**Problem Set #7 Solutions**

**25.1 (a)** The analytical solution can be derived by separation of variables

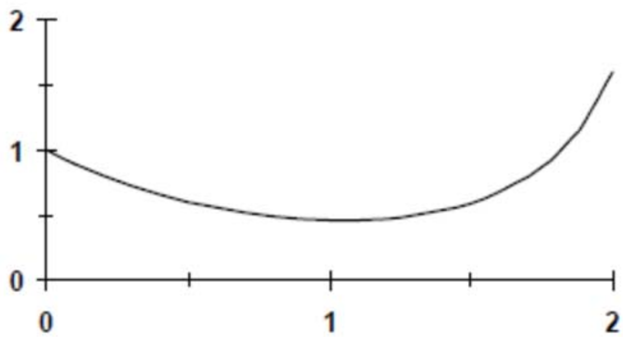
$$\int \frac{dy}{y} = \int t^2 - 1.1\, dt$$

$$\ln y = \frac{t^3}{3} - 1.1t + C$$

Substituting the initial conditions yields $C = 0$. Taking the exponential gives the final result

$$y = e^{\frac{t^3}{3} - 1.1t}$$

The result can be plotted as



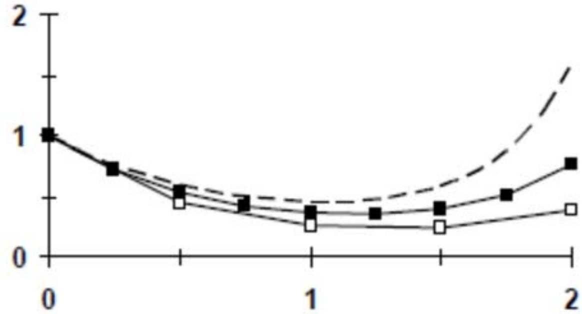

**(b)** Euler's method with $h = 0.5$

| $t$ | $y$ | $dy/dt$ |
|---|---|---|
| 0 | 1 | -1.1 |
| 0.5 | 0.45 | -0.3825 |
| 1 | 0.25875 | -0.02588 |
| 1.5 | 0.245813 | 0.282684 |
| 2 | 0.387155 | 1.122749 |

Euler's method with $h = 0.25$ gives

| $t$ | $y$ | $dy/dt$ |
|---|---|---|
| 0 | 1 | -1.1 |
| 0.25 | 0.725 | -0.75219 |
| 0.5 | 0.536953 | -0.45641 |
| 0.75 | 0.422851 | -0.22728 |
| 1 | 0.36603 | -0.0366 |
| 1.25 | 0.356879 | 0.165057 |
| 1.5 | 0.398143 | 0.457865 |
| 1.75 | 0.51261 | 1.005997 |
| 2 | 0.764109 | 2.215916 |

The results can be plotted along with the analytical solution as



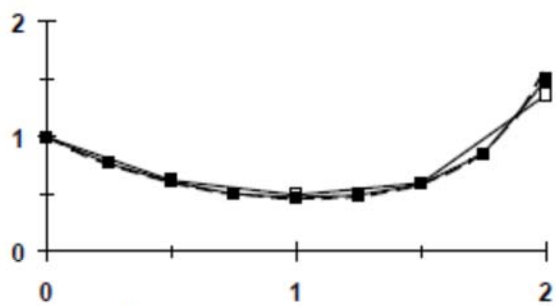

**(c)** The midpoint method with $h = 0.5$

| $t$ | $y$ | $dy/dt$ | $t_m$ | $y_m$ | $dy/dt$-mid |
|---|---|---|---|---|---|
| 0 | 1 | -1.1 | 0.25 | 0.725 | -0.75219 |
| 0.5 | 0.623906 | -0.53032 | 0.75 | 0.491326 | -0.26409 |
| 1 | 0.491862 | -0.04919 | 1.25 | 0.479566 | 0.221799 |
| 1.5 | 0.602762 | 0.693176 | 1.75 | 0.776056 | 1.52301 |
| 2 | 1.364267 | 3.956374 | 2.25 | 2.35336 | 9.32519 |

with $h = 0.25$ gives

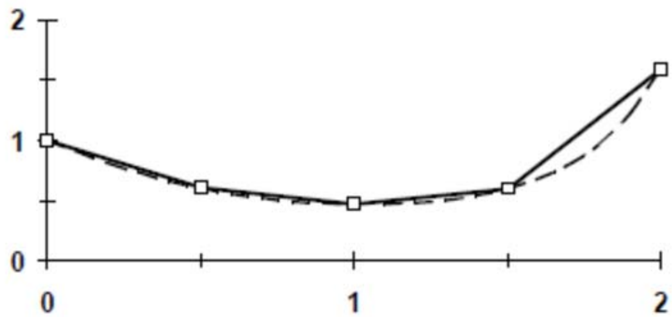
| $t$ | $y$ | $dy/dt$ | $t_m$ | $y_m$ | $dy/dt$-mid |
|---|---|---|---|---|---|
| 0 | 1 | -1.1 | 0.125 | 0.8625 | -0.93527 |
| 0.25 | 0.766182 | -0.79491 | 0.375 | 0.666817 | -0.63973 |
| 0.5 | 0.60625 | -0.51531 | 0.625 | 0.541836 | -0.38436 |
| 0.75 | 0.510158 | -0.27421 | 0.875 | 0.475882 | -0.15912 |
| 1 | 0.470378 | -0.04704 | 1.125 | 0.464498 | 0.076932 |
| 1.25 | 0.489611 | 0.226445 | 1.375 | 0.517916 | 0.409478 |
| 1.5 | 0.59198 | 0.680777 | 1.625 | 0.677077 | 1.043122 |
| 1.75 | 0.852761 | 1.673543 | 1.875 | 1.061954 | 2.565282 |
| 2 | 1.494081 | 4.332836 | 2.125 | 2.035686 | 6.953139 |

The results can be plotted along with the analytical solution as




**(d)** The 4th-order RK method with $h = 0.5$ gives

| $t$ | $y$ | $k_1$ | $y_m$ | $k_2$ | $y_m$ | $k_3$ | $y_e$ | $k_4$ | $\phi$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | -1.1000 | 0.725 | -0.7522 | 0.8120 | -0.8424 | 0.5788 | -0.4920 | -0.7969 |
| 0.5 | 0.6016 | -0.5113 | 0.4737 | -0.2546 | 0.5379 | -0.2891 | 0.4570 | -0.0457 | -0.2741 |
| 1 | 0.4645 | -0.0465 | 0.4529 | 0.2095 | 0.5169 | 0.2391 | 0.5841 | 0.6717 | 0.2537 |
| 1.5 | 0.5914 | 0.6801 | 0.7614 | 1.4943 | 0.9649 | 1.8937 | 1.5382 | 4.4609 | 1.9861 |
| 2 | 1.5845 | 4.5949 | 2.7332 | 10.8302 | 4.2920 | 17.0071 | 10.0880 | 51.9532 | 18.7038 |




**25.3** The second-order ODE is transformed into a pair of first-order ODEs as in

$$\frac{dy}{dt} = z \qquad\qquad y(0) = 2$$

$$\frac{dz}{dt} = 0.5t - y \qquad\qquad z(0) = 0$$

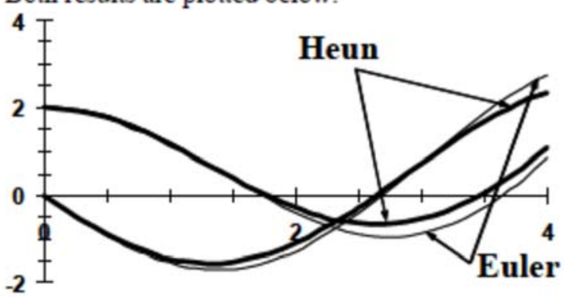
(*a*) The first few steps of Euler's method are

| *t* | *y* | *z* | *dy*/*dt* | *dz*/*dt* |
|---|---|---|---|---|
| 0 | 2 | 0 | 0 | -2 |
| 0.1 | 2 | -0.2 | -0.2 | -1.95 |
| 0.2 | 1.98 | -0.395 | -0.395 | -1.88 |
| 0.3 | 1.9405 | -0.583 | -0.583 | -1.7905 |
| 0.4 | 1.8822 | -0.76205 | -0.76205 | -1.6822 |
| 0.5 | 1.805995 | -0.93027 | -0.93027 | -1.556 |

(*b*) For Heun (without iterating the corrector) the first few steps are

| *t* | *y* | *z* | *dy*/*dt* | *dz*/*dt* | $y_{end}$ | $z_{end}$ | *dy*/*dt* | *dz*/*dt* | avg slope |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 2 | 0 | 0 | -2 | 2 | -0.2 | -0.2 | -1.95 | -0.1 |
| 0.1 | 1.99 | -0.1975 | -0.1975 | -1.94 | 1.97025 | -0.3915 | -0.3915 | -1.87025 | -0.2945 |
| 0.2 | 1.96055 | -0.38801 | -0.38801 | -1.86055 | 1.921749 | -0.57407 | -0.57407 | -1.77175 | -0.48104 |
| 0.3 | 1.912446 | -0.56963 | -0.56963 | -1.76245 | 1.855483 | -0.74587 | -0.74587 | -1.65548 | -0.65775 |
| 0.4 | 1.846671 | -0.74052 | -0.74052 | -1.64667 | 1.772619 | -0.90519 | -0.90519 | -1.52262 | -0.82286 |
| 0.5 | 1.764385 | -0.89899 | -0.89899 | -1.51439 | 1.674486 | -1.05043 | -1.05043 | -1.37449 | -0.97471 |

Both results are plotted below:



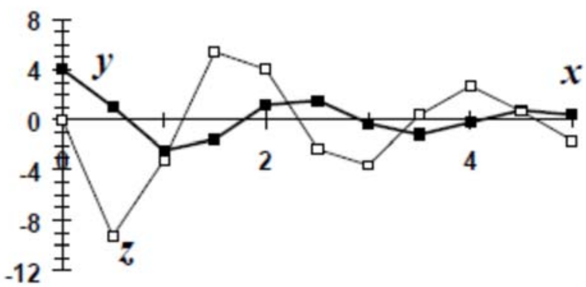
**25.4** The second-order ODE is transformed into a pair of first-order ODEs as in

$$\frac{dy}{dx} = z \qquad y(0) = 4$$

$$\frac{dz}{dx} = -0.6z - 8y \qquad z(0) = 0$$

The results for the 4$^{th}$-order RK method are tabulated and plotted below:

| $x$ | $y$ | $z$ | $k_{11}$ | $k_{12}$ | $y_{mid}$ | $z_{mid}$ | $k_{21}$ | $k_{22}$ | $y_{mid}$ | $z_{mid}$ | $k_{31}$ | $k_{32}$ | $y_{end}$ | $z_{end}$ | $k_{41}$ | $k_{42}$ | $\phi_1$ | $\phi_2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 4.0000 | 0.0000 | 0.00 | -32.00 | 4.00 | -8.00 | -8.00 | -27.20 | 2.00 | -6.80 | -6.80 | -11.92 | 0.60 | -5.96 | -5.96 | -1.22 | -5.93 | -18.58 |
| 0.5 | 1.0367 | -9.2887 | -9.29 | -2.72 | -1.29 | -9.97 | -9.97 | 16.27 | -1.46 | -5.22 | -5.22 | 14.78 | -1.57 | -1.90 | -1.90 | 13.74 | -6.93 | 12.18 |
| 1 | -2.4276 | -3.1969 | -3.20 | 21.34 | -3.23 | 2.14 | 2.14 | 24.53 | -1.89 | 2.94 | 2.94 | 13.38 | -0.96 | 3.49 | 3.49 | 5.58 | 1.74 | 17.12 |
| 1.5 | -1.5571 | 5.3654 | 5.37 | 9.24 | -0.22 | 7.67 | 7.67 | -2.88 | 0.36 | 4.65 | 4.65 | -5.68 | 0.77 | 2.53 | 2.53 | -7.64 | 5.42 | -2.59 |
| 2 | 1.1539 | 4.0719 | 4.07 | -11.67 | 2.17 | 1.15 | 1.15 | -18.07 | 1.44 | -0.44 | -0.44 | -11.27 | 0.93 | -1.56 | -1.56 | -6.51 | 0.65 | -12.81 |
| 2.5 | 1.4810 | -2.3334 | -2.33 | -10.45 | 0.90 | -4.95 | -4.95 | -4.21 | 0.24 | -3.39 | -3.39 | 0.07 | -0.21 | -2.30 | -2.30 | 3.08 | -3.55 | -2.61 |
| 3 | -0.2935 | -3.6375 | -3.64 | 4.53 | -1.20 | -2.50 | -2.50 | 11.13 | -0.92 | -0.86 | -0.86 | 7.87 | -0.72 | 0.30 | 0.30 | 5.59 | -1.68 | 8.02 |
| 3.5 | -1.1319 | 0.3723 | 0.37 | 8.83 | -1.04 | 2.58 | 2.58 | 6.76 | -0.49 | 2.06 | 2.06 | 2.66 | -0.10 | 1.70 | 1.70 | -0.22 | 1.89 | 4.58 |
| 4 | -0.1853 | 2.6602 | 2.66 | -0.11 | 0.48 | 2.63 | 2.63 | -5.42 | 0.47 | 1.31 | 1.31 | -4.56 | 0.47 | 0.38 | 0.38 | -3.97 | 1.82 | -4.01 |
| 4.5 | 0.7241 | 0.6564 | 0.66 | -6.19 | 0.89 | -0.89 | -0.89 | -6.57 | 0.50 | -0.99 | -0.99 | -3.42 | 0.23 | -1.05 | -1.05 | -1.21 | -0.69 | -4.56 |
| 5 | 0.3782 | -1.6258 | -1.63 | -2.05 | -0.03 | -2.14 | -2.14 | 1.51 | -0.16 | -1.25 | -1.25 | 2.00 | -0.25 | -0.63 | -0.63 | 2.34 | -1.50 | 1.22 |




**25.17** The MATLAB program shown below performs the Euler Method and displays the time just before the cylindrical tank empties ($y < 0$).

```
%prob2517.m
dt=0.5;
t=0;
y=3;
i=1;
while(1)
  y=y+dydt(t,y)*dt;
  if y<0, break, end
  t=t+dt;
  i=i+1;
end

function dy=dydt(t,y);
dy=-0.06*sqrt(y);
```

The result is 56 minutes as shown below

```
>> prob2517
t =
    56
```

**25.19** The two differential equations to be solved are

$$\frac{dv}{dt} = g - \frac{c_d}{m} v^2$$

$$\frac{dx}{dt} = -v$$

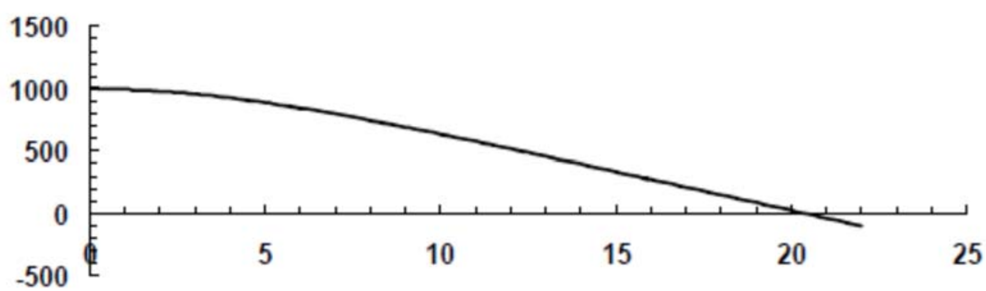
**(a)** Here are the first few steps of Euler's method with a step size of $h = 0.2$.

| *t* | *x* | *v* | *dx/dt* | *dv/dt* |
|---|---|---|---|---|
| 0 | 1000 | 0 | 0 | 9.81 |
| 0.2 | 1000 | 1.962 | -1.962 | 9.800376 |
| 0.4 | 999.6076 | 3.922075 | -3.92208 | 9.771543 |
| 0.6 | 998.8232 | 5.876384 | -5.87638 | 9.72367 |
| 0.8 | 997.6479 | 7.821118 | -7.82112 | 9.657075 |
| 1 | 996.0837 | 9.752533 | -9.75253 | 9.57222 |

**(b)** Here are the results of the first few steps of the 4th-order RK method with a step size of $h = 0.2$.

| *t* | *x* | *v* |
|---|---|---|
| 0 | 1000 | 0 |
| 0.2 | 999.8038 | 1.961359 |
| 0.4 | 999.2157 | 3.918875 |
| 0.6 | 998.2368 | 5.868738 |
| 0.8 | 996.869 | 7.807195 |
| 1 | 995.1149 | 9.730582 |

The results for $x$ of both methods are displayed graphically on the following plots. Because the step size is sufficiently small the results are in close agreement. Both indicate that the parachutist would hit the ground at a little after 20 s. The more accurate 4th-order RK method indicates that the solution reaches the ground between $t$ = 20.2 and 20.4 s.



**27.1** The solution can be assumed to be $T = e^{\lambda x}$. This, along with the second derivative $T'' = \lambda^2 e^{\lambda x}$, can be substituted into the differential equation to give

$$\lambda^2 e^{\lambda x} - 0.15 e^{\lambda x} = 0$$

which can be used to solve for

$$\lambda^2 - 0.15 = 0$$
$$\lambda = \pm\sqrt{0.15}$$

Therefore, the general solution is

$$T = Ae^{\sqrt{0.15}\,x} + Be^{-\sqrt{0.15}\,x}$$

The constants can be evaluated by substituting each of the boundary conditions to generate two equations with two unknowns,

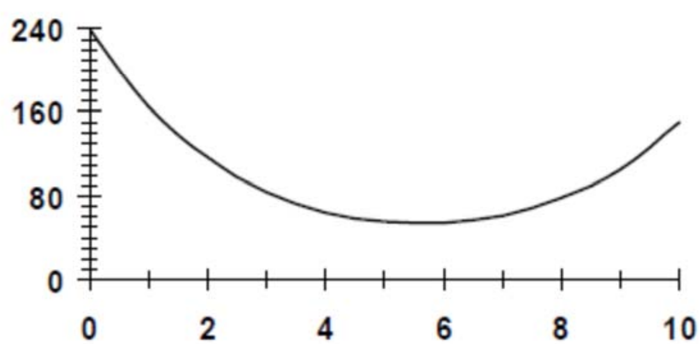
$$240 = A + B$$
$$150 = 48.08563A + 0.020796B$$

which can be solved for $A = 3.016944$ and $B = 236.9831$. The final solution is, therefore,

$$T = 3.016944e^{\sqrt{0.15}\,x} + 236.9831e^{-\sqrt{0.15}\,x}$$

which can be used to generate the values below:

| *x* | *T* |
|---|---|
| 0 | 240 |
| 1 | 165.329 |
| 2 | 115.7689 |
| 3 | 83.79237 |
| 4 | 64.54254 |
| 5 | 55.09572 |
| 6 | 54.01709 |
| 7 | 61.1428 |
| 8 | 77.55515 |
| 9 | 105.7469 |
| 10 | 150 |



**27.2** Reexpress the second-order equation as a pair of ODEs:

$$\frac{dT}{dx} = z$$

$$\frac{dz}{dx} = 0.15T$$

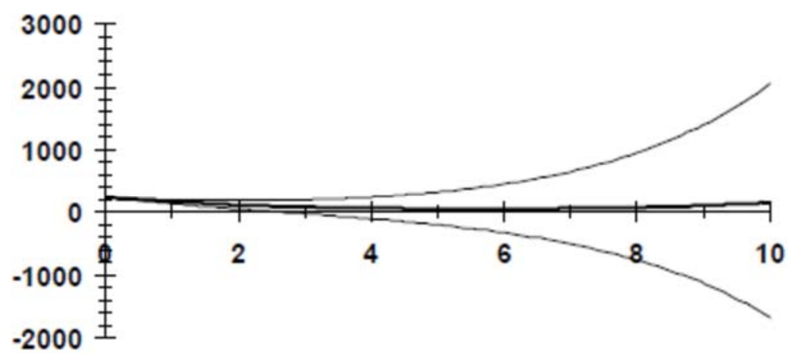
The solution was then generated on the Excel spreadsheet using the Heun method (without iteration) with a step-size of 0.01. An initial condition of $z = -120$ was chosen for the first shot. The first few calculation results are shown below.

| $x$ | $T$ | $z$ | $k_{11}$ | $k_{12}$ | $T_{end}$ | $z_{end}$ | $k_{21}$ | $k_{22}$ | $\phi_1$ | $\phi_2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 240.000 | -120.000 | -120.000 | 36.000 | 228.000 | -116.400 | -116.400 | 34.200 | -118.200 | 35.100 |
| 0.1 | 228.180 | -116.490 | -116.490 | 34.227 | 216.531 | -113.067 | -113.067 | 32.480 | -114.779 | 33.353 |
| 0.2 | 216.702 | -113.155 | -113.155 | 32.505 | 205.387 | -109.904 | -109.904 | 30.808 | -111.529 | 31.657 |
| 0.3 | 205.549 | -109.989 | -109.989 | 30.832 | 194.550 | -106.906 | -106.906 | 29.183 | -108.447 | 30.007 |
| 0.4 | 194.704 | -106.988 | -106.988 | 29.206 | 184.006 | -104.068 | -104.068 | 27.601 | -105.528 | 28.403 |
| 0.5 | 184.152 | -104.148 | -104.148 | 27.623 | 173.737 | -101.386 | -101.386 | 26.061 | -102.767 | 26.842 |

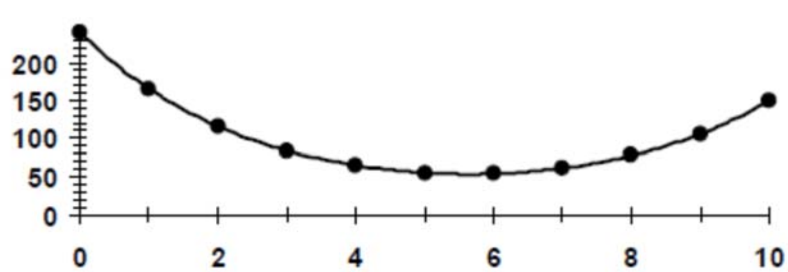
The resulting value at $x = 10$ was $T(10) = -1671.817$. A second shot using an initial condition of $z(0) = -60$ was attempted with the result at $x = 10$ of $T(10) = 2047.766$. These values can then be used to derive the correct initial condition,

$$z(0) = -120 + \frac{-60+120}{2047.766-(-1671.817)}(150-(-1671.817)) = -90.6126$$

The resulting fit, along with the two "shots" are displayed below:




The final shot along with the analytical solution (displayed as filled circles) shows close agreement:



**27.3** A centered finite difference can be substituted for the second derivative to give,

$$\frac{T_{i-1} - 2T_i + T_{i+1}}{h^2} - 0.15T_i = 0$$

or for $h = 1$,

$$-T_{i-1} + 2.15T_i - T_{i+1} = 0$$

The first node would be

$$2.15T_1 - T_2 = 240$$

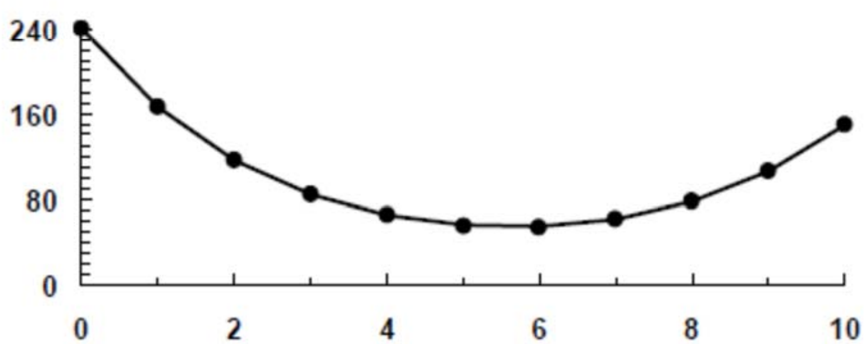
and the last node would be

$$-T_9 + 2.15T_{10} = 150$$

The tridiagonal system can be solved with the Thomas algorithm or Gauss-Seidel for (the analytical solution is also included)

| *x* | *T* | Analytical |
|---|---|---|
| 0 | 240 | 240 |
| 1 | 165.7573 | 165.3290 |
| 2 | 116.3782 | 115.7689 |
| 3 | 84.4558 | 83.7924 |
| 4 | 65.2018 | 64.5425 |
| 5 | 55.7281 | 55.0957 |
| 6 | 54.6136 | 54.0171 |
| 7 | 61.6911 | 61.1428 |
| 8 | 78.0223 | 77.5552 |
| 9 | 106.0569 | 105.7469 |
| 10 | 150 | 150 |

The following plot of the results (with the analytical shown as filled circles) indicates close agreement.



**27.4** The second-order ODE can be expressed as the following pair of first-order ODEs,

$$\frac{dy}{dx} = z$$

$$\frac{dz}{dx} = \frac{2z + y - x}{7}$$

These can be solved for two guesses for the initial condition of $z$. For our cases we used –1 and –0.5. We solved the ODEs with the Heun method without iteration using a step size of 0.125. The results are

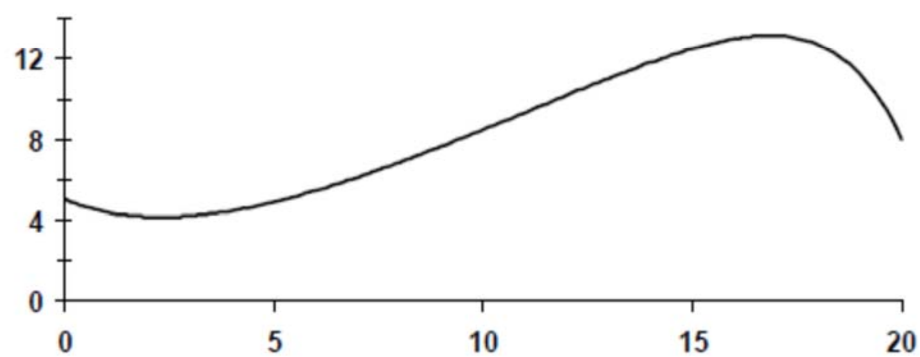
| $z(0)$ | –1 | –0.5 |
|---|---|---|
| $y(20)$ | –11,837.64486 | 22,712.34615 |

Clearly, the solution is quite sensitive to the initial conditions. These values can then be used to derive the correct initial condition,

$$z(0) = -1 + \frac{-0.5 + 1}{22712.34615 - (-11837.64486)}(8 - (-11837.64486)) = -0.82857239$$

The resulting fit is displayed below:

| $x$ | $y$ |
|---|---|
| 0 | 5 |
| 2 | 4.151601 |
| 4 | 4.461229 |
| 6 | 5.456047 |
| 8 | 6.852243 |
| 10 | 8.471474 |
| 12 | 10.17813 |
| 14 | 11.80277 |
| 16 | 12.97942 |
| 18 | 12.69896 |
| 20 | 8 |



**27.5** Centered finite differences can be substituted for the second and first derivatives to give,

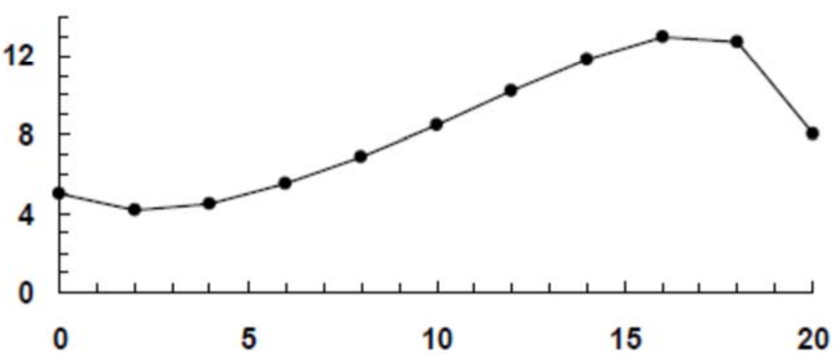
$$7\frac{y_{i+1}-2y_i+y_{i-1}}{\Delta x^2}-2\frac{y_{i+1}-y_{i-1}}{2\Delta x}-y_i+x_i=0$$

or substituting $\Delta x = 2$ and collecting terms yields

$$-2.25y_{i-1}+4.5y_i-1.25y_{i+1}=x_i$$

This equation can be written for each node and solved with methods such as the Tridiagonal solver, the Gauss-Seidel method or LU Decomposition. The following solution was computed using Excel's Minverse and Mmult functions:

| *x* | *y* |
|---|---|
| 0 | 5 |
| 2 | 4.199592 |
| 4 | 4.518531 |
| 6 | 5.507445 |
| 8 | 6.893447 |
| 10 | 8.503007 |
| 12 | 10.20262 |
| 14 | 11.82402 |
| 16 | 13.00176 |
| 18 | 12.7231 |
| 20 | 8 |




**27.23** Boundary Value Problem

1. $x$-spacing

at $x = 0$, i = 1; and at $x = 2$, $i = n$

$$\Delta x=\frac{2-0}{n-1}$$

2. Finite Difference Equation

$$\frac{d^2u}{dx^2}+6\frac{du}{dx}-u=2$$

Substitute finite difference approximations:

$$\frac{u_{i+1}-2u_i+u_{i-1}}{\Delta x^2}+6\frac{u_{i+1}-u_{i-1}}{2\Delta x}-u_i=2$$

$$[1-3(\Delta x)]u_{i-1}+[-2-\Delta x^2]u_i+[1+3(\Delta x)]u_{i+1}=2\Delta x^2$$

Coefficients:

$a_i = 1 - 3\Delta x$ $\quad b_i = -2 - \Delta x^2$ $\quad c_i = 1 + 3\Delta x$ $\quad d_i = 2\Delta x^2$

3. End point equations

<u>$i = 2$:</u>

$$[1-3(\Delta x)]10+[-2-\Delta x^2]u_2+[1+3(\Delta x)]u_3=2\Delta x^2$$

Coefficients:

$a_2 = 0$ $\quad b_2 = -2 - \Delta x^2$ $\quad c_2 = 1 + 3\Delta x$ $\quad d_2 = 2\Delta x^2 - 10(1 - 3(\Delta x))$

<u>$i = n - 1$:</u>

$$[1-3(\Delta x)]u_{n-2}+[-2-\Delta x^2]u_{n-1}+[1+3(\Delta x)]1=2\Delta x^2$$

Coefficients:

$a_2 = 1 - 3\Delta x$ $\quad b_2 = -2 - \Delta x^2$ $\quad c_2 = 0$ $\quad d_2 = 2\Delta x^2 - (1 - 3(\Delta x))$

```
% Boundary Value Problem
% u[xx]+6u[x]-u=2
% BC: u(x=0)=10  u(x=2)=1
% i=spatial index from 1 to n
% numbering for points is i=1 to i=21 for 20 dx spaces
% u(1)=10 and u(n)=1

n=41; xspan=2.0;

% Constants
dx=xspan/(n-1);
dx2=dx*dx;

% Sizing matrices
u=zeros(1,n); x=zeros(1,n);
a=zeros(1,n); b=zeros(1,n); c=zeros(1,n); d=zeros(1,n);
ba=zeros(1,n); ga=zeros(1,n);

% Coefficients and Boundary Conditions
x=0:dx:2;
u(1)=10; u(n)=1;
b(2)=-2-dx2;
c(2)=1+3*dx;
d(2)=2*dx2-(1-3*dx)*10;
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for i=3:n-2
  a(i)=1-3*dx;
  b(i)=-2-dx2;
  c(i)=1+3*dx;
  d(i)=2*dx2;
end
a(n-1)=1-3*dx;
b(n-1)=-2-dx2;
d(n-1)=2*dx2-(1+3*dx);

% Solution by Thomas Algorithm
ba(2)=b(2);
ga(2)=d(2)/b(2);
for i=3:n-1
  ba(i)=b(i)-a(i)*c(i-1)/ba(i-1);
  ga(i)=(d(i)-a(i)*ga(i-1))/ba(i);
end
% back substitution
u(n-1)=ga(n-1);
for i=n-2:-1:2
  u(i)=ga(i)-c(i)*u(i+1)/ba(i);
end

% Plot
plot(x,u)
title('u[xx]+6u[x]-u=2; u(x=0)=10, u(x=2)=1')
xlabel('x-Independent Variable Range 0 to 2');ylabel('u-Dependent  Variable')
grid
```

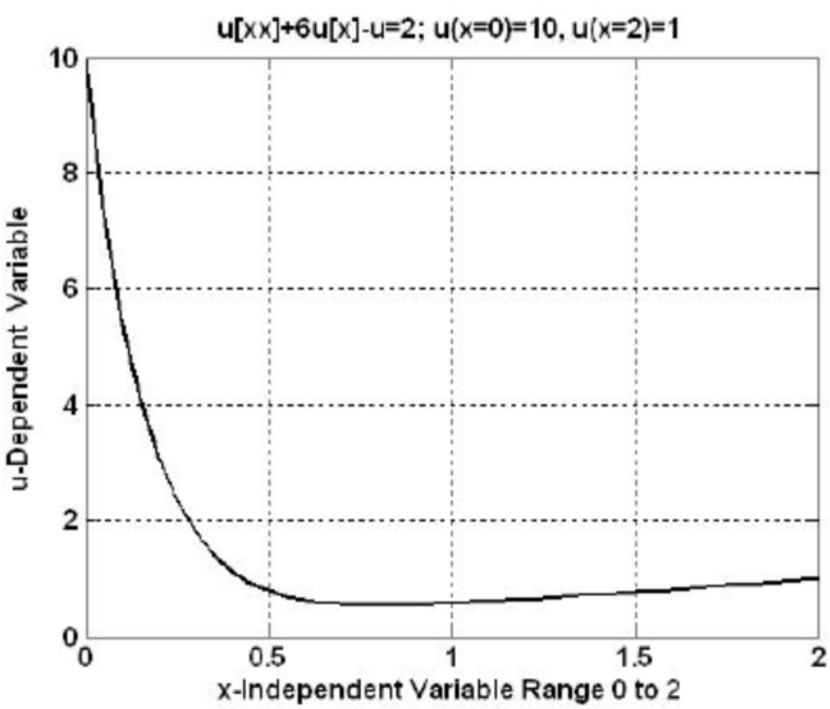

**27.24**

1. Divide the radial coordinate into $n$ finite points.

$$\Delta r = \frac{1}{n-1}$$

2. The finite difference approximations for the general point $i$

$$\frac{d^2T}{dr^2} = \frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta r^2}$$

$$\frac{dT}{dr} = \frac{T_{i+1} - T_{i-1}}{2\Delta r}$$

$$r = \Delta r(i-1)$$

3. Substituting in the finite difference approximations for the derivatives

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} + S = 0$$

$$\frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta r^2} + \frac{1}{\Delta r(i-1)}\frac{T_{i+1} - T_{i-1}}{2\Delta r} + S = 0$$

4. Collecting like terms results in the general finite difference equation at point $i$

$$-\left[1 - \frac{1}{2(i-1)}\right]T_{i-1} + 2T_i - \left[1 + \frac{1}{2(i-1)}\right]T_{i+1} = \Delta r^2 S$$

5. End point equation at $i = 1$

$$\left.\frac{dT}{dr}\right|_{r=0} = 0$$

Substituting in the FD approximation gives

$$\frac{T_2 - T_0}{2\Delta r} = 0$$

where $T_0$ is a fictitious point. Thus, we see that $T_0 = T_2$ for zero slope at $r = 0$. Writing out the general equation at point $i = 1$ gives:

$$-\left[1 - \frac{1}{2(i-1)}\right]T_2 + 2T_1 - \left[1 + \frac{1}{2(i-1)}\right]T_2 = \Delta r^2 S$$

Collecting terms gives

$$2T_1 - 2T_2 = \Delta r^2 S$$

6. End point equation at $i = n - 1$

$$T(r=1) = 1$$

$$-\left[1 - \frac{1}{2(i-1)}\right] T_{n-2} + 2T_{n-1} = \Delta r^2 S + \left[1 + \frac{1}{2(i-1)}\right]$$

7. Solve the resulting tridiagonal system of algebraic equations using the Thomas Algorithm.

Here is a plot of all the results for the 3 cases: