

**University of Waterloo  
Mechatronics Engineering**

**Sensors and Instrumentation  
MTE 220**

**Lecture Notes**

**by**

**J.A. Barby**

**Fall 2018**

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2007,2008,2009,2010,2011,2012,2013,2014,2015,2016,2017,2018

**Note: These notes are for the sole use of students registered in  
MTE 220 and may not be used for any other purpose.**

## Operational Issues

### Why not the easy way out?

The Butterfly..... One day a small opening appeared on a cocoon. A man sat and watched for the butterfly for several hours as it struggled to force its body through that little hole. Then it seemed to stop making any progress. It appeared as if it had gotten as far as it could and it could go no further. So the man decided to help the butterfly, he took a pair of scissors and snipped off the remaining bit of the cocoon. The butterfly then emerged easily. But it had a swollen body and small, shriveled wings. The man continued to watch the butterfly because he expected that, at any moment, the wings would enlarge and expand to be able to support the body, which would contract in time. Neither happened! In fact, the butterfly spent the rest of its life crawling around with a swollen body and shriveled wings. It never was able to fly. What the man in his kindness and haste did not understand was that the restricting cocoon and the struggle required for the butterfly to get through the tiny opening were God's way of forcing fluid from the body of the butterfly into its wings so that it would be ready for flight once it achieved its freedom from the cocoon. Sometimes struggles are exactly what we need in our life. If God allowed us to go through our life without any obstacles, it would cripple us. We would not be as strong as what we could have been. We could never fly.

### Faculty of Engineering Course Responsibilities

Refer to

<http://uwaterloo.ca/engineering/current-undergraduate-students/academic-support/course-responsibilities>

for the various policies and procedures associated with courses offered in the Faculty of Engineering.

### In Case of Absence

- Course attendance is not marked in MTE 220
- You must obtain a medical certificate if you miss an assignment deadline or an examination for health reasons
- Medical certificates can be obtained from Health Services

<https://uwaterloo.ca/health-services/sites/ca.health-services/files/uploads/files/VIF-online.pdf>

- If you must miss a laboratory session or an examination to participate in a co-op interview, it is your responsibility to inform your lab instructor / teaching assistants prior to the absence
- When in doubt, consult your instructor / lab instructor / teaching assistants

### University Rules and Policies

For descriptions of academic programs and rules, refer to the Online University Calendar

<http://ugradcalendar.uwaterloo.ca>

- Policy 71

<http://www.adm.uwaterloo.ca/infosec/Policies/policy71.htm>

provides some examples of academic offences: infringing unreasonably on work of others, violation of safety regulations, cheating, impersonating another student, plagiarism, obtaining materials by improper means, falsifying academic records, oral or written misrepresentations.

## Course overview and how it fits into the Mechatronics program

### calendar description

Review of circuit theory; input-output relationships, transfer functions and frequency response of linear systems; operational amplifiers, operational amplifier circuits using negative or positive feedback; diodes, operational amplifier circuits using diodes; analog signal detection, conditioning and conversion systems; transducers and sensors, difference and instrumentation amplifiers, active filters.

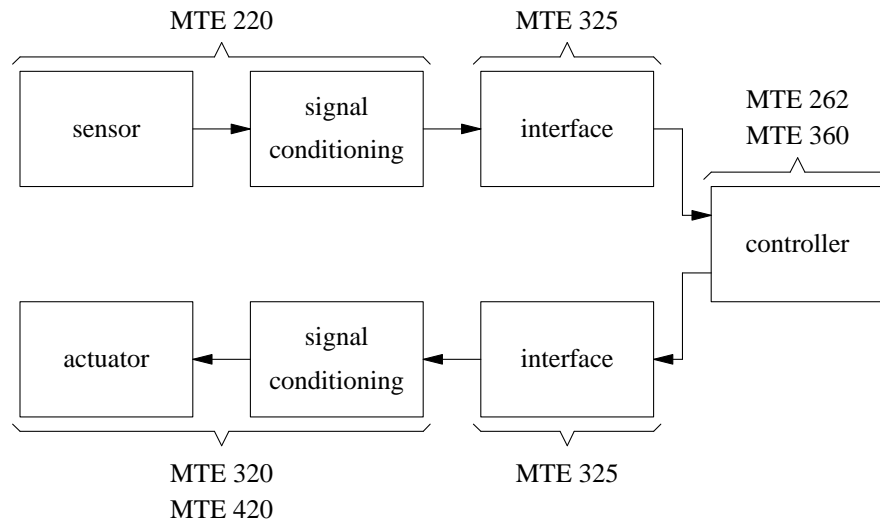
### course learning objectives

Students will be able to design and build signal conditioning circuits for robot sensors and actuators.

### course outline

(1) linear circuits, (2) operational amplifier circuits, (3) electronic devices, (4) filters, (5) sensors & process control, (6) analog signal conditioning, (7) thermal sensors, (8) mechanical sensors, (9) optical sensors

### mechatronics system



### types of sensors (transducers)

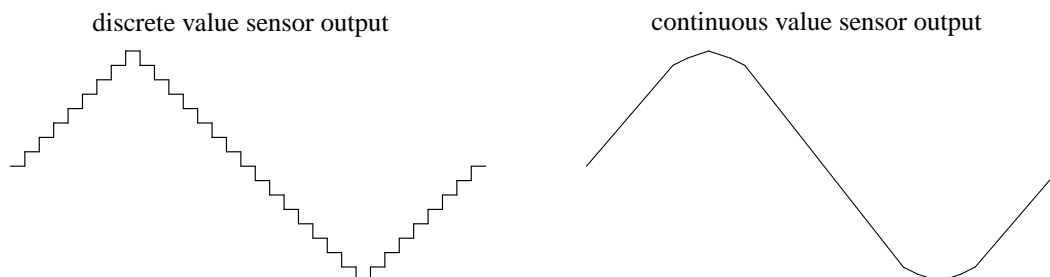
proximity: mechanical, electrical, magnetic

encoders: digital sensors

motion: infrared

temperature: thermocouples

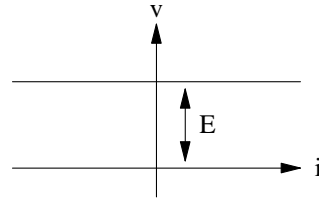
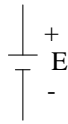
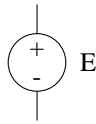
chemical: pH meters



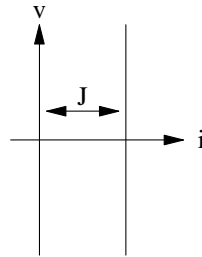
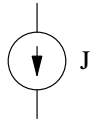
## Linear Circuit Fundamentals

### linear circuit components

- electric current
  - flow of electrons
  - measured in amperes, denoted by A
- voltage
  - a force which causes electrons to flow
  - measured in volts, denoted by V
  - by definition, positive current flows from “+” to “-”
- independent voltage source

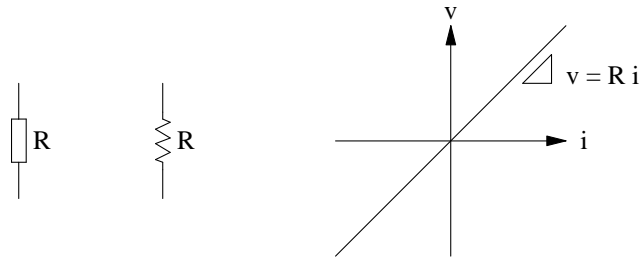


- specified voltage but arbitrary current
  - parallel connect: identical voltages only
- independent current source

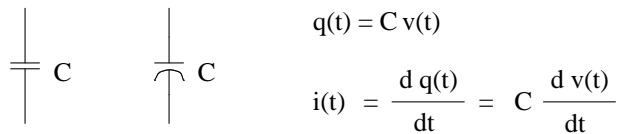


- specified current but arbitrary voltage
- series connected: identical currents only

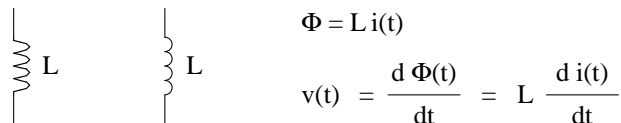
- resistor



- element which resists the flow of current, denoted by  $R$
- measured in ohms, denoted by  $\Omega$
- conductance
  - inverse of resistance, denoted by  $G$
  - $G = \frac{1}{R}$  so  $i = G v$
  - measured in siemens, denoted by  $S$
- capacitor

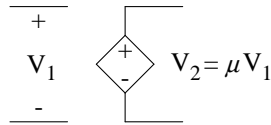


- element that stores energy in the form of charge
- formed by two conductors separated by an insulator
- Charge is denoted by  $Q$  when independent of time, or  $q(t)$  when a function of time; measured in coulombs, denoted by  $C$
- inductor

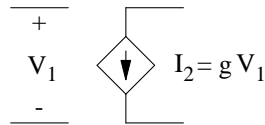


- element that stores energy in the form of flux
- formed by coil of insulated wire, possibly placed into a core of magnetic material
- flux is denoted by  $\Phi$   
measured in webers, denoted by  $Wb$

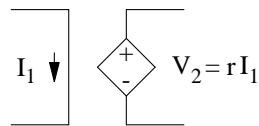
- voltage-controlled voltage source (VV)
  - voltage-to-voltage transducer (VVT)



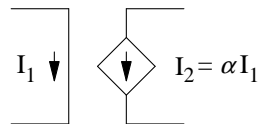
- an ideal voltage amplifier
  - $\mu$  is unitless
- voltage-controlled current source (VC)
  - voltage-to-current transducer (VCT)



- also referred to as a transconductance
  - $g$  is measured in siemens
- current-controlled voltage source (CV)
  - current-to-voltage transducer (CVT)

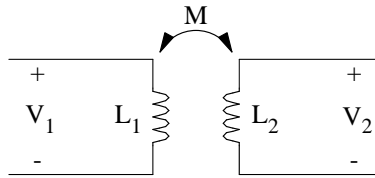


- also referred to as a transresistance
  - $r$  is measured in ohms
- current-controlled current source (CC)
  - current-to-current transducer (CCT)



- an ideal current amplifier
  - $\alpha$  is unitless

- transformer

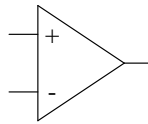


- contains two (or more) coils (inductors) whose magnetic fields interact

$$v_1 = L_1 \frac{di_1(t)}{dt} \pm M \frac{di_2(t)}{dt}$$

$$v_2 = L_2 \frac{di_2(t)}{dt} \pm M \frac{di_1(t)}{dt}$$

- operational amplifier (opamp)



- a voltage amplifier with infinite gain in the ideal case
- actual gain is finite and frequency dependent

**power and energy**

- energy is the amount of work a source can deliver
  - it is denoted by  $W$  if constant or  $w(t)$  if time varying, measured in joules, denoted by  $J$
- power is the change in stored energy
  - it is denoted by  $P$  if constant or  $p(t)$  if time varying, measured in watts, denoted by  $W$

$$p(t) = \frac{d w(t)}{dt}$$

$$w(t) = \int_{t_1}^{t_2} p(t) dt$$

- note  $1 W = 1 \frac{J}{s}$
- power, voltage and current in a resistive circuit are related by

$$p(t) = v(t) \cdot i(t)$$

$$p(t) = \frac{v^2(t)}{R} = i^2(t) \cdot R$$

$$p(t) = v^2(t) \cdot G = \frac{i^2(t)}{G}$$

**Kirchhoff's laws**

- Kirchhoff current law (KCL)  
sum of currents leaving a node is zero
  - assume positive current is flowing away from a node
  - a node is a connection of several elements/components
  - used for **nodal** analysis/formulation
- Kirchhoff voltage law (KVL)  
sum of voltage drops around a loop is zero
  - recall positive current flows  
from “+” to “-”
  - a loop is a connection of elements/components that we can walk around and get back to the starting point
  - used for **mesh** analysis/formulation



**impedance for  $v(t) = V_o e^{st}$  and  $i(t) = I_o e^{st}$**

- if  $v(t)$  &  $i(t)$  are exponential functions

$$i(t) = I_o e^{st} \rightarrow \frac{d i(t)}{dt} = s I_o e^{st} = s i(t)$$

$$v(t) = V_o e^{st} \rightarrow \frac{d v(t)}{dt} = s V_o e^{st} = s v(t)$$

$$v_L(t) = L \frac{d i_L(t)}{dt} = s L i_L(t) \rightarrow Z_L = \frac{v_L(t)}{i_L(t)} = s L$$

$$Y_L = \frac{i_L(t)}{v_L(t)} = \frac{1}{s L}$$

$$i_C(t) = C \frac{d v_C(t)}{dt} = s C v_C(t) \rightarrow Y_C = \frac{i_C(t)}{v_C(t)} = s C$$

$$Z_C = \frac{v_C(t)}{i_C(t)} = \frac{1}{s C}$$

**sinusoidal  $v(t)$  and  $i(t)$**

- recall

$$\frac{d}{dx} (\sin(u)) = \cos(u) \frac{du}{dx} \quad ; \quad \frac{d}{dx} (\cos(u)) = -\sin(u) \frac{du}{dx}$$

$$\cos(\omega t) = \sin(\omega t + \frac{\pi}{2}) \quad ; \quad \sin(\omega t) = \cos(\omega t - \frac{\pi}{2})$$

- if  $v(t)$  &  $i(t)$  are sinusoidal functions

$$v(t) = V_p \cos(\omega t) \rightarrow \frac{d v(t)}{dt} = -\omega V_p \sin(\omega t) = -\omega V_p \cos(\omega t - \frac{\pi}{2}) = \omega V_p \cos(\omega t + \frac{\pi}{2})$$

$$i(t) = I_p \cos(\omega t) \rightarrow \frac{d i(t)}{dt} = -\omega I_p \sin(\omega t) = -\omega I_p \cos(\omega t - \frac{\pi}{2}) = \omega I_p \cos(\omega t + \frac{\pi}{2})$$

- recall

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} \quad ; \quad \cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \quad ; \quad e^{j\theta} = \cos(\theta) + j\sin(\theta) \quad ; \quad e^{-j\theta} = \cos(\theta) - j\sin(\theta)$$

$$\vec{w} = a + j b = M(\cos(\theta) + j\sin(\theta)) = M e^{j\theta} = M \underline{\angle \theta} \quad ; \quad \text{where } M = \sqrt{a^2 + b^2} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

giving us

$$w(t) = M e^{j(\omega t + \theta)}$$

- recall

$$\operatorname{Re}\{e^{j(\omega t + \theta)}\} = \cos(\omega t + \theta) \quad ; \quad \operatorname{Im}\{e^{j(\omega t + \theta)}\} = \sin(\omega t + \theta)$$

- let

$$v(t) = V_p \cos(\omega t + \theta) = \sqrt{2} V_{\text{rms}} \cos(\omega t + \theta) = \operatorname{Re}\{V_p e^{j(\omega t + \theta)}\} = \operatorname{Re}\{(V_{\text{rms}} e^{j\theta})(\sqrt{2} e^{j\omega t})\}$$

- phasor

$$\vec{V} = V_{\text{rms}} e^{j\theta} = V_{\text{rms}} \underline{\angle \theta}$$

**impedance for phasors  $v(t) = \operatorname{Re}\{V_p e^{j(\omega t + \theta)}\}$**

Since

$$\operatorname{Re}\{(\sqrt{2} e^{j\omega t})\}$$

is common to all waveforms for a single tone linear network, we set it aside and only use

$$\operatorname{Re}\{(V_{\text{rms}} e^{j\theta})\}$$

for calculation purposes.

$$\vec{V} = V_{\text{rms}} e^{j\theta} = V_{\text{rms}} \underline{\angle \theta}$$

- voltage source

$$v(t) = V_p \cos(\omega t + \theta) \rightarrow \vec{V} = \frac{V_p}{\sqrt{2}} \underline{\angle \theta}$$

- current source

$$i(t) = I_p \cos(\omega t + \theta) \rightarrow \vec{I} = \frac{I_p}{\sqrt{2}} \underline{\angle \theta}$$

- inductor

$$v_L(t) = L \frac{di_L(t)}{dt} = \sqrt{2} \omega L I_L \cos(\omega t + \frac{\pi}{2}) = \sqrt{2} V_L \cos(\omega t + \frac{\pi}{2})$$

$$Z_L = \frac{V_L / \pi/2}{I_L / 0} = \frac{\omega L I_L / \pi/2}{I_L / 0} = \omega L / \pi/2 = j \omega L$$

$$Y_L = \frac{1}{Z_L} = \frac{1}{j \omega L}$$

- capacitor

$$i_C(t) = C \frac{dv_C(t)}{dt} = \sqrt{2} \omega C V_C \cos(\omega t + \frac{\pi}{2}) = \sqrt{2} I_C \cos(\omega t + \frac{\pi}{2})$$

$$Y_C = \frac{I_C / \pi/2}{V_C / 0} = \frac{\omega C V_C / \pi/2}{V_C / 0} = \omega C / \pi/2 = j \omega C$$

$$Z_C = \frac{1}{Y_C} = \frac{1}{j \omega C}$$

- resistor

$$v_R(t) = R i_R(t) = \sqrt{2} V_R \cos(\omega t)$$

$$Z_R = \frac{v_R(t)}{i_R(t)} = \frac{V_R / 0}{I_R / 0} = \frac{R I_R / 0}{I_R / 0} = R$$

$$Y_R = \frac{1}{Z_R} = \frac{1}{R} = G$$

**time invariant vs. time varying**

- time invariant:  $R(t) = \text{constant}$
- time varying:  $R(t) = f(t)$  : varies with time

**linear vs. nonlinear**

- linear:  $v = R \cdot i$
- nonlinear  $v = R \cdot i^2$

**passive vs. active elements**

- passive: net power consumed from network
- active: net power supplied to network

**response of a single time constant circuit**

- recall that if the initial conditions for a single time constant circuit are not zero, then

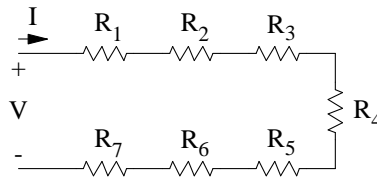
$$y(t) = y(\infty) - \left( y(\infty) - y(0^+) \right) e^{-(t/\tau)}$$

**parallel/series connected resistor circuits**

- effective resistance of series connected resistors is

$$R_{\text{effective}} = \sum_{i=1}^n R_i$$

$$G_{\text{effective}} = \frac{1}{\sum_{i=1}^n \frac{1}{G_i}}$$

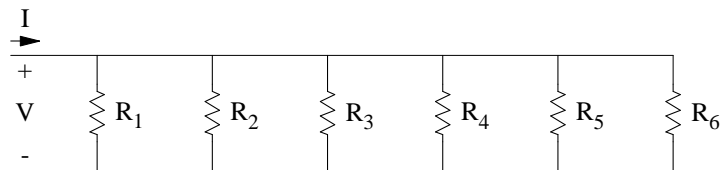


$$R_{\text{effective}} = \frac{V}{I} = \sum_{i=1}^7 R_i$$

- effective resistance of parallel connected resistors is

$$G_{\text{effective}} = \sum_{i=1}^n G_i$$

$$R_{\text{effective}} = \frac{1}{\sum_{i=1}^n \frac{1}{R_i}}$$

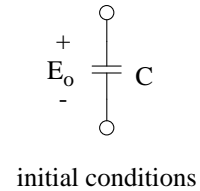
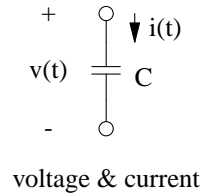
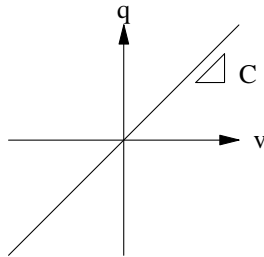


$$G_{\text{effective}} = \frac{I}{V} = \sum_{i=1}^6 G_i = \sum_{i=1}^6 \frac{1}{R_i}$$

$$R_{\text{effective}} = \frac{V}{I} = \frac{1}{G_{\text{effective}}}$$

**capacitors**

- stores energy in the form of charge
- formed by two conductors separated by an insulator
- Charge is denoted by  $Q$  when independent of time, or  $q(t)$  when a function of time; measured in Coulombs, denoted by  $C$



- linear capacitor charge and current

$$i(t) = \frac{dq(t)}{dt} \rightarrow q(t) = \int_{-\infty}^t i(\tau) d\tau = \int_{-\infty}^0 i(\tau) d\tau + \int_0^t i(\tau) d\tau = q_o + \int_0^t i(\tau) d\tau$$

for the case of a linear capacitor

$$q(t) = C v(t) \rightarrow i(t) = C \frac{dv(t)}{dt}$$

- linear capacitor voltage

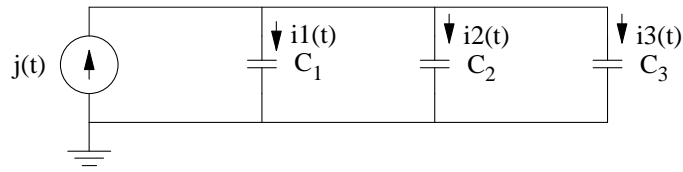
$$v(t) = \frac{1}{C} \int_{-\infty}^t i(\tau) d\tau = \frac{1}{C} \int_{-\infty}^0 i(\tau) d\tau + \frac{1}{C} \int_0^t i(\tau) d\tau = E_o + \frac{1}{C} \int_0^t i(\tau) d\tau$$

- stored energy

given  $p(t) = v(t) i(t)$  the energy stored in a capacitor is

$$w_C = \int_{-\infty}^t p(\tau) d\tau = C \int_{-\infty}^t v(\tau) \frac{dv(\tau)}{d\tau} d\tau = C \int_{-\infty}^t v(\tau) dv(\tau) = \frac{1}{2} C v^2(t)$$

- parallel connected capacitors

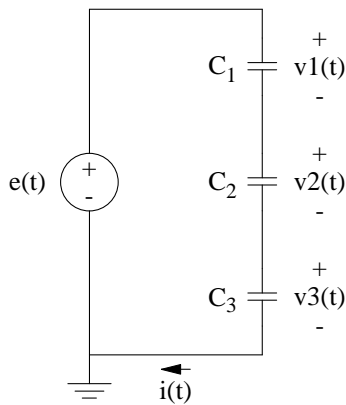


$$j(t) = i_1(t) + i_2(t) + i_3(t) = C_1 \frac{dv(t)}{dt} + C_2 \frac{dv(t)}{dt} + C_3 \frac{dv(t)}{dt} = (C_1 + C_2 + C_3) \frac{dv(t)}{dt} = C_{\text{parallel}} \frac{dv(t)}{dt}$$

for n capacitors connected in parallel

$$C_{\text{parallel}} = \sum_{i=1}^n C_i$$

- series connected capacitors



assuming the capacitors are initially discharged

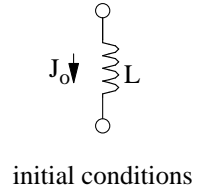
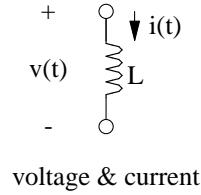
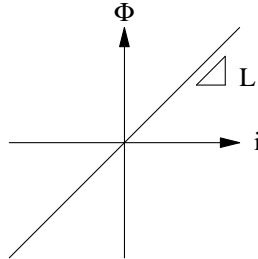
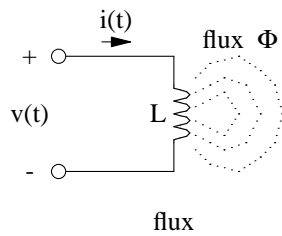
$$e(t) = v_1(t) + v_2(t) + v_3(t) = \frac{1}{C_1} \int_0^t i(\tau) d\tau + \frac{1}{C_2} \int_0^t i(\tau) d\tau + \frac{1}{C_3} \int_0^t i(\tau) d\tau = \left( \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \right) \int_0^t i(\tau) d\tau = \left( \frac{1}{C_{\text{series}}} \right) \int_0^t i(\tau) d\tau$$

for n capacitors connected in series

$$\frac{1}{C_{\text{series}}} = \sum_{i=1}^n \frac{1}{C_i}$$

**inductors**

- stores energy in the form of flux
- formed by coil of insulated wire, possibly placed into a core of magnetic material
- flux is denoted by  $\Phi$   
measured in Webers, denoted by Wb



- linear inductor flux and voltage

$$v(t) = \frac{d\Phi(t)}{dt} = L \frac{di(t)}{dt} \quad \rightarrow \quad \Phi(t) = \int_{-\infty}^t v(\tau) d\tau = \int_{-\infty}^0 v(\tau) d\tau + \int_0^t v(\tau) d\tau = \Phi_o + \int_0^t v(\tau) d\tau$$

for the case of a linear inductor

$$\Phi(t) = L i(t) \quad ; \quad v(t) = L \frac{di(t)}{dt}$$

- linear inductor current

$$i(t) = \frac{1}{L} \int_{-\infty}^t v(\tau) d\tau = \frac{1}{L} \int_{-\infty}^0 v(\tau) d\tau + \frac{1}{L} \int_0^t v(\tau) d\tau = J_o + \frac{1}{L} \int_0^t v(\tau) d\tau$$

- stored energy

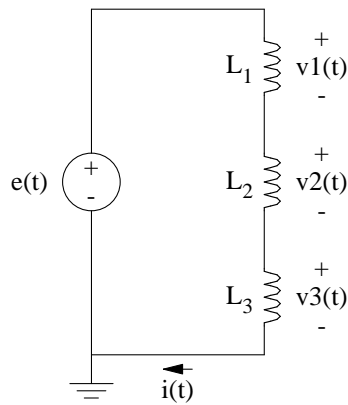
given  $p(t) = v(t) i(t)$

the energy stored in an inductor is

$$w_C = \int_{-\infty}^t p(\tau) d\tau = L \int_{-\infty}^t i(\tau) \frac{di(\tau)}{d\tau} d\tau = L \int_{-\infty}^t i(\tau) di(\tau) = \frac{1}{2} L i^2(t)$$



- series connected inductors

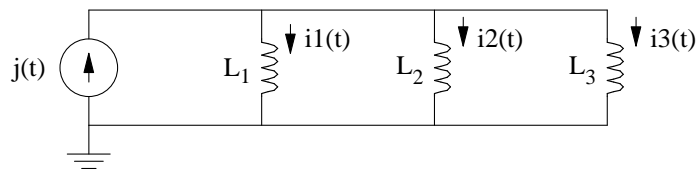


$$e(t) = v_1(t) + v_2(t) + v_3(t) = L_1 \frac{di(t)}{dt} + L_2 \frac{di(t)}{dt} + L_3 \frac{di(t)}{dt} = (L_1 + L_2 + L_3) \frac{di(t)}{dt} = L_{\text{series}} \frac{di(t)}{dt}$$

for n inductors connected in series

$$L_{\text{series}} = \sum_{i=1}^n L_i$$

- parallel connected inductors






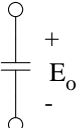
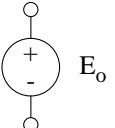




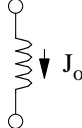
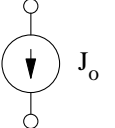

assuming the inductors are initially discharged

$$j(t) = i_1(t) + i_2(t) + i_3(t) = \frac{1}{L_1} \int_0^t v(\tau) d\tau + \frac{1}{L_2} \int_0^t v(\tau) d\tau + \frac{1}{L_3} \int_0^t v(\tau) d\tau = \left( \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} \right) \int_0^t v(\tau) d\tau = \left( \frac{1}{L_{\text{parallel}}} \right) \int_0^t v(\tau) d\tau$$

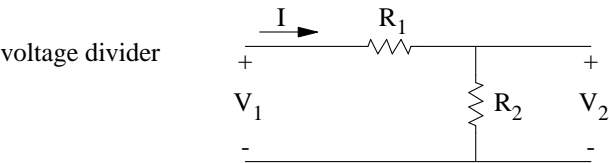
for n inductors connected in parallel

$$\frac{1}{L_{\text{parallel}}} = \sum_{i=1}^n \frac{1}{L_i}$$

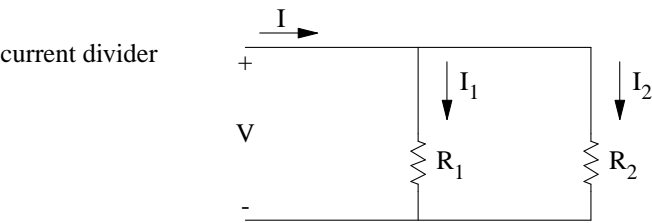
initial and final values

	$t = 0$	$t \rightarrow \infty$
	 short circuit	 open circuit
 $E_o$	 $E_o$	 open circuit
	 open circuit	 short circuit
 $J_o$	 $J_o$	 short circuit

voltage and current dividers



$$V_2 = R_2 I = R_2 \frac{V_1}{R_1 + R_2} = \frac{R_2}{R_1 + R_2} V_1$$

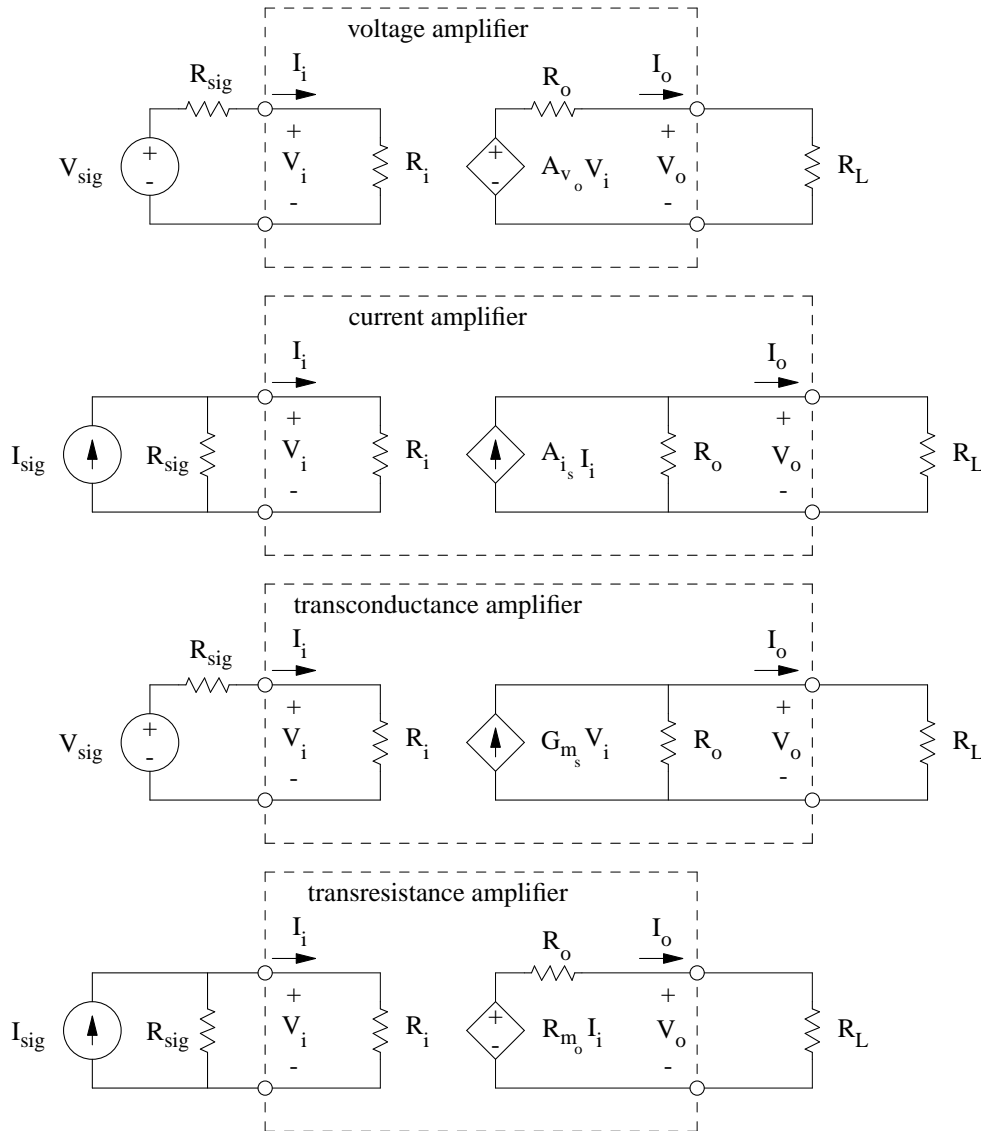


$$I_2 = \frac{V}{R_2} = \frac{\left( \frac{I}{\frac{1}{R_1} + \frac{1}{R_2}} \right)}{R_2} = \frac{\left( \frac{R_1 R_2}{R_1 + R_2} \right) I}{R_2} = \frac{R_1}{R_1 + R_2} I$$

## amplifier types and parameters

$$A_v = \frac{v_o}{v_i} \quad ; \quad G_v = \frac{v_o}{v_{sig}} \quad ; \quad A_i = \frac{i_o}{i_i} \quad ; \quad A_p = A_v A_i$$

$$R_{in} = \frac{v_i}{i_i} \quad ; \quad R_i = \frac{v_i}{i_i} \bigg|_{R_L = \infty} \quad ; \quad R_{out} = \frac{v_x}{i_x} \bigg|_{v_{sig} = 0} \quad ; \quad R_o = \frac{v_x}{i_x} \bigg|_{v_i = 0}$$



$$V_o = \frac{R_L}{R_L + R_o} A_v \frac{R_i}{R_i + R_{sig}} V_{sig}$$

$$A_{v_o} = \frac{V_o}{V_i} \bigg|_{R_L = \infty}$$

$$I_o = \frac{R_o}{R_L + R_o} A_{i_s} \frac{R_{sig}}{R_i + R_{sig}} I_{sig}$$

$$A_{i_s} = \frac{I_o}{I_i} \bigg|_{R_L = 0}$$

$$I_o = \frac{R_o}{R_L + R_o} G_{m_s} \frac{R_i}{R_i + R_{sig}} V_{sig}$$

$$G_{m_s} = \frac{I_o}{V_i} \bigg|_{R_L = 0}$$

$$V_o = \frac{R_L}{R_L + R_o} R_{m_o} \frac{R_{sig}}{R_i + R_{sig}} I_{sig}$$

$$R_{m_o} = \frac{V_o}{I_i} \bigg|_{R_L = \infty}$$

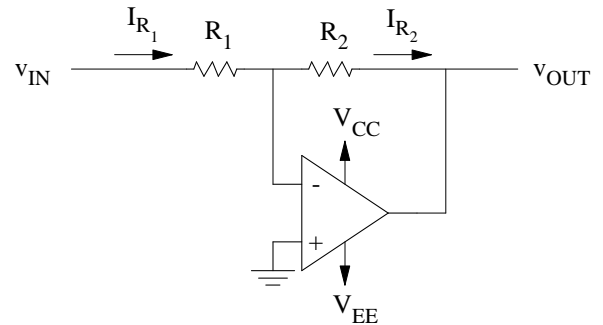
## Opamp Circuits

## inverting amplifier

$$v_- \approx v_+ \approx 0 \rightarrow i_{R_1} = i_{R_2} = \frac{v_{IN}}{R_1}$$

$$v_{OUT} = -I_{R_2} R_2 = -\frac{v_{IN}}{R_1} R_2 = -\frac{R_2}{R_1} v_{IN}$$

$$A_v = \frac{v_{OUT}}{v_{IN}} = -\frac{R_2}{R_1}$$



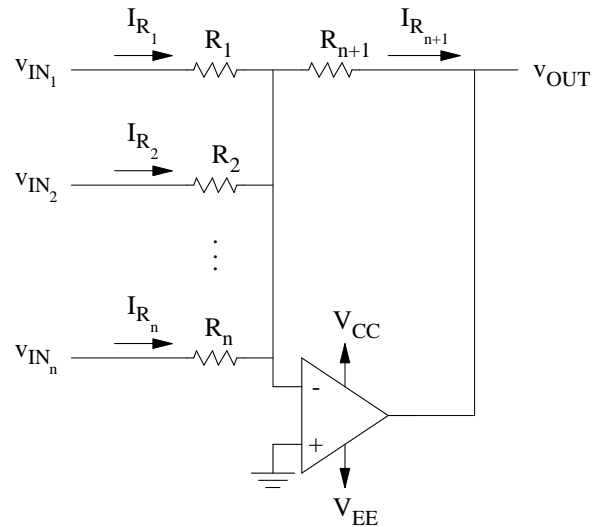
## summer

$$v_- \approx v_+ \approx 0$$

$$i_{R_1} = \frac{v_{IN_1}}{R_1} \quad ; \quad i_{R_2} = \frac{v_{IN_2}}{R_2} \quad \dots \quad i_{R_n} = \frac{v_{IN_n}}{R_n}$$

$$i_{R_{n+1}} = \sum_{k=1}^{k=n} i_{R_k} = \sum_{k=1}^{k=n} \frac{v_{IN_k}}{R_k}$$

$$v_{OUT} = -I_{R_{n+1}} R_{n+1} = -\sum_{k=1}^{k=n} \frac{v_{IN_k}}{R_k} R_{n+1} = -\sum_{k=1}^{k=n} \frac{R_{n+1}}{R_k} v_{IN_k}$$

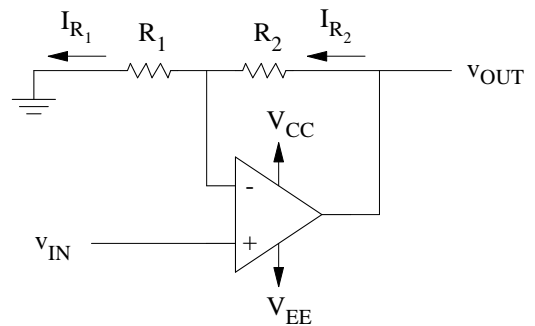


## noninverting amplifier

$$v_- \approx v_+ = v_{IN} \rightarrow i_{R_2} = i_{R_1} = \frac{v_{IN}}{R_1}$$

$$v_{OUT} = (R_2 + R_1) i_{R_1} = (R_2 + R_1) \frac{v_{IN}}{R_1} = \frac{R_2 + R_1}{R_1} v_{IN}$$

$$A_v = \frac{v_{OUT}}{v_{IN}} = \frac{R_2 + R_1}{R_1} = 1 + \frac{R_2}{R_1}$$



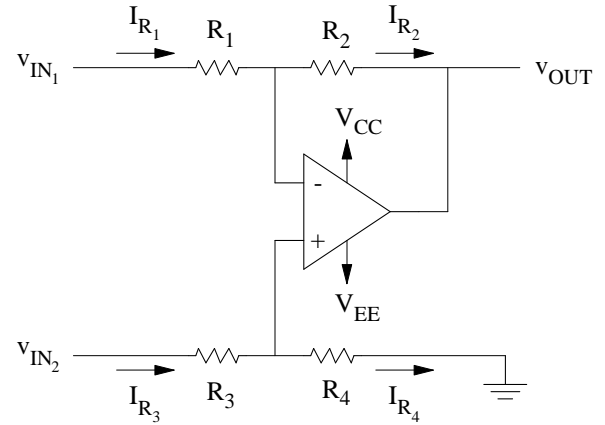
**difference amplifier**

$$v_- \approx v_+ = \frac{R_4}{R_3 + R_4} v_{IN2}$$

$$i_{R2} = i_{R1} = \frac{v_{IN1} - v_+}{R_1} = \frac{v_{IN1}}{R_1} - \frac{v_+}{R_1} = \frac{v_{IN1}}{R_1} - \frac{\frac{R_4}{R_3 + R_4} v_{IN2}}{R_1}$$

$$\begin{aligned} v_{OUT} &= v_- - \left( R_2 i_{R2} \right) \\ &= \frac{R_4}{R_3 + R_4} v_{IN2} - \frac{v_{IN1} - \frac{R_4}{R_3 + R_4} v_{IN2}}{R_1} R_2 \\ &= \left( \left( \frac{R_1 + R_2}{R_1} \right) \left( \frac{R_4}{R_3 + R_4} \right) \right) v_{IN2} - \left( \frac{R_2}{R_1} \right) v_{IN1} \end{aligned}$$

$$R_4 = R_2 \quad \text{and} \quad R_3 = R_1 \quad \rightarrow \quad v_{OUT} = \frac{R_2}{R_1} \left( v_{IN2} - v_{IN1} \right)$$

**instrumentation amplifier**

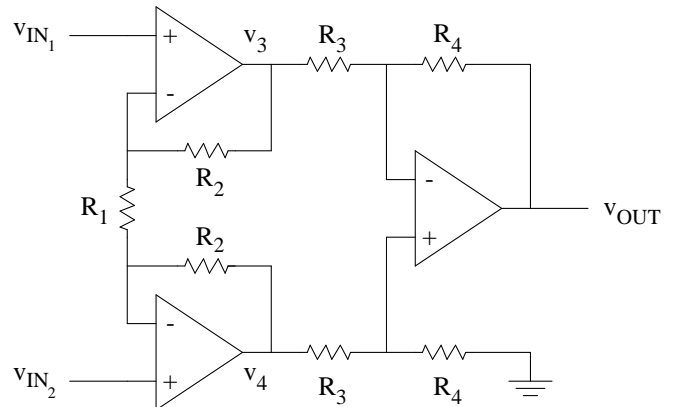
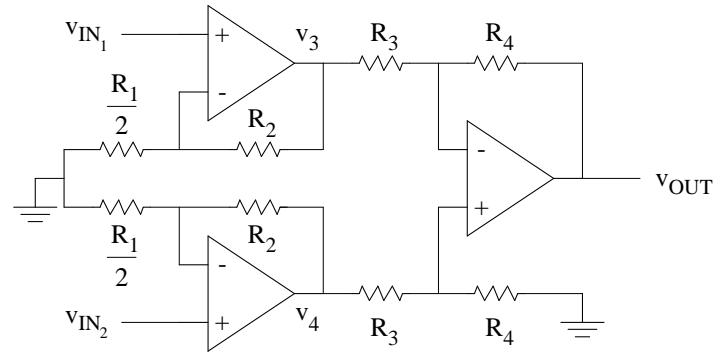
$$v_{OUT} = \frac{R_3 + R_4}{R_3} \frac{R_4}{R_3 + R_4} v_4 - \frac{R_4}{R_3} v_3 = \frac{R_4}{R_3} (v_4 - v_3)$$

$$v_4 = v_{IN2} + \frac{\frac{v_{IN2} - v_{IN1}}{R_1}}{\frac{2}{R_1}} R_2 \quad ; \quad v_3 = v_{IN1} - \frac{\frac{v_{IN2} - v_{IN1}}{R_1}}{\frac{2}{R_1}} R_2$$

$$\begin{aligned} v_4 - v_3 &= \left( v_{IN2} + \frac{v_{IN2} - v_{IN1}}{R_1} R_2 \right) - \left( v_{IN1} - \frac{v_{IN2} - v_{IN1}}{R_1} R_2 \right) \\ &= \left( v_{IN2} - v_{IN1} \right) + \frac{2R_2}{R_1} \left( v_{IN2} - v_{IN1} \right) \\ &= \left( 1 + \frac{2R_2}{R_1} \right) \left( v_{IN2} - v_{IN1} \right) \end{aligned}$$

$$v_{OUT} = \frac{R_4}{R_3} \left( 1 + \frac{2R_2}{R_1} \right) \left( v_{IN2} - v_{IN1} \right)$$

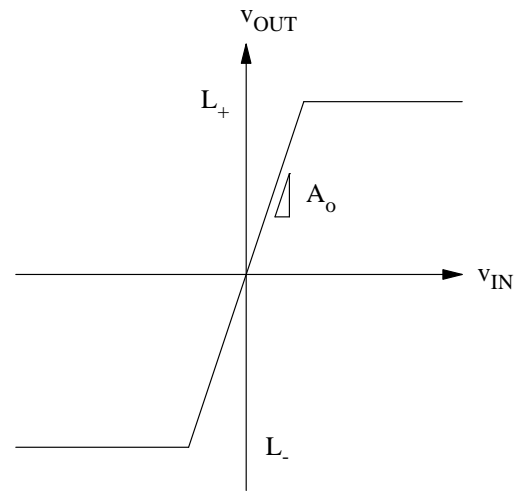
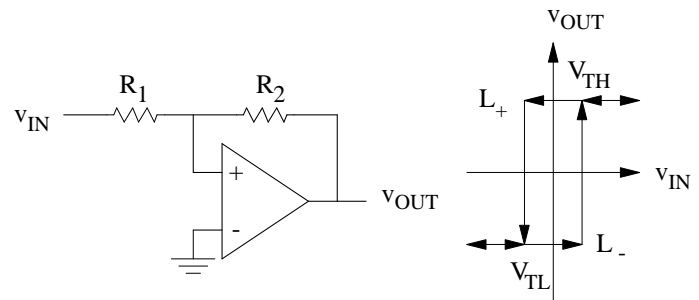
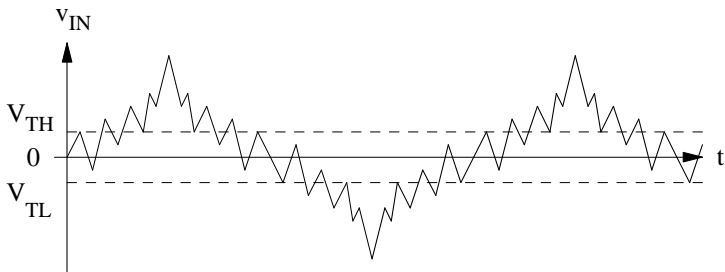
$R_1$  is the gain control resistor, all others require matching.



**nonideal opamp parameters**

$V_{OS}$	input offset voltage
$I_B$	input bias current
$I_{OS}$	input offset current
$A_o$	DC open loop voltage gain
$f_t$	unity gain frequency
$L_+$	maximum output voltage
$L_-$	minimum output voltage

$$SR \quad \text{slew rate} \quad \max \left( \left| \frac{d v_{OUT}}{dt} \right| \right)$$

**zero reference Schmidt trigger**

$$v_+ > v_- \rightarrow v_{OUT} = L_+ \quad ; \quad v_+ < v_- \rightarrow v_{OUT} = L_- \quad ; \quad v_+ = \frac{R_2}{R_1 + R_2} v_{IN} + \frac{R_1}{R_1 + R_2} v_{OUT}$$

if  $v_{OUT} = L_+$  to switch state we need  $v_+ = 0$

$$0 = \frac{R_2}{R_1 + R_2} v_{IN} + \frac{R_1}{R_1 + R_2} L_+$$

$$-\frac{R_2}{R_1 + R_2} v_{IN} = \frac{R_1}{R_1 + R_2} L_+$$

$$-R_2 v_{IN} = R_1 L_+$$

$$v_{IN} = -\left(\frac{R_1}{R_2}\right)L_+ = V_{TL}$$

if  $v_{OUT} = L_-$  to switch state we need  $v_+ = 0$

$$0 = \frac{R_2}{R_1 + R_2} v_{IN} + \frac{R_1}{R_1 + R_2} L_-$$

$$-\frac{R_2}{R_1 + R_2} v_{IN} = \frac{R_1}{R_1 + R_2} L_-$$

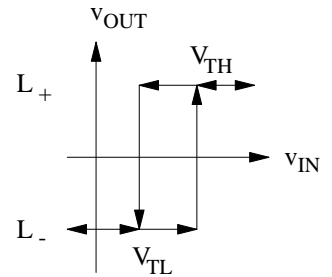
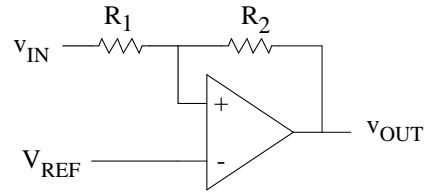
$$-R_2 v_{IN} = R_1 L_-$$

$$v_{IN} = -\left(\frac{R_1}{R_2}\right)L_- = V_{TH}$$

**nonzero reference Schmidt trigger**

$$v_+ = V_{\text{REF}} = \frac{R_2}{R_1 + R_2} v_{\text{IN}} + \frac{R_1}{R_1 + R_2} v_{\text{OUT}}$$

$$V_{\text{REF}} - \frac{R_1}{R_1 + R_2} v_{\text{OUT}} = \frac{R_2}{R_1 + R_2} v_{\text{IN}}$$

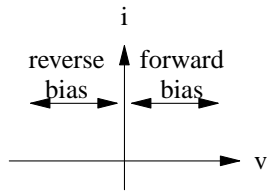
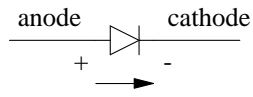


$$v_{\text{IN}} = \frac{V_{\text{REF}} - \frac{R_1}{R_1 + R_2} v_{\text{OUT}}}{\frac{R_2}{R_1 + R_2}} = \frac{R_1 + R_2}{R_2} V_{\text{REF}} - \frac{R_1}{R_2} v_{\text{OUT}}$$

$$V_{\text{TH}} = \frac{R_1 + R_2}{R_2} V_{\text{REF}} - \frac{R_1}{R_2} L_- \quad ; \quad V_{\text{TL}} = \frac{R_1 + R_2}{R_2} V_{\text{REF}} - \frac{R_1}{R_2} L_+$$

## Diode Circuits

### ideal diode model

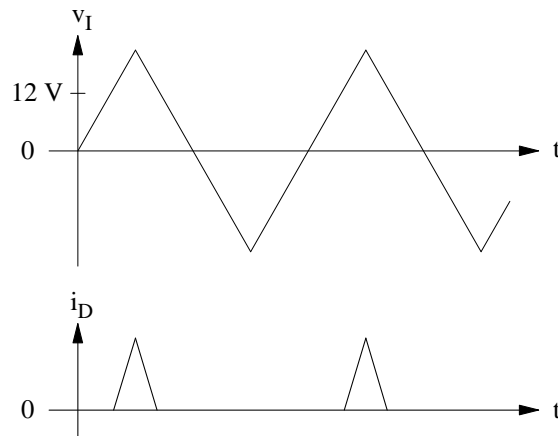
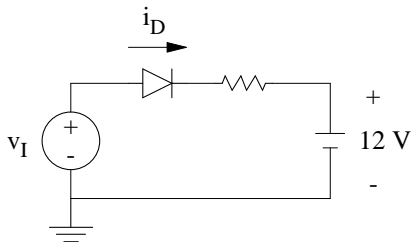
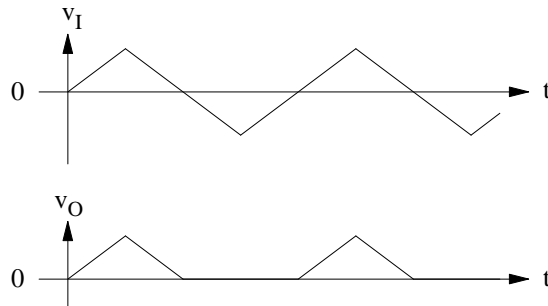
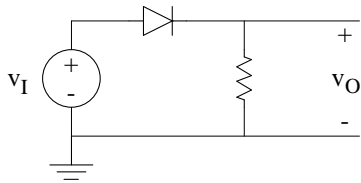
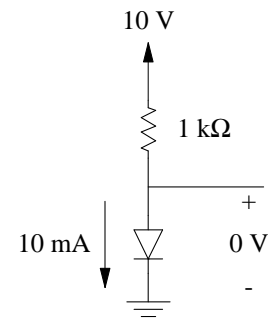
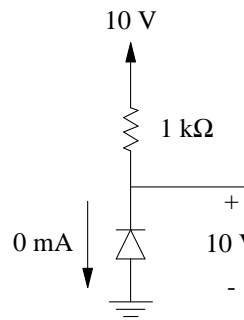


ideal diode

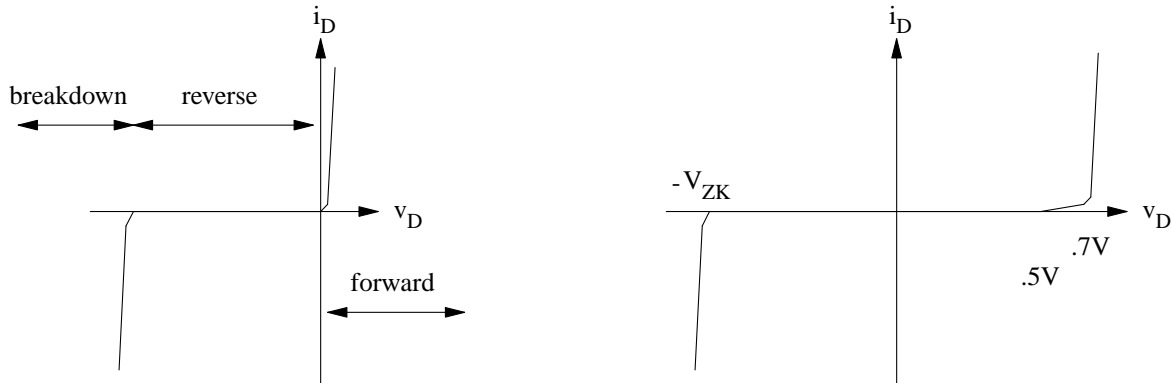
$$i_D = 0 \text{ for } v_D < 0$$

$$v_D = 0 \text{ for } i_D > 0$$

for ideal diode circuits start by assuming the diode current is zero and solve for the diode voltage, if the diode voltage is negative then the assumption was correct, otherwise set the diode voltage to zero and solve for the diode current.





**nonideal diode model****forward bias**

$$i_D = I_S \left( e^{\frac{v_D}{n V_T}} - 1 \right) \quad ; \quad V_T = \frac{k T}{q} \quad ; \quad \begin{array}{l} k = \text{Boltzmann's constant} = 1.38 \times 10^{-23} \text{ joules/kelvin} \\ T = \text{temperature in kelvin} \\ q = \text{electron charge} = 1.60 \times 10^{-19} \text{ coulomb} \end{array}$$

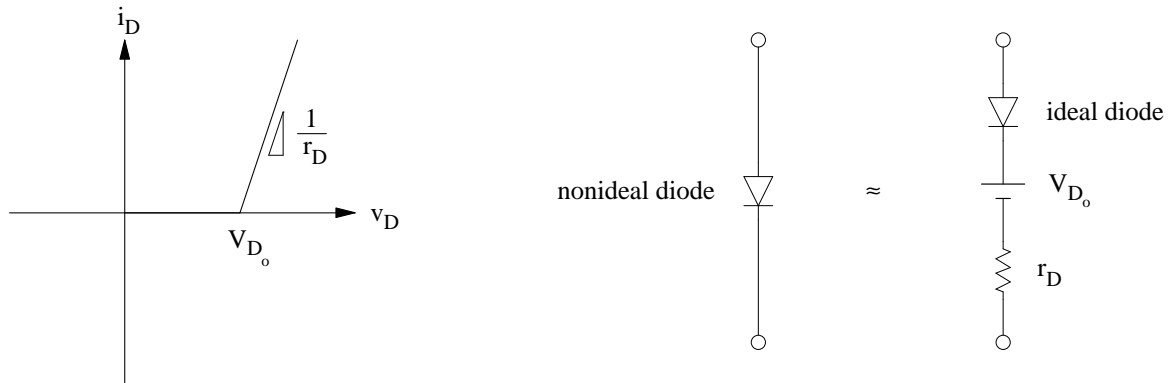
$$i_D \approx I_S e^{\frac{v_D}{n V_T}} \quad \rightarrow \quad v_D = n V_T \ln \left( \frac{i_D}{I_S} \right)$$

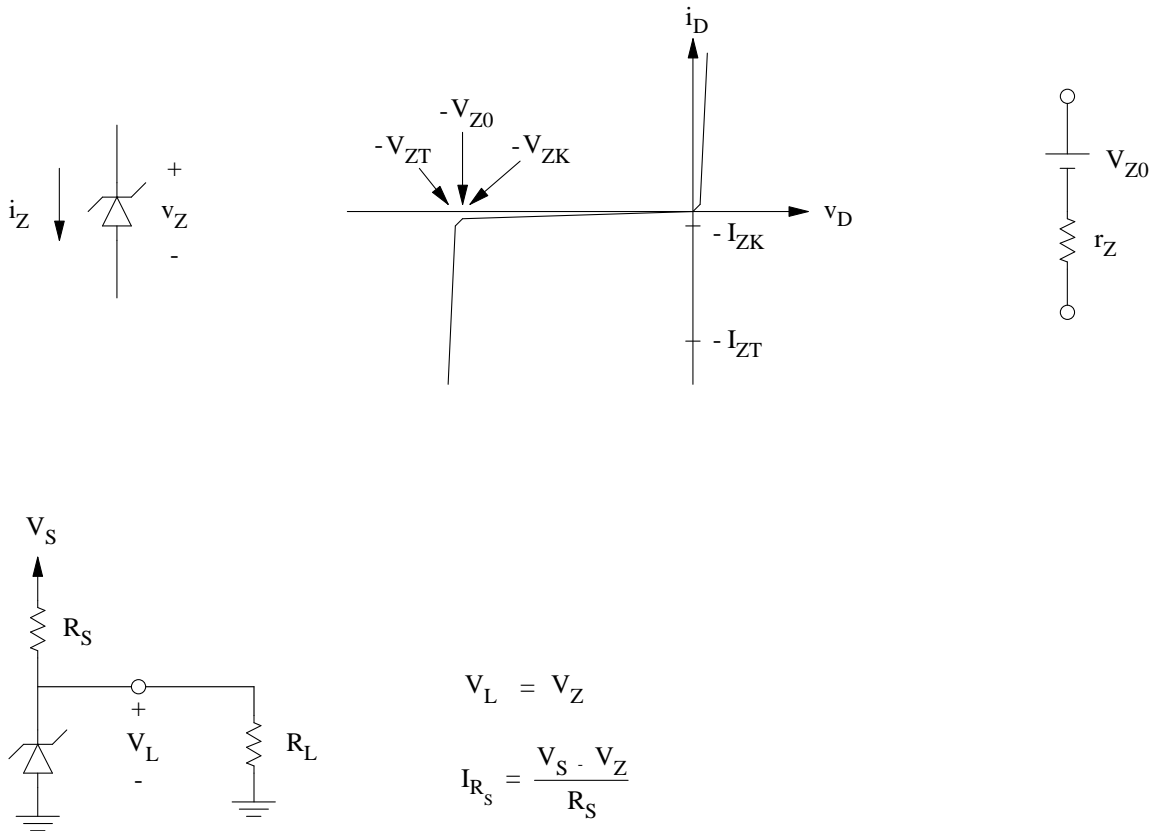
**reverse bias**

$$i_D \approx -I_S$$

**breakdown**

complex function dependent on whether the diode is in zener or avalanche breakdown

**piecewise linear diode model**

**zener diode****special diodes****Schottky-Barrier Diode (SBD)**

turn on voltage lower (0.3 V to 0.5 V), switches off faster

**Varactors**

reverse bias diode as a voltage controlled capacitor

**Photodiode**

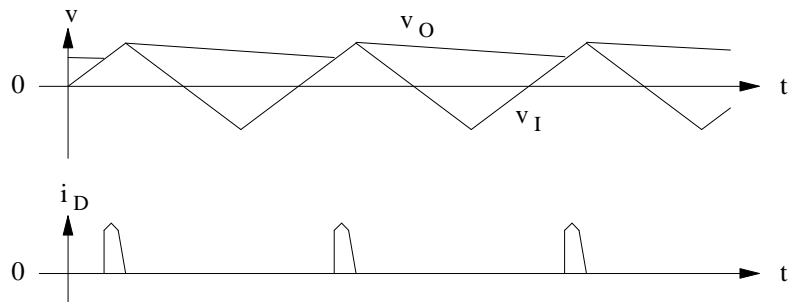
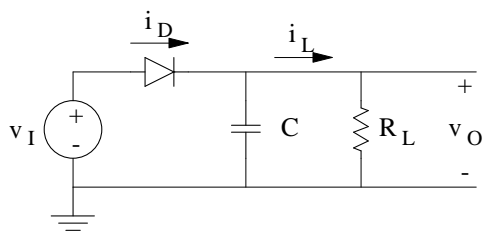
$I_S$  is a function of light resulting in diode current being a function of light shining on the diode junction material

**Light-Emitting Diode (LED)**

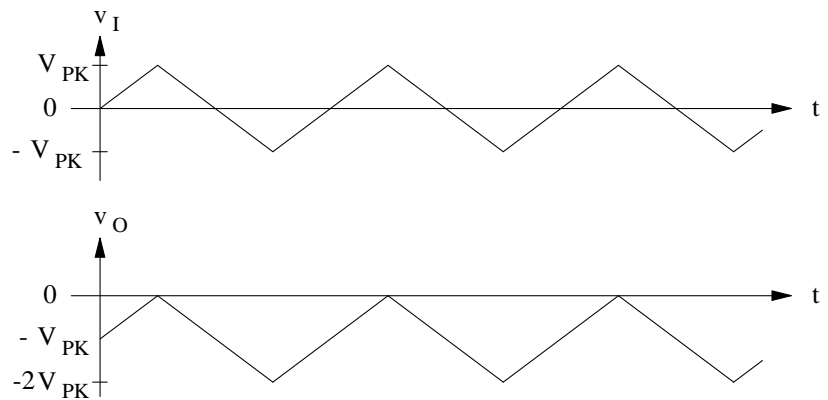
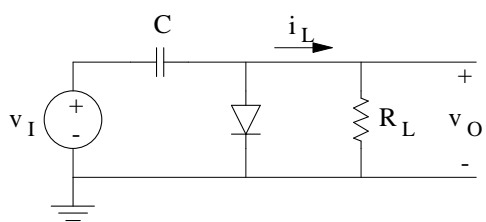
converts current into light

## diode signal conditioning circuits

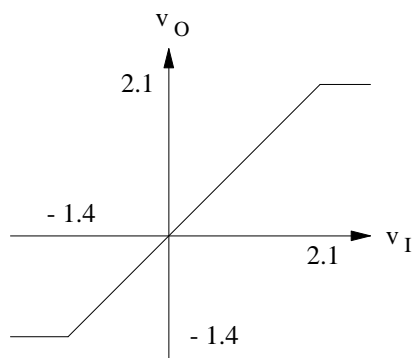
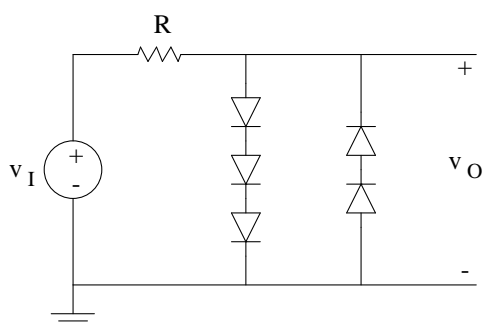
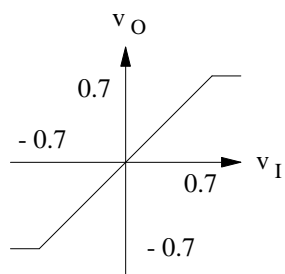
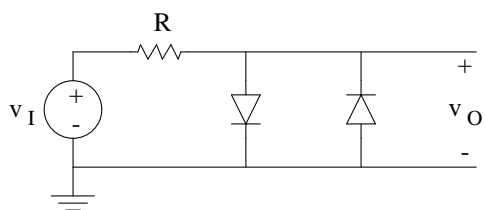
peak detector



clamp



limiters



**BJT**

terminals

B is the base, C is the collector, E is the emitter

forward active region

$$v_{BE} \geq 0.6 \text{ and } v_{BC} < 0.4$$

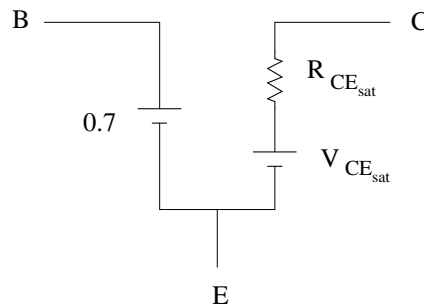
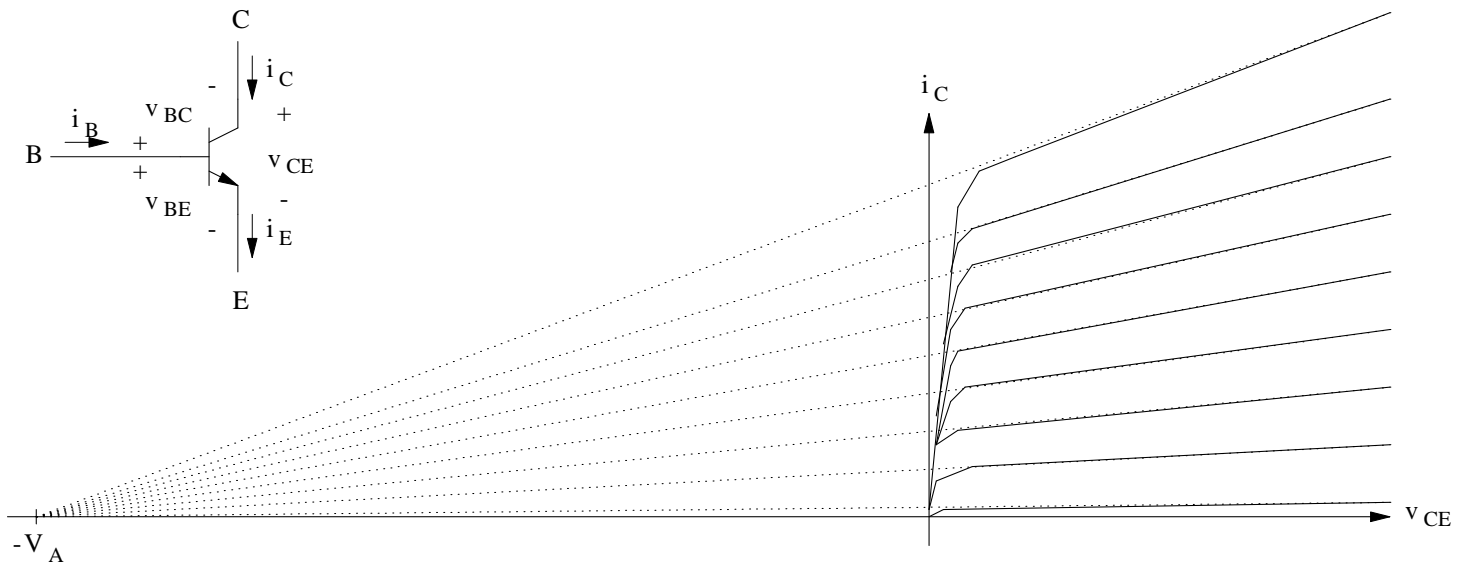
$$i_C = I_S e^{\frac{v_{BE}}{V_T}} \left( 1 + \frac{v_{CE}}{V_A} \right) ; \quad i_B = \frac{1}{\beta} i_C ; \quad i_E = i_C + i_B$$

saturation region

$$v_{BE} \geq 0.6 \text{ and } v_{BC} \geq 0.4 \text{ resulting in } v_{CE} = 0.2 \text{ V}$$

cut-off

$$v_{BE} < 0.6 \text{ results in } i_C = 0$$



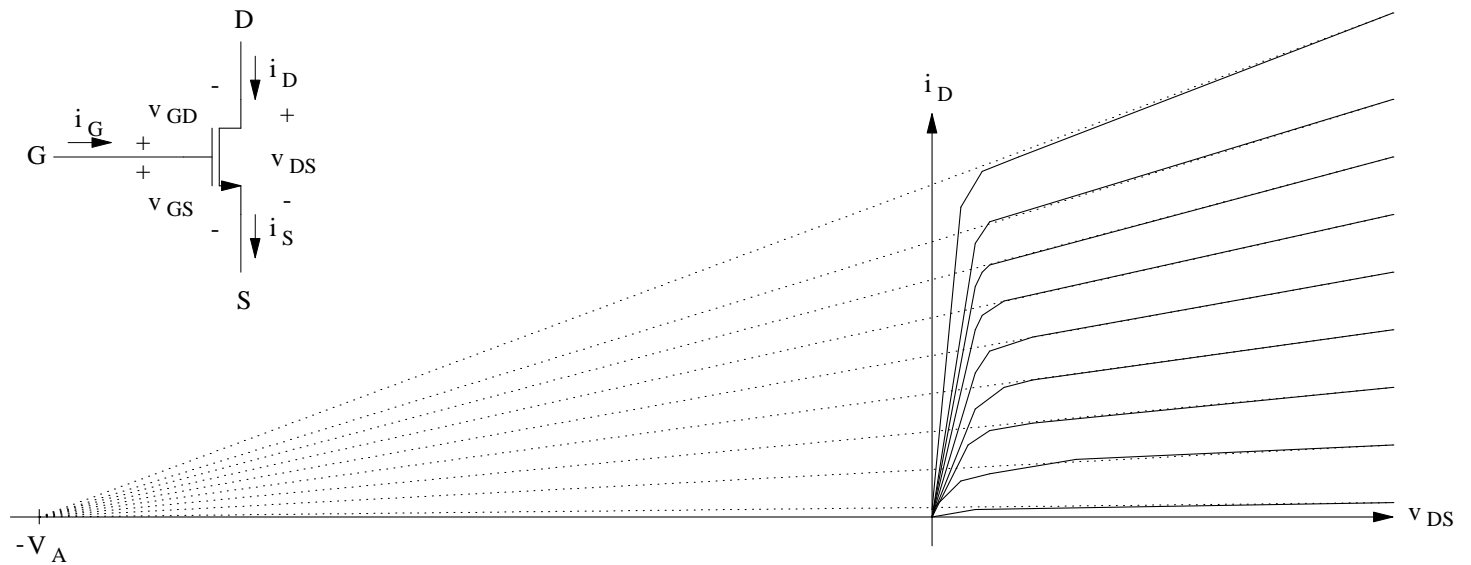
applications

voltage-to-current transducer

current-to-current transducer

photon-to-current transducer

## MOSFET



cut-off

$$v_{GS} \leq V_t \rightarrow i_D = 0$$

$$\text{triode } v_{GS} > V_t \text{ and } v_{GD} > V_t \rightarrow i_D = k_n \left( (v_{GS} - V_t) v_{DS} - \frac{1}{2} v_{DS}^2 \right)$$

saturation

$$v_{GS} > V_t \text{ and } v_{GD} \leq V_t \rightarrow i_D = \frac{1}{2} k_n (v_{GS} - V_t)^2 \left( 1 + \frac{v_{DS}}{V_A} \right)$$

applications

voltage-to-voltage transducer

voltage-controlled-resistor

## Filters

### Laplace transform basics

Laplace operator

$$L(f(t)) = F(s)$$

$$L^{-1}(F(s)) = f(t)$$

$$L\left(\frac{df(t)}{dt}\right) = sF(s) - f(0)$$

$$L\left(\int_0^t f(\tau) d\tau\right) = \frac{1}{s} F(s)$$

special functions of time

$$\delta(t) = \text{dirac impulse ; } L(\delta(t)) = 1$$

$$u(t) = \text{unit step ; } L(u(t)) = \frac{1}{s}$$

$$t = \text{ramp ; } L(t) = \frac{1}{s^2}$$

$$\text{sine ; } L(\sin(\omega t)) = \frac{\omega}{s^2 + \omega^2}$$

$$\text{cosine ; } L(\cos(\omega t)) = \frac{s}{s^2 + \omega^2}$$

$$\text{exponential ; } L(e^{-at}) = \frac{1}{s + a}$$

$$L(e^{-at} \sin(\omega t)) = \frac{\omega}{(s + a)^2 + \omega^2}$$

$$L(e^{-at} \cos(\omega t)) = \frac{s + a}{(s + a)^2 + \omega^2}$$

function scaling and shifting

$$\text{time scaling ; } L(f(kt)) = \frac{1}{k} F\left(\frac{s}{k}\right)$$

$$\text{value scaling ; } L(kf(t)) = kF(s)$$

$$\text{time shifting ; } L(f(t - T)) = e^{-sT} F(s)$$

pulse: sum of time-shifted unit steps

pulse is

$$f(t) = u(t) - u(t - T)$$

which transforms to

$$L(u(t) - u(t - T)) = \frac{1}{s} - \frac{1}{s} e^{-sT}$$

partial fractions

$$F(s) = K_o \frac{\prod_{i=1}^M (s - z_i)}{\prod_{i=1}^N (s - p_i)}$$

$$= \sum_{i=1}^N \frac{K_i}{s - p_i} \quad \text{for } N > M$$

$$= K_o + \sum_{i=1}^N \frac{K_i}{s - p_i} \quad \text{for } N = M$$

$$L^{-1}(F(s)) = \left( \sum_{i=1}^N K_i e^{-p_i t} \right) u(t) \quad \text{for } N > M$$

$$= \left( K_o \delta(t) + \sum_{i=1}^N K_i e^{-p_i t} \right) u(t) \quad \text{for } N = M$$

**partial fraction using residue method**

$$K_i = \left( (s - p_i) F(s) \right) \bigg|_{s=p_i}$$

example

$$F(s) = \frac{(s+2)(s+4)}{(s+1)(s+3)(s+5)} = \frac{K_1}{s+1} + \frac{K_2}{s+3} + \frac{K_3}{s+5}$$

$$K_1 = ((s+1)F(s)) \bigg|_{s=-1} = \frac{(-1+2)(-1+4)}{(-1+3)(-1+5)} = \frac{(1)(3)}{(2)(4)} = \frac{3}{8}$$

$$K_2 = ((s+3)F(s)) \bigg|_{s=-3} = \frac{(-3+2)(-3+4)}{(-3+1)(-3+5)} = \frac{(-1)(1)}{(-2)(2)} = \frac{1}{4}$$

$$K_3 = ((s+5)F(s)) \bigg|_{s=-5} = \frac{(-5+2)(-5+4)}{(-5+1)(-5+3)} = \frac{(-3)(-1)}{(-4)(-2)} = \frac{3}{8}$$

resulting in

$$L^{-1}(F(s)) = f(t) = \left( \frac{3}{8} e^{-t} + \frac{1}{4} e^{-3t} + \frac{3}{8} e^{-5t} \right) u(t)$$

**frequency domain**

generalised frequency:

$$s = \sigma + j\omega \quad ; \quad e^{st} = e^{(\sigma + j\omega)t} = e^{\sigma t} \cdot e^{j\omega t} = e^{\sigma t} \{ \cos(\omega t) + j \sin(\omega t) \}$$

physical frequency

$$s = j\omega$$

impedances and admittances

the differential equations for L and C transform to

$$i(t) = C \frac{dv(t)}{dt} \quad \rightarrow \quad I(s) = s C V(s)$$

or just

$$i = C \frac{dv}{dt} \quad \rightarrow \quad I = s C V$$

$$v = L \frac{di}{dt} \quad \rightarrow \quad V = s L I$$

the integral equations for L and C transform to

$$v(t) = \frac{1}{C} \int_0^t i(\tau) d\tau \quad \rightarrow \quad V = \frac{1}{s C} I$$

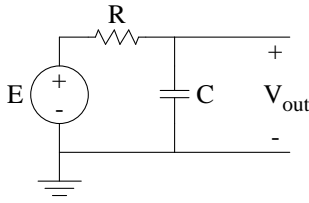
$$i(t) = \frac{1}{L} \int_0^t v(\tau) d\tau \quad \rightarrow \quad I = \frac{1}{s L} V$$

element	admittance Y	impedance Z
resistor	G	R
capacitor	s C	$\frac{1}{s C}$
inductor	$\frac{1}{s L}$	s L



**RC networks**

Given a series connected R-C network



a KVL gives us  $\left(R + \frac{1}{sC}\right)I(s) = E \rightarrow I(s) = \frac{sC}{sCR + 1} E$

$$V_{\text{out}}(s) = Z_C(s) I(s) = \frac{1}{sCR + 1} E = \frac{\frac{1}{RC}}{s + \frac{1}{RC}} E = \frac{1}{s\tau + 1} E$$

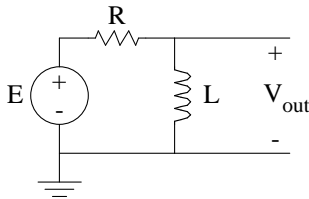
substituting  $E=1$  and  $s = j\omega$  gives  $V_{\text{out}}(j\omega) = \frac{1}{j\omega\tau + 1}$

the absolute value (or magnitude) is  $|V_{\text{out}}| = \frac{1}{\left(\omega^2\tau^2 + 1\right)^{\frac{1}{2}}}$

the phase angle (or phase) is  $\phi = -\arctan(\omega\tau)$

**RL networks**

Given a series connected R-L network



a KVL gives us  $(R + sL)I(s) = E \rightarrow I(s) = \frac{1}{sL + R} E$

$$V_{\text{out}}(s) = Z_L I(s) = \frac{sL}{sL + R} E = \frac{s}{s + \frac{R}{L}} E = \frac{s\tau}{s\tau + 1} E$$

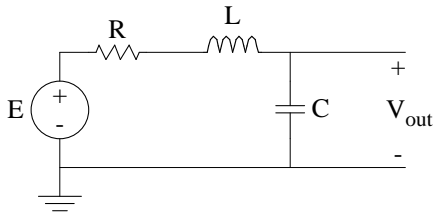
substituting  $E=1$  and  $s = j\omega$  gives  $V_{\text{out}}(j\omega) = \frac{j\omega\tau}{j\omega\tau + 1}$

the absolute value (or magnitude) is  $|V_{\text{out}}| = \frac{\left(\omega^2\tau^2\right)^{\frac{1}{2}}}{\left(\omega^2\tau^2 + 1\right)^{\frac{1}{2}}}$

the phase angle (or phase) is  $\phi = \frac{\pi}{2} - \arctan(\omega\tau)$

**RLC networks**

Given a series connected RLC network



a KVL gives us  $\left(R + sL + \frac{1}{sC}\right)I(s) = E$

$$I(s) = \left( \frac{1}{R + sL + \frac{1}{sC}} \right) E = \frac{sC}{s^2 LC + sCR + 1} E$$

$$= \frac{\frac{1}{L}}{s^2 + s \frac{R}{L} + \frac{1}{LC}} E$$

from which we get

$$V_{out}(s) = Z_C I(s) = \frac{1}{s^2 LC + sCR + 1} E = \frac{\frac{1}{LC}}{s^2 + s \frac{R}{L} + \frac{1}{LC}} E = \frac{\omega_o^2}{s^2 + s \frac{\omega_o}{Q_o} + \omega_o^2} E$$

from which we see that  $\omega_o = \frac{1}{(LC)^{\frac{1}{2}}}$  and  $\frac{\omega_o}{Q_o} = \frac{R}{L}$  so  $Q_o = \frac{\omega_o}{\left(\frac{R}{L}\right)} = \omega_o \frac{L}{R} = \frac{1}{(LC)^{\frac{1}{2}}} \frac{L}{R} = \frac{1}{R} \left(\frac{L}{C}\right)^{\frac{1}{2}}$

at the resonant frequency,  $\omega_o$ , we have  $V_L(\omega_o) + V_C(\omega_o) = 0$

as seen by  $sL I(s) = -\frac{1}{sC} I(s) \rightarrow s^2 = -\frac{1}{LC} \rightarrow (j\omega_o)^2 = -\omega_o^2 = -\frac{1}{LC} \rightarrow \omega_o = \frac{1}{(LC)^{\frac{1}{2}}}$

so  $V_r(\omega_o) = E$  and  $I(\omega_o) = I_r(\omega_o) = \frac{E}{R}$

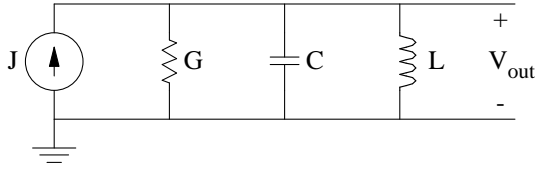
Q is the quality factor of an energy storage circuit.  $Q = 2\pi \frac{(\text{maximum energy stored per period})}{(\text{energy dissipated per period})}$

$Q_o$  is the tuned circuit quality factor at the resonant frequency.

At resonance in a series tuned circuit  $V_c(s) = \frac{I(s)}{sC}$  substituting  $s = j\omega_o$

$$|V_c| = \frac{\left(\frac{E}{R}\right)}{\omega_o C} = \frac{\left(\frac{1}{R}\right)}{\left(\frac{1}{(LC)^{\frac{1}{2}}} C\right)} E = \frac{\left(\frac{1}{R}\right)}{\left(\frac{C}{L}\right)^{\frac{1}{2}}} E = \frac{1}{R} \left(\frac{L}{C}\right)^{\frac{1}{2}} E = Q_o E$$

Given a parallel connected RLC network



a KCL gives us  $(G + sC + \frac{1}{sL}) V_{\text{out}}(s) = J$

$$V_{\text{out}}(s) = \frac{1}{G + sC + \frac{1}{sL}} J = \frac{sL}{s^2 LC + sLG + 1} J$$

$$= \frac{s \frac{1}{C}}{s^2 + s \frac{1}{RC} + \frac{1}{LC}} J = \left( \frac{1}{C} \right) \left( \frac{s}{s^2 + s \frac{\omega_o}{Q_o} + \omega_o^2} \right) J$$

from which we see again that  $\omega_o = \frac{1}{(LC)^{\frac{1}{2}}}$  and  $\frac{\omega_o}{Q_o} = \frac{G}{C}$  so

$$Q_o = \frac{\omega_o}{\left( \frac{G}{C} \right)} = \omega_o \frac{C}{G} = \frac{1}{(LC)^{\frac{1}{2}}} \frac{C}{G} = \frac{1}{G} \left( \frac{C}{L} \right)^{\frac{1}{2}}$$

at the resonant frequency,  $\omega_o$ , we have  $I_l(\omega_o) + I_c(\omega_o) = 0$

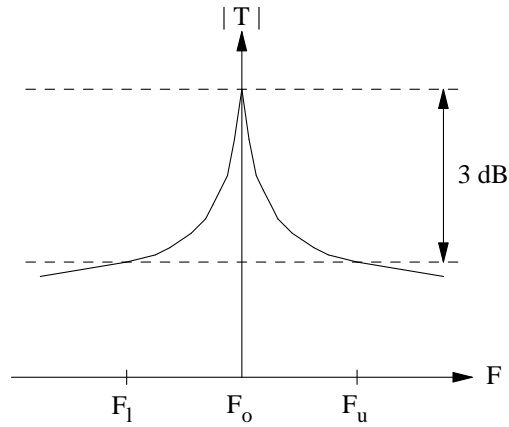
as seen by  $sC V_{\text{out}}(s) = -\frac{1}{sL} V_{\text{out}}$   $s^2 = -\frac{1}{LC} \rightarrow (j\omega_o)^2 = -\omega_o^2 = -\frac{1}{LC} \rightarrow \omega_o = \frac{1}{(LC)^{\frac{1}{2}}}$

so  $I_r(\omega_o) = J$  and  $V_{\text{out}}(\omega_o) = V_r(\omega_o) = \frac{J}{G}$

at resonance in a parallel tuned circuit  $I_l = \frac{V_{\text{out}}}{sL}$  substituting  $s = j\omega_o$

$$|I_l| = \frac{\left( \frac{J}{G} \right)}{\omega_o L} = \frac{\left( \frac{1}{G} \right)}{\left( \frac{1}{(LC)^{\frac{1}{2}}} L \right)} J = \frac{\left( \frac{1}{G} \right)}{\left( \frac{L}{C} \right)^{\frac{1}{2}}} J = \frac{1}{G} \left( \frac{C}{L} \right)^{\frac{1}{2}} J = Q_o J$$

returning to  $V_{\text{out}}(s) = \left( \frac{1}{C} \right) \left( \frac{s}{s^2 + s \frac{\omega_o}{Q_o} + \omega_o^2} \right) J$  we see that we have a bandpass transfer function



recalling that

$$20 \log(\sqrt{2}) = 3 \text{ dB}$$

we see that

$$\frac{|V_{\text{out}}(j\omega_o)|}{|V_{\text{out}}(j\omega_l)|} = \frac{|V_{\text{out}}(j\omega_o)|}{|V_{\text{out}}(j\omega_u)|} = \sqrt{2}$$

Note that

$$V_{\text{out}}(j\omega_o) = \left(\frac{1}{C}\right) \left( \frac{j\omega_o}{(j\omega_o)^2 + j\omega_o \frac{\omega_o}{Q_o} + \omega_o^2} \right) J = \left(\frac{1}{C}\right) \left( \frac{j\omega_o}{-\omega_o^2 + j\omega_o \frac{\omega_o}{Q_o} + \omega_o^2} \right) J = \left(\frac{1}{C}\right) \left( \frac{Q_o}{\omega_o} \right) J$$

After a number of algebraic operations, we get

$$\omega_l = -\frac{\omega_o}{2Q_o} + \omega_o \left( \frac{1}{4Q_o^2} + 1 \right)^{\frac{1}{2}} \quad \text{and} \quad \omega_u = \frac{\omega_o}{2Q_o} + \omega_o \left( \frac{1}{4Q_o^2} + 1 \right)^{\frac{1}{2}}$$

$$3 \text{ dB bandwidth} = \Delta \omega = \omega_u - \omega_l = \frac{\omega_o}{Q_o} \quad \text{and} \quad Q_o = \frac{\omega_o}{\Delta \omega} = \frac{f_o}{\Delta f}$$

**network functions**

- (1) network must be linear and all Cs & Ls have zero initial conditions.
- (2) connect source (E or J) to the input
- (3) solve for desired output & divide by source
- (4) avoid 1/s terms in network functions
- (5) for frequency domain response: use phasor for E or J and set  $s = j\omega$
- (6) for time domain response: use Laplace transform of E or J and inverse Laplace transform network function

$$\text{network function} = \frac{\text{output}}{\text{input}}$$

$$\text{voltage transfer function: } T_v = \frac{V_{\text{out}}}{E}$$

$$\text{current transfer function: } T_i = \frac{I_{\text{out}}}{J}$$

$$\text{transfer impedance: } Z_{\text{tr}} = \frac{V_{\text{out}}}{J}$$

$$\text{transfer admittance: } Y_{\text{tr}} = \frac{I_{\text{out}}}{E}$$

$$\text{input impedance: } Z_{\text{in}} = \frac{V_{\text{in}}}{J}$$

$$\text{input admittance: } Y_{\text{in}} = \frac{I_{\text{in}}}{E}$$

**poles and zeros**

$$F(s) = \frac{N(s)}{D(s)} = \frac{\sum_{i=0}^m A_i s^i}{\sum_{i=0}^n B_i s^i} = K_0 \frac{\prod_{i=1}^m (s - z_i)}{\prod_{i=1}^n (s - p_i)} \quad \text{where} \quad K_0 = \frac{A_m}{B_n} \quad \text{for practical circuits} \quad z_i = \alpha_i + j\beta_i \quad \text{and} \quad p_i = \gamma_i + j\zeta_i$$

**stability**

$$F(s) = \sum_{i=0}^n \frac{K_i}{s - p_i} \quad \rightarrow \quad f(t) = \sum_{i=0}^n K_i e^{t p_i}$$

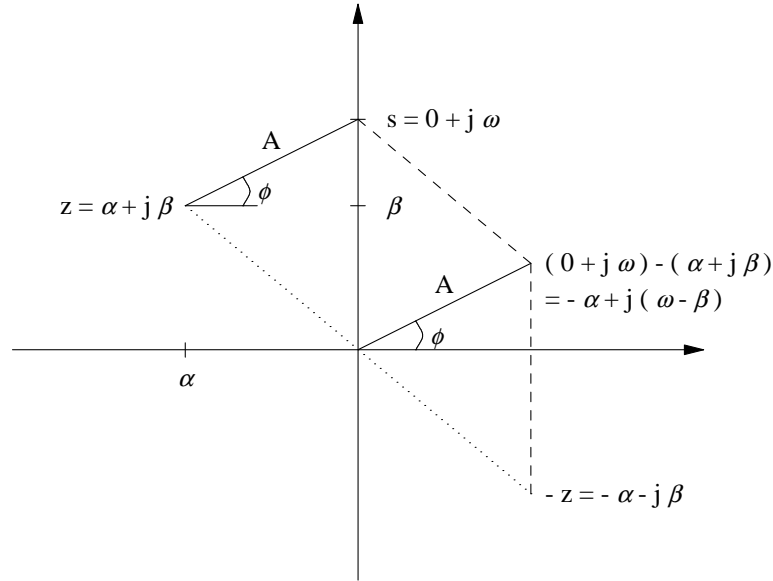
if  $\text{Re}(p_i) > 0$  then  $f(t)$  unstable. ; if  $\text{Re}(p_i) = 0$  then  $f(t)$  is an oscillator. ; if  $\text{Re}(p_i) < 0$  then  $f(t)$  stable.

**frequency response from poles & zeros**

$$\text{recall that} \quad F(s) = K_0 \frac{\prod_{i=1}^m (s - z_i)}{\prod_{i=1}^n (s - p_i)}$$

note that  $s$ ,  $z_i$ , and  $p_i$  are complex numbers which can be viewed as vectors.  $(s - z) = (0 + j\omega) - (\alpha + j\beta) = A e^{j\phi}$

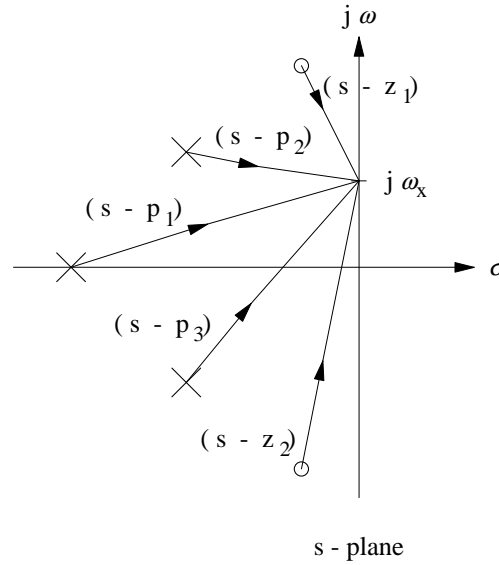
$$\text{where} \quad A = \left( (\omega - \beta)^2 + \alpha^2 \right)^{\frac{1}{2}} \quad \phi = \arctan\left( \frac{\omega - \beta}{-\alpha} \right)$$



$$|F(j\omega)| = |K_0| \frac{\prod_{i=1}^m A_{z,i}}{\prod_{i=1}^n A_{p,i}} = \left( K_0^2 \frac{\prod_{i=1}^m \left( (\omega - \beta_i)^2 + \alpha_i^2 \right)}{\prod_{i=1}^n \left( (\omega - \zeta_i)^2 + \gamma_i^2 \right)} \right)^{\frac{1}{2}}$$

$$20 \log(|F(j\omega)|) = 20 \log(|K_0|) + \sum_{i=1}^m 20 \log(|j\omega - z_i|) - \sum_{i=1}^n 20 \log(|j\omega - p_i|)$$

$$\begin{aligned} \phi(j\omega) &= \phi(K_0) + \sum_{i=1}^m \phi_{z,i} - \sum_{i=1}^n \phi_{p,i} \\ &= \phi(K_0) + \sum_{i=1}^m \arctan\left(\frac{\omega - \beta_i}{-\alpha_i}\right) - \sum_{i=1}^n \arctan\left(\frac{\omega - \zeta_i}{-\gamma_i}\right) \\ &= \phi(K_0) + \sum_{i=1}^m \arctan\left(\frac{\beta_i - \omega}{\alpha_i}\right) - \sum_{i=1}^n \arctan\left(\frac{\zeta_i - \omega}{\gamma_i}\right) \end{aligned}$$



$$F(s) = \frac{(s - z_1)(s - z_2)}{(s - p_1)(s - p_2)(s - p_3)} \quad ; \quad z_2 = \bar{z}_1 \quad \text{and} \quad p_3 = \bar{p}_2$$

### Bode plots

- (1) the dB magnitude plot is the summation of the dB magnitude plots of the individual  $K_0$ ,  $(s - z_i)$  and  $\frac{1}{s - p_i}$  terms.
- (2) the phase plot is the summation of the phase plots of the individual  $K_0$ ,  $(s - z_i)$  and  $\frac{1}{s - p_i}$  terms.

these terms are one of:

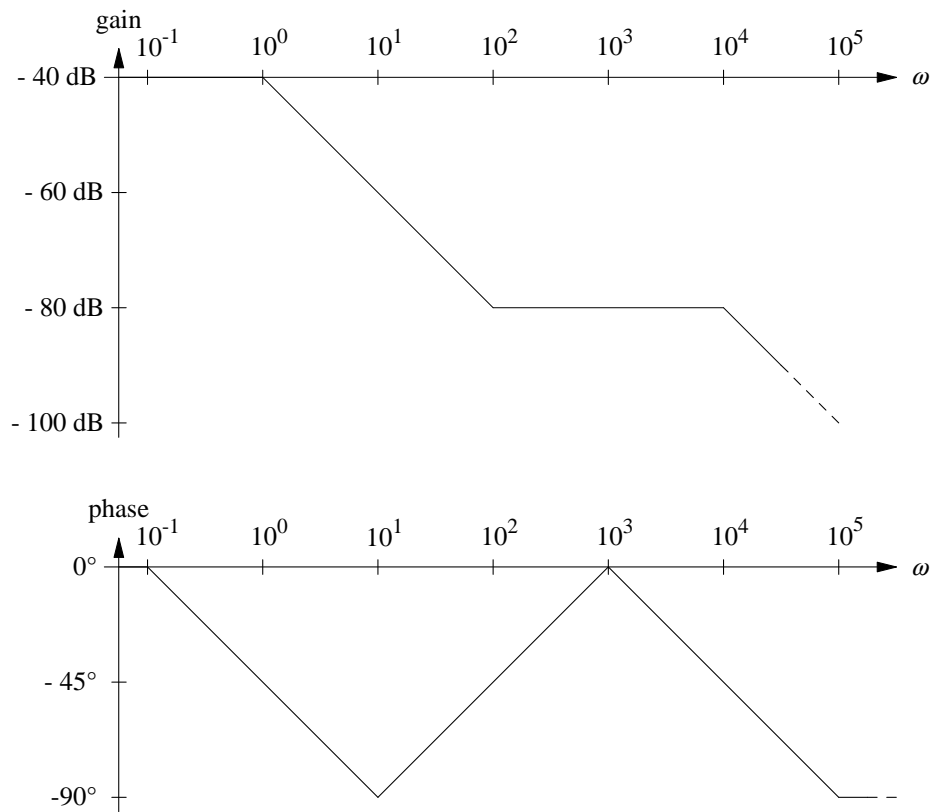
- (1)  $T(s) = K_0 \quad ; \quad T(j\omega) = K_0$
- (2)  $T(s) = \tau s = \frac{s}{\omega_o} \quad ; \quad T(j\omega) = \frac{j\omega}{\omega_o}$
- (3)  $T(s) = \frac{1}{\tau s} = \frac{\omega_o}{s} \quad ; \quad T(j\omega) = \frac{\omega_o}{j\omega} = -j \frac{\omega_o}{\omega}$
- (4)  $T(s) = (\tau s + 1) = \left( \frac{s}{\omega_o} + 1 \right) \quad ; \quad T(j\omega) = \left( j \frac{\omega}{\omega_o} + 1 \right)$
- (5)  $T(s) = \frac{1}{\tau s + 1} = \frac{1}{\frac{s}{\omega_o} + 1} \quad ; \quad T(j\omega) = \frac{1}{j \frac{\omega}{\omega_o} + 1}$
- (6)  $T(s) = \frac{1}{\omega_o^2} \left( s^2 + \frac{\omega_o}{Q_o} s + \omega_o^2 \right) = \left( \frac{s^2}{\omega_o^2} + \frac{s}{\omega_o Q_o} + 1 \right) \quad ; \quad T(j\omega) = \left( \frac{-\omega^2}{\omega_o^2} + \frac{j\omega}{\omega_o Q_o} + 1 \right)$
- (7)  $T(s) = \omega_o^2 \left( \frac{1}{s^2 + \frac{\omega_o}{Q_o} s + \omega_o^2} \right) = \frac{1}{\left( \frac{s^2}{\omega_o^2} + \frac{s}{\omega_o Q_o} + 1 \right)} \quad ; \quad T(j\omega) = \frac{1}{\left( \frac{-\omega^2}{\omega_o^2} + \frac{j\omega}{\omega_o Q_o} + 1 \right)}$

example 1

$$T(s) = \frac{(s + 10^2)}{(s + 1)(s + 10^4)} = \frac{10^2}{10^4} \frac{\left(\frac{s}{10^2} + 1\right)}{(s + 1)\left(\frac{s}{10^4} + 1\right)}$$

$$|T(j0)| = \frac{10^2}{10^4} = 10^{-2} \rightarrow -40 \text{ dB} \quad ; \quad \phi(T(j0)) = 0^\circ$$

$$|T(j\infty)| = 0 \rightarrow -\infty \text{ dB} \quad ; \quad \phi(T(j\infty)) = -90^\circ$$





example 2

$$T(s) = \frac{10^5 s}{(s + 10^2)(s + 10^5)} = \frac{\frac{s}{10^2}}{\left(\frac{s}{10^2} + 1\right)\left(\frac{s}{10^5} + 1\right)}$$

$$|T(j0)| = 0 \rightarrow -\infty \text{ dB} \quad ; \quad \phi(T(j0)) = +90^\circ$$

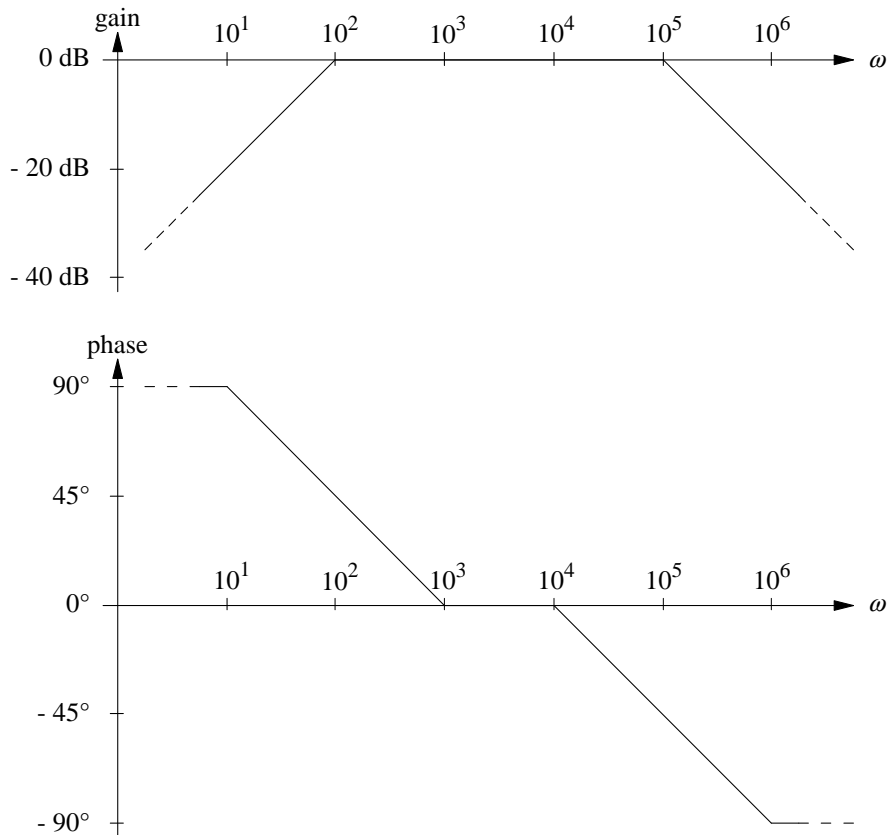
$$|T(j\infty)| = 0 \rightarrow -\infty \text{ dB} \quad ; \quad \phi(T(j\infty)) = -90^\circ$$

pick a point 3 decades before the pass band

$$|T(j10^{-1})| = |j10^{-3}| \rightarrow -60 \text{ dB}$$

gain rising at 20 dB / decade, therefore the gain at the first corner is

$$-60 \text{ dB} + (3 \text{ decades} \times 20 \text{ dB /decade}) = 0 \text{ dB}$$



As an aside

$$T(s) = \frac{10^5 s}{(s + 10^2)(s + 10^5)} = \frac{10^5 s}{s^2 + (10^5 + 10^2)s + (10^5 10^2)} \rightarrow \omega_o^2 = 10^5 10^2 \rightarrow \omega_o = 3162 \text{ Rad/s}$$

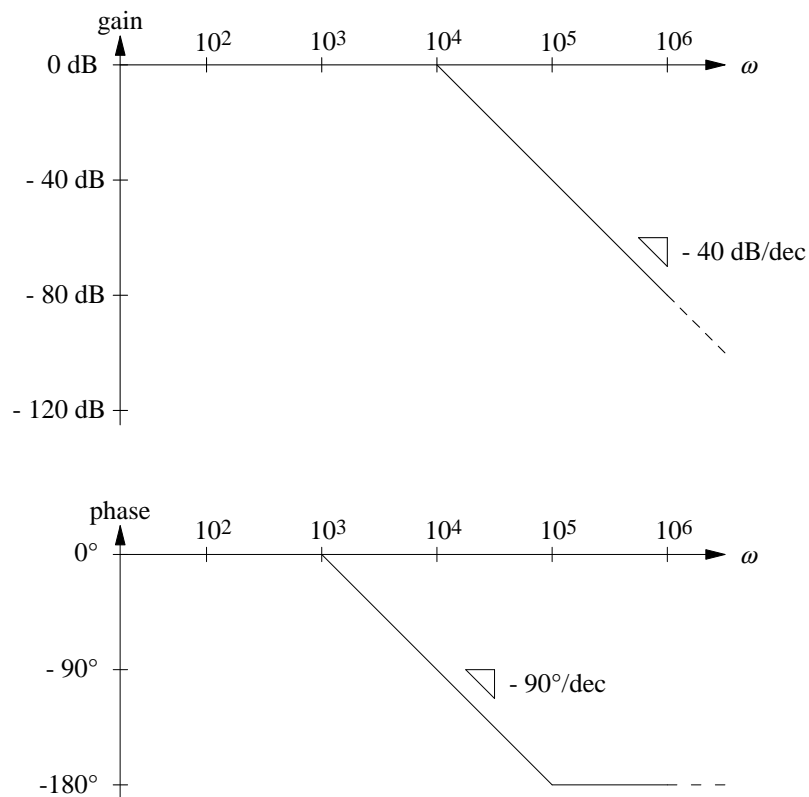
$$\frac{\omega_o}{Q_0} = 10^5 + 10^2 \rightarrow Q_0 = \frac{\omega_o}{10^5 + 10^2} = \frac{(10^7)^{\frac{1}{2}}}{10^5 + 10^2} = 0.03159 \quad ; \quad Q_0 = \frac{f_o}{\text{BW}} = \frac{\sqrt{f_{lp} f_{hp}}}{f_{lp} - f_{hp}}$$

example 3

$$T(s) = \frac{(10^4)^2}{(s + 10^4)^2} = \frac{1}{\left(\frac{s}{10^4} + 1\right)^2}$$

$$|T(j0)| = 1 \rightarrow 0 \text{ dB} \quad ; \quad \phi(T(j0)) = 0^\circ$$

$$|T(j\infty)| = 0 \rightarrow -\infty \text{ dB} \quad ; \quad \phi(T(j\infty)) = -180^\circ$$



As an aside  $\omega_o = 10^4$  and  $Q_o = 0.5$

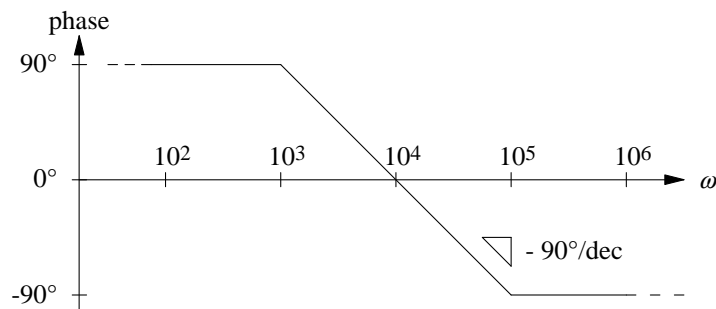
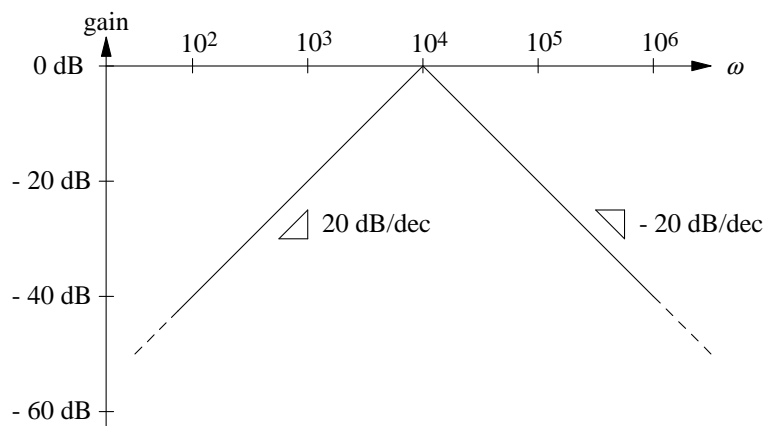
example 4

$$T(s) = \frac{10^4 s}{(s + 10^4)^2} = \frac{\frac{s}{10^4}}{\left(\frac{s}{10^4} + 1\right)^2}$$

$$|T(j0)| = 0 \rightarrow -\infty \text{ dB} \quad ; \quad \phi(T(j0)) = +90^\circ$$

$$|T(j\infty)| = 0 \rightarrow -\infty \text{ dB} \quad ; \quad \phi(T(j\infty)) = -90^\circ$$

$$|T(j10)| = |j10^{-3}| \rightarrow -60 \text{ dB} \quad \text{the gain at the first corner is} \quad -60 \text{ dB} + (3 \text{ decades} \times 20 \text{ dB/decade}) = 0 \text{ dB}$$



As an aside  $\omega_o = 10^4$  and  $Q_o = 0.5$

**Bode plot for quadratic terms with  $Q_o > .5$** 

dB plot

start with the  $Q_o = 0.5$  case, sketch resonant peak on top, resonant peak at  $\omega_o \left( \frac{1}{4Q_o^2} + 1 \right)^{\frac{1}{2}}$ ,  
 resonant peak is  $20 \log(Q_o)$  higher/lower than  $Q_o = 0.5$  case, 3 dB bandwidth of peak is  $\frac{\omega_o}{Q_o}$

phase plot

starting/ending phase same as  $Q_o = 0.5$  case, starting phase  $\pm 45^\circ$  at lower 3 dB frequency of resonant peak, starting phase  $\pm 90^\circ$  at resonant peak, starting phase  $\pm 135^\circ$  at upper 3 dB frequency of resonant peak

upper/lower 3 dB frequencies

lower 3 dB freq is  $\omega_l = -\frac{\omega_o}{2Q_o} + \omega_o \left( \frac{1}{4Q_o^2} + 1 \right)^{\frac{1}{2}}$

upper 3 dB freq is  $\omega_u = \frac{\omega_o}{2Q_o} + \omega_o \left( \frac{1}{4Q_o^2} + 1 \right)^{\frac{1}{2}}$

example 5

$$T(s) = \frac{(10^4)^2}{s^2 + 10^3 s + (10^4)^2} = \frac{1}{\frac{s^2}{(10^4)^2} + \frac{10^3}{(10^4)^2} s + 1} = \frac{1}{10^{-8} s^2 + 10^{-5} s + 1}$$

$$|T(j0)| = 1 \rightarrow 0 \text{ dB} \quad ; \quad \phi(T(j0)) = 0^\circ$$

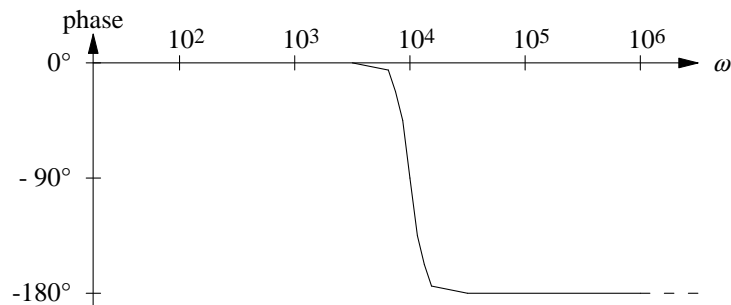
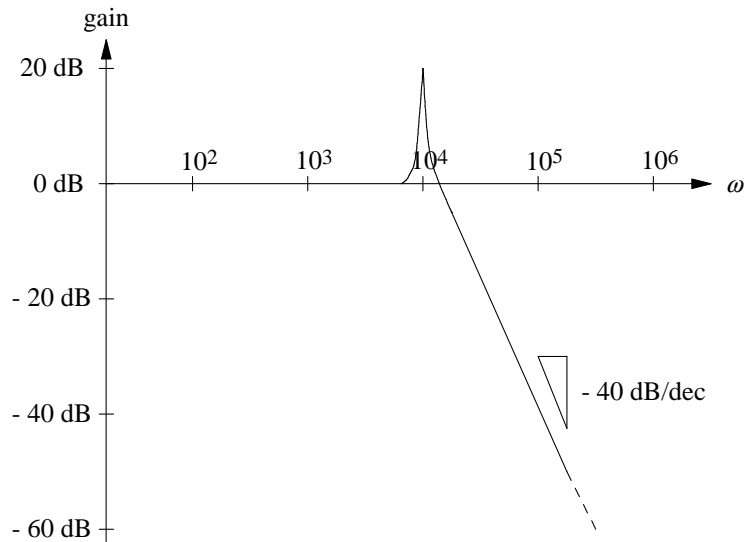
$$|T(j\infty)| = 0 \rightarrow -\infty \text{ dB} \quad ; \quad \phi(T(j\infty)) = -180^\circ$$

$$(\omega_o)^2 = (10^4)^2 \rightarrow \omega_o = 10^4 \text{ Rad/s} \quad ; \quad \frac{\omega_o}{Q_o} = 10^3 \rightarrow Q_o = \frac{\omega_o}{10^3} = \frac{10^4}{10^3} = 10$$

from example 3 we know that the peak for the  $Q_o = 0.5$  case is 0 dB

resonant peak is  $0 \text{ dB} + 20 \log(Q_o) = 0 \text{ dB} + 20 \log(10) = 20 \text{ dB}$

3 dB bandwidth is  $\frac{\omega_o}{Q_o} = \frac{10^4}{10} = 10^3 \text{ Rad/s}$



example 6

$$T(s) = \frac{10^4 s}{s^2 + 10^3 s + (10^4)^2} = \frac{\frac{s}{10^4}}{\frac{s^2}{(10^4)^2} + \frac{10^3}{(10^4)^2} s + 1} = \frac{10^{-4} s}{10^{-8} s^2 + 10^{-5} s + 1}$$

$$|T(j0)| = 0 \rightarrow -\infty \text{ dB} \quad ; \quad \phi(T(j0)) = +90^\circ$$

$$|T(j\infty)| = 0 \rightarrow -\infty \text{ dB} \quad ; \quad \phi(T(j\infty)) = -90^\circ$$

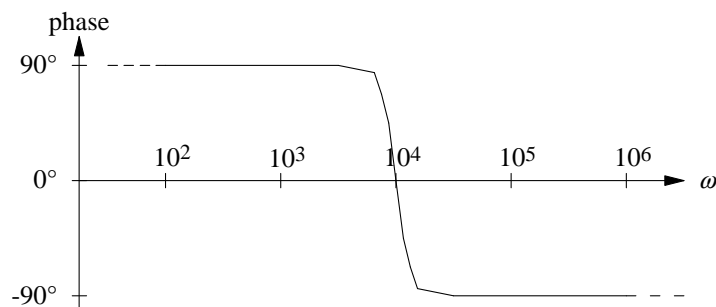
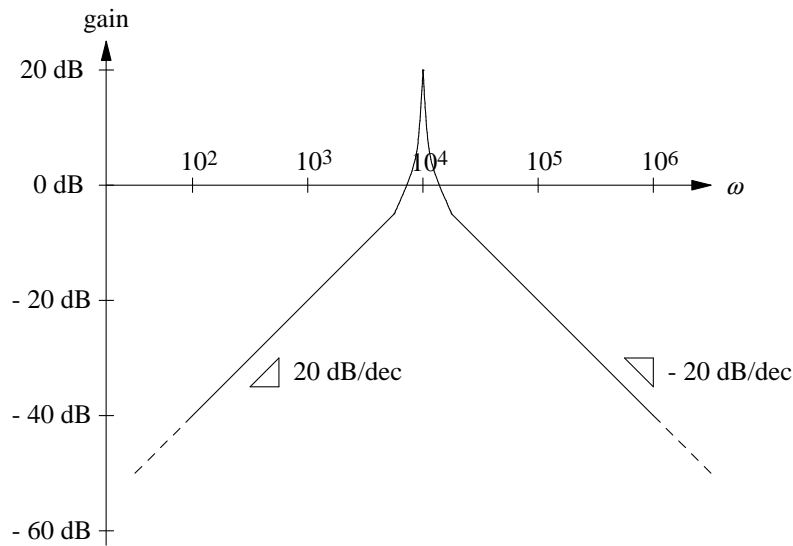
$$|T(j10)| = |j10^{-3}| \rightarrow -60 \text{ dB}$$

the gain at the first corner is  $-60 \text{ dB} + (3 \text{ decades} \times 20 \text{ dB/decade}) = 0 \text{ dB}$

$$(\omega_o)^2 = (10^4)^2 \rightarrow \omega_o = 10^4 \text{ Rad/s} \quad ; \quad \frac{\omega_o}{Q_o} = 10^3 \rightarrow Q_o = \frac{\omega_o}{10^3} = \frac{10^4}{10^3} = 10$$

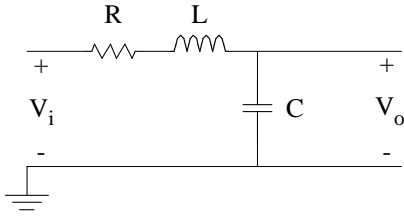
from example 4 we know that the peak for the  $Q_o = 0.5$  case is 0 dB so the resonant peak is

$$0 \text{ dB} + 20 \log(Q_o) = 0 \text{ dB} + 20 \log(10) = 20 \text{ dB} \quad \text{and} \quad 3 \text{ dB bandwidth is } \frac{\omega_o}{Q_o} = \frac{10^4}{10} = 10^3 \text{ Rad/s}$$



**RLC passive filters**

low-pass filter

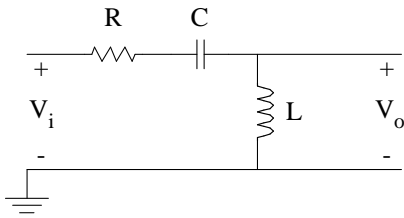


$$A_v = \frac{V_o}{V_i} = \frac{\frac{1}{L C}}{s^2 + \frac{R}{L} s + \frac{1}{L C}}$$

$$\omega_o = \frac{1}{\sqrt{L C}}$$

$$Q_o = \frac{1}{R} \sqrt{\frac{L}{C}}$$

high-pass filter

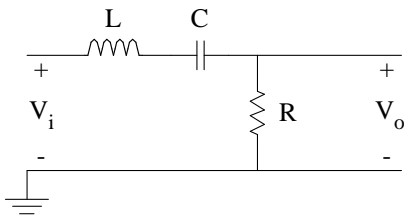


$$A_v = \frac{V_o}{V_i} = \frac{s^2}{s^2 + \frac{R}{L} s + \frac{1}{L C}}$$

$$\omega_o = \frac{1}{\sqrt{L C}}$$

$$Q_o = \frac{1}{R} \sqrt{\frac{L}{C}}$$

band-pass filter

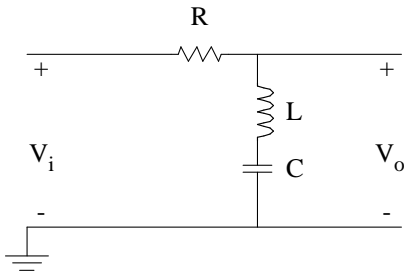


$$A_v = \frac{V_o}{V_i} = \frac{\frac{R}{L} s}{s^2 + \frac{R}{L} s + \frac{1}{L C}}$$

$$\omega_o = \frac{1}{\sqrt{L C}}$$

$$Q_o = \frac{1}{R} \sqrt{\frac{L}{C}}$$

band-stop filter



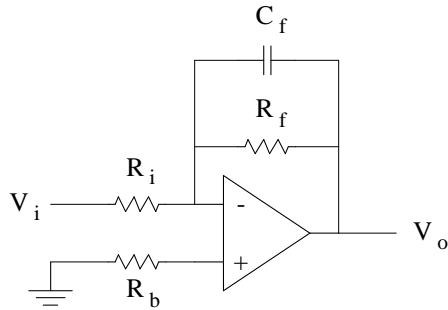
$$A_v = \frac{V_o}{V_i} = \frac{s^2 + \frac{1}{L C}}{s^2 + \frac{R}{L} s + \frac{1}{L C}}$$

$$\omega_o = \frac{1}{\sqrt{L C}}$$

$$Q_o = \frac{1}{R} \sqrt{\frac{L}{C}}$$

## single opamp active filters

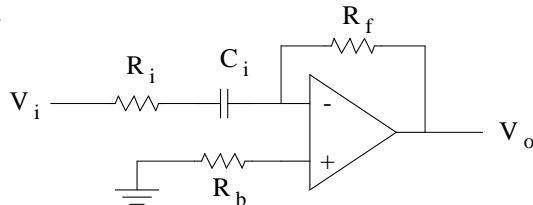
low-pass filter



$$R_b = R_f \parallel R_i$$

$$A_v = \frac{V_o}{V_i} = \frac{-\frac{R_f}{R_i}}{sR_fC_f + 1}$$

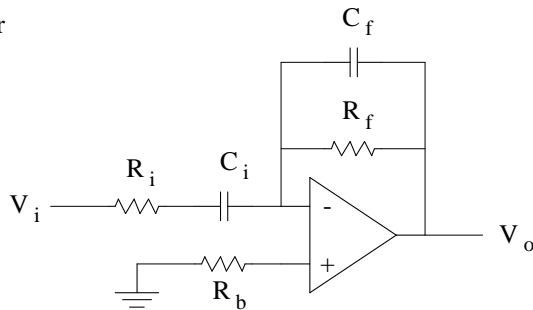
high-pass filter



$$R_b = R_f$$

$$A_v = \frac{V_o}{V_i} = \frac{-\frac{R_f}{R_i} sR_iC_i}{sR_iC_i + 1}$$

band-pass filter

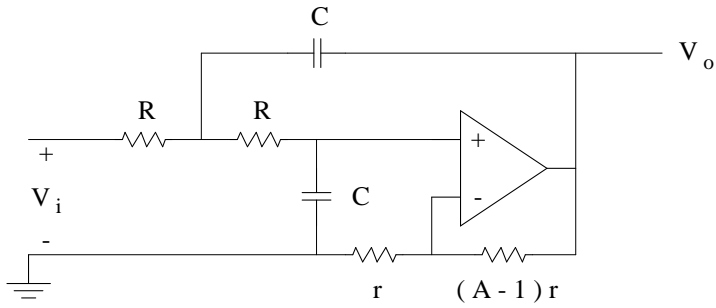


$$R_b = R_f$$

$$A_v = \frac{V_o}{V_i} = \frac{-\frac{R_f}{R_i} sR_iC_i}{\left(sR_iC_i + 1\right)\left(sR_fC_f + 1\right)}$$



## Sallen-Key single opamp biquadratic filters

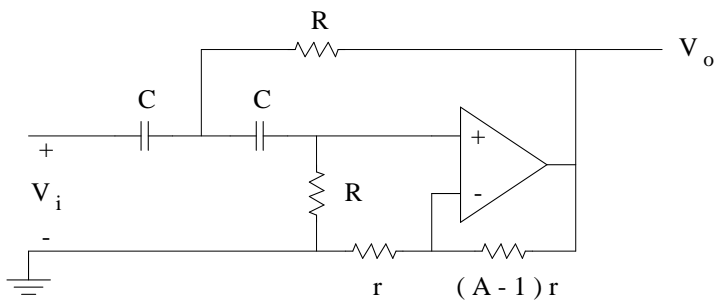


$$\omega_o = \frac{1}{RC}$$

$$A_v = \frac{V_o}{V_i} = A \frac{\omega_o^2}{s^2 + \frac{\omega_o}{Q_o} s + \omega_o^2}$$

$$Q_o = \frac{1}{3-A}$$

$$2R = r \parallel ((A-1)r)$$

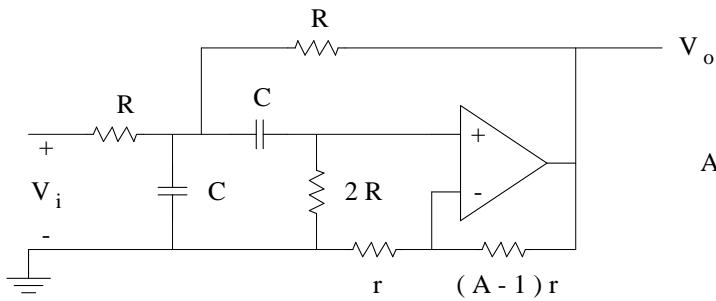


$$\omega_o = \frac{1}{RC}$$

$$A_v = \frac{V_o}{V_i} = A \frac{s^2}{s^2 + \frac{\omega_o}{Q_o} s + \omega_o^2}$$

$$Q_o = \frac{1}{3-A}$$

$$R = r \parallel ((A-1)r)$$

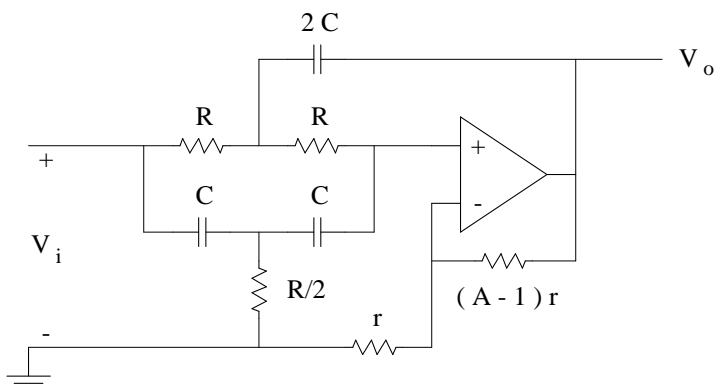


$$\omega_o = \frac{1}{RC}$$

$$A_v = \frac{V_o}{V_i} = A Q_o \frac{\frac{\omega_o}{Q_o} s}{s^2 + \frac{\omega_o}{Q_o} s + \omega_o^2}$$

$$Q_o = \frac{1}{3-A}$$

$$2R = r \parallel ((A-1)r)$$



$$\omega_o = \frac{1}{RC}$$

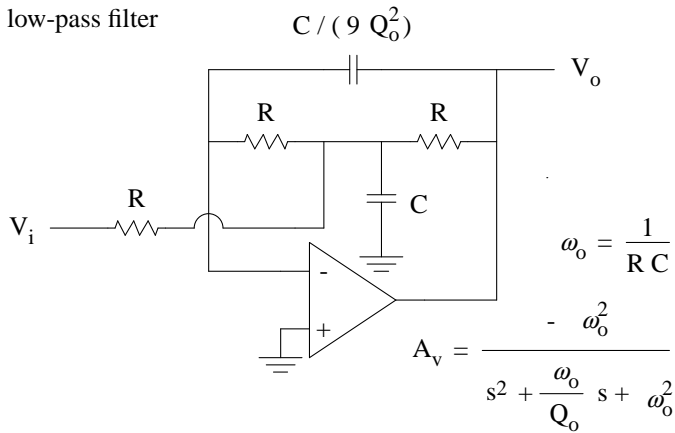
$$A_v = \frac{V_o}{V_i} = A \frac{s^2 + \omega_o^2}{s^2 + \frac{\omega_o}{Q_o} s + \omega_o^2}$$

$$Q_o = \frac{1}{4-2A}$$

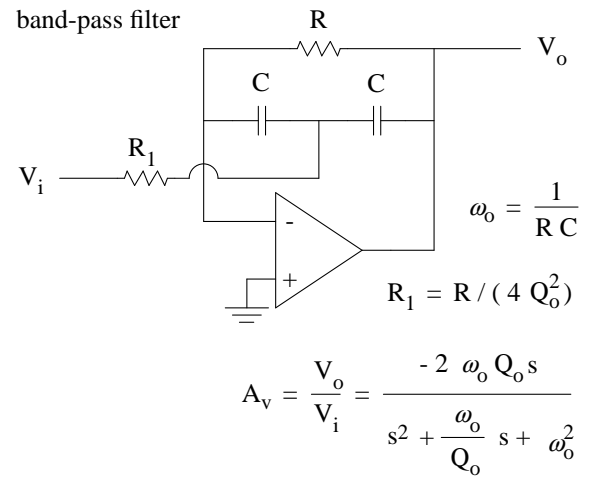
$$2R = r \parallel ((A-1)r)$$

## other single opamp biquadratic filters

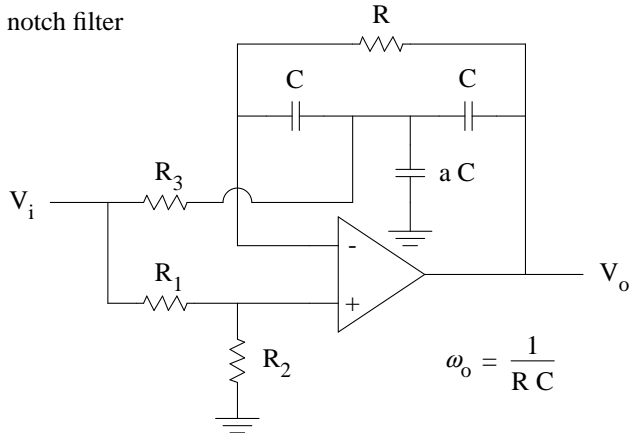
low-pass filter



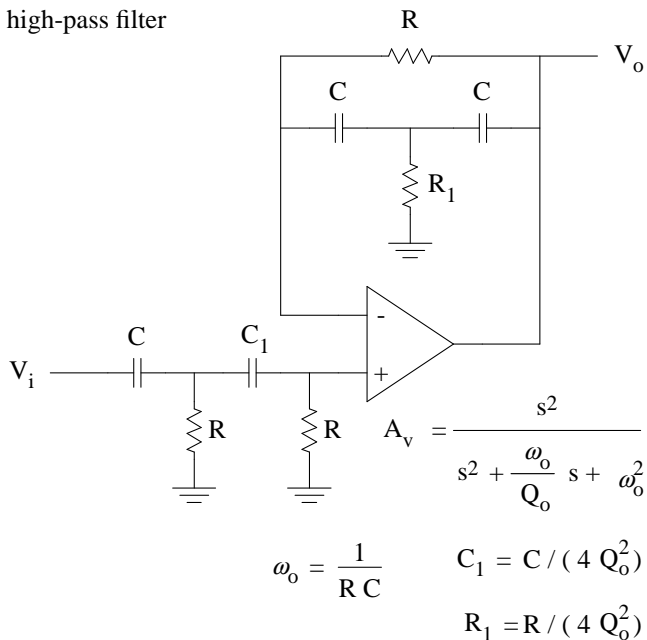
band-pass filter



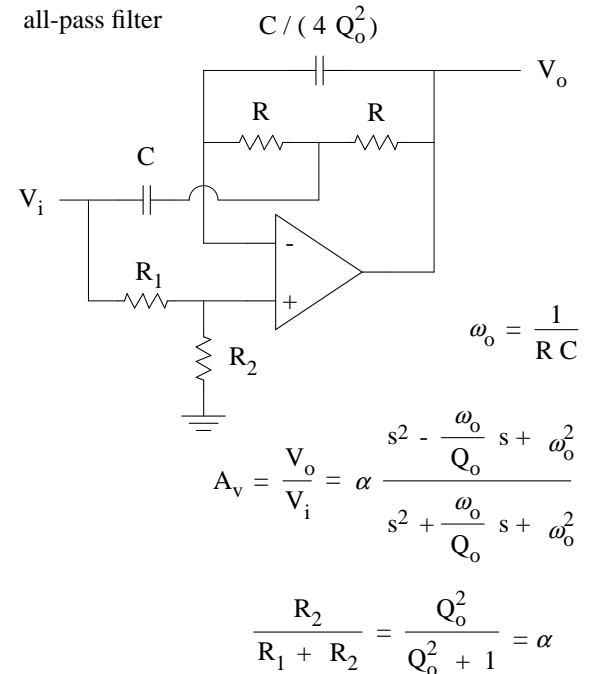
notch filter



high-pass filter

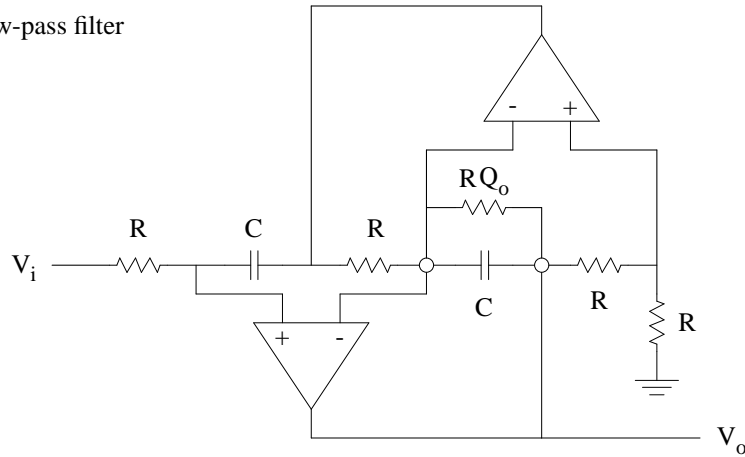


all-pass filter



## two opamp biquadratic filters

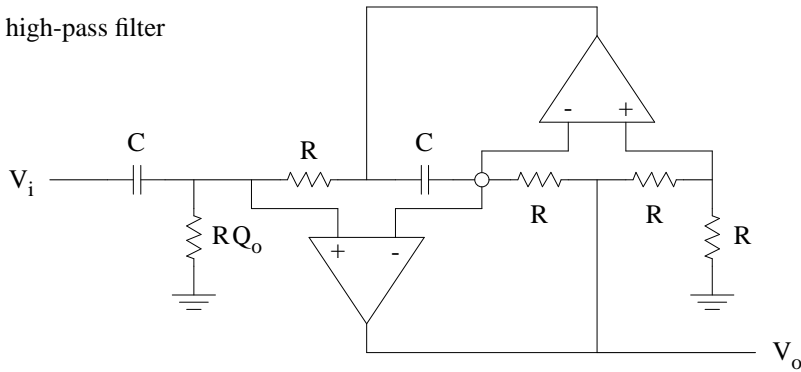
low-pass filter



$$A_v = \frac{V_o}{V_i} = 2 \frac{\omega_o^2}{s^2 + \frac{\omega_o}{Q_o} s + \omega_o^2}$$

$$\omega_o = \frac{1}{RC}$$

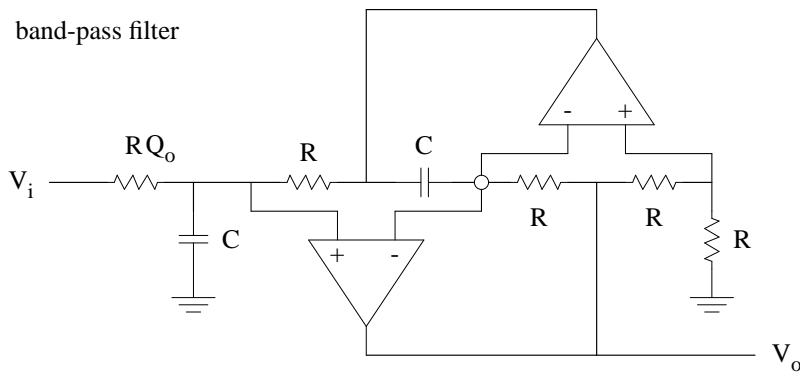
high-pass filter



$$A_v = \frac{V_o}{V_i} = 2 \frac{s^2}{s^2 + \frac{\omega_o}{Q_o} s + \omega_o^2}$$

$$\omega_o = \frac{1}{RC}$$

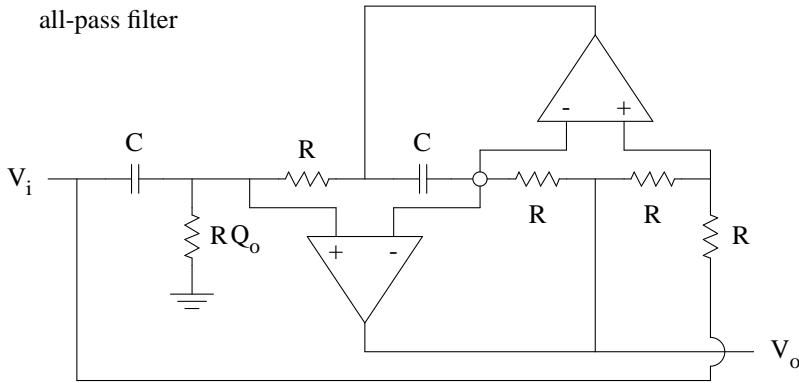
band-pass filter



$$A_v = \frac{V_o}{V_i} = 2 \frac{\frac{\omega_o}{Q_o} s}{s^2 + \frac{\omega_o}{Q_o} s + \omega_o^2}$$

$$\omega_o = \frac{1}{RC}$$

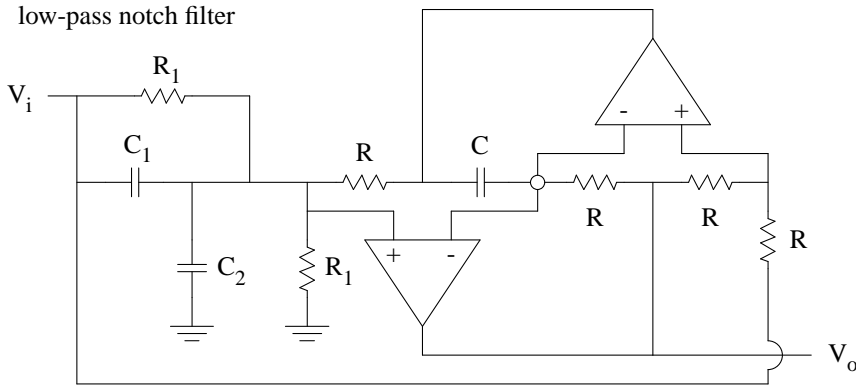
all-pass filter



$$A_v = \frac{V_o}{V_i} = \frac{s^2 - \frac{\omega_o}{Q_o} s + \omega_o^2}{s^2 + \frac{\omega_o}{Q_o} s + \omega_o^2}$$

$$\omega_o = \frac{1}{RC}$$

low-pass notch filter



$$A_v = \frac{V_o}{V_i} = \left( \frac{\omega_o}{\omega_n} \right)^2 \frac{s^2 + \omega_n^2}{s^2 + \frac{\omega_o}{Q_o} s + \omega_o^2}$$

$$\alpha = \frac{\omega_n}{\omega_o} \geq 1$$

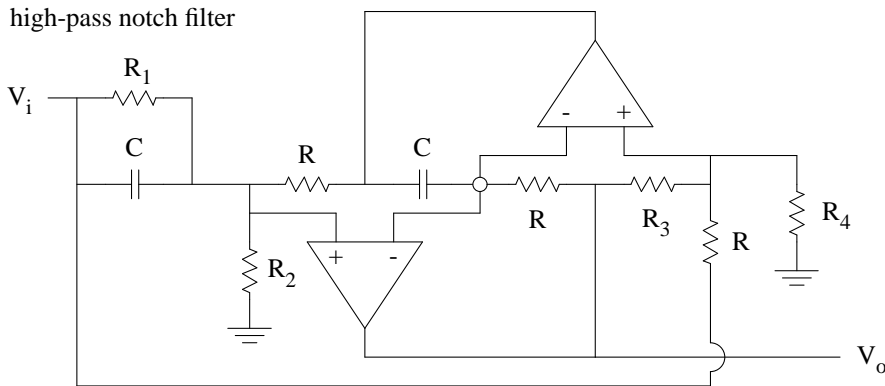
$$R_1 = 2RQ_o$$

$$\omega_o = \frac{1}{RC}$$

$$C_1 = \frac{\alpha^2 + 1}{2\alpha^2} C$$

$$C_2 = \frac{\alpha^2 - 1}{2\alpha^2} C$$

high-pass notch filter



$$A_v = \frac{V_o}{V_i} = 2 \frac{2 - \alpha^2}{3 - \alpha^2} \frac{s^2 + \omega_n^2}{s^2 + \frac{\omega_o}{Q_o} s + \omega_o^2}$$

$$\alpha = \frac{\omega_n}{\omega_o} \leq 1$$

$$R_1 = \frac{\alpha^2 + 1}{\alpha^2} Q_o R$$

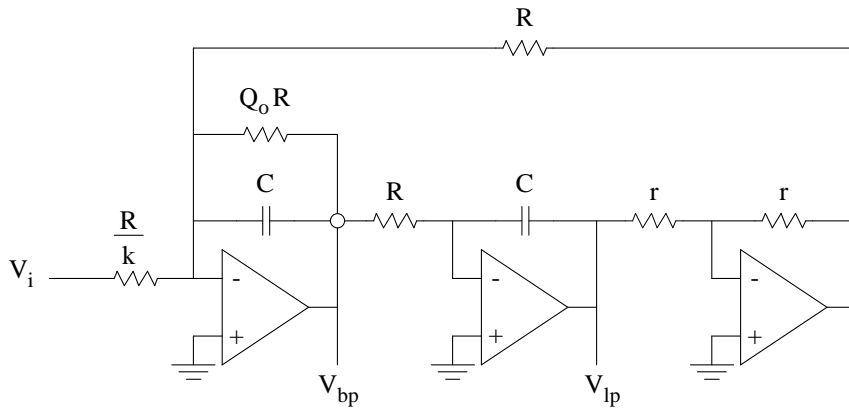
$$\omega_o = \frac{1}{RC}$$

$$R_2 = (\alpha^2 + 1) Q_o R$$

$$R_3 = \frac{2\alpha^2}{1 + \alpha^2} R$$

$$R_4 = \frac{2\alpha^2}{1 - \alpha^2} R$$

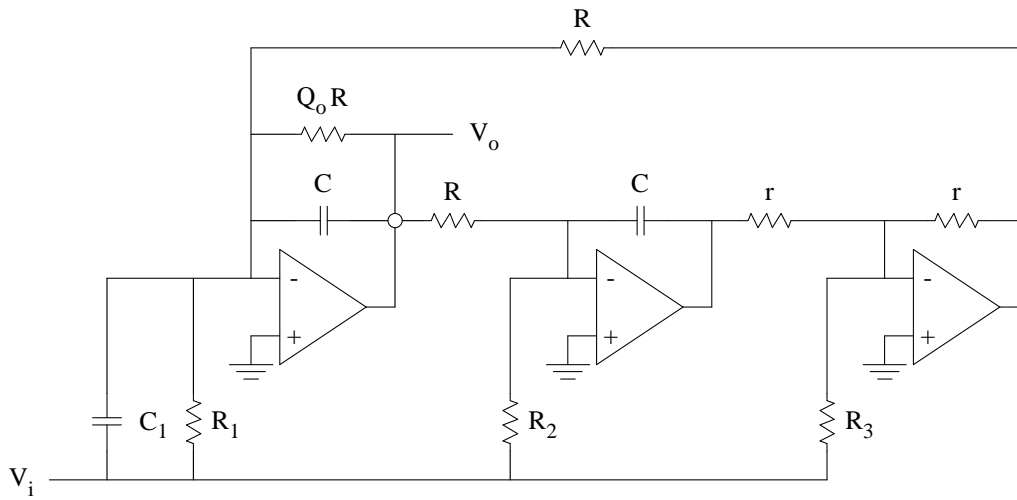
## three opamp biquadratic filters



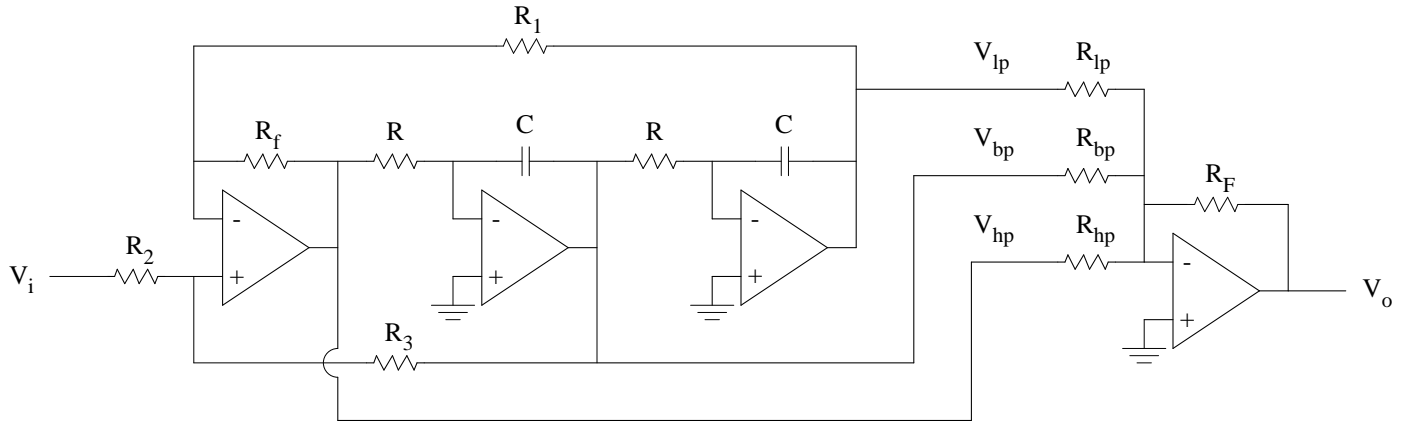
$$A_{lp} = \frac{V_{lp}}{V_i} = k \frac{\omega_o^2}{s^2 + \frac{\omega_o}{Q_o} s + \omega_o^2}$$

$$A_{bp} = \frac{V_{bp}}{V_i} = k \frac{\omega_o s}{s^2 + \frac{\omega_o}{Q_o} s + \omega_o^2}$$

$$\omega_o = \frac{1}{RC}$$



$$A_v = \frac{V_o}{V_i} = \frac{\frac{C_1}{C} s^2 + \frac{1}{C} \left( \frac{1}{R_1} - \frac{r}{R_3 R} \right) s + \frac{1}{R_2 R C^2}}{s^2 + \frac{1}{Q_o R C} s + \frac{1}{C^2 R^2}}$$



$$A_{hp} = \frac{V_{hp}}{V_i} = k \frac{s^2}{s^2 + \frac{\omega_o}{Q_o} s + \omega_o^2}$$

$$A_{bp} = \frac{V_{bp}}{V_i} = k \frac{\frac{\omega_o}{Q_o} s}{s^2 + \frac{\omega_o}{Q_o} s + \omega_o^2}$$

$$A_{lp} = \frac{V_{lp}}{V_i} = k \frac{\omega_o^2}{s^2 + \frac{\omega_o}{Q_o} s + \omega_o^2}$$

$$k = 2 - \frac{1}{Q_o}$$

$$\omega_o = \frac{1}{RC}$$

$$\frac{R_f}{R_1} = 1$$

$$\frac{R_3}{R_2} = 2Q_o - 1$$

$$V_o = - \left( \frac{R_F}{R_{lp}} V_{lp} + \frac{R_F}{R_{bp}} V_{bp} + \frac{R_F}{R_{hp}} V_{hp} \right)$$

## Sensors and Process Control

### analog data representation

measurement errors

difference between the actual value and the measured value

measurement accuracy

measured variable  $\pm$  accuracy

percentage error of full scale

percentage of instrument span

percentage of actual reading

system accuracy

take into account all the individual errors

sensitivity

change in output signal for a change in input signal

hysteresis

the relationship between the input and output signals depends on whether the input signal is increasing or decreasing

reproducibility

same output sequence each time the input sequence repeats

resolution

minimum measurable value of the input signal

linearity

for a given input, one unique output value  $c_m = m \cdot c + c_o$

### sensor time response

first-order response

sensor response is dominated by a first-order differential equation

second-order response

sensor response is dominated by a second-order differential equation

### sensors and transducers

transducer

converts physical measured energy (radiant, mechanical, thermal, electrical, magnetic, chemical) into electrical energy

passive sensors

passive sensors modify an applied voltage or current

active sensors

active sensors produce an output signal (voltage or current) directly

physical effects

resistance  $R = \rho \frac{L}{A}$

capacitance  $C = \epsilon_o \epsilon_r \frac{A}{d}$

inductance  $L = \mu_o \frac{N^2 A}{d}$

## Analog Signal Conditioning

### analog signal conditioning terms

change signal level (amplify or attenuate)

change signal bias or offset value

linearization: straight line approximation of the sensor response

conversions: e.g. change in R seen as a change in V

signal transmission: transport signals

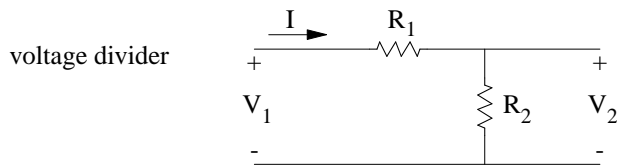
filtering

impedance match between sensor and signal conditioning circuit and/or load

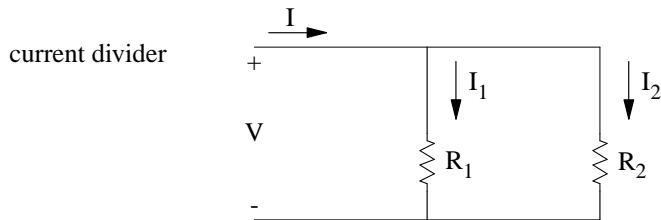
sensor loading by signal conditioning circuit

signal conditioning circuit loaded by process controller input

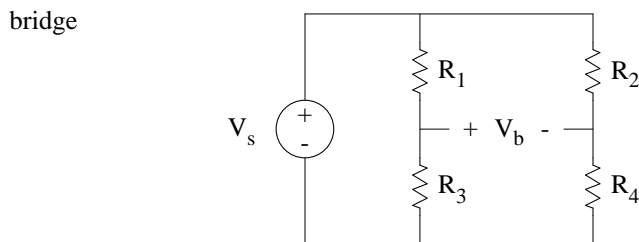
### passive circuits



$$V_2 = R_2 I = R_2 \frac{V_1}{R_1 + R_2} = \frac{R_2}{R_1 + R_2} V_1$$



$$I_2 = \frac{V}{R_2} = \frac{\left( \frac{I}{\frac{1}{R_1} + \frac{1}{R_2}} \right)}{R_2} = \frac{\left( \frac{R_1 R_2}{R_1 + R_2} \right) I}{R_2} = \frac{R_1}{R_1 + R_2} I$$

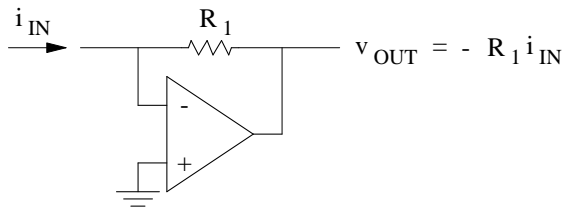


$$V_b = \frac{R_3 R_2 - R_1 R_4}{\left( R_1 + R_3 \right) \left( R_2 + R_4 \right)} V_s$$

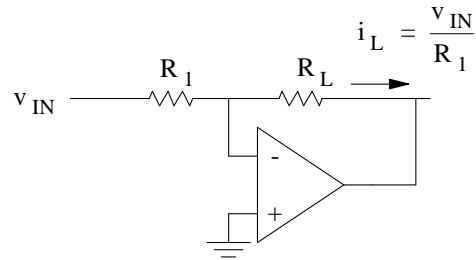


**operational amplifier circuits**

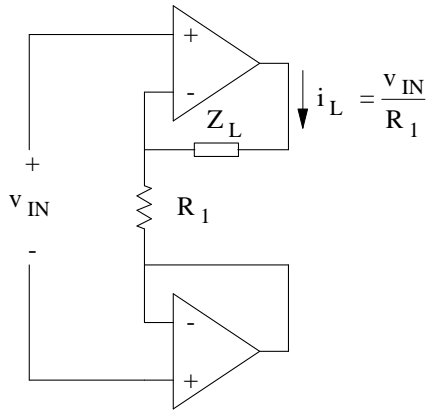
current-to-voltage converter



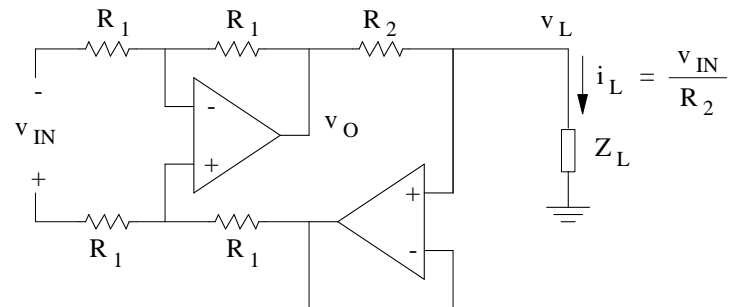
voltage-to-current converter with floating load

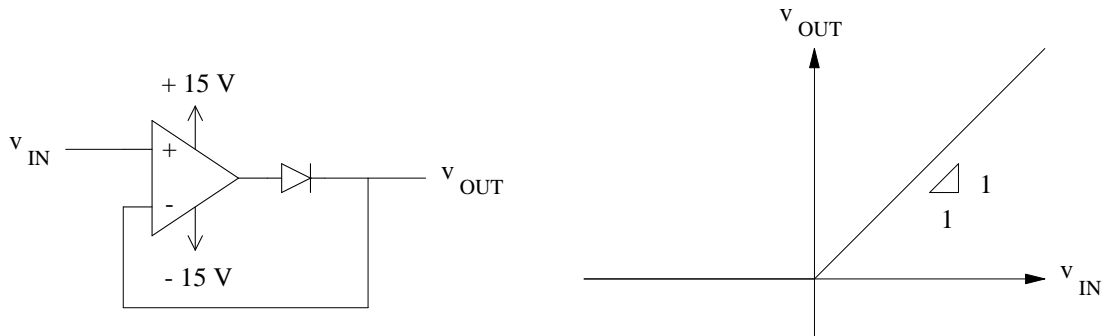


voltage-to-current converter with floating load

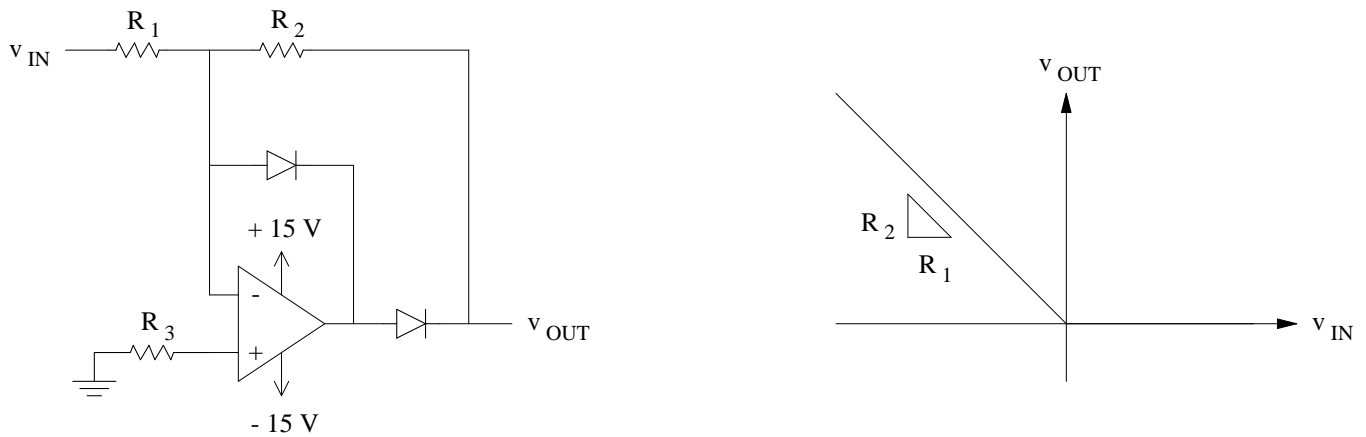


voltage-to-current converter with non-floating load



**active diode circuits****super-diode**

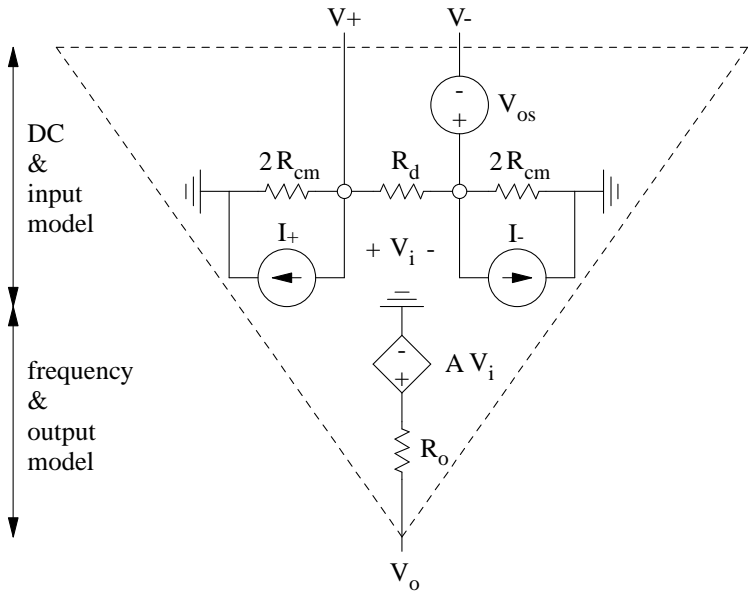
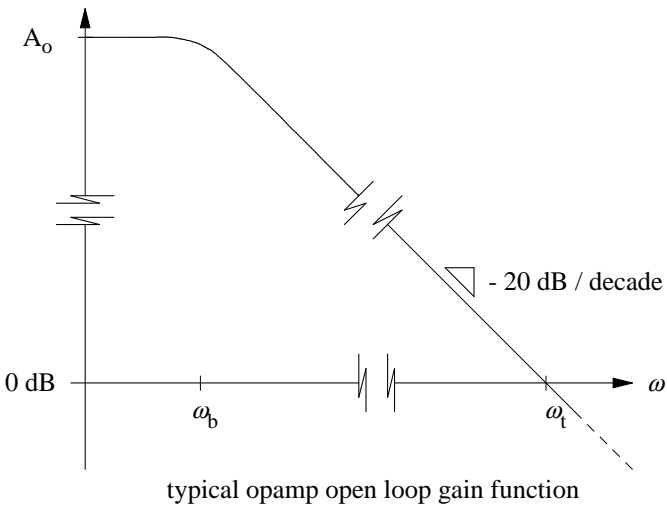
opamp goes open-loop when diode is off resulting in a  $\text{dead\_time} = \frac{15\text{ V}}{0.5\text{ V}/\mu\text{s}} = 30\text{ }\mu\text{s}$  while the opamp slews from  $-15\text{ V}$  to  $0\text{ V}$  for the diode to turn back on. If the bandwidth of the signals being handled is less than  $3\text{ kHz}$ , then fine, if not precision rectifier



is required.

nonideal frequency dependent operational amplifier model

opamp data book values using ±15 V supplies			
description	values		
	min	typical	max
input offset voltage		2 mV	3 mV
input bias current		80 nA	500 nA
input offset current		20 nA	200 nA
differential input resistance	300 kΩ	2 MΩ	
output resistance		75 Ω	150 Ω
output swing	±12 V	±14 V	
slew rate	0.3 $\frac{\text{V}}{\mu\text{s}}$	0.5 $\frac{\text{V}}{\mu\text{s}}$	
unity gain frequency	0.4 MHz	1.0 MHz	
DC voltage gain	90 dB	106 dB	

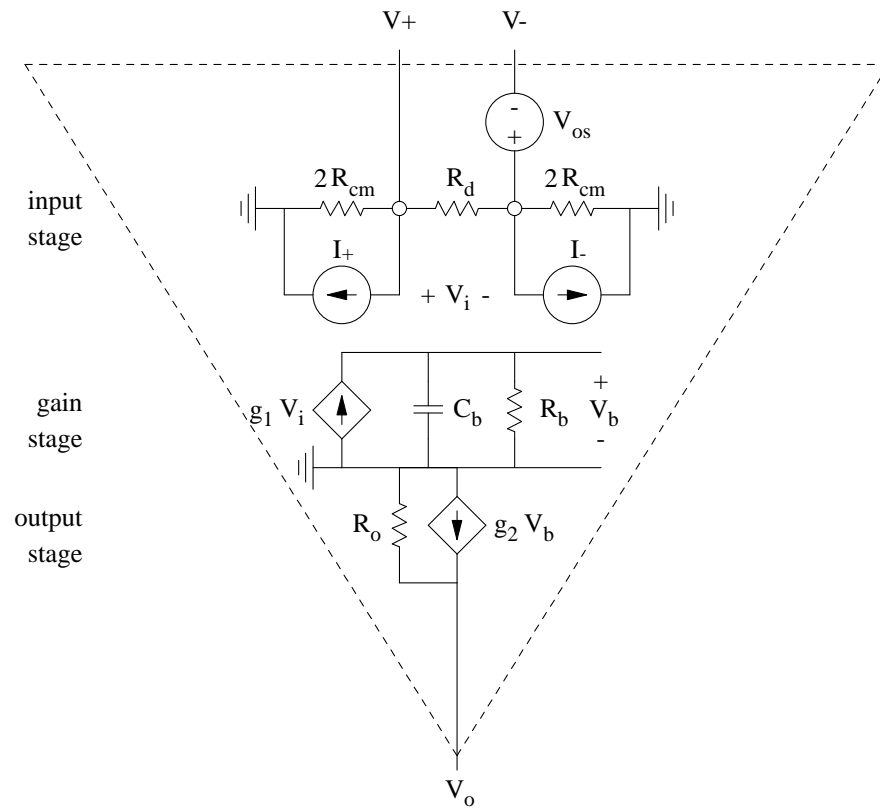


$$I_+ = I_B + 0.5 I_{OS}$$

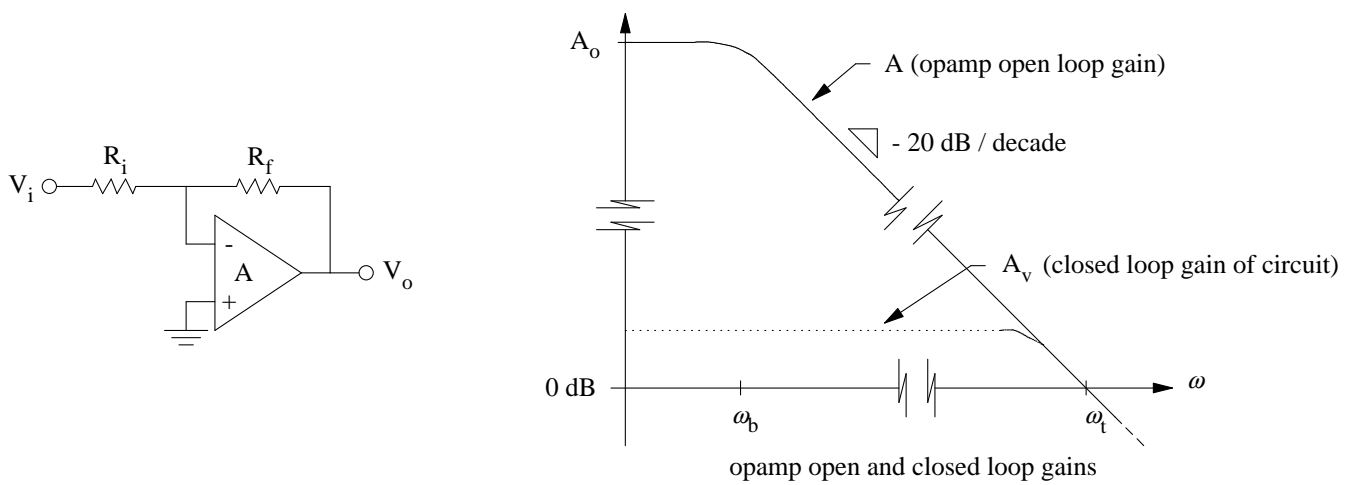
$$I_- = I_B - 0.5 I_{OS}$$

$$A = \frac{A_o}{\frac{s}{\omega_b} + 1}$$

model using frequency dependent VVT



model using frequency independent components



Note  $V_o = -A V_-$ .

Define  $\beta$  to be the fraction of the opamp's output fed back to the opamp's input. For the above circuit

$$\beta = \frac{R_i}{R_i + R_f} = \frac{1}{\frac{R_i + R_f}{R_i}} = \frac{1}{1 + \frac{R_f}{R_i}}$$

The gain around the feed-forward, feed-back loop is  $-A\beta$ .

For the above circuit start with a KCL at the opamps' inverting input terminal

$$\frac{V_- - V_i}{R_i} + \frac{V_- - V_o}{R_f} = 0$$

$$\frac{V_i}{R_i} = \left( \frac{1}{R_i} + \frac{1}{R_f} \right) V_- - \frac{V_o}{R_f} = \left( \frac{1}{R_i} + \frac{1}{R_f} \right) \left( -\frac{V_o}{A} \right) - \frac{V_o}{R_f} = - \left( \left( \frac{1}{R_i} + \frac{1}{R_f} \right) \frac{1}{A} + \frac{1}{R_f} \right) V_o$$

$$A_v = \frac{V_o}{V_i} = \frac{-\frac{1}{R_i}}{\left( \frac{1}{R_i} + \frac{1}{R_f} \right) \frac{1}{A} + \frac{1}{R_f}} = \frac{-\frac{R_f}{R_i}}{\left( \frac{R_f}{R_i} + 1 \right) \frac{1}{A} + 1} = \frac{-\frac{R_f}{R_i}}{1 + \frac{1}{A\beta}} = \frac{\text{ideal\_gain}}{1 + \frac{1}{A\beta}}$$

Given

$$A = \frac{A_o}{\frac{s}{\omega_b} + 1} \quad \rightarrow \quad A \bigg|_{s = j\omega \gg j\omega_b} \approx \frac{A_o}{\frac{s}{\omega_b}} = \frac{A_o \omega_b}{s} = \frac{\omega_t}{s}$$

then

$$A_v = \frac{-\frac{R_f}{R_i}}{1 + \frac{1}{A\beta}} = \frac{-\frac{R_f}{R_i}}{1 + \frac{1}{\frac{A_o}{\frac{s}{\omega_b} + 1}\beta}} = \frac{-\frac{R_f}{R_i}}{1 + \frac{\frac{s}{\omega_b} + 1}{A_o\beta}} = \frac{-\frac{R_f}{R_i}}{\frac{s}{\omega_b A_o\beta} + \left( 1 + \frac{1}{A_o\beta} \right)} \approx \frac{-\frac{R_f}{R_i}}{\frac{s}{\omega_b A_o\beta} + 1} = \frac{-\frac{R_f}{R_i}}{\frac{s}{\omega_{-3dB}} + 1}$$

The frequency dependent finite gain has an impact on the maximum stage gain in a multi-stage amplifier. There are different ways the specifications may be supplied:

- (1) step response: maximum settling time will be specified
- (2) filtering function: bandwidth specification

Given the opamp's open loop voltage gain is dropping off at 20 dB/decade, the approximation  $BW_{\text{stage}} = \frac{f_t}{A_{v_{\text{stage}}}}$  holds.

If the maximum settling time is specified, then  $\tau \approx \frac{t_{st}}{5}$  resulting in the multi-stage over-all bandwidth requirement of

$$f_{-3dB} = \frac{1}{2\pi\tau} = \frac{5}{2\pi t_{st}} = \frac{f_t}{A_v} \rightarrow A_v = \frac{f_t}{f_{-3dB}} = \frac{f_t}{\left(\frac{5}{2\pi t_{st}}\right)} = \frac{f_t 2\pi t_{st}}{5}$$

Basically, the settling time specification drives the maximum overall voltage gain independent of the number of stages. However, to meet that settling time specification the stage bandwidth must be higher to compensate for the -3 dB loss at the corner in each stage of the multistage amplifier. If we pad the bandwidth requirement by 10×, then the  $f_{-3dB}$  corner loss per stage will be

$$20 \log \left( \frac{1}{\sqrt{0.1^2 + 1^2}} \right) = -0.04 \text{ dB}$$

rather than

$$20 \log \left( \frac{1}{\sqrt{1^2 + 1^2}} \right) = -3.01 \text{ dB}$$

As such, the standard calculation would be

$$\tau = \frac{t_{st}}{5} \quad ; \quad f_{-3dB} = \frac{1}{2\pi\tau} \quad ; \quad A_{v_{\text{stage ceiling}}} = \frac{f_t}{10 \cdot f_{-3dB}}$$

**short-circuit time-constant method**

When solving for the approximate  $-3\text{ dB}$  high-pass frequency of a large/complex network,

- + replace all high-pass capacitors with a short-circuit
- + then for each high-pass capacitor,
  - set it to its value and solve the resulting  $-3\text{ dB}$  frequency
  - set it back to a short-circuit

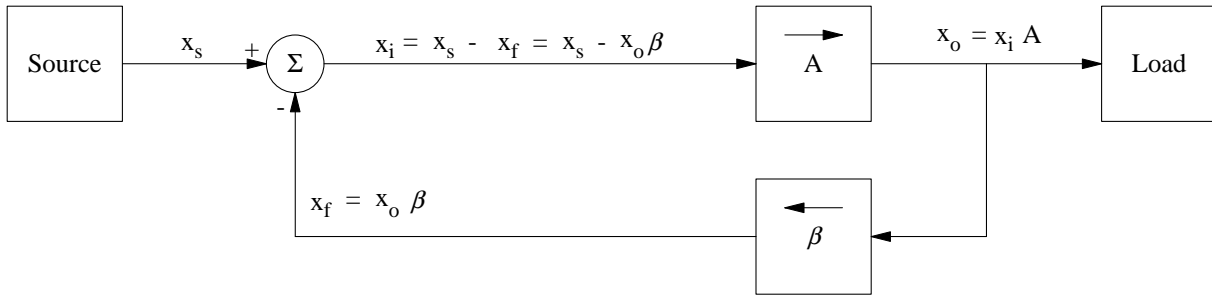
$$+ f_{\text{hp}} \approx \sum_{i=1}^{i=n} f_i$$

**open-circuit time-constant method**

When solving for the approximate  $-3\text{ dB}$  low-pass frequency of a large/complex network,

- + replace all low-pass capacitors with an open-circuit
- + then for each low-pass capacitor,
  - set it to its value and solve the resulting time constant  $\tau$
  - set it back to an open-circuit

$$+ \tau_{\text{lp}} \approx \sum_{i=1}^{i=n} \tau_i$$



$$x_o = A x_i = A (x_s - \beta x_o) \rightarrow (1 + A \beta) x_o = A x_s \rightarrow A_f \equiv \frac{x_o}{x_s} = \frac{A}{1 + A \beta} = \frac{A}{1 + A \beta} \frac{\frac{1}{A \beta}}{\frac{1}{A \beta}} = \frac{\frac{1}{\beta}}{1 + \frac{1}{A \beta}} \approx \frac{1}{\beta}$$

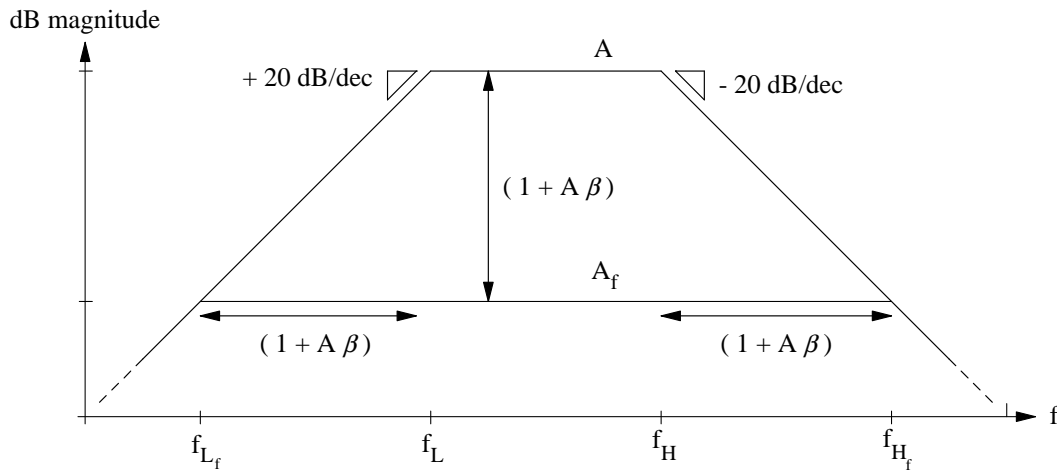
$$x_f = x_o \beta = \left( \frac{A}{1 + A \beta} x_s \right) \beta = \frac{A \beta}{1 + A \beta} x_s$$

$$x_i = x_s - x_f = x_s - \frac{A \beta}{1 + A \beta} x_s = \frac{(1 + A \beta) x_s}{1 + A \beta} - \frac{A \beta}{1 + A \beta} x_s = \frac{1}{1 + A \beta} x_s \rightarrow x_s = (1 + A \beta) x_i$$

gain desensitivity

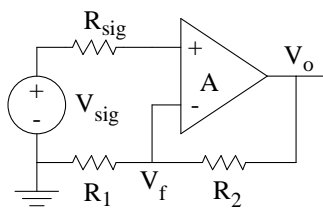
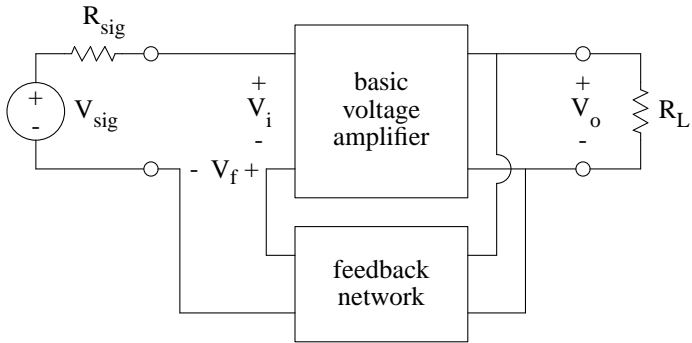
$$\frac{d A_f}{d A} = \frac{1}{(1 + A \beta)^2} \quad ; \quad d A_f = \frac{d A}{(1 + A \beta)^2} \quad ; \quad \frac{d A_f}{A_f} = \frac{d A}{(1 + A \beta)^2} \frac{1 + A \beta}{A} = \frac{1}{1 + A \beta} \frac{d A}{A}$$

bandwidth extension

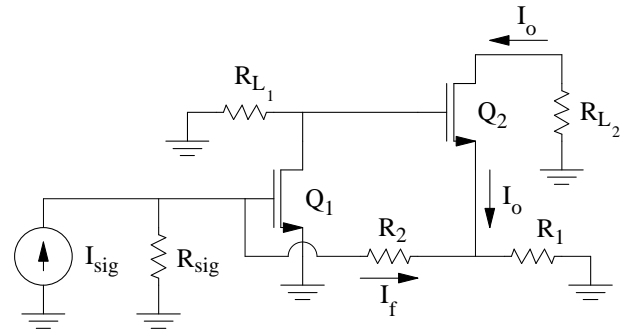
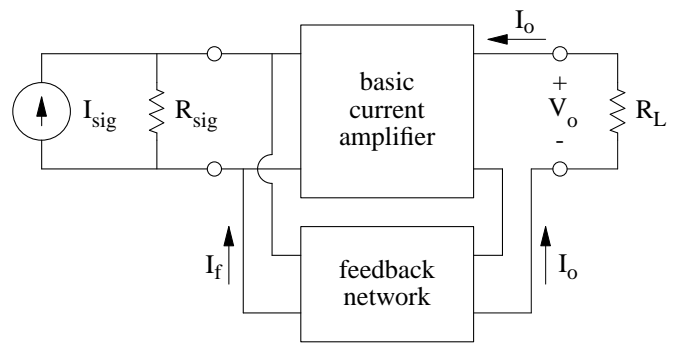


$$f_{L_f} = \frac{f_L}{1 + A \beta} \quad ; \quad f_{H_f} = f_H (1 + A \beta)$$

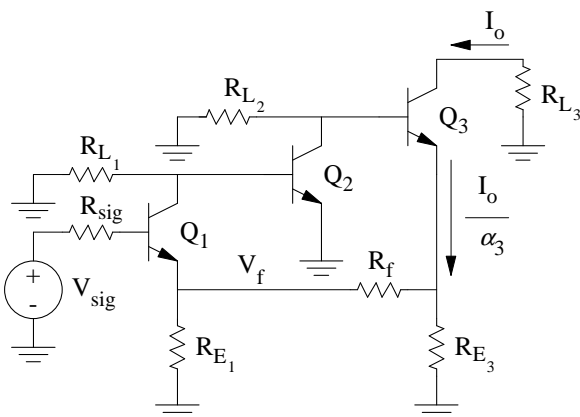
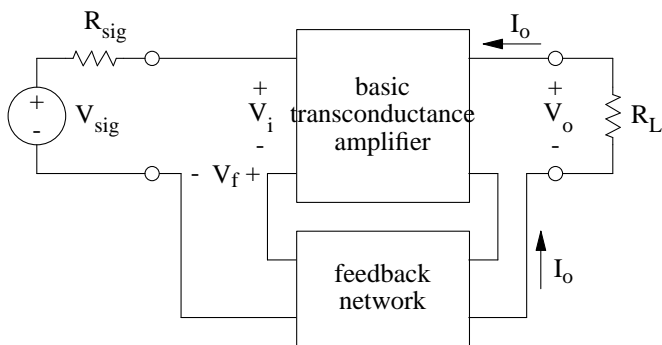




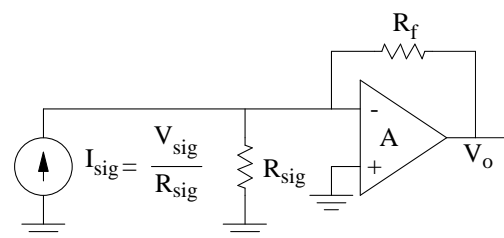
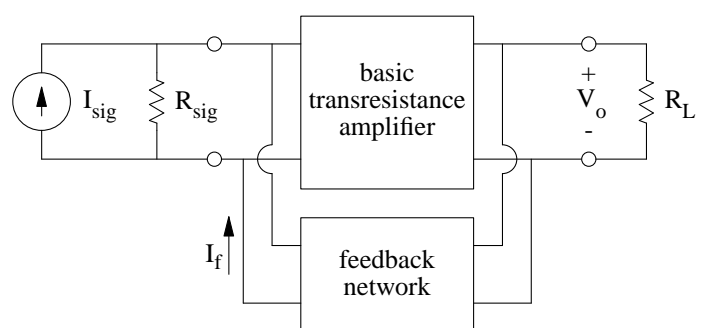
series-shunt topology



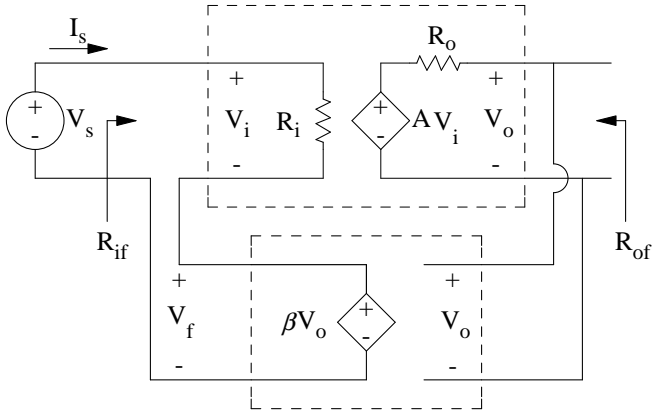
shunt-series topology



series-series topology



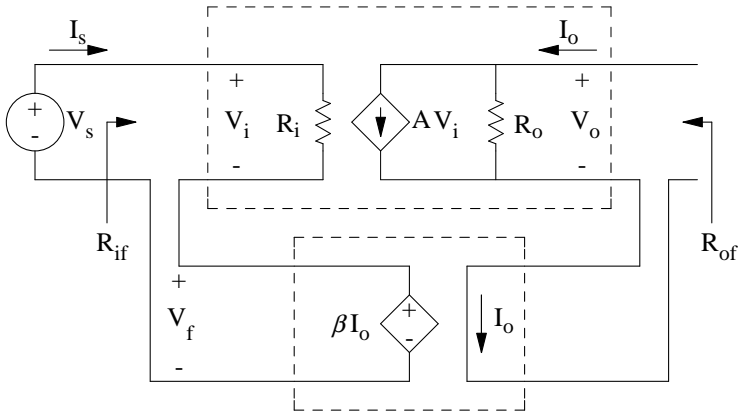
shunt-shunt topology

**voltage mixing, voltage sampling**

$$R_{if} = \frac{V_s}{I_s} = \frac{V_s}{V_i / R_i} = \frac{V_s}{V_i} R_i = \frac{(1 + A \beta) V_i}{V_i} R_i = (1 + A \beta) R_i$$

$$V_s = 0 \rightarrow V_i = -\beta V_o$$

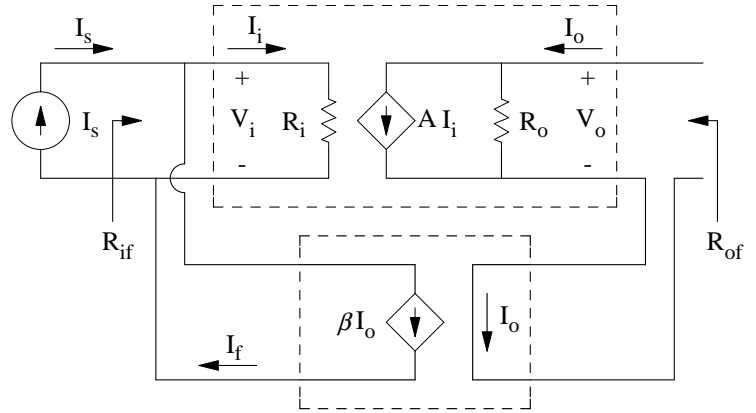
$$R_{of} = \frac{V_o}{I_o} = \frac{V_o}{V_o - A V_i} = \frac{V_o}{V_o - A(-\beta V_o)} = \frac{R_o}{1 + A \beta}$$

**voltage mixing, current sampling**

$$R_{if} = \frac{V_s}{I_s} = \frac{V_s}{V_i / R_i} = \frac{V_s}{V_i} R_i = \frac{(1 + A \beta) V_i}{V_i} R_i = (1 + A \beta) R_i$$

$$V_s = 0 \rightarrow V_i = -\beta I_o$$

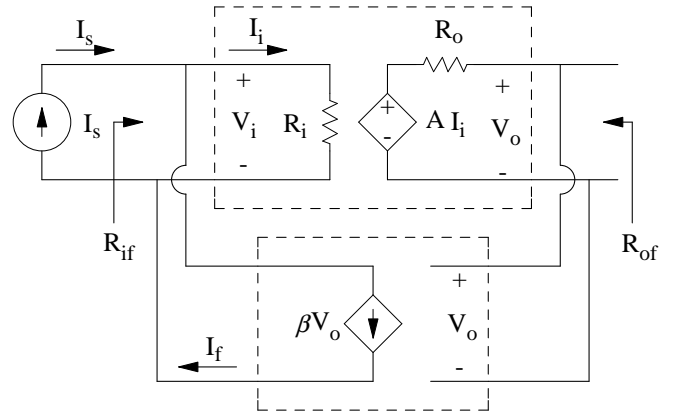
$$R_{of} = \frac{V_o}{I_o} = \frac{(I_o - A V_i) R_o}{I_o} = \frac{I_o - A(-\beta I_o)}{I_o} R_o = (1 + A \beta) R_o$$

**current mixing, current sampling**

$$R_{if} = \frac{V_s}{I_s} = \frac{V_i}{I_s} = \frac{I_i R_i}{I_s} = \frac{I_i R_i}{(1 + A \beta) I_i} = \frac{R_i}{1 + A \beta}$$

$$I_s = 0 \rightarrow I_i = -\beta I_o$$

$$R_{of} = \frac{V_o}{I_o} = \frac{(I_o - A I_i) R_o}{I_o} = \frac{I_o - A(-\beta I_o)}{I_o} R_o = (1 + A \beta) R_o$$

**current mixing, voltage sampling**

$$R_{if} = \frac{V_s}{I_s} = \frac{V_i}{I_s} = \frac{I_i R_i}{I_s} = \frac{I_i R_i}{(1 + A \beta) I_i} = \frac{R_i}{1 + A \beta}$$

$$I_s = 0 \rightarrow I_i = -\beta V_o$$

$$R_{of} = \frac{V_o}{I_o} = \frac{V_o}{V_o - A I_i} = \frac{V_o}{V_o - A(-\beta V_o)} = \frac{R_o}{1 + A \beta}$$

**summary for feedback systems**

general

$$A_f \equiv \frac{x_o}{x_s} = \frac{A}{1 + A\beta} = \frac{\frac{1}{\beta}}{1 + \frac{1}{A\beta}} \approx \frac{1}{\beta}$$

bandwidth increases by  $1 + A\beta$  as gain decreases by  $1 + A\beta$ .

voltage mixing, voltage sampling

$$R_{if} = (1 + A\beta) R_i$$

$$R_{of} = \frac{1}{1 + A\beta} R_o$$

current mixing, current sampling

$$R_{if} = \frac{1}{1 + A\beta} R_i$$

$$R_{of} = (1 + A\beta) R_o$$

voltage mixing, current sampling

$$R_{if} = (1 + A\beta) R_i$$

$$R_{of} = (1 + A\beta) R_o$$

current mixing, voltage sampling

$$R_{if} = \frac{1}{1 + A\beta} R_i$$

$$R_{of} = \frac{1}{1 + A\beta} R_o$$

**Fourier Series**

given

$$\begin{aligned}
 f(t) &= f(t + nT) \quad n = \pm 1, \pm 2, \pm 3, \dots \\
 &= a_0 + \sum_{n=1}^N a_n \cos(n \omega_0 t) + \sum_{n=1}^N b_n \sin(n \omega_0 t) \\
 &= c_0 + \sum_{n=1}^N c_n \cos(n \omega_0 t + \theta_n)
 \end{aligned}$$

then

symmetry	Fourier coefficients
odd function $f(t) = -f(-t)$	$a_n = 0$ for all $n$ $b_n = \frac{4}{T} \int_0^{T/2} f(t) \sin(n \omega_0 t) dt$
even function $f(t) = f(-t)$	$a_n = \frac{4}{T} \int_0^{T/2} f(t) \cos(n \omega_0 t) dt$ $b_n = 0$ for all $n$
half-wave symmetry $f(t) = -f\left(t + \frac{T}{2}\right)$	$a_0 = 0$ $a_n = 0$ for even $n$ $b_n = 0$ for even $n$ $a_n = \frac{4}{T} \int_0^{T/2} f(t) \cos(n \omega_0 t) dt$ for odd $n$ $b_n = \frac{4}{T} \int_0^{T/2} f(t) \sin(n \omega_0 t) dt$ for odd $n$

square wave with peak A and period T  $f(t) = \frac{4A}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin((2n-1)\omega_0 t)$

full-wave rectified with peak A and period T  $f(t) = \frac{2A}{\pi} - \frac{4A}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2-1} \cos(n\omega_0 t)$

saw-tooth wave with peak A and period T  $f(t) = \frac{A}{2} - \frac{A}{\pi} \sum_{n=1}^{\infty} \frac{\sin(n\omega_0 t)}{n}$

triangular wave with peak A and period T  $f(t) = \frac{A}{2} - \frac{4A}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos((2n-1)\omega_0 t)$

**large-signal vs small-signal**

If  $\frac{d v_{OUT}}{dt} \leq SR = \text{slew\_rate}$  for all frequencies of interest in the Fourier Series, then small signal analysis can be used (which assumes linear systems response). That is, if we can make the amplitude small enough, that above can be satisfied (avoiding unacceptable harmonic distortion) small-signal analysis can be used, otherwise large signal operation occurs and the opamp's slew rate restricts the amplitude and/or frequency.

sine wave

$$\frac{d V_{pk} \sin(\omega t)}{dt} = \omega V_{pk} \cos(\omega t) \leq \omega V_{pk} \leq SR = 0.5 \text{ V}/\mu\text{s}$$

triangular wave

$$SR = \frac{V_{pk}}{T/4} \rightarrow V_{pk} = SR \frac{T}{4} = \frac{SR}{4f} \rightarrow f = \frac{SR}{4V_{pk}}$$

**analog signal-conditioning design guidelines**

define the measurement objective

parameter

What is the nature of the measured variable: pressure, temperature, flow, level, voltage, current, resistance, ... ?

input range

What is the range of the measurement?

accuracy

What is the required measurement accuracy and/or resolution?

linearity

Is the measured output signal to be linear?

measurement noise

What is the noise level and frequency spectrum of the measurement environment?

select a sensor

parameter

What is the relationship between the sensor output and the measured variable: resistance, voltage, current, ... ?

transfer function

What is the relationship between the sensor output and the measured variable?

time response

What is the time response of the sensor: first-order time constant, second-order damping, frequency, ... ?

sensor output range

What is the range of the sensor parameter output for the given measurement range?

power requirement

What is the power specification of the sensor: resistive dissipation maximum, current draw, ... ?

design the analog signal-conditioning

parameter

What is the nature of the desired output: voltage, current, frequency, ... ?

circuit output range

What is the desired range of the output parameter?

input impedance

What input impedance should the signal-conditioning circuit present to the sensor?

output impedance

What output impedance should the signal-conditioning circuit present to the output load circuit?

analog signal-conditioning tips

- (a) If the input is a resistance change and a bridge or driver must be used, be sure to consider both the effect of output voltage nonlinearity with resistance and the effect of current through the resistive sensor.
- (b) For the opamp portion of the design, the easiest design approach is to develop an equation for the output versus input. From this equation, it will be clear what types of circuits may be used. This equation represents the static transfer function of the signal-conditioning.
- (c) Always consider any possible loading of voltage sources by the signal-conditioning. Such loading is a direct error in the measurement system.

## Thermal Sensors

### metal temperature versus temperature devices

energy bands

metal

The conduction and valence bands overlap so there are free electrons to conduct current independent of temperature.

semiconductor

There is a small energy gap between the conduction and valence bands so a small amount of energy is required for electrons to make the jump and conduct current.

insulator

There is a very large energy gap between the conduction and valence bands so a very large amount of energy is required for electrons to make the jump and conduct current.

resistance-temperature detector (RTD)

$$R(T) = \rho \frac{l}{A} (T)$$

where

$R$  = resistance to electric current ( $\Omega$ )

$\rho$  = resistivity ( $\Omega \cdot m$ )

$l$  = length (m)

$A$  = cross-sectional area ( $m^2$ )

$T$  = temperature which is constant

temperature versus temperature approximation

$\rho$  is a nonlinear function of temperature for most metals.

over a small enough temperature range we can linearize  $\rho$  vs  $T$

$$R(T) = R(T_0) \left[ 1 + \alpha_0 \cdot \Delta T \right]$$

where

$R(T)$  = approximation of  $R$  at temperature  $T$

$R(T_0)$  =  $R$  at temperature  $T_0$

$\Delta T = T - T_0$

$\alpha_0$  = slope of  $R$  vs  $T$  curve at  $T_0$

also use a quadratic approximation

$$R(T) = R(T_0) \left[ 1 + \alpha_1 \cdot \Delta T + \alpha_2 \cdot (\Delta T)^2 \right]$$

where values of  $\alpha_0$ ,  $\alpha_1$ , and  $\alpha_2$  are available in tables for a given metal

resistance-temperature detectors (RTD)

sensitivity  $\frac{\partial \rho}{\partial T}$

response time is 0.5 s to 5 s for metal to stabilize at new  $T$  and is a function of thermal conductivity

RTD is a length of wire possibly a coil of wire to increase sensitivity

use a bridge with compensation lines to detect  $\Delta R$

bridge results in  $I^2 R$  losses heating up the RTD changing  $R$  known as self-heating, as such to correct for this

$$\Delta T = \frac{P}{P_D}$$

where

$\Delta T$  = temperature rise due to self-heating ( $^{\circ}C$ )

$P$  = power dissipated in RTD (W)

$P_D$  = dissipation constant ( $W/^\circ C$ )

RTD have a range of  $-180^\circ C$  to  $300^\circ C$

### **thermistors (semiconductors)**

RTD resistance increases with temperature

thermistors resistance decreases with temperature

thermistor resistance is a highly nonlinear function of temperature

sensitivity: 10% per  $^\circ C$

construction: bulk semiconductor material doped to adjust resistance range  
range

melting: less than  $300^\circ C$

package: thermistor package material limits temperature range

nonlinear:  $-80^\circ C$   $R > 3 M\Omega$

response time: 0.5 s to 10 s or higher depending on the package material

signal conditioning: must keep power dissipation low

self-heating:  $1 mW/^\circ C$  to  $10 mW/^\circ C$

### **thermocouples (TC)**

RTD and themistors are passive devices

thermocouples are active devices, emf varies with temperature

thermoelectric devices with emf approximately linear in T

based on electron movement vs temperature dependent on metal type

sensitivity:  $6 \mu V/^\circ C$  to  $50 \mu V/^\circ C$

range:  $-150^\circ C$  to  $1765^\circ C$

response time:

10 s to 20 s for large industrial units

10 ms to 20 ms for small-gauge wire units

signal conditioning:

voltage amplification required as measured voltage  $\sim 50 mV$

noise issues given high impedance  $\rightarrow$  differential amplifier required

reference compensation

controlled temperature reference block

reference compensation circuits

software reference correction

noise issues

low voltage, high impedance, ... in noisy industrial environment

use twisted pair in a grounded metal or foil sheath

measurement junction is 0 V (ground)

instrumentation amplifier with high common-mode rejection

### **other thermal sensors**

bimetal strips

metals expand differing amounts for the same temperature

bimetallic sensor as an on/off switch

gas thermometers

if volume constant, pressure varies with temperature



vapour-pressure thermometers

converts temperature measurement into pressure measurement

liquid-expansion thermometers

use liquid expansion with temperature in a small column to measure temperature

$$V(T) = V(T_o) \left[ 1 + \beta \cdot \Delta T \right]$$

where

$V(T)$  = volume at temperature  $T$

$V(T_o)$  = volume at temperature  $T_o$

$\Delta T = T - T_o$

$\beta$  = volume thermal expansion coefficient

solid-state temperature sensors

range:  $-50^{\circ}\text{C}$  to  $150^{\circ}\text{C}$

response time: 1 s to 5 s

dissipation constant:  $2 \text{ mW}/^{\circ}\text{C}$  to  $20 \text{ mW}/^{\circ}\text{C}$

accurate to  $1^{\circ}\text{C}$

application; reference temperature sensor for thermocouples

### thermal sensor system design consideration

1. identify the nature of the measurement
2. identify the required output signal
3. select an appropriate sensor
4. design the required signal conditioning

## Mechanical Sensors

### displacement, location, position sensors

potentiometric sensors

linear position translated into resistance and then into a voltage

problems with: wear, electronic noise, limited resolution

capacitive sensors

$$C = \epsilon_0 \epsilon_r \frac{A}{d}$$

where

$\epsilon_0$  = permittivity of free space = 8.85 pF/m

$\epsilon_r$  = relative permittivity of dielectric (dielectric constant K)

A = plate common area

d = plate separation

can vary C by

1. vary d
2. vary A
3. vary  $\epsilon_r$

use an impedance bridge to measure  $\Delta C$

inductive sensors

$$L = \mu_0 \mu_r \frac{N^2 A}{d}$$

where

$\mu_0$  = permeability of free space

$\mu_r$  = relative permeability of core

N = number of turns

A = cross sectional area

d = distance

example: eddy current inspection instrument

variable-reluctance sensors

LVDT: linear variable differential transformer

### level sensors

mechanical level sensors

float and linkage to a variable R or LVDT

electrical level sensors

use liquid level to vary capacitor dielectric

ultrasonic level sensors

transmit and receive the reflection

use length of time from transmit to receive as a gauge of distance to surface level

**strain sensors**

## terms

deformation: change of shape/dimension due to force

stress: applied force

strain: resulting deformation

tensile stress-strain

$$\text{tensile stress} = \frac{F}{A}$$

where

F = applied force in N

A = cross-sectional area of sample in m<sup>2</sup>

tensile stress: elongate or pull apart

$$\text{tensile strain} = \frac{\Delta l}{l}$$

where

$\Delta l$  = change in length in m

l = original length in m

compression stress-strain

$$\text{compression stress} = \frac{F}{A}$$

where

F = applied force in N

A = cross-sectional area of sample in m<sup>2</sup>

compression stress: compress or push together

$$\text{compression strain} = \frac{\Delta l}{l}$$

where

$\Delta l$  = change in length in m

l = original length in m

shear stress-strain

$$\text{shear stress} = \frac{F}{A}$$

where

F = applied force in N

A = cross-sectional area of sample in m<sup>2</sup>

shear stress: push/pull perpendicular

$$\text{shear strain} = \frac{\Delta x}{l}$$

where

$\Delta x$  = deformation in m

l = original width of sample in m

stress-strain curve

$$E = \frac{\text{stress}}{\text{strain}} = \frac{F/A}{\Delta l/l}$$

where E is the modulus of elasticity for tensile or compression in N/m<sup>2</sup>

$$M = \frac{\text{stress}}{\text{strain}} = \frac{F/A}{\Delta x/l}$$

where M is the modulus of elasticity for shear in  $\text{N/m}^2$

strain gauge principles

$$R_o = \rho \frac{l_o}{A_o}$$

where

$R_o$  = sample resistance in  $\Omega$

$\rho$  = sample resistivity in  $\Omega \cdot \text{m}$

$l_o$  = length in m

$A_o$  = cross-sectional area in  $\text{m}^2$

$V = l_o A_o$  is volume in  $\text{m}^3$

$$V = l_o A_o = (l_o + \Delta l)(A_o - \Delta A)$$

$$R = \rho \frac{l_o + \Delta l}{A_o - \Delta A} \approx \rho \frac{l_o}{A_o} \left( 1 + 2 \frac{\Delta l}{l_o} \right)$$

$$\Delta R \approx 2 R_o \frac{\Delta l}{l_o}$$

$$\text{example: } R_{\text{nom}} = 120 \Omega, \frac{\Delta l}{l_o} = \frac{1 \text{ mm}}{\text{m}}, \Delta R = 0.24 \Omega$$

temperature effects

$$R(T) = R(T_o) \left[ 1 + \alpha_o \cdot \Delta T \right]$$

$$\Delta R_T = R_o \cdot \alpha_o \cdot \Delta T$$

where

$\Delta R_T$  = resistance change due to  $\Delta T$

$\alpha_o = 0.004/^\circ\text{C}$

$\Delta T$  = change in temperature

$R(T_o)$  = nominal R at reference temperature T

$$\text{example: } R_{\text{nom}} = 120 \Omega, \alpha_o = 0.004/^\circ\text{C}, \Delta T = 1^\circ\text{C}, \Delta R_T = 0.48 \Omega$$

$\Delta R_T$  can be double  $\Delta R$  due to strain

metal strain gauges

$$\text{Gauge Factor} = GF = \frac{\Delta R/R}{\text{strain}} = \frac{\Delta R/R}{\Delta l/l} = \frac{\text{fractional change in resistance}}{\text{fractional change in length}}$$

GF tends to be around 2 but could go as high as 10

$R_{\text{nom}} = \{ 60, 120, 240, 350, 500, 1000 \} \Omega$ , typically  $R_{\text{nom}} = 120 \Omega$

use active and dummy strain gauge for temperature compensation bridge measurement

one-arm bridge

$$\Delta V = - \frac{V_s}{4} GF \frac{\Delta l}{l}$$

two-arm bridge

$$\Delta V = - \frac{V_s}{2} GF \frac{\Delta l}{l}$$

four-arm bridge

$$\Delta V = - \frac{V_s}{1} GF \frac{\Delta l}{l}$$

semiconductor strain gauges

higher (negative) gauge factor but highly nonlinear

signal conditioning uses a bridge but still must correct nonlinearity

### motion sensors

$$v(t) = \frac{dx(t)}{dt} \rightarrow a(t) = \frac{dv(t)}{dt} = \frac{d^2x(t)}{dt^2} \rightarrow v(t) = v(0) + \int_0^t a(t) dt \rightarrow x(t) = x(0) + \int_0^t v(t) dt$$

$g \approx 9.8 \text{ m/s}^2$  = acceleration of gravity

spring-mass system

$$m \cdot a = k \cdot \Delta x \rightarrow a = \frac{k}{m} \cdot \Delta x$$

where

$k$  = spring constant in N/m

$\Delta x$  = spring extension in m

$m$  = mass in kg

$a$  = acceleration in  $\text{m/s}^2$

natural frequency and damping

$$f_N = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$$

where

$f_N$  = natural frequency in Hz

$k$  = spring constant in N/m

$m$  = seismic mass in kg

giving rise to

$$X_T(t) = X_0 e^{-\alpha t} \sin(2\pi \cdot f_N \cdot t)$$

where

$X_T(t)$  = transient mass position

$\alpha$  = damping coefficient

vibration effects

$$a(t) = -\omega^2 \cdot X_0 \cdot \sin(\omega t) \quad ; \quad \Delta X = -\frac{m \cdot X_0}{k} \omega^2 \sin(\omega t) \quad ; \quad \text{valid up to } \frac{f_N}{2.5}$$

accelerometers

potentiometric: spring mass attached to wiper arm of potentiometer (30 Hz)

LVDT: LVDT core itself is the seismic mass (80 Hz)

variable reluctance: geophone (100 Hz)

piezoelectric: spring mass attached to piezoelectric crystal (5 kHz) but output in millivolt range

### pressure sensors

terms

$$\text{pressure} = \text{force per unit area} \quad \frac{F}{A} \left[ \frac{\text{N}}{\text{m}^2} \right]$$

statics pressure: fluid is not moving

dynamic pressure: fluid is in motion

$$\text{pascal (Pa)} \quad 1 \text{ Pa} = 1 \frac{\text{N}}{\text{m}^2}$$

$$1 \text{ psi} \approx 6.895 \text{ kPa}$$

atmosphere (atm)    1 atm = 101.325 kPa  $\approx$  14.7 psi  
 gauge pressure

$$p_g = p_{abs} - p_{at}$$

where

$p_g$  - gauge pressure

$p_{abs}$  = absolute pressure

$p_{at}$  = atmospheric pressure

head pressure

$$p = \rho \cdot g \cdot h$$

where

$p$  = pressure in Pa

$\rho$  = density in  $\frac{\text{kg}}{\text{m}^3}$

$g$  = acceleration due to gravity ( $9.8 \frac{\text{m}}{\text{s}^2}$ )

$h$  = depth in liquid in m

or

$$p = \rho_w \cdot h$$

where

$p$  = pressure in  $\frac{\text{lb}}{\text{ft}^2}$

$\rho_w$  = weight density in  $\frac{\text{lb}}{\text{ft}^3}$

$h$  = depth in ft

sensor types for  $p > 1$  atmosphere  
 diaphragm pressure sensors

$$F = (p_2 - p_1) \cdot A$$

Bellows pressure sensors

Bourdon tube

solid-state pressure sensors

sensitivity 10 to 100  $\frac{\text{mV}}{\text{kPa}}$

response time 10 ms (10% to 90% response time)

linear voltage vs pressure

3-terminals: DC power, ground, sensor output voltage

sensor types for  $p < 1$  atmosphere

Pirani gauge

thermocouple

ionization gauge

range:  $10^{-3}$  atm to  $10^{-13}$  atm

## flow sensors

sensor types

solid flow

$$Q = \frac{WR}{L}$$

where

$$Q = \text{flow in } \frac{\text{kg}}{\text{min}} \text{ or } \frac{\text{lb}}{\text{min}}$$

$W$  = weight of material on section  $L$  in kg or lb

$$R = \text{conveyor speed in } \frac{\text{m}}{\text{min}} \text{ of } \frac{\text{ft}}{\text{min}}$$

$L$  = length of weighing platform in m or ft

liquid flow

$$V = \frac{Q}{A}$$

where

$V$  = flow velocity

$Q$  = volume flow rate

$A$  = cross-sectional area

and

$$F = \rho \cdot Q$$

where

$F$  = mass or weight flow rate

$\rho$  = mass density or weight density

$Q$  = volume flow rate

restriction flow sensors

$$Q = k \sqrt{\Delta p}$$

where

$Q$  = volume flow rate

$K$  = a constant for the pipe and liquid type

$\Delta p$  = drop in pressure across the restriction

pitot tube

obstruction flow sensors

magnetic flow meter

## Optical Sensors

terms

speed of propagation

$$c = \lambda f$$

where

$$c = 2.998 \cdot 10^8 \text{ m/s} \approx 3 \cdot 10^8 \text{ m/s in a vacuum}$$

$$\lambda = \text{wave length in m}$$

$$f = \text{frequency in Hz}$$

index of refraction

$$n = \frac{c}{v}$$

where

$$v = \text{velocity of EM in a given material in m/s}$$

photon

$$W_p = h \cdot f = \frac{h \cdot c}{\lambda}$$

where

$$\text{Planck's constant } h = 6.63 \cdot 10^{-34} \text{ J} \cdot \text{s}$$

$$W_p = \text{photon energy in J}$$

$$1 \text{ eV} = 1.602 \cdot 10^{-19} \text{ J}$$

light intensity

$$I = \frac{P}{A}$$

where

$$I = \text{intensity in } \frac{\text{W}}{\text{m}^2}$$

$$P = \text{power}$$

$$A = \text{beam cross-sectional area in m}^2$$

divergence

$\theta$  angle the beam diverges

maximum divergence

$$I = \frac{P}{4 \pi R^2}$$

where

$$R = \text{radius of sphere}$$

## photodetectors

photoconductive detectors

$$E_p = \frac{h \cdot c}{\lambda_{\max}} = \Delta W_g \quad \rightarrow \quad \lambda_{\max} = \frac{h \cdot c}{\Delta W_g}$$

where

$$\lambda_{\max} = \text{maximum detectable radiation wave length in m}$$

photovoltaic detectors

$$V_c = V_0 \ln (1 + I_R)$$

where



$V_c$  = open-circuit cell voltage

$V_0$  = material constant

$I_R$  = light intensity

$I_{sc}$  varies linearly with light intensity

photodiode detectors

phototransistors

photoemissive detectors

photomultiplier tube

gains of  $10^5$  to  $10^7$

### **pyrometry**

total radiation

$$E \propto T^4$$

E in J/s per unit area or  $\text{W/m}^2$

T temperature in K

wide band for noncontact temperature measurement

### **optical sources**

LASER = light amplification by stimulated emission radiation

**large signal and small signal concept**

$$v_{DS} = V_{DS} + v_{ds}$$

$$i_{DS} = I_{DS} + i_{ds}$$

where

$v_{DS}$  is the time domain total drain-to-source voltage

$V_{DS}$  is the time domain DC drain-to-source voltage

$v_{ds}$  is the time domain small-signal drain-to-source voltage

$V_{ds}$  is the frequency domain small-signal drain-to-source voltage

and

$i_{DS}$  is the time domain total drain-to-source current

$I_{DS}$  is the time domain DC drain-to-source current

$i_{ds}$  is the time domain small-signal drain-to-source current

$I_{ds}$  is the frequency domain small-signal drain-to-source current

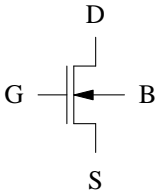
**Taylor series expansion of the nonlinear circuit**

The "DC" circuit is the 0<sup>th</sup> term of the Taylor series expansion.

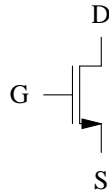
The "ac" circuit or small-signal circuit is the 1<sup>st</sup> term of the Taylor series expansion.

The signal voltage is assumed small enough that the balance of the Taylor series component are approximately zero.

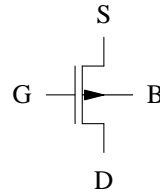
We start with a nonlinear circuit model and replace it with the sum of two separate circuit models, the "DC" circuit model plus the "ac" circuit model. The solution of the "DC" circuit model supplies the parameter values for the "ac" (small-signal) model.

**MOS models, biasing, negative feedback**

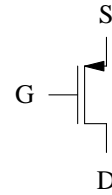
4-terminal NMOS symbol



3-terminal NMOS symbol



4-terminal PMOS symbol



3-terminal PMOS symbol

mode	voltage criteria	drain current
cut-off	$v_{GS} < V_t$	$i_D = 0$
triode	$v_{GS} > V_t$ and $v_{DS} < v_{GS} - V_t = v_{OV}$ , $v_{GD} > V_t$	$i_D = \mu_n C_{ox} \frac{W}{L} \left( (v_{GS} - V_t) v_{DS} - \frac{1}{2} v_{DS}^2 \right)$
saturation	$v_{GS} > V_t$ and $v_{DS} > v_{GS} - V_t = v_{OV}$ , $v_{GD} < V_t$	$i_D = \frac{1}{2} \mu_n C_{ox} \frac{W}{L} (v_{GS} - V_t)^2 (1 + \lambda v_{DS})$

$C_{ox} = \frac{\epsilon_{ox}}{t_{ox}} \left[ \frac{F}{m^2} \right]$  is the gate oxide capacitance per unit area

$\epsilon_{ox} \left[ \frac{F}{m} \right]$  is the permittivity of silicon dioxide

$t_{ox}$  [ m] is the oxide thickness

$\mu_n$  is the electron mobility

$W$  [ m] is the gate width

$L$  [ m] is the gate length

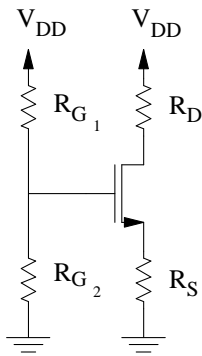
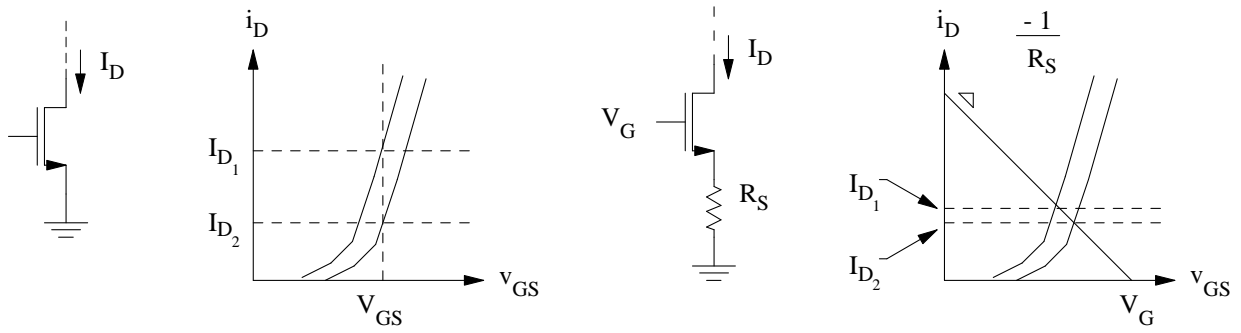
$\lambda = \frac{1}{V_A}$  where  $V_A$  is the Early voltage

$k_n' = \mu_n C_{ox}$

$k_n = \mu_n C_{ox} \frac{W}{L}$

$r_{DS} = \frac{v_{DS}}{i_{DS}} = \frac{1}{\mu_n C_{ox} \frac{W}{L} (v_{GS} - V_t)}$  for  $v_{DS} \ll 2(v_{GS} - V_t)$

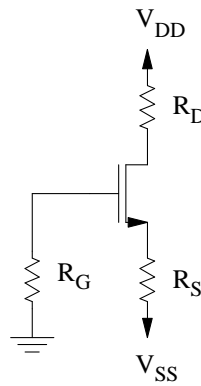
Use the 3-terminal model when the source is connected to supply (rail) voltage, otherwise use the 4-terminal model.



$$V_G = \frac{R_{G_2}}{R_{G_1} + R_{G_2}} V_{DD}$$

$$V_{GS} = V_G - I_D R_S$$

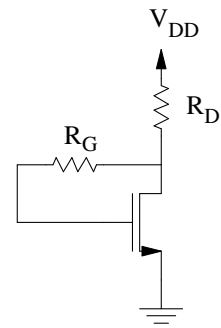
$$V_{DS} = V_{DD} - I_D (R_D + R_S)$$



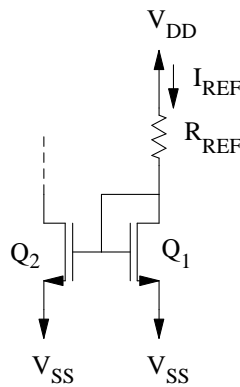
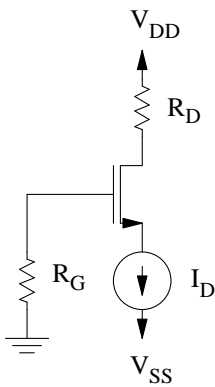
$$V_G = 0$$

$$V_{GS} = -(I_D R_S + V_{SS})$$

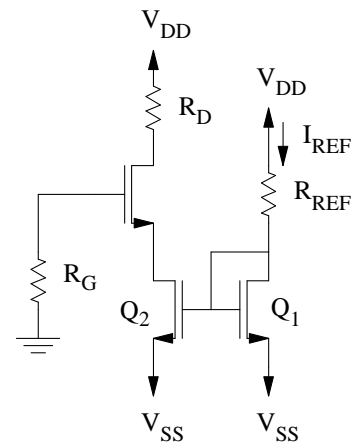
$$V_{DS} = V_{DD} - V_{SS} - I_D (R_D + R_S)$$



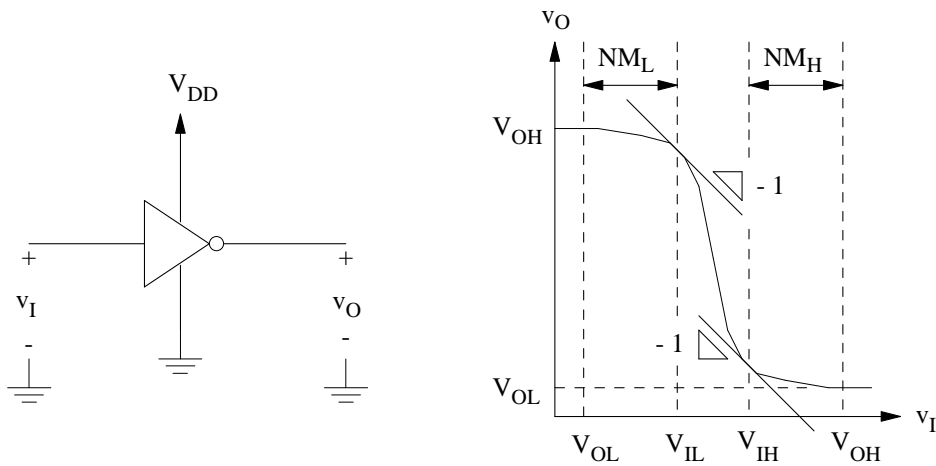
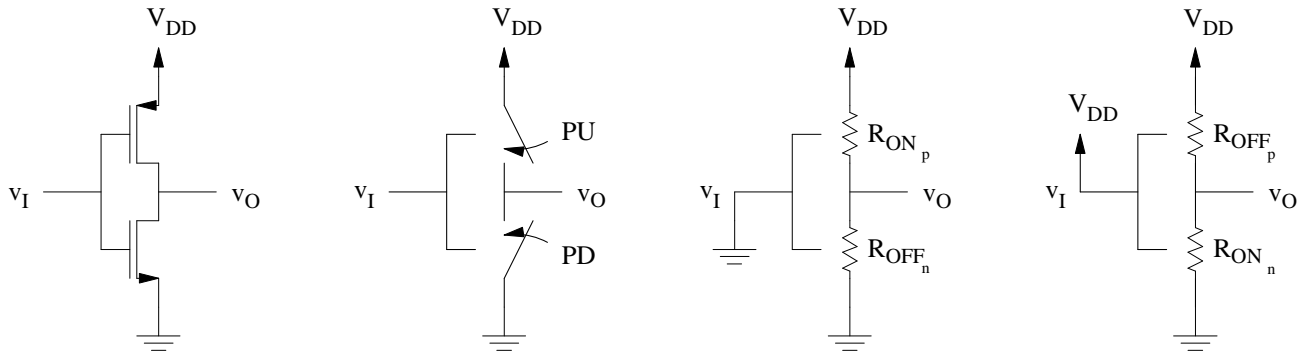
$$V_{GS} = V_{DS} = V_{DD} - I_D R_D$$



$$I_{D_2} = I_{REF} \frac{(W/L)_2}{(W/L)_1}$$



## CMOS inverter



$V_{OH}$  = output high level

$V_{OL}$  = output low level

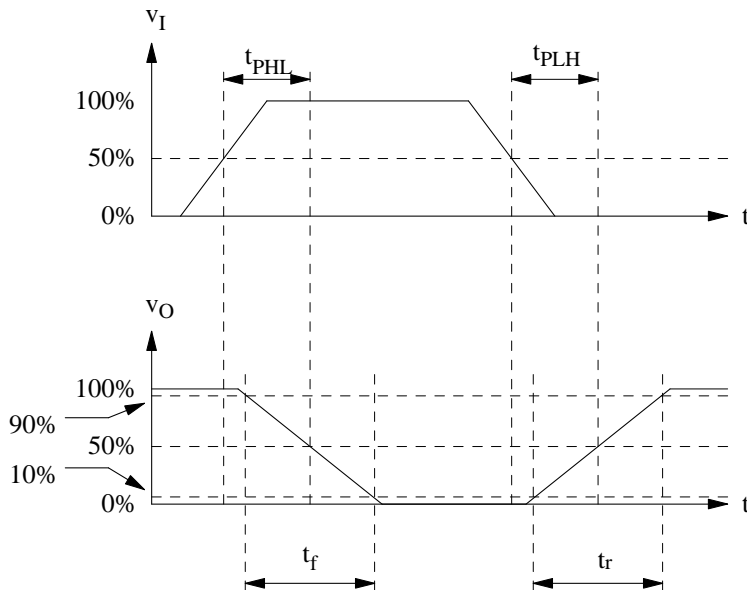
$V_{IH}$  = max input for high output

$V_{IL}$  = min input for low output

$NM$  = noise margin

$$NM_H = V_{OH} - V_{IH}$$

$$NM_L = V_{IL} - V_{OL}$$

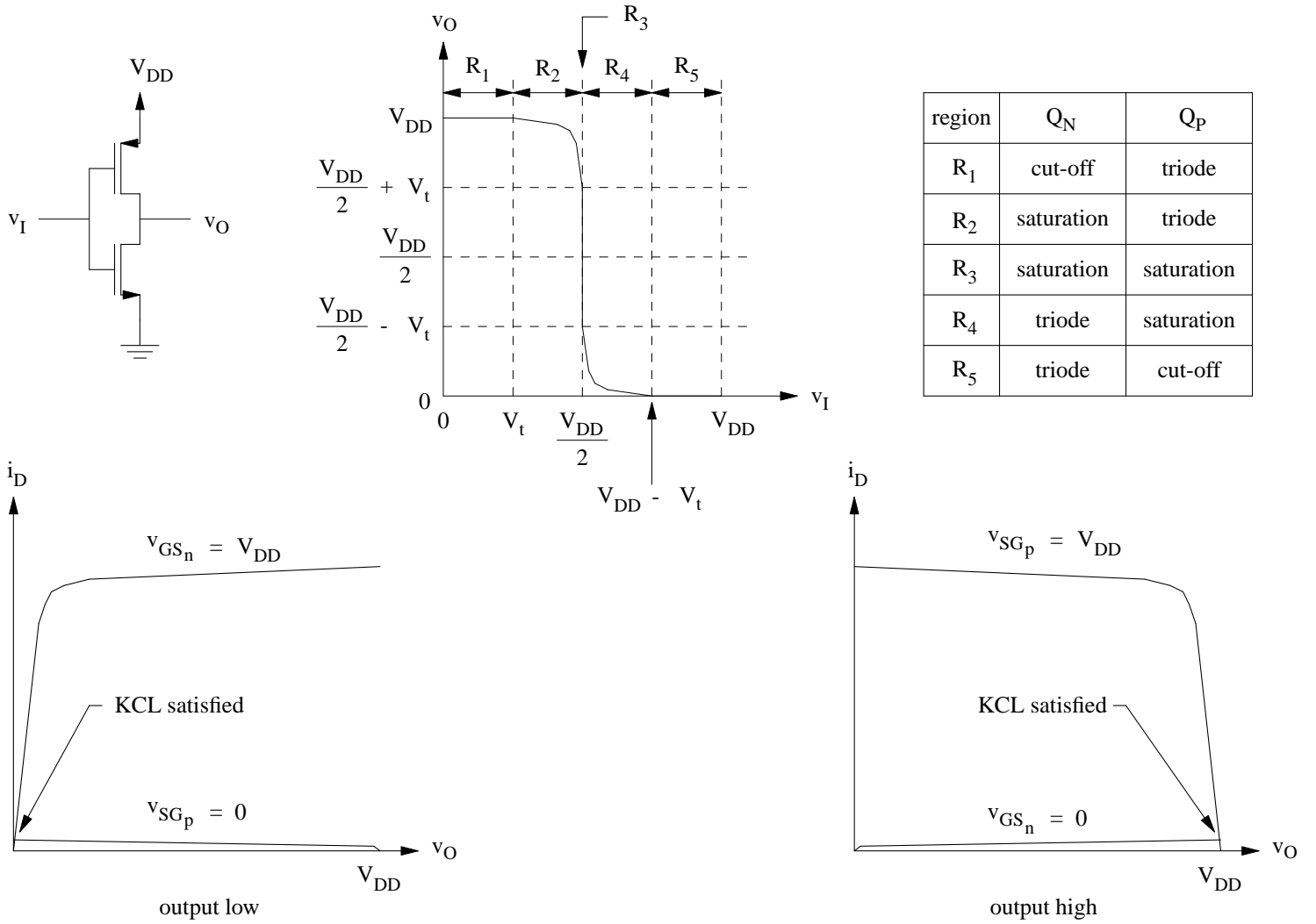


$t_{PHL}$  = high-to-low propagation delay

$t_{PLH}$  = low-to-high propagation delay

$t_r$  = 10% to 90% rise time

$t_f$  = 90% to 10% fall time



$$\text{for } v_O \leq v_I - V_{t_n} \quad i_{D_n} = k_n \left( (v_I - V_{t_n})v_O - \frac{1}{2}v_O^2 \right)$$

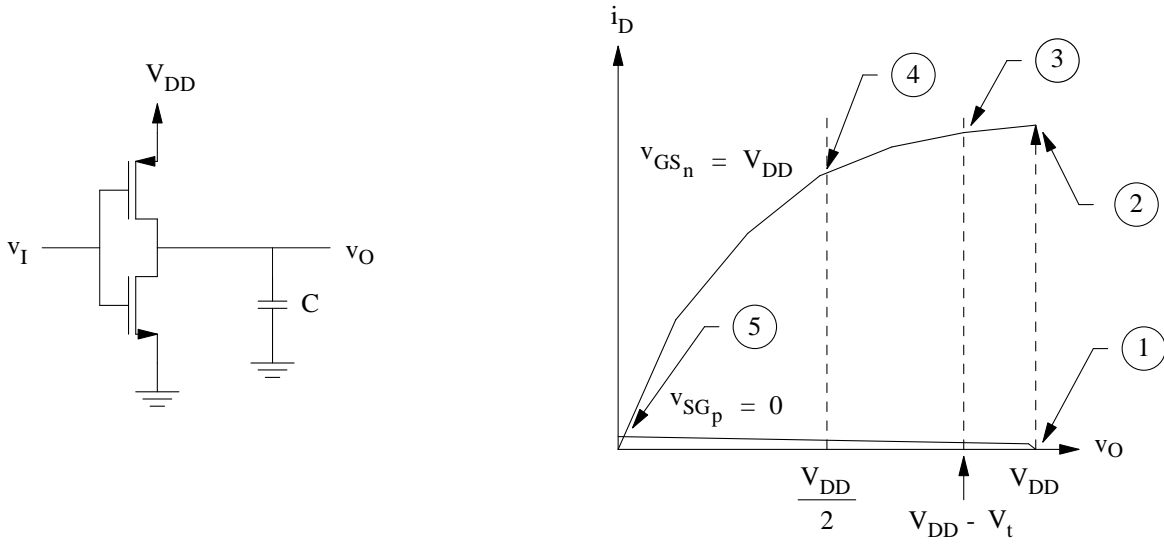
$$\text{for } v_O \geq v_I - V_{t_n} \quad i_{D_n} = \frac{1}{2} k_n (v_I - V_{t_n})^2$$

$$\text{for } v_O \geq v_I + |V_{t_p}| \quad i_{D_p} = k_p \left( (V_{DD} - v_I - |V_{t_p}|)(V_{DD} - v_O) - \frac{1}{2}(V_{DD} - v_O)^2 \right)$$

$$\text{for } v_O \leq v_I + |V_{t_p}| \quad i_{D_p} = \frac{1}{2} k_p (V_{DD} - v_I - |V_{t_p}|)^2$$

...

$$NM_H = V_{OH} - V_{IH} = \frac{1}{8} (3V_{DD} - 2V_{t_l}) \quad ; \quad NM_L = V_{IL} - V_{OL} = \frac{1}{8} (3V_{DD} - 2V_{t_l})$$



at  $t=0^+$ ,  $v_I$  goes from 0 to  $V_{DD}$  and the NMOS device goes from point 1 to point 2

from point 2 to point 3 the NMOS device is operating in saturation region

from point 3 to point 4 to point 5 the NMOS device is operating in the triode region

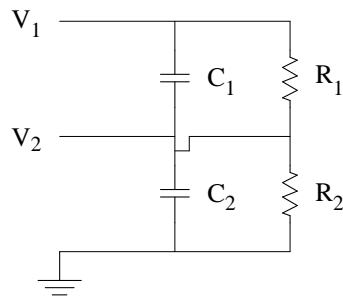
That is, from point 2 to point 3 the capacitor is discharged with a constant current, while from point 3 to point 5 the capacitor discharge current is decreasing with the output voltage.

$$v_t = 0.2 V_{DD} \rightarrow t_{PHL} = \frac{1.6 C}{k_n V_{DD}} \quad \text{and} \quad t_{PLH} = \frac{1.6 C}{k_p V_{DD}}$$

Energy to charge/discharge capacitor

$$Q = C V \rightarrow \frac{dq}{dt} = C \frac{dv}{dt} \rightarrow i_{D_n} = -C \frac{dv}{dt}$$

$$\frac{1}{2} C V_{DD}^2 \quad \text{to charge } C \text{ to } V_{DD} \quad ; \quad \frac{1}{2} C V_{DD}^2 \quad \text{to discharge } C \text{ to } 0 \rightarrow P_{\text{dynamic}} = f C V_{DD}^2$$



$$\begin{aligned}
 \frac{V_2}{V_1} &= \frac{\frac{1}{G_2 + sC_2}}{\frac{1}{G_1 + sC_1} + \frac{1}{G_2 + sC_2}} \\
 &= \frac{G_1 + sC_1}{G_1 + sC_1 + G_2 + sC_2} \\
 &= \frac{G_1 + sC_1}{(G_1 + G_2) + s(C_1 + C_2)} \\
 &= \frac{G_1}{G_1 + G_2} \frac{1 + s \frac{C_1}{G_1}}{1 + s \frac{C_1 + C_2}{G_1 + G_2}}
 \end{aligned}$$

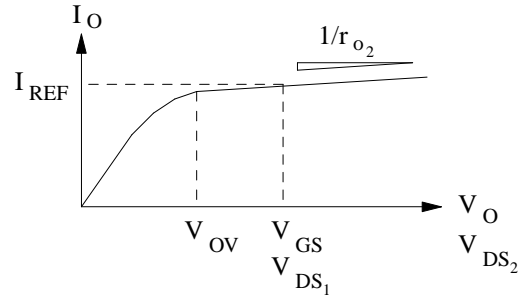
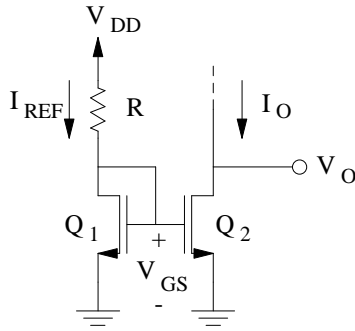
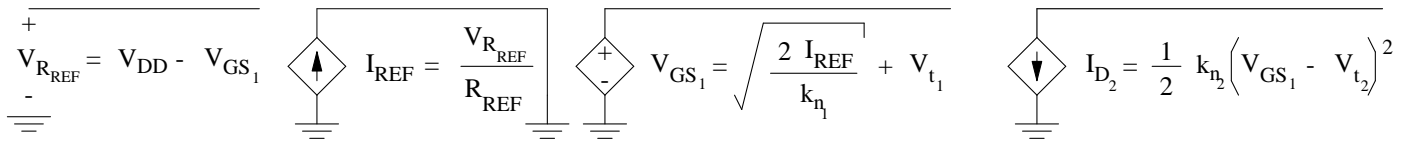
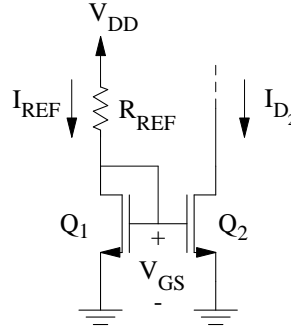
$$\frac{C_1}{G_1} = \frac{C_1 + C_2}{G_1 + G_2}$$

$$\frac{V_2}{V_1} = \frac{G_1}{G_1 + G_2} = \frac{\frac{1}{R_1}}{\frac{1}{R_1} + \frac{1}{R_2}} = \frac{R_2}{R_2 + R_1}$$

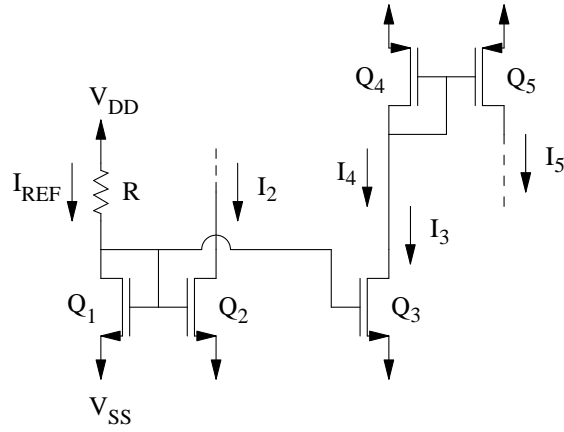


**MOS current mirrors**

$$I_{\text{REF}} = I_D = \frac{1}{2} k_n V_{\text{OV}}^2 \rightarrow V_{\text{OV}}^2 = \frac{2 I_{\text{REF}}}{k_n} \rightarrow V_{\text{OV}} = \left( \frac{2 I_{\text{REF}}}{k_n} \right)^{\frac{1}{2}} ; \quad V_{\text{GS}} = V_{\text{OV}} + V_t = \left( \frac{2 I_{\text{REF}}}{k_n} \right)^{\frac{1}{2}} + V_t$$



$$I_O = \left( \frac{W}{L} \right)_2^2 \cdot I_{\text{REF}} \cdot \left( 1 + \frac{V_{\text{DS}2} - V_{\text{DS}1}}{V_{\text{A}2}} \right) = \left( \frac{W}{L} \right)_2^2 \cdot I_{\text{REF}} \cdot \left( 1 + \frac{V_O - V_{\text{GS}}}{V_{\text{A}2}} \right)$$



$Q_1, Q_2, Q_3$  sink current.  $Q_4, Q_5$  source current.

$$I_2 = \frac{\left(\frac{W}{L}\right)_2}{\left(\frac{W}{L}\right)_1} \cdot I_{\text{REF}} \cdot \left(1 + \frac{V_{DS_2} - V_{GS_1}}{V_{A_2}}\right) ; \quad I_4 = I_3 = \frac{\left(\frac{W}{L}\right)_3}{\left(\frac{W}{L}\right)_1} \cdot I_{\text{REF}} \cdot \left(1 + \frac{V_{DS_3} - V_{GS_1}}{V_{A_3}}\right)$$

$$I_5 = \frac{\left(\frac{W}{L}\right)_5}{\left(\frac{W}{L}\right)_4} \cdot I_4 \cdot \left(1 + \frac{V_{SD_5} - V_{SG_4}}{V_{A_5}}\right)$$

## MOS small signal models

## low frequency model in saturation region

$$i_D = \frac{1}{2} k_n (v_{GS} - V_t)^2 = \frac{1}{2} k_n (V_{GS} + v_{gs} - V_t)^2 = \frac{1}{2} k_n (V_{OV} + v_{gs})^2 = \frac{1}{2} k_n V_{OV}^2 \left( 1 + \frac{v_{gs}}{V_{OV}} \right)^2$$

$$i_D = I_D \left( 1 + \frac{v_{gs}}{V_{OV}} \right)^2 = I_D \left( 1 + 2 \frac{v_{gs}}{V_{OV}} + \left( \frac{v_{gs}}{V_{OV}} \right)^2 \right)$$

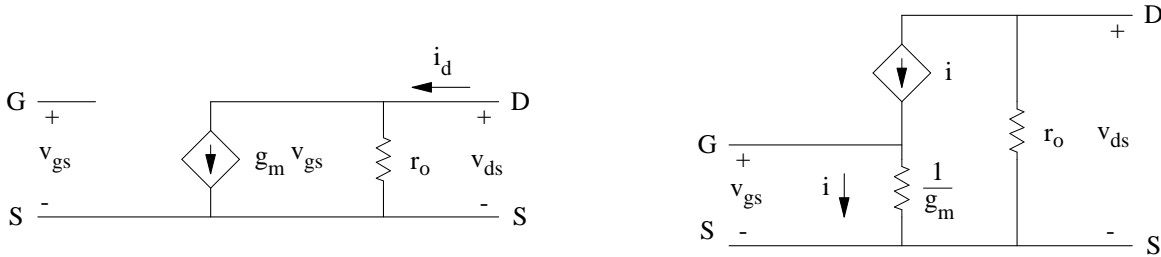
small-signal:  $\left( \frac{v_{gs}}{V_{OV}} \right)^2 \ll 2 \frac{v_{gs}}{V_{OV}} \quad \text{or} \quad v_{gs} \ll 2 V_{OV}$

$$i_D \approx I_D \left( 1 + 2 \frac{v_{gs}}{V_{OV}} \right) = I_D + 2 \frac{I_D}{V_{OV}} v_{gs} = I_D + g_m v_{gs} = I_D + i_d \quad \rightarrow \quad g_m = \frac{2 I_D}{V_{OV}} = \frac{I_D}{\left( \frac{V_{OV}}{2} \right)}$$

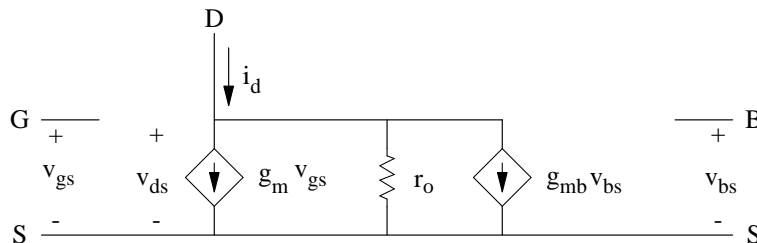
$$v_{DS} = V_{DD} - i_D R_D = V_{DD} - (I_D + i_d) R_D = V_{DD} - I_D R_D - i_d R_D = V_{DS} + v_{ds}$$

$$v_{ds} = -i_d R_D = -g_m R_D v_{gs} \quad \rightarrow \quad A_v = \frac{v_{ds}}{v_{gs}} = -g_m R_D$$

$$i_D = \frac{1}{2} k_n v_{OV}^2 \left( 1 + \frac{v_{DS}}{|V_A|} \right) \quad \rightarrow \quad g_o = \frac{d i_D}{d v_{DS}} \bigg|_Q = \frac{I_D}{|V_A|} \quad \rightarrow \quad r_o = \frac{1}{g_o}$$



$$g_{mb} = \frac{\partial i_D}{\partial v_{BS}} \bigg|_{\substack{v_{GS} = V_{GS} \\ v_{DS} = V_{DS}}}$$



**MOSFET high-frequency model**

triode region:

$$C_{gs} = C_{gd} = \frac{1}{2} W L C_{ox}$$

saturation region:

$$C_{gs} = \frac{2}{3} W L C_{ox}$$

$$C_{gd} = 0$$

cut-off region:

$$C_{gs} = C_{gd} = 0$$

$$C_{gb} = W L C_{ox}$$

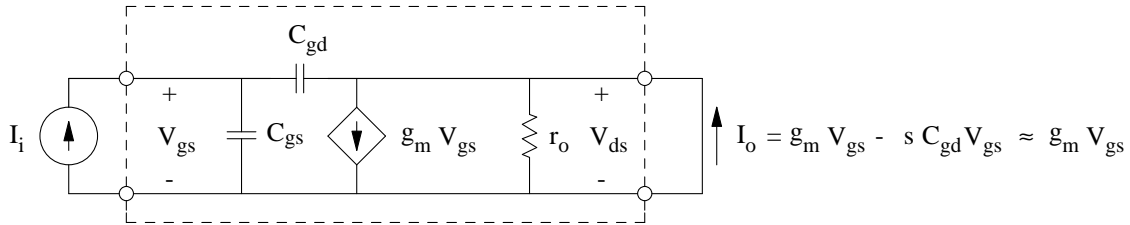
overlap capacitance:

$$C_{ov} = W L_{ov} C_{ox} \quad ; \quad 0.05 L \leq L_{ov} \leq 0.1 L$$

PN junction capacitance:

$$C_{sb} = \frac{C_{sb_0}}{\left(1 + \frac{V_{SB}}{V_\phi}\right)^{\frac{1}{2}}}$$

$$C_{db} = \frac{C_{db_0}}{\left(1 + \frac{V_{SB}}{V_\phi}\right)^{\frac{1}{2}}}$$

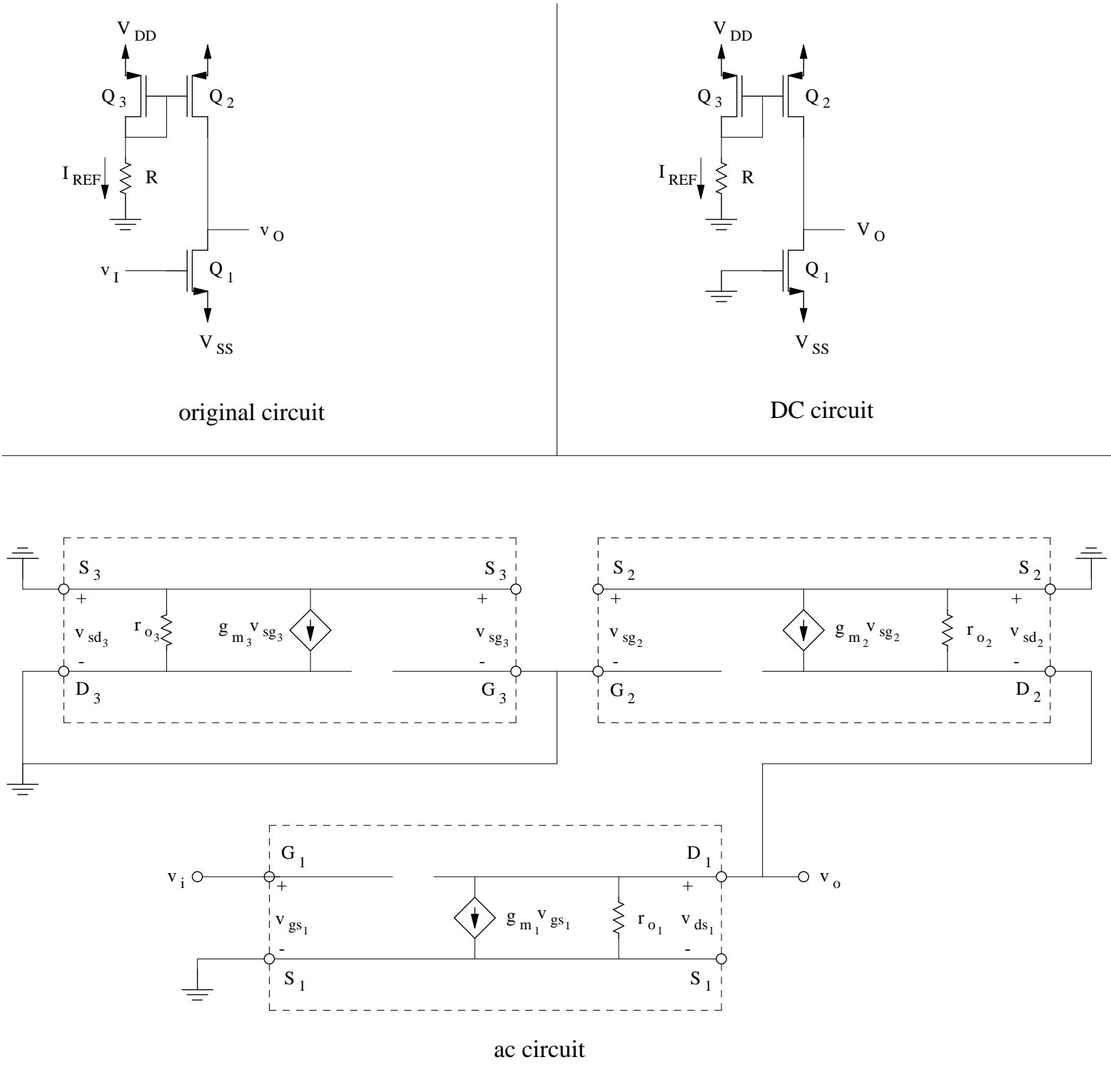


$$V_{gs} = \frac{I_i}{s(C_{gs} + C_{gd})} \quad ; \quad I_o \approx g_m V_{gs} = g_m \frac{I_i}{s(C_{gs} + C_{gd})} \quad ; \quad \frac{I_o}{I_i} = \frac{g_m}{s(C_{gs} + C_{gd})}$$

$$\text{for } \left| \frac{I_o}{I_i} \right| = 1 = \frac{g_m}{2\pi f_T (C_{gs} + C_{gd})} \quad \rightarrow \quad f_T = \frac{g_m}{2\pi (C_{gs} + C_{gd})}$$

MOS transistor active load amplifiers

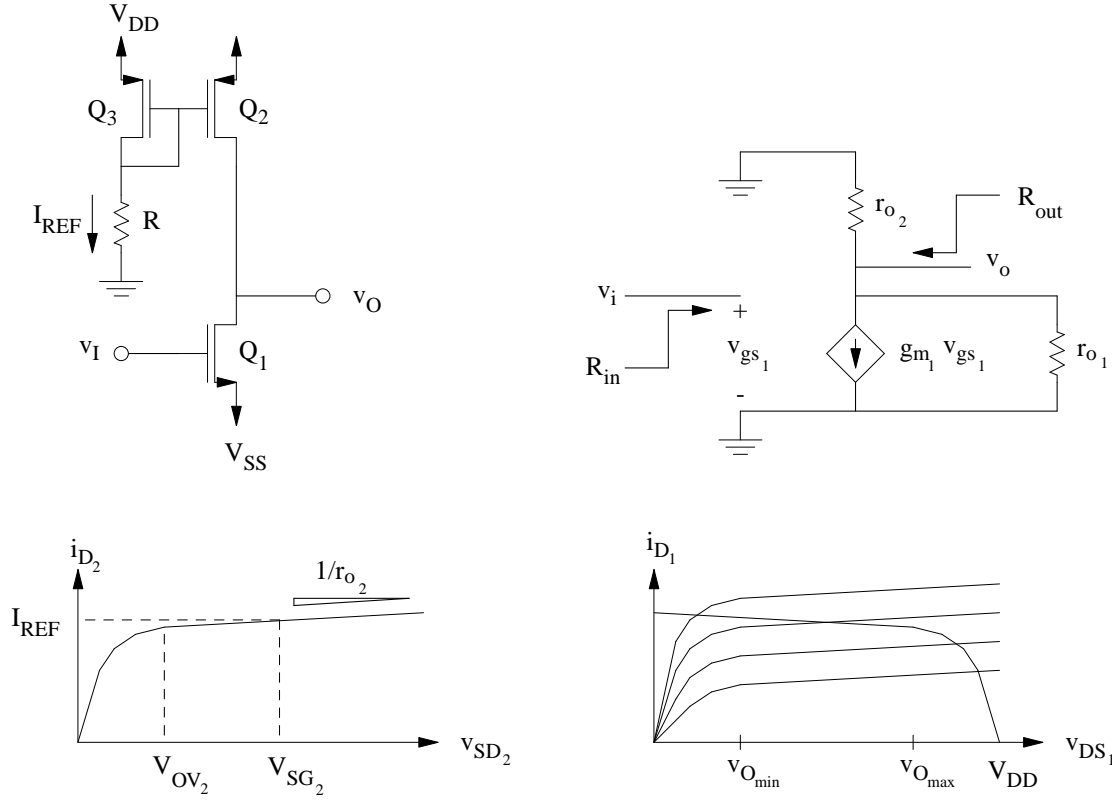
low frequency common source configuration



**process linking DC and ac circuits**

- (1) solve the DC circuit for:  $I_{D_1}$ ,  $I_{D_2}$ ,  $I_{D_3}$ ,  $V_{OV_1}$ ,  $V_{OV_2}$ ,  $V_{OV_3}$ ,  $V_{DS_1}$ ,  $V_{SD_2}$ ,  $V_{SD_3}$ .
- (2) Using these DC values, solve for small signal:  $g_{m_1}$ ,  $g_{m_2}$ ,  $g_{m_3}$ ,  $r_{o_1}$ ,  $r_{o_2}$ ,  $r_{o_3}$ .
- (3) solve the ac circuit for: low frequency voltage gain, input resistance, output resistance, (amplifier's - 3 dB frequency).

## low frequency common source configuration



assume  $Q_2$  and  $Q_3$  are matched ;  $v_{Omax} = V_{DD} - |V_{OV2}|$  ;  $v_{Omin} = V_{OV1}$

$$r_{o1} = \frac{|V_{A1}|}{I_{REF}} ; \quad r_{o2} = \frac{|V_{A2}|}{I_{REF}} ; \quad V_{OV} = \left( \frac{2 \cdot I_D}{\mu_n \cdot C_{ox} \cdot \frac{W}{L}} \right)^{\frac{1}{2}} ; \quad R_{in} = \infty ; \quad R_{out} = r_{o1} \parallel r_{o2}$$

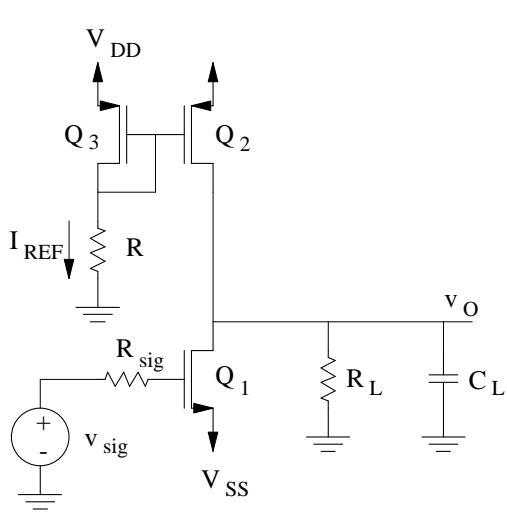
$$A_{v_o} = -g_{m1} \cdot r_{o1} = -A_0 \rightarrow A_v = A_{v_o} \frac{r_{o2}}{r_{o1} + r_{o2}} = -g_{m1} \cdot r_{o1} \frac{r_{o2}}{r_{o1} + r_{o2}} = -g_{m1} \cdot \left( r_{o1} \parallel r_{o2} \right)$$

$$I_D = \frac{1}{2} k_n V_{OV}^2 \rightarrow g_m = \frac{\partial i_D}{\partial v_{GS}} = k_n V_{OV} = k_n V_{OV} \frac{I_D}{\frac{1}{2} k_n V_{OV}^2} = \frac{I_D}{\frac{V_{OV}}{2}} \rightarrow A_0 = g_{m1} \cdot r_{o1} = \frac{I_D}{\frac{V_{OV}}{2}} \cdot \frac{V_A}{I_D} = \frac{V_A}{\frac{V_{OV}}{2}} = 2 \frac{V_A' L}{V_{OV}}$$

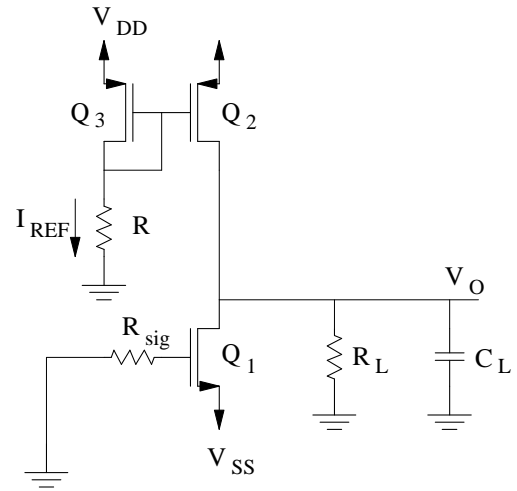
$$I_D = \frac{1}{2} k_n V_{OV}^2 ; \quad V_{OV} = \left( \frac{2 I_D}{k_n} \right)^{\frac{1}{2}} ; \quad g_m = k_n V_{OV} = k_n \left( \frac{2 I_D}{k_n} \right)^{\frac{1}{2}} = (2 k_n I_D)^{\frac{1}{2}}$$

$$A_0 = g_{m1} \cdot r_{o1} = \sqrt{2 k_n I_D} \cdot \frac{V_A}{I_D} = \sqrt{2 k_n} \frac{V_A}{\sqrt{I_D}}$$

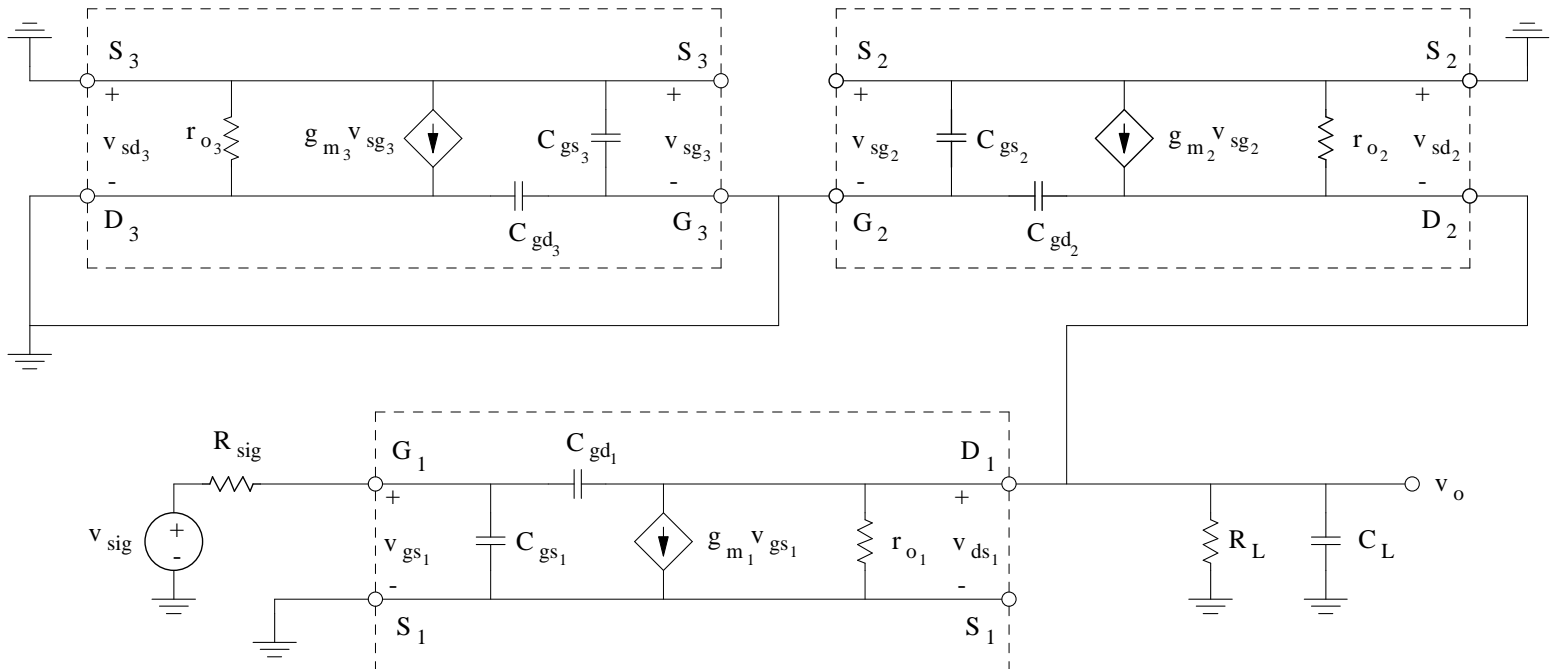
## high frequency common source configuration



original circuit



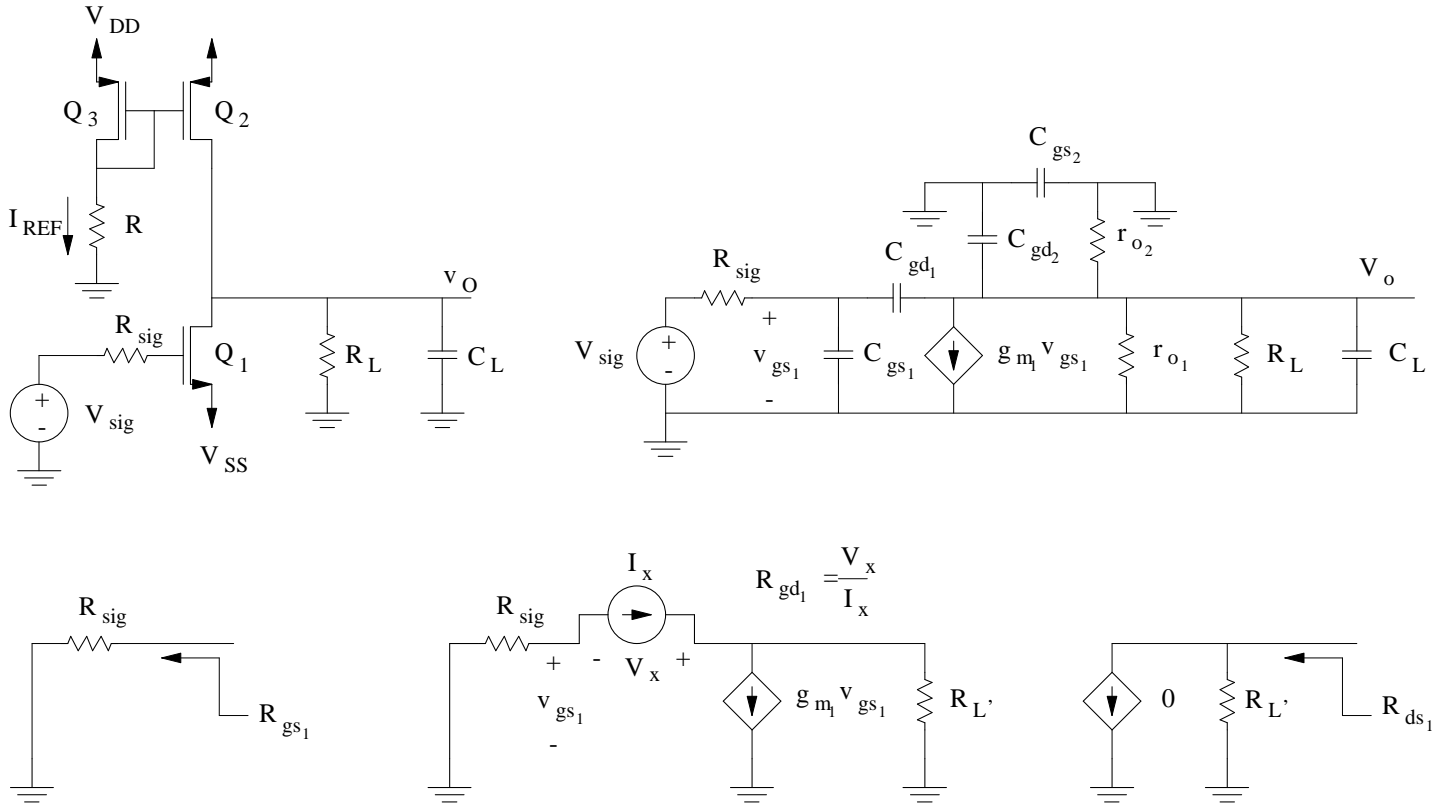
DC circuit



ac circuit



## high frequency common source configuration



$$A_M = \frac{V_o}{V_{sig}} = -g_{m1} \cdot \left( r_{o1} \parallel r_{o2} \parallel R_L \right) = -g_{m1} \cdot R_L'$$

$$\tau_H = C_{gs1} R_{gs1} + C_{gd1} R_{gd1} + \left( C_{gd2} + C_L \right) R_{ds1} \quad ; \quad R_{gs1} = R_{sig} \quad ; \quad R_{ds1} = R_L'$$

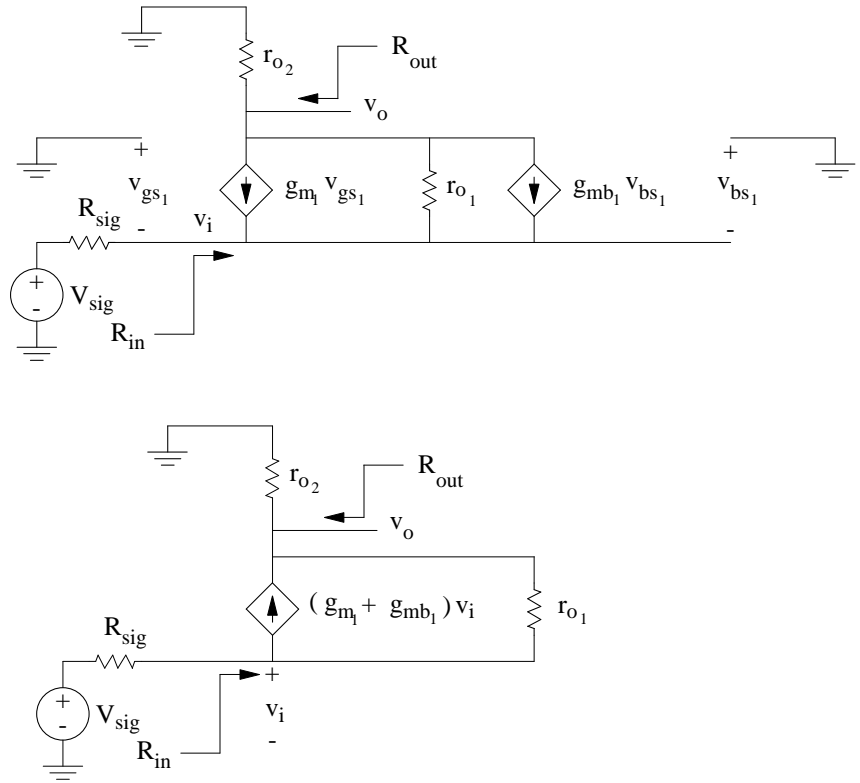
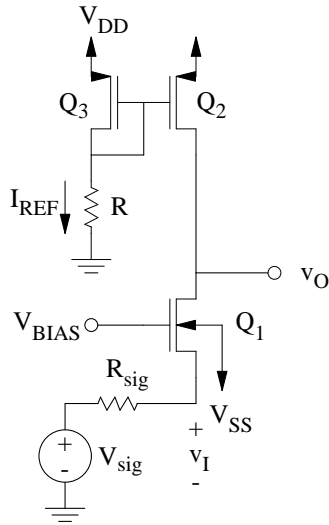
$$v_{gs1} = -i_x R_{sig} \quad ; \quad i_{ds1} = g_{m1} v_{gs1} = g_{m1} \left( -i_x R_{sig} \right) \quad ; \quad v_{ds1} = \left( i_x - i_{ds1} \right) R_L' = \left( i_x - \left( -g_{m1} i_x R_{sig} \right) \right) R_L' = i_x \left( R_L' + g_{m1} R_{sig} R_L' \right)$$

$$v_x = v_{dg1} = v_{ds1} - v_{gs1} = \left( i_x \left( R_L' + g_{m1} R_{sig} R_L' \right) \right) - \left( -i_x R_{sig} \right) = i_x \left( R_L' + g_{m1} R_{sig} R_L' + R_{sig} \right) = i_x \left( \left( 1 + g_{m1} R_L' \right) R_{sig} + R_L' \right)$$

$$r_{gd1} = \frac{v_x}{i_x} = \left( 1 + g_{m1} R_L' \right) R_{sig} + R_L'$$

$$\tau_H = C_{gs1} R_{sig} + C_{gd1} \left( \left( 1 + g_{m1} R_L' \right) R_{sig} + R_L' \right) + \left( C_{gd2} + C_L \right) R_L' = \left( C_{gs1} + C_{gd1} \left( 1 + g_{m1} R_L' \right) \right) R_{sig} + \left( C_{gd1} + C_{gd2} + C_L \right) R_L'$$

## low frequency common gate configuration



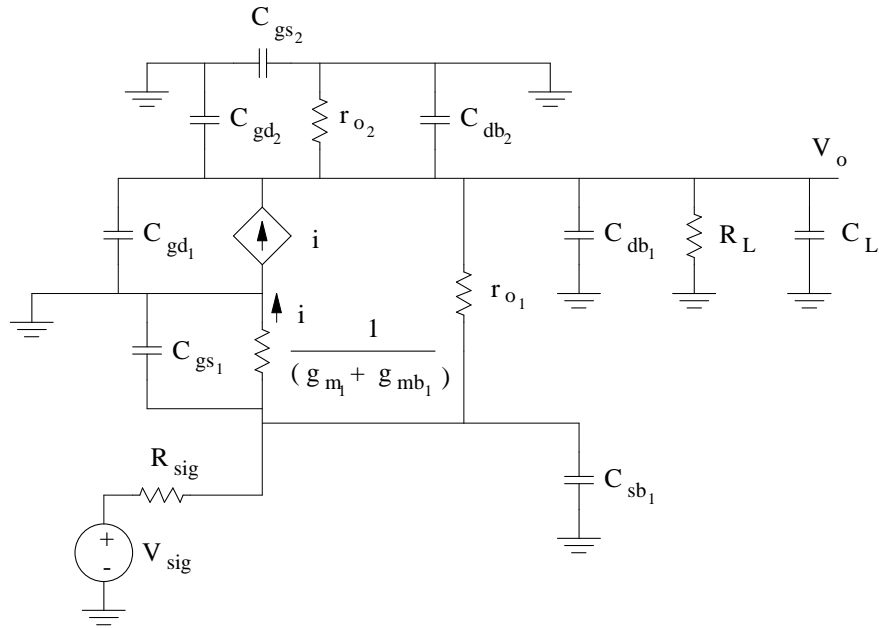
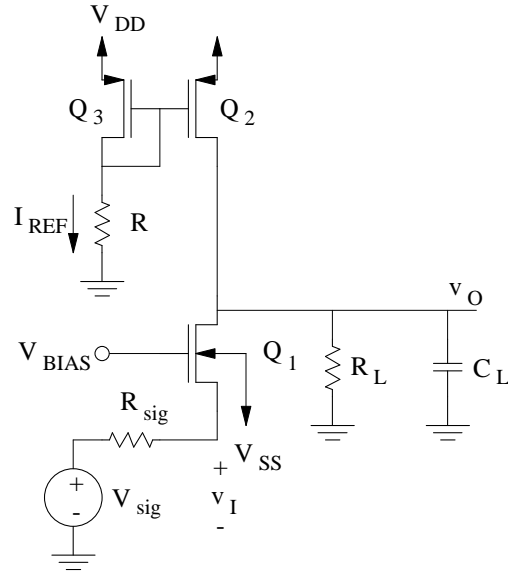
$$A_{v_o} = \frac{v_i + (g_{m1} + g_{mb1}) v_i r_{o1}}{v_i} = 1 + (g_{m1} + g_{mb1}) r_{o1}$$

$$A_v = A_{v_o} \frac{r_{o2}}{r_{o1} + r_{o2}} = \left( 1 + (g_{m1} + g_{mb1}) r_{o1} \right) \frac{r_{o2}}{r_{o1} + r_{o2}} \approx (g_{m1} + g_{mb1}) (r_{o1} \parallel r_{o2})$$

$$R_{in} = \frac{v_i}{i_i} \bigg|_{R_L < \infty} = \frac{v_i}{\frac{v_o}{r_{o2}}} = \frac{v_i}{\frac{A_v v_i}{r_{o2}}} = \frac{r_{o2}}{A_v} = \frac{r_{o2}}{A_{v_o} \frac{r_{o2}}{r_{o1} + r_{o2}}} = \frac{r_{o1} + r_{o2}}{A_{v_o}}$$

$$R_{out} = \frac{v_o}{i_o} \bigg|_{v_{sig} = 0} = \left( r_{o1} + \left( 1 + (g_{m1} + g_{mb1}) r_{o1} \right) R_{sig} \right) \parallel r_{o2} = \left( r_{o1} + A_{v_o} R_{sig} \right) \parallel r_{o2}$$

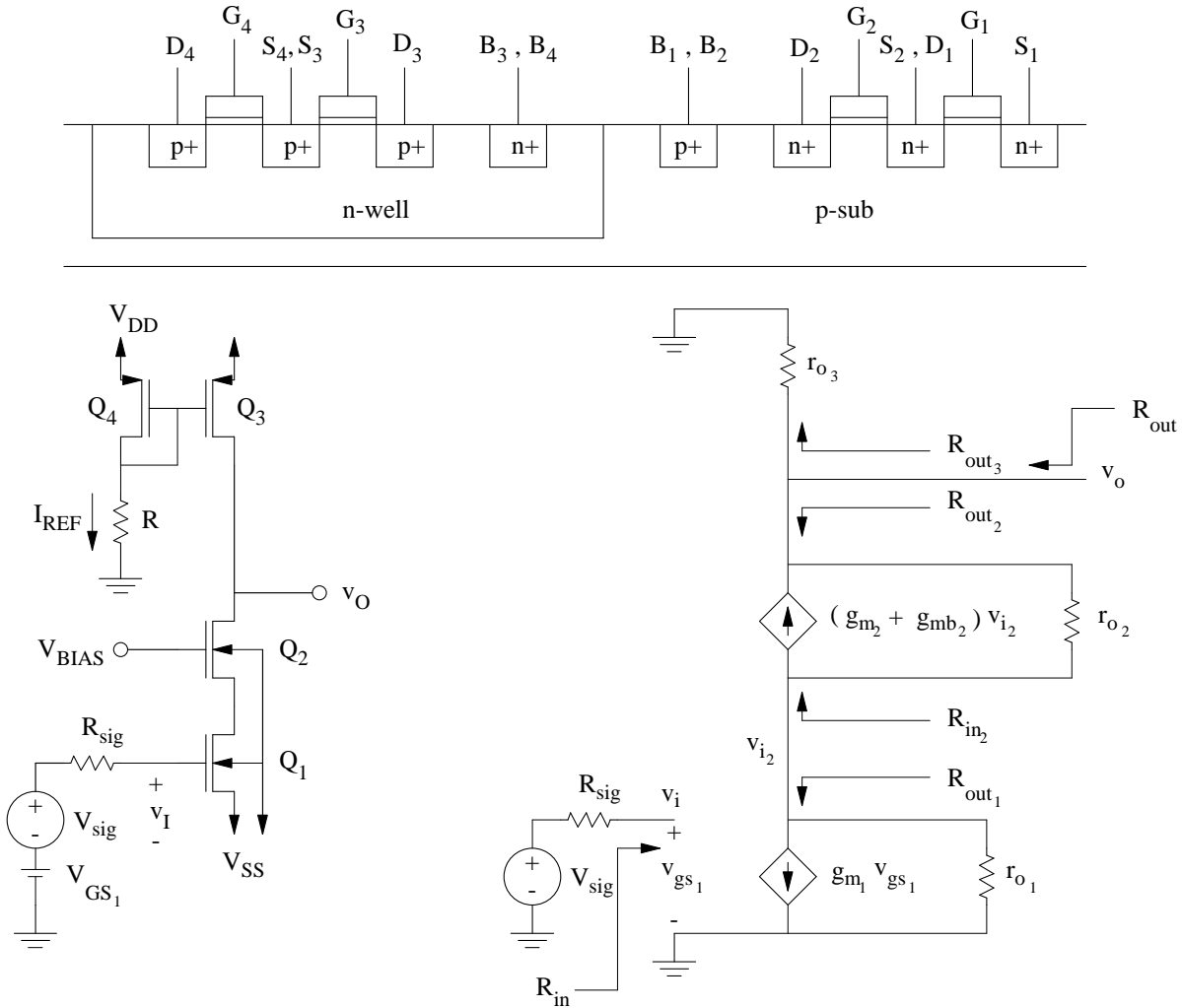
$$R_i = \frac{v_i}{i_i} \bigg|_{R_L = \infty} = \infty \quad ; \quad R_o = \frac{v_o}{i_o} \bigg|_{v_i = 0} = r_{o1} \parallel r_{o2}$$

**high frequency common gate configuration**

$$\tau_H = \left( C_{gs1} + C_{sb1} \right) \left( R_{sig} \parallel R_{in} \right) + \left( C_{gd1} + C_{gd2} + C_{db2} + C_{db1} + C_L \right) (R_{out} \parallel R_L)$$

$$= C_{s1} \left( R_{sig} \parallel R_{in} \right) + C_{d1} (R_{out} \parallel R_L)$$

## low frequency cascode configuration



$$R_{out3} = r_{o3} \quad ; \quad R_{out2} = \left( r_{o2} + A_{vo2} r_{o1} \right) \quad ; \quad R_{in2} = \frac{r_{o2} + r_{o3}}{1 + (g_{m2} + g_{mb2}) r_{o2}} \approx \frac{2}{g_{m2} + g_{mb2}} \quad ; \quad R_{out1} = r_{o1}$$

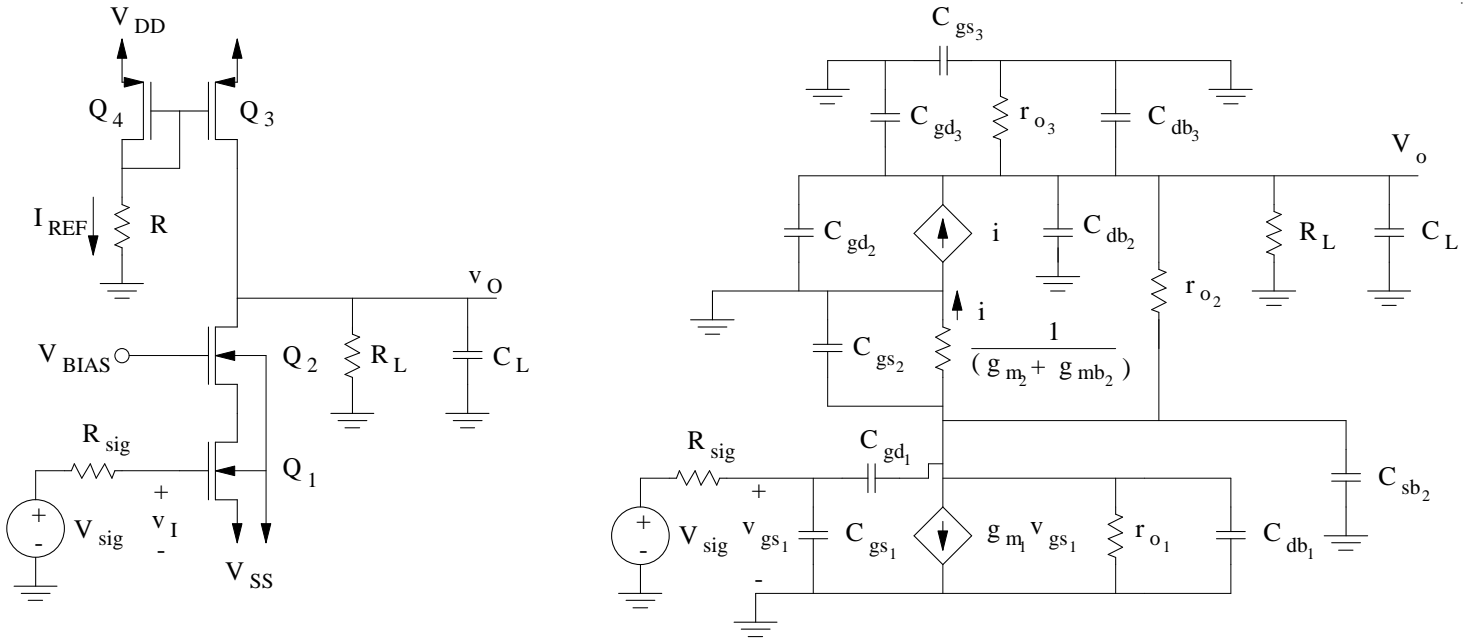
$$R_{d1} = R_{out1} \parallel R_{in2} \approx r_{o1} \parallel \frac{2}{g_{m2} + g_{mb2}} \approx \frac{2}{g_{m2} + g_{mb2}} \quad ; \quad R_{in} = \infty \quad ; \quad R_{out} = R_{out3} \parallel R_{out2} = r_{o3} \parallel \left( r_{o2} + A_{vo2} r_{o1} \right)$$

$$A_v = A_{vo1} A_{vo2} \frac{R_L}{R_{out} + R_L} = \left( -g_{m1} r_{o1} \right) \left( 1 + \left( g_{m2} + g_{mb2} \right) r_{o2} \right) \frac{R_L}{R_{out} + R_L}$$

$$A_v = -g_{m1} (R_{out} \parallel R_L) = -g_{m1} \left( \left( r_{o2} + A_{vo2} r_{o1} \right) \parallel r_{o3} \parallel R_L \right) \approx -g_{m1} \left( \left( r_{o2} + g_{m2} r_{o2} r_{o1} \right) \parallel r_{o3} \parallel R_L \right) \approx -g_{m1} (r_{o3} \parallel R_L)$$

$$A_{v1} = -g_{m1} R_{d1} \approx -g_{m1} \frac{2}{g_{m2} + g_{mb2}} \approx -\frac{2 g_{m1}}{1.2 g_{m2}} \approx -\frac{2 g_m}{1.2 g_m} = -1.67$$

## high frequency cascode configuration



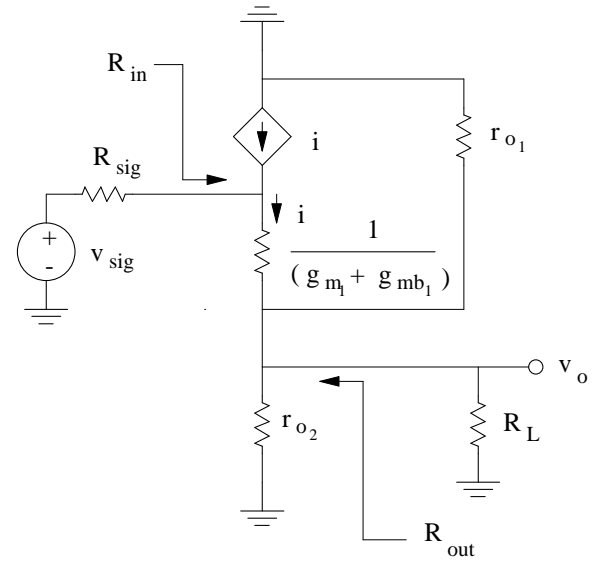
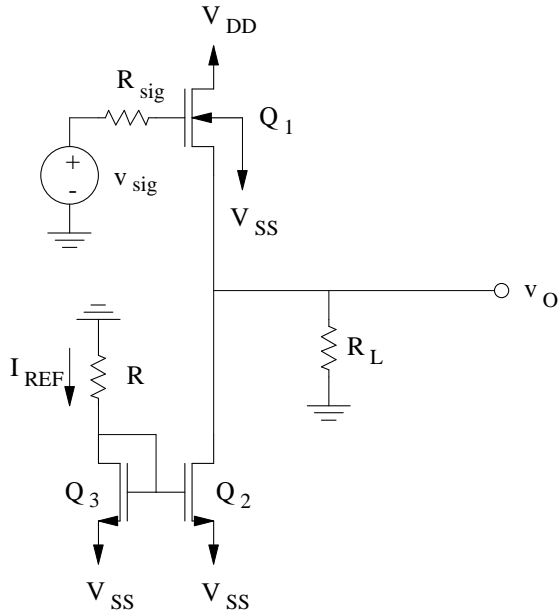
$$\begin{aligned}
 \tau_H &= C_{gs1} R_{gs1} + C_{gd1} R_{gd1} + \left( C_{gs2} + C_{db1} + C_{sb2} \right) R_{d1} + \left( C_L + C_{db2} + C_{gd2} + C_{gd3} + C_{db3} \right) (R_{out} \parallel R_L) \\
 &= C_{gs1} R_{gs1} + C_{gd1} R_{gd1} + C_{d1} R_{d1} + C_{d2} (R_{out} \parallel R_L) \\
 &= C_{gs1} R_{sig} + C_{gd1} \left( \left( 1 + g_{m1} R_{d1} \right) R_{sig} + R_{d1} \right) + C_{d1} R_{d1} + C_{d2} (R_{out} \parallel R_L) \\
 &= R_{sig} \left( C_{gs1} + C_{gd1} \left( 1 + g_{m1} R_{d1} \right) \right) + R_{d1} \left( C_{gd1} + C_{d1} \right) + (R_{out} \parallel R_L) C_{d2} \\
 &\approx R_{sig} \left( C_{gs1} + C_{gd1} \left( 1 + \frac{2g_{m1}}{g_{m2} + g_{mb2}} \right) \right) + \frac{2}{g_{m2} + g_{mb2}} \left( C_{gd1} + C_{d1} \right) + (R_{out} \parallel R_L) C_{d2}
 \end{aligned}$$

if  $R_{sig} \approx R_{out}$  and  $R_L \approx \infty$  then

$$\begin{aligned}
 \tau_H &= R_{out} \left( C_{gs1} + C_{gd1} \left( 1 + \frac{2g_{m1}}{g_{m2} + g_{mb2}} \right) \right) + \frac{2}{g_{m2} + g_{mb2}} \left( C_{gd1} + C_{d1} \right) + (R_{out} \parallel \infty) C_{d2} \\
 &= R_{out} \left( \left( C_{gs1} + C_{gd1} \left( 1 + \frac{2g_{m1}}{g_{m2} + g_{mb2}} \right) \right) + C_{d2} \right) + \frac{2}{g_{m2} + g_{mb2}} \left( C_{gd1} + C_{d1} \right)
 \end{aligned}$$

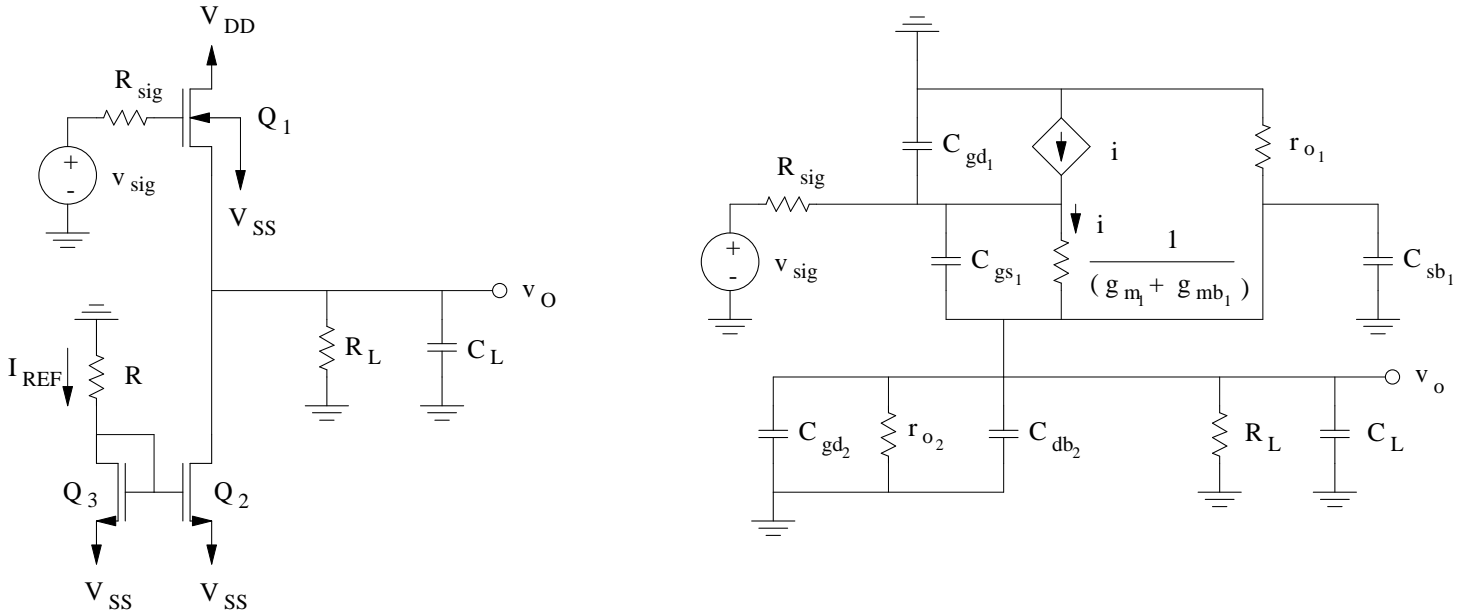
$$f_T = f_H A_{v_{DC}}$$

## low frequency common drain (source follower) configuration



$$R_{in} = \infty \quad ; \quad R_{out} \approx \frac{1}{g_{m1} + g_{mb1}} \quad ; \quad R_L' \approx R_L \parallel r_{o1} \parallel r_{o2}$$

$$A_v \approx \frac{R_L'}{R_L' + \frac{1}{g_{m1} + g_{mb1}}} \approx 1$$

**high frequency common drain (source follower) configuration**

$$R_{out} = \frac{1}{g_{m1} + g_{mb1}} \parallel r_{o1} \parallel r_{o2} \parallel R_L \approx \frac{1}{g_{m1} + g_{mb1}}$$

$$R'_L = r_{o1} \parallel r_{o2} \parallel R_L$$

$$R_{gs1} = \frac{R_{sig} + R'_L}{1 + (g_{m1} + g_{mb1}) R'_L}$$

$$\begin{aligned} \tau_H &= \left( C_{gd1} \right) R_{sig} + \left( C_{gs1} \right) R_{gs1} + \left( C_{gd2} + C_{db2} + C_{sb1} + C_L \right) R_{out} \\ &\approx \left( C_{gd1} \right) R_{sig} \end{aligned}$$

**amplifier comparison**

amplifier type	CS	CG	cascode	CD
input resistance	high	low	high	high
output resistance	high	high	high	low
voltage gain inverting	yes	no	yes	no
voltage gain magnitude	high	high	high	unity
frequence response	poor	good	good	good



**amplifier output stage design**

three main measures of an amplifier output stage performance

- (1) power-conversion efficiency

$$\eta \equiv \frac{\text{load\_power } (P_L)}{\text{supply\_power } (P_S)}$$

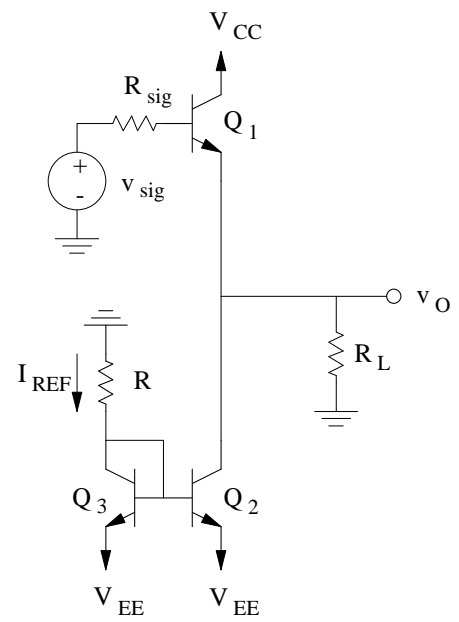
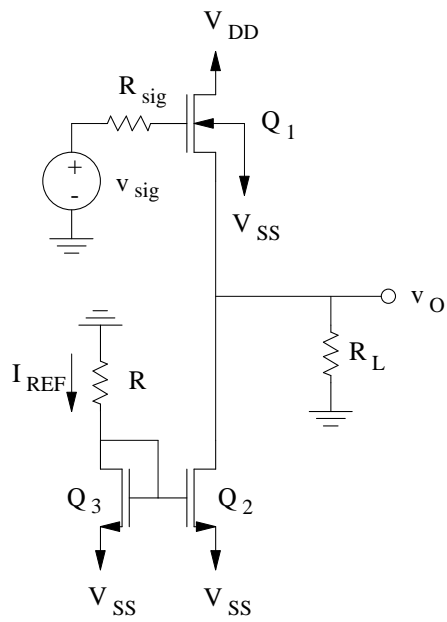
- (2) output voltage swing

- (3) total harmonic distortion (THD) in the load voltage

$$\text{THD} \equiv \frac{\sum_{i=2}^{i=\infty} \text{rms\_power\_of\_harmonic\_i}}{\text{rms\_power\_of\_fundamental}}$$

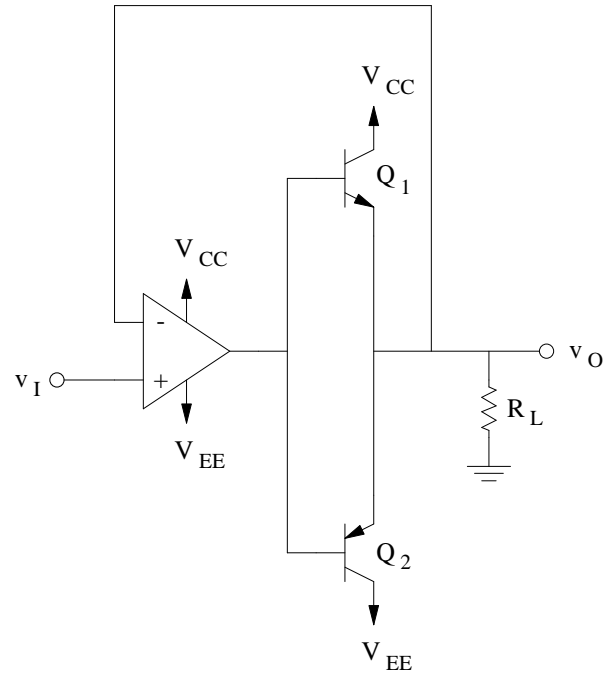
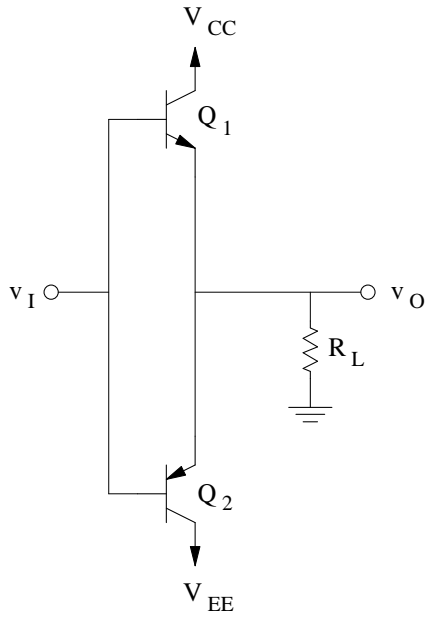
**class A output stage**

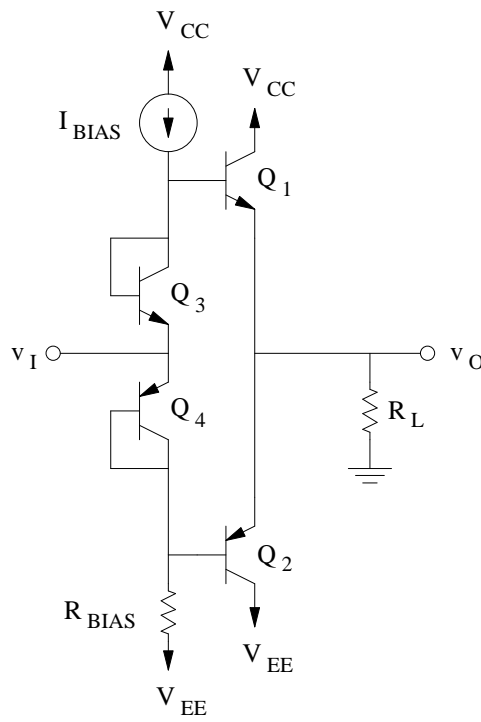
$$\eta_{\max} = 25\%$$



**class B output stage**

$\eta_{\max} = 50\%$  to  $78.5\%$  depending on voltage swing

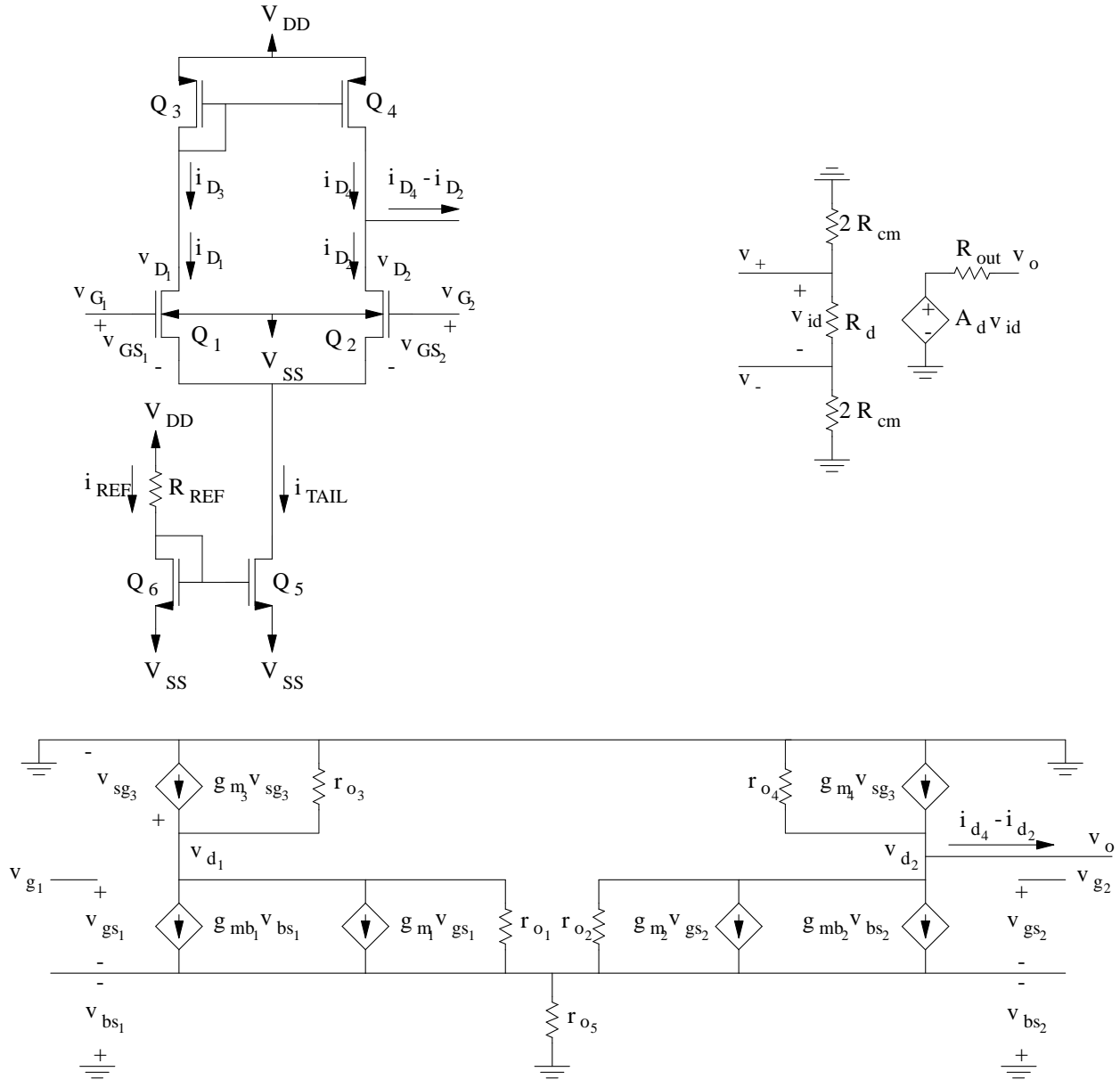


**BJT class AB output stage**



## MOS transistor differential amplifiers

### low frequency operation with active load



Note  $v_{bs_1} = v_{bs_2} = 0$  for differential operation.

for differential operation (ignoring  $r_{o_1}, r_{o_2}, r_{o_3}, r_{o_4}$  on the current mirror action).

$$v_{id} = v_{g_1} - v_{g_2} \quad ; \quad v_{g_1} = +\frac{v_{id}}{2} \quad ; \quad v_{g_2} = -\frac{v_{id}}{2} \quad ; \quad i_{d_3} = i_{d_1} = g_{m_1} \left( +\frac{v_{id}}{2} \right) \quad ; \quad i_{d_2} = g_{m_2} \left( -\frac{v_{id}}{2} \right)$$

$$v_{sg_3} = \frac{i_{d_3}}{g_{m_3}} = \frac{g_{m_1} \left( \frac{v_{id}}{2} \right)}{g_{m_3}} = \frac{g_{m_1}}{g_{m_3}} \frac{v_{id}}{2} \quad ; \quad i_{d_4} = g_{m_4} v_{sg_3} = g_{m_4} \frac{g_{m_1}}{g_{m_3}} \frac{v_{id}}{2} = g_{m_1} \frac{g_{m_4}}{g_{m_3}} \frac{v_{id}}{2} = g_{m_1} \frac{v_{id}}{2}$$

$$i_o = i_{d_4} - i_{d_2} = \left( g_{m_1} \frac{v_{id}}{2} \right) - \left( g_{m_2} \left( -\frac{v_{id}}{2} \right) \right) = \left( g_{m_1} \frac{v_{id}}{2} \right) + \left( g_{m_2} \frac{v_{id}}{2} \right) = 2 g_{m_1} \frac{v_{id}}{2} = g_{m_1} v_{id}$$

$$A_d \equiv \frac{v_o}{v_{id}} = g_{m_1} \left( r_{o_2} \parallel r_{o_4} \right)$$

for common mode operation (ignoring  $r_{o_1}, r_{o_2}, r_{o_3}, r_{o_4}$  on the current mirror action)

$$v_{icm} = \frac{v_{g_1} + v_{g_2}}{2} \quad ; \quad R_{o_1} = r_{o_1} + \left( 1 + g_{m_1} r_{o_1} \right) 2 r_{o_5} \quad ; \quad R_{o_2} = r_{o_2} + \left( 1 + g_{m_2} r_{o_2} \right) 2 r_{o_5} \quad ; \quad R_{o_3} = \frac{1}{g_{m_3}} \quad ; \quad R_{o_4} = r_{o_4}$$

$$R_{out} = R_{o_4} \parallel R_{o_2} = \left( r_{o_4} \right) \parallel \left( r_{o_2} + \left( 1 + g_{m_2} r_{o_2} \right) 2 r_{o_5} \right) \approx r_{o_4}$$

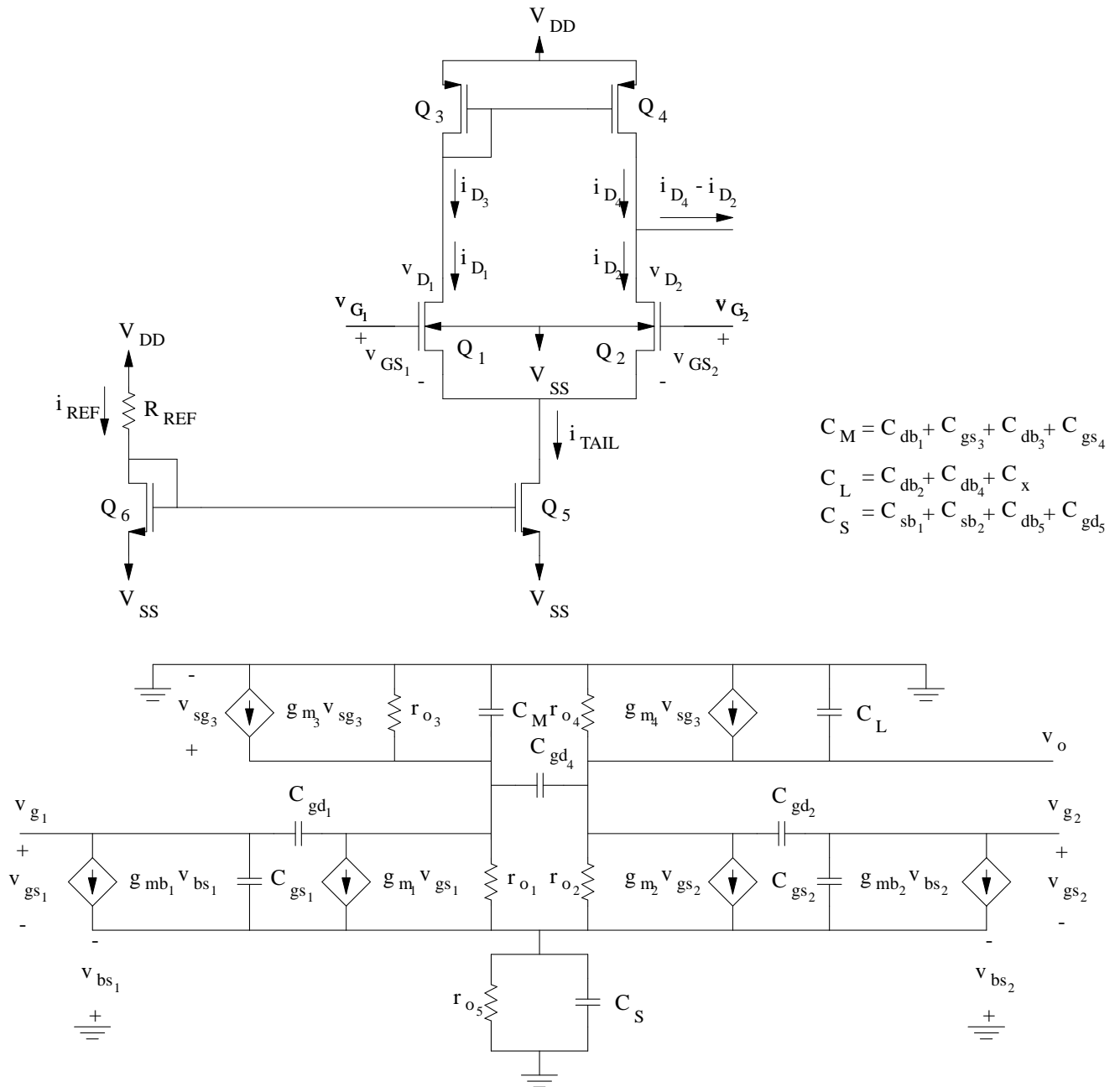
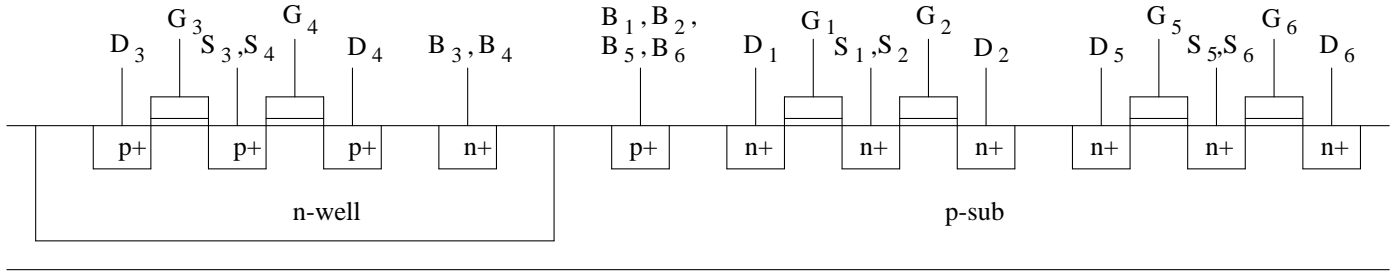
$$i_{d_3} = i_{d_1} = \frac{v_{icm}}{\frac{1}{g_{m_1}} + 2 r_{o_5}} = \frac{g_{m_1} v_{icm}}{1 + 2 g_{m_1} r_{o_5}} \quad ; \quad i_{d_2} = \frac{v_{icm}}{\frac{1}{g_{m_2}} + 2 r_{o_5}} = \frac{g_{m_2} v_{icm}}{1 + 2 g_{m_2} r_{o_5}} \quad ; \quad i_{d_4} = \frac{g_{m_4}}{g_{m_3}} i_{d_3} = \frac{g_{m_4}}{g_{m_3}} \frac{g_{m_1} v_{icm}}{1 + 2 g_{m_1} r_{o_5}}$$

$$i_o = i_{d_4} - i_{d_2} = \left( \frac{g_{m_4}}{g_{m_3}} \frac{g_{m_1} v_{icm}}{1 + 2 g_{m_1} r_{o_5}} \right) - \left( \frac{g_{m_2} v_{icm}}{1 + 2 g_{m_2} r_{o_5}} \right) = \left( \left( \frac{g_{m_4}}{g_{m_3}} \frac{g_{m_1}}{1 + 2 g_{m_1} r_{o_5}} \right) - \left( \frac{g_{m_2}}{1 + 2 g_{m_2} r_{o_5}} \right) \right) v_{icm}$$

$$v_o = i_o R_{out} \quad ; \quad A_{cm} \equiv \frac{v_o}{v_{icm}} = \left( \left( \frac{g_{m_4}}{g_{m_3}} \frac{g_{m_1}}{1 + 2 g_{m_1} r_{o_5}} \right) - \left( \frac{g_{m_2}}{1 + 2 g_{m_2} r_{o_5}} \right) \right) r_{o_4}$$

$$CMRR \equiv \frac{A_d}{A_{cm}} = \frac{\left( g_{m_1} \frac{g_{m_4}}{g_{m_3}} \right) \left( r_{o_2} \parallel r_{o_4} \right)}{\left( \left( \frac{g_{m_4}}{g_{m_3}} \frac{g_{m_1}}{1 + 2 g_{m_1} r_{o_5}} \right) - \left( \frac{g_{m_2}}{1 + 2 g_{m_2} r_{o_5}} \right) \right) r_{o_4}} \approx \frac{g_{m_1} \left( r_{o_2} \parallel r_{o_4} \right)}{\left( \frac{1}{2 r_{o_5}} - \frac{1}{2 r_{o_5}} \right) r_{o_4}} = \infty$$

## high frequency operation with active load



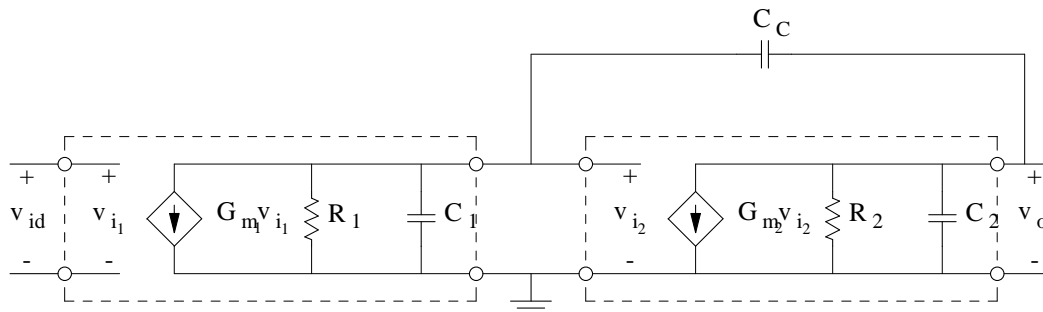
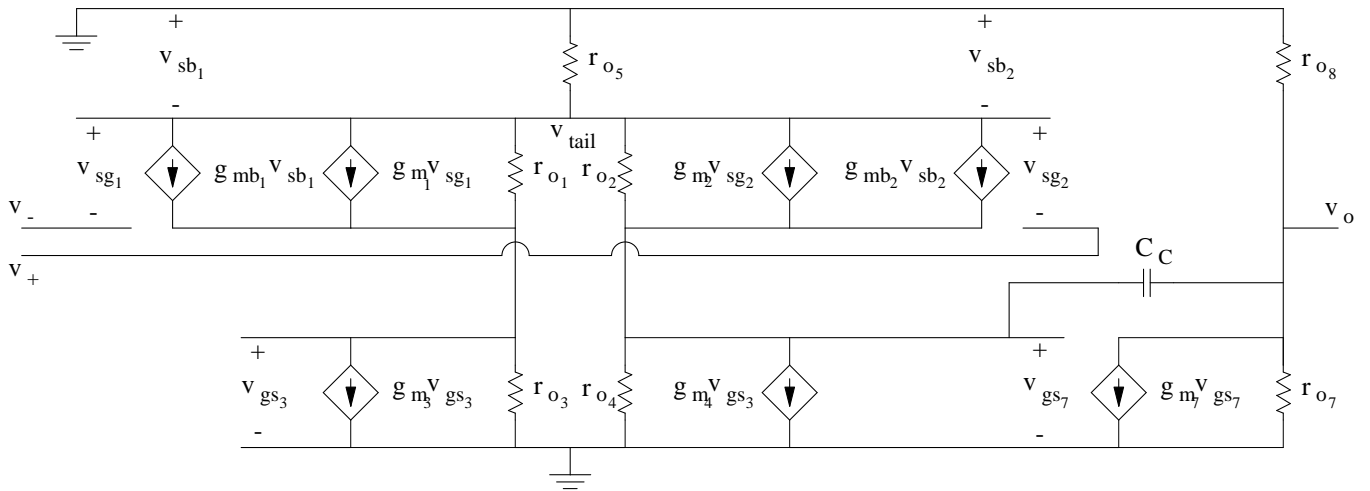
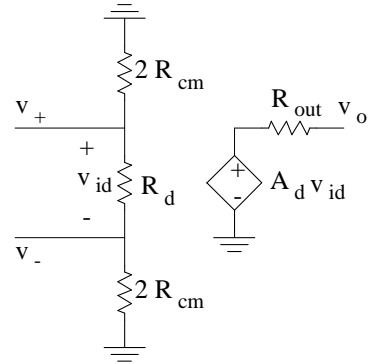
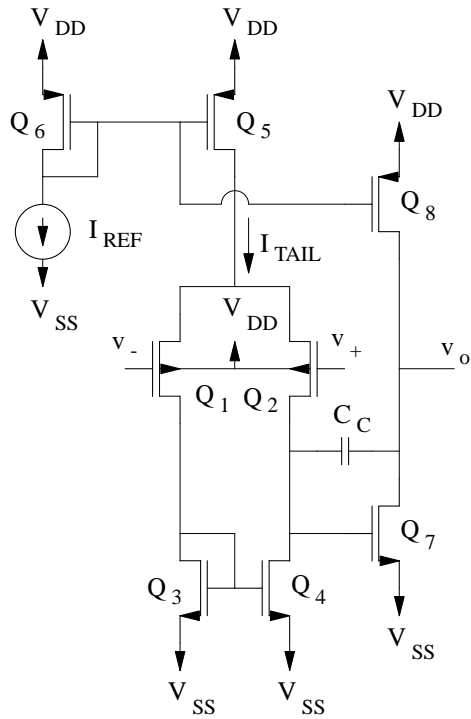


in general  $C_M$  and  $C_L$  dominate the frequency response,  $f_{p_1} \approx \frac{1}{2\pi C_L R_{out}}$  and  $f_{p_2} \approx \frac{1}{2\pi C_M \frac{1}{g_{m_3}}} = \frac{g_{m_3}}{2\pi C_M}$ .

$C_M$  also results in a pole-zero combination associated with the  $Q_3$  and  $Q_4$  current mirror  $f_z \approx \frac{2g_{m_3}}{2\pi C_M}$ .

Recall that the transistors unity current gain frequency is  $f_T \approx \frac{g_m}{2\pi(C_{gs} + C_{gd})}$  which is around  $f_z$ . This zero lift's the opamp's phase as it approaches  $f_T$  to avoid unwanted positive feedback.

## multi-stage amplifiers



$$\frac{I_{D_7}}{\left(\frac{W}{L}\right)_7} = \frac{I_{TAIL}}{2} \quad ; \quad \frac{I_{D_8}}{\left(\frac{W}{L}\right)_8} = \frac{I_{TAIL}}{\left(\frac{W}{L}\right)_5} \quad ; \quad I_{D_8} = I_{D_7} \quad \rightarrow \quad \frac{\left(\frac{W}{L}\right)_7}{\left(\frac{W}{L}\right)_4} = 2 \frac{\left(\frac{W}{L}\right)_8}{\left(\frac{W}{L}\right)_5}$$

$$V_{SS} + V_{OV_4} + V_{t_2} \leq v_+ \leq V_{DD} - V_{OV_5} - V_{t_2} - V_{OV_2}$$

$$V_{SS} + V_{OV_3} + V_{t_1} \leq v_- \leq V_{DD} - V_{OV_5} - V_{t_1} - V_{OV_1}$$

$$V_{SS} + V_{OV_7} \leq v_O \leq V_{DD} - V_{OV_8}$$

cascaded transconductance amplifier

$$R_d = \infty \quad ; \quad R_{cm} = \infty \quad ; \quad R_{out} = r_{O_7} \parallel r_{O_8}$$

$$A_d = A_1 A_2 = \left( G_{m_1} R_1 \right) \left( G_{m_2} R_2 \right) = \left( g_{m_1} (r_{O_2} \parallel r_{O_4}) \right) \left( g_{m_7} (r_{O_7} \parallel r_{O_8}) \right)$$

$$C_1 \approx C_{gd_2} + C_{db_2} + C_{gd_4} + C_{db_4} + C_{gs_7} \quad ; \quad C_2 \approx C_{db_7} + C_{db_8} + C_{gd_8} + C_{load}$$

$$\omega_z \approx \frac{G_{m_2}}{C_C}$$

$$\omega_{p_1} \approx \frac{1}{R_1 \left( C_1 + C_C \left( 1 + G_{m_2} R_2 \right) \right) + R_2 (C_C + C_2)} \approx \frac{1}{R_1 \left( C_1 + C_C \left( 1 + G_{m_2} R_2 \right) \right)} \approx \frac{1}{R_1 C_C G_{m_2} R_2}$$

$$\omega_{p_2} \approx \frac{G_{m_2} C_C}{C_1 C_2 + C_C (C_1 + C_2)} \approx \frac{G_{m_2} C_C}{C_C (C_1 + C_2)} \approx \frac{G_{m_2}}{C_1 + C_2} \approx \frac{G_{m_2}}{C_2}$$

$$\omega_t = \left( G_{m_1} R_1 G_{m_2} R_2 \right) \omega_{p_1} = A_o \omega_{p_1}$$

Want  $\omega_t < \omega_z$  and  $\omega_t < \omega_{p_2}$

opamp transistor grouping