SYDE252 - lecture notes

09/01/18

Presented by: John Zelek
Systems Design Engineering
note: some material (figures) borrowed from various sources

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2. Systems

09/09/18

Presented by: John Zelek
Department of Engineering

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inspiration

- “ knowing reality means constructing systems of transformations that correspond, more or less adequately, to reality” Jean Piaget

- “the way to build a complex system that works is to build it from very simple systems that work” Kevin Kelly

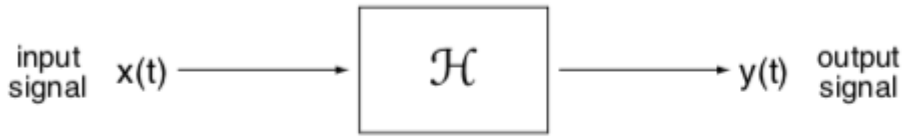
- “The overall name of these interrelated structures is system. The motorcycle is a system. A real system …There’s so much talk about the system. And so little understanding. That’s all a motorcycle is, a system of concepts worked out in steel. There’s no part in it, no shape in it that is not in someone’s mind. I’ve noticed that people who have never worked with steel have trouble seeing this - that the motorcycle is primarily a mental phenomenon.” Robert Pirsig, Zen and the Art of Motorcycle Maintenance

systems

- Systems are used to process signals to modify or extract information. A system is characterized by (i) inputs; (ii) outputs; and (iii) rules of operation (mathematical) model of systems. A study of systems involves mathematical modelling, analysis and design. Analysis is how to determine the system output given the input and a system mathematical model. The design or synthesis is how to design a system that will produce the desired set of outputs for given inputs.

- We only deal with LTI (Linear Time-Invariant) Systems
  - A system is LTI if it is linear and time invariant.
  - A system is linear if it has the scaling + additivity properties.

  - g(x) = H[f(x)], x is the independent variable; H is the System operator or transfer function variable



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scaling property

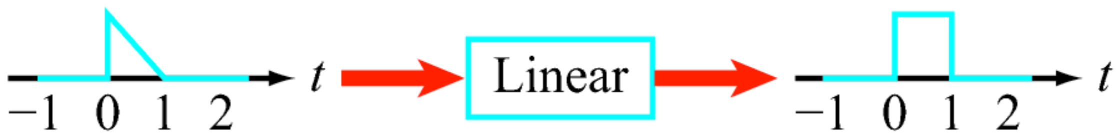
$x \longrightarrow \boxed{\mathrm{L}} \longrightarrow y \qquad \alpha x \longrightarrow \boxed{\mathrm{L}} \longrightarrow \alpha y$

Given: $x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t),$

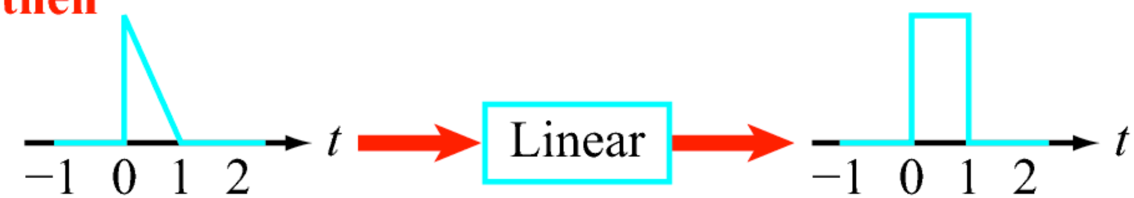
then the system is ***scalable*** (has the scaling property) if

$c\ x(t) \longrightarrow \boxed{\text{System}} \longrightarrow c\ y(t).$

If



then



example

Example 1: System described by Diff. Eq.

$$\frac{d^2y}{dt^2} + 2\,\frac{dy}{dt} + 3y = 4\,\frac{dx}{dt} + 5x. \qquad (2.1)$$

Upon replacing $x(t)$ with $c\,x(t)$ and $y(t)$ with $c\,y(t)$ in all terms, we end up with

$$\frac{d^2}{dt^2}(cy) + 2\,\frac{d}{dt}(cy) + 3(cy) = 4\,\frac{d}{dt}(cx) + 5(cx).$$

Since c is constant, we can rewrite the expression as

$$c\left[\frac{d^2y}{dt^2} + 2\,\frac{dy}{dt} + 3y\right] = c\left[4\,\frac{dx}{dt} + 5x\right], \qquad (2.2)$$

which is identical to the original equation, but multiplied by the constant c. Hence, since the response to $c\,x(t)$ is $c\,y(t)$, the system is scalable and has the scaling property.

example

- Example 2: System described by Diff. Eq.

$$\frac{d^2 y}{dt^2} + 2\,\frac{dy}{dt} + 3y = 4\,\frac{dx}{dt} + 5x + 6$$

Because of the constant term on the far right-hand side of the equation, this system is NOT scalable, and therefore NOT linear.

adding property

$x_1 \rightarrow \boxed{L} \rightarrow y_1$ $\quad$ $x_2 \rightarrow \boxed{L} \rightarrow y_2$ $\quad$ $x_1 + x_2 \rightarrow \boxed{L} \rightarrow y_1 + y_2$

If the system responses to N inputs $x_1(t),\ x_2(t), \ldots, x_N(t)$ are respectively $y_1(t),\ y_2(t), \ldots, y_N(t)$, then the system is ***additive*** if

$$\sum_{i=1}^{N} x_i(t) \rightarrow \boxed{\text{System}} \rightarrow \sum_{i=1}^{N} y_i(t). \qquad (2.4)$$

That is, *the response to the sum is the sum of the responses*.

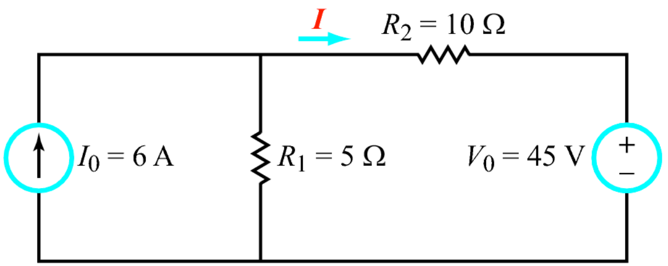
▶ The combination of scalability and additivity is also known as the ***superposition principle***. ◀

superposition example

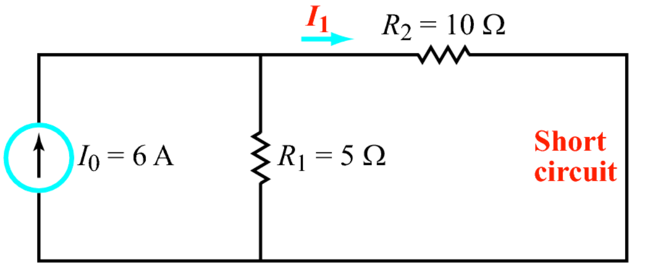
$$I_1 = \frac{I_0 R_1}{R_1 + R_2} = \frac{6 \times 5}{5 + 10} = 2\text{ A}.$$

$$I_2 = \frac{-V_0}{R_1 + R_2} = \frac{-45}{5 + 10} = -3\text{ A}.$$

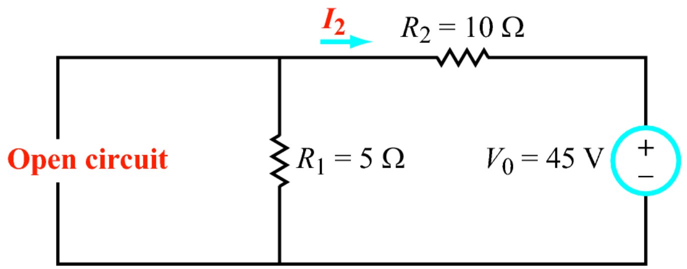
$$I = I_1 + I_2 = 2 - 3 = -1\text{ A}.$$



(a) **Original circuit**



(b) **Source I_0 alone generates I_1**



(c) **Source V_0 alone generates I_2**

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systems - linearity

- $g(x) = H[f(x)]$, x is the independent variable; H is the System operator or transfer function variable
- discrete systems require analog to digital and vice versa
- let $g_i(x) = H[f_i(x)]$ where $f_i(x)$ is an arbitrary input in the class of all inputs $\{f_n(x)\}$ and $g(x)$ is the corresponding output
  if $H[\alpha_i f_i(x) + \alpha_j f_j(x)] = \alpha_i H[f_i(x)] + \alpha_j H[f_j(x)] = \alpha_i g_i(x) + \alpha_j g_j(x)$, then the system H is called a linear system. A linear system has the properties of additivity and homogeneity.

systems - response of a system

- response of a linear system

The total response is equal to the zero input response plus the zero state response. The output of a system for $t \geq 0$ is the result of 2 independent causes: the inital conditions of the system (or system state) at $t = 0$ and the input for $f(t)$ when $t \geq 0$. Beccause of linearity, the total response is the sum of the responses due to those causes, that being (1) zero-input response is only due to initial conditions; and (2) zero-state response is only due to input. This is called the decomposition property.

systems linearity - example 1

(a) system $y(t) = 2\pi x(t)$? is it linear?

① $a x_1(t) \xrightarrow{H} 2\pi a x_1(t) = a(2\pi x_1(t))$

② $b x_2(t) \xrightarrow{H} 2\pi b x_2(t) = b(2\pi x_2(t))$

③ $a x_1(t) + b x_2(t) \xrightarrow{H} 2\pi(a x_1(t) + b x_2(t))$

RHS① + RHS② = RHS③

$\therefore$ system is linear

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systems linearity - example 2

(b) $y[n] = (x[2n])^2$ is not linear?

① $ax_1[n] \rightarrow y_1[n] = a^2 (x_1[2n])^2$

② $bx_2[n] \rightarrow y_2[n] = b^2 (x_2[2n])^2$

③ $ax_1[n] + bx_2[n] \rightarrow y_c[n]$

$$y_c[n] = (ax_1[2n] + bx_2[2n])^2$$
$$= a^2 x_1^2[2n] + b^2 x_2^2[2n]$$
$$+ 2ab\,x_1[2n]\,x_2[2n]$$

RHS① + RHS② $\neq$ RHS③ $\rightarrow$ ∴ NOT LINEAR

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systems - time-invariance

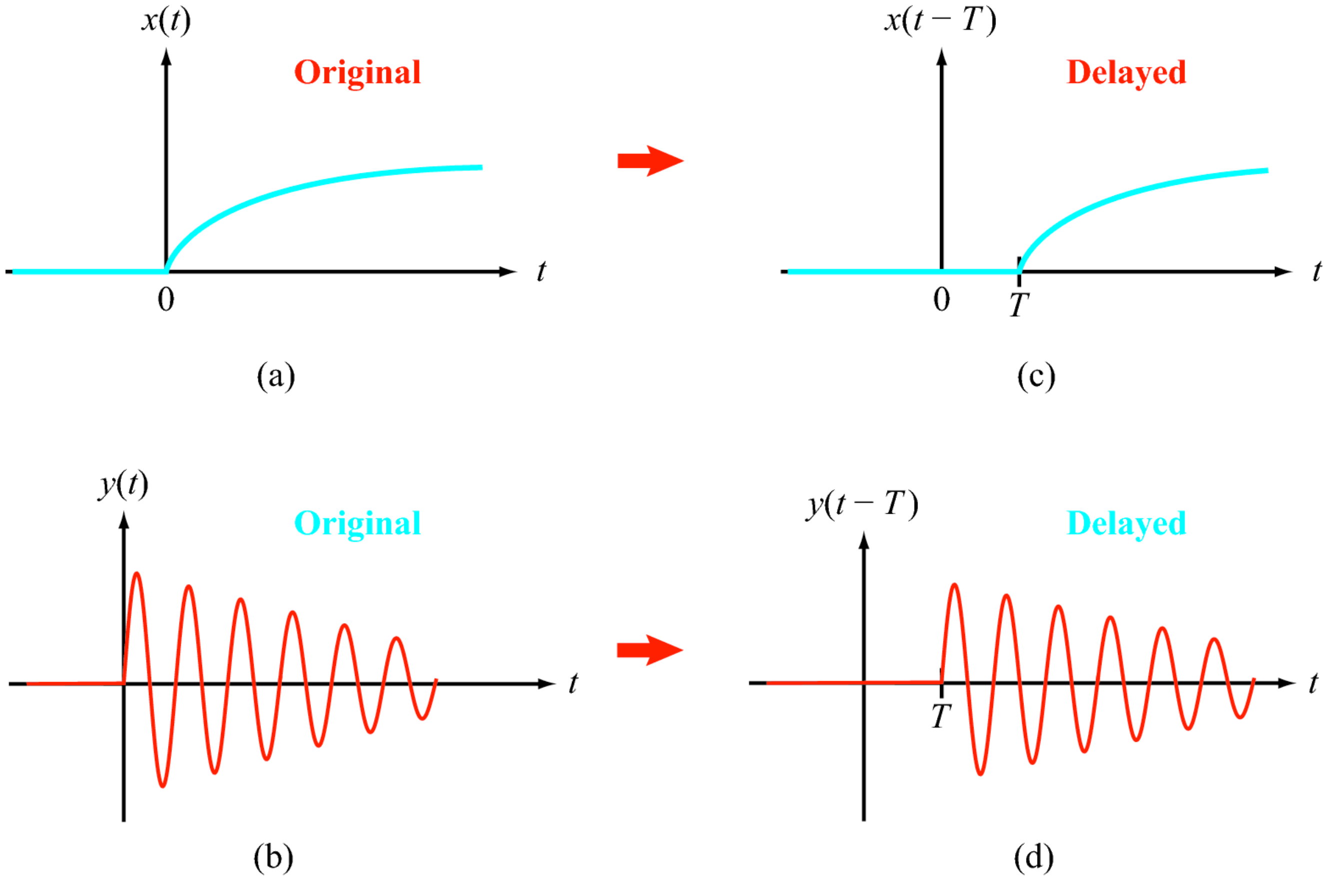
▶ A system is ***time-invariant*** if delaying the input signal $x(t)$ by any constant T generates the same output $y(t)$, but delayed by exactly T. ◀

$$\text{Given: } x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t),$$

then the system is ***time-invariant*** if

$$x(t-T) \longrightarrow \boxed{\text{System}} \longrightarrow y(t-T).$$

systems - time-invariance examples



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systems - time-invariance

- time-invariance formal definition

A system is time-invariant if a time shift in the input produces the same time shift in the output. For a system F s.t. $y(t) = Fx(t)$, implies that $y(t-\tau) = Fx(t-\tau)$ for any time shift τ.

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systems - time-invariance example 1

$$x(t) \rightarrow \boxed{S} \rightarrow y(t)$$

$$x(t) \rightarrow \sin(x(t))$$

$$x(t-t_0) \rightarrow \sin(x(t-t_0))$$

(a) $y(t) = \sin(x(t))$ is time invariant

$$y(t) = \sin(x(t))$$
$$\Rightarrow \sin(x(t-t_0))$$

$$x(t-t_0) \rightarrow y(t-t_0)$$

for any $t_0 \in \mathbb{R}$ (real)

let $y_1(t)$ be output of $x_1(t)$

$$y_1(t) = \sin[x_1(t)] \Rightarrow \sin[x_1(t-t_0)]$$

now check

$$y(t) = y(t-t_0)$$
$$= \sin[x(t-t_0)]$$

$\therefore$ TI

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systems - time-invariance example 2

(b) $y[n] = n x[n]$ is not time invariant

show by counter example

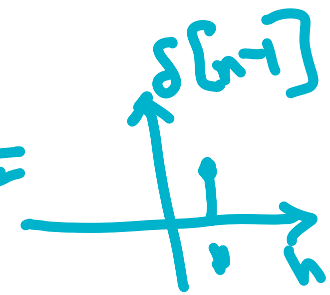
let $x[n] = \delta[n]$

then $y[n] = n\,\delta[n] = 0, \forall n$

let $x_1[n] = x[n-1] = \delta[n-1]$

then $y_1[n] = n x_1[n] = n\,\delta[n-1]$

$= \delta[n-1]$ =



however

$y[n-1] = (n-1)x[n-1] = (n-1)\,\delta(n-1) = 0, \forall n$

$\therefore$ not TI

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systems - linearity & time-invariance examples

Example 3 Linearity & Time-Invariance

	LINEAR	T.I.
$y(t) = Ax(t)$	Y	Y
$y(t) = Ax(t) + B,\ B \neq 0$	N	Y
$y[n] = n\,x[n]$	Y	N
$y(t) = x(t)\cos(\omega_c t)$	* Y	N *
$y[n] = x[-n]$	Y	N *
$y(t) = x^2(t-1)$	N	Y
$y[n] = \dfrac{1}{1 - x[n-2]}$	N	Y *
$y(t) = e^{3x(t)}$	N	Y *

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systems - linearity & time-invariance more examples

- Time-Invariant

(a) $y_1(t) = 3\,\dfrac{d^2x}{dt^2}$,

(b) $y_2(t) = \sin[x(t)]$

(c) $y_3(t) = \dfrac{x(t+2)}{x(t-1)}$

Not Time-Invariant

(d) $y_4(t) = t\;x(t)$,

(e) $y_5(t) = x(t^2)$,

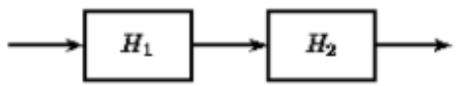
(f) $y_6(t) = x(-t)$.

Note1: Systems b and c are time-invariant, but not linear.

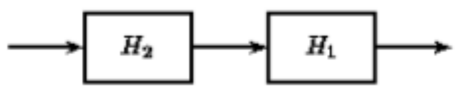
Note2: Systems d to f are linear, but not time-invariant.

systems - linearity & time-invariance more examples

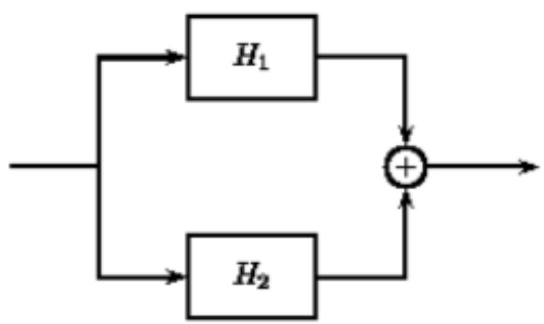
Cascaded (series) LTI systems order cannot be changed without changing the result of the output.

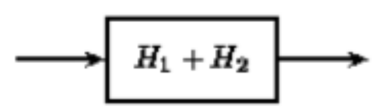


(a)



Parallel systems can be condensed into the sum of systems.





systems - causality

We define a ***causal system*** as a system for which the present value of the output $y(t)$ can only depend on present and past values of the input $\{x(\tau), \ \tau \leq t\}$. For a noncausal system, the present output could depend on future inputs. Noncausal systems are also called ***anticipatory*** systems, since they anticipate the future.

A physical *system* must *be causal*, because a noncausal system must have the ability to see into the future! For example, the noncausal system $y(t) = x(t+2)$ must know the input two seconds into the future to deliver its output at the present time. This is clearly impossible in the real world.

we will see this later when we talk about impulse response of a system:

▶ An LTI system is causal *if and only if* its impulse response is a causal function: $h(t) = 0$ for $t < 0$. ◀

systems - causality examples

Examples

CAUSAL?

1. $y(t) = Ax(t)$ Y
2. $y(t) = Ax(t) + B, B \neq 0$ Y
3. $y[n] = (n+1)x[n]$ Y
4. $y(t) = x(t)\cos(\omega_c(t+1))$ Y
5. $y[n] = x[-n]$ N
6. $y[n] = \frac{1}{3}(x[n+1] + x[n] + x[n-1])$ N
7. $y[n] = \frac{3}{1 - x[n+2]}$ N
8. $y(t) = e^{3x(t)}$ Y

A causal (also known as a physical or nonanticipative) system is one for which the output at any instant t_0 depends only on the value of the input $x(t)$ for $t \leq t_0$.

systems - stability

A linear system H is said to be stable if its response to any bounded input is bounded. That is,
$|f(x)| < K$, implies that $|g(x)| < cK$ where $g(x) + H[f(x)]$, and K and c are constants.

systems - stability

A signal $x(t)$ is ***bounded*** if there is a constant C so that $|x(t)| \le C$ for all t. Examples of bounded signals include:

- $\cos(3t)$, $7e^{-2t}\,u(t)$, and $e^{2t}\,u(1-t)$.

Examples of unbounded signals include:

- t^2, $e^{2t}\,u(t)$, e^{-t}, and $1/t$.

A system is ***BIBO (bounded input/bounded output) stable*** every bounded input $x(t)$ results in a bounded output $y(t)$,

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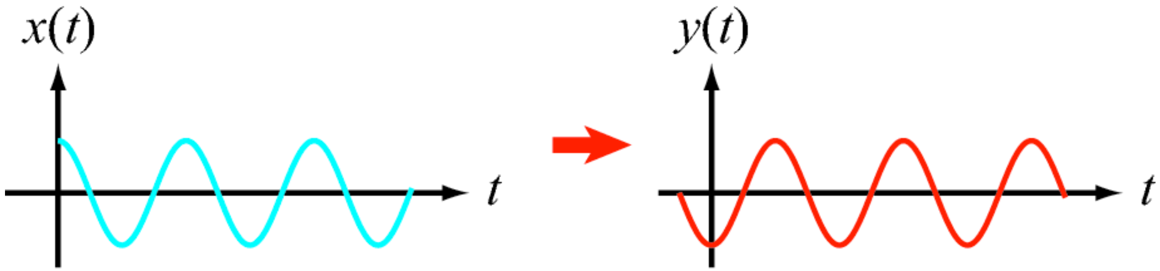
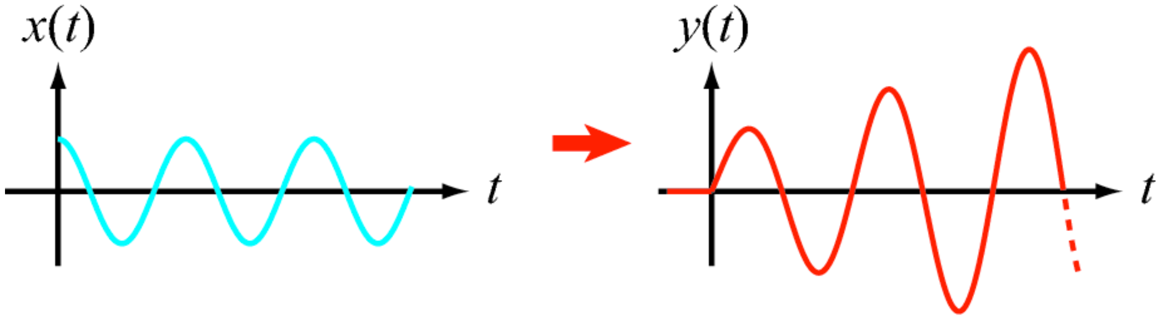
systems - stability examples



(a) BIBO-stable system



(b) Non-BIBO-stable system

systems - stability examples

2-6.4 BIBO Stability of System with Decaying Exponentials

Consider a causal system with an impulse response

$$h(t) = Ce^{\gamma t}\, u(t), \tag{2.93}$$

where C is a finite constant and γ is, in general, a finite complex coefficient given by

$$\gamma = \alpha + j\beta, \quad \alpha = \mathfrak{Re}[\gamma], \quad \text{and} \quad \beta = \mathfrak{Im}[\gamma]. \tag{2.94}$$

Such a system is BIBO stable if and only if $\alpha < 0$ (i.e., $h(t)$ is a one-sided exponential with an exponential coefficient whose real part is negative). To verify the validity of this statement, we test to see if $h(t)$ is absolutely integrable. Since $|e^{j\beta t}| = 1$ and $e^{\alpha t} > 0$,

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{0}^{\infty} |Ce^{\alpha t} e^{j\beta t}|\, dt$$
$$= |C| \int_{0}^{\infty} e^{\alpha t}\, dt. \tag{2.95}$$

(a) $\boldsymbol{\alpha} < 0$

If $\alpha < 0$, we can rewrite it as $\alpha = -|\alpha|$ in the exponential, which leads to

$$\int_{-\infty}^{\infty} |h(t)|\, dt = |C| \int_{0}^{\infty} e^{-|\alpha| t}\, dt = \frac{|C|}{|\alpha|} < \infty. \tag{2.96}$$

Hence, $h(t)$ is absolutely integrable and the system is BIBO stable.

(b) $\boldsymbol{\alpha} \geq 0$

If $\alpha \geq 0$, Eq. (2.95) becomes

$$\int_{-\infty}^{\infty} |h(t)|\, dt = |C| \int_{0}^{\infty} e^{\alpha t}\, dt \rightarrow \infty,$$

thereby proving that the system is not BIBO stable when $\alpha \geq 0$.

systems - stability examples

Example 6

STABLITY – are BIBO? stable?

1. $y(t) = A x(t)$, $|A| < \infty$ Y

2. $y(t) = A x(t) + B$
$|A|, |B| < \infty$, $B \neq 0$ Y

3. $y[n] = n x[n]$ N

4. $y(t) = x(t) \cos(\omega_c t)$ Y

5. $y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2])$ Y

6. $y[n] = r^n x[n]$, $|r| < 1$ N

if $n < 0$ $y[n] \to \infty$

7. $y[n] = \dfrac{1}{1 - x[n+2]}$ N

as $x[n+2] \to 1$
$y[n] \to \infty$

8. $y(t) = e^{3x(t)}$ Y

systems - memoryless

A system is memory-less if it only depends on the input signal value at that same time t. A memoryless system is always causal, though the reverse is not always true.

systems - memoryless examples

Example 4

which have memory?

1. $y(t) = Ax(t)$ N
2. $y(t) = \frac{1}{C} \int_{-\infty}^{t} x(\tau) d\tau$ Y
3. $y[n] = nx[n]$ N
4. $y(t) = x(t) \cos(\omega_c(t-1))$ N
5. $y[n] = x[-n]$ Y
6. $y(t) = a_0 + a_1 x(t) + a_2 x^2(t) + \ldots$ N
7. $y[n] = \frac{1}{3}[x[n] + x[n-1] + x[n-2]]$ Y
8. $y(t) = e^{3x(t)}$ N

systems - invertible

An input can always be recovered from the output $x(t) = H^{inv}\{y(t)\} = H^{inv}\{Hx(t)\}$

systems - invertible examples

$$y(t) \rightarrow \boxed{H^{-1}} \rightarrow x(t)$$

Example 7 INVERTIBILITY

invertible??

1. $y(t) = A\,x(t)$, $A \neq 0$ — Y — $x(t) = \frac{y(t)}{A}$

2. $y(t) = Ax(t) + B$, $A, B \neq 0$ — Y — $x(t) = \frac{y(t) - B}{A}$

3. $y[n] = n\,x[n]$ — N — $x[n] = \frac{y[n]}{n}$ but let $x_1 =$

4. $y(t) = \frac{1}{L}\int_{-\infty}^{t} x(\tau)\,d\tau$ — Y — integral needs to converge — $x(t) = L\frac{dy(t)}{dt}$

5. $y[n] = x[-n]$ — Y — $x[n] = y[-n]$

6. $y(t) = x^2(t-1)$ — N — $x_1(t) = 1$, $x_2(t) = -1$ $\Rightarrow y(t) = 1$

7. $y[n] = \sum_{k=-\infty}^{n} x[k]$ — Y — $x[n] = y[n] - y[n-1]$

8. $y(t) = e^{3x(t)}$ — Y — $x(t) = \frac{\ln y(t)}{3}$

$$x(t) \rightarrow \boxed{H} \rightarrow y(t)$$

Linear Differential Equations

- define a system

$$\frac{d^N y}{dt^N} + a_1 \frac{d^{N-1} y}{dt^{N-1}} + \ldots + a_{N-1} \frac{dy}{dt} = b_{N-M} \frac{d^M x}{dt^M} + b_{N-M+1} \frac{d^{M-1} x}{dt^{M-1}} + \ldots + b_{N-1} \frac{dx}{dt} + b_N x(t)$$

we can use compact notation D operator for $\frac{d}{dt}$

$$(D^N + a_1 D^{N-1} + \ldots a_{N-1} D + a_N) y(t) = (b_{N-M} D^M + b_{N-M+1} D^{M-1} + \ldots + b_{N-1} D + b_N)$$ OR

$$Q(D)y(t) = P(D)x(t)$$
$$Q(D) = (D^N + a_1 D^{N-1} + \ldots a_{N-1} D + a_N)$$
$$P(D) = (b_{N-M} D^M + b_{N-M+1} D^{M-1} + \ldots + b_{N-1} D + b_N)$$

Impulse Response of a System

- define a system

The ***impulse response*** $h(t)$ of a system is (logically enough) the response of the system to an impulse $\delta(t)$. Similarly, the ***step response*** $y_{\text{step}}(t)$ is the response of the system to a unit step $u(t)$. In symbolic form:

$$\delta(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow h(t) \tag{2.7a}$$

and

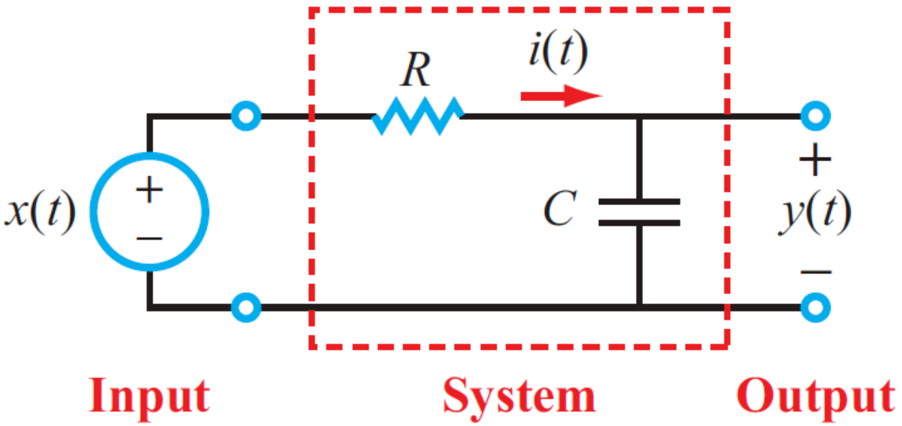
$$u(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y_{\text{step}}(t). \tag{2.7b}$$

System Response: computing h(t) & s(t) (step response)

$$R\, i(t) + y(t) = x(t)$$

$$i(t) = C\,\frac{dy}{dt}$$

$$\frac{dy}{dt} + \frac{1}{RC}\,y(t) = \frac{1}{RC}\,x(t)$$



(a) RC circuit

System Response: h(t) (impulse response)

To compute the ***impulse response***, we label $x(t) = \delta(t)$ and $y(t) = h(t)$ and obtain

$$\frac{dh}{dt} + \frac{1}{RC}\,h(t) = \frac{1}{RC}\,\delta(t). \tag{2.11}$$

Next, we introduce the ***time constant*** $\tau_c = RC$ and multiply both sides of the differential equation by the ***integrating factor*** e^{t/τ_c}. The result is

$$\frac{dh}{dt}\,e^{t/\tau_c} + \frac{1}{\tau_c}\,e^{t/\tau_c}\,h(t) = \frac{1}{\tau_c}\,e^{t/\tau_c}\,\delta(t). \tag{2.12}$$

The left side of Eq. (2.12) is recognized as

$$\frac{dh}{dt}\,e^{t/\tau_c} + \frac{1}{\tau_c}\,e^{t/\tau_c}\,h(t) = \frac{d}{dt}\left[h(t)\,e^{t/\tau_c}\right], \tag{2.13a}$$

and the sampling property of the impulse function given by Eq. (1.27) reduces the right-hand side of Eq. (2.12) to

$$\frac{1}{\tau_c}\,e^{t/\tau_c}\,\delta(t) = \frac{1}{\tau_c}\,\delta(t). \tag{2.13b}$$

Incorporating these two modifications in Eq. (2.12) leads to

$$\frac{d}{dt}\left[h(t)\,e^{t/\tau_c}\right] = \frac{1}{\tau_c}\,\delta(t). \tag{2.14}$$

Integrating both sides from 0^- to t gives

$$\int_{0^-}^{t} \frac{d}{d\tau}\left[h(\tau)\,e^{\tau/\tau_c}\right] d\tau = \frac{1}{\tau_c}\int_{0^-}^{t} \delta(\tau)\,d\tau, \tag{2.15}$$

$$\frac{dy}{dt} + \frac{1}{RC}\,y(t) = \frac{1}{RC}\,x(t)$$

$$h(t) = \frac{1}{\tau_c}\,e^{-t/\tau_c}\,u(t)$$

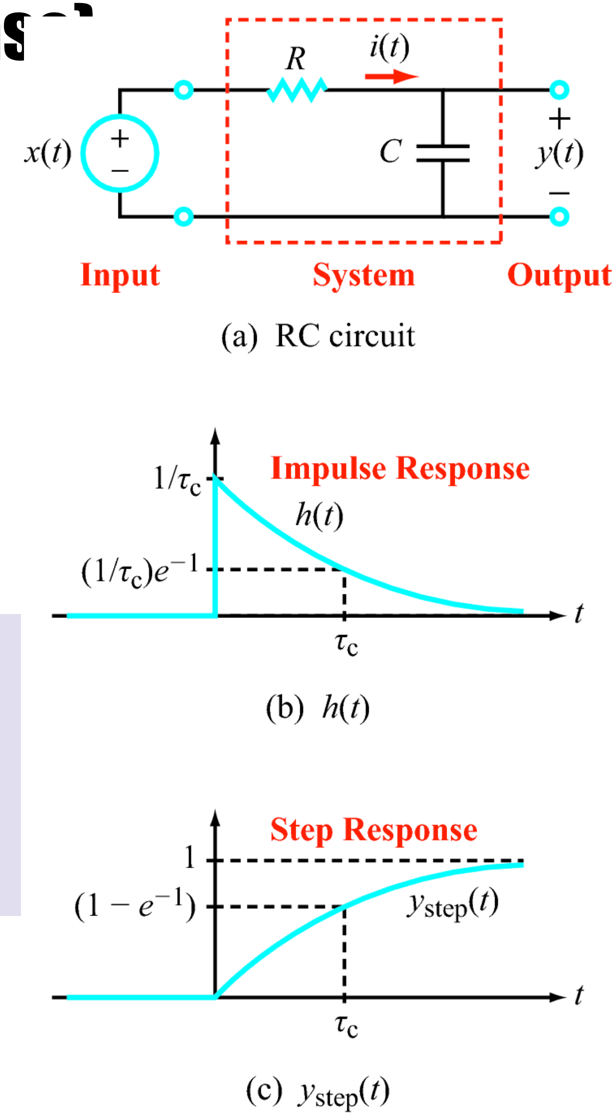
System Response: h(t) (impulse response)

$$h(t) = \frac{1}{\tau_c} e^{-t/\tau_c} u(t).$$

(impulse response of the RC circuit)

$$y_{\text{step}}(t) = [1 - e^{-t/\tau_c}]\, u(t).$$

(step response of the RC circuit)



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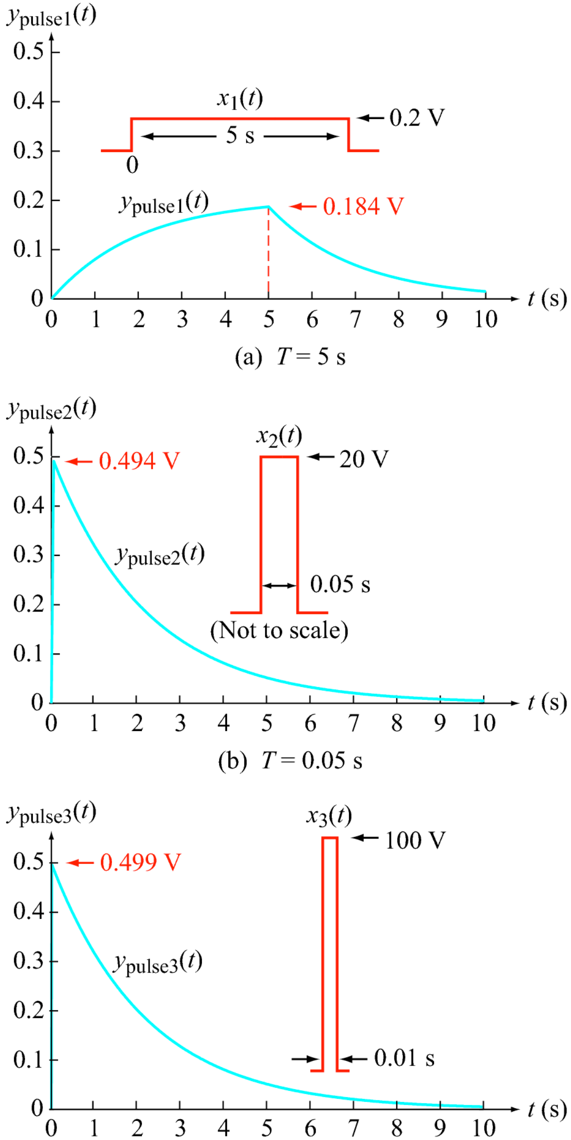
System Response: simulating response with RC = 2 sec



(a) $T = 5$ s

(b) $T = 0.05$ s

(c) $T = 0.01$ s

For a perfect impulse at the input, h(t)=0.5 at t=0.

notice, narrowing step width, response starts to look more like impulse response

impulse response from step response

Step 1: Physically *measure* the step response $y_{\text{step}}(t)$.

Step 2: Differentiate it to obtain

$$h(t) = \frac{dy_{\text{step}}}{dt}.$$

$$\frac{du}{dt} = \delta(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow h(t) = \frac{dy_{\text{step}}}{dt}$$

ramp response from step response

$$r(t) = \int_{-\infty}^{t} u(\tau)\, d\tau$$

$$r(t) = \int_{-\infty}^{t} u(\tau)\, d\tau \longrightarrow \boxed{\text{LTI}} \longrightarrow y_{\text{ramp}}(t)$$

ramp response from step response - example

$$y_{\text{ramp}}(t) = \int_{-\infty}^{t} y_{\text{step}}(\tau)\, d\tau$$

$$= \int_{-\infty}^{t} (1 - e^{-\tau/\tau_{\text{c}}})\, u(\tau)\, d\tau$$

$$= \int_{0}^{t} (1 - e^{-\tau/\tau_{\text{c}}})\, d\tau$$

$$= [t - \tau_{\text{c}}(1 - e^{-t/\tau_{\text{c}}})]\, u(t).$$

$$y_{\text{step}}(t) = [1 - e^{-t/\tau_{\text{c}}}]\, u(t).$$

(step response of the RC circuit)

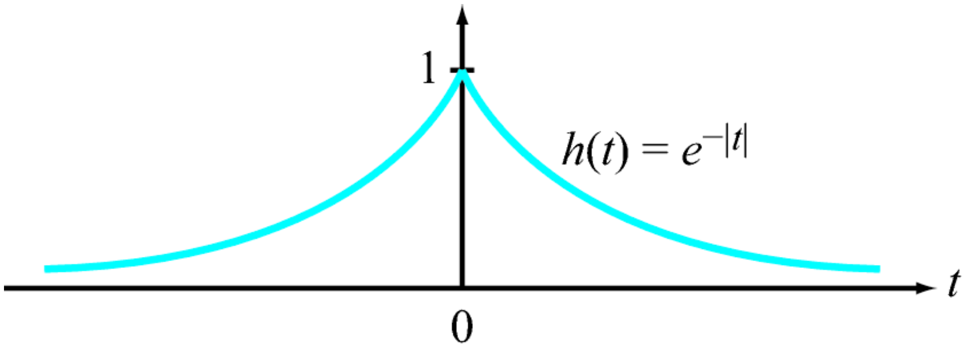
system properties revisited

▶ An LTI system is causal ***if and only if*** its impulse response is a causal function: $h(t) = 0$ for $t < 0$. ◀

▶ An LTI system is BIBO stable *if and only if* its impulse response $h(t)$ is ***absolutely integrable*** (i.e., if $\int_{-\infty}^{\infty} |h(t)|\, dt$ is finite). ◀

system properties revisited - h(t) & BIBO

Example



$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-\infty}^{\infty} e^{-|t|}\, dt = 2\int_{0}^{\infty} e^{-t}\, dt = 2.$$

Hence, the system is BIBO stable.

system properties revisited - h(t) & BIBO

2-6.4 BIBO Stability of System with Decaying Exponentials

Consider a causal system with an impulse response

$$h(t) = Ce^{\gamma t}\, u(t), \qquad (2.93)$$

where C is a finite constant and γ is, in general, a finite complex coefficient given by

$$\gamma = \alpha + j\beta, \quad \alpha = \mathfrak{Re}[\gamma], \quad \text{and} \quad \beta = \mathfrak{Im}[\gamma]. \qquad (2.94)$$

Such a system is BIBO stable if and only if $\alpha < 0$ (i.e., $h(t)$ is a one-sided exponential with an exponential coefficient whose real part is negative). To verify the validity of this statement, we test to see if $h(t)$ is absolutely integrable. Since $|e^{j\beta t}| = 1$ and $e^{\alpha t} > 0$,

$$\begin{aligned}\int_{-\infty}^{\infty} |h(t)|\, dt &= \int_{0}^{\infty} |Ce^{\alpha t} e^{j\beta t}|\, dt \\ &= |C| \int_{0}^{\infty} e^{\alpha t}\, dt. \qquad (2.95)\end{aligned}$$

(a) $\boldsymbol{\alpha} < 0$

If $\alpha < 0$, we can rewrite it as $\alpha = -|\alpha|$ in the exponential, which leads to

$$\int_{-\infty}^{\infty} |h(t)|\, dt = |C| \int_{0}^{\infty} e^{-|\alpha| t}\, dt = \frac{|C|}{|\alpha|} < \infty. \qquad (2.96)$$

Hence, $h(t)$ is absolutely integrable and the system is BIBO stable.

(b) $\boldsymbol{\alpha} \geq 0$

If $\alpha \geq 0$, Eq. (2.95) becomes

$$\int_{-\infty}^{\infty} |h(t)|\, dt = |C| \int_{0}^{\infty} e^{\alpha t}\, dt \to \infty,$$

thereby proving that the system is not BIBO stable when $\alpha \geq 0$.

system properties revisited - h(t) & BIBO

▶ By extension, for any positive integer N, an impulse response composed of a linear combination of N exponential signals

$$h(t) = \sum_{i=1}^{N} C_i e^{\gamma_i t}\, u(t) \qquad (2.97)$$

is absolutely integrable, and its LTI system is BIBO stable, if and only if all of the exponential coefficients γ_i have negative real parts. This is a fundamental attribute of LTI system theory. ◀

system response to complex exponential

▶ By extension, for any positive integer N, an impulse response composed of a linear combination of N exponential signals

$$h(t) = \sum_{i=1}^{N} C_i e^{\gamma_i t} u(t) \tag{2.97}$$

is absolutely integrable, and its LTI system is BIBO stable, if and only if all of the exponential coefficients γ_i have negative real parts. This is a fundamental attribute of LTI system theory. ◀

system response - solving differential equations

for a linear system: the **total response** is equal to the **zero input response** plus **zero state response**
This is like adding the homogeneous solution to the particular solution however the zero input response and the zero state response have a physical meaning whereas the homogeneous and particular solutions do not. The zero input response $y_0(t)$ is the solution of the system equation when $x(t) = 0$ or putting it in another way, when the input is zero! The zero state response is the response of the system with the input applied and we should know how to do that now? or do we?

NOTE: zero input response = zero input and IC;
zero state response = input with 0 IC

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ZI

system response - example 1

$Q(D)y(t) = P(D)x(t)$

$(D^2+4D+3)y(t) = (D+1)x(t)$

(a) $Q(D) = D^2 + 4D + 3$

$y(0^-) = 0$ $\quad y(0^-)=0$

$\dot{y}(0^-) = 1$ $\quad \dot{y}(0^-)=1$

let $Q(D)=0$, $x(t)=0$

$\lambda^2+4\lambda+3=0 \Rightarrow \lambda_1=-1, \lambda_2=-3$

$y_{ZI}(t) = (c_1e^{-t} + c_2e^{-3t})u(t)$

$$\boxed{y_{ZI}(t) = c_1e^{-t}+c_2e^{-3t}, \quad t\geq 0}$$

$y(0^-)=0=c_1+c_2$

$\dot{y}(0^-)=1=-c_1-3c_2$

$\Rightarrow c_2 = -\frac{1}{2}$, $c_1 = \frac{1}{2}$

$$y_{ZI}(t) = \left(\frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t}\right)u(t)$$

system response - example 2

$$P(D)x(t) = Q(D)y(t)$$

ⓑ $Q(D) = (D+1)^2(D+2)$ $\lambda_{1,2} = -1$ $\lambda_3 = -2$

$$y(0^-) = \dot{y}(0^-) = 0, \quad \ddot{y}(0^-) = 1$$

$$y_{zI}(t) = \left[(c_1 + c_2 t)e^{-t} + c_3 e^{-2t}\right]u(t)$$

$(c_1 + c_2 t + c_3 t^2)$

using IC

$$\left.\begin{aligned} 0 &= c_1 + c_3 \\ 0 &= -c_1 + c_2 - 2c_3 \\ 1 &= c_1 + 2c_2 + 4c_3 \end{aligned}\right\} \quad c_1 = -\frac{1}{5}, \quad c_2 = \frac{1}{5} = c_3$$

$$y_{zI}(t) = \left(-\frac{1}{5} + \frac{t}{5}\right)e^{-t} + \frac{e^{-2t}}{5}, \quad t \geq 0$$

$$= \left[\left(-\frac{1}{5} + \frac{t}{5}\right)e^{-t} + \frac{e^{-2t}}{5}\right]u(t)$$

system response - example 3

(c) $Q(D) = D^2(D+2)$ $\qquad \lambda_{1,2} = 0$, $\lambda_3 = -2$

$y(0^-) = \dot{y}(0^-) = 0 \qquad \ddot{y}(0^-) = 1$

$y_{zi}(t) = (c_1 + c_2 t) + c_3 e^{-2t}, \; t \geq 0$

$$\left.\begin{aligned} 0 &= c_1 + c_3 \\ 0 &= c_2 - 2c_3 \\ 1 &= 4c_3 \end{aligned}\right\} \quad c_1 = -\frac{1}{4} \quad c_2 = \frac{1}{2} \quad c_3 = \frac{1}{4}$$

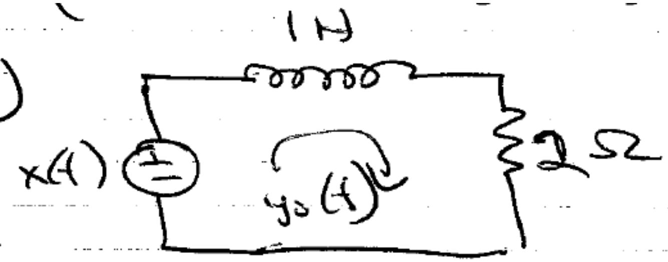
$\therefore \; y_{zI}(t) = \frac{1}{4}\left(2t - 1 + e^{-2t}\right) u(t)$

system response - example 4

(d) $(D+2)y(t) = x(t)$

$\lambda = -2$

$\therefore y(t) = ce^{-2t}$



$$x(t) = L\frac{dy}{dt} + Ry(t)$$

$$= \frac{dy}{dt} + 2y(t)$$

$$= \frac{d}{dt}\left(ce^{-2t}\right) + 2ce^{-2t}$$

$$= -2ce^{-2t} + 2ce^{-2t} = 0$$

system response - example 5

ⓔ find $y_{zi}(t)$ for LTI

$$(D^2+4D+40)y(t) = (D+2)x(t)$$

w/ IC $y_0(0)=2$, $\dot{y}_0(0)=16.78$

char^n poly^n $\lambda^2+4\lambda+40 = (\lambda^2+4\lambda+4)+36$

$$= (\lambda+2)^2+(6)^2$$

$$= (\lambda+2-j6)(\lambda+2+j6)$$

complex roots $\lambda_{1,2} = -2 \pm j6$

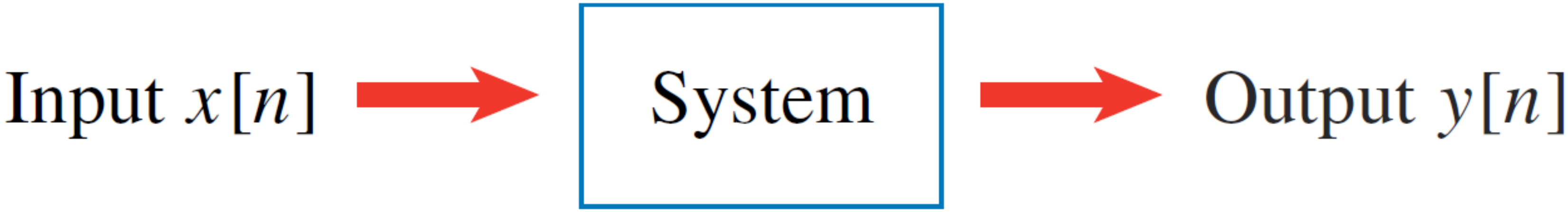
$$y_{zI}(t) = ce^{-2t}\cos(6t+\theta)$$

$$\dot{y}_{zi}(t) = -2ce^{-2t}\cos(6t+\theta) - 6ce^{-2t}\sin(6t+\theta)$$

$$\left.\begin{aligned} &① \quad 2 = c\cos\theta \\ &② \quad 16.78 = -2c\cos\theta - 6c\sin\theta \end{aligned}\right\} \quad \begin{aligned} &c = 4 \\ &\theta = \tan^{-1}\left(\frac{-3.463}{2}\right) = \frac{-\pi}{3} \end{aligned}$$

$$y_{zi}(t) = 4e^{-2t}\cos\left(6t - \frac{\pi}{3}\right)$$

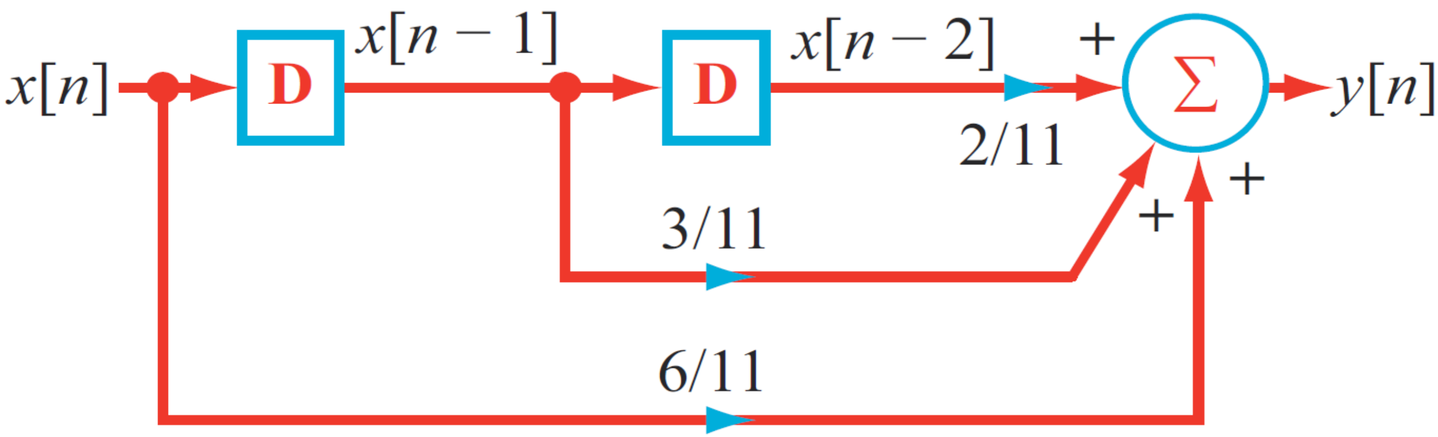
system response - discrete- difference equations



Example: Weighted Moving-Average (MA) of 3 most recent inputs:

$$y[n] = \frac{6}{11}\,x[n] + \frac{3}{11}\,x[n-1] + \frac{2}{11}\,x[n-2].$$

$y[n] = \left(\frac{6}{11} + \frac{3}{11}D + \frac{2}{11}D^2\right)x[n]$



system response - discrete vs continuous equations

Discrete Time:

Difference Equation

$$\sum_{i=0}^{N} a_i \; y[n-i] = \sum_{i=0}^{M} b_i \; x[n-i].$$

Continuous Time:

Differential Equation

$$\sum_{i=0}^{N} c_{N-i} \frac{d^i y}{dt^i} = \sum_{i=0}^{M} d_{M-i} \frac{d^i x}{dt^i}$$

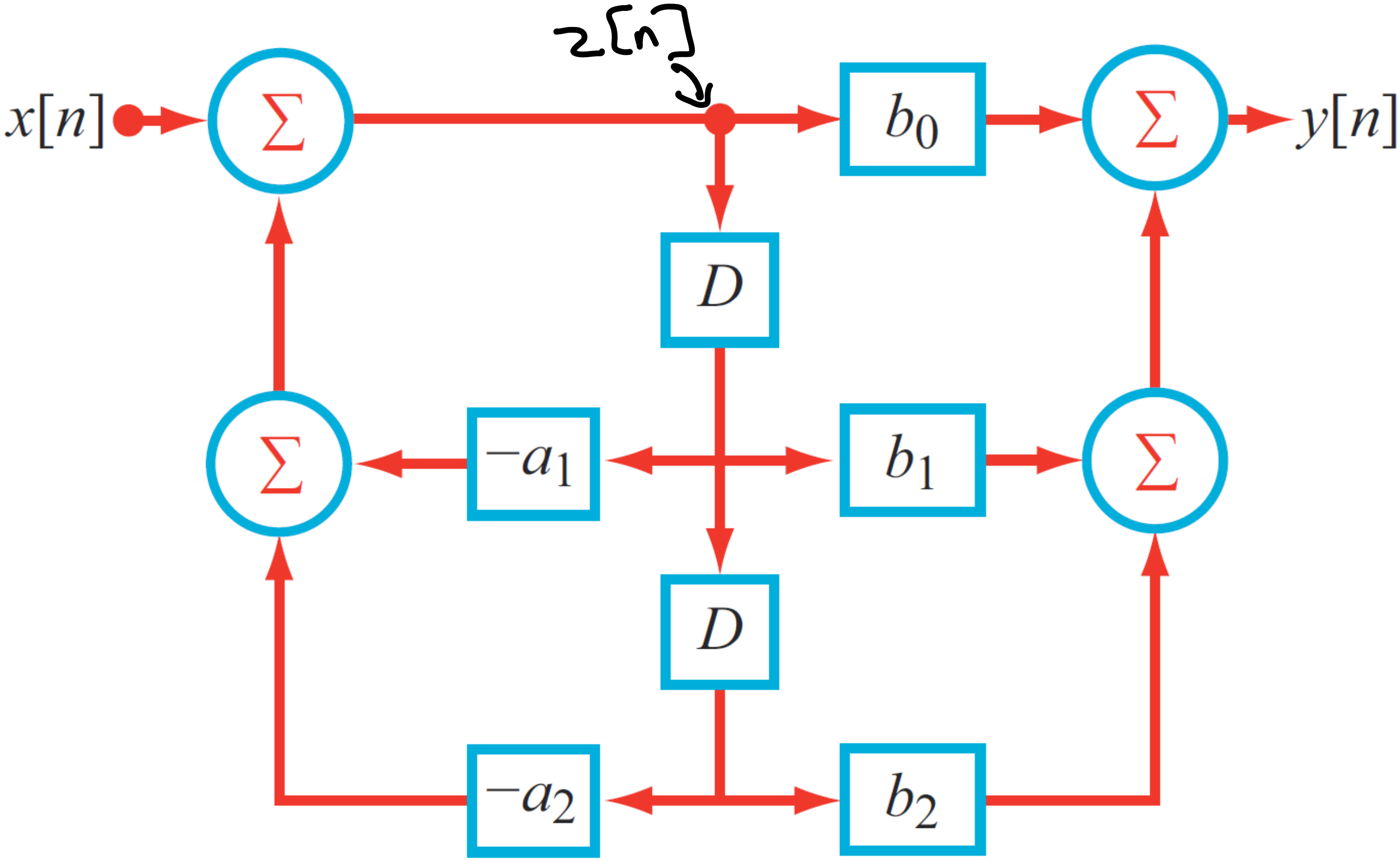
An ARMA difference equation has two parts:

$$\sum_{i=0}^{N} a_i \; y[n-i] = x[n] \qquad \textbf{(autoregressive)}.$$

$$y[n] = \sum_{i=0}^{M} b_i \; x[n-i] \qquad \textbf{(moving average)}.$$

system response - ARMA difference equation realization

$$y[n] + a_1\, y[n-1] + a_2\, y[n-2] = b_0\, x[n] + b_1\, x[n-1] + b_2\, x[n-2].$$



discrete time LTI

Definition of an LTI (Linear Time-Invariant) system is the same in discrete time as in continuous time:

A system is linear if it has the scaling and additivity properties.

A system is time-invariant if delaying the input delays the output.

A system is LTI if it is both linear and time-invariant.

ARMA difference equations with constant coefficients are LTI.

$$\sum_{i=0}^{N} a_i \; y[n-i] = \sum_{i=0}^{M} b_i \; x[n-i].$$

This describes an LTI system.

impulse response of a MA

The impulse response h[n] of a system is its response to an impulse:

$$\delta[n] \longrightarrow \boxed{\text{System}} \longrightarrow h[n].$$

We can read off the impulse response of an MA difference equation:

$$y[n] = \sum_{i=0}^{M} b_i \, x[n-i] \longleftrightarrow h[n] = \{\underline{b_0}, b_1, \ldots, b_M\}.$$

Example: Impulse response of the system described by the MA difference equation

$$y[n] = 2x[n] - 3x[n-1] - 4x[n-2]. \qquad h[n] = \{\underline{2}, -3, -4\}$$

discrete vs cts. properties

Property:	Discrete Time:	Continuous Time:
Causality:	$h[n] = 0$ for all $n < 0$	$h(t) = 0$ for $t < 0$
Stability:	$\sum_{n=-\infty}^{\infty} \lvert h[n] \rvert$ is finite	$\int_{-\infty}^{\infty} \lvert h(t) \rvert \, dt$ is finite
Impulse response has the form	$h[n] = \sum_{i=1}^{N} \mathbf{C}_i \mathbf{p}_i^n \, u[n].$	$h(t) = \sum_{i=1}^{N} C_i e^{\gamma_i t} \, u(t)$
Stability:	$\lvert \mathbf{p}_i \rvert < 1$	Real[ ]<0.

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discrete vs cts. properties

Property:	Discrete Time:	Continuous Time:
Causality:	$h[n] = 0$ for all $n < 0$	$h(t) = 0$ for $t < 0$
Stability:	$\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert$ is finite	$\int_{-\infty}^{\infty} \lvert h(t)\rvert \, dt$ is finite
Impulse response has the form	$h[n] = \sum_{i=1}^{N} \mathbf{C}_i \mathbf{p}_i^n \; u[n].$	$h(t) = \sum_{i=1}^{N} C_i e^{\gamma_i t} \; u(t)$
Stability:	$\lvert \mathbf{p}_i \rvert < 1$	Real[ ]<0.

formulating a difference equation

$y[n] = x[n] + \frac{1}{3} y[n-1]$

$y[-1] = 2$

$x[n] = 2$

Joe likes coffee, and he drinks his coffee according to a very particular routine. He begins by adding two teaspoons of sugar to his mug which he then fills to the brim with hot coffee. He drinks $\frac{2}{3}$ of the mug's contents, adds another two teaspoons of sugar, and tops the mug off with steaming hot coffee. This refill procedure continues, sometimes for many, many cups of coffee. Joe has noted that his coffee tends to taste sweeter with the number of refills.

Let independent variable n designate the coffee refill number. In this way, $n = 0$ indicates the first cup of coffee, $n = 1$ is the first refill and so forth. Let $x[n]$ represent the sugar (measured in teaspoons) added into the system (a coffee mug) on refill n. Let $y[n]$ designate the amount of sugar (again, teaspoons) contained in the mug on refill n.

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formulating a difference equation (2)

formulating a difference equation (2)

Let us say you have share data prices, e.g., ibm. To calculate the average share price over the last 5 days written as a difference equation is:

$y_n = \frac{1}{5}(x_n + x_{n-1} + x_{n-2} + x_{n-3} + n_{n-4})$

This is a digital filter and is called a moving average.

What happens to the original input signal?, What is removed?

The moving average is a form of a low pass filter.

difference equations - filters

IF the impulse response has finite duration (non-zero values do not last forever), this kind of filter is called a **Finite Impulse Response** filter of FIR filter.

general form is

$y_n = \sum_{k=0}^{N-1} b_k x_{n-k}$

IIR Filters is where the output of the filter depends on both the current and previous inputs as well as previous outputs.

The general form of an IIR filter is

$\sum_{m=0}^{M-1} a_m y_{n-m} = \sum_{k=0}^{N-1} b_k x_{n-k}$

difference equations - filters

recursively

not all ~~recruveively~~ defined filters will be IIR, consider
$y_n = y_{n-1} + \frac{1}{3}x_n - \frac{1}{3}x_{n-3}$

note to get a step response, set $u_n = \sum_{k=0}^{n} \delta_n$ as the input and
see the response, the step response will be of the form
$g_n = \sum_{k=0}^{n} h_k$

LCCDE (Linear Constant Coefficent Difference Equation)
$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$
I.C. $y[-1], y[-2], ..., y[-N]$

difference equations - examples 1

e.g., $y_n = \frac{1}{3}\left[x_n + x_{n-1} + x_{n-2}\right]$ for $n = 4, 5, 6, 4, 3, 7$

$$y_0 = \frac{4}{3} = 1.3$$
$$y_1 = \frac{1}{3}(5+4) = 3$$
$$y_2 = \frac{1}{3}(6+5+4) = 5$$
$$y_3 = \frac{1}{3}(4+6+5) = 5$$
$$y_4 = \frac{1}{3}(3+4+6) = \frac{13}{3} = 4.3$$
$$y_5 = \frac{1}{3}(7+3+4) = \frac{14}{3} = 4.7$$

4	5	6	4	3	7
1.3	3	5	5	4.3	4.7

difference equations - examples 1

$y_n = \frac{1}{3}\left[x_n + x_{n-1} + x_{n-2}\right]$

impulse response

what is impulse response of a filter?

$x_n = \delta_n = 1, 0, 0, 0, \ldots$

$y_0 = \frac{1}{3}(1 + 0 + 0) = \frac{1}{3}$

$y_1 = \frac{1}{3}(0 + 1 + 0) = \frac{1}{3}$

$y_2 = \frac{1}{3}(0 + 0 + 1) = \frac{1}{3}$

$y_3 = y_4 = \ldots = 0$

$\therefore h_n = \left\{ \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, \ldots \right\}$

difference equations - examples 2

FIR filters

- how about a weighted avg filter

$$y_n = \frac{1}{4} x_n + \frac{1}{2} x_{n-1} + \frac{1}{4} x_{n-2}$$

OR// $$y_n = \sum_{k=0}^{2} b_k x_{n-k}$$

where filter length is 3 & coefficients are [1, 2, 1]/4

$$h_n = \left\{ \frac{1}{4}, \frac{1}{2}, \frac{1}{4} \right\}$$

- in general we can write a FIR filter in the form

$$y_n = \sum_{k=0}^{N-1} b_k x_{n-k}$$

- impulse response of a FIR filter is just a sequence of the coefficients

$$h_0 = \sum_{k=0}^{N-1} b_k x_{0-k} = b_0 x_0 + b_1 x_{-1} + \dots = b_0$$

$$h_1 = \sum_{k=0}^{N-1} b_k x_{1-k} = b_0 x_1 + b_1 x_0 + \dots = b_1$$

$$h_2 = \sum_{k=0}^{N-1} b_k x_{2-k} = b_0 x_2 + b_1 x_1 + b_2 x_0 + \dots = b_2$$

difference equations - examples 2

FIR filters

- how about a weighted avg filter

$$y_n = \frac{1}{4} x_n + \frac{1}{2} x_{n-1} + \frac{1}{4} x_{n-2}$$

OR// $$y_n = \sum_{k=0}^{2} b_k x_{n-k}$$

where filter length is 3 & coefficients are [1, 2, 1]/4

$$h_n = \{ \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{4} \}$$

- in general we can write a FIR filter in the form

$$y_n = \sum_{k=0}^{N-1} b_k x_{n-k}$$

- impulse response of a FIR filter is just a sequence of the coefficients

$$h_0 = \sum_{k=0}^{N-1} b_k x_{0-k} = b_0 x_0 + b_1 x_{-1} + \dots = b_0$$

$$h_1 = \sum_{k=0}^{N-1} b_k x_{1-k} = b_0 x_1 + b_1 x_0 + \dots = b_1$$

$$h_2 = \sum_{k=0}^{N-1} b_k x_{2-k} = b_0 x_2 + b_1 x_1 + b_2 x_0 + \dots = b_2$$

difference equations - examples 3

IIR filters & Impulse:

e.g., $y_n = \frac{1}{2}\left[y_{n-1} + x_n\right]$ ← recursive

let $x_n = \underset{\uparrow}{4}, 3, 2, 1, 0, 0$

$\therefore y_0 = \frac{1}{2}(0) + \frac{1}{2}(4) = 2$

$y_1 = 2.5$

$y_2 = 2.25$

$y_3 = 1.625$

$\vdots$

difference equations - examples 4

e.g., $y_n = y_{n-1} + \frac{1}{3}x_n - \frac{1}{3}x_{n-3}$

impulse $h_0 = 0 + \frac{1}{3} - 0 = \frac{1}{3}$

$h_1 = \frac{1}{3}$

$h_2 = \frac{1}{3} + 0 + 0 = \frac{1}{3}$

$h_3 = h_4 \ldots = 0$

$h_3 = \frac{1}{3} + 0 - \frac{1}{3} = 0$


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