Part 3. Linear Algebraic Equations
Ch 9. Gaussian Elimination

Lecture 10

Pitfalls of Elimination; Pivoting and Scaling

9.3, 9.4

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Gaussian Elimination

- Direct method for solving systems of linear equations

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

Forward Elimination

$$\begin{bmatrix} a_{11} & a_{,2} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b'_2 \\ b''_3 \end{Bmatrix}$$

Back Substitution

$$x_1 \leftarrow x_2 \leftarrow \boxed{x_3 = b_3'' / a_{33}''}$$

Gaussian Elimination

- Exact solution in one pass
- Straight forward algorithm

- Not very efficient –lots of computations
- Divided by zero terms on diagonal
- Round-off errors

Operation Counting

Execution time, efficiency of method

Floating Point Operations (FLOPs): number of mathematical operations performed

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

Row # 2 = Row # 2 – (a_{21} / a_{11}) Row #1

Total Number of Flops Per System

Forward Elimination: $2n^3/3 + O(n^2)$ ← of order n^2, neglect for large n

Backward Substitution: $n^2 + O(n)$

n	Elimination	Back substitution	Total flops	$\frac{2n^3}{3}$	% due to Elimination
10	705	100	805	667	87.58
100	671550	10000	681550	666667	98.53
1000	6.67×10^8	$1\text{x}10^6$	6.67×10^8	6.67×10^8	99.85

Number of Flops for Gauss Elimination.

- Large systems greatly increase computational effort
- Most effort is in elimination step

Pivoting

Rearrange the order of equations to avoid division by zero

Example 1. Pivot element

$a_{11} = 0$

$$\begin{bmatrix} 0 & 1 & 2 \\ 0.01 & 3 & 3 \\ -10 & 7 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \\ 5 \end{Bmatrix}$$

Row # 2 = Row # 2 – (a_{21} / a_{11}) Row # 1

Partial pivoting interchanges row # 1 with row that has largest coefficient value, ∴ Row # 3

$$\begin{bmatrix} -10 & 7 & 1 \\ 0.01 & 3 & 3 \\ 0 & 1 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 5 \\ 2 \\ 1 \end{Bmatrix}$$

Remember to pivot 5

Same equation set, same solutions

Partial Pivoting

- Useful for small (non-zero) pivot element values; can lead to round-off errors (operations with small, large values)

Scaling

- Change equations; so maximum coefficient value is 1 (normalizing)

Example 2. Use Gaussian elimination with 3 significant digits

$$\begin{cases} 2x_1 + 100{,}000x_2 = 100{,}000 \\ x_1 + x_2 = 2 \end{cases}$$

Row # 2 = Row # 2 – (a_{21} / a_{11}) Row # 1

$a_{22} = 1 - (½)(100{,}000) = -50{,}000$

$b_2 = 2 - (½)(100{,}000) = -50{,}000$

$$\left.\begin{aligned} 2x_1 + 100{,}000x_2 &= 100{,}000 \\ -50{,}000x_2 &= -50{,}000 \end{aligned}\right\} \Rightarrow \begin{aligned} x_2 &= 1 \\ x_1 &= 0 \end{aligned}$$

incorrect

Example 3. Solve using scaling and pivoting with 3 significant digits

$$\begin{cases} 2x_1 + 100{,}000x_2 = 100{,}000 \\ x_1 + \quad x_2 = 2 \end{cases}$$

1) Scale: Divide Eq. 1 by 100,000 → $\begin{cases} 0.00002x_1 + \ x_2 = 1 \\ x_1 + \ x_2 = 2 \end{cases}$

2) Pivot:

$$x_1 + \ x_2 = 2$$
$$0.00002x_1 + \ x_2 = 1$$

3) Forward Elimination:

~ 0

Row # 2 = Row # 2 – (0.00002 / 1) Row # 1

$$\boldsymbol{x_2 = 1}$$

$$\boldsymbol{x_1 = 1}$$

- Include scaling and pivoting

- Use as many digits as possible to reduce round-off errors

Notes