N_0-1} x[n]e^{-jn\omega_0 T}$ where N_0 is the number of samples, T is the sample period and $\omega_0 = \frac{2\pi}{T}$. The purpose of the DFT is to convert finite discrete-time signal to the discrete frequency domain. We deal in discrete time, otherwise it is similar to the FT.


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