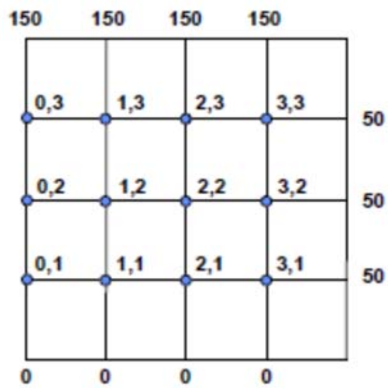
Problem Set #8 Solutions

29.1 The new representation of the plate is



Because the left edge is insulated, the finite-difference equations for the nodes on that edge are written as

(0, 3): $4T_{0,3} - 2T_{1,3} - T_{0,2} = 150$

(0, 2): $4T_{0,2} - T_{0,3} - 2T_{1,2} - T_{0,1} = 0$

(0, 1): $4T_{0,1} - 2T_{1,1} - T_{0,2} = 0$

All the other nodes are represented by Eq. (29.11). The first two iterations of Liebmann's method are

150	150	150	150	
45	58.5	62.55	84.615	50
0	0	0	19.5	50
0	0	0	15	50
0	0	0	0	

150	150	150	150	
75.15	81.09	91.935	86.2509	50
13.5	21.6	32.445	51.978	50
0	0	4.5	19.2	50
0	0	0	0	

After 10 iterations, the maximum approximate error is 0.754% with the result

150	150	150	150	
109.8519	108.8637	104.6117	91.54351	50
71.75664	70.97995	68.0271	61.55182	50
35.49834	35.32963	34.98164	36.63082	50
0	0	0	0	

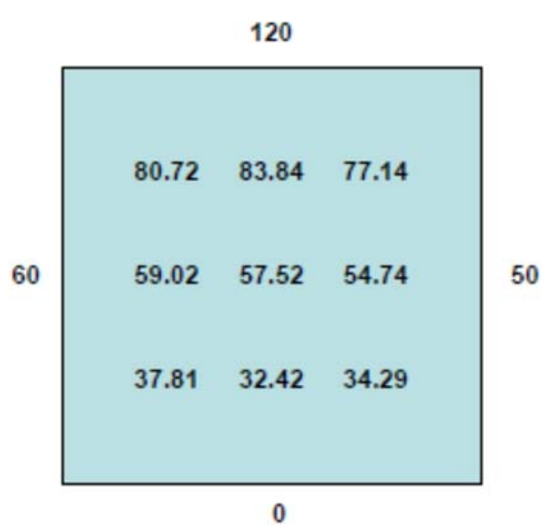
Note that the ultimate result is

150	150	150	150	
109.9655	108.9359	104.6497	91.55766	50
71.9903	71.12847	68.10531	61.58093	50
35.73874	35.48233	35.0621	36.66076	50
0	0	0	0	

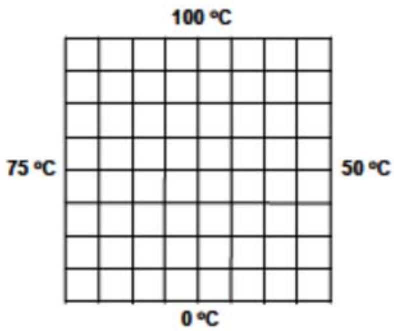
29.2 Here are the results of using Liebmann's method to obtain the solution. Notice that after 6 iterations all the relative error estimates have fallen below 1% and the computation is terminated.

```
iteration = 1
          18                5.4               16.62
          23.4              8.64              22.578
          61.02             56.898            74.8428
ea:
          100               100               100
          100               100               100
          100               100               100
iteration = 2
          23.04             13.41             22.4724
          41.13             38.4768           51.222
          71.2044           79.9776           75.39132
ea:
          21.875            59.73154362       26.04261227
          43.10722101       77.54491018       55.92128382
          14.30304869       28.85758012       0.727563863
•
•
•
iteration = 6
          37.80666066       32.41632148       34.29437766
          59.01952801       57.51727394       54.73884006
          80.71921628       83.84433834       77.14262488
ea:
          0.688181746       0.130204454       0.019913358
          0.010933694       0.020874012       0.049145839
          0.033767672       0.037176168       0.02465349
```

Therefore, the results are:



29.4 The plate is redrawn below



After 15 iterations of the Liebmann method, the result is

	100	100	100	100	100	100	100	
75	85.32617	88.19118	88.54443	87.79909	86.06219	82.39736	73.69545	50
75	78.10995	78.88691	78.1834	76.58771	74.05069	69.82967	62.38146	50
75	73.23512	71.06672	68.71675	66.32057	63.72554	60.48986	55.99875	50
75	68.75568	63.42793	59.30269	56.25934	54.04625	52.40787	51.1222	50
75	63.33804	54.57569	48.80562	45.37425	43.79945	43.97646	46.08048	50
75	54.995	42.71618	35.95756	32.62971	31.80514	33.62176	39.22063	50
75	38.86852	25.31308	19.66293	17.3681	17.16645	19.48972	27.17735	50
	0	0	0	0	0	0	0	

with percent approximate errors of

	0	0	0	0	0	0	0	
0	0.012%	0.011%	0.007%	0.005%	0.004%	0.004%	0.003%	0
0	0.011%	0.012%	0.008%	0.005%	0.006%	0.006%	0.005%	0
0	0.024%	0.010%	0.001%	0.001%	0.004%	0.007%	0.006%	0
0	0.054%	0.016%	0.007%	0.011%	0.002%	0.007%	0.008%	0
0	0.101%	0.040%	0.008%	0.007%	0.003%	0.008%	0.011%	0
0	0.234%	0.119%	0.063%	0.033%	0.012%	0.010%	0.015%	0
0	0.712%	0.292%	0.219%	0.126%	0.030%	0.001%	0.014%	0

29.5 The solution is identical to Prob. 29.4, except that now the bottom edge must be modeled. This means that the nodes along the bottom edge are simulated with equations of the form

$$4T_{i,j} - T_{i-1,j} - T_{i+1,j} - 2T_{i,j+1} = 0$$

The resulting simulation (after 15 iterations) yields

	100	100	100	100	100	100	100	
75	86.4529	90.2627	91.2337	90.6948	88.7323	84.4436	74.8055	50
75	80.5554	83.3649	83.9691	82.8006	79.7785	74.2277	64.7742	50
75	77.4205	78.6771	78.4736	76.7481	73.3375	67.8999	60.0537	50
75	75.5117	75.4794	74.5080	72.3743	68.9073	63.9608	57.5247	50
75	74.2631	73.2996	71.7406	69.3405	65.9436	61.4870	56.0597	50
75	73.4348	71.8433	69.8853	67.3320	64.0357	59.9572	55.1934	50
75	72.8998	70.9218	68.7359	66.1171	62.9167	59.0892	54.7153	50
75	72.5345	70.4401	68.1797	65.5685	62.4467	58.7477	54.5354	50

with percent approximate errors of

	0	0	0	0	0	0	0	
0	0.009%	0.018%	0.024%	0.024%	0.018%	0.011%	0.004%	0
0	0.042%	0.066%	0.074%	0.069%	0.053%	0.034%	0.015%	0
0	0.079%	0.140%	0.155%	0.140%	0.110%	0.071%	0.032%	0
0	0.113%	0.224%	0.260%	0.239%	0.187%	0.124%	0.057%	0
0	0.133%	0.327%	0.388%	0.359%	0.284%	0.190%	0.089%	0
0	0.173%	0.471%	0.544%	0.502%	0.395%	0.261%	0.124%	0
0	0.289%	0.628%	0.709%	0.651%	0.508%	0.330%	0.153%	0
0	0.220%	0.665%	0.779%	0.756%	0.620%	0.407%	0.180%	0

29.18

120	**90.05528**	**74.61385**	**66.58249**	**63.96664**				
100	**82.80365**	**70.9088**	**63.87474**	**61.35079**				
80	**70.2505**	**62.34297**	**56.65689**	**53.68704**				
60	**55.85537**	**51.5557**	**46.72278**	**40.08361**	**26.69476**	**16.97098**	**8.328404**	**0**
40	**41.61528**	**41.30169**	**38.59493**	**33.22986**	**24.86222**	**16.43038**	**8.171316**	**0**
20	**29.30405**	**33.44084**	**33.12539**	**29.37867**	**23.09389**	**15.71702**	**7.926474**	**0**
0	**22.16008**	**30.03222**	**31.08714**	**28.06553**	**22.41766**	**15.41731**	**7.817566**	**0**

30.3 The solution for $\Delta t = 0.1$ is (as computed in Example 30.1),

t	$x = 0$	$x = 2$	$x = 4$	$x = 6$	$x = 8$	$x = 10$
0	100	0	0	0	0	50
0.1	100	2.0875	0	0	1.04375	50
0.2	100	4.087847	0.043577	0.021788	2.043923	50

For $\Delta t = 0.05$, it is

t	$x = 0$	$x = 2$	$x = 4$	$x = 6$	$x = 8$	$x = 10$
0	100	0.000000	0.000000	0.000000	0.000000	50
0.05	100	1.043750	0.000000	0.000000	0.521875	50
0.1	100	2.065712	0.010894	0.005447	1.032856	50
0.15	100	3.066454	0.032284	0.016228	1.533227	50
0.2	100	4.046528	0.063786	0.032229	2.023265	50

To assess the differences between the results, we performed the simulation a third time using a more accurate approach (the Heun method) with a much smaller step size ($\Delta t = 0.001$). It was assumed that this more refined approach would yield a prediction close to true solution. These values could then be used to assess the relative errors of the two Euler solutions. The results are summarized as

	$x = 0$	$x = 2$	$x = 4$	$x = 6$	$x = 8$	$x = 10$
Heun (h = 0.001)	100	4.006588	0.083044	0.042377	2.003302	50
Euler (h = 0.1)	100	4.087847	0.043577	0.021788	2.043923	50
Error relative to Heun		2.0%	47.5%	48.6%	2.0%	
Euler (h = 0.05)	100	4.046528	0.063786	0.032229	2.023265	50
Error relative to Heun		1.0%	23.2%	23.9%	1.0%	

Notice, that as would be expected for Euler's method, halving the step size approximately halves the global relative error.

30.5 The solution is identical to Example 30.3, but with 9 interior nodes. Thus, the simultaneous equations to be solved at the first step are

$$\begin{bmatrix} 2.167 & -0.0835 & & & & & & & \\ -0.0835 & 2.167 & -0.0835 & & & & & & \\ & -0.0835 & 2.167 & -0.0835 & & & & & \\ & & -0.0835 & 2.167 & -0.0835 & & & & \\ & & & -0.0835 & 2.167 & -0.0835 & & & \\ & & & & -0.0835 & 2.167 & -0.0835 & & \\ & & & & & -0.0835 & 2.167 & -0.0835 & \\ & & & & & & -0.0835 & 2.167 & -0.0835 \\ & & & & & & & -0.0835 & 2.167 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{Bmatrix} = \begin{Bmatrix} 16.7 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 8.35 \end{Bmatrix}$$

which can be solved for

$$\begin{Bmatrix} 7.717983 \\ 0.297836 \\ 0.011493 \\ 0.000444 \\ 0.000026 \\ 0.000222 \\ 0.005747 \\ 0.148918 \\ 3.858992 \end{Bmatrix}$$

For the second step, the right-hand side is modified to reflect these computed values of T at $t = 0.1$,

$$\begin{bmatrix} 2.167 & -0.0835 & & & & & & & \\ -0.0835 & 2.167 & -0.0835 & & & & & & \\ & -0.0835 & 2.167 & -0.0835 & & & & & \\ & & -0.0835 & 2.167 & -0.0835 & & & & \\ & & & -0.0835 & 2.167 & -0.0835 & & & \\ & & & & -0.0835 & 2.167 & -0.0835 & & \\ & & & & & -0.0835 & 2.167 & -0.0835 & \\ & & & & & & -0.0835 & 2.167 & -0.0835 \\ & & & & & & & -0.0835 & 2.167 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{Bmatrix} = \begin{Bmatrix} 30.8719 \\ 1.1913 \\ 0.04597 \\ 0.001775 \\ 0.000103 \\ 0.00089 \\ 0.002299 \\ 0.59567 \\ 15.436 \end{Bmatrix}$$

which can be solved for

$$\begin{Bmatrix} 14.28889 \\ 1.102814 \\ 0.063836 \\ 0.003288 \\ 0.000238 \\ 0.001650 \\ 0.031918 \\ 0.551407 \\ 7.144443 \end{Bmatrix}$$