SYDE252 - lecture notes

09/01/18

Presented by: John Zelek
Systems Design Engineering
note: some material (figures) borrowed from various sources

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6. Fourier Applications

09/11/18

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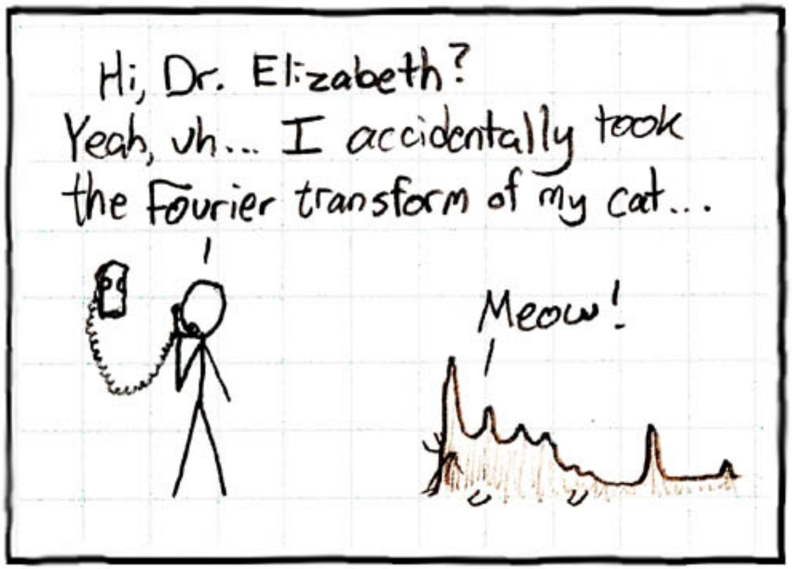
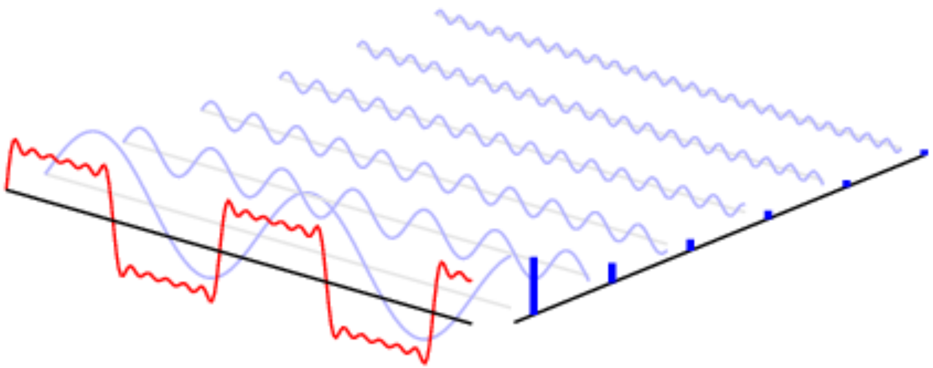
inspiration

- ““=Despots prefer the friendship of the dog, who, unjustly mistreated and debased, still loves and serves the man who wronged him. The method of doubt must be applied to civilization; we must doubt its necessity, its excellence, and its permanence”

— **Charles Fourier**





http://math.sfsu.edu/beck/quotes.html

- http://pgfplots.net/tikz/examples/fourier-transform/

Fourier Applications

- modulation
- sampling
- filtering
-

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Fourier Applications

$$x_c(t) = A\cos(2\pi f_c t)$$

$$\Updownarrow$$

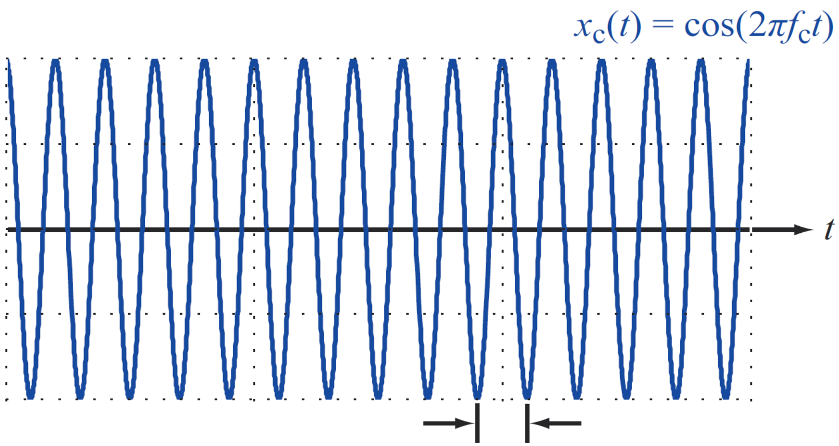
$$\mathbf{X}_c(f) = \frac{A}{2}[\delta(f - f_c) + \delta(f + f_c)]$$

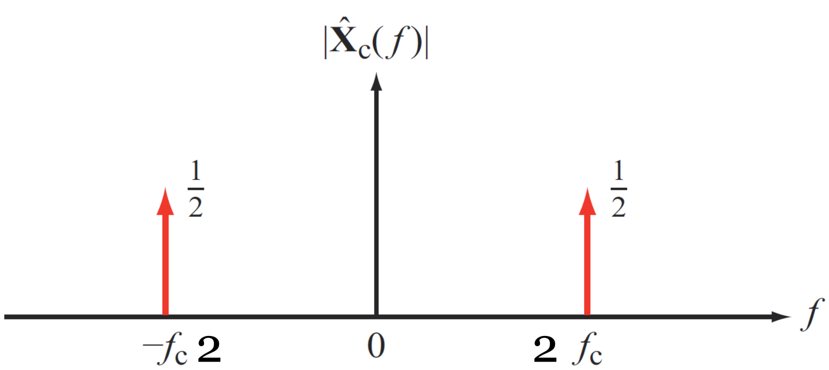
$$y_m(t) = x(t)\cos(2\pi f_c t)$$

$$\Updownarrow$$

$$\mathbf{Y}_m(f) = \frac{1}{2}[\mathbf{X}(f - f_c) + \mathbf{X}(f + f_c)].$$

(DSB modulation)





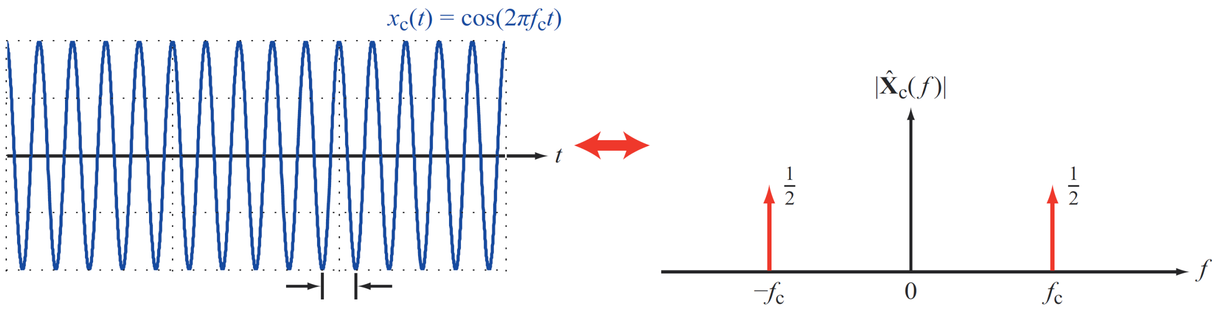
Fourier transforms - DSB (double side band) modulation

DSB Modulation of x(t): $y_m(t) = x(t)\cos(2\pi f_c t)$

DSB Demodulation of y(t): $y_d(t) = y_m(t)\cos(2\pi f_c t)$

$$y_d(t) = x(t)\cos^2(2\pi f_c t) = \frac{1}{2}x(t) + \frac{1}{2}x(t)\cos(4\pi f_c t).$$

$$\hat{\mathbf{Y}}_d(f) = \frac{1}{2}\hat{\mathbf{X}}(f) + \frac{1}{4}\left[\hat{\mathbf{X}}(f-2f_c) + \hat{\mathbf{X}}(f+2f_c)\right]$$



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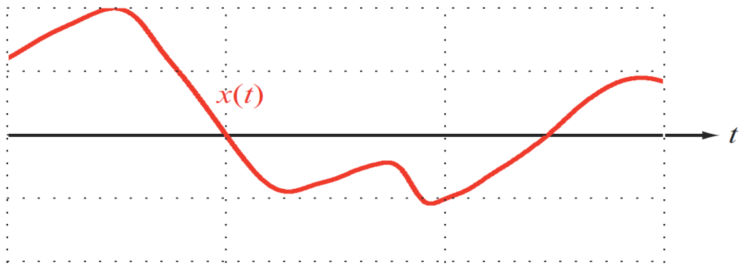
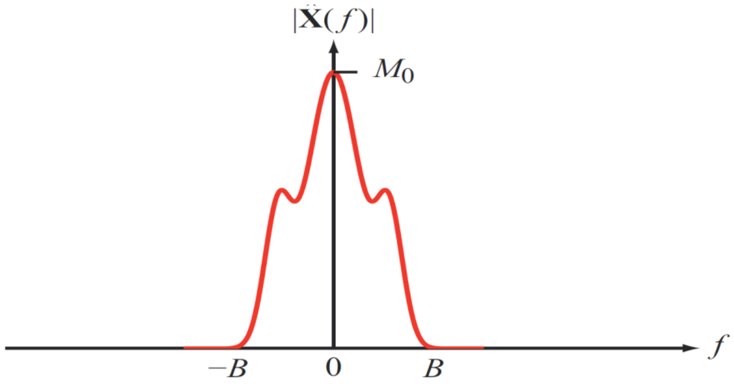
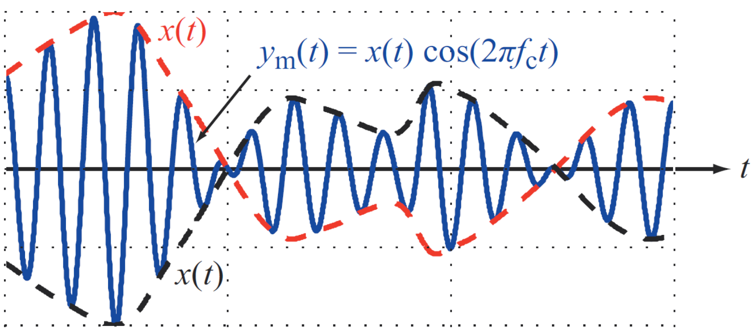
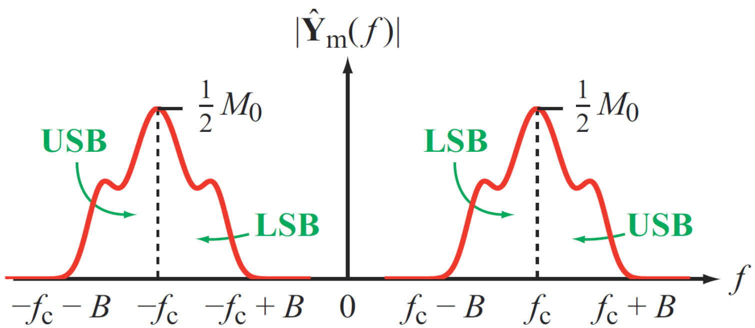
Fourier transforms - DSB modulation





Multiplication of a signal by a sinusoid shifts its spectrum up and down by the frequency of the multiplying sinusoid:





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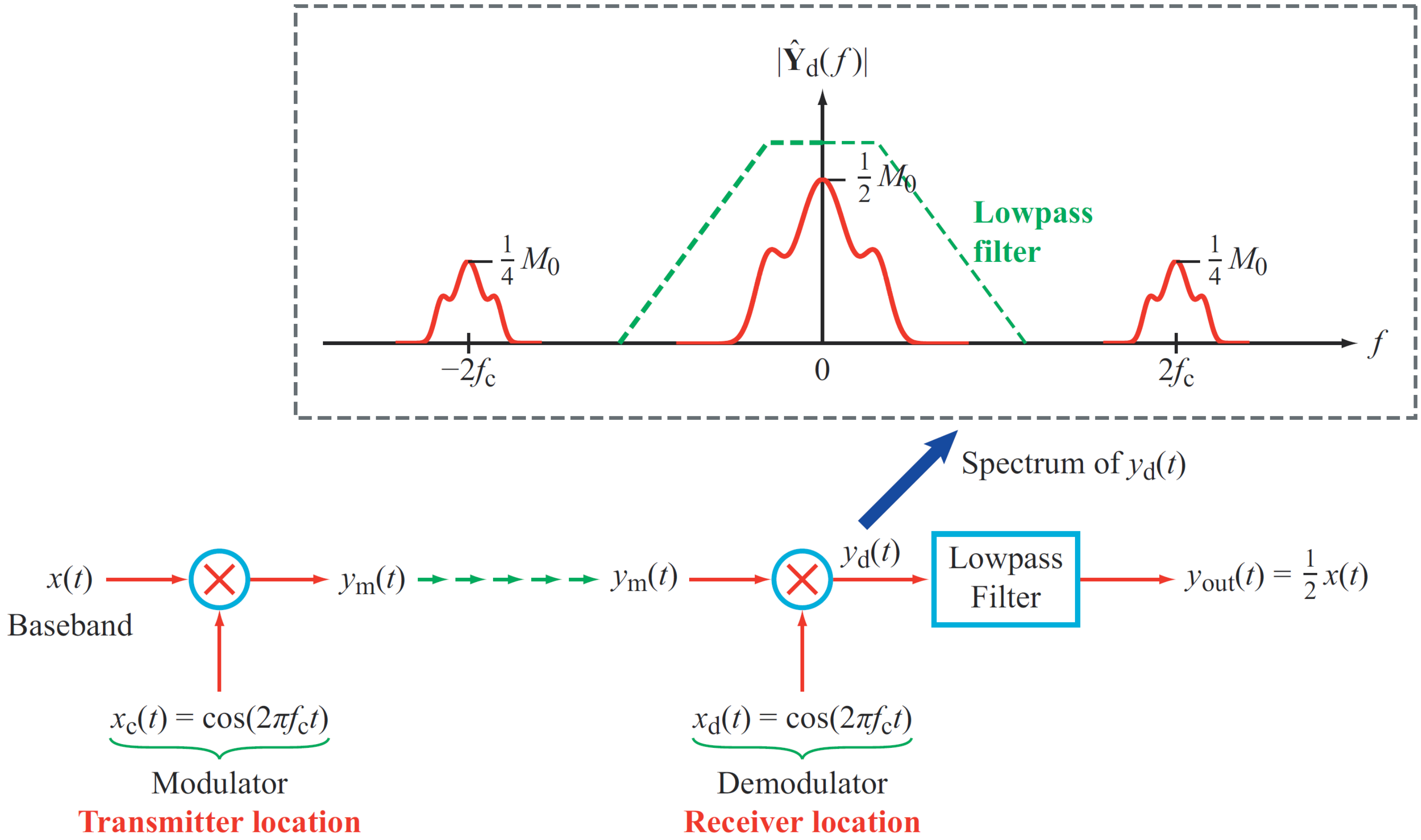
Fourier transforms - DSB modulation - recovery of signal



Fourier transforms - DSB example

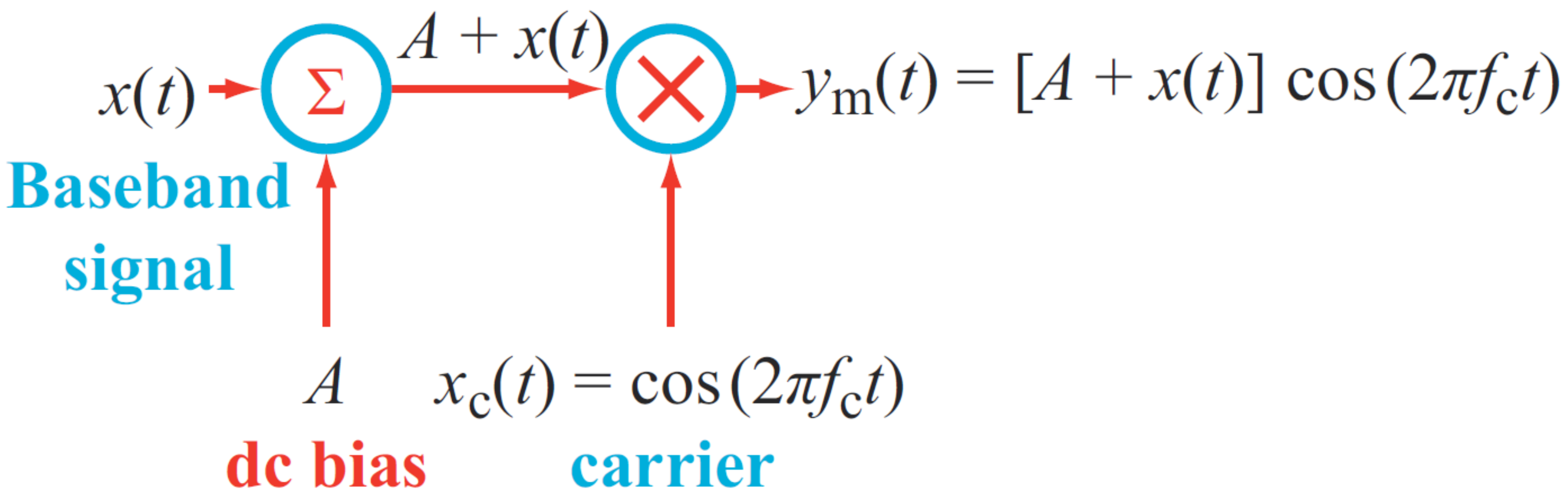
Given signals $x_1(t) = 4\cos(8\pi t)$, $x_2(t) = 6\cos(6\pi t)$, and $x_3(t) = 4\cos(4\pi t)$, generate the spectrum of $y(t) = x_1(t) + x_2(t)\cos(20\pi t) + x_3(t)\cos(40\pi t)$.

Solution:

Fourier transforms - AM

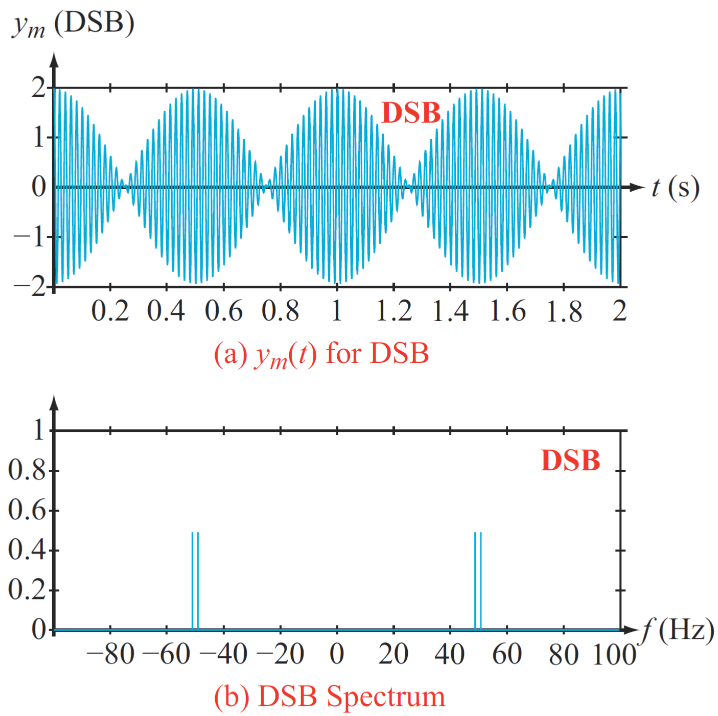
AM: Add a copy of the carrier to the DSB modulated signal.
Why? Can now use envelope detection to recover the signal.
This is much simpler than using DSB demodulation, as above.

$$y_m(t) = [A + x(t)]\cos(2\pi f_c t)$$

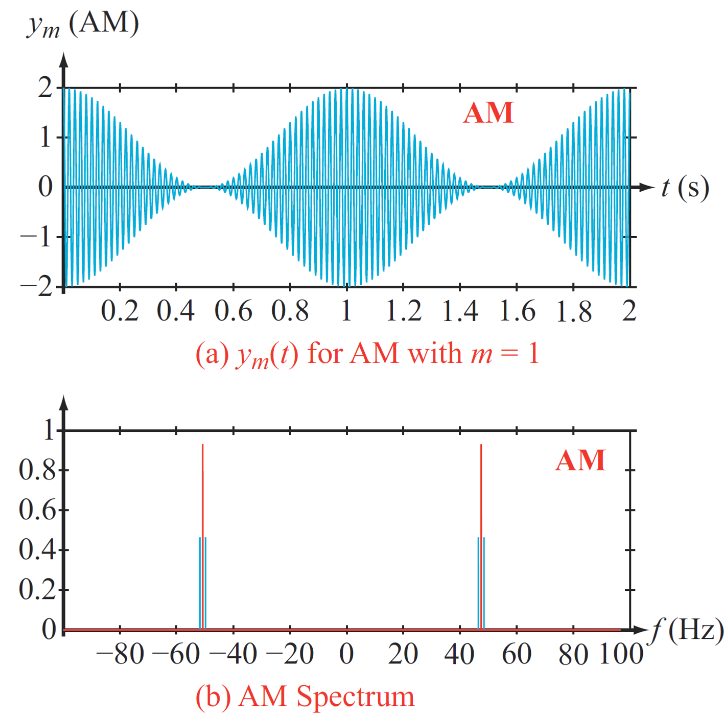


Fourier transforms - AM

<u>Envelope:</u> DSB: $|x(t)|$. AM: $|A+x(t)|=A+x(t)$ if $|x(t)|<A$.
Can recover envelope using envelope detection (next slide).



Waveform and line spectrum of DSB modulated



Waveform and line spectrum of AM sinusoid.

Fourier transforms - FM

Alter carrier frequency in a manner proportional to the signal x(t)

$$\omega(t) = \omega_c + k_f x(t)$$

Consider phase modulation:

$$y(t) = \cos \omega_c t + \theta(t)$$

Phase angle is proportional to the time integral of x(t)

$$\theta(t) = \theta_0 + k_f \int_0^t x(\tau) d\tau$$

Now suppose that the signal is a pure tone

$$x(t) = A \cos(\omega_m t)$$

Fourier transforms - FM

A is the amplitude of the tone and w_m is its frequency, then phase angle is

$$\theta(t) = \theta_0 + k_f \int_0^t A\cos(\omega_m t)dt$$

$$= \theta_0 + \frac{k_f A}{\omega_m}\sin(\omega_m t)$$

There is no need for phase bias so IC is zero and phase angle becomes

$$\theta(t) = \frac{k_f A}{\omega_m}\sin(\omega_m t)$$

Substituting back for y(t) gives

$$y(t) = \cos(\omega_c t + \frac{k_f A}{\omega_m}\sin(\omega_m t)\,.\,)$$

Modulation index is $$m = \frac{k_f A}{\omega_m}$$

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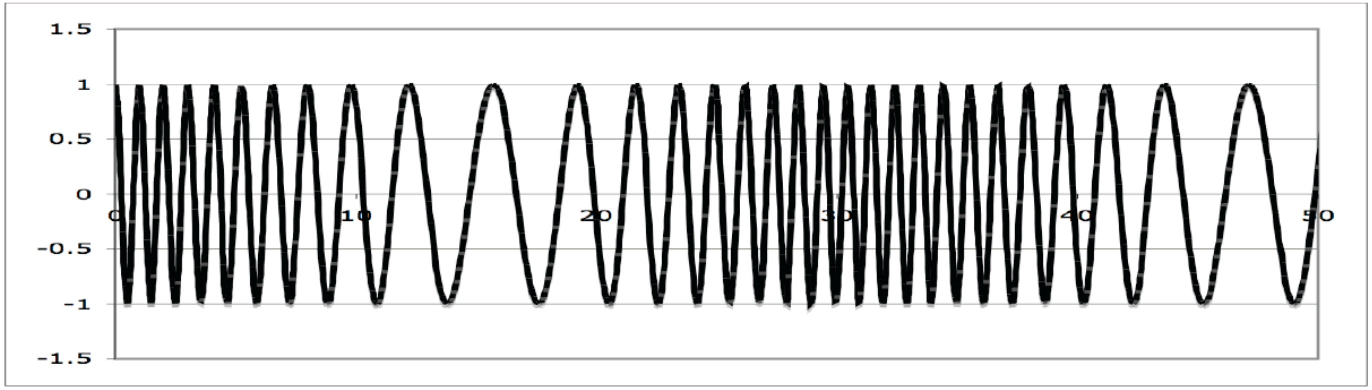
Fourier transforms - FM

Now y(t) can be expanded as

$$y(t) = \cos(\omega_c t)\cos(m\sin(\omega_m t)) - \sin(\omega_c t)\sin(m\sin(\omega_m t))$$

This characterizes frequency modulation to be interpreted as the sum of 2 amplitude modulated signals: first term is an amplitude modulation of the cosine of the carrier and the 2nd term is an amplitude modulation of the sine carrier

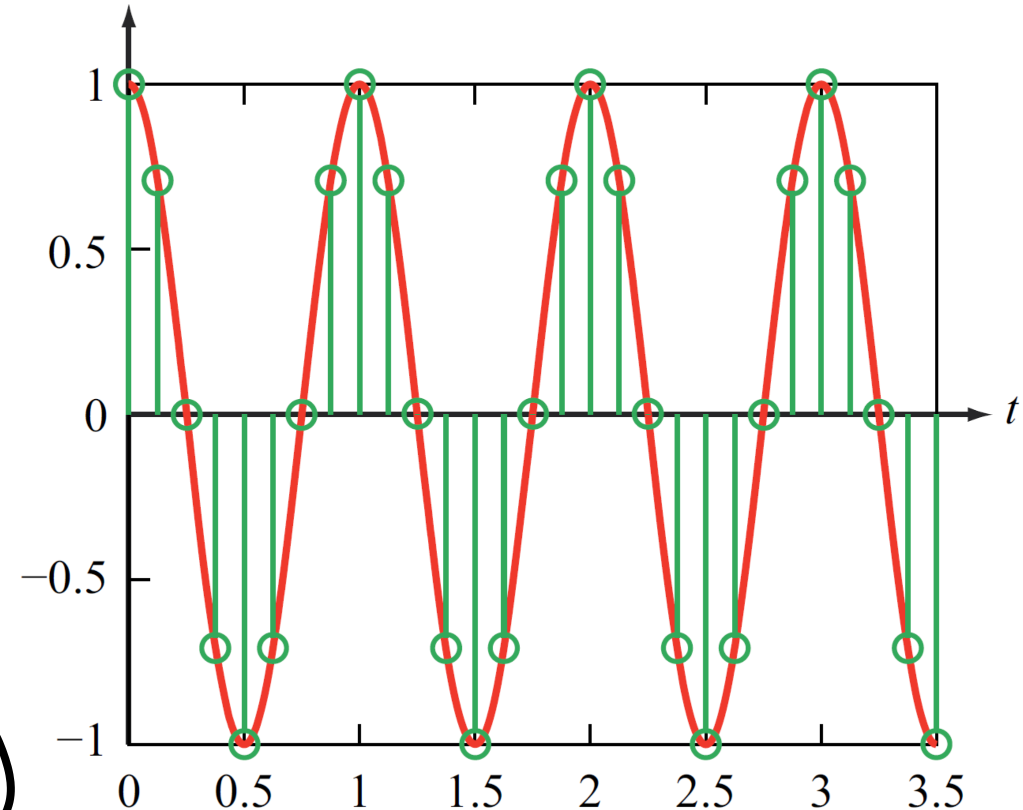
The following diagram illustrates such a frequency modulated signal



Fourier transforms - sampling (cts to discrete)

$$x[n] = x(nT_s)$$

Example: Sample a 1 kHz sinusoid at 8000 samples/s.



Solution:

$$x(t) = \cos(2\pi \cdot 1000\, t)$$

$$T_s = 1/8000 \text{ s}$$

$$x[n] = x\left(t = \frac{n}{8000}\right)$$

$$= \cos\left(2\pi \frac{1000 n}{8000}\right) = \cos\left(\frac{\pi}{4} n\right)$$

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Fourier transforms - sampling theorem

Sampling Theorem

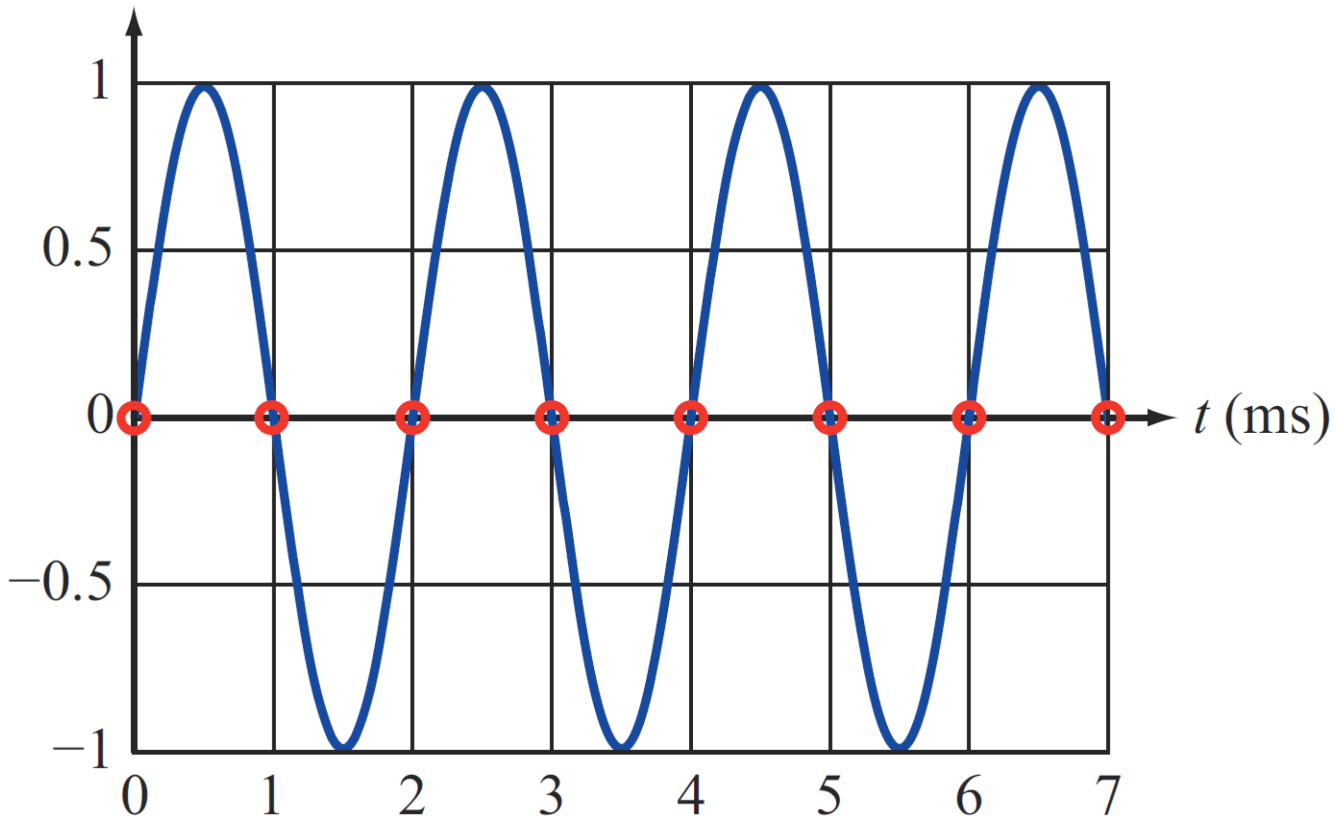
- Let $x(t)$ be a real-valued, continuous-time, lowpass signal *bandlimited* to B Hz.

- Let $x[n] = x(nT_s)$ be the sequence of numbers obtained by *sampling* $x(t)$ at a sampling rate of f_s samples per second, that is, every $T_s = 1/f_s$ seconds.

- Then $x(t)$ can be *uniquely* reconstructed from its samples $x[n]$ if and only if $f_s > 2B$. The sampling rate must exceed double the bandwidth.

Fourier transforms - sampling theorem > 2 Max frequency

Do We Need $f_s > 2B$ or $f_s \geq 2B$?



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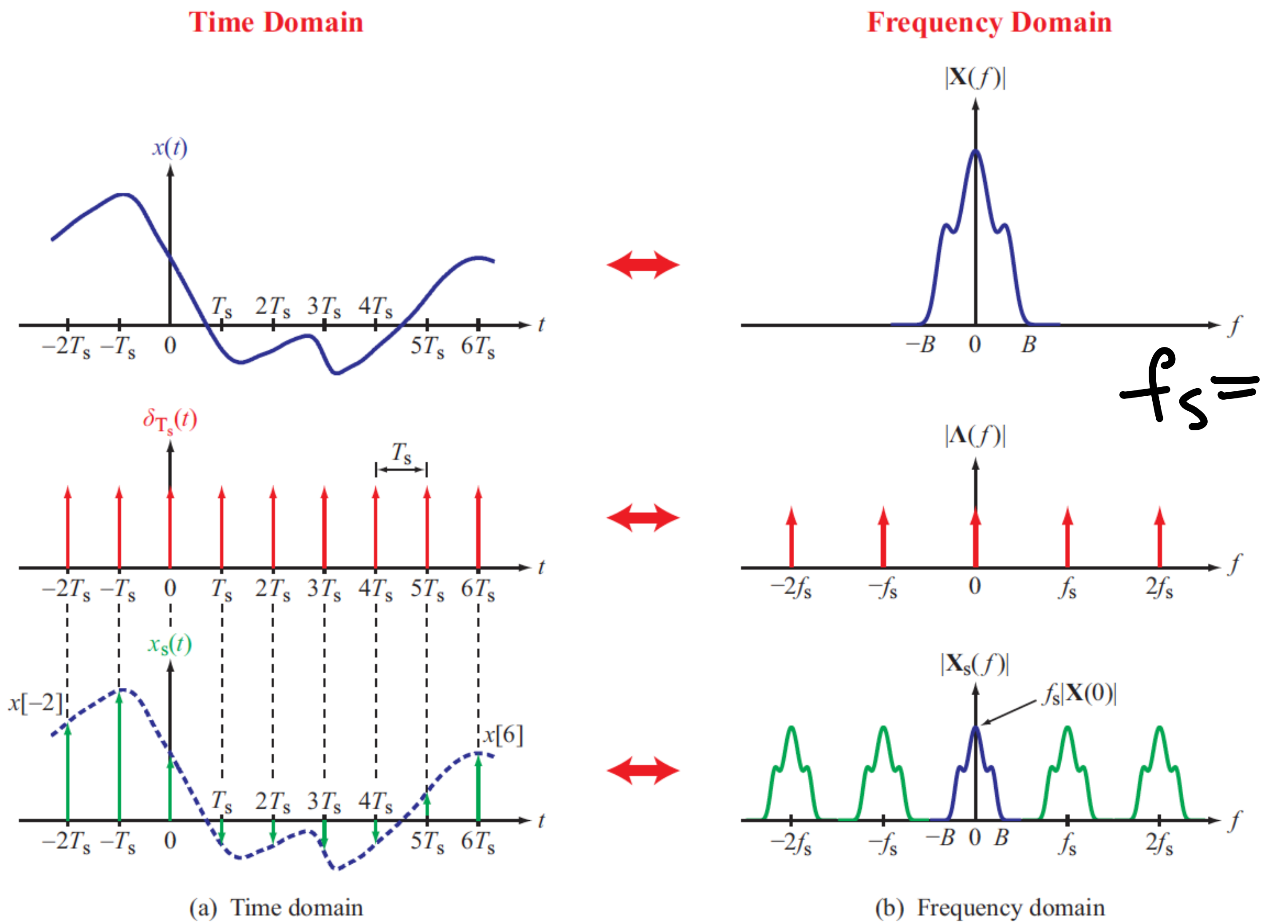
Fourier transforms - sampling theorem derivation



(a) Time domain

(b) Frequency domain

$f_s = \frac{1}{T_s}$

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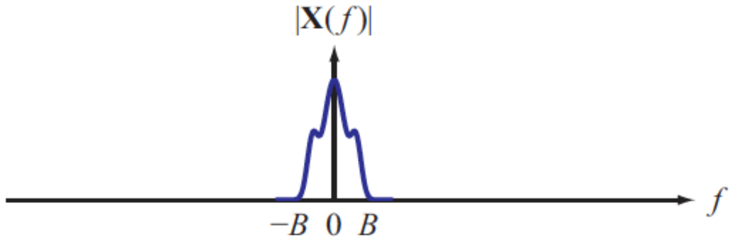
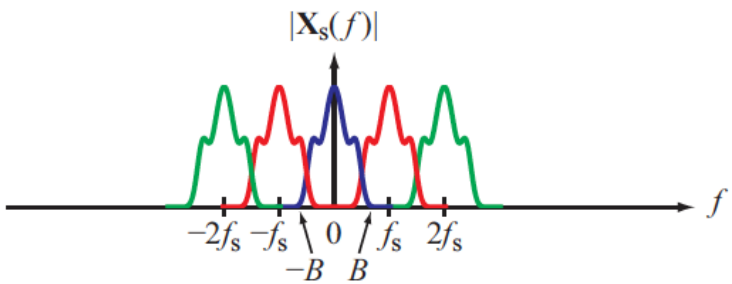
Fourier transforms - sampling over & under

Undersampling: Copies of spectra overlap, so can't recover original.

Oversampling: Can recover original signal with a filter



(a) Original spectrum



(b) Undersampled ($f_s < 2B$)

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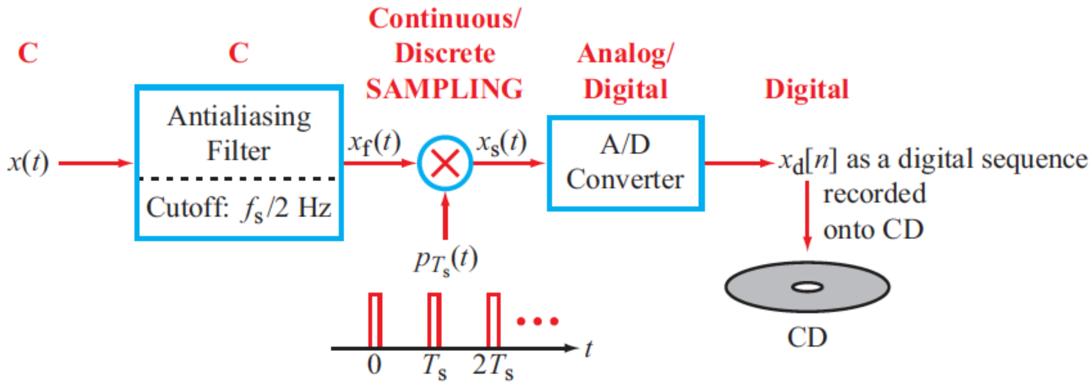
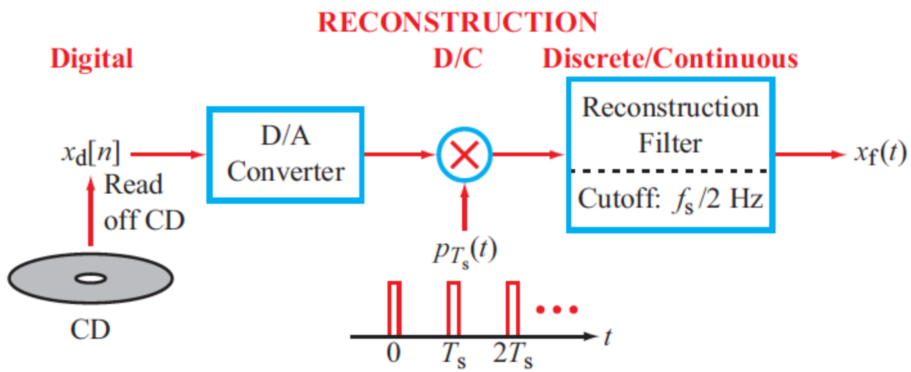
Fourier transforms - sampling & reconstruction



(a) Sampling sequence



(b) Reconstruction sequence

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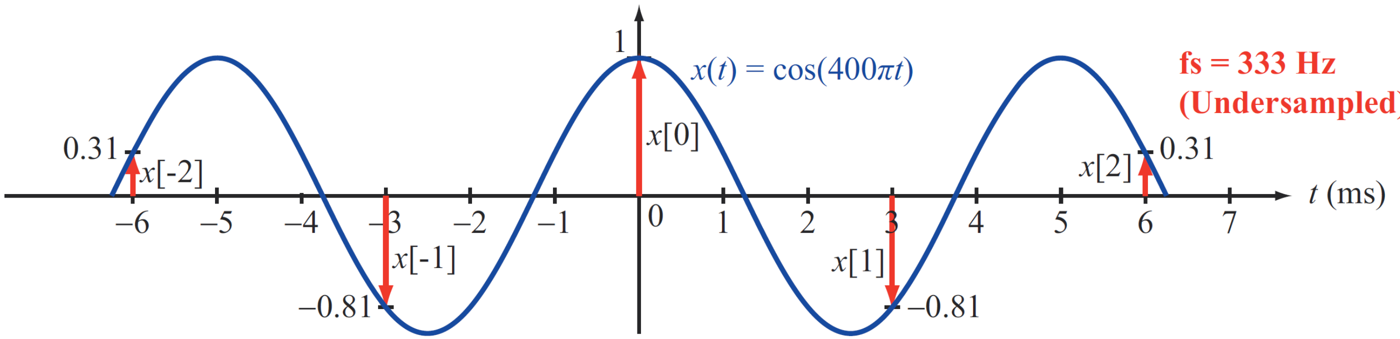
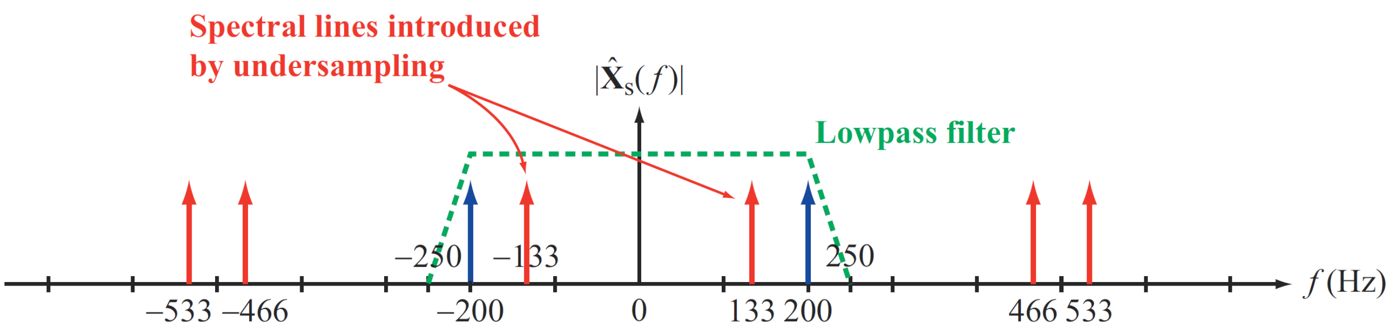
Fourier transforms - undersampling



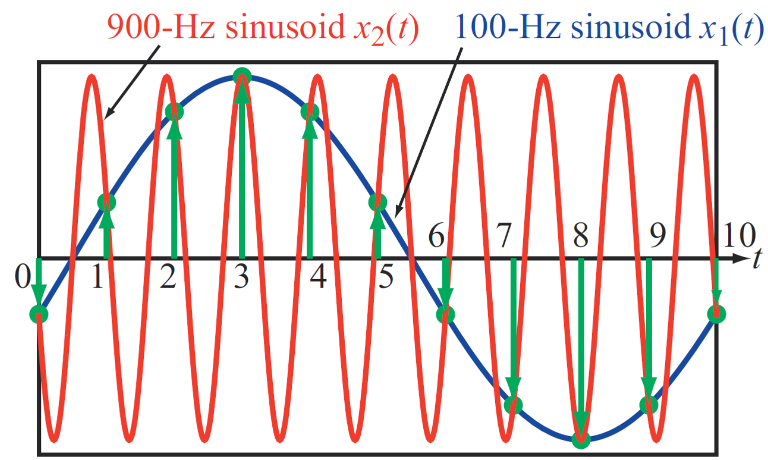
(a) $x(t)$ and $x_s(t)$ at $f_s = 333$ Hz

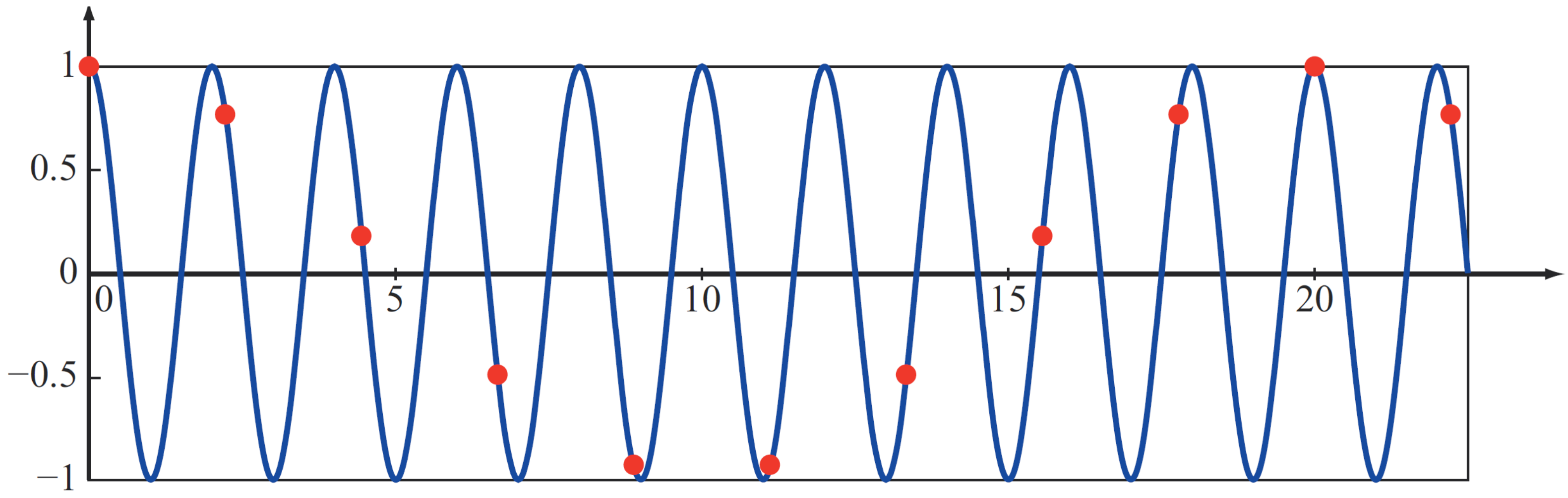


(b) Spectrum of $\hat{\mathbf{X}}_s(f)$ [blue = spectrum of $x(t)$; red = image spectra]

Fourier transforms - aliasing in time domain

The reconstruction lowpass filter
assumes the samples came from
the lower-frequency sinusoids.
Your eyes and brain do, also.





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