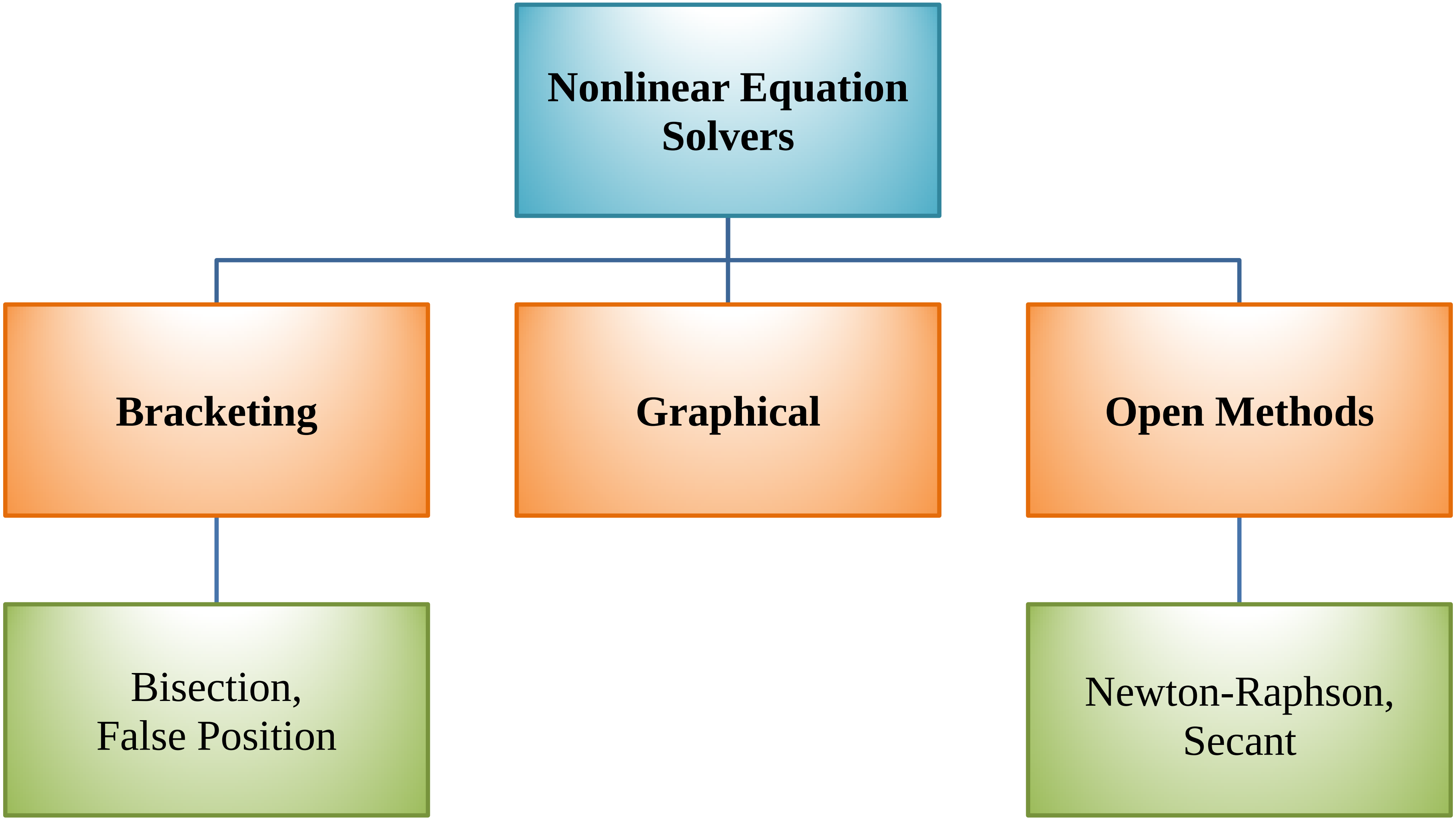
Part 2. Roots of Equations
Chapter 5. Bracketing Methods

Lecture 5

Bisection Method

5.2

Homeyra Pourmohammadali


Nonlinear Equation Solvers
Bracketing
Graphical
Open Methods
Bisection, False Position
Newton-Raphson, Secant

Roots of equation

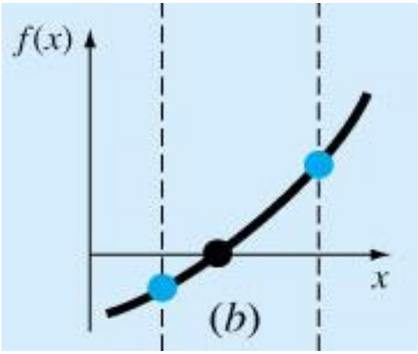
- values that satisfy homogeneous equation, solutions to implicit equation

Graphical Method

- graph function, locate roots where curve crosses x axis → predict # of roots, approximate location

Recall

- general "rules of thumb" for root prediction based on graphical methods: f(x) changes sign on the opposite side the root



Incremental Search Methods

- Locate, divide an interval where function changes sign
- One approach is bisection method (make interval in half)

Bracketing Method

- 2 initial guesses for the root are required.
- These guesses must bracket (be on either side of) the root.

- If one root of a real and continuous function, $f(x)=0$, is bounded by values $x = x_l$, $x = x_u$ then:

$$\boxed{f(x_l) \,.\, f(x_u) < 0}$$

(The function changes sign on opposite sides of the root)

Algorithm for Bisection Method

For the arbitrary equation of one variable, $f(x)=0$

1. Pick x_l and x_u such that they bound the root of interest, check if $f(x_l).f(x_u) < 0$.

2. Estimate the root by evaluating $f[(x_l + x_u) / 2]$
   (evaluating at midpoint)

3. Find the pair:

Algorithm for Bisection Method

- If $\mathbf{f(x_l) \,.\, f[(x_l + x_u)/2] < 0}$

  root lies in the lower interval, then $x_u = (x_l + x_u)/2$ and go to step 2.

- If $\mathbf{f(x_l) \,.\, f[(x_l + x_u)/2] > 0}$

  root lies in the upper interval, then $x_l = [(x_l + x_u)/2$, go to step 2.

- If $\mathbf{f(x_l) \,.\, f[(x_l + x_u)/2] = 0}$

  then root is the midpoint at $(x_l + x_u)/2$, and terminate.

Bisection Method

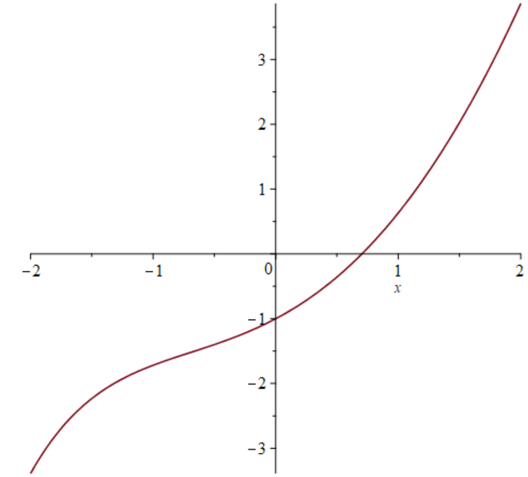
4. Compare ε_s with ε_a

5. If $\varepsilon_a < \varepsilon_s$, stop. Otherwise repeat the process.

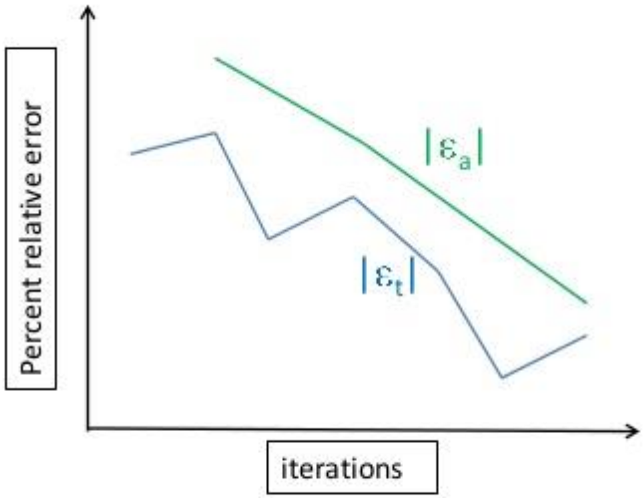
$$\left\{ \begin{array}{l} \dfrac{\left| x_l - \dfrac{x_l + x_u}{2} \right|}{\left| \dfrac{x_l + x_u}{2} \right|} \prec 100\% \\ or \\ \dfrac{\left| x_u - \dfrac{x_l + x_u}{2} \right|}{\left| \dfrac{x_l + x_u}{2} \right|} \prec 100\% \end{array} \right\}$$

Example 1. Bisection Method

Find roots of $f(x) = x^2 - e^{-x}$ using the bisection method and calculate the approximate relative error up to 4 iterations



Compare relative approximate error to relative true error



https://www.slideshare.net/batuhanyil23/es272-ch3a

Number of iterations to achieve desired error level

- Absolute error (0 iteration) $E_a^0 = \Delta X^o = X_u^0 - X_l^0$
- After 1 iteration $E_a^1 = \Delta X^o / 2$
- After 2 iterations $E_a^2 = \Delta X^1 / 2 = \Delta X_o / 2^2$
- After n iterations $E_a^n = \Delta X^o / 2^n$
- Solve for n:

$$n = \frac{\log_{10} (\Delta X^o / E_a^n)}{\log_{10} (2)}$$

Example 2. Bisection Method-Number of Iterations

For $f(x) = x^2 - e^{-x}$ find the number of iterations in bisection method that is required to reach to 0.001 desired level of error (choose the same boundary (0, 1))

$$n = \frac{\log_{10}(\Delta X^{o} / E_a^{n})}{\log_{10}(2)}$$

iteration #	x low	x high	x r	f(x low)	f(x high)	f(x r)
0	0	1	0.5	-1	0.632121	-0.35653
1	0.5	1	0.75	-0.35653	0.632121	0.090133
2	0.5	0.75	0.625	-0.35653	0.090133	-0.14464
3	0.625	0.75	0.6875	-0.14464	0.090133	-0.03018
4	0.6875	0.75	0.71875	-0.03018	0.090133	0.02924
5	0.6875	0.71875	0.703125	-0.03018	0.02924	-0.00065
6	0.703125	0.71875	0.710938	-0.00065	0.02924	0.014249
7	0.703125	0.710938	0.707031	-0.00065	0.014249	0.006787
8	0.703125	0.707031	0.705078	-0.00065	0.006787	0.003065
9	0.703125	0.705078	0.704102	-0.00065	0.003065	0.001206
10	0.703125	0.704102	0.703613	-0.00065	0.001206	0.000277

Evaluation of Bisection Method

Pros

- Easy & robust
- Always find root
- Number of iterations required to attain an absolute error can be computed a priori.

Cons

- Slow convergence
- Cannot handle multiple roots
- Take no account on magnitude of $f(x_l)$ and $f(x_u)$, if $f(x_l)$ is closer to zero, then root may be closer to x_l than x_u.

Part 2. Roots of Equations
Chapter 6. Open Methods

Lecture 6

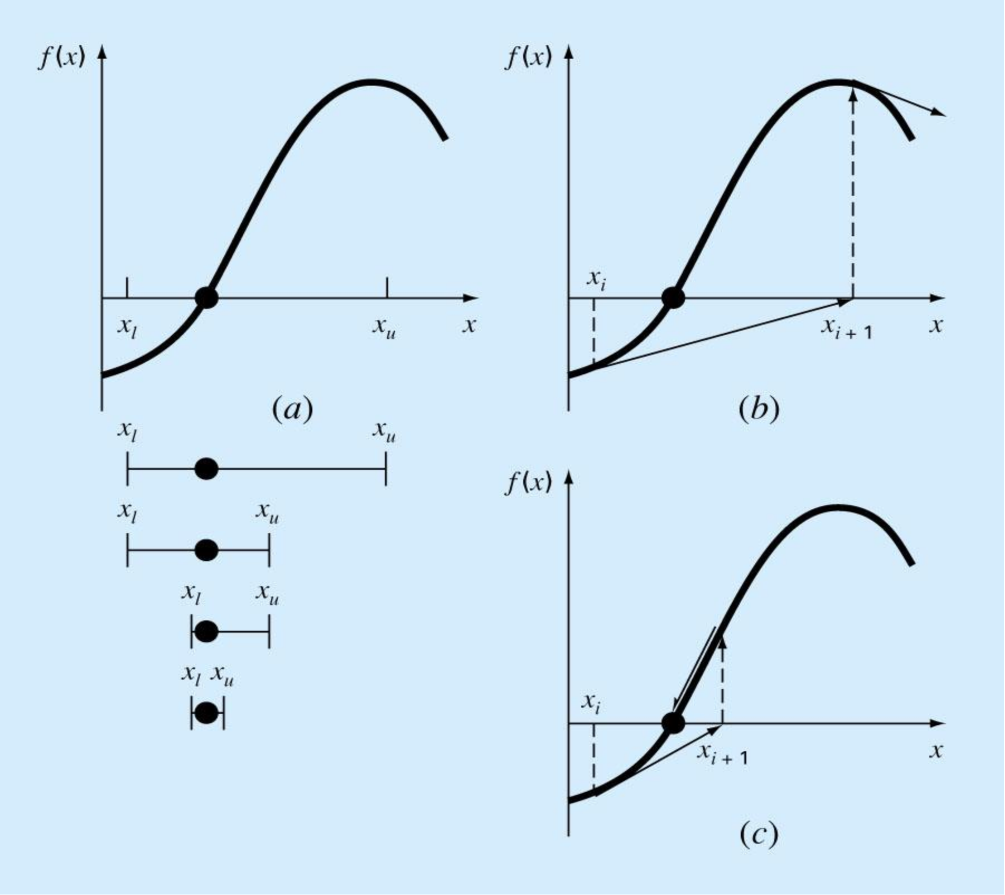
Fixed-Point Iteration & Newton-Raphson

6.1, 6.2

Homeyra Pourmohammadali

Open Methods

Open methods are based on formulas that require only a single starting value of x or two starting values that do not necessarily bracket the root.



a) Bracketing methods, b) & c) Open methods

Simple Fixed-point Iteration

Rearrange the function so that x is on the left side of the equation:

$$f(x) = 0 \quad \Rightarrow \quad g(x) = x$$
$$x_k = g(x_{k-1}) \qquad x_o \text{ given, k} = 1, 2, \ldots$$

- Bracketing methods are "convergent".
- Fixed-point methods may sometime "diverge", depending on the stating point (initial guess) and how the function behaves.

Convergence: Simple Fixed-Point Iteration

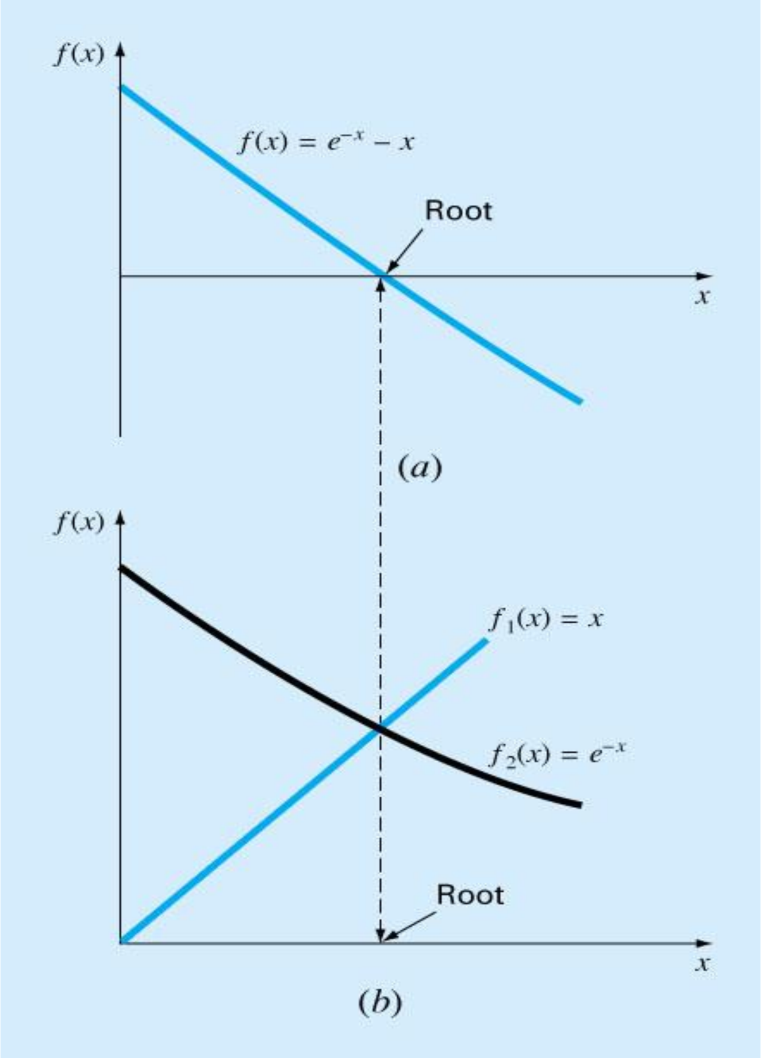
- $x=g(x)$ can be expressed as a pair of equations:

  $y_1 = x$

  $y_2 = g(x)$

(component equations)

- Plot them separately.



Convergence: Simple Fixed-Point Iteration

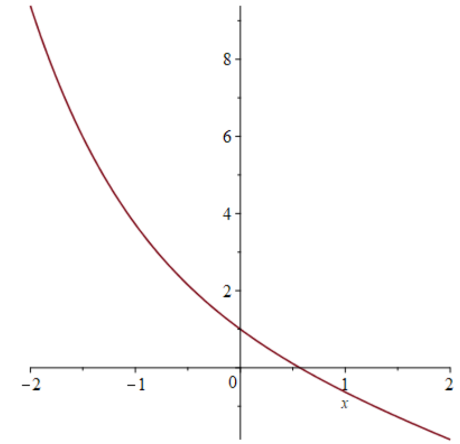
- Fixed-point iteration converges if

$$\left|g'(x)\right| \prec 1 \quad \text{(slope of the line } f(x) = x\text{)}$$

•When the method converges, the error is roughly proportional to or less than the error of the previous step, therefore it is called "linearly convergent."

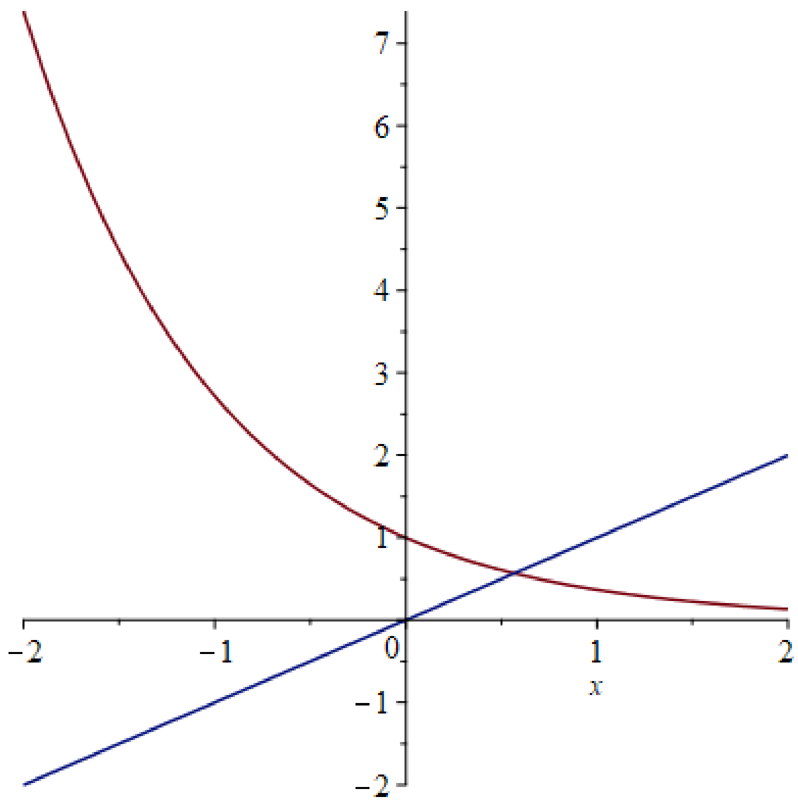
Example. Fixed-point Iteration

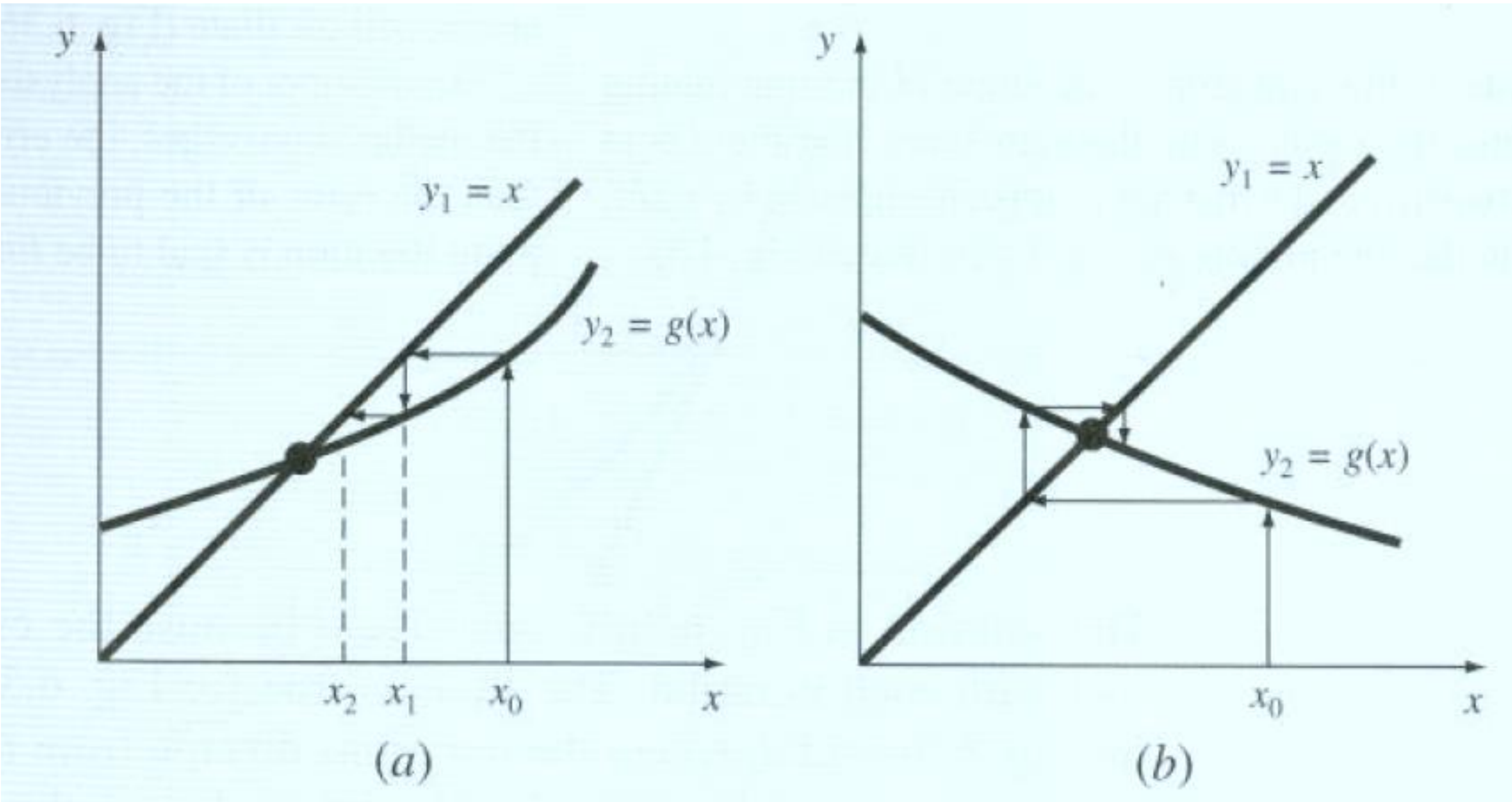
$$f(x) = e^{-x} - x$$



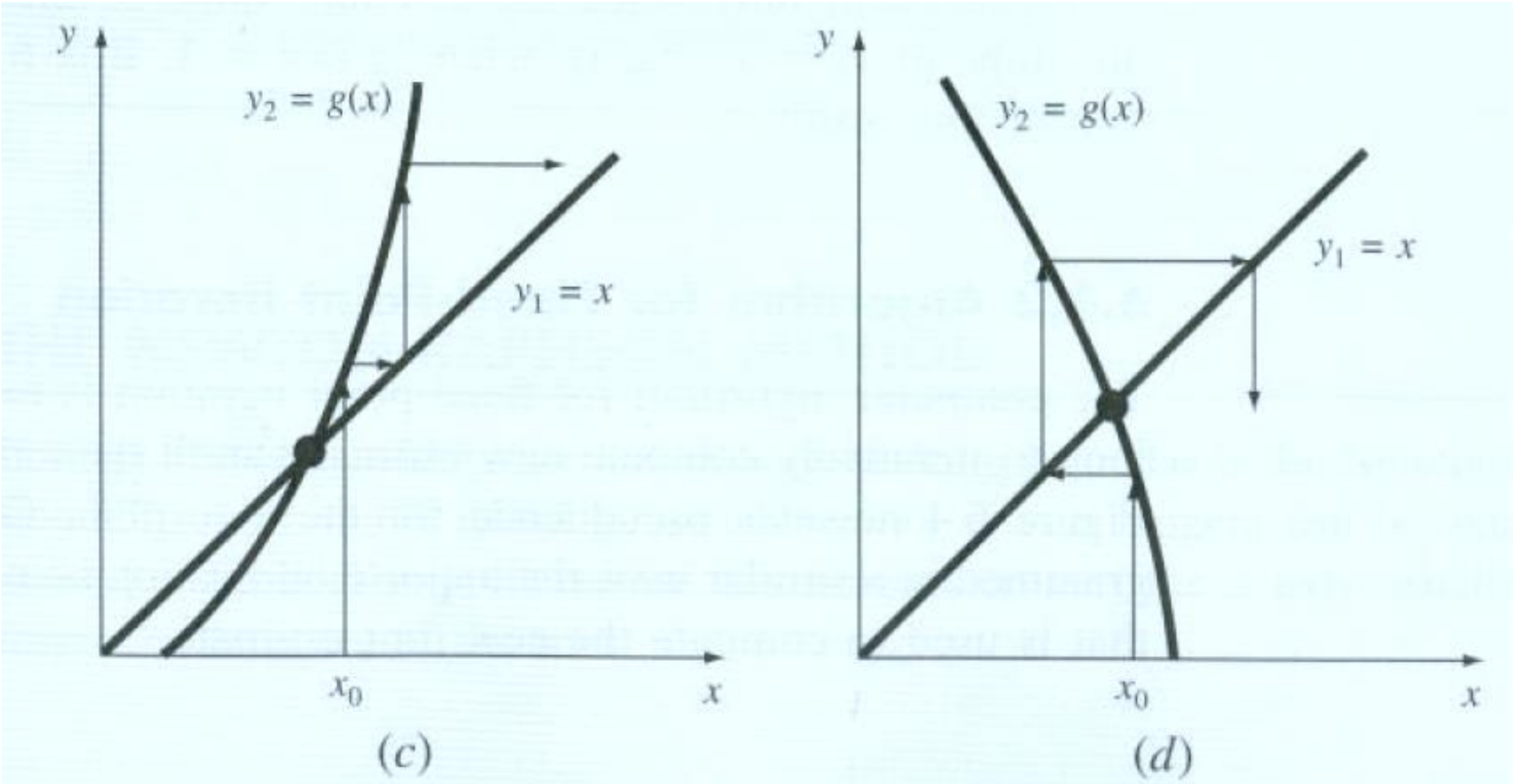
What about different initial value? Does method always converge?

$$f(x) = e^{-x} - x$$





y
$y_1 = x$
$y_2 = g(x)$
x_2
x_1
x_0
x
(a)
y
$y_1 = x$
$y_2 = g(x)$
x_0
x
(b)


y
$y_2 = g(x)$
$y_1 = x$
x_0
x
(c)
y
$y_2 = g(x)$
$y_1 = x$
x_0
x
(d)

Newton-Raphson Method

- Most widely used method.
- Based on Taylor series expansion:

$$f(x_{i+1}) = f(x_i) + f'(x_i)\Delta x + f''(x_i)\frac{\Delta x^2}{2!} + O\Delta x^3$$

The root is the value of $\mathrm{x_{i+1}}$ when $\mathrm{f(x_{i+1}) = 0}$

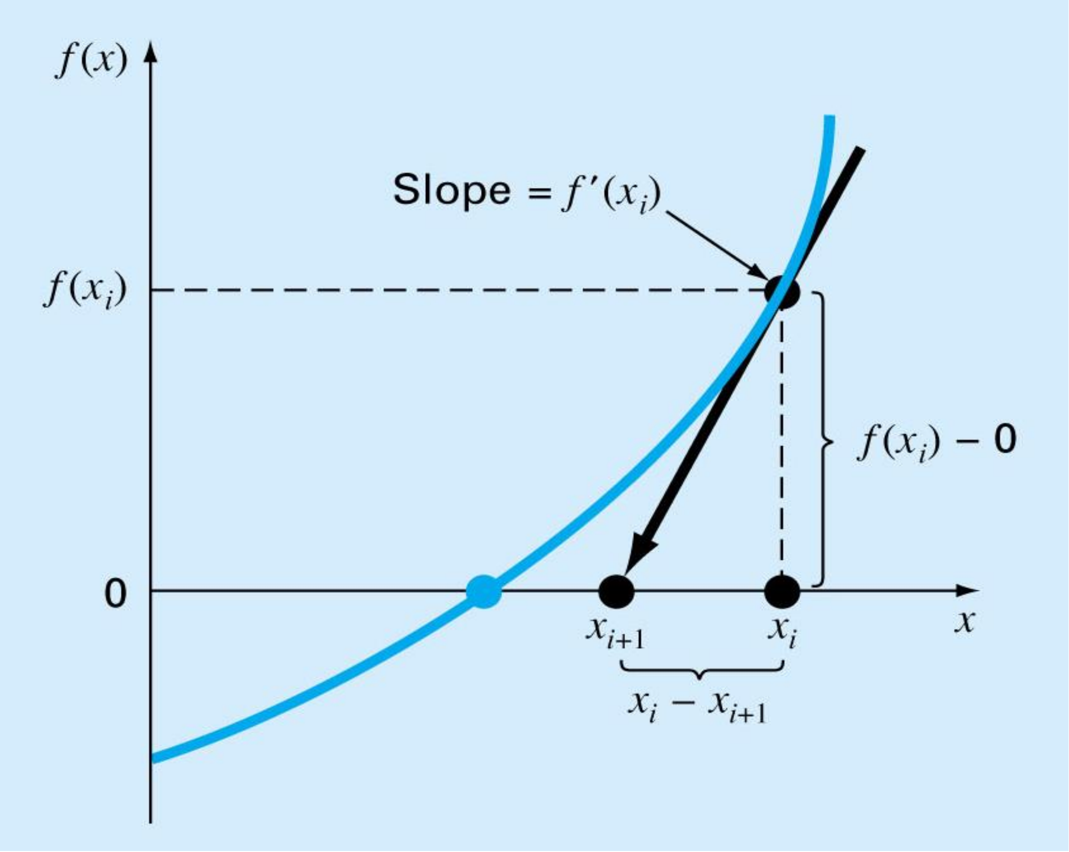
Rearranging,

$$0 = f(x_i) + f'(x_i)(x_{i+1} - x_i)$$

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

Newton-Raphson Method

- Convenient for functions whose derivatives can be evaluated analytically.
- Not convenient for functions whose derivatives cannot be evaluated analytically.



Algorithm for Newton-Raphson

- **Step 1.** Choose one initial guess near the root
- **Step 2.** Find slope of f(x)
- **Step 3.** Extend f'(x) t x-axis to find x i+1 and apply:

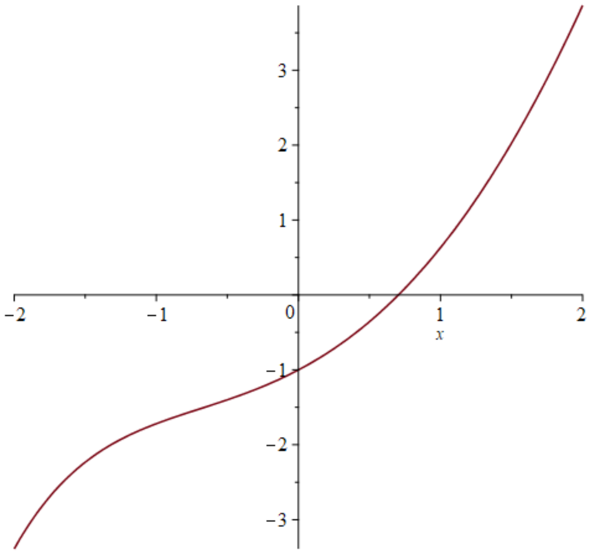
$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

(follow tangent line to x-axis)

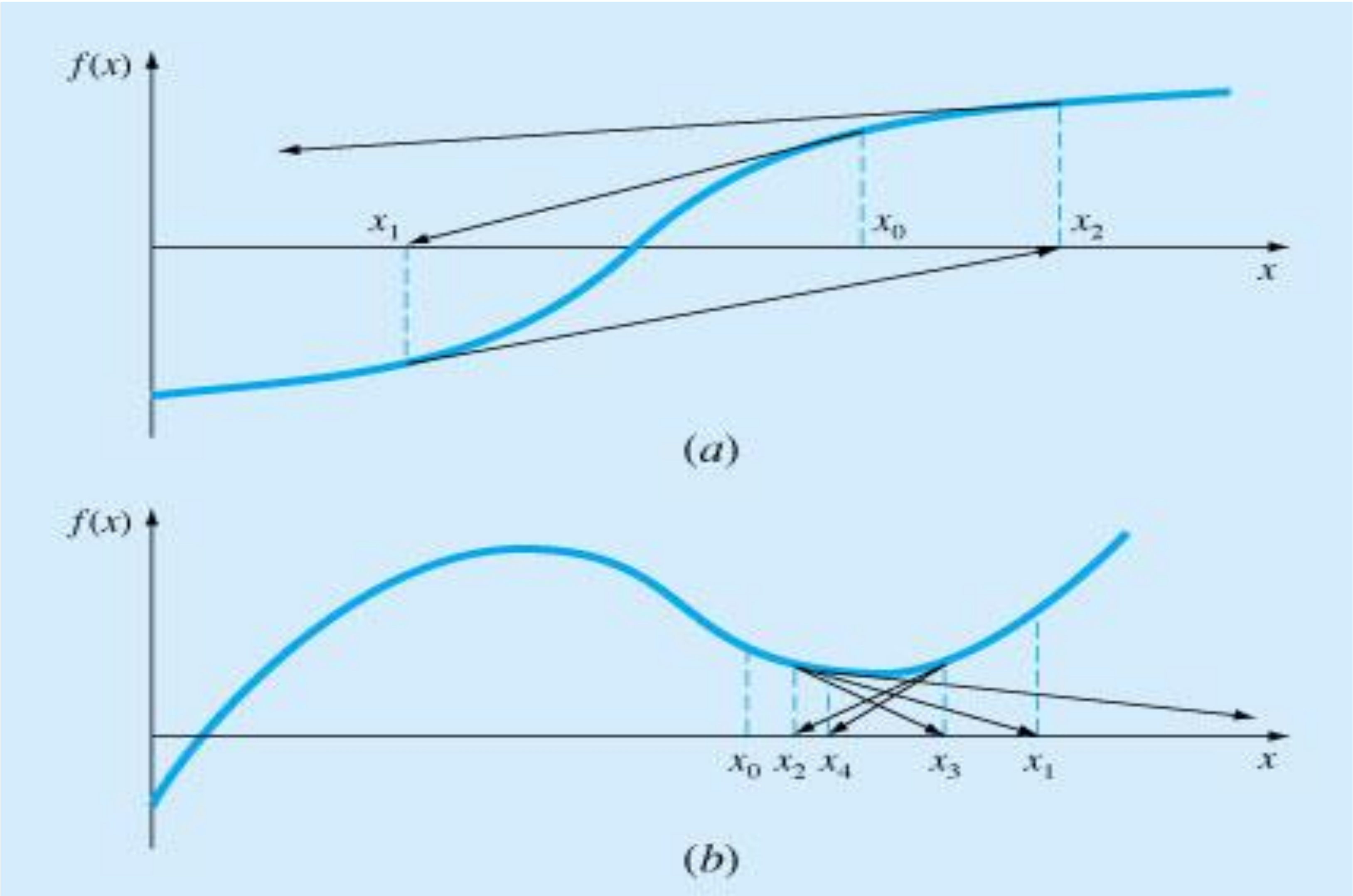
- **Step 4.** Repeat (until converged)

Example. Newton-Raphson Method

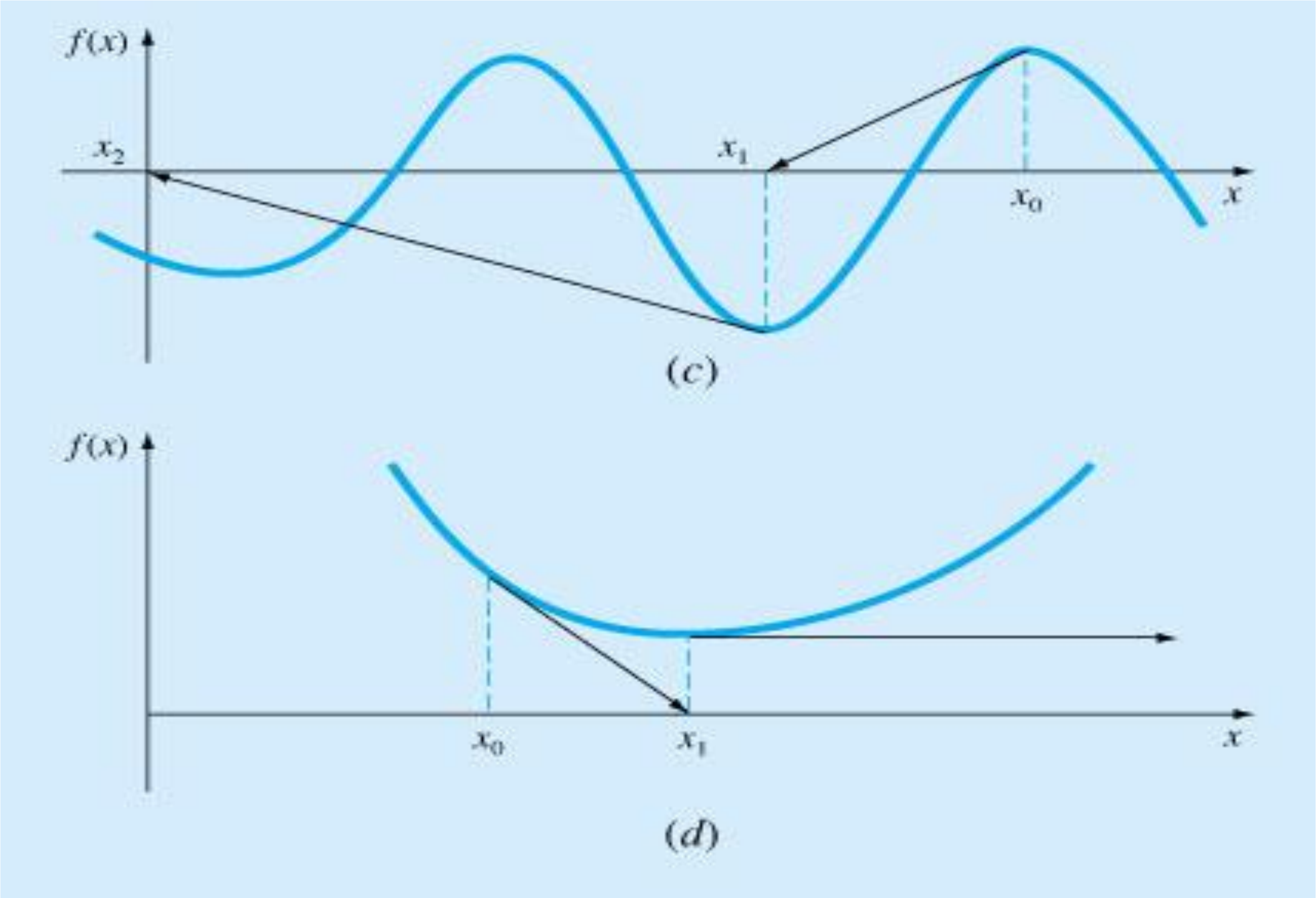
$$f(x) = x^2 - e^{-x}$$



Potential problems: Newton-Raphson



Potential problems: Newton-Raphson



Notes: Newton-Raphson

- Faster convergence – compared to other methods
- Simple algorithm compared to bi-section (no if else)
- Cannot handle multiple roots, can diverge if initial value not chosen properly