MTE 203 – Advanced Calculus
Homework 3 - Solutions

Differentiation and Integration of Vectors

Problem 1 [11.9, Prob. 7, 19]

If $f(t) = t^2 + 3$, $g(t) = 2t^3 + 3t$, $\boldsymbol{u}(t) = t\hat{\boldsymbol{\imath}} - t^2\hat{\boldsymbol{\jmath}} + 2t\hat{\boldsymbol{k}}$, and $\boldsymbol{v}(t) = \hat{\boldsymbol{\imath}} - 2t\hat{\boldsymbol{\jmath}} + 3t^2\hat{\boldsymbol{k}}$, find the scalar or the components of the vector in the following exercises:

a. $\frac{d}{dt}[f(t)\boldsymbol{v}(t)]$
b. $\int[f(t)\boldsymbol{u}\cdot\boldsymbol{v}]dt$

Solution:

a.
$$\frac{d}{dt}[f(t)\mathbf{v}(t)] = f'(t)\mathbf{v}(t) + f(t)\mathbf{v}'(t) = 2t(\hat{\mathbf{i}} - 2t\hat{\mathbf{j}} + 3t^2\hat{\mathbf{k}}) + (t^2+3)(-2\hat{\mathbf{j}} + 6t\hat{\mathbf{k}})$$
$$= 2t\hat{\mathbf{i}} - 6(t^2+1)\hat{\mathbf{j}} + 6t(2t^2+3)\hat{\mathbf{k}}$$

b.
$$\int [f(t)\mathbf{u}\cdot\mathbf{v}]\,dt = \int (t^2+3)(t+2t^3+6t^3)\,dt = \int (8t^5+25t^3+3t)\,dt = \frac{4t^6}{3} + \frac{25t^4}{4} + \frac{3t^2}{2} + C$$

Problem 2 [11.9, Prob. 25]:

Prove that if a differentiable function $\mathbf{v}(t)$ has constant length, then at any point at which $\frac{d\mathbf{v}}{dt} \neq 0$, the vector $\frac{d\mathbf{v}}{dt}$ is perpendicular to $\mathbf{v}$

Solution:

If $\mathbf{v}$ has constant length, then $\mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|^2 = \text{constant}$. Differentiation with 11.59b gives

$$0 = \frac{d\mathbf{v}}{dt}\cdot\mathbf{v} + \mathbf{v}\cdot\frac{d\mathbf{v}}{dt} = 2\left(\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}\right).$$

But this implies that $\mathbf{v}$ and $d\mathbf{v}/dt$ are perpendicular.

Arc Length and Length of Curves

Problem 3 [S. 11.11, Prob. 11, 13]:

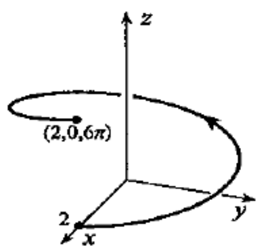
Find the length of the following curves. Draw the curves.

a. $x = 2\cos t\,,\ y = 2\sin t\,,\ z = 3t,\ 0 \le t \le 2\pi$
b. $x = t^3,\ y = t^2,\ z = t^3,\ 0 \le t \le 1$

Solution:

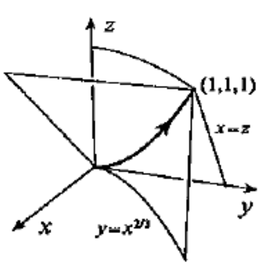
a. With equation 11.78,

$$L = \int_0^{2\pi} \sqrt{(-2\sin t)^2 + (2\cos t)^2 + 9}\, dt$$
$$= \sqrt{13}\Big\{t\Big\}_0^{2\pi} = 2\sqrt{13}\pi.$$



b. With equation 11.78,

$$L = \int_0^1 \sqrt{(3t^2)^2 + (2t)^2 + (3t^2)^2}\, dt$$
$$= \int_0^1 t\sqrt{4 + 18t^2}\, dt.$$
$$= \left\{\frac{(4 + 18t^2)^{3/2}}{54}\right\}_0^1 = \frac{11\sqrt{22} - 4}{27}.$$



Displacement, Velocity and Acceleration

Problem 4 [11.13, Prob. 13]

A particle moves along the curve $x(t) = t, \quad y(t) = t^3 - 3t^2 + 2t, \quad 0 \le t \le 5$ in the xy plane (where t is the time). Is there any point at which its velocity is parallel to its displacement?

Solution:

Velocity and displacement will be parallel if for some value of λ,

$$\mathbf{v} = \lambda \mathbf{r} \quad \Longrightarrow \quad \mathbf{v} = \hat{\mathbf{i}} + (3t^2 - 6t + 2)\hat{\mathbf{j}} = \lambda[t\hat{\mathbf{i}} + (t^3 - 3t^2 + 2t)\hat{\mathbf{j}}].$$

When we equate components, $1 = \lambda t$, $3t^2 - 6t + 2 = \lambda(t^3 - 3t^2 + 2t)$. Substituting $\lambda = 1/t$ into the second leads to the equation $2t^2 - 3t = 0$ with solutions $t = 0, 3/2$. Since $\mathbf{r}(0) = \mathbf{0}$, we cannot discuss parallelism at $t = 0$. The position of the particle at $t = 3/2$ is $(3/2, -3/8)$.

Problem 5 [11.13, Prob. 17]

A particle travels around the circle $x^2 + y^2 = 4$ counterclockwise at constant speed, making 2 revolutions each second. If x and y are measured in meters, what is the velocity of the particle when it is at the point $\left(1, -\sqrt{3}\right)$

Solution:

If we choose $t \ge 0$ and $t = 0$ when the particle is at the point $(2, 0)$, its position can be described by $x = 2\cos(4\pi t)$, $y = 2\sin(4\pi t)$. It is at the point $(1, -\sqrt{3})$ when $1 = 2\cos(4\pi t)$, and $-\sqrt{3} = 2\sin(4\pi t)$. This happens for the first time at $t = 5/12$ s. The velocity of the particle at this time is

$$\mathbf{v}(5/12) = -8\pi \sin\left(\frac{5\pi}{3}\right)\hat{\mathbf{i}} + 8\pi \cos\left(\frac{5\pi}{3}\right)\hat{\mathbf{j}} = 4\pi(\sqrt{3}\hat{\mathbf{i}} + \hat{\mathbf{j}}) \text{ m/s}.$$

Problem 6 [11.13, Prob. 21]

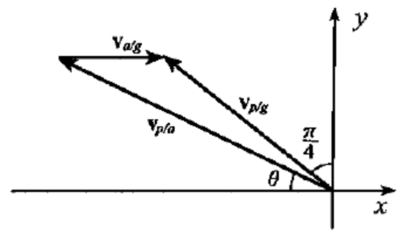
A plane flies with speed 600 km/h in still air. The plane is to fly in a straight line from city A to city B, where B is 1000 km northwest of A. What should be its bearing if the wind is blowing from the west at 50 km/h? How long will the trip take?

Solution:

Let $\mathbf{v}_{p/a}$ be the velocity of the plane with respect to the air, $\mathbf{v}_{a/g}$ the velocity of the air with respect to the ground, and $\mathbf{v}_{p/g}$ the velocity of the plane with respect to the ground. According to Exercise 19, $\mathbf{v}_{p/a} + \mathbf{v}_{a/g} = \mathbf{v}_{p/g}$, where

$$\mathbf{v}_{p/a} = 600[-\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}], \qquad \mathbf{v}_{a/g} = 50\hat{\mathbf{i}},$$

$$\mathbf{v}_{p/g} = v[-\cos(\pi/4)\hat{\mathbf{i}} + \sin(\pi/4)\hat{\mathbf{j}}] = \frac{v}{\sqrt{2}}(-\hat{\mathbf{i}} + \hat{\mathbf{j}}),$$



where v is the speed of the plane with respect to the ground. When we substitute these into the above equation

$$600(-\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) + 50\hat{\mathbf{i}} = \frac{v}{\sqrt{2}}(-\hat{\mathbf{i}} + \hat{\mathbf{j}}),$$

and equate components, $-600\cos\theta + 50 = -\frac{v}{\sqrt{2}}$, $600\sin\theta = \frac{v}{\sqrt{2}}$. Eliminating v leads to the equation $\cos\theta - \sin\theta = 1/12$, which we square

$$\cos^2\theta - 2\sin\theta\cos\theta + \sin^2\theta = \frac{1}{144} \quad\Longrightarrow\quad \sin 2\theta = \frac{143}{144}.$$

The appropriate angle satisfying this equation (between 0 and $\pi/4$) is 0.726 radians. The plane should therefore take the bearing of west 0.726 radians north. Ground speed of the plane is $600\sqrt{2}\sin(0.726)$, which when divided into 1000 results in a trip time of 1.8 hours.

Extra Practice Problems

Solutions to these problems can be found at the back of your textbook

1. S. 11.9, Probs. 10, 18, 24
2. S. 11.11, Probs. 12,14
3. S. 11.13, Probs. 4, 14, 38