SYDE252 - lecture notes

09/01/18

Presented by: John Zelek
Systems Design Engineering
note: some material (figures) borrowed from various sources

UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING

4. Laplace

09/11/18

Presented by: John Zelek
Department of Engineering

UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING

inspiration

- “You can certainly use the Laplace transform on this problem, but that would be akin to slicing your bread with a chainsaw: it’s overkill, it’s difficult to set up, and if you let your 5-year-old try it someone is going to end up calling 911. Well, maybe not that last one.” — Electrical engineering professor

Laplace

The Laplace transform is a tool to map signal and system behaviours from the time domain into the frequency domain. We break $x(t)$ into exponential components of the form e^{st}, where s is the complex frequency $s = \alpha + j\omega$. The Laplace transform applies to general signals and not just sinusoids (e.g., the Fourier only applies to sinusoids and is a subset of the Laplace).

The Laplace transform is:

$$X(\alpha,\omega) = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

and the inverse Laplace transform is

$$x(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}ds$$

where c is a constant chosen to ensure convergence of the first integral. The path of integration along $c + j\omega$ with ω varying from $-\infty$ to ∞ must lie in the ROC (Region of Convergence).

Let us assume that all signals we will be dealing with are **causal**, meaning the they are on-sided (or unilateral) Laplace transform.

ιω

s-Domain

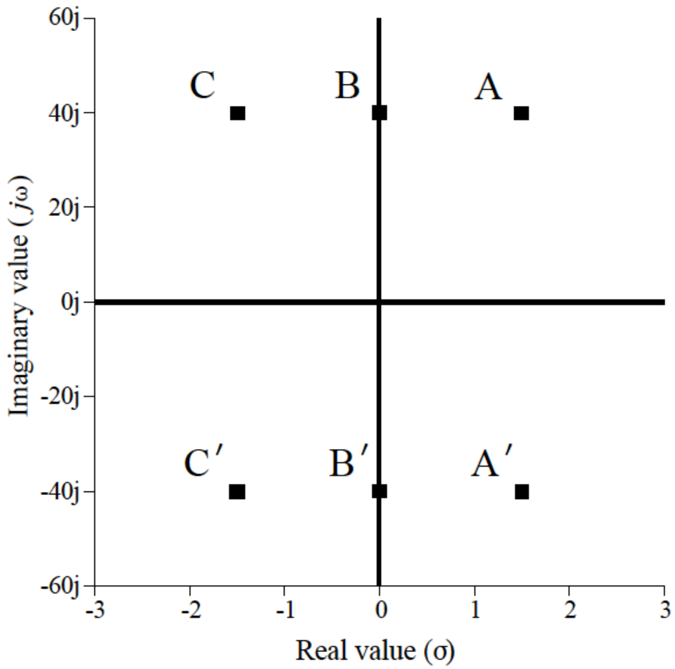
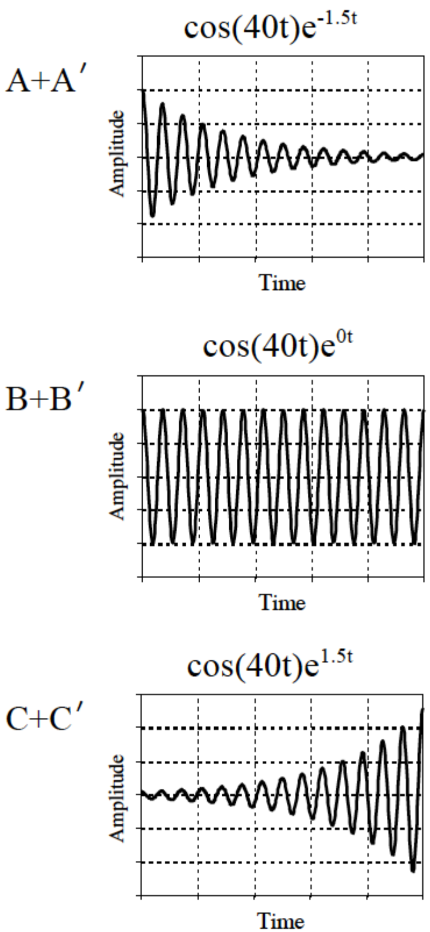


FIGURE 32-2
Waveforms associated with the s-domain. Each location in the s-domain is identified by two parameters: σ and ω. These parameters also define two waveforms associated with each location. If we only consider *pairs* of points (such as: A&A′, B&B′, and C&C′), the two waveforms associated with each location are sine and cosine waves of frequency ω, with an exponentially changing amplitude controlled by σ.

Associated Waveforms



dspguide.com

UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING

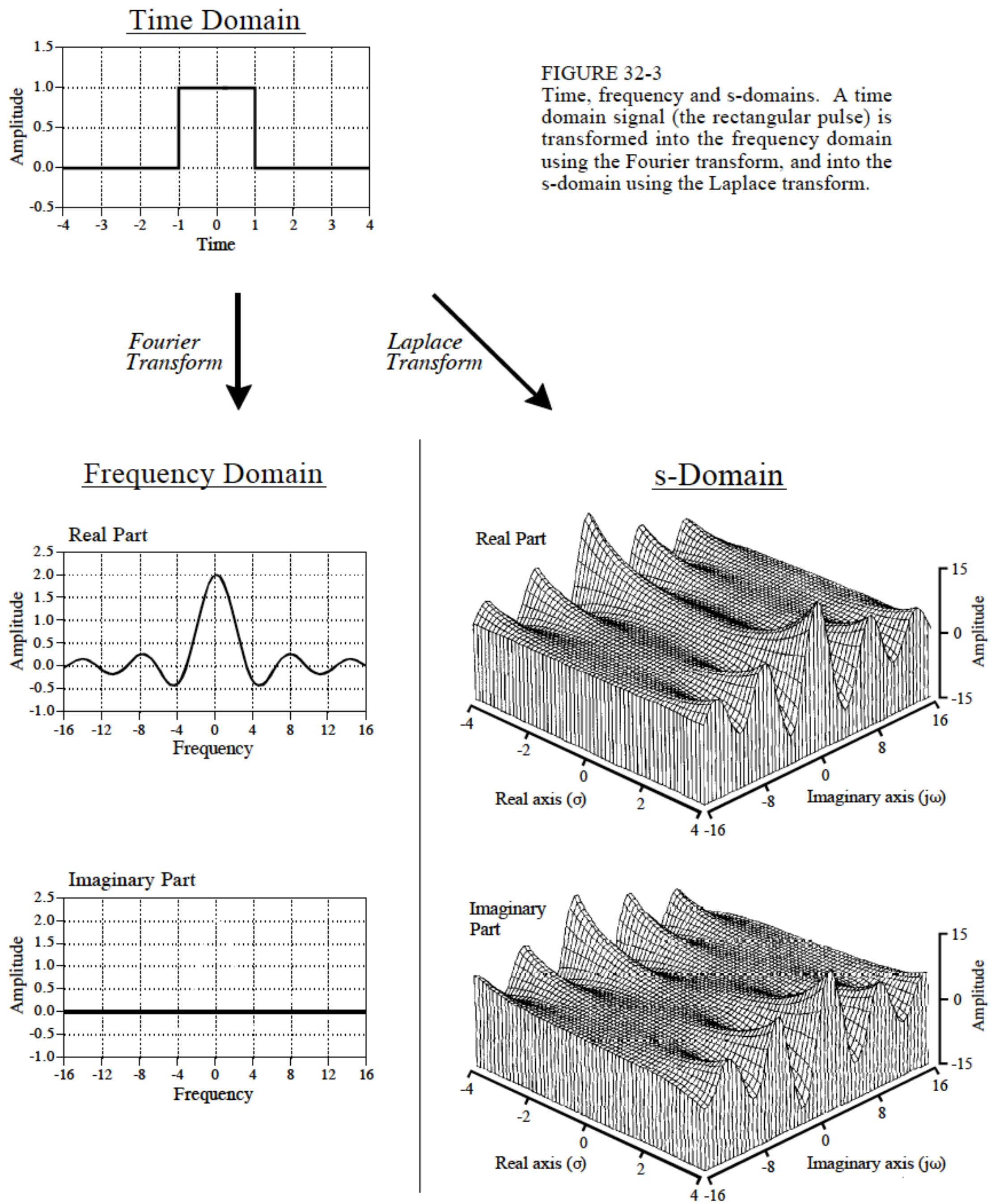
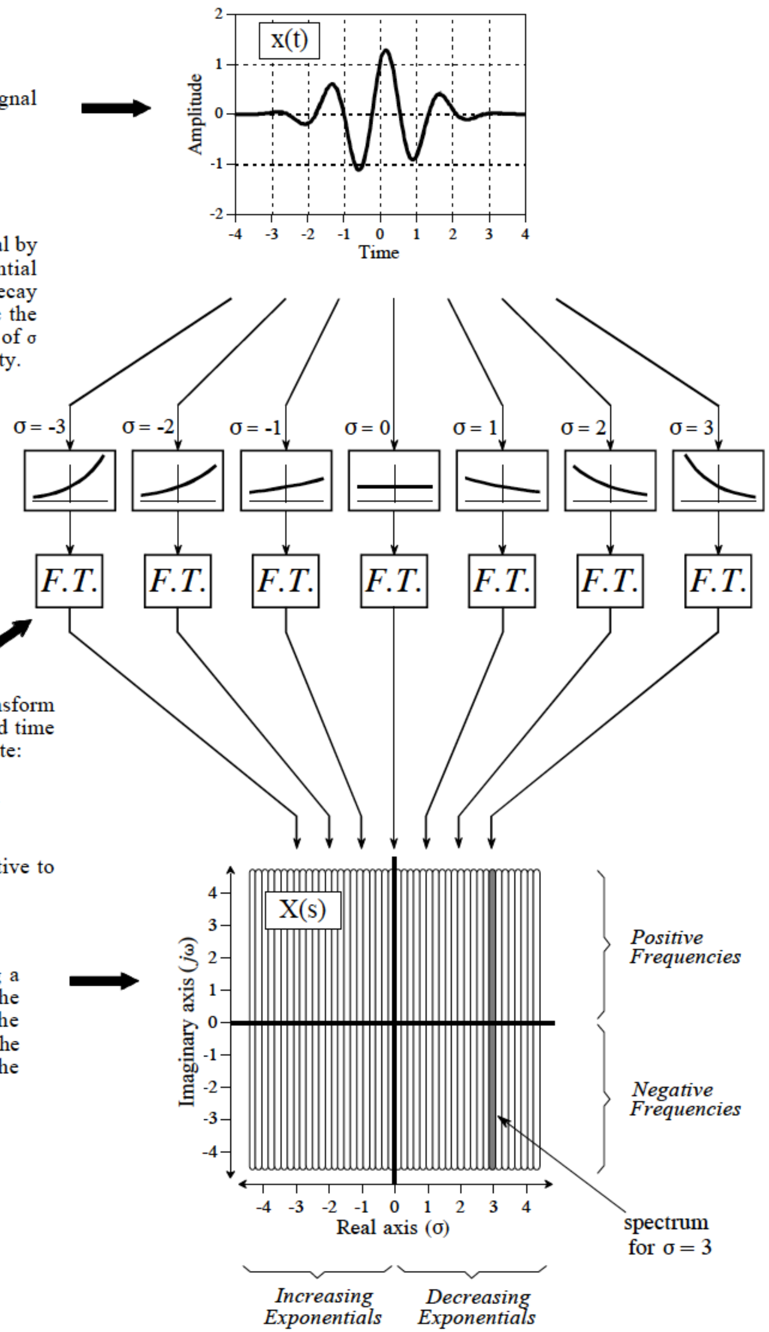


FIGURE 32-3
Time, frequency and s-domains. A time domain signal (the rectangular pulse) is transformed into the frequency domain using the Fourier transform, and into the s-domain using the Laplace transform.

dspguide.com

STEP 1
Start with the time domain signal called $x(t)$

STEP 2
Multiply the time domain signal by an infinite number of exponential curves, each with a different decay constant, σ. That is, calculate the signal: $x(t)\, e^{-\sigma t}$ for each value of σ from negative to positive infinity.



STEP 3
Take the complex Fourier Transform of each exponentially weighted time domain signal. That is, calculate:

$$\int_{-\infty}^{\infty} [\, x(t)\, e^{-\sigma t}\,]\, e^{-j\omega t}\, dt$$

for each value of σ from negative to positive infinity.

STEP 4
Arrange each spectrum along a vertical line in the s-plane. The positive frequencies are in the upper half of the s-plane while the negative frequencies are in the lower half.

dspguide.com

UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING

Laplace transform

The Laplace transform is:

$$X(\alpha, \omega) = X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$s = \alpha + j\omega$

and the inverse Laplace transform is

$$x(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(s) e^{st} ds$$

where c is a constant chosen to ensure convergence of the first integral. The path of integration along $c + j\omega$ with

ω varying from $-\infty$ to ∞ must lie in the ROC (Region of Convergence).

Let us assume that all signals we will be dealing with are **causal**, meaning the they are on-sided (or unilateral) Laplace transform.

UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING

Laplace transform

time

$y(t) = x(t) * h(t)$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$Y(s) = X(s)H(s)$

freq ω

$$x(t) \longrightarrow \boxed{\mathcal{L}} \xrightarrow{X(s)} \boxed{H(s)} \xrightarrow{Y(s)} \boxed{\mathcal{L}^{-1}} \longrightarrow y(t)$$

Laplace transform examples

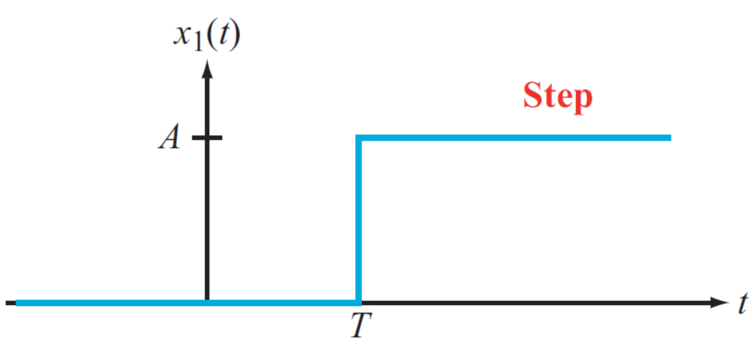
Example 3-1: Laplace Transforms of Singularity Functions

Determine the Laplace transforms of the signal waveforms displayed in Fig. 3-1.

Solution:

(a) The step function in Fig. 3-1(a) is given by

$$x_1(t) = A\, u(t - T).$$

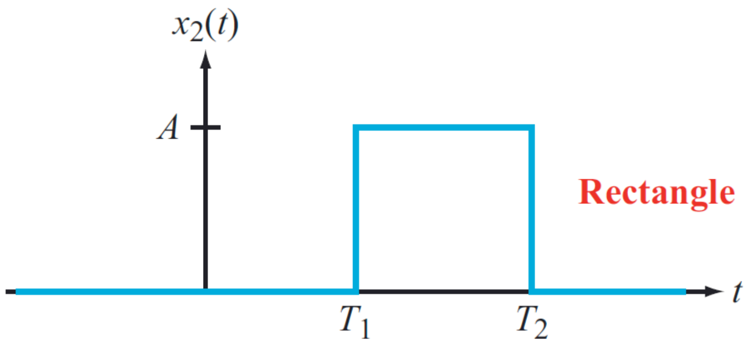


$$\begin{aligned}
\mathbf{X}_1(\mathbf{s}) &= \int_{0^-}^{\infty} x_1(t)\, e^{-\mathbf{s}t}\, dt \\
&= \int_{0^-}^{\infty} A\, u(t-T)\, e^{-\mathbf{s}t}\, dt \\
&= A \int_{T}^{\infty} e^{-\mathbf{s}t}\, dt \\
&= -\frac{A}{\mathbf{s}}\, e^{-\mathbf{s}t}\Big|_{T}^{\infty} = \frac{A}{\mathbf{s}}\, e^{-\mathbf{s}T}.
\end{aligned}$$

For the special case where $A = 1$ and $T = 0$ (i.e., the step occurs at $t = 0$), the transform pair becomes

$$\boxed{u(t) \leftrightarrow \frac{1}{\mathbf{s}}.} \qquad (3.6)$$

Laplace transform examples



(b)

(b) The rectangle function in Fig. 3-1(b) can be constructed as the sum of two step functions:

$$x_2(t) = A[u(t - T_1) - u(t - T_2)],$$

and its Laplace transform is

$$\begin{aligned}\mathbf{X}_2(\mathbf{s}) &= \int_{0^-}^{\infty} A[u(t - T_1) - u(t - T_2)]e^{-\mathbf{s}t}\, dt \\ &= A\int_{0^-}^{\infty} u(t - T_1)\, e^{-\mathbf{s}t}\, dt - A\int_{0^-}^{\infty} u(t - T_2)\, e^{-\mathbf{s}t}\, dt \\ &= \frac{A}{\mathbf{s}}[e^{-\mathbf{s}T_1} - e^{-\mathbf{s}T_2}].\end{aligned}$$

UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING

Laplace transform examples

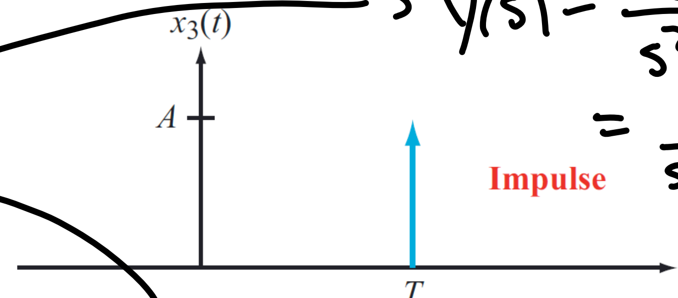
$(D^2+D)y(t) = D\,x(t)$

$(s^2+s)Y(s) = sX(s)$

$Y(s) = \frac{s}{s^2+s} = \frac{s}{s(s+1)} = \frac{1}{s+1}$

$\mathcal{L}^{-1}$

$h(t)$



(c)

$H(s) = \frac{Y(s)}{X(s)}$

$H(s) = \frac{Y(s)}{1}$

$h(t) = \mathcal{L}^{-1}\,H(s)$

(c) The impulse function in Fig. 3-1(c) is given by

$$x_3(t) = A\,\delta(t-T),$$

and the corresponding Laplace transform is

$$\mathbf{X}_3(\mathbf{s}) = \int_{0^-}^{\infty} A\,\delta(t-T)\,e^{-\mathbf{s}t}\,dt = Ae^{-\mathbf{s}T},$$

defined by Eq. (1.29). For the special case where $A = 1$ and $T = 0$, the Laplace transform pair simplifies to

$$\delta(t) \leftrightarrow 1. \qquad (3.7)$$

Laplace transform examples

Example 3-2: Laplace Transform Pairs

Obtain the Laplace transforms of (a) $x_1(t) = e^{-at}\ u(t)$ and (b) $x_2(t) = [\cos(\omega_0 t)]\ u(t)$.

Solution:

(a) Application of Eq. (3.1) gives

$$\mathbf{X}_1(\mathbf{s}) = \int_{0^-}^{\infty} e^{-at}\ u(t)\ e^{-\mathbf{s}t}\ dt$$

$$= \left.\frac{e^{-(\mathbf{s}+a)t}}{-(\mathbf{s}+a)}\right|_0^{\infty} = \frac{1}{\mathbf{s}+a}\ .$$

Hence,

$$e^{-at}\ u(t) \leftrightarrow \frac{1}{\mathbf{s}+a}\ . \qquad (3.8)$$

(b) We start by expressing $\cos(\omega_0 t)$ in the form

$$\cos(\omega_0 t) = \frac{1}{2}[e^{j\omega_0 t} + e^{-j\omega_0 t}].$$

Next, we take advantage of Eq. (3.8):

$$\begin{aligned}
\mathbf{X}_2(\mathbf{s}) &= \mathcal{L}[\cos(\omega_0 t)\ u(t)] \\
&= \frac{1}{2}\ \mathcal{L}[e^{j\omega_0 t}\ u(t)] + \frac{1}{2}\ \mathcal{L}[e^{-j\omega_0 t}\ u(t)] \\
&= \frac{1}{2}\ \frac{1}{\mathbf{s} - j\omega_0} + \frac{1}{2}\ \frac{1}{\mathbf{s} + j\omega_0} \\
&= \frac{\mathbf{s}}{\mathbf{s}^2 + \omega_0^2}\ .
\end{aligned}$$

Hence,

$$[\cos(\omega_0 t)]\ u(t) \leftrightarrow \frac{\mathbf{s}}{\mathbf{s}^2 + \omega_0^2}\ .$$

Laplace transform examples (1)

$x(t) = e^{-at}u(t)$, a is real

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}\,dt = \int_{0^+}^{\infty} e^{-(s+a)t}\,dt$$

$$= -\frac{1}{s+a} e^{-(s+a)t}\Big|_{0^+}^{\infty} = \frac{1}{s+a}, \ \mathrm{Re}(s) > a$$

because $\lim_{t \to \infty} e^{-(s+a)t} = 0$ only if $\mathrm{Re}(s+a) > 0$

or $\mathrm{Re}(s) > -a$

UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING

Laplace transform examples (2)

$$x(t) = -e^{-at}u(-t) \qquad \text{, a is real}$$

$$X(s) = \frac{1}{s+a} \quad , \quad Re(s) < -a$$

Laplace transform examples (3+)

$x(t) = \delta(t)$ $\quad \mathcal{L}[\delta(t)] = 1 \qquad \delta(t) \leftrightarrow 1$

$x(t) = u(t)$ $\quad u(t) \leftrightarrow \frac{1}{s} \quad \mathrm{Re}\, s > 0$

$x(t) = e^{at}u(t)$ $\quad e^{at}u(t) \leftrightarrow \frac{1}{s-a}$

$x(t) = \cos\omega_0 t u(t)$ $\quad \cos\omega_0 t\, u(t) \leftrightarrow \frac{s}{s^2+\omega_0^2}$

Laplace transform pairs

Laplace transform pairs $\boxed{x(t) \leftrightarrow X(s)}$

$\delta(t) \Leftrightarrow 1$

$u(t) \Leftrightarrow \frac{1}{s}$

$tu(t) \Leftrightarrow \frac{1}{s^2}$

$t^n u(t) \Leftrightarrow \frac{n!}{s^{n+1}}$

$e^{\lambda t} u(t) \Leftrightarrow \frac{1}{s-\lambda}$

$t^n e^{\lambda t} u(t) \Leftrightarrow \frac{n!}{(s-\lambda)^{n+1}}$

Inverse Laplace is best done by using the tables available.

You will need to use partial fraction expansion before you can use the tables. ~~See Appendix B for a review of partial fraction expansion.~~ If the order of the numerator is equal to order of the denominator then you will need to add coefficient of the highest power in the numerator for partial fraction expansion.

UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING

Table 3-2: Examples of Laplace transform pairs. Note that $x(t) = 0$ for $t < 0^-$ and $T \geq 0$.

Laplace Transform Pairs			
	$x(t)$		$\mathbf{X}(\mathbf{s}) = \mathcal{L}[x(t)]$
1	$\delta(t)$	$\leftrightarrow$	1
1a	$\delta(t-T)$	$\leftrightarrow$	$e^{-T\mathbf{s}}$
2	$u(t)$	$\leftrightarrow$	$\frac{1}{\mathbf{s}}$
2a	$u(t-T)$	$\leftrightarrow$	$\frac{e^{-T\mathbf{s}}}{\mathbf{s}}$
3	$e^{-at}\,u(t)$	$\leftrightarrow$	$\frac{1}{\mathbf{s}+a}$
3a	$e^{-a(t-T)}\,u(t-T)$	$\leftrightarrow$	$\frac{e^{-T\mathbf{s}}}{\mathbf{s}+a}$
4	$t\,u(t)$	$\leftrightarrow$	$\frac{1}{\mathbf{s}^2}$
4a	$(t-T)\,u(t-T)$	$\leftrightarrow$	$\frac{e^{-T\mathbf{s}}}{\mathbf{s}^2}$
5	$t^2\,u(t)$	$\leftrightarrow$	$\frac{2}{\mathbf{s}^3}$
6	$te^{-at}\,u(t)$	$\leftrightarrow$	$\frac{1}{(\mathbf{s}+a)^2}$
7	$t^2e^{-at}\,u(t)$	$\leftrightarrow$	$\frac{2}{(\mathbf{s}+a)^3}$
8	$t^{n-1}e^{-at}\,u(t)$	$\leftrightarrow$	$\frac{(n-1)!}{(\mathbf{s}+a)^n}$
9	$\sin(\omega_0 t)\,u(t)$	$\leftrightarrow$	$\frac{\omega_0}{\mathbf{s}^2+\omega_0^2}$
10	$\sin(\omega_0 t+\theta)\,u(t)$	$\leftrightarrow$	$\frac{\mathbf{s}\sin\theta+\omega_0\cos\theta}{\mathbf{s}^2+\omega_0^2}$
11	$\cos(\omega_0 t)\,u(t)$	$\leftrightarrow$	$\frac{\mathbf{s}}{\mathbf{s}^2+\omega_0^2}$
12	$\cos(\omega_0 t+\theta)\,u(t)$	$\leftrightarrow$	$\frac{\mathbf{s}\cos\theta-\omega_0\sin\theta}{\mathbf{s}^2+\omega_0^2}$
13	$e^{-at}\sin(\omega_0 t)\,u(t)$	$\leftrightarrow$	$\frac{\omega_0}{(\mathbf{s}+a)^2+\omega_0^2}$
14	$e^{-at}\cos(\omega_0 t)\,u(t)$	$\leftrightarrow$	$\frac{\mathbf{s}+a}{(\mathbf{s}+a)^2+\omega_0^2}$
15	$2e^{-at}\cos(bt-\theta)\,u(t)$	$\leftrightarrow$	$\frac{e^{j\theta}}{\mathbf{s}+a+jb}+\frac{e^{-j\theta}}{\mathbf{s}+a-jb}$
15a	$e^{-at}\cos(bt-\theta)\,u(t)$	$\leftrightarrow$	$\frac{(\mathbf{s}+a)\cos\theta+b\sin\theta}{(\mathbf{s}+a)^2+b^2}$
16	$\frac{2t^{n-1}}{(n-1)!}e^{-at}\cos(bt-\theta)\,u(t)$	$\leftrightarrow$	$\frac{e^{j\theta}}{(\mathbf{s}+a+jb)^n}+\frac{e^{-j\theta}}{(\mathbf{s}+a-jb)^n}$

inverse Laplace (1,2)

$$X(s) = \frac{7}{s^2}$$

$$\mathcal{L}^{-1}\left\{\frac{7}{s^2}\right\} = 7\,\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

(entry 4) $= \frac{7}{2!}t^2 u(t) = \frac{7}{2}t^2 u(t)$

$$X(s) = \frac{2s-1}{s^2+4}$$

$$\mathcal{L}^{-1}\left\{\frac{2s-1}{s^2+4}\right\} = 2\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$

table entry 8a, 8b

$$= \left(2\cos 2t - \frac{1}{2}\sin 2t\right)u(t)$$

UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING

inverse Laplace (3)

$$X(s) = \frac{7s-6}{s^2-s-6}$$

$$\mathcal{L}^{-1}\left\{\frac{7s-6}{s^2-s-6}\right\}$$

$$X(s) = \frac{7s-6}{(s+2)(s-3)} = \frac{k_1}{s+2} + \frac{k_2}{s-3}$$

use partial fraction expansion

$$k_1 = \left.\frac{7s-6}{\cancel{(s+2)}(s-3)}\right|_{s=-2} = 4$$

$$k_2 = \left.\frac{7s-6}{(s+2)\cancel{(s-3)}}\right|_{s=3} = -3$$

$$\therefore X(s) = \frac{4}{s+2} - \frac{3}{s-3}$$

check $s=0$ both sides

$$\frac{7(0)-6}{(0^2-0-6)} = 1 \checkmark$$

$$\frac{4}{0+2} + \frac{3}{0-3} = 1 \checkmark$$

UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING

inverse Laplace (3) (ctn)

$$X(s) = \frac{7s - 6}{s^2 - s - 6}$$

check

$$x(t) = 4e^{-2t}u(t) - 3e^{3t}u(t)$$

UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING

inverse Laplace (4)

$$X(s) = \frac{2s^2+5}{(s+1)(s+2)}$$

$$\mathcal{L}^{-1}\left\{\frac{2s^2+5}{(s+1)(s+2)}\right\}$$

$N \geq D$

$$\therefore X(s) = 2 + \frac{k_1}{s+1} + \frac{k_2}{s+2}$$

$$k_1 = \frac{2s^2+5}{\cancel{(s+1)}(s+2)}\Bigg|_{s=-1} = 7 \qquad k_2 = \frac{2s^2+5}{(s+1)\cancel{(s+2)}}\Bigg|_{s=-2} = -13$$

$$\therefore X(s) = 2 + \frac{7}{s+1} - \frac{13}{s+2}$$

$$\boxed{\begin{array}{l} LHS \rightarrow \frac{0+5}{1(2)} = \frac{5}{2} = 2.5 \checkmark \\ RHS \rightarrow 2 + 7 - 6.5 = 2.5 \checkmark \end{array}}$$

use $\delta(t) \leftrightarrow 1 \qquad e^{\lambda t}u(t) \leftrightarrow \frac{1}{s-\lambda}$

$$\therefore x(t) = 2\delta(t) + \left(7e^{-t} - 13e^{-2t}\right)u(t)$$

Laplace properties

Time Shifting

if $x(t) \Leftrightarrow X(s)$
then $x(t-t_0) \Leftrightarrow X(s)e^{-st_0}$

Frequency Shifting

if $x(t) \Leftrightarrow X(s)$
then $x(t)e^{s_o t} \Leftrightarrow X(s-s_0)$

Time Differentiation

if $x(t) \Leftrightarrow X(s)$
then $\frac{dx}{dt} \Leftrightarrow sX(s) - x(0^-)$
and $\frac{d^2x}{dt^2} \Leftrightarrow s^2X(s) - sx(0^-) - \dot{x}(0^-)$
and $\frac{d^nx}{dt^n} \Leftrightarrow s^nX(s) - \sum_{k=1}^{n} s^{n-k}x^{k-1}(0^-)$

Frequency Differentiation

if $x(t) \Leftrightarrow X(s)$
then $tx(t) \Leftrightarrow -\frac{d}{ds}X(s)$

Laplace properties (2)

Time Integration

if $x(t) \Leftrightarrow X(s)$
then $\int_{0-}^{t} x(\tau)d\tau \Leftrightarrow \frac{X(s)}{s}$

Scaling

if $x(t) \Leftrightarrow X(s)$
then $x(at) \Leftrightarrow \frac{1}{a}X(\frac{s}{a})$ for all a

Time Convolution

if $x_1(t) \Leftrightarrow X_1(s)$ and $x_2(t) \Leftrightarrow X_2(s)$
then $x_1(t) * x_2(t) \Leftrightarrow X_1(s)X_s(s)$

Frequency Convolution

if $x_1(t) \Leftrightarrow X_1(s)$ and $x_2(t) \Leftrightarrow X_2(s)$
then $x_1(t)x_2(t) \Leftrightarrow \frac{1}{2\pi j}X_1(s) * X_s(s)$

Laplace properties (3)

Initial Value Theorem

if $x(t) \Leftrightarrow X(s)$

$$\text{then } \lim_{t \to 0} x(t) = x(0^{+}) = \lim_{s \to \infty} sX(s)$$

Final Value Theorem

if $x(t) \Leftrightarrow X(s)$

$$\text{then } \lim_{t \to \infty} x(t) = x(0^{+}) = \lim_{s \to 0} sX(s)$$

Laplace properties - examples (1) - time shift

$$x(t) = (t-1)[u(t-1) - u(t-2)] + [u(t-2) - u(t-4)]$$

Laplace properties - examples (2) - frequency shift

$$x(t) = e^{-at}\cos(bt)u(t)$$

Laplace properties - examples (3) - time differentiation

$$x(t) = t[u(t) - u(t-2)] + (3-t)[u(t-2) - u(t-3)]$$

Laplace properties - examples (3) - time differentiation

$$x(t) = t[u(t) - u(t-2)] + (3-t)[u(t-2) - u(t-3)]$$

Laplace properties - examples (4) - time convolution

$$c(t) = e^{at}u(t) * e^{bt}u(t)$$

UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING

Laplace properties - examples (5) - initial, final value

$$Y(s) = \frac{5(3s+1)}{s(s+3)(s+2)}$$

Laplace - solving differential equations - ZI +ZS

$$\frac{d^2y}{dt^2} + 9y(y) = e^{4t}u(t)$$

I.C. $\quad y(0) = 1$

$$\dot{y}(0) = 0$$

Laplace - solving differential equations - ZI +ZS (2)

$$\frac{d^2y}{dt^2} + 9y(y) = e^{4t}u(t)$$

I.C. $\quad y(0) = 1$

$$\dot{y}(0) = 0$$

UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING

Laplace - solving differential equations - ZI +ZS (3)

$$\frac{d^2y}{dt^2} + 9y(y) = e^{4t}u(t)$$

I.C. $\quad y(0) = 1$

$$\dot{y}(0) = 0$$

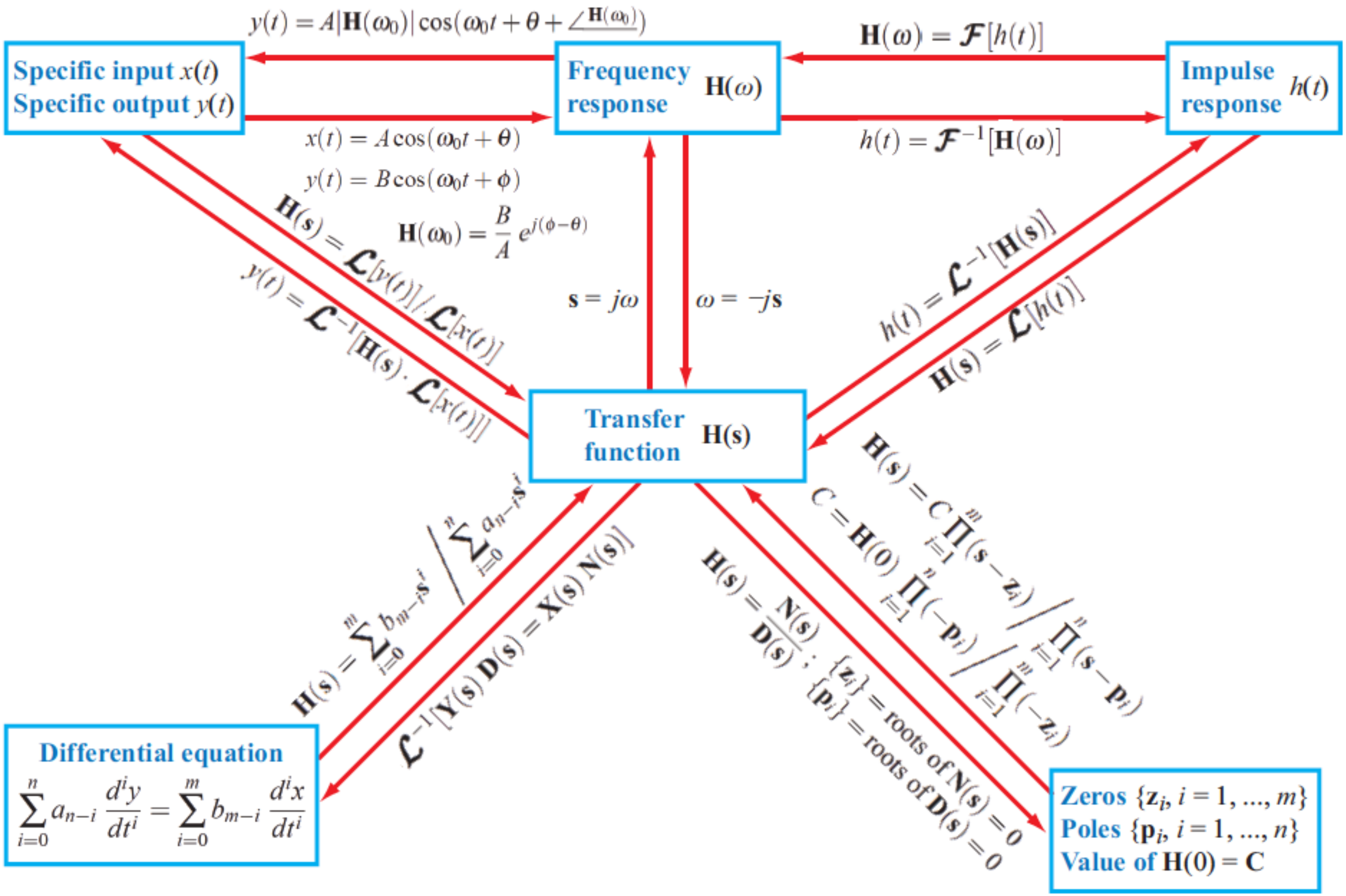
Laplace - solving differential equations - ZI +ZS - example 2

$$(D^2 + 5D + 6)y(t) = (D + 1)x(t)$$

I.C. $\quad y(0) = 2 \qquad \dot{y}(0) = 1$

$$x(t) = e^{-4t}u(t)$$

System Description relationships




UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING