# MTE 203 – Advanced Calculus
# Homework 12 (Solutions)

**<u>Surface Integrals involving Vector Fields</u>**

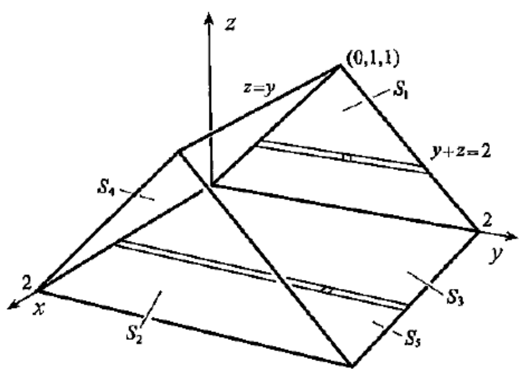
**Problem 1: [S. 14.8, Prob.9]**

Evaluate the surface integral.

$\oiint_S (yz\hat{\boldsymbol{\imath}} + xz\hat{\boldsymbol{\jmath}} + xy\hat{\boldsymbol{k}}) \cdot \hat{\boldsymbol{n}}\, dS$, where S is the surface enclosing the volume defined by $x = 0,\ x = 2,\ z = 0,\ z = y,\ y + z = 2$ *and* $\hat{\boldsymbol{n}}$ is the unit outer normal to S.

**Solution:**

We start by plotting the surface S:




On $S_1$, $\hat{\mathbf{n}} = -\hat{\mathbf{i}}$, and therefore

$$\iint_{S_1} (yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS$$

$$= \iint_{S_1} -yz\, dA = \int_0^1 \int_z^{2-z} -yz\, dy\, dz$$

$$= -\int_0^1 \left\{ \frac{y^2 z}{2} \right\}_z^{2-z} dz = -\frac{1}{2} \int_0^1 (4z - 4z^2)\, dz$$

$$= -\frac{1}{2} \left\{ 2z^2 - \frac{4z^3}{3} \right\}_0^1 = -\frac{1}{3}.$$

On $S_2$, $\hat{\mathbf{n}} = \hat{\mathbf{i}}$, and therefore

$$\iint_{S_2} (yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = \iint_{S_2} yz\, dS = \iint_{S_1} yz\, dA = \frac{1}{3}.$$

On $S_3$, $\mathbf{n} = (0, 1, 1)/\sqrt{2}$, and therefore

$$\iint_{S_3} (yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = \iint_{S_3} \left(\frac{xz + xy}{\sqrt{2}}\right) dS = \frac{1}{\sqrt{2}} \iint_{S_{xy}} x(2)\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

$$= \sqrt{2} \iint_{S_{xy}} x\sqrt{1 + (-1)^2}\, dA = 2\int_0^2 \int_1^2 x\, dy\, dx$$

$$= 2\int_0^2 \left\{xy\right\}_1^2 dx = 2\int_0^2 x\, dx = 2\left\{\frac{x^2}{2}\right\}_0^2 = 4.$$

On $S_4$, $\mathbf{n} = (0, -1, 1)/\sqrt{2}$, and therefore

$$\iint_{S_4} (yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = \iint_{S_4} \left(\frac{-xz + xy}{\sqrt{2}}\right) dS = \frac{1}{\sqrt{2}} \iint_S x(0)\, dS = 0.$$

On $S_5$, $\mathbf{n} = -\hat{\mathbf{k}}$, and therefore

$$\iint_{S_5} (yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = \iint_{S_5} -xy\, dS = \int_0^2 \int_0^2 -xy\, dy\, dx = -\int_0^2 \left\{\frac{xy^2}{2}\right\}_0^2 dx = -\frac{1}{2}\int_0^2 4x\, dx$$

$$= -2\left\{\frac{x^2}{2}\right\}_0^2 = -4.$$

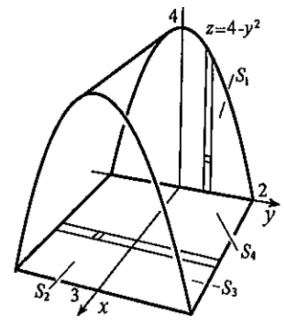
Thus, $\displaystyle\oiint_S (yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = -\frac{1}{3} + \frac{1}{3} + 4 + 0 - 4 = 0.$

**Problem 2: [S. 14.8, Prob. 13]**

Evaluate the surface integral

$\oiint_S \vec{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}}\, \boldsymbol{d\sigma}$, where $\hat{F} = (z^2 - x)\hat{\imath} - xy\hat{\jmath} + 3z\hat{k}$, S is the surface enclosing the volume defined by $z = 4 - y^2, x = 0, x = 3, z = 0$ and the vector $\hat{\boldsymbol{n}}$ is the unit outer normal to S.

**Solution:**




**13.** On $S_1$, $\hat{\mathbf{n}} = -\hat{\mathbf{i}}$, and therefore

$$\iint_{S_1} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_{S_1} (x-z^2)\,dS = \iint_{S_1} -z^2\,dA$$
$$= -2\int_0^2\int_0^{4-y^2} z^2\,dz\,dy = -2\int_0^2 \left\{\frac{z^3}{3}\right\}_0^{4-y^2} dy$$
$$= -\frac{2}{3}\int_0^2 (64-48y^2+12y^4-y^6)\,dy$$
$$= -\frac{2}{3}\left\{64y-16y^3+\frac{12y^5}{5}-\frac{y^7}{7}\right\}_0^2 = -\frac{4096}{105}.$$

On $S_2$, $\hat{\mathbf{n}} = \hat{\mathbf{i}}$, and therefore

$$\iint_{S_2} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_{S_2} (z^2-x)\,dS = \iint_{S_2} z^2\,dS - \iint_{S_2} x\,dS = -\iint_{S_1} z^2\,dS - \iint_{S_1} 3\,dS$$
$$= \frac{4096}{105} - 6\int_0^2\int_0^{4-y^2} dz\,dy = \frac{4096}{105} - 6\int_0^2 (4-y^2)\,dy = \frac{4096}{105} - 6\left\{4y-\frac{y^3}{3}\right\}_0^2 = \frac{4096}{105} - 32.$$

On $S_3$, $\hat{\mathbf{n}} = -\hat{\mathbf{k}}$, and therefore $\displaystyle\iint_{S_3} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_{S_3} -3z\,dS = 0.$

On $S_4$, $\hat{\mathbf{n}} = \dfrac{\nabla(y^2+z-4)}{|\nabla(y^2+z-4)|} = \dfrac{(0,2y,1)}{\sqrt{4y^2+1}}$, and therefore

$$\iint_{S_4} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_{S_4} \frac{(-2xy^2+3z)}{\sqrt{1+4y^2}}dS = \iint_{S_{xy}} \frac{(-2xy^2+12-3y^2)}{\sqrt{1+4y^2}}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$
$$= \int_0^3\int_{-2}^2 (12-2xy^2-3y^2)\,dy\,dx = \int_0^3 \left\{12y-\frac{2xy^3}{3}-y^3\right\}_{-2}^2 dx$$
$$= \int_0^3 \left(24-\frac{16x}{3}-8+24-\frac{16x}{3}-8\right)dx = \frac{32}{3}\int_0^3 (3-x)\,dx = \frac{32}{3}\left\{3x-\frac{x^2}{2}\right\}_0^3 = 48.$$

Thus, $\displaystyle\oiint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = -\frac{4096}{105}+\frac{4096}{105}-32+48 = 16.$

## The Divergence Theorem

**Problem 3: [S. 14.9, Prob.7]**

Use the divergence theorem to evaluate the surface integral:

$\oiint_S \left(z\hat{\boldsymbol{\imath}} - x\hat{\boldsymbol{\jmath}} + y\hat{\boldsymbol{k}}\right)\cdot\hat{\boldsymbol{n}}\,dS$ where $S$ is the surface enclosing the volume defined by the surface $z = \sqrt{4-x^2-y^2}$, $z=0$, and $\hat{\boldsymbol{n}}$ is the unit outer normal to $S$.

**Solution:**

We define function $\boldsymbol{F}$ as:

$$\boldsymbol{F} = \left(z\hat{\boldsymbol{\imath}} - x\hat{\boldsymbol{\jmath}} + y\hat{\boldsymbol{k}}\right)$$

Using divergence theorem, we have:

$$\nabla \cdot \boldsymbol{F} = \frac{dz}{dx} + \frac{d(-x)}{dy} + \frac{dy}{dz} = 0 + 0 + 0 = 0$$

Thus:

By the divergence theorem, $\oiint_S (z\hat{\mathbf{i}} - x\hat{\mathbf{j}} + y\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = \iiint_V (0+0+0)\, dV = 0.$

**Problem 4: [S. 14.9, Prob.11]**

Use the divergence theorem to evaluate the surface integral:

$\oiint_S \left(y\hat{\boldsymbol{\imath}} - xy\hat{\boldsymbol{\jmath}} + zy^2\widehat{\boldsymbol{k}}\right) \cdot \widehat{\boldsymbol{n}}\, dS$ where $S$ is the surface enclosing the volume defined by $y^2 - x^2 - z^2 = 4$, $y = 4$, and $\widehat{\boldsymbol{n}}$ is the unit inner normal to $S$.

**Solution:**

The divergence theorem requires the normal to S be towards the outside. Hence a minus sign is added in front of the volume integral.

By the divergence theorem, $\oiint_S (y\hat{\mathbf{i}} - xy\hat{\mathbf{j}} + zy^2\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = -\iiint_V (-x + y^2)\, dV$. Since $x$ is an odd function of $x$ and $V$ is symmetric about the $yz$-plane, this term contributes nothing to the integral. If we introduce polar coordinates $x = r\cos\theta$ and $z = r\sin\theta$ in the $xz$-plane, then

$$\begin{aligned}
\oiint_S (y\hat{\mathbf{i}} - xy\hat{\mathbf{j}} + zy^2\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS &= -4\int_0^{\pi/2}\int_0^{2\sqrt{3}}\int_{\sqrt{4+r^2}}^{4} y^2 r\, dy\, dr\, d\theta \\
&= -4\int_0^{\pi/2}\int_0^{2\sqrt{3}} \left\{\frac{ry^3}{3}\right\}_{\sqrt{4+r^2}}^{4} dr\, d\theta \\
&= -\frac{4}{3}\int_0^{\pi/2}\int_0^{2\sqrt{3}} [64r - r(4+r^2)^{3/2}]\, dr\, d\theta \\
&= -\frac{4}{3}\int_0^{\pi/2} \left\{32r^2 - \frac{1}{5}(4+r^2)^{5/2}\right\}_0^{2\sqrt{3}} d\theta = -\frac{3712}{15}\left\{\theta\right\}_0^{\pi/2} = -\frac{1856\pi}{15}.
\end{aligned}$$

Alternatively, if x is not canceled out:

$$-\iiint_V (-x + y^2)dV = -\int_0^{2\pi}\int_0^{\sqrt{12}}\int_{\sqrt{4+r^2}}^{4} (-r\cos\theta + y^2)r\; dydrd\theta$$

$$= -\int_0^{2\pi}\int_0^{\sqrt{12}} -r^2 cos\theta y + \frac{y^3 r}{3}\Bigg|_{\sqrt{4+r^2}}^{4} dr\, d\theta$$

$$= -\int_0^{2\pi}\int_0^{\sqrt{12}} -4r^2 cos\theta + \frac{64r}{3} + \underbrace{r^2 cos\theta\sqrt{4+r^2}}_{Schaum\ 17.9.10} - \underbrace{\frac{r(4+r^2)^{\frac{3}{2}}}{3}}_{substitution\ u=r^2} dr\, d\theta$$

$$= \cdots Matlab\ symbolic\ toolbox \ldots = -\frac{1856\pi}{15}$$

**Problem 5: [S. 14.9, Prob.13] - Challenging**

Use the divergence theorem to evaluate the surface integral.

$\oiint_S \left(x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\widehat{\boldsymbol{k}}\right)\cdot\widehat{\boldsymbol{n}}\, dS$ where $S$ is the top half of the ellipsoid $x^2 + 4y^2 + 9z^2 = 36$, and $\widehat{\boldsymbol{n}}$ is the unit outer normal to $S$.

**Hint 1:** Since the surface $S$ needs to enclose a volume, you need to introduce an additional surface ($\acute{S}$).

**Hint 2:** The volume of an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{B^2} + \frac{Z^2}{C^2} = 1$ is $V = 4\pi abc/3$ (see exercise 27 in section 13.9 for the proof).

**Solution:**

As, it is mentioned in hint 1, the surface $S$ is enclosing a volume, we need to introduce an additional surface ($\acute{S}$) to be enclosing a volume. By introducing this new surface, the problem we are solving becomes:

$$\oiint_{S+\acute{S}}\left(x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\widehat{\boldsymbol{k}}\right)\cdot\widehat{\boldsymbol{n}}\, dS = \iiint_V \nabla\cdot\left(x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\widehat{\boldsymbol{k}}\right)\, dV$$

and

$$\oiint_S \left(x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\widehat{\boldsymbol{k}}\right)\cdot\widehat{\boldsymbol{n}}\, dS = \iiint_V \nabla\cdot\left(x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\widehat{\boldsymbol{k}}\right)\, dV - \oiint_{\acute{S}} \left(x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\widehat{\boldsymbol{k}}\right)\cdot\widehat{\boldsymbol{n}}\, dS$$

If we let $S'$ be that part of the $xy$-plane bounded by $x^2 + 4y^2 = 36$, then

$$\oiint_{S+S'} (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = \iiint_V (1+1+1)\, dV = 3 \iiint_V dV.$$

Therefore, $\oiint_S (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = 3 \iiint_V dV - \iint_{S'} (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS.$

Using the given formula in hint 2:

$$\iiint_V \ dV = (2\pi/3)(6)(3)(2)$$

Since in $\oiint_{\acute{S}} \ (x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}) \cdot \hat{\boldsymbol{n}}\, dS$, we know that $\hat{\boldsymbol{n}} = -\hat{\boldsymbol{k}}$ we have:

$$\iint_S (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS = 3\left(\frac{2\pi}{3}\right)(6)(3)(2) - \iint_{S'} -z\, dS = 72\pi.$$

<u>**Stoke's Theorem**</u>

**Problem 6: [S. 14.10, Prob.1]**

Use Stoke's theorem to evaluate the line integral

$\oint_C \ x^2ydx + y^2zdy + z^2xdz$ where $C$ is the curve $z = x^2 + y^2$, and $x^2 + y^2 = 4$, directed counterclockwise as viewed from the origin.

**Solution:**

According to Stokes's theorem, $\oint_C x^2y\, dx + y^2z\, dy + z^2x\, dz = \iint_S \nabla \times (x^2y, y^2z, z^2x) \cdot \hat{\mathbf{n}}\, dS$ where S is any surface with $C$ as boundary. Now,

$$\nabla \times (x^2y, y^2z, z^2x) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & y^2z & z^2x \end{vmatrix} = (-y^2, -z^2, -x^2).$$

If we choose $S$ as that part of the plane $z = 4$ inside $C$, then $\hat{\mathbf{n}} = -\hat{\mathbf{k}}$, and

$$\oint_C x^2y\,dx + y^2z\,dy + z^2x\,dz = \iint_S (-y^2, -z^2 - x^2) \cdot (-\hat{\mathbf{k}})\,dS = \iint_S x^2\,dS$$
$$= \iint_{S_{xy}} x^2\,dA = 4\int_0^{\pi/2}\int_0^2 r^2\cos^2\theta\,r\,dr\,d\theta = 4\int_0^{\pi/2}\left\{\frac{r^4}{4}\cos^2\theta\right\}_0^2 d\theta$$
$$= 16\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)d\theta = 8\left\{\theta + \frac{1}{2}\sin 2\theta\right\}_0^{\pi/2} = 4\pi.$$

**Problem 7: [S. 14.10, Prob. 7]**

$\oint_C zy^2dx + xydy + (x^2 + z^2)dz$, where C is the curve $x^2 + z^2 = 9$, $y = (x^2 + z^2)^{\frac{1}{2}}$ directed counterclockwise as viewed from origin.

**Solution:**

**7.** According to Stokes's theorem, $\oint_C zy^2\,dx + xy\,dy + (y^2 + z^2)\,dz = \iint_S \nabla \times (zy^2, xy, y^2 + z^2) \cdot \hat{\mathbf{n}}\,dS$ where S is any surface with $C$ as boundary. Now,

$$\nabla \times (zy^2, xy, y^2 + z^2) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ zy^2 & xy & y^2 + z^2 \end{vmatrix} = (2y, y^2, y - 2yz).$$

If we choose $S$ as that part of the plane $y = 3$ inside $C$, then $\hat{\mathbf{n}} = -\hat{\mathbf{j}}$, and

$$\oint_C zy^2\,dx + xy\,dy + (y^2 + z^2)\,dz = \iint_S (2y, y^2, y - 2yz) \cdot (-\hat{\mathbf{j}})\,dS$$
$$= \iint_S -y^2\,dS = -9\iint_S dS = -9(9\pi) = -81\pi.$$

**Problem 8: [S. 14.10, Prob.13]**

Use Stoke's theorem to evaluate the line integral

$\oint_C \; z(x+y)^2 dx + (y-x)^2 dy + z^2 dz$ where $C$ is the smooth curve of intersection of the surfaces $x^2+z^2=a^2$, and $y^2+z^2=a^2$ which has a portion in the first octant, directed so that $z$ decreases in the first octant.

**Solution:**

If $S$ is that part of the plane $y=x$ inside $C$, then according to Stokes's theorem,

$$I = \oint_C z(x+y)^2\,dx + (y-x)^2\,dy + z^2\,dz = \iint_S \nabla \times \left(z(x+y)^2, (y-x)^2, z^2\right) \cdot \hat{\mathbf{n}}\,dS.$$

Now, $\nabla \times \left(z(x+y)^2, (y-x)^2, z^2\right) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z(x+y)^2 & (y-x)^2 & z^2 \end{vmatrix} = \left(0, (x+y)^2, 2(x-y-xz-yz)\right).$

Since $\hat{\mathbf{n}} = (-1,1,0)/\sqrt{2}$,

$$\begin{aligned} I &= \iint_S \left(0, (x+y)^2, 2(x-y-xz-yz)\right) \cdot \frac{(-1,1,0)}{\sqrt{2}}\,dS = \frac{1}{\sqrt{2}} \iint_S (x+y)^2\,dS \\ &= \frac{1}{\sqrt{2}} \iint_{S_{xz}} (x+x)^2 \sqrt{1+(1)^2}\,dA = 4 \iint_{S_{xz}} x^2\,dA. \end{aligned}$$

If we set up polar coordinates $x = r\cos\theta$ and $z = r\sin\theta$ in the $xz$-plane,

$$\begin{aligned} I &= 16 \int_0^{\pi/2} \int_0^a r^2 \cos^2\theta \, r\,dr\,d\theta = 16 \int_0^{\pi/2} \left\{ \frac{r^4}{4} \cos^2\theta \right\}_0^a d\theta \\ &= 4a^4 \int_0^{\pi/2} \left( \frac{1+\cos 2\theta}{2} \right) d\theta = 2a^4 \left\{ \theta + \frac{1}{2}\sin 2\theta \right\}_0^{\pi/2} = \pi a^4. \end{aligned}$$