Part 8. Partial Differential Equations
Chapter 29. Finite Difference: Elliptic Equations

Lecture 29 & 30

The Laplace Equation and Solution Technique

29.1, 29.2

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Elliptic PDE's

- The general form for a 2nd order linear PDE with 2 independent variables (x, y) and one dependent variable (u) is

$$\boxed{A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0}$$

- Criteria of elliptic equations: $B^2 - 4AC < 0$

- Laplace equation given by $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$

where A=1, B=0 , C=1

$$B^2 - 4AC = 0 - 4(1)(1) = -4 < 0$$

Then Laplace equation is in the class of elliptic PDEs.

The Laplace Equation

- In 2D diffusion equation: $$\frac{\partial T(x,y,t)}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

- As t → infinity , LHS approach to zero at steady-state condition and then 2D diffusion equation becomes 2D Laplace equation:

$$\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0}$$

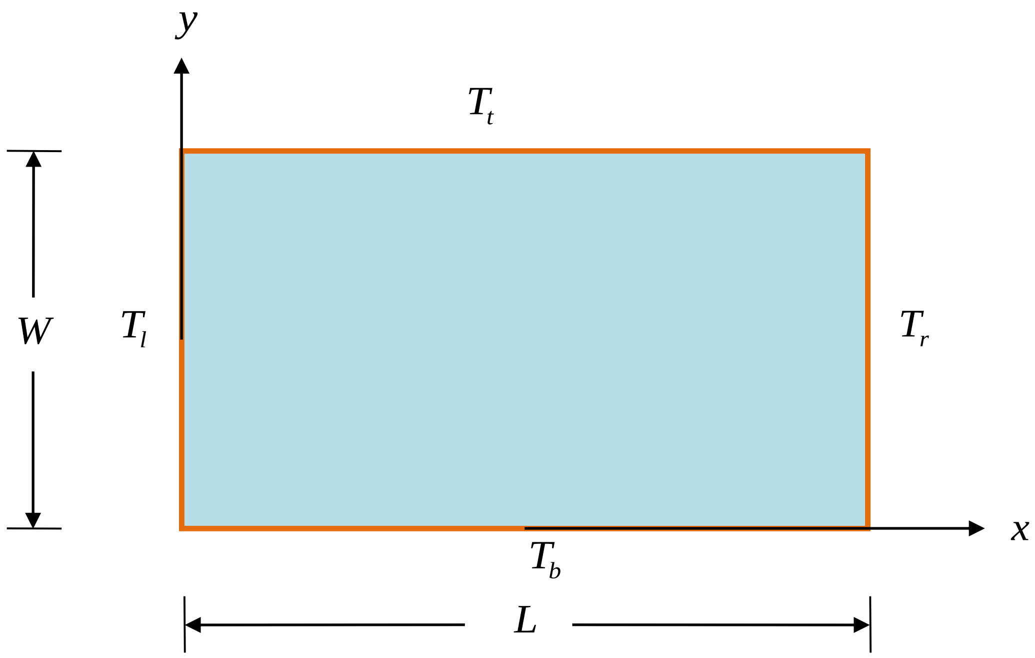
where A=1, B=0 , C=1

$$B^2 - 4AC = 0 - 4(1)(1) = -4 < 0$$

- Then Laplace equation is in the class of elliptic PDEs.

Solution Technique

Elliptic PDE of Plate

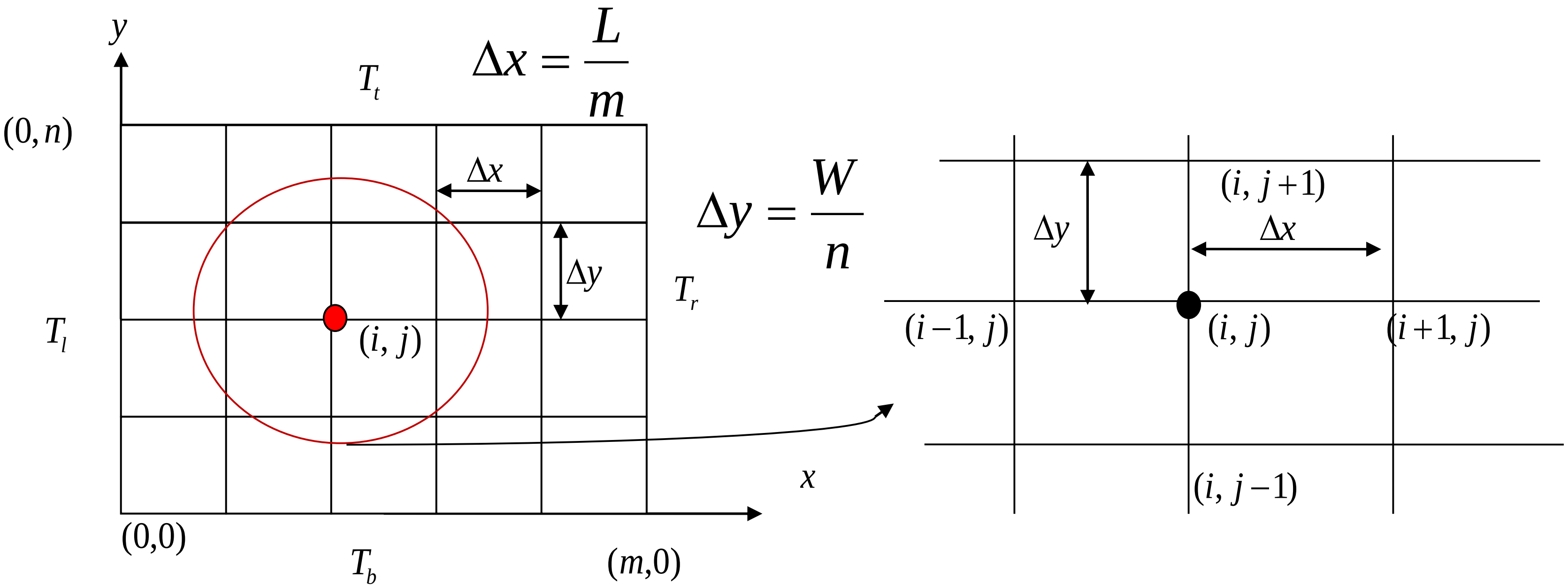


A plate with specified temperature boundary conditions

The Laplace equation governs the temperature:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

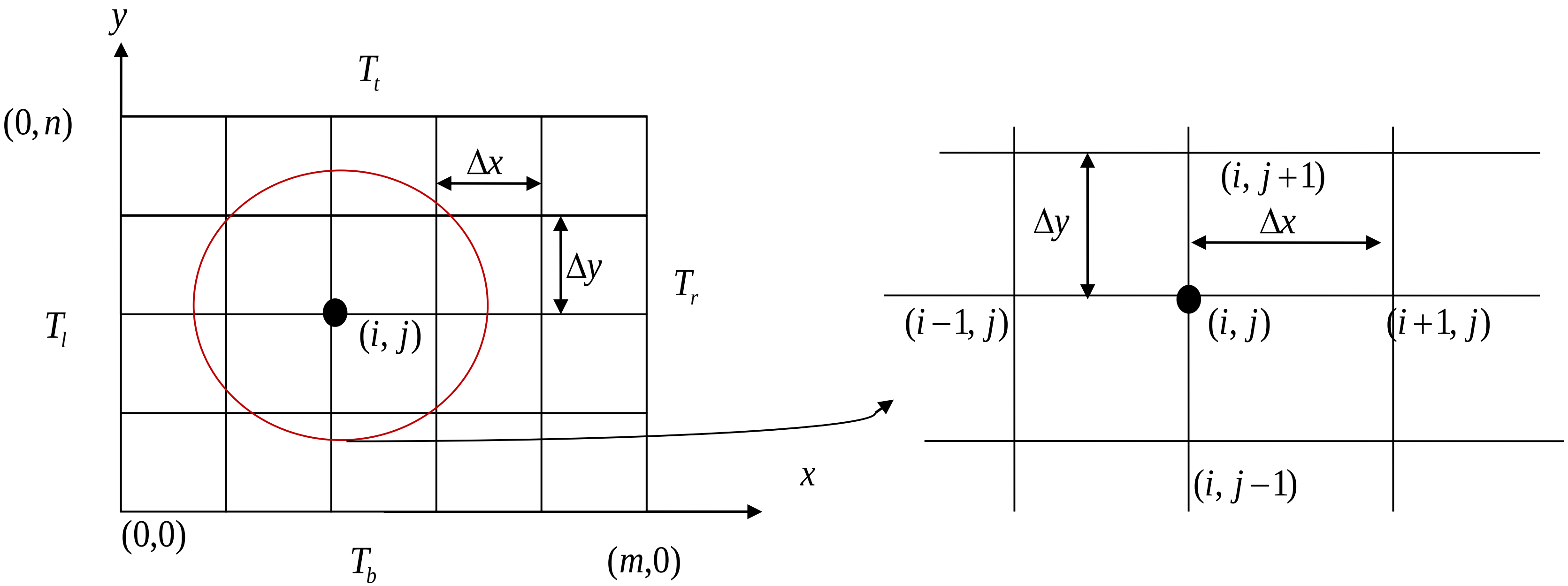
Elliptic PDE of Plate: Discretization



$$\frac{\partial^2 T}{\partial x^2}(x,y) \cong \frac{T(x+\Delta x,y) - 2T(x,y) + T(x-\Delta x,y)}{(\Delta x)^2}$$

$$\frac{\partial^2 T}{\partial y^2}(x,y) \cong \frac{T(x,y+\Delta y) - 2T(x,y) + T(x,y-\Delta y)}{(\Delta y)^2}$$

Elliptic PDE of Plate: Discretization



$$\frac{\partial^2 T}{\partial x^2}(x,y) \cong \frac{T(x+\Delta x, y) - 2T(x,y) + T(x-\Delta x, y)}{(\Delta x)^2} \Longrightarrow \left.\frac{\partial^2 T}{\partial x^2}\right|_{i,j} \cong \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2}$$

$$\frac{\partial^2 T}{\partial y^2}(x,y) \cong \frac{T(x, y+\Delta y) - 2T(x,y) + T(x, y-\Delta y)}{(\Delta y)^2} \Longrightarrow \left.\frac{\partial^2 T}{\partial y^2}\right|_{i,j} \cong \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2}$$

The Laplacian Difference Equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Substituting these approximations into the Laplace equation yields:

$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2} = 0$$

if, $\Delta x = \Delta y$ the Laplace equation can be rewritten as

$$\boxed{T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0}$$

Laplacian Difference Equation

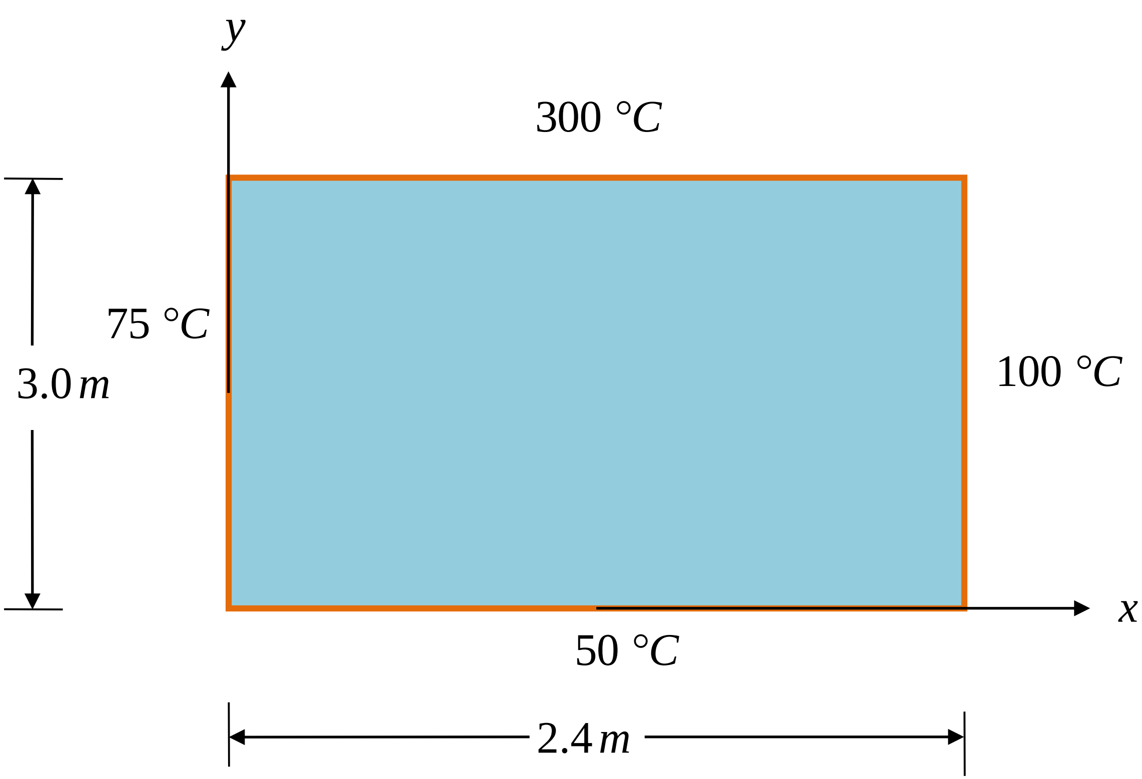
$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0$$

- This relationship which holds for all interior points on the plate, is referred to as Laplacian equation.

- This discretized form allows for application of several numerical methods for solving the problem (e.g. Gauss-Seidel)

Example 1. Laplace Equation. Plate Example

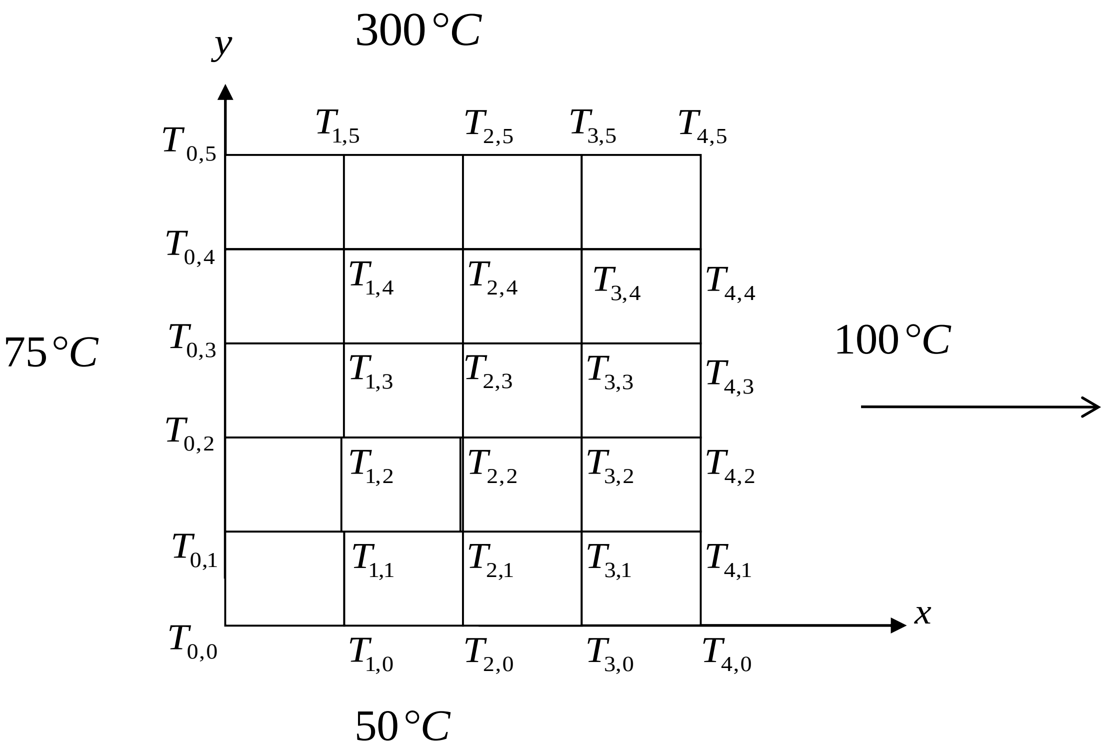
Consider a plate 2.4 m in 3 m that is subjected to the boundary conditions shown below. Find the temperature at the interior nodes using a square grid with a length of 0.6 m by using the direct method..



Example 1. Laplace Equation. Plate Example

1. Discretize: $\Delta x = \Delta y = 0.6m$

2. The nodal temperatures at the boundary nodes:



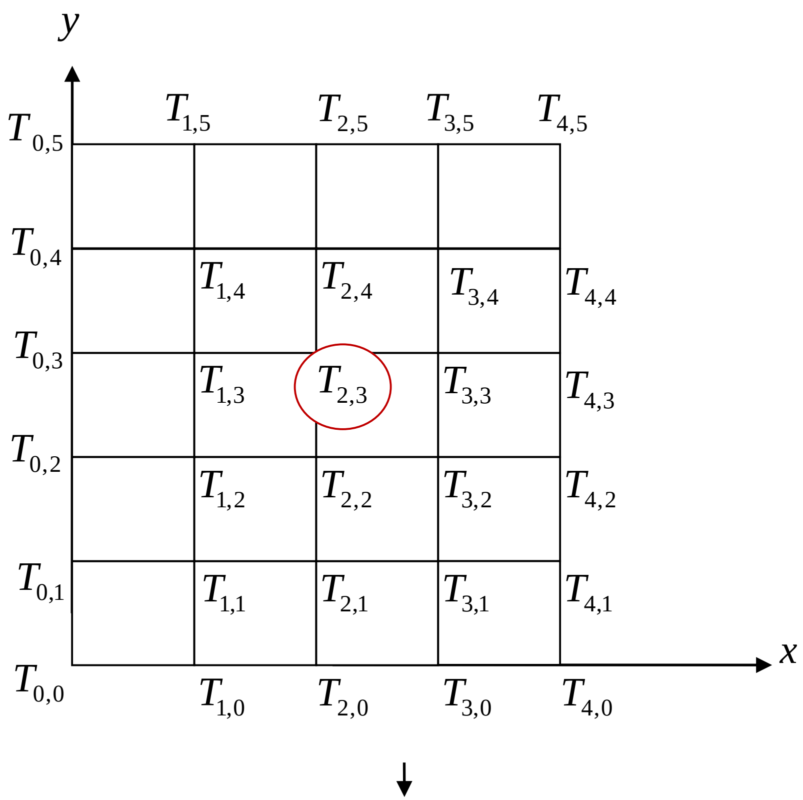
$$T_{0,j} = 75,\ j = 1,2,3,4$$

$$T_{4,j} = 100,\ j = 1,2,3,4$$

$$T_{i,0} = 50,\ i = 1,2,3$$

$$T_{i,5} = 300,\ i = 1,2,3$$

Example 1. Laplace Equation. Plate Example



For instance: equation for the temperature at the node (2,3)

i=2 and *j*=3

$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0$$

$$T_{3,3} + T_{1,3} + T_{2,4} + T_{2,2} - 4T_{2,3} = 0$$

$$T_{1,3} + T_{2,2} - 4T_{2,3} + T_{2,4} + T_{3,3} = 0$$

We can develop similar equations for every interior node leaving us with an equal number of equations and unknowns.

Example 1. Laplace Equation. Plate Example

How many equations would this generate?

Solution: $$\begin{bmatrix} T_{1,1} \\ T_{1,2} \\ T_{1,3} \\ T_{1,4} \\ T_{2,1} \\ T_{2,2} \\ T_{2,3} \\ T_{2,4} \\ T_{3,1} \\ T_{3,2} \\ T_{3,3} \\ T_{3,4} \end{bmatrix} = \begin{bmatrix} 73.8924 \\ 93.0252 \\ 119.907 \\ 173.355 \\ 77.5443 \\ 103.302 \\ 138.248 \\ 198.512 \\ 82.9833 \\ 104.389 \\ 131.271 \\ 182.446 \end{bmatrix} {}^\circ C$$