Problem Set #3 Solutions

10.2 (a) The coefficient a_{21} is eliminated by multiplying row 1 by $f_{21} = -3/10 = -0.3$ and subtracting the result from row 2. a_{31} is eliminated by multiplying row 1 by $f_{31} = 1/10 = 0.1$ and subtracting the result from row 3. The factors f_{21} and f_{31} can be stored in a_{21} and a_{31}.

$$\begin{bmatrix} 10 & 2 & -1 \\ -0.3 & -5.4 & 1.7 \\ 0.1 & 0.8 & 5.1 \end{bmatrix}$$

a_{32} is eliminated by multiplying row 2 by $f_{32} = 0.8/(-5.4) = -0.14815$ and subtracting the result from row 3. The factor f_{32} can be stored in a_{32}.

$$\begin{bmatrix} 10 & 2 & -1 \\ -0.3 & -5.4 & 1.7 \\ 0.1 & -0.14815 & 5.351852 \end{bmatrix}$$

Therefore, the LU decomposition is

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.1 & -0.14815 & 1 \end{bmatrix} \qquad [U] = \begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.351852 \end{bmatrix}$$

These two matrices can be multiplied to yield the original system. For example, using MATLAB to perform the multiplication gives

```
>> L=[1 0 0;-.3 1 0;0.1 -.14815 1];
>> U=[10 2 -1;0 -5.4 1.7;0 0 5.351852];
>> L*U
ans =
   10.0000    2.0000   -1.0000
   -3.0000   -6.0000    2.0000
    1.0000    1.0000    5.0000
```

(b) Forward substitution: $[L]\{D\} = \{B\}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.1 & -0.14815 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 27 \\ -61.5 \\ -21.5 \end{Bmatrix}$$

Solving yields $d_1 = 27$, $d_2 = -53.4$, and $d_3 = -32.1111$.

Back substitution:

$$\begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.351852 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 27 \\ -53.4 \\ -32.1111 \end{Bmatrix}$$

$$x_3 = \frac{-32.1111}{5.351852} = -6$$

$$x_2 = \frac{-53.4 - 1.7(-6)}{-5.4} = 8$$

$$x_1 = \frac{27 - (-1)(-6) - 2(8)}{10} = 0.5$$

(c) Forward substitution: $[L]\{D\} = \{B\}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.1 & -0.14815 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 12 \\ 18 \\ -6 \end{Bmatrix}$$

Solving yields $d_1 = 12$, $d_2 = 21.6$, and $d_3 = -4$.

Back substitution:

$$\begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.351852 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 12 \\ 21.6 \\ -4 \end{Bmatrix}$$

$$x_3 = \frac{-4}{5.351852} = -0.7474$$

$$x_2 = \frac{21.6 - 1.7(-0.7474)}{-5.4} = -4.23529$$

$$x_1 = \frac{12 - (-1)(-0.7474) - 2(-4.23529)}{10} = 1.972318$$

10.3 (a) The coefficient a_{21} is eliminated by multiplying row 1 by $f_{21} = -2/8 = -0.25$ and subtracting the result from row 2. a_{31} is eliminated by multiplying row 1 by $f_{31} = 2/8 = 0.25$ and subtracting the result from row 3. The factors f_{21} and f_{31} can be stored in a_{21} and a_{31}.

$$\begin{bmatrix} 8 & 4 & -1 \\ -0.25 & 6 & 0.75 \\ 0.25 & -2 & 6.25 \end{bmatrix}$$

a_{32} is eliminated by multiplying row 2 by $f_{32} = -2/6 = -0.33333$ and subtracting the result from row 3. The factor f_{32} can be stored in a_{32}.

$$\begin{bmatrix} 8 & 4 & -1 \\ -0.25 & 6 & 0.75 \\ 0.25 & -0.33333 & 6.5 \end{bmatrix}$$

Therefore, the *LU* decomposition is

$$[L]=\begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0.25 & -0.33333 & 1 \end{bmatrix} \qquad [U]=\begin{bmatrix} 8 & 4 & -1 \\ 0 & 6 & 0.75 \\ 0 & 0 & 6.5 \end{bmatrix}$$

Forward substitution: $[L]\{D\} = \{B\}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0.25 & -0.33333 & 1 \end{bmatrix}\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix}=\begin{Bmatrix} 11 \\ 4 \\ 7 \end{Bmatrix}$$

Solving yields $d_1 = 11$, $d_2 = 6.75$, and $d_3 = 6.5$.

Back substitution:

$$\begin{bmatrix} 8 & 4 & -1 \\ 0 & 6 & 0.75 \\ 0 & 0 & 6.5 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}=\begin{Bmatrix} 11 \\ 6.75 \\ 6.5 \end{Bmatrix}$$

$$x_3 = \frac{6.5}{6.5} = 1$$

$$x_2 = \frac{6.75 - 0.75(1)}{6} = 1$$

$$x_1 = \frac{11 - (-1)(1) - 4(1)}{8} = 1$$

(b) The first column of the inverse can be computed by using $[L]\{D\} = \{B\}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0.25 & -0.33333 & 1 \end{bmatrix}\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix}=\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

This can be solved for $d_1 = 1$, $d_2 = 0.25$, and $d_3 = -0.16667$. Then, we can implement back substitution

$$\begin{bmatrix} 8 & 4 & -1 \\ 0 & 6 & 0.75 \\ 0 & 0 & 6.5 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}=\begin{Bmatrix} 1 \\ 0.25 \\ -0.16667 \end{Bmatrix}$$

to yield the first column of the inverse

$$\{X\} = \begin{Bmatrix} 0.099359 \\ 0.0448718 \\ -0.025641 \end{Bmatrix}$$

For the second column use $\{B\}^T = \{0\ 1\ 0\}$ which gives $\{D\}^T = \{0\ 1\ 0.33333\}$. Back substitution then gives $\{X\}^T = \{-0.073718\ 0.160256\ 0.051282\}$.

For the third column use $\{B\}^T = \{0\ 0\ 1\}$ which gives $\{D\}^T = \{0\ 0\ 1\}$. Back substitution then gives $\{X\}^T = \{0.028846\ -0.01923\ 0.153846\}$.

Therefore, the matrix inverse is

$$[A]^{-1} = \begin{bmatrix} 0.099359 & -0.073718 & 0.028846 \\ 0.044872 & 0.160256 & -0.019231 \\ -0.025641 & 0.051282 & 0.153846 \end{bmatrix}$$

We can verify that this is correct by multiplying $[A][A]^{-1}$ to yield the identity matrix. For example, using MATLAB,

```
>> A=[8 4 -1;-2 5 1;2 -1 6];
>> AI=[0.099359 -0.073718 0.028846;
0.044872 0.160256 -0.019231;
-0.025641 0.051282 0.153846]
>> A*AI
ans =
    1.0000   -0.0000   -0.0000
    0.0000    1.0000   -0.0000
         0         0    1.0000
```

10.6 First, we compute the *LU* decomposition. The coefficient a_{21} is eliminated by multiplying row 1 by $f_{21} = -3/10 = -0.3$ and subtracting the result from row 2. a_{31} is eliminated by multiplying row 1 by $f_{31} = 1/10 = 0.1$ and subtracting the result from row 3. The factors f_{21} and f_{31} can be stored in a_{21} and a_{31}.

$$\begin{bmatrix} 10 & 2 & -1 \\ -0.3 & -5.4 & 1.7 \\ 0.1 & 0.8 & 5.1 \end{bmatrix}$$

a_{32} is eliminated by multiplying row 2 by $f_{32} = 0.8/(-5.4) = -0.148148$ and subtracting the result from row 3. The factor f_{32} can be stored in a_{32}.

$$\begin{bmatrix} 10 & 2 & -1 \\ -0.3 & -5.4 & 1.7 \\ 0.1 & -0.148148 & 5.351852 \end{bmatrix}$$

Therefore, the *LU* decomposition is

$$[L]=\begin{bmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.1 & -0.148148 & 1 \end{bmatrix} \qquad [U]=\begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.351852 \end{bmatrix}$$

The first column of the inverse can be computed by using $[L]\{D\} = \{B\}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.1 & -0.148148 & 1 \end{bmatrix}\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix}=\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

This can be solved for $d_1 = 1$, $d_2 = 0.3$, and $d_3 = -0.055556$. Then, we can implement back substitution

$$\begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.351852 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}=\begin{Bmatrix} 1 \\ 0.3 \\ -0.055556 \end{Bmatrix}$$

to yield the first column of the inverse

$$\{X\}=\begin{Bmatrix} 0.110727 \\ -0.058824 \\ -0.0103806 \end{Bmatrix}$$

For the second column use $\{B\}^T = \{0\ 1\ 0\}$ which gives $\{D\}^T = \{0\ 1\ 0.148148\}$. Back substitution then gives $\{X\}^T = \{0.038062\ -0.176471\ 0.027682\}$.

For the third column use $\{B\}^T = \{0\ 0\ 1\}$ which gives $\{D\}^T = \{0\ 0\ 1\}$. Back substitution then gives $\{X\}^T = \{0.00692\ 0.058824\ 0.186851\}$.

Therefore, the matrix inverse is

$$[A]^{-1}=\begin{bmatrix} 0.110727 & 0.038062 & 0.006920 \\ -0.058824 & -0.176471 & 0.058824 \\ -0.010381 & 0.027682 & 0.186851 \end{bmatrix}$$

We can verify that this is correct by multiplying $[A][A]^{-1}$ to yield the identity matrix. For example, using MATLAB,

```
>> A=[10 2 -1;-3 -6 2;1 1 5];
>> AI=[0.110727 0.038062 0.006920;
-0.058824 -0.176471 0.058824;
-0.010381 0.027682 0.186851];
>> A*AI
ans =
    1.0000   -0.0000   -0.0000
    0.0000    1.0000   -0.0000

  -0.0000    0.0000    1.0000
```

10.8 (a) Using MATLAB, the matrix inverse can be computed as

```
>> A=[15 -3 -1;-3 18 -6;-4 -1 12];
>> AI=inv(A)
AI =
    0.0725    0.0128    0.0124
    0.0207    0.0608    0.0321
    0.0259    0.0093    0.0902
```

(b)

```
>> B=[3800;1200;2350];
>> C=AI*B

C =
  320.2073
  227.2021
  321.5026
```

(c) $\Delta W_3 = \dfrac{\Delta c_1}{a_{13}^{-1}} = \dfrac{10}{0.012435} = 804.1667$

(d) $\Delta c_3 = a_{31}^{-1}\Delta W_1 + a_{32}^{-1}\Delta W_2 = 0.025907(-500) + 0.009326(-250) = -15.285$