MTE 203 – Advanced Calculus
Homework 9 (Solutions)

Problem 1: [13.1, Prob. 19]

Evaluate the double iterated integral $\int_0^1 \int_0^x \frac{1}{\sqrt{1-y^2}} dy\, dx$

Solution:

$$\int_0^1 \int_0^x \frac{1}{\sqrt{1-y^2}} dy\, dx = \int_0^1 \left\{ \mathrm{Sin}^{-1} y \right\}_0^x dx = \int_0^1 \mathrm{Sin}^{-1} x\, dx$$

If we set $u = \mathrm{Sin}^{-1} x$, $dv = dx$, $du = \frac{1}{\sqrt{1-x^2}} dx$, $v = x$, and use integration by parts,

$$\int_0^1 \int_0^x \frac{1}{\sqrt{1-y^2}} dy\, dx = \left\{ x\, \mathrm{Sin}^{-1} x \right\}_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}} dx = \frac{\pi}{2} + \left\{ \sqrt{1-x^2} \right\}_0^1 = \frac{\pi}{2} - 1.$$

Problem 2: [S. 13.1, Prob. 33] Application Problem for Double Integrals

In two-dimensional steady state, incompressible flow, the velocity $\mathbf{v} = u(x,y)\hat{\mathbf{i}} + v(x,y)\hat{\mathbf{j}}$, which must satisfy the *continuity equation,* $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$.

If $u(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$, find all possible functions $v(x,y)$.

Solution:

In order to find $v(x,y)$, In the first step, we can use the *continuity* equation to find the derivative of v as follows:

From the *continuity* equation we have:

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\frac{\partial \left(\tan^{-1} \frac{y}{x}\right)}{\partial x} = -\frac{1}{1 + \frac{y^2}{x^2}} \left(\frac{-y}{x^2}\right)$$

Therefore we have:

$$\frac{\partial v}{\partial y} = -\frac{1}{1 + \frac{y^2}{x^2}} \left(\frac{-y}{x^2}\right) = \frac{y}{x^2 + y^2}$$

By integrating the above equation with respect to y, we have:

$$v(x,y) = \frac{1}{2}\ln(x^2 + y^2) + f(x)$$

Note that $f(x)$ can be any differentiable function of x and $\left(\frac{\partial f(x)}{\partial y} = 0\right)$

Problem 3: [13.1, Prob. 39] Application Problem

Stream functions $\psi(x,y)$ for two dimensional, steady state, incompressible flow satisfy

$$\frac{\partial \psi}{\partial x} = -v(x,y)\ , \qquad \frac{\partial \psi}{\partial y} = u(x,y)$$

where $\mathbf{v} = u(x,y)\hat{\mathbf{i}} + v(x,y)\hat{\mathbf{j}}$ is the velocity of the flow. Find all stream functions for the flow with

$$\mathbf{v} = -\cos x \sin y\, \hat{\mathbf{i}} + (\sin x \cos y + x)\hat{\mathbf{j}}$$

Solution:

Stream functions must satisfy

$$\frac{\partial \psi}{\partial x} = -\sin x \cos y - x, \qquad \frac{\partial \psi}{\partial y} = -\cos x \sin y.$$

Integration of the second gives $\psi(x,y) = \cos x \cos y + f(x)$, where $f(x)$ is any differentiable function of x. Substitution of this into the first equation requires

$$-\sin x \cos y + f'(x) = -\sin x \cos y - x \quad \Longrightarrow \quad f(x) = -\frac{x^2}{2} + C,$$

where C is a constant. Thus, $\psi(x,y) = \cos x \cos y - x^2/2 + C$.

<u>Evaluation of Double Integrals by Double Iterated Integrals</u>

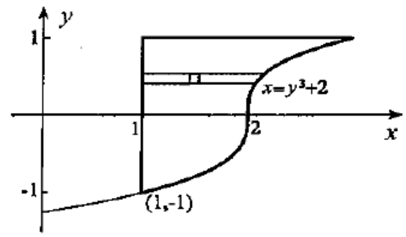
Problem 4: [13.2, Prob. 3]

Evaluate the double integral over the region

$\iint_R (x+y)\, dA$ where R is bounded by $x = y^3 + 2$ and $x = 1$ and $y = 1$

Solution:

$$\iint_R (x+y)\,dA = \int_{-1}^{1}\int_{1}^{y^3+2}(x+y)\,dx\,dy$$
$$= \int_{-1}^{1}\left\{\frac{x^2}{2}+xy\right\}_1^{y^3+2} dy$$
$$= \frac{1}{2}\int_{-1}^{1}(y^6+2y^4+4y^3+2y+3)\,dy$$
$$= \frac{1}{2}\left\{\frac{y^7}{7}+\frac{2y^5}{5}+y^4+y^2+3y\right\}_{-1}^{1} = \frac{124}{35}$$



Problem 5: [13.2, Prob. 17]

Evaluate the double iterated integral by <u>reversing the order of the integral.</u>

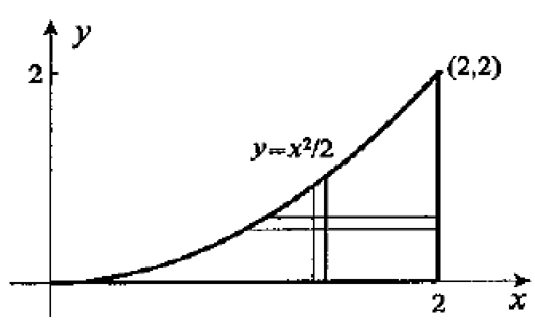
$$\int_0^2\int_0^{\frac{x^2}{2}}\frac{x}{\sqrt{1+x^2+y^2}}\,dy\,dx$$

Hint1: after revising the order use integral by substitution method ($y=\sqrt{5}\ \tan\theta$)

Hint2: $\int(\sec\theta)^3\,d\theta = \frac{1}{2}(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|)$

Solution:

$$\int_0^2\int_0^{x^2/2}\frac{x}{\sqrt{1+x^2+y^2}}dy\,dx = \int_0^2\int_{\sqrt{2y}}^{2}\frac{x}{\sqrt{1+x^2+y^2}}dx\,dy$$
$$= \int_0^2\left\{\sqrt{1+x^2+y^2}\right\}_{\sqrt{2y}}^{2}dy$$
$$= \int_0^2\left[\sqrt{5+y^2}-(1+y)\right]dy$$



We can solve the integral using integral by substitution method:

If we set $y = \sqrt{5}\tan\theta$, then $dy = \sqrt{5}\sec^2\theta\, d\theta$, and

$$\int_0^2 \int_0^{x^2/2} \frac{x}{\sqrt{1+x^2+y^2}}\, dy\, dx = \int_0^{\mathrm{Tan}^{-1}(2/\sqrt{5})} \sqrt{5}\sec\theta\,\sqrt{5}\sec^2\theta\, d\theta - \left\{ y + \frac{y^2}{2} \right\}_0^2$$

$$= 5\int_0^{\mathrm{Tan}^{-1}(2/\sqrt{5})} \sec^3\theta\, d\theta - 4$$

$$= \frac{5}{2}\left\{ \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| \right\}_0^{\mathrm{Tan}^{-1}(2/\sqrt{5})} - 4$$

$$= \frac{5}{4}\ln 5 - 1$$

Double Iterated Integrals in Polar Coordinates

Problem 6: [13.7, Prob. 25]

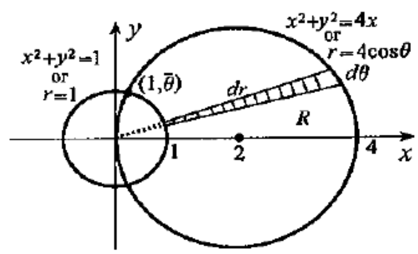
Find the area inside the circle $x^2 + y^2 = 4x$ and outside the circle $x^2 + y^2 = 1$.

Solution:

If R is the region bounded by these circles and above the x-axis, then the required area is

$$2\iint_R dA.$$

Since the curves intersect in the first quadrant at a point where $\theta = \bar{\theta} = \mathrm{Cos}^{-1}(\frac{1}{4})$, then



$$\text{area} = 2\int_0^{\bar{\theta}}\int_1^{4\cos\theta} r\, dr\, d\theta = 2\int_0^{\bar{\theta}} \left\{ \frac{r^2}{2} \right\}_1^{4\cos\theta} d\theta = \int_0^{\bar{\theta}} (16\cos^2\theta - 1) d\theta$$

$$= \int_0^{\bar{\theta}} \left[16\left(\frac{1+\cos 2\theta}{2}\right) - 1 \right] d\theta = \int_0^{\bar{\theta}} (7 + 8\cos 2\theta) d\theta = \{7\theta + 4\sin 2\theta\}_0^{\bar{\theta}}$$

$$= 7\bar{\theta} + 4\sin 2\bar{\theta} = 7\mathrm{Cos}^{-1}(\tfrac{1}{4}) + 8\cos\bar{\theta}\sin\bar{\theta}$$

$$= 7\,\mathrm{Cos}^{-1}(\tfrac{1}{4}) + 8(\tfrac{1}{4})\sqrt{1 - \tfrac{1}{16}} = 7\,\mathrm{Cos}^{-1}(\tfrac{1}{4}) + \sqrt{15}/2.$$

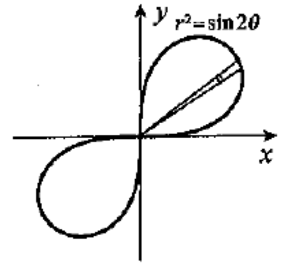
Problem 7: [13.7, Prob. 29]

Find the area of the region bounded by the curve $(x^2 + y^2)^2 = 2xy$

Solution:

The equation of the curve in polar coordinates is $r^4 = 2r^2 \sin\theta\cos\theta \implies r^2 = \sin 2\theta$.

$$A = 2\int_0^{\pi/2}\int_0^{\sqrt{\sin 2\theta}} r\,dr\,d\theta$$
$$= 2\int_0^{\pi/2}\left\{\frac{r^2}{2}\right\}_0^{\sqrt{\sin 2\theta}} d\theta$$
$$= \int_0^{\pi/2} \sin 2\theta\,d\theta = \left\{-\frac{1}{2}\cos 2\theta\right\}_0^{\pi/2} = 1$$



Triple Integrals and Triple Iterated Integrals

Problem 8: [S.13.8, Prob. 3]

Evaluate the triple integral over the region:

$\iiint_V \sin(y+z)\,dV$ Where V is bounded by $z = 0$, $y = 2x$, $y = 0$, $x = 1$, $z = x + 2y$

Solution:

$$\iiint_V \sin(y+z)\,dV = \int_0^1\int_0^{2x}\int_0^{x+2y} \sin(y+z)\,dz\,dy\,dx = \int_0^1\int_0^{2x}\left\{-\cos(y+z)\right\}_0^{x+2y} dy\,dx$$
$$= \int_0^1\int_0^{2x}\left[\cos y - \cos(x+3y)\right]dy\,dx = \int_0^1\left\{\sin y - \frac{1}{3}\sin(x+3y)\right\}_0^{2x} dx$$
$$= \int_0^1\left(\sin 2x - \frac{1}{3}\sin 7x + \frac{1}{3}\sin x\right)dx = \left\{-\frac{1}{2}\cos 2x + \frac{1}{21}\cos 7x - \frac{1}{3}\cos x\right\}_0^1$$
$$= (2\cos 7 - 14\cos 1 - 21\cos 2 + 33)/42$$

Problem 9: [13.8, Prob. 17]

Setup, but do not evaluate, a triple iterated integral for the triple integral.

$\iiint_V x^2y^2z^2\,dV$ where V is bounded by $x = y^2 + z^2$ and $x + 1 = (y^2+z^2)^2$

Solution:

The surfaces intersect in a plane parallel to the yz-plane defined by $x+1=x^2$, from which $x=(1\pm\sqrt{1+4})/2=(1\pm\sqrt{5})/2$, only the positive result being acceptable. The equation of the projection of the curve in the yz-plane is $y^2+z^2=(1+\sqrt{5})/2$. Hence,

$$\iiint_V x^2y^2z^2\,dV = 4\int_0^{\sqrt{(1+\sqrt{5})/2}}\int_0^{\sqrt{(1+\sqrt{5})/2-y^2}}\int_{(y^2+z^2)^2-1}^{y^2+z^2} x^2y^2z^2\,dx\,dz\,dy.$$

Volumes

Problem 10: [13.9, Prob. 19]

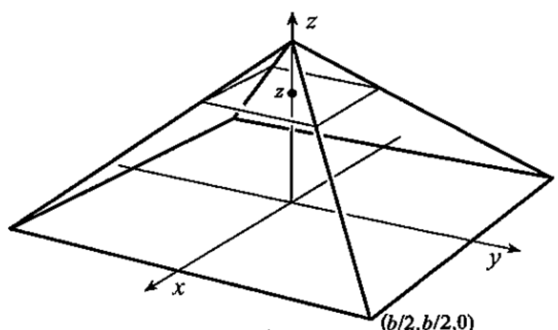
A pyramid has a square base with side length *b* and has height *h* at its center.

(a) Find its volume by taking cross-sections parallel to the base (see section 7.9).
(b) Find its volume using triple integrals.

Solution:

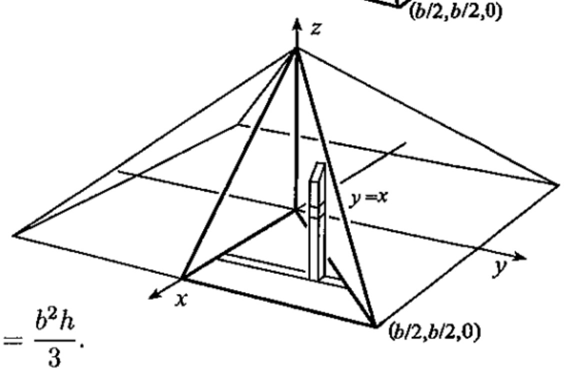
(a) The square cross section at height z has sides of length $b(h-z)/h$. Consequently, the area of the cross section is $b^2(h-z)^2/h^2$, and the volume of the pyramid is

$$V=\int_0^h \frac{b^2}{h^2}(h-z)^2\,dz=\frac{b^2}{h^2}\left\{-\frac{1}{3}(h-z)^3\right\}_0^h=\frac{b^2h}{3}.$$



(b) Since the equation of the face of the pyramid containing the point $(b/2,0,0)$ is $2x/b+z/h=1$,

$$\begin{aligned}V&=8\int_0^{b/2}\int_0^x\int_0^{h(1-2x/b)}dz\,dy\,dx\\&=8\int_0^{b/2}\int_0^x h\left(1-\frac{2x}{b}\right)dy\,dx\\&=\frac{8h}{b}\int_0^{b/2}\left\{(b-2x)y\right\}_0^x dx\\&=\frac{8h}{b}\int_0^{b/2}(bx-2x^2)\,dx=\frac{8h}{b}\left\{\frac{bx^2}{2}-\frac{2x^3}{3}\right\}_0^{b/2}=\frac{b^2h}{3}.\end{aligned}$$



Problem 11: [13.9, Prob. 21] Application problem for Average - Cartesian Coordinates

Find the average value [$\bar{f} = \frac{1}{V}\iiint_V f(x,y,z)dV$] if $f(x,y,z) = x + y + z$ over the region in the first octant bounded by the surfaces $z = 9 - x^2 - y^2$, $z = 0$, and for which $0 \le x \le 1$, $0 \le y \le 1$.

Solution:

$$\text{Since V} = \int_0^1 \int_0^1 \int_0^{9-x^2-y^2} dz\,dy\,dx = \int_0^1 \int_0^1 (9 - x^2 - y^2)\,dy\,dx = \int_0^1 \left\{9y - x^2y - \frac{y^3}{3}\right\}_0^1 dx$$
$$= \int_0^1 \left(9 - x^2 - \frac{1}{3}\right) dx = \left\{\frac{26x}{3} - \frac{x^3}{3}\right\}_0^1 = \frac{25}{3},$$

$$\overline{f} = \frac{3}{25}\int_0^1 \int_0^1 \int_0^{9-x^2-y^2} (x + y + z)\,dz\,dy\,dx = \frac{3}{25}\int_0^1 \int_0^1 \left\{(x+y)z + \frac{z^2}{2}\right\}_0^{9-x^2-y^2} dy\,dx$$
$$= \frac{3}{50}\int_0^1 \int_0^1 (81 + 18x + 18y - 18x^2 - 18y^2 - 2x^3 - 2y^3 - 2xy^2 - 2x^2y + x^4 + y^4 + 2x^2y^2)\,dy\,dx$$
$$= \frac{3}{50}\int_0^1 \left\{81y + 18xy + 9y^2 - 18x^2y - 6y^3 - 2x^3y - \frac{y^4}{2} - \frac{2xy^3}{3} - x^2y^2 + x^4y + \frac{y^5}{5} + \frac{2x^2y^3}{3}\right\}_0^1 dx$$
$$= \frac{3}{50}\int_0^1 \left(\frac{837}{10} + \frac{52x}{3} - \frac{55x^2}{3} - 2x^3 + x^4\right) dx = \frac{3}{50}\left\{\frac{837x}{10} + \frac{26x^2}{3} - \frac{55x^3}{9} - \frac{x^4}{2} + \frac{x^5}{5}\right\}_0^1 = \frac{1934}{375}.$$