SYDE252 - lecture notes

09/01/18

Presented by: John Zelek
Systems Design Engineering
note: some material (figures) borrowed from various sources

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6. Fourier

09/11/18

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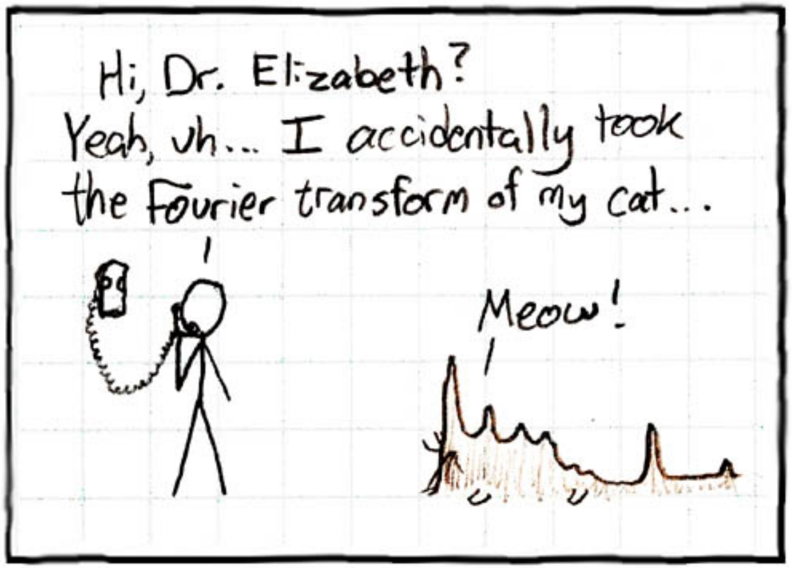
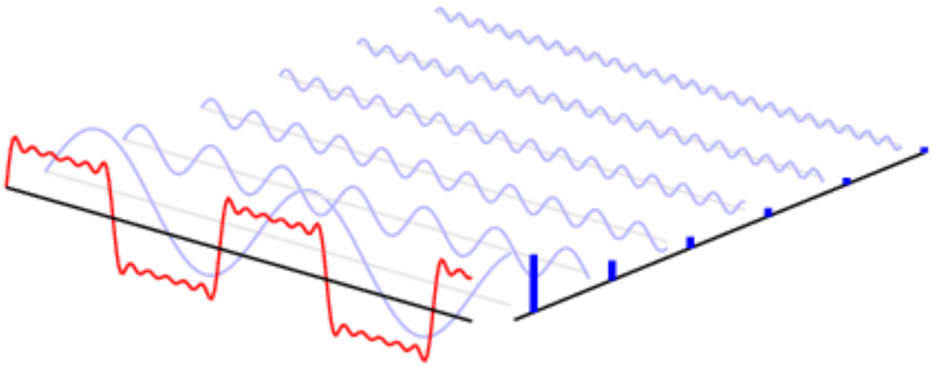
inspiration

- ““=Despots prefer the friendship of the dog, who, unjustly mistreated and debased, still loves and serves the man who wronged him. The method of doubt must be applied to civilization; we must doubt its necessity, its excellence, and its permanence”

— **Charles Fourier**





http://math.sfsu.edu/beck/quotes.html

- http://pgfplots.net/tikz/examples/fourier-transform/

Fourier Analysis: basics - Phasors

- phasor associated with the signal

$$\upsilon(t) = V_0 \cos(\omega t + \phi)$$

- is the complex number

$$\mathbf{V} = V_0 e^{j\phi}$$

- $\upsilon(t) = V_0 \cos(\omega t + \phi) \quad \longleftrightarrow \quad \mathbf{V} = V_0 e^{j\phi}$

Fourier Analysis: basics - Phasors

$x(t)$		$\mathbf{X}$
$A\cos\omega t$	$\leftrightarrow$	A
$A\cos(\omega t+\phi)$	$\leftrightarrow$	$Ae^{j\phi}$
$-A\cos(\omega t+\phi)$	$\leftrightarrow$	$Ae^{j(\phi\pm\pi)}$
$A\sin\omega t$	$\leftrightarrow$	$Ae^{-j\pi/2}=-jA$
$A\sin(\omega t+\phi)$	$\leftrightarrow$	$Ae^{j(\phi-\pi/2)}$
$-A\sin(\omega t+\phi)$	$\leftrightarrow$	$Ae^{j(\phi+\pi/2)}$
$\frac{d}{dt}[A\cos(\omega t+\phi)]$	$\leftrightarrow$	$j\omega Ae^{j\phi}$
$\int A\cos(\omega t'+\phi)\,dt'$	$\leftrightarrow$	$\frac{1}{j\omega}Ae^{j\phi}$

Fourier Analysis: basics - Phasors

- Use phasor analysis to solve the differential equation (a=300 and b=50000) $$\frac{d^2y}{dt^2} + a\,\frac{dy}{dt} + by = 10\sin(100t + 60^\circ)$$
- The phasor associated with the input is $\mathbf{X} = 10e^{-j30^\circ}$
- The differential equation becomes algebraic
- The phasor associated with the output is then found as
-

Fourier Analysis: Fourier series

A spectrum of a signal is a way of describing a signal as a sum of sinusoids.

A signal can be described in 4 different ways:

1. a mathematical definition $x(t) = A_0 + \sum_{k=1}^{N} A_k \cos\left(2\pi f_k t + \phi_k\right)$
2. a plot of $x(t)$ versus time
3. a list of amplitudes, phases and frequencies such as $(A_0, (A_1, \phi_1, f_1), \cdots (A_N, \phi_N, f_N))$
4. as a plot of the amplitudes and phases against frequencies.

Fourier Analysis: Fourier series

A Fourier series is a way of breaking of the signal into a weighted sum of a family of signals. We want to identify a family of signals such that $\{x_k(t)\}$ such that

1. every signal in the family passes through LTI system with only a scale change: $x_k(t) \rightarrow \lambda_k x_k(t)$

2. any signal can be represented as a linear combination of signals in their family: $x(t) = \sum_{k=-\infty}^{\infty} a_k x_k(t)$ and therefore $x(t) \rightarrow \sum_{k=-\infty}^{\infty} a_k x_k(t)$

Fourier Analysis: Fourier series

Definition: for an LTI system, if the output is a scaled version of the input, then the input function is called an eigenfunction of the system.

suppose: $x(t) = e^{st}$, see example: to see that this function is an eigenfunction.

lets consider $x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}$ and recall that $e^{s_i t} \rightarrow H(s_i) e^{s_i t}$, therefore

$y(t) = a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + a_3 H(s_3) e^{s_3 t}$

if the input is a linear combination of exponentials, so is the output!

Generally if $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{s_k t}$ (an infinite sum of complex exponentials), so is the output $y(t) = \sum_{k=-\infty}^{\infty} a_k H(s_k) e^{s_k t}$

Fourier Analysis: Fourier series of periodic signals

- Let x(t) be periodic:
- Then x(t) can be expressed as a linear combination of sinusoids at harmonic frequencies:

$$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)]$$

(sine/cosine representation)

$$= c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \phi_n)$$

(amplitude/phase representation)

Fourier Analysis: Fourier series of periodic signals

- Let x(t) be periodic:
- Now x(t) need not be real-valued
- Then x(t) can be expressed as a linear combination of complex exponentials at harmonic frequencies:

$$x(t) = \sum_{n=-\infty}^{\infty} \mathbf{x}_n e^{jn\omega_0 t},$$

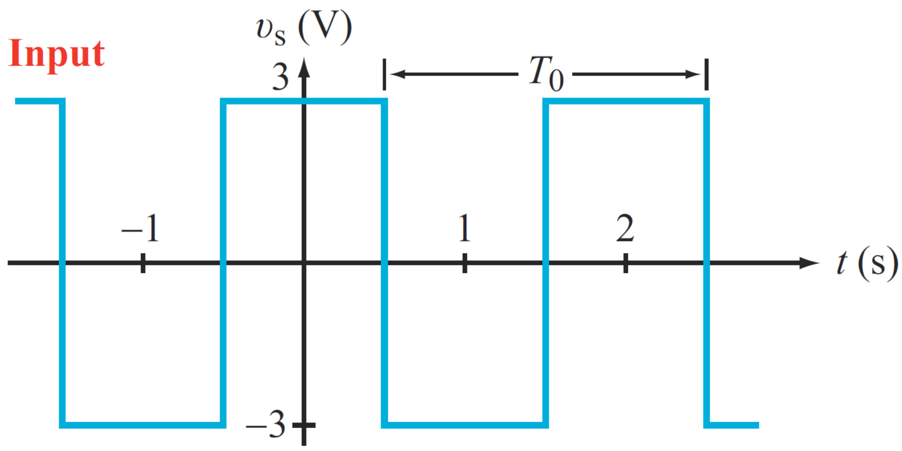
(exponential representation)

Fourier Analysis: Fourier series of periodic signals

- Fundamental angular frequency: $\omega_0 = 2\pi / T_0$
- dc or average term: a_0 or c_0
- Fundamental term: $c_1 \cos(\omega_0 t + \phi_1)$
- Harmonics: $c_n \cos(n\omega_0 t + \phi_n)$
- Fundamental has same period as x(t)
- In music: harmonics are known as overtones
- $\{a_n, b_n, c_n, \mathbf{x}_n\}$ are Fourier coefficients

Fourier Analysis: Fourier series example

- The Fourier series expansion of the periodic signal

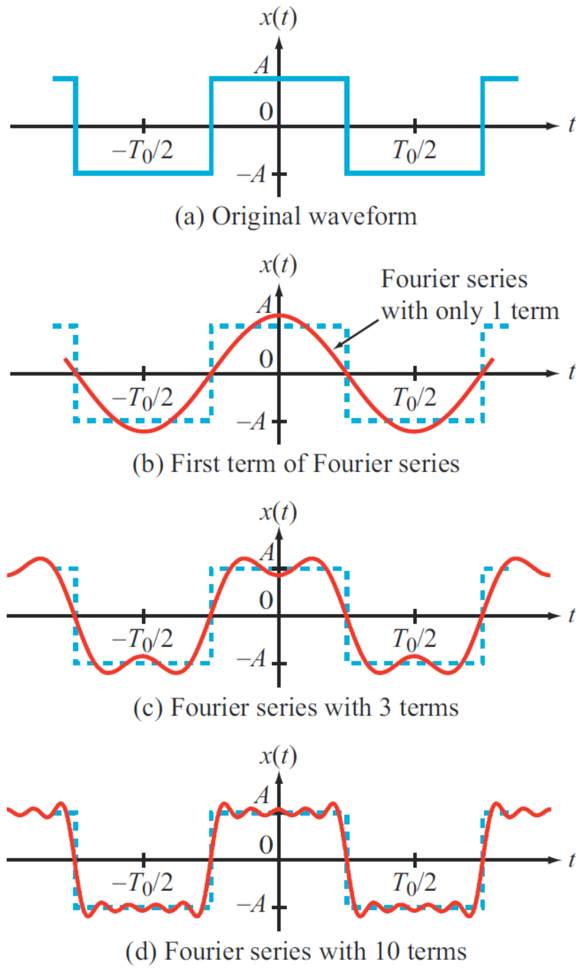


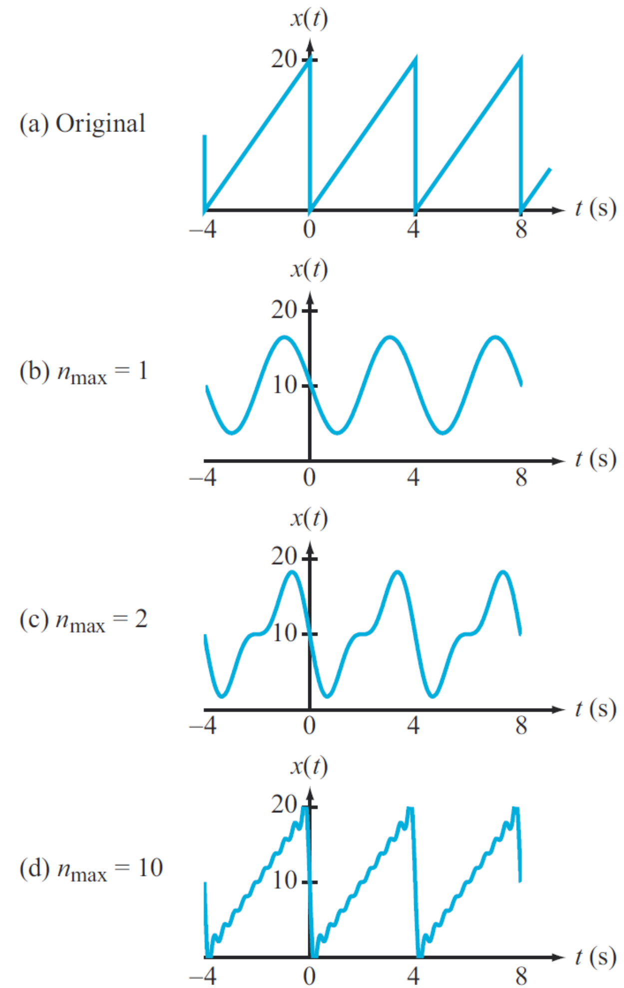
- is the infinite series

$$v_s(t) = \frac{12}{\pi}\left(\cos\omega_0 t - \frac{1}{3}\cos 3\omega_0 t + \frac{1}{5}\cos 5\omega_0 t - \cdots\right)$$

Fourier Analysis: Fourier series more examples

- Adding more terms makes the Fourier series resemble more closely the original signal:





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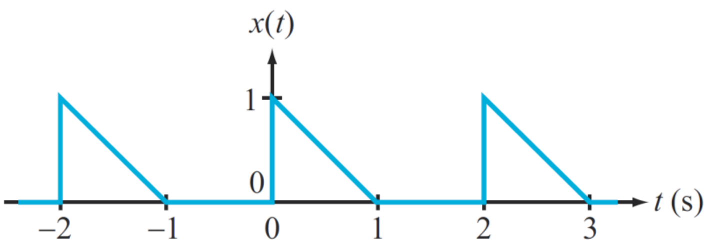
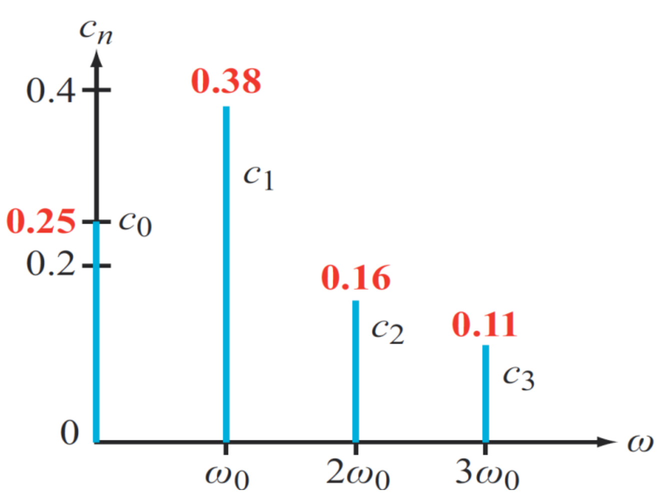
Fourier Analysis: Fourier series - one sided line spectrum

- Plot of amplitudes and phases vs. frequency .
- Example: x(t) has period=2.

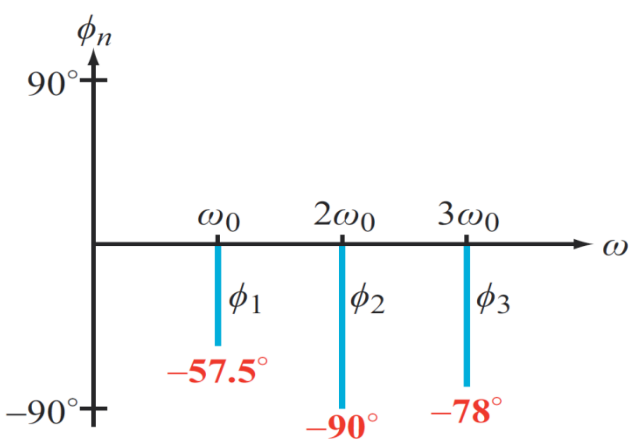
$$c_n = \begin{cases} \left(\dfrac{4}{n^4\pi^4} + \dfrac{1}{n^2\pi^2} \right)^{1/2} & \text{for } n = \text{odd}, \\ \dfrac{1}{n\pi} & \text{for } n = \text{even} \end{cases}$$



- Amplitude spectrum:



Phase spectrum:



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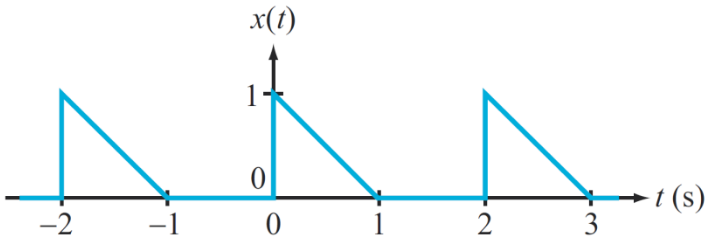
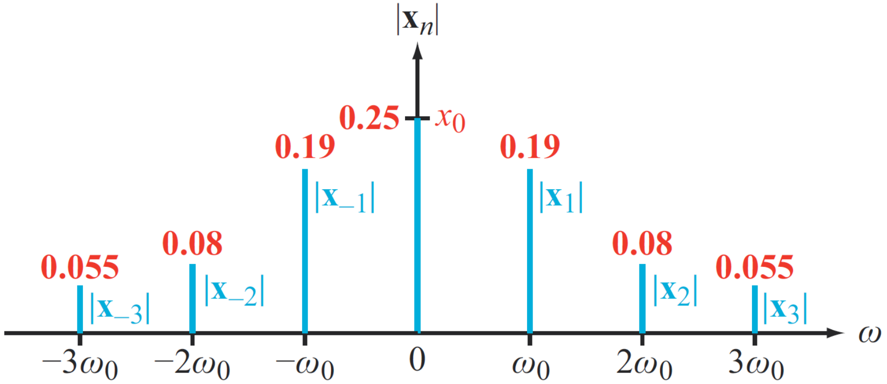
Fourier Analysis: Fourier series - two sided line spectrum

- Plot of magnitudes and phases vs. frequency .
- Example: x(t) has period=2.

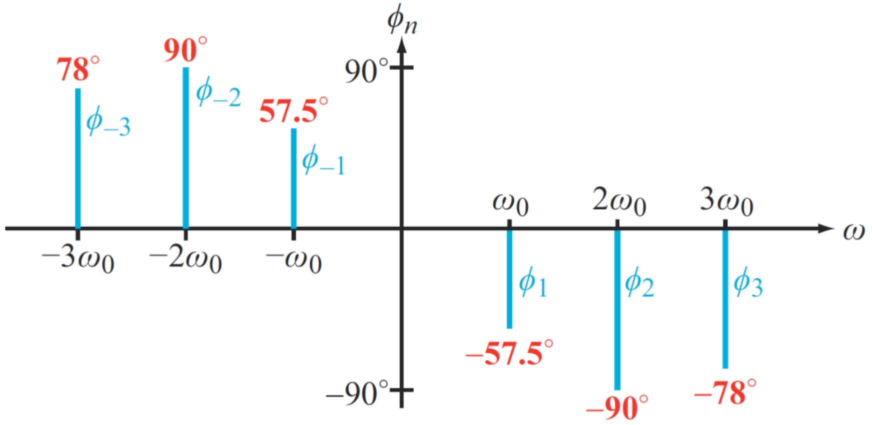
$$c_n = \begin{cases} \left(\dfrac{4}{n^4\pi^4} + \dfrac{1}{n^2\pi^2} \right)^{1/2} & \text{for } n = \text{odd}, \\ \dfrac{1}{n\pi} & \text{for } n = \text{even} \end{cases}$$



- Magnitude spectrum:

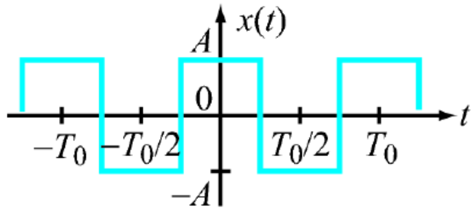
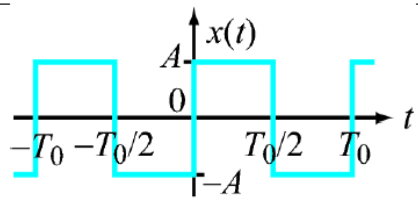
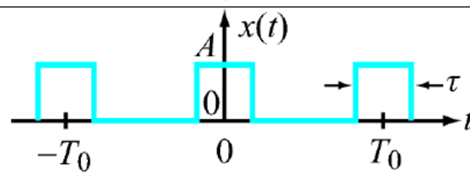
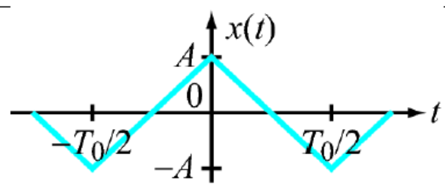


Phase spectrum:



Fourier Analysis: Fourier series examples for some periodic functions

Fourier series expressions for a select set of periodic waveforms.

<table>
<tr><th></th><th>Waveform</th><th>Fourier Series</th></tr>
<tr><td>1. Square Wave</td><td></td><td>$$x(t) = \sum_{n=1}^{\infty} \frac{4A}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos\left(\frac{2n\pi t}{T_0}\right)$$</td></tr>
<tr><td>2. Time-Shifted Square Wave</td><td></td><td>$$x(t) = \sum_{\substack{n=1 \\ n=\text{odd}}}^{\infty} \frac{4A}{n\pi} \sin\left(\frac{2n\pi t}{T_0}\right)$$</td></tr>
<tr><td>3. Pulse Train</td><td></td><td>$$x(t) = \frac{A\tau}{T_0} + \sum_{n=1}^{\infty} \frac{2A}{n\pi} \sin\left(\frac{n\pi\tau}{T_0}\right) \cos\left(\frac{2n\pi t}{T_0}\right)$$</td></tr>
<tr><td>4. Triangular Wave</td><td></td><td>$$x(t) = \sum_{\substack{n=1 \\ n=\text{odd}}}^{\infty} \frac{8A}{n^2\pi^2} \cos\left(\frac{2n\pi t}{T_0}\right)$$</td></tr>
</table>

Fourier Analysis: 3 forms of Fourier series expansion

$$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)].$$

(sine/cosine representation)

$$x(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \phi_n),$$

(amplitude/phase representation)

$$x(t) = \sum_{n=-\infty}^{\infty} \mathbf{x}_n e^{jn\omega_0 t},$$

(exponential representation)

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Fourier Series: computing coefficients for sine/cosine form

$$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)].$$

(sine/cosine representation)

Compute coefficients from a single period of x(t) using the formulae

$$a_0 = \frac{1}{T_0} \int_0^{T_0} x(t)\, dt,$$

$$a_n = \frac{2}{T_0} \int_0^{T_0} x(t)\, \cos(n\omega_0 t)\, dt,$$

$$\text{and} \quad b_n = \frac{2}{T_0} \int_0^{T_0} x(t)\, \sin(n\omega_0 t)\, dt.$$

Fourier Series: computing coefficients for amplitude/phase form

$$x(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \phi_n),$$

(amplitude/phase representation)

Compute coefficients from a single period of x(t) using the formula

$$c_n = \sqrt{a_n^2 + b_n^2}$$

and

$$\phi_n = \begin{cases} -\tan^{-1}\left(\dfrac{b_n}{a_n}\right), & a_n > 0 \\ \pi - \tan^{-1}\left(\dfrac{b_n}{a_n}\right), & a_n < 0 \end{cases}$$

Fourier Series: computing coefficients for exponential form

$$x(t) = \sum_{n=-\infty}^{\infty} \mathbf{x}_n e^{jn\omega_0 t},$$

(exponential representation).

Compute coefficients from a single period of x(t) using the formula

$$\mathbf{x}_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)\, e^{-jn\omega_0 t}\, dt.$$

Note this is much simpler than using the other two representations.

Fourier Series: 3 representations summary

Cosine/Sine	Amplitude/Phase	Complex Exponential
$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)]$	$x(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \phi_n)$	$x(t) = \sum_{n=-\infty}^{\infty} \mathbf{x}_n e^{jn\omega_0 t}$
$a_0 = \frac{1}{T_0} \int_0^{T_0} x(t)\, dt$	$c_n e^{j\phi_n} = a_n - jb_n$	$\mathbf{x}_n = \lvert\mathbf{x}_n\rvert e^{j\phi_n};\ \mathbf{x}_{-n} = \mathbf{x}_n^*;\ \phi_{-n} = -\phi_n$
$a_n = \frac{2}{T_0} \int_0^{T_0} x(t)\ \cos n\omega_0 t\ dt$	$c_n = \sqrt{a_n^2 + b_n^2}$	$\lvert\mathbf{x}_n\rvert = c_n/2;\ x_0 = c_0$
$b_n = \frac{2}{T_0} \int_0^{T_0} x(t)\ \sin n\omega_0 t\ dt$	$\phi_n = \begin{cases} -\tan^{-1}(b_n/a_n), & a_n > 0 \\ \pi - \tan^{-1}(b_n/a_n), & a_n < 0 \end{cases}$	$\mathbf{x}_n = \frac{1}{T_0} \int_0^{T_0} x(t)\ e^{-jn\omega_0 t}\ dt$
$a_0 = c_0 = x_0;\ a_n = c_n \cos\phi_n;\ b_n = -c_n \sin\phi_n;\ \mathbf{x}_n = \frac{1}{2}(a_n - jb_n)$		

Fourier Series: 3 representations summary - symmetry simplifies

Even Symmetry: $x(t) = x(-t)$

$$a_0 = \frac{2}{T_0} \int_0^{T_0/2} x(t)\, dt$$

$$a_n = \frac{4}{T_0} \int_0^{T_0/2} x(t)\, \cos(n\omega_0 t)\, dt$$

$$b_n = 0$$

$$c_n = |a_n|, \qquad \phi_n = \begin{cases} 0 & \text{if } a_n > 0 \\ 180^\circ & \text{if } a_n < 0 \end{cases}$$

Odd Symmetry: $x(t) = -x(-t)$

$$a_0 = 0 \qquad a_n = 0$$

$$b_n = \frac{4}{T_0} \int_0^{T_0/2} x(t)\, \sin(n\omega_0 t)\, dt$$

$$c_n = |b_n| \qquad \phi_n = \begin{cases} -90^\circ & \text{if } b_n > 0 \\ 90^\circ & \text{if } b_n < 0 \end{cases}$$

Fourier Series: circuit analysis

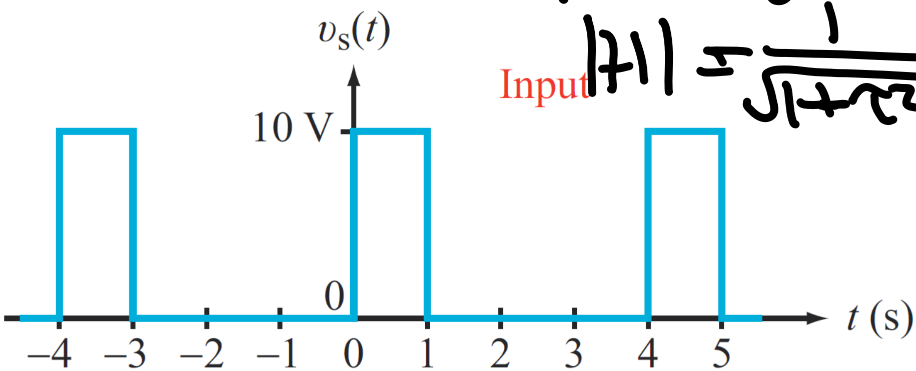
- Compute the voltage response of the RC circuit shown to the input square wave shown, using Fourier series. The time constant RC of the circuit is RC=2 s.

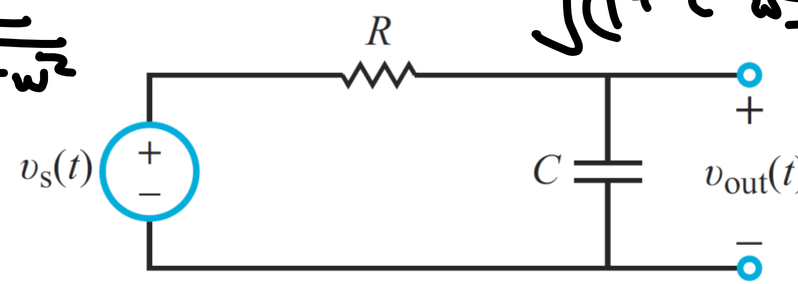
$$H(j\omega) = \frac{1-\tau j\omega}{(1+\tau j\omega)(1-\tau j\omega)} = \frac{1-\tau j\omega}{1+\tau^2\omega^2}$$

$$\angle = \tan^{-1}(-\tau\omega)$$

$$|H| = \sqrt{\frac{1+\tau^2\omega^2}{(1+\tau^2\omega^2)^2}}$$

$$|H| = \frac{1}{\sqrt{1+\tau^2\omega^2}}$$





u

$$H(s) = \frac{1}{1+\tau s} \qquad s = j\omega$$

$$H(j\omega) = \frac{1}{1+\tau j\omega}$$

Fourier Series: circuit analysis

- The input waveform has the Fourier series expansion

$$v_s(t) = a_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \phi_n)$$

$$c_n = a_n - jb_n = \frac{10}{n\pi}\left[\sin\frac{n\pi}{2} - j\left(1 - \cos\frac{n\pi}{2}\right)\right]$$

- The RC circuit has the transfer function (RC=2 s)

$$\hat{\mathbf{H}}(\omega) = \frac{1}{\sqrt{1+4\omega^2}}\, e^{-j\tan^{-1}(2\omega)}$$

$$e^{\angle H(j\omega)}$$

$$|H(j\omega)| \angle H(j\omega)$$

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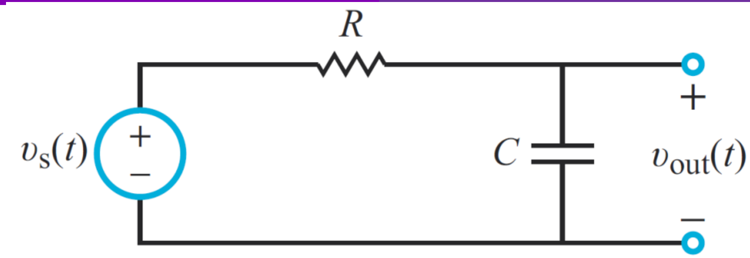
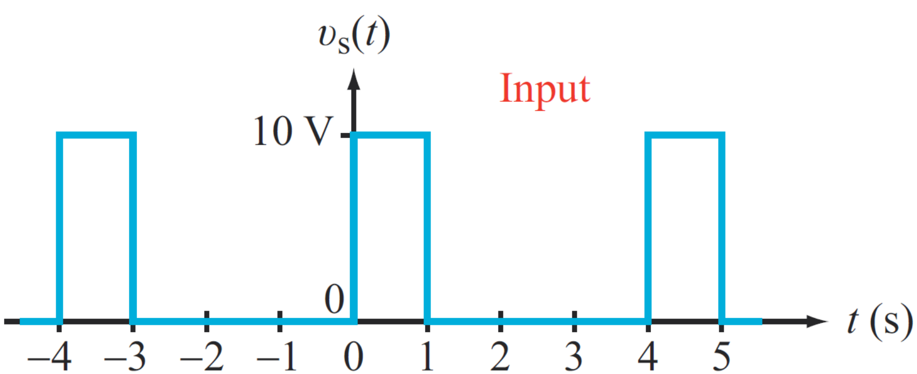
Fourier Series: circuit analysis

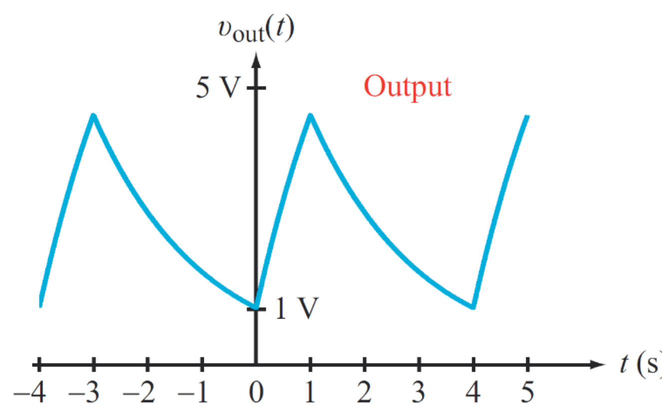


- The output waveform has the Fourier series expansion

$$v_{out}(t) = \underbrace{2.5}_{c_0} + \sum_{n=1}^{\infty} \mathrm{Re}\left[c_n \frac{1}{\sqrt{1+4n^2\omega_0^2}} e^{j(n\omega_0 t + \phi_0 - \tan^{-1} 2n\omega_0)} \right]$$

- Plot of the output waveform





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Fourier Series: basic harmonics

② e.g., $x(t) = 1 + \frac{1}{2}\cos 2\pi t + \sin 3\pi t$

$T_0 = ?$

$T = 2$

$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$

$$x(t) = 1 + \frac{1}{4}\left[e^{j2\pi t} + e^{-j2\pi t}\right] + \frac{1}{2j}\left[e^{j3\pi t} - e^{-j3\pi t}\right]$$

$a_0 = 1$, $\underbrace{a_1 = 0}_{= a_{-1}}$, $\underbrace{a_2 = \frac{1}{4}}_{= a_{-2}}$

$a_3 = \frac{1}{2j}$ $a_{-3} = -\frac{1}{2j}$

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Fourier Series: basic harmonics

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u ③ e.g., a) $x(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$ impulse train

fundamental period of $x(t)$ is T

FS coefficients $a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\,dt = \frac{1}{T}$

$\omega_0 = \frac{2\pi}{T}$

$$\boxed{\therefore x(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{jk\frac{2\pi}{T}t}}$$

Fourier Series: basic harmonics

GCD

e.g., $x(t) = 7\cos(\frac{1}{2}t + \theta_1) + 3\cos(\frac{2}{3}t + \theta_2)$
$+ 5\cos(\frac{7}{6}t + \theta_3)$

greatest common divisors

find harmonic freq^n

freq^n of $\frac{1}{2}, \frac{2}{3}, \frac{7}{6}$ is this periodic

ratios $\frac{1/2}{2/3} = \frac{3}{4}$ ✓, $\frac{1/2}{7/6} = \frac{3}{7}$ ✓ $\frac{2/3}{7/6} = \frac{4}{7}$ ✓ all are rational

$\omega_0 = \frac{1}{6}$

thus $n = 3, 4, 7$ harmonics are present in signal

Fourier Series: basic harmonics

e.g., c) $x(t) = \cos(2t + \theta_1) + 5\sin(\pi t + \theta_2)$

$$\boxed{\frac{\omega_0}{\omega_1} = \frac{2}{\pi} \neq \text{rational } \# \therefore \text{ not periodic}}$$

e.g., d) $x(t) = 3\sin(3\sqrt{2}t + \theta_1) + 7\cos(6\sqrt{2}t + \theta_2)$

$$\boxed{\begin{gathered}\frac{\omega_1}{\omega_2} = \frac{3\cancel{\sqrt{2}}}{6\cancel{\sqrt{2}}} = \frac{1}{2} \checkmark \text{ rational, } \therefore \text{ periodic} \\ 2 \text{ harmonics}, \; n = 1, 2 \text{ present}\end{gathered}}$$

$\omega_0 = 3\sqrt{2}$

Fourier Series: fundamental frequency

Fundamental Frequency

The fundamental frequency ω_0 can be determined knowing the harmonics

- a sinusoid is periodic
- a periodic signal can be written in terms of a sum of sinusoids

Note: every frequency of a periodic signal is an integer multiple of the fundamental frequency

- ratio of any two frequencies must be of the form $\frac{m}{a}$ where m and a are integers
- ratio of any 2 frequencies must be a rational number
- frequencies must be harmonically related
- fundamental frequency is the GCF (greatest common factor) ~~a~~ all sinusoid frequencies

of

Fourier Series: representation requirements

Not every signal $x(t)$ can be decomposed as a linear combination of complex exponentials. One class of signals that decomposes are periodic signals. $x(t+T) = x(t)$. The conditions that need to be satisfied are:

1. over any period, $x(t)$ must be absolutely integrable $\int_T |x(t)|dt < \infty$
2. in any finite interval of time, $x(t)$ is of bounded variation: single period has finite amount of minima and maxima
3. in any finite interval of time, there are a finite number of discontinuities

for this class of signals
$x(t) = \sum_{-\infty}^{\infty} a_k e^{jk\omega_0 t}$
ω_0 is the fundamental frequency and recall that $\omega_0 = \frac{2\pi}{T}$

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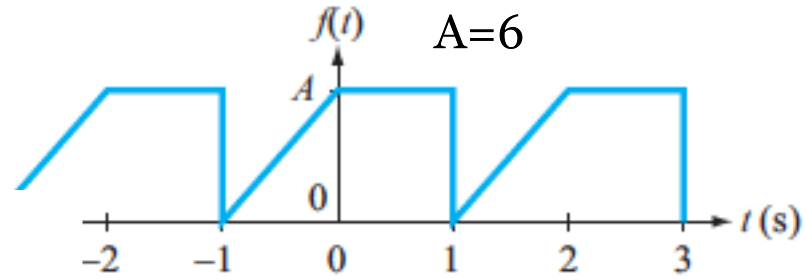
Fourier Series: example



A=6

Solution:

(a) No symmetry

(b) period $T = 2$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi \text{ rad/sec}$$

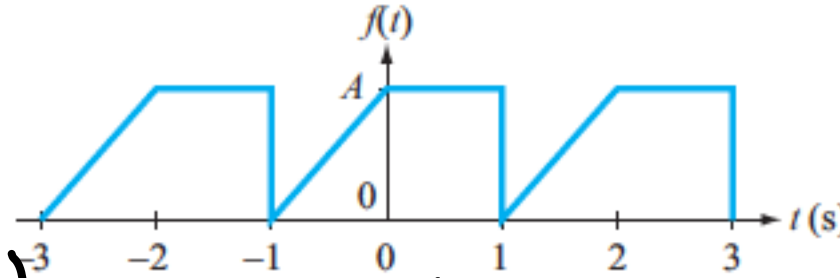
$$f(t) = \begin{cases} 6(t+1) & -1 \le t \le 0\,s \\ 6 & 0 \le t \le 1\,s \end{cases}$$

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{2}\left[\int_{-1}^{0} 6(t+1)\,dt + \int_0^1 6\,dt\right]$$

$$= \frac{1}{2}\left[(3t^2+6t)\Big|_{-1}^{0} + (6t)\Big|_0^1\right]$$

$$= \frac{9}{2}$$

Fourier Series: example



$$a_n = \frac{2}{T}\int_{-1}^{1} f(t)\cos(n\pi t)\,dt$$

$$= \int_{-1}^{0} 6(t+1)\cos(n\pi t)\,dt + \int_{0}^{1} 6\cos(n\pi t)\,dt$$

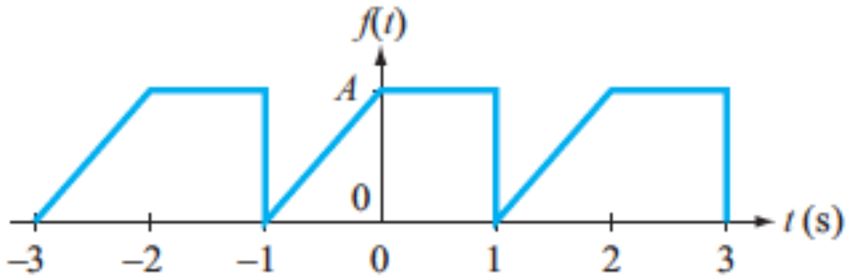
$$a_n = \frac{6}{(n\pi)^2}\left[1-\cos(n\pi)\right]$$

$$b_n = \frac{2}{T}\int_{-1}^{1} f(t)\sin(n\pi t)\,dt$$

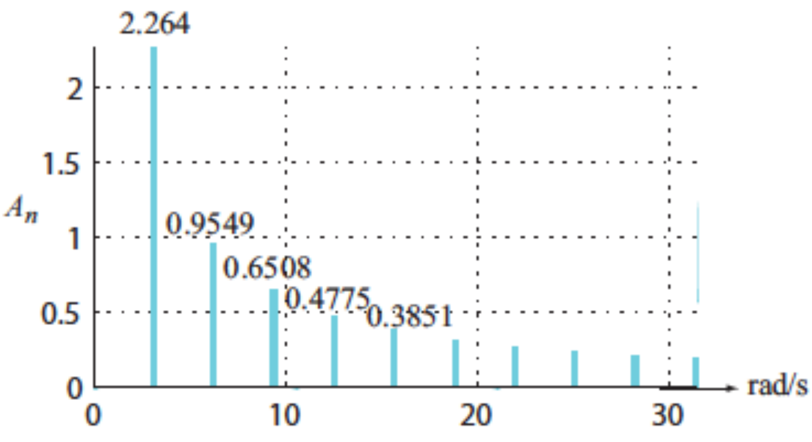
$$= -\frac{6\cos n\pi}{n\pi}$$

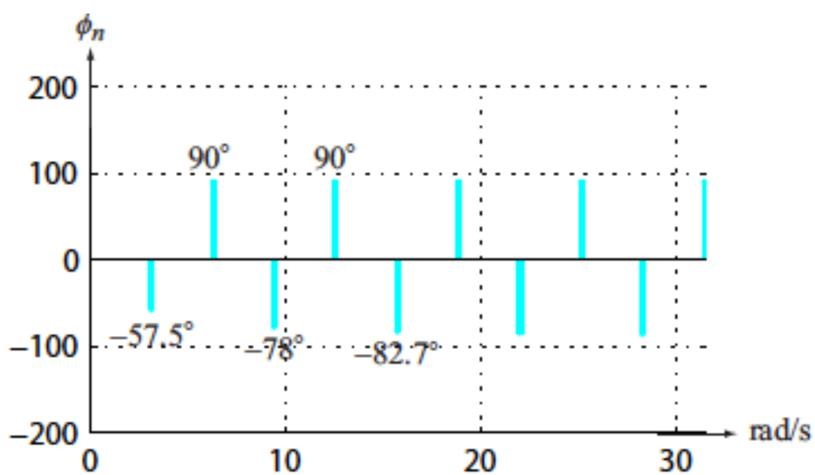
$$f(t) = \frac{a}{2} + \sum_{n=1}^{\infty}\left\{\frac{6}{(n\pi)^2}\left[1-\cos(n\pi)\right]\cos(n\pi t) - \frac{6}{n\pi}\cos(n\pi)\cdot\sin(n\pi t)\right\}$$

Fourier Series: example



A_n ϕ_n





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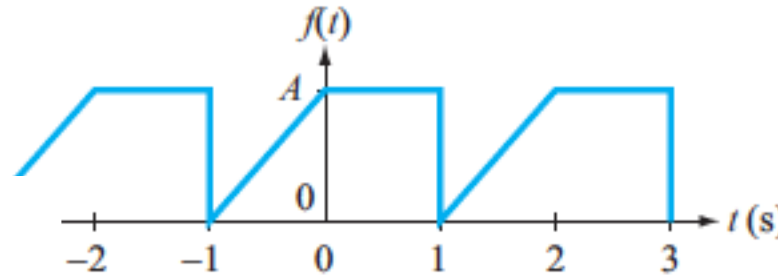
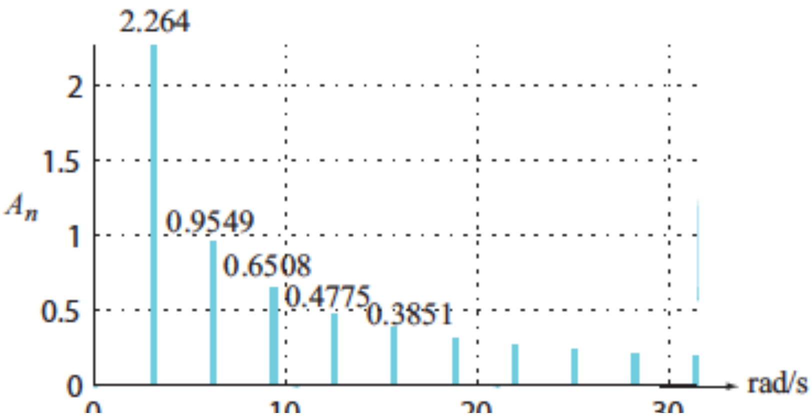
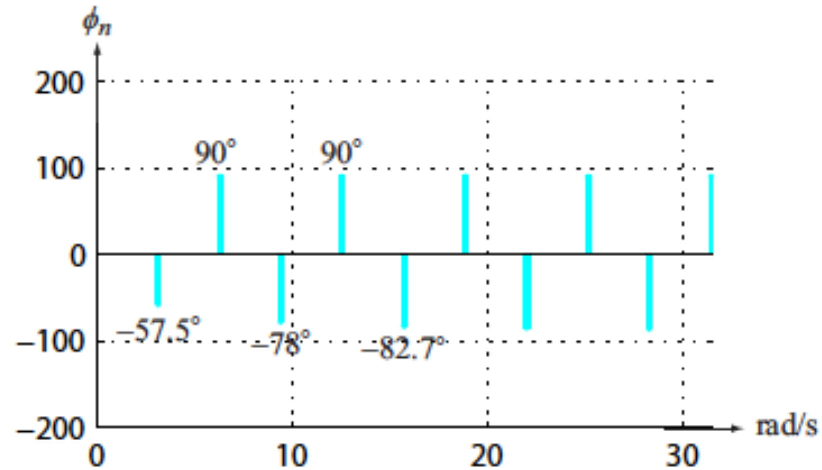
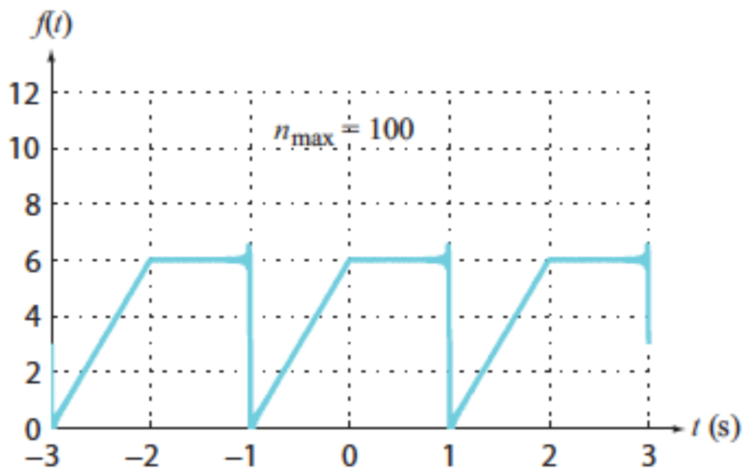
Fourier Series: example









Fourier Series: Parseval's theorem

Average Power is the same whether computed in the time domain or frequency domain

$$P_x = \frac{1}{T_0}\int_0^{T_0} |x(t)|^2\, dt = a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)/2,$$

$$P_x = \frac{1}{T_0}\int_0^{T_0} |x(t)|^2\, dt = c_0^2 + \sum_{n=1}^{\infty} c_n^2/2,$$

$$P_x = \frac{1}{T_0}\int_0^{T_0} |x(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |\mathbf{x}_n|^2.$$

Recall that average power of $c_n \cos(n\omega_0 t + \phi_n)$ is $c_n^2/2$

Fourier transform

- For non-periodic signals
- Obtain from Fourier series as period$\rightarrow\infty$
- Compute forward Fourier transform from x(t):

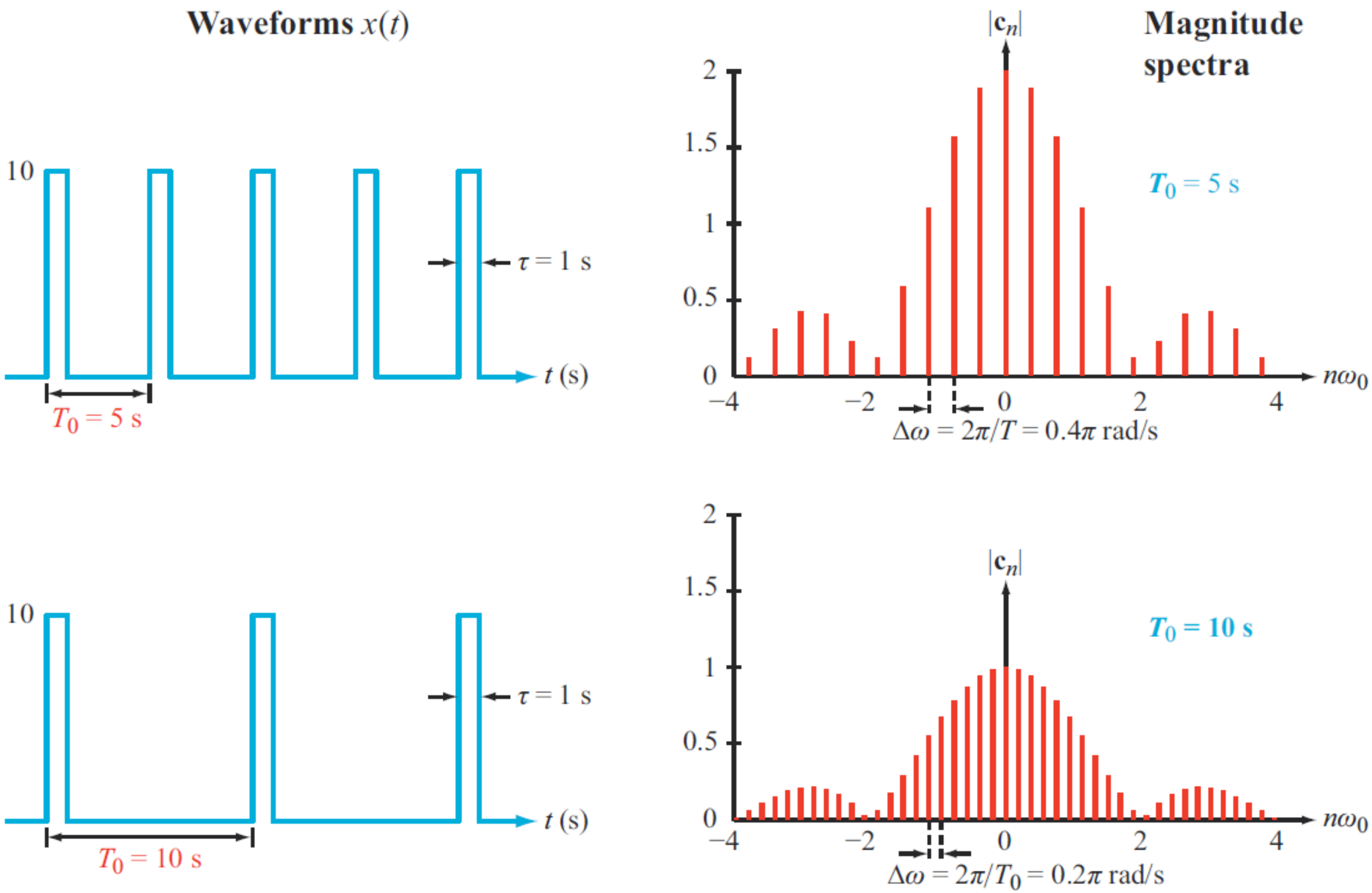
$$\mathbf{X}(\omega) = \mathfrak{F}[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

- Compute inverse Fourier transform: recover x(t):

$$x(t) = \mathfrak{F}^{-1}[\mathbf{X}(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{X}(\omega)\, e^{j\omega t}\, d\omega$$

Fourier series -> Fourier transform

- Consider a periodic pulse train. Let its period → ∞. Then its two-sided magnitude line spectrum looks like:



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Fourier series -> Fourier transform

- Exponential Fourier series:

$$x(t) = \sum_{n=-\infty}^{\infty} \mathbf{x}_n e^{jn\omega_0 t}$$

- Coefficients computed using:

$$\mathbf{x}_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t')\, e^{-jn\omega_0 t'}\, dt'$$

- Spacing between harmonics:

$$\Delta\omega = (n+1)\omega_0 - n\omega_0 = \omega_0 = \frac{2\pi}{T_0}$$

- Insert coefficient formula into exponential series:

$$x(t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-T_0/2}^{T_0/2} x(t')\, e^{-jn\omega_0 t'}\, dt' \right] e^{jn\omega_0 t}\, \Delta\omega.$$

- Let the period $\rightarrow\infty$. Bracketed quantity is $\hat{\mathbf{X}}(\omega)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(t')\, e^{-j\omega t'}\, dt' \right] e^{j\omega t}\, d\omega.$$

Fourier transform - important transfer pairs

- Impulses in time and frequency:

$$\delta(t) \longleftrightarrow 1.$$

$$1 \longleftrightarrow 2\pi\,\delta(\omega).$$

- Causal exponential signals:

$$Ae^{-at}\,u(t) \longleftrightarrow \frac{A}{a+j\omega} \quad \text{for } a>0.$$

- Eternal sinusoidal signals:

$$\cos\omega_0 t \longleftrightarrow \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)],$$

$$\sin\omega_0 t \longleftrightarrow j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)].$$

Fourier transform - pairs (1)

No.	$x(t)$	$X(\omega)$	
1	$e^{-at}u(t)$	$\dfrac{1}{a+j\omega}$	$a>0$
2	$e^{at}u(-t)$	$\dfrac{1}{a-j\omega}$	$a>0$
3	$e^{-a\lvert t\rvert}$	$\dfrac{2a}{a^2+\omega^2}$	$a>0$
4	$te^{-at}u(t)$	$\dfrac{1}{(a+j\omega)^2}$	$a>0$
5	$t^n e^{-at}u(t)$	$\dfrac{n!}{(a+j\omega)^{n+1}}$	$a>0$
6	$\delta(t)$	1	
7	1	$2\pi\delta(\omega)$	
8	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	
9	$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	
10	$\sin\omega_0 t$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$	
11	$u(t)$	$\pi\delta(\omega)+\dfrac{1}{j\omega}$	

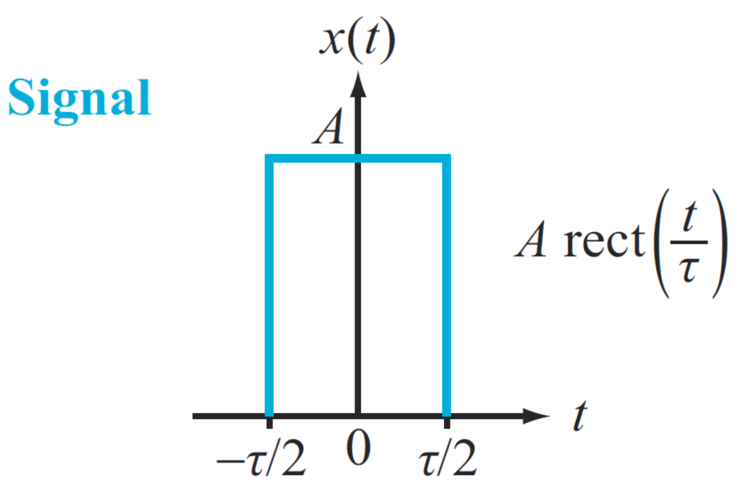
Fourier transform - pairs (1)

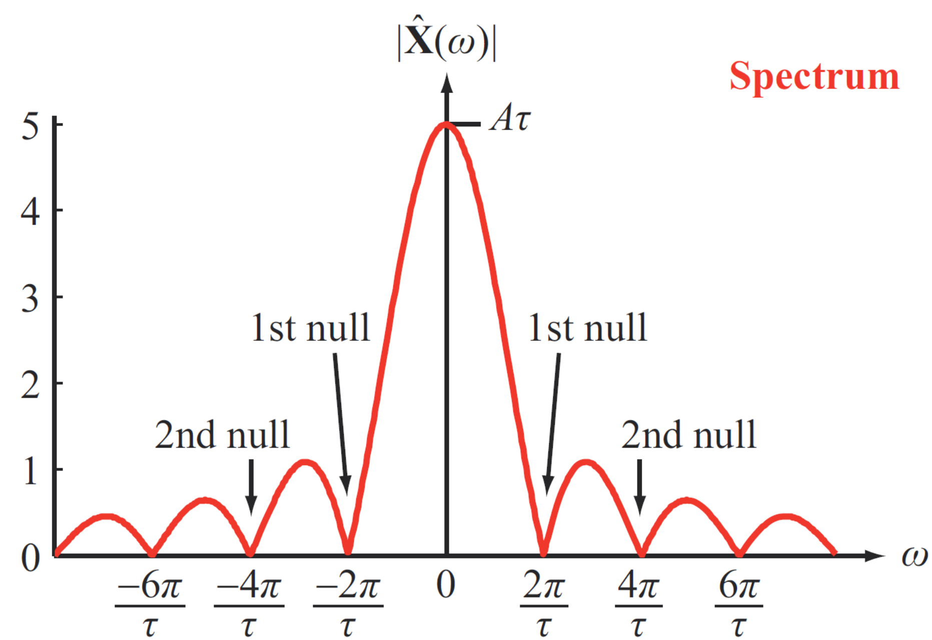
12	$\operatorname{sgn} t$	$\dfrac{2}{j\omega}$	
13	$\cos \omega_0 t\, u(t)$	$\dfrac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\dfrac{j\omega}{\omega_0^2-\omega^2}$	
14	$\sin \omega_0 t\, u(t)$	$\dfrac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]+\dfrac{\omega_0}{\omega_0^2-\omega^2}$	
15	$e^{-at}\sin \omega_0 t\, u(t)$	$\dfrac{\omega_0}{(a+j\omega)^2+\omega_0^2}$	$a>0$
16	$e^{-at}\cos \omega_0 t\, u(t)$	$\dfrac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$	$a>0$
17	$\operatorname{rect}\left(\dfrac{t}{\tau}\right)$	$\tau \operatorname{sinc}\left(\dfrac{\omega\tau}{2}\right)$	
18	$\dfrac{W}{\pi}\operatorname{sinc}(Wt)$	$\operatorname{rect}\left(\dfrac{\omega}{2W}\right)$	
19	$\Delta\left(\dfrac{t}{\tau}\right)$	$\dfrac{\tau}{2}\operatorname{sinc}^2\left(\dfrac{\omega\tau}{4}\right)$	
20	$\dfrac{W}{2\pi}\operatorname{sinc}^2\left(\dfrac{Wt}{2}\right)$	$\Delta\left(\dfrac{\omega}{2W}\right)$	
21	$\displaystyle\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\displaystyle\omega_0\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$	$\omega_0=\dfrac{2\pi}{T}$
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}\,e^{-\sigma^2\omega^2/2}$	

Fourier transform of a pulse

$x(t) = A\,\mathrm{rect}(t/\tau)$ leads to $\hat{\mathbf{X}}(\omega) = A\tau \dfrac{\sin(\omega\tau/2)}{(\omega\tau/2)} = A\tau\,\mathrm{sinc}\left(\dfrac{\omega\tau}{2}\right)$

The sinc function is defined as $\boxed{\mathrm{sinc}(\theta) = \dfrac{\sin\theta}{\theta}}$ $\quad \mathrm{sinc}(0) = \left.\dfrac{\sin(\theta)}{\theta}\right|_{\theta=0} = 1$





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Fourier transform - important properties

$$x'(t) \longleftrightarrow j\omega\, \mathbf{X}(\omega).$$

(derivative property)

$$K_1\, x_1(t) + K_2\, x_2(t) \longleftrightarrow K_1\, \mathbf{X}_1(\omega) + K_2\, \mathbf{X}_2(\omega),$$

(linearity property) (5.90)

$$\mathbf{X}(-\omega) = \mathbf{X}^*(\omega).$$

(reversal property)

$$x(at) \longleftrightarrow \frac{1}{|a|}\mathbf{X}\left(\frac{\omega}{a}\right), \qquad \text{for any } a.$$

(scaling property)

$$x(t)\cos(\omega_0 t) \longleftrightarrow \frac{1}{2}[\mathbf{X}(\omega-\omega_0) + \mathbf{X}(\omega+\omega_0)].$$

(modulation property) (5.109)

Fourier transform - properties

Operation	$x(t)$	$X(\omega)$
Scalar multiplication	$kx(t)$	$kX(\omega)$
Addition	$x_1(t) + x_2(t)$	$X_1(\omega) + X_2(\omega)$
Conjugation	$x^*(t)$	$X^*(-\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Scaling (a real)	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{\omega}{a}\right)$
Time shifting	$x(t - t_0)$	$X(\omega)e^{-j\omega t_0}$
Frequency shifting (ω_0 real)	$x(t)e^{j\omega_0 t}$	$X(\omega - \omega_0)$
Time convolution	$x_1(t) * x_2(t)$	$X_1(\omega)X_2(\omega)$
Frequency convolution	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1(\omega) * X_2(\omega)$
Time differentiation	$\frac{d^n x}{dt^n}$	$(j\omega)^n X(\omega)$
Time integration	$\int_{-\infty}^{t} x(u)\,du$	$\frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$

Fourier transform - DUAL properties

$$e^{j\omega_0 t}\, x(t) \longleftrightarrow \mathbf{X}(\omega - \omega_0),$$

(frequency-shift property)

$$x(t - t_0) \longleftrightarrow e^{-j\omega t_0}\, \mathbf{X}(\omega).$$

(time-shift property)

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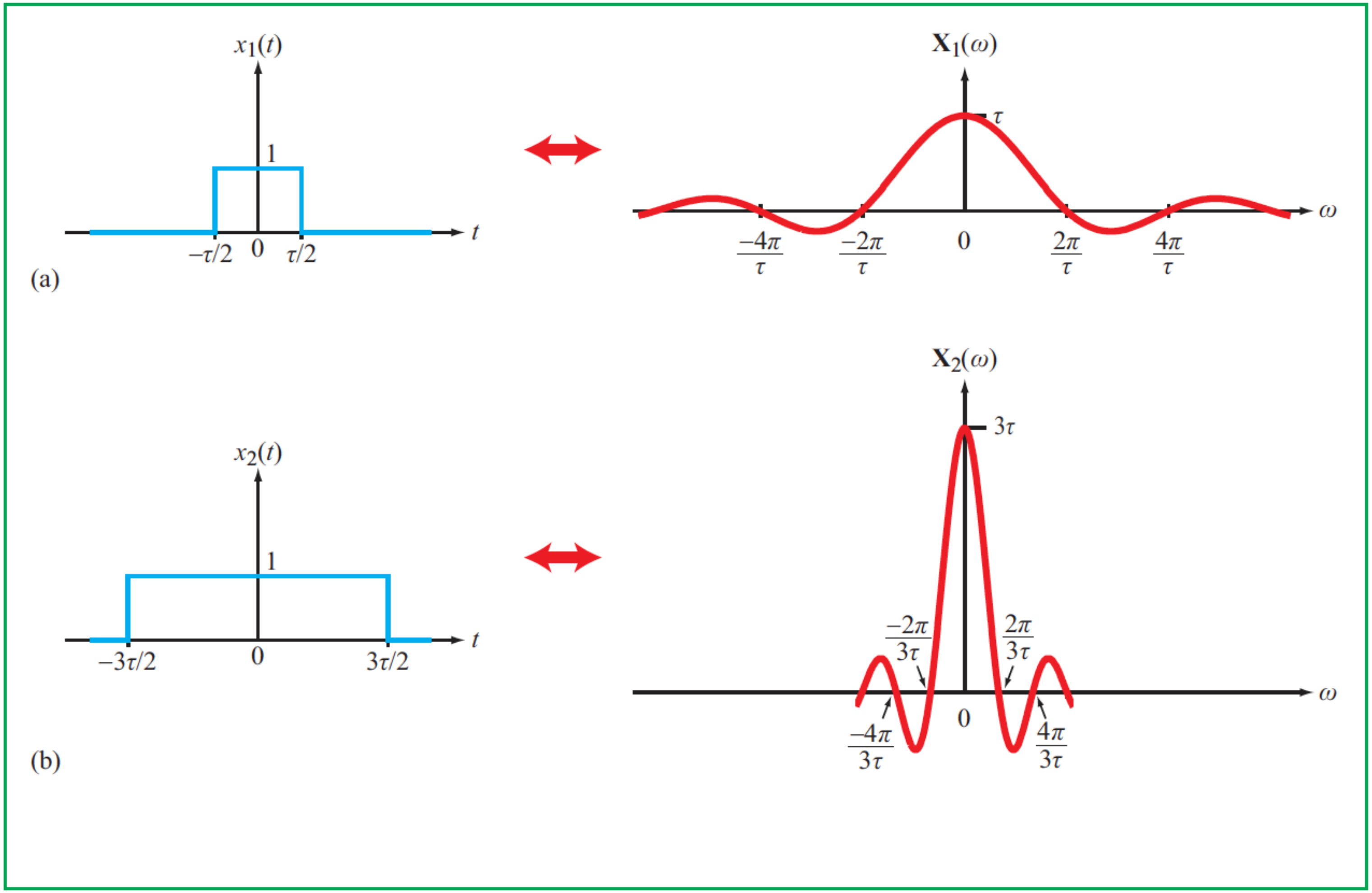
Fourier transform - Parseval's theorem

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\mathbf{X}(\omega)|^2 \, d\omega.$$

(Parseval's theorem)

same in time domain or frequency domain!

Fourier transform - time scaling property



Fourier transform -Parseval example

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{\mathbf{X}}(\omega)|^2 \, d\omega$$

$$Ae^{-at} \, u(t) \longleftrightarrow \frac{A}{a + j\omega} \quad \text{for } a > 0$$

$$x(t) = e^{-at} \, u(t)$$

Energy in time domain:

$$\int_{0}^{\infty} |e^{-at}|^2 \, dt = \int_{0}^{\infty} e^{-2at} \, dt = \frac{1}{2a} \, .$$

Energy in frequency domain:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{1}{a + j\omega} \right|^2 d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{a^2 + \omega^2} \, d\omega = \frac{1}{2a}$$

Fourier transform - periodic signal

$$x(t) = \sum_{n=-\infty}^{\infty} \mathbf{x}_n e^{jn\omega_0 t}$$

$$\mathcal{F}\left\{e^{j\omega_0 t}\right\} = 2\pi\ \delta(\omega - \omega_0)$$

$$\mathcal{F}\{x(t)\} = \sum_{n=-\infty}^{\infty} \mathbf{x}_n 2\pi\ \delta(\omega - n\omega_0)$$

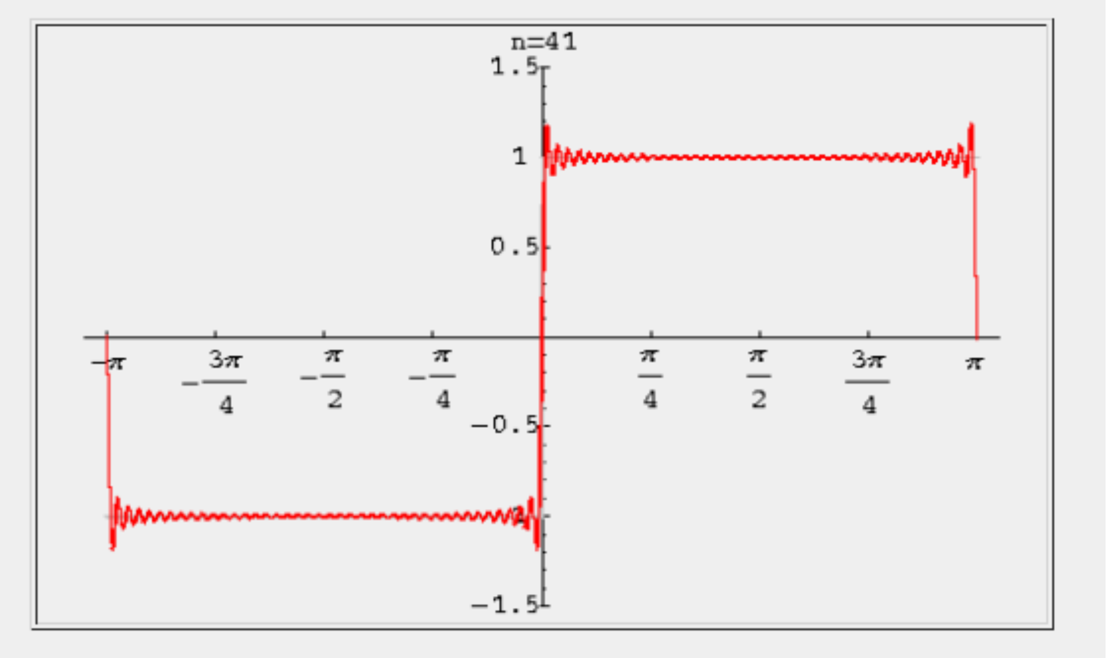
Fourier transform - Gibb's phenomenon

http://www.sosmath.com/fourier/fourier3/gibbs.html

$$f(x) = \begin{cases} 1 & 0 \leq x \leq \pi \\ -1 & -\pi \leq x < 0. \end{cases}$$

$$b_n = \frac{2}{\pi}\frac{(1-(-1)^n)}{n}, \quad \text{for } n \geq 1.$$

$$f_{2n-1}(x) = \frac{4}{\pi}\left(\sin(x) + \frac{\sin(3x)}{3} + \ldots + \frac{\sin((2n-1)x)}{(2n-1)}\right)$$



this is because of the discontinuity in x(t)

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