SYDE252 - lecture notes

09/01/18

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UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING

1. SIGNALS

09/01/18

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inspiration

- You may delay, but time will not. \footnote{Confucius}.
- The most important question a human being has to face... What is it? The question, Why are we here?.\\--Elie Wiesel
- To study and not think is a waste. To think and not study is dangerous. \footnote{Confucius}.

signals

- A signal can be represented in the time or frequency domain
- it can also be be a discrete or continuous signal
- continuous signal
  - to go from a time representation to a frequency representation we use the Laplace or Fourier transform
- discrete signal
  - To go from a discrete signal time representation to a frequency representation we either use the Z transform or the discrete Fourier transform
- system is represented by a differential equation
- other concepts covered include convolution & filtering

what is a signal

- any kind of physical variable subject to variations represents a signal
- it is a function
- it has independent and dependent variables
- examples
  - in time (t) y=f(t)
  - in space (x,y,z) $y = f(x_1, x_2)$ OR $y = f(x_1, x_2, x_3)$

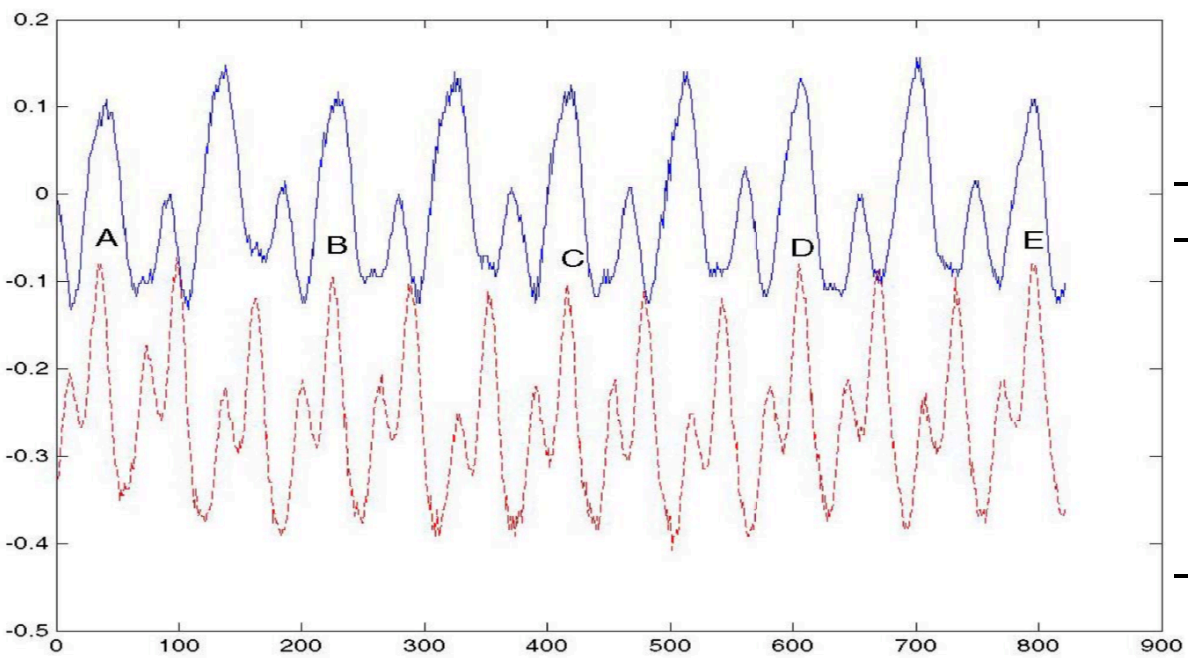
what is a signal (2)

- 1D signal; EEG, voice, music, accelerometer
- 2D signal; images
- 3D signals; video sequences (x1, x2, t) or volumetric data (x1, x2, x3), pixels, voxels
-

musical signals

- female singer singing do (solid blue), and soh (dotted red shifted down)

- (from Mumford notes (Brown)).



- fairly close to being periodic but not exactly periodic
- blue curve for do shows 9 periods with major peaks interspersed with minor peaks
- red curve shows 13 peaks
- Look at the points marked A,B,C,D and E. At each letter both curves have peaks but between each pair, there is one extra peak for do and two for sol
- In other words, two periods of do match three periods of sol
-

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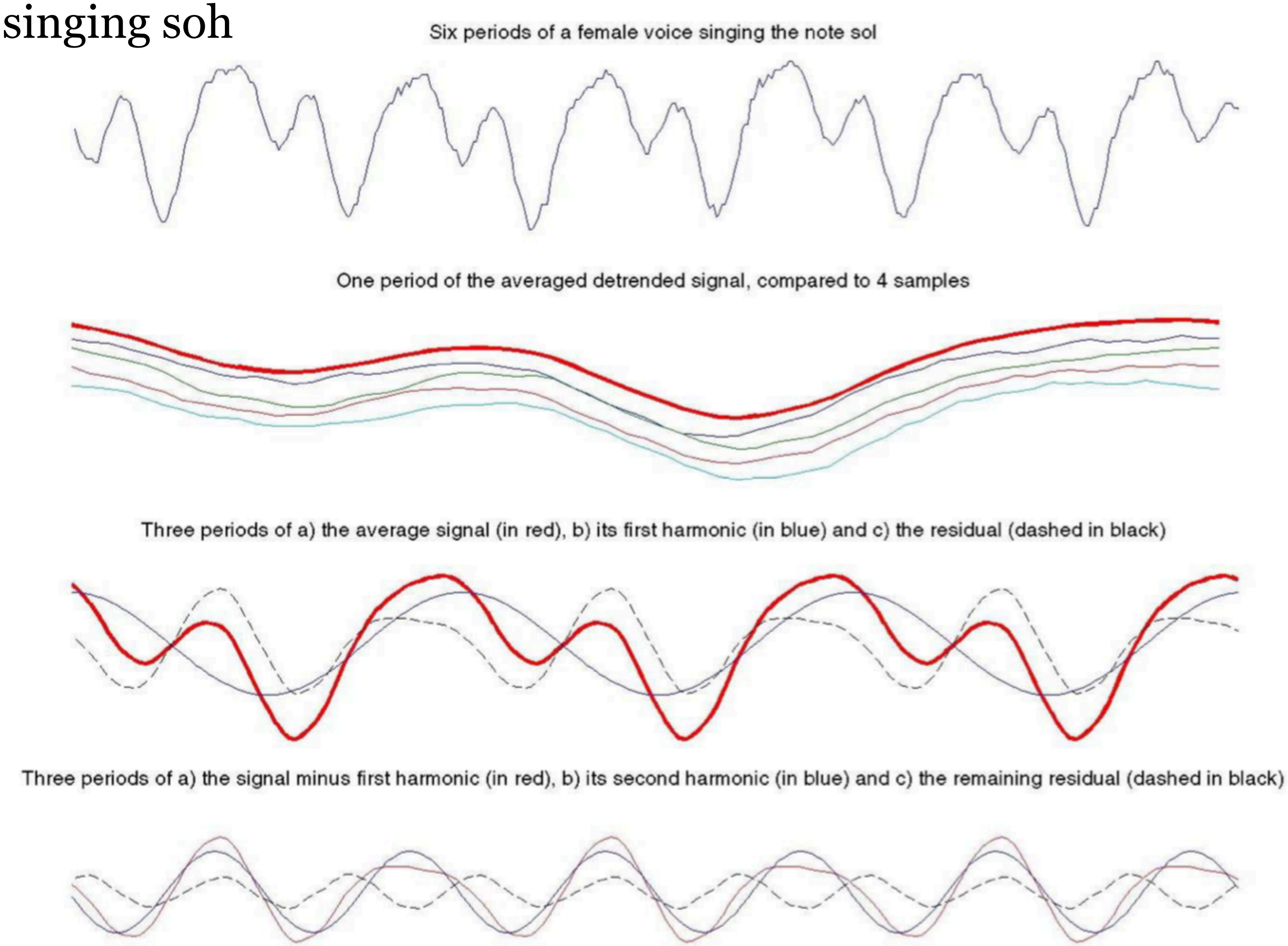
musical signals

- female singing soh



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musical signals

Let's put this in formulas. Let $P(t)$ be the air pressure as a function of time. Then we model this by an exactly periodic function $Q(t)$, i.e. there is a period p such that $Q(t+p) \equiv Q(t)$, all t. P and Q will be very close to each other. We write Q as a sum of sinusoids like this:

$$\boxed{Q(t) = C_0 + C_1 \sin(2\pi f t + D_1) + C_2 \sin(4\pi f t + D_2)}$$

$$+ C_3 \sin(6\pi f t + D_3) + \ldots$$

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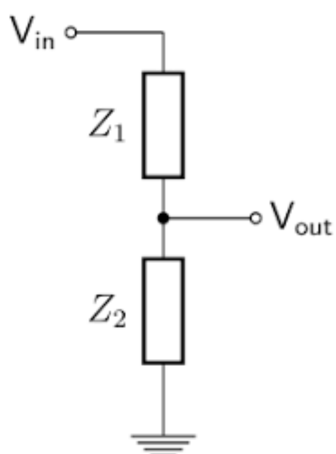
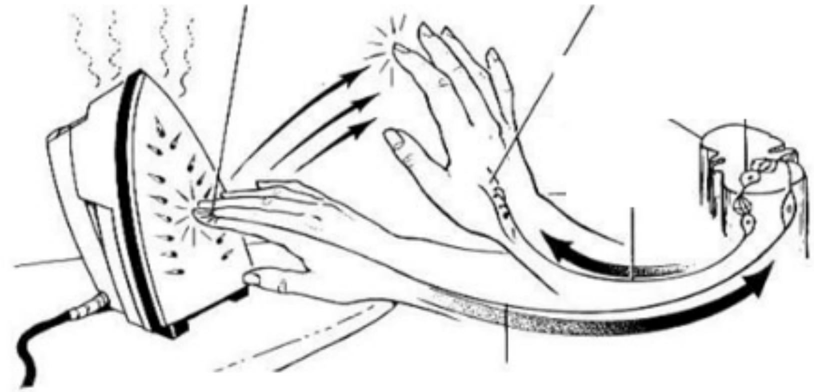
what is a system

- A process for transforming the input signal x(t) into an output signal y(t)
- continuous functions of real independent variable, e.g., audio, ECG, images; 1D f = f(x); 2D f = f(x,y)
- real valued functions of discrete variables; e.g., sampled signals or digital signals; 1D f = f[k]; 2D f = f[i,j]
- discrete functions of discrete variables (sampled and quantized); 1D $f^d = f^d[k]$ 2D $f^d = f^d[i,j]$





signal classifications

- continuous time vs. discrete time
- analog vs. digital
- periodic vs. aperiodic
- real vs. complex
- causal vs. non causal signals vs. anti-causal
- even vs. odd signals
- deterministic vs. probabilistic signals
- finite length vs. infinite length signals

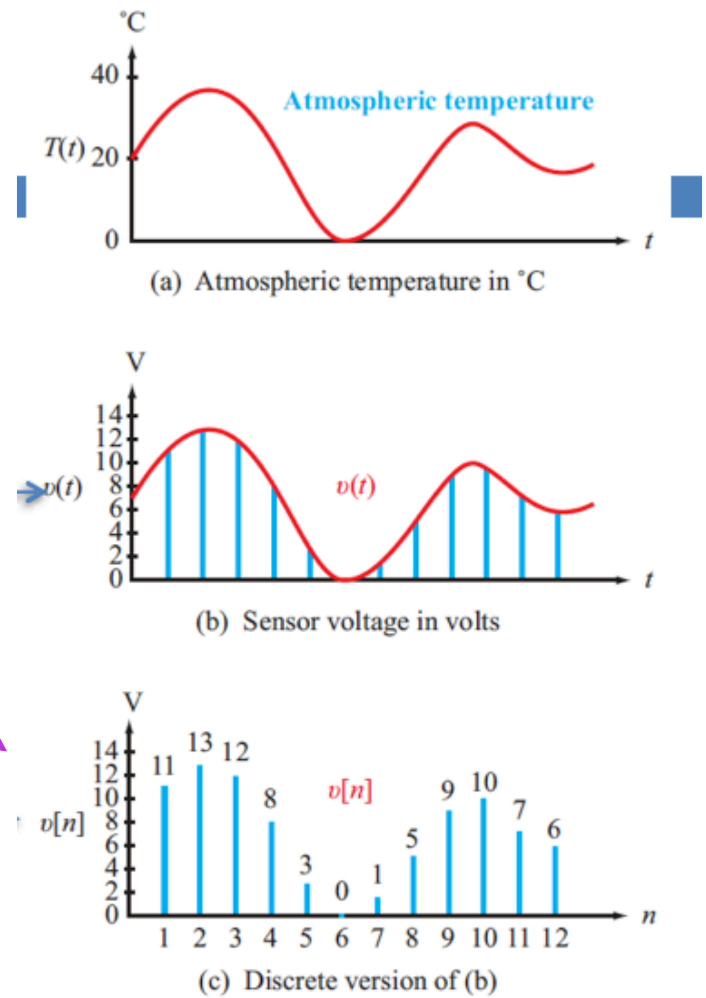
continuous time vs. discrete time

- continuous-time - a signal is specified for every real value of the independent variable

- discrete-time - a signal specified for only discrete values of the independent variable

- horizontal axis is its or discrete

- digital signals



(a) Atmospheric temperature in °C

(b) Sensor voltage in volts

(c) Discrete version of (b)

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analog vs. digital

- analog signal is where the amplitude (output, dependent variable) can take any value in a continuous range

- digital signal is where the amplitude can only take on a finite number of values (Quantized)

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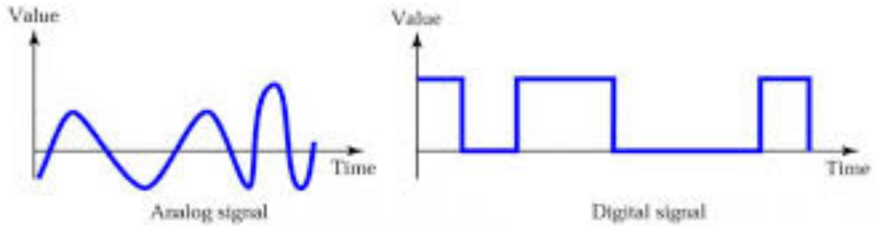

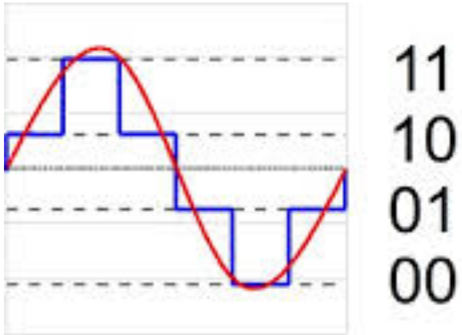

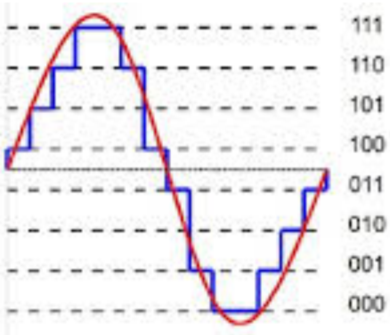

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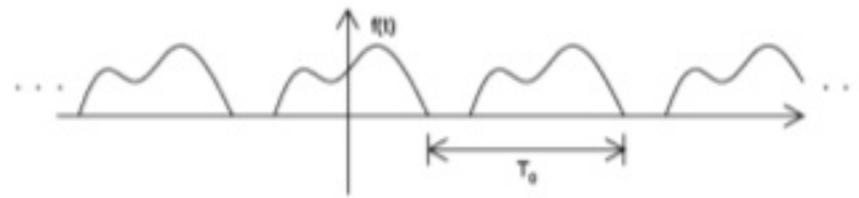
periodic vs. aperiodic

- a signal $f(t)$ is periodic if there exists a positive constant T_0 s.t. (such that) $f(t + T_0) = f(t), \forall t$
- smallest T_0 is called the period of the function $f(t)$
- a periodic signal remains unchanged when time-shifted integer multiples of period
- an aperiodic signal is not periodic

Periodic signals repeat with some *period T*.



A signal is called aperiodic if it is not periodic.

real vs. complex

- a signal can have a real and complex component

$$x(t) = x_r(t) + x_i(t)$$

$$x(t) = |x(t)| \exp^{j\angle x(t)}$$

$$|x(t)| = |x_r^2(t) + x_i^2(t)|^{\frac{1}{2}}$$

$$\angle x(t) = \tan^{-1}[\frac{x_i(t)}{x_r(t)}]$$

$$x_i(t) = |x(t)| \sin(\angle x(t))$$

$$x_r(t) = |x(t)| \cos(\angle x(t))$$

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causal vs. non-causal

causal signals are zero for all negative time (or space) $f(t) = 0, \forall t < 0$

anticausal signals are zero for all positive time (or space) $f(t) = 0, \forall t \geq 0$

non causal signals have non zero values in both positive and negative time $\exists t_1, f(t_1) \neq 0, \forall t_1 < 0$ and $\exists t_2, f(t_2) \neq 0, \forall t_2 > 0$

Causal signals are signals that are zero for all negative time, while **anticausal** are signals that are zero for all positive time. **Noncausal** signals are signals that have nonzero values in both positive and negative time

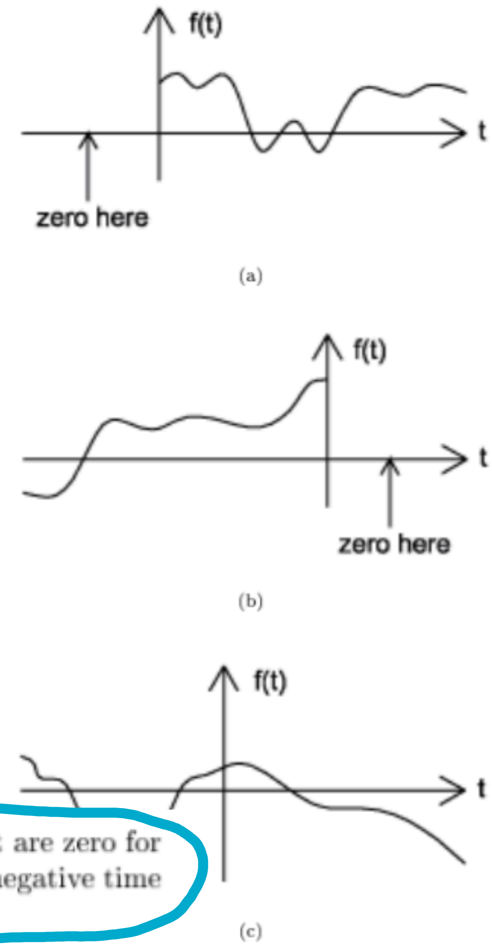


Figure 1.4: (a) A causal signal (b) An anticausal signal (c) A noncausal signal

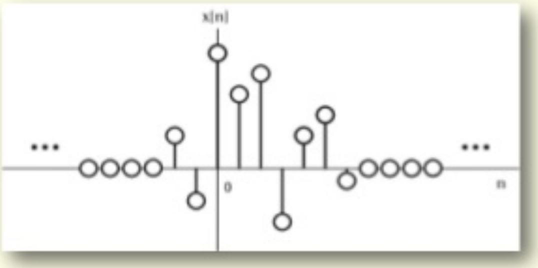
finite vs infinite length signals

- a finite signal is non zero over a finite set of values of the independent variable $f = f(t), \forall t; t_1 \leq t \leq t_2; t_1 > -\infty; t_2 < +\infty$

- – an infinite signal is non zero over an infinite set of values of the independent variable ; e.g., $f(t) = \sin(wt)$

-

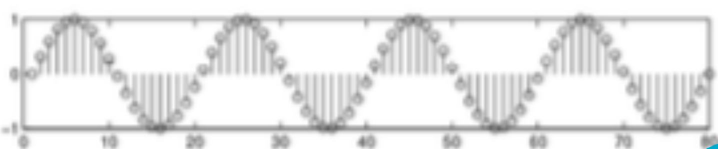
Finite-length signal: nonzero over a finite interval $t_{min} < t < t_{max}$



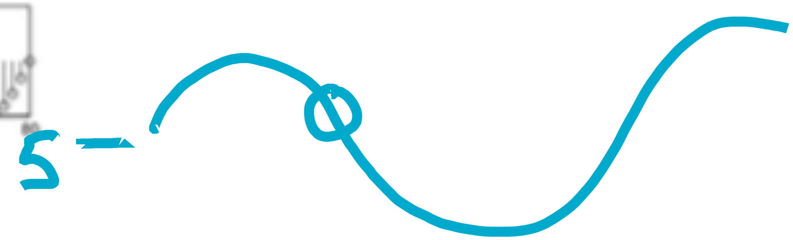
Infinite-length signal: nonzero over all real numbers

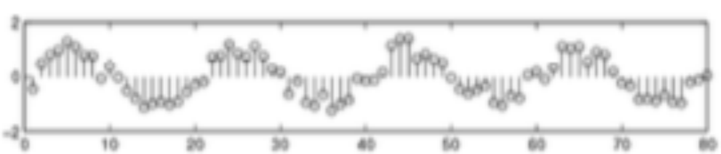
deterministic vs. probabilistic signals

- deterministic signal - signal whose physical description is known completely
- probabilistic signal - signal has amplitude val- ues that cannot be predicted precisely but only in terms of probabilistic descriptors



(a)





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even vs. odd signals

- an even signal is any signal f s.t. f(t) = f (–t)
- an odd signal is any signal f s.t. f(t) = –(f (–t))
- we can use even and odd signals to decompose any signal

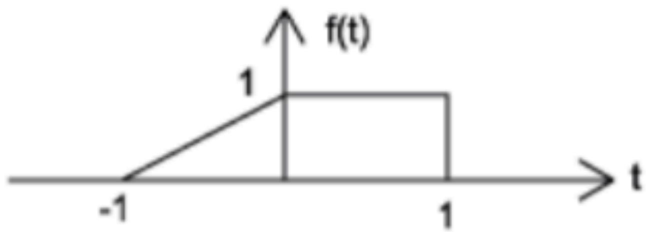
$$f(t) = \boxed{\frac{1}{2}\left(f(t)+f(-t)\right)} + \frac{1}{2}\left(f(t)-f(-t)\right)$$

- even signal $\boxed{\frac{1}{2}\left(f(t)+f(-t)\right)}$
- odd signal

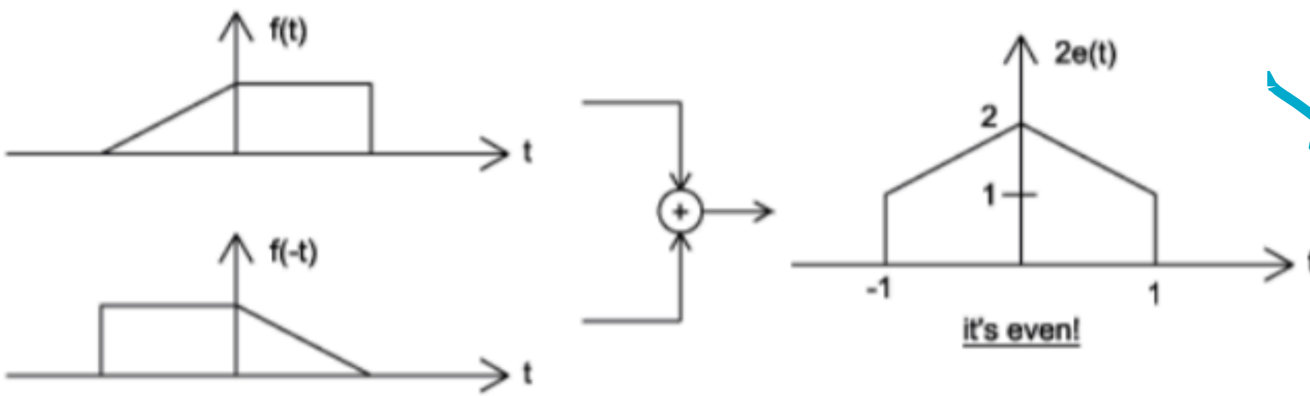
$$\boxed{\frac{1}{2}\left(f(t)-f(-t)\right)}$$

- even function multiplied by an odd function = odd function
- odd function multiplied by an odd function = even function
- even function multiplied by an even function = even function
- $$\int_{-a}^{a} f_e(t)\,dt = 2\int_{0}^{a} f_e(t)\,dt$$
  $$\int_{-a}^{a} f_o(t)\,dt = 0$$

even vs. odd signals



(a)



$\times \frac{1}{2}$

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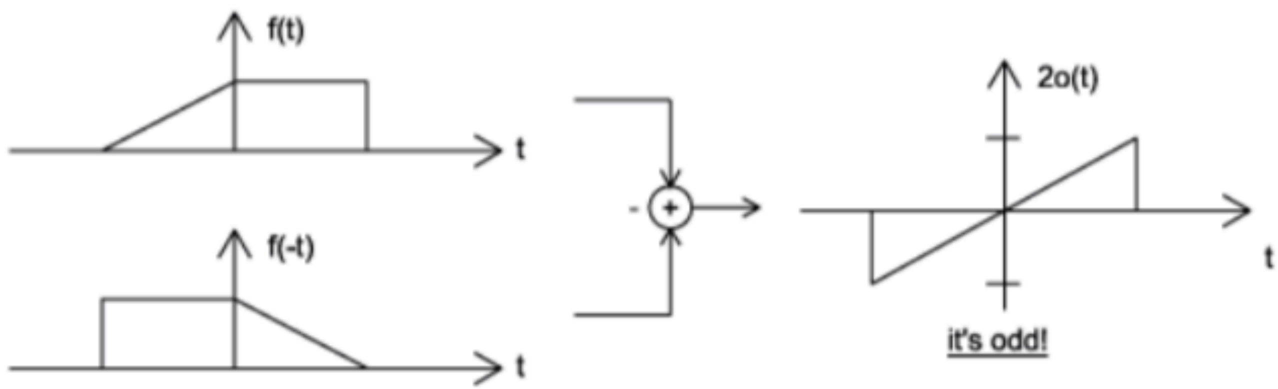
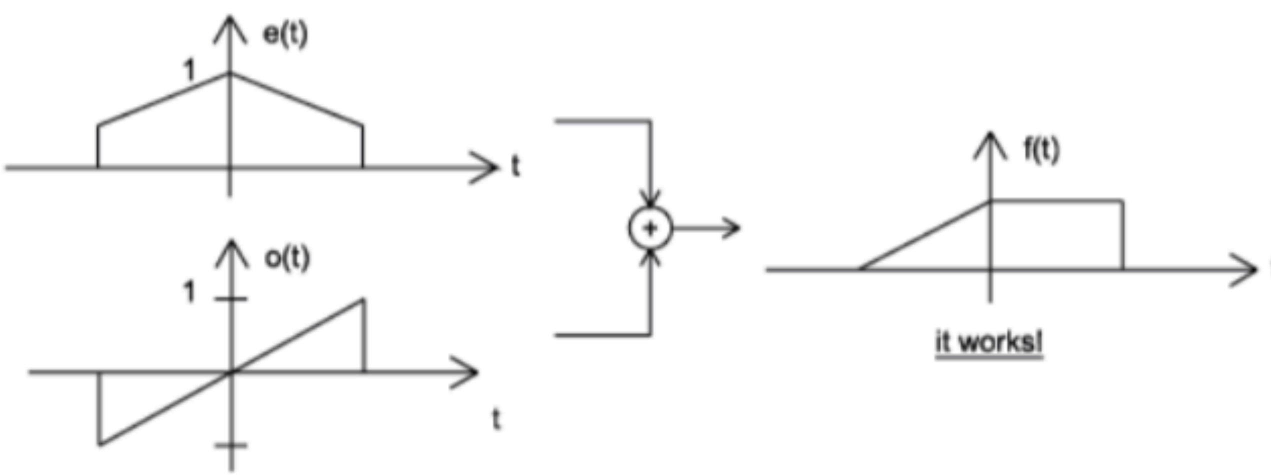
even vs. odd signals



(c)



energy & power of a signal

- Norms: size is the largeness or strength
- energy is represented by the area under the curve of the squared signal
- signal energy: $E_f = \int_{-\infty}^{\infty} f^2(t)\,dt = \int_{-\infty}^{\infty} |f(t)|^2\,dt$
- signal power: $P_f = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{T/2} f^2(t)\,dt$
- signal to noise ratio: $SNR = 20\log_{10}\sqrt{\left(\frac{P_{signal}}{P_{noise}}\right)}$
- a signal with finite energy is an energy signal; the amplitude must go to zero as independent variable goes to ∞
- a signal with finite and different from zero power is called a power signal. The mean of an entity averaged over an infinite interval exists if either the entity is periodic or it has some statistical regularity
- all practical signals have finite energy and thus are energy signals. It is impossible to have infinite du- ration signals
-

energy & power of a signal

$$E = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |x(t)|^2 \, dt,$$

$$P_{\text{av}} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} p(t) \, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt.$$

$$P_{\text{av}} = \lim_{T\to\infty} \frac{E}{T}$$

Power signal - finite & different from zero power

Energy signal - finite energy

- P_{av} and E define three classes of signals:

(a) *Power signals*: P_{av} finite, $E \to \infty$

(b) *Energy signals*: E finite, $P_{av} = 0$

(c) *Non-physical si*[gnals]: $P_{av} \to \infty$, $E \to \infty$

energy & power of a signal (example)

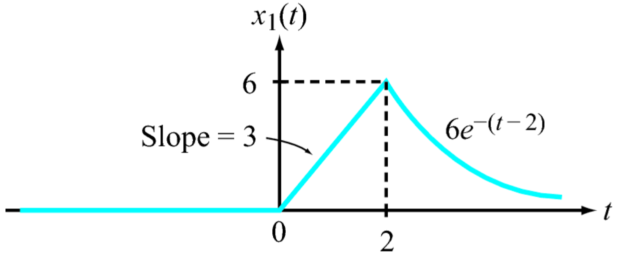
Example [redacted] Power and Energy

Evaluate P_{av} and E for each of the three signals displayed in Fig. [redacted]

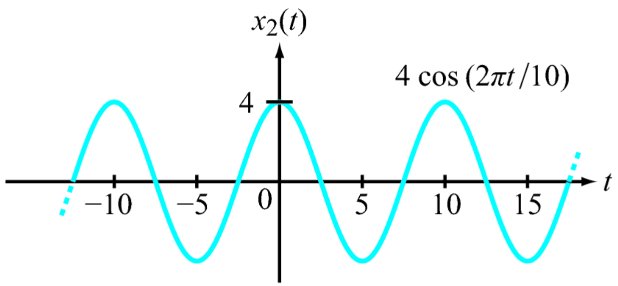
Solution:

(a) Signal $x_1(t)$ is given by

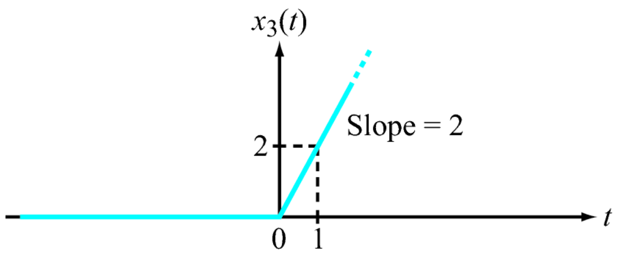
$$x_1(t) = \begin{cases} 0 & \text{for } t \leq 0, \\ 3t & \text{for } 0 \leq t \leq 2, \\ 6e^{-(t-2)} & \text{for } t \geq 2. \end{cases}$$



(a) $x_1(t)$



(b) $x_2(t)$



(c) $x_3(t)$

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energy & power of a signal (example)

$$E_1 = \int_0^2 (3t)^2\,dt + \int_2^\infty \left[6e^{-(t-2)}\right]^2 dt$$

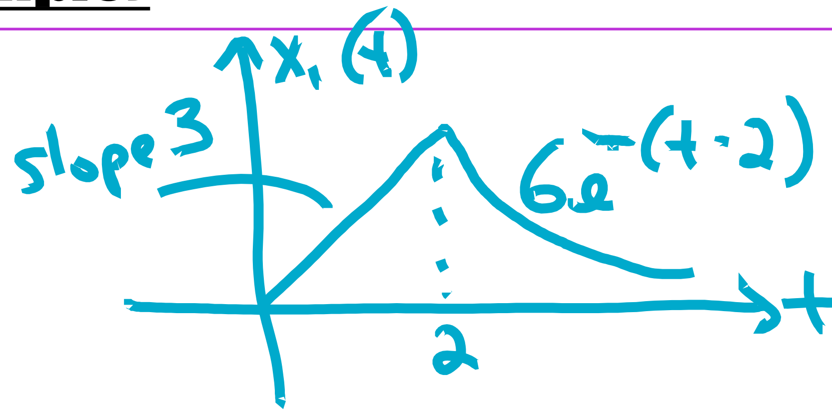
$$= \int_0^2 9t^2\,dt + \int_2^\infty 36e^{-2(t-2)}\,dt$$

$$= \frac{9t^3}{3}\Big|_0^2 + 36e^4 \int_2^\infty e^{-2t}\,dt$$

$$= 24 + 36e^4 \left(-\frac{e^{-2t}}{2}\Big|_2^\infty\right)$$

$$= 42$$

Since E_1 finite, $P_{av} = 0$



$$P_{av} = \lim_{T \to \infty} \frac{E}{T} = 0$$

energy & power of a signal (example)

$$x(t) = 4\cos(2\pi t/10)$$

$$E_2 \to \infty$$

$$P_{av} = \frac{1}{10}\int_{-5}^{5}\left[4\cos\left(\frac{2\pi t}{10}\right)\right]^2 dt$$

$$= \frac{1}{10}\int_{-5}^{5} 16\cos^2\left(\frac{2\pi t}{10}\right)dt$$

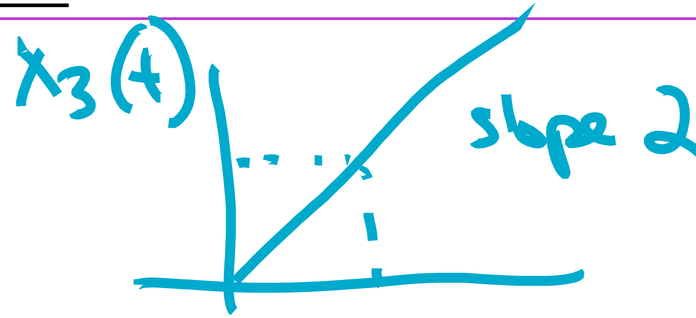
$$= 8$$

P_{av} finite,

$E_2 \to \infty$

energy & power of a signal (example)

$$x_3(t) = \begin{cases} 0 & t \le 0 \\ 2t & t > 0 \end{cases}$$



$$P_{av} = \lim_{T \to \infty} \frac{1}{T} \int_0^{T/2} 4t^2 \, dt$$

$$= \lim_{T \to \infty} \frac{1}{T} \left[\frac{4t^3}{3} \Big|_0^{T/2} \right]$$

$$= \lim_{T \to \infty} \left[\frac{1}{T} \times \frac{4T^3}{24} \right]$$

$$= \lim_{T \to \infty} \left[\frac{T^2}{6} \right] \to \infty$$

also

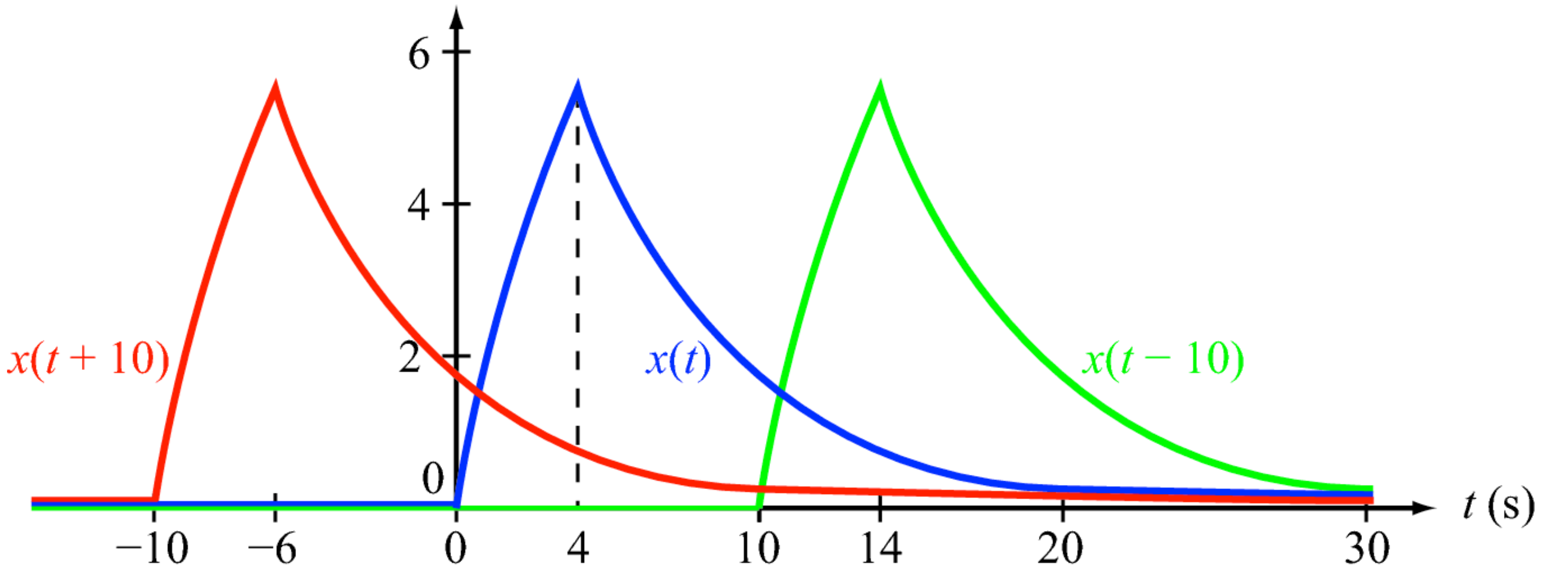
$$E_3 \to \infty$$

signal operations

- time shifting
- time scaling
- time inversion
- amplitude shift
- addition
- multiplication
- combined operations

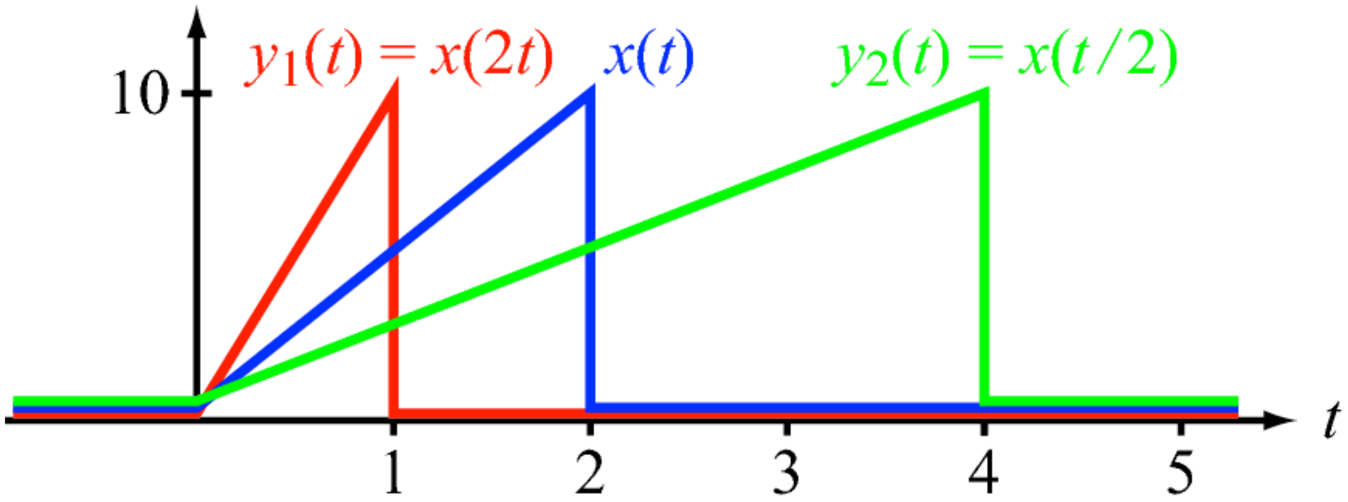
signal operations - time shifting

- shifting: f(t) ; anticipated f(t + T) ; delayed f(t - T)



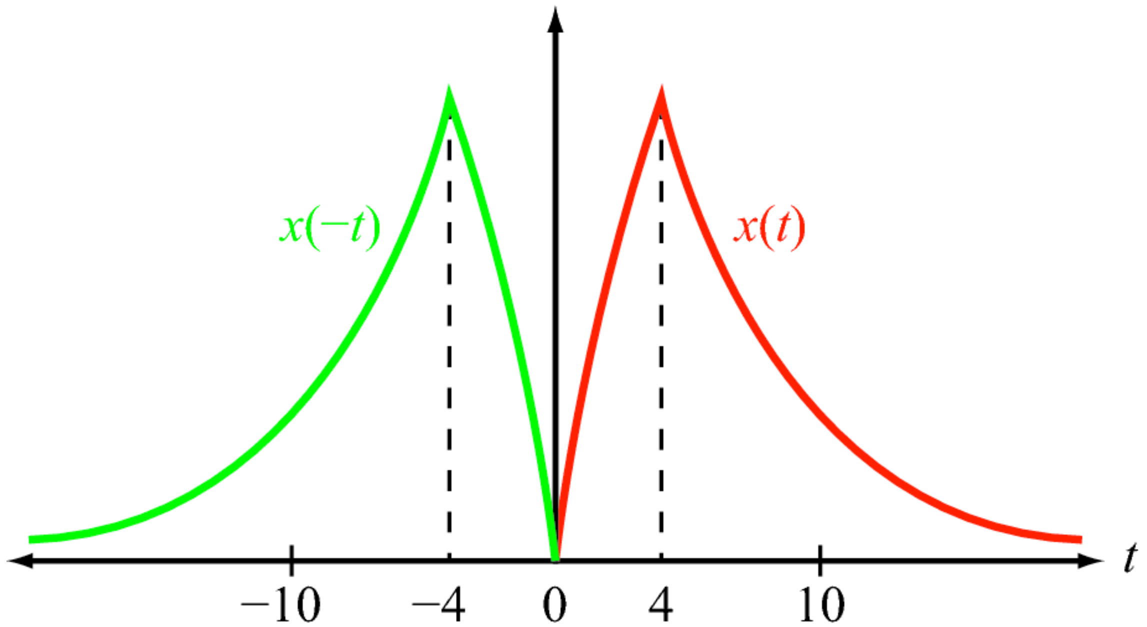
signal operations - time scaling

- time scaling: f(t) ; compression f(at) ; expansion (dilated) f(t/a); a > 1 ; note that a is a real constant,
- 0 < a < 1 is time stretching;
- a = 0 is trivial ; however x(t) cannot be recovered from y(t) when a = 0 ;
- For a > 1 this is compression.



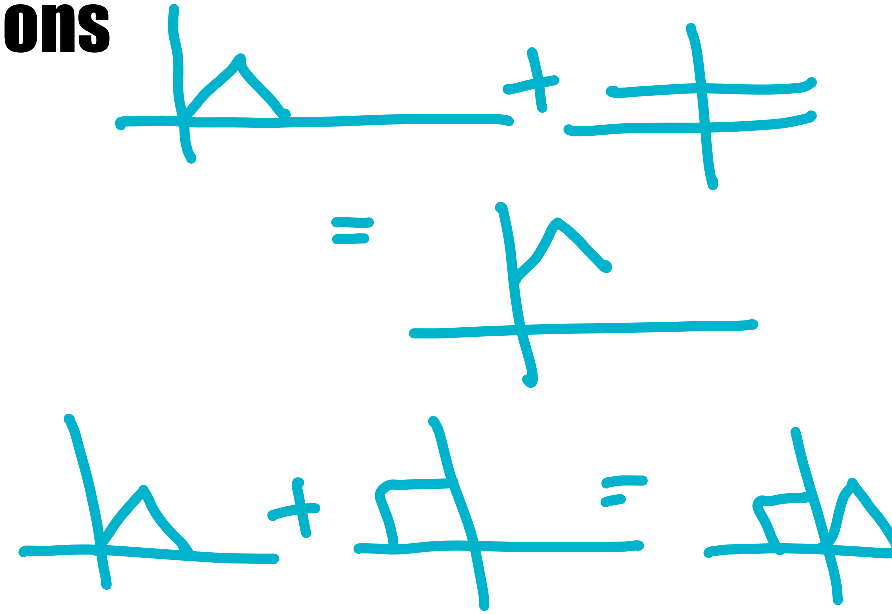
signal operations - time inversion

- time inversion f(t) ; f(-t) ; inverted about f(t) axis



signal operations - other operations

- amplitude shift y(t) = x(t) + b
- addition y(t) = x(t) + z(t)
- multiplication y(t) = x(t) z(t)



signal operations - combined operations

- transformations can be combined into a single transformation
- one way to handle this is to check by BRUTE force however a systematic approach is another way to handle this
- in the expression below, recognize that T=b/a. T is the time shift and a is the compression/expansion factor. The sign of a can also be an inversion factor.

$$y(t) = x(at - b) = x\left(a\left(t - \frac{b}{a}\right)\right) = x(a(t - T))$$

signal operations - multiple transformations

example:

For signal $x(t)$ profiled in Fig. 1-10(a), generate the corresponding profile of $y(t) = x(-2t + 6)$.

Solution:

$y(t) = x(-2t+6)$

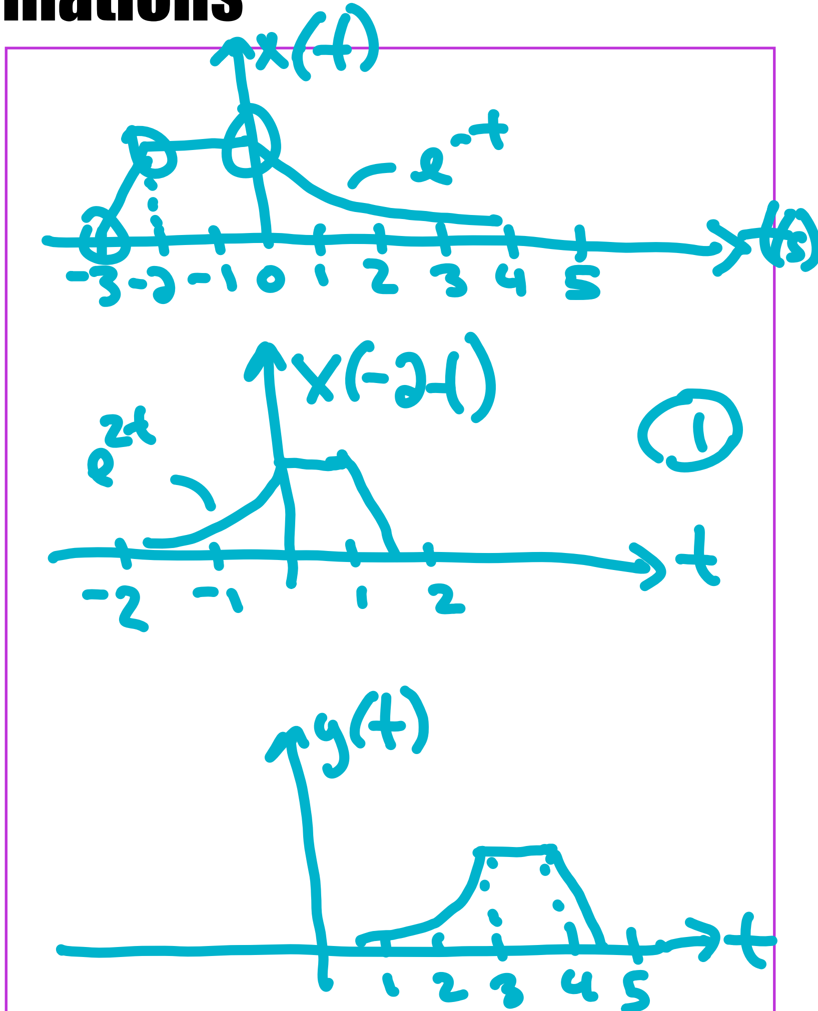
$y(t) = x\left(-2\left(t - \frac{6}{2}\right)\right)$

$= x(-2(t-3))$

reversal compression time shift

① Scale by $-2t$

② Delay waveform by 3 s



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signals - useful functions - step

- unit step

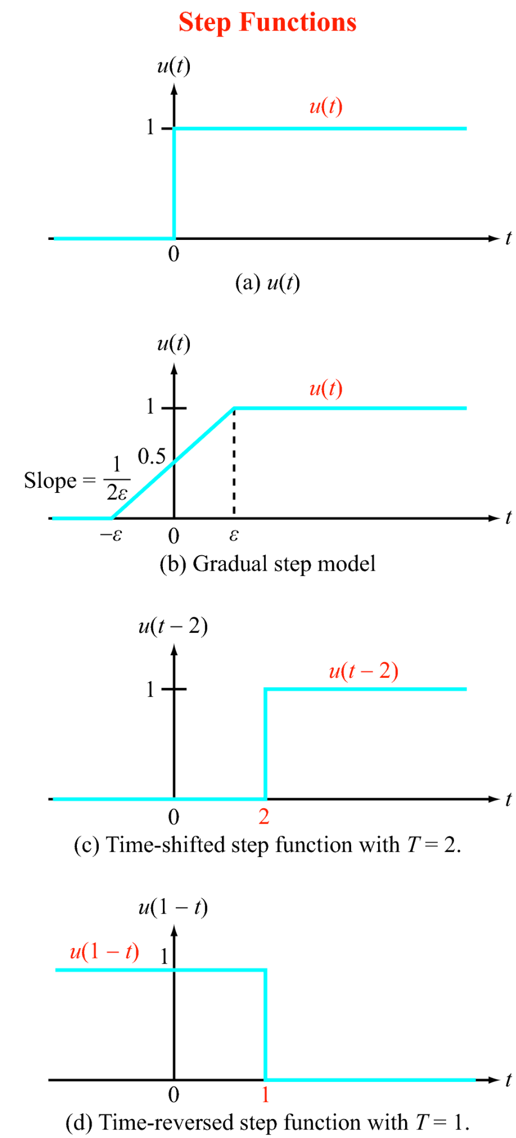
$$\begin{aligned} u(t) &= 1, t \geq 0 \\ &= 0, t < 0 \end{aligned}$$

- realistic step

$$u(t) = \lim_{\epsilon \to 0} \begin{cases} 0 & \text{for } t \leq -\epsilon \\ \left[\frac{1}{2}\left(\frac{t}{\epsilon} + 1\right)\right] & \text{for } -\epsilon \leq t \leq \epsilon \\ 1 & \text{for } t \geq \epsilon, \end{cases}$$

- time shifted step

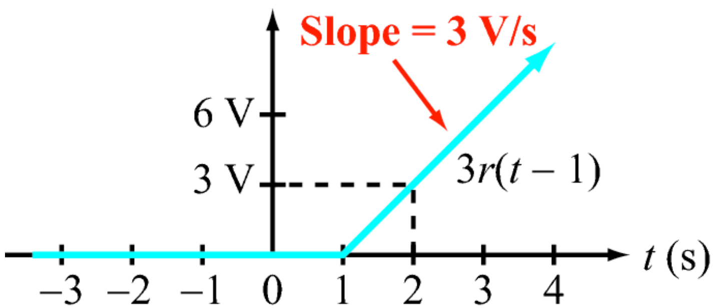
$$u(T - t) = \begin{cases} 1 & \text{for } t < T, \\ 0 & \text{for } t > T. \end{cases}$$



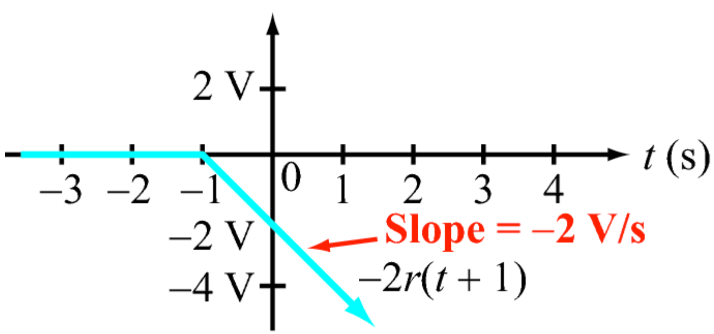
signals - useful functions - ramp

- ramp

Ramp Functions



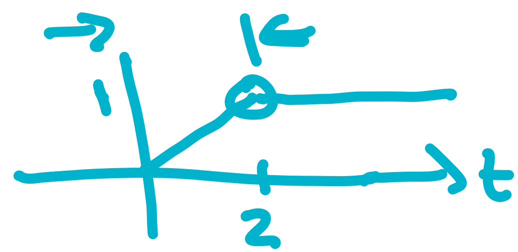
(a)



(b)

$$
\begin{aligned}
r(t, t_0) &= 0, t < 0 \\
&= \frac{t}{t_0}, 0 \leq t \leq t_0 \\
&= 1, t > t_0
\end{aligned}
$$

$$
\begin{aligned}
r(t) &= t, t \geq 0 \\
&= 0, t < 0
\end{aligned}
$$



$\frac{1}{2}\left(r(t)u(t) - r(t)u(t-2)\right) + u(t-2)$

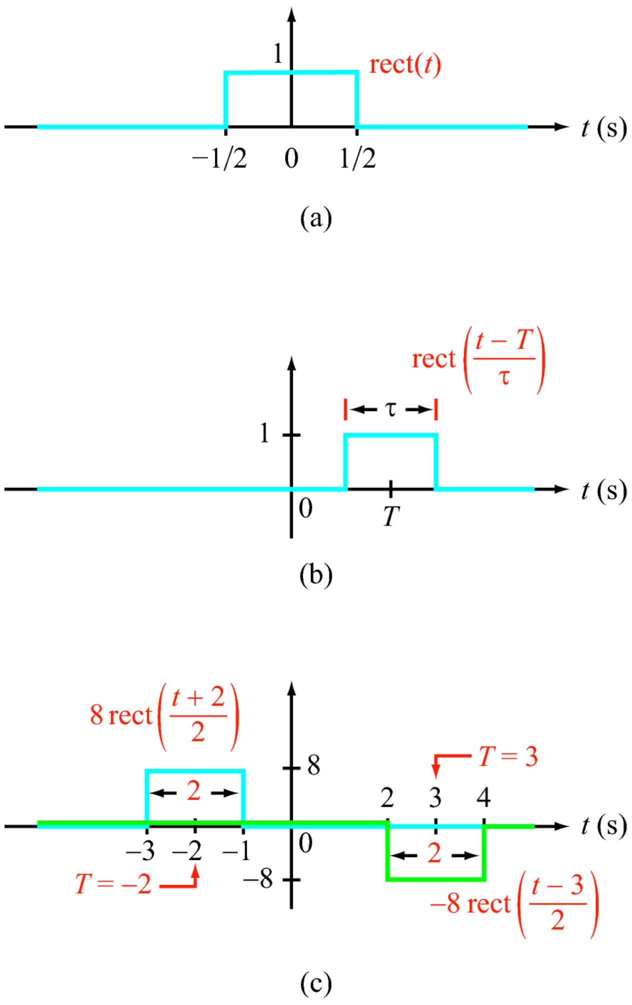
signals - useful functions - unit pulse

- pulse

$$\Pi(t) = 1, |t| < \frac{1}{2}$$
$$= 0, t > \frac{1}{2}$$

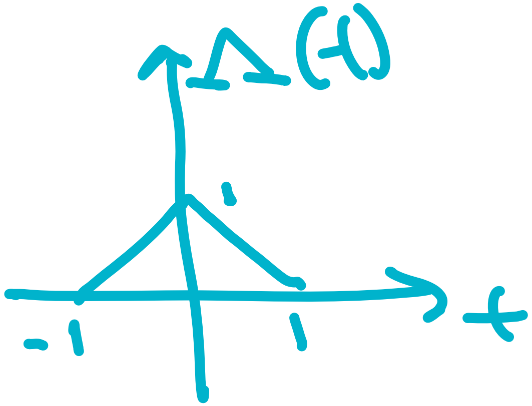
$$\text{rect}\left(\frac{t-T}{\tau}\right) = \begin{cases} 0 & \text{for } t < (T - \tau/2), \\ 1 & \text{for } (T - \tau/2) < t < (T + \tau/2), \\ 0 & \text{for } t > (T + \tau/2). \end{cases}$$

Rectangular Pulses



signals - useful functions - triangle

- unit triangle function

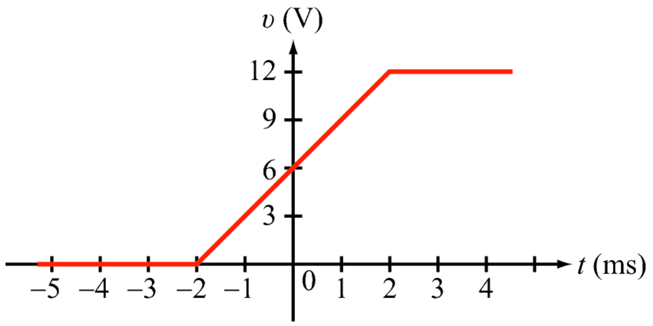


$$
\begin{aligned}
\Lambda(t) &= t+1, -1 \leq t < 0 \\
&= 1-t, 0 \leq t < 1 \\
&= 0, otherwise
\end{aligned}
$$

signals - useful functions - waveform synthesis

- various combinations

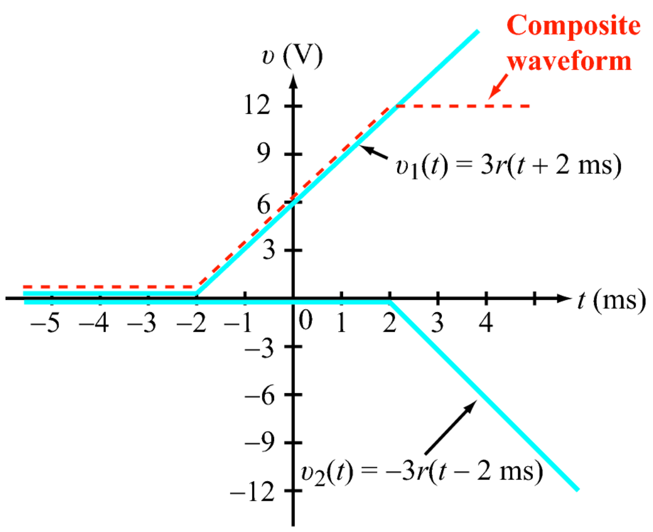
$$v(t) = v_1(t) + v_2(t)$$
$$= 3r(t + 2\text{ ms}) - 3r(t - 2\text{ ms})$$



(a) Original function

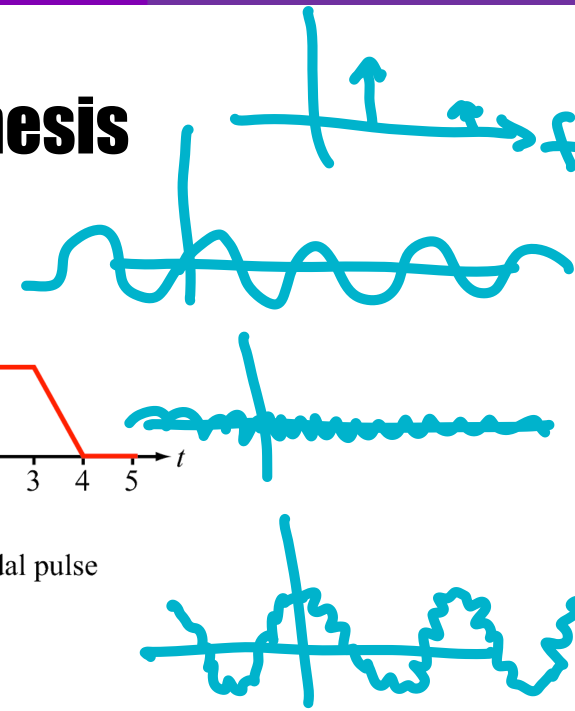
- equivalently

$$v(t) = 3(t + 2\text{ ms})\, u(t + 2\text{ ms})$$
$$- 3(t - 2\text{ ms})\, u(t - 2\text{ ms})$$

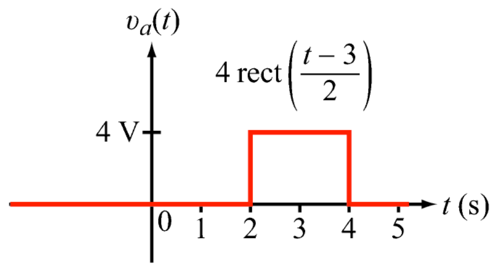


(b) As sum of two time-shifted ramp functions

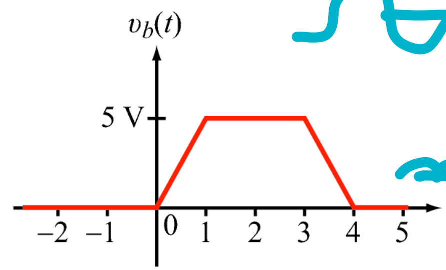
signals - useful functions - waveform synthesis



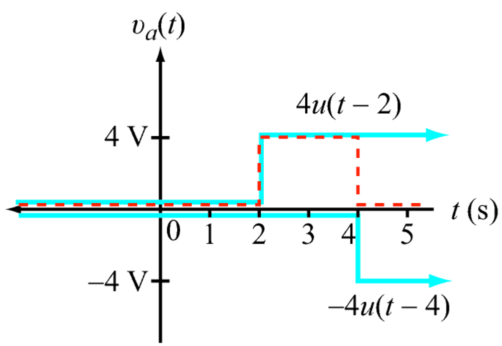
Waveform Synthesis



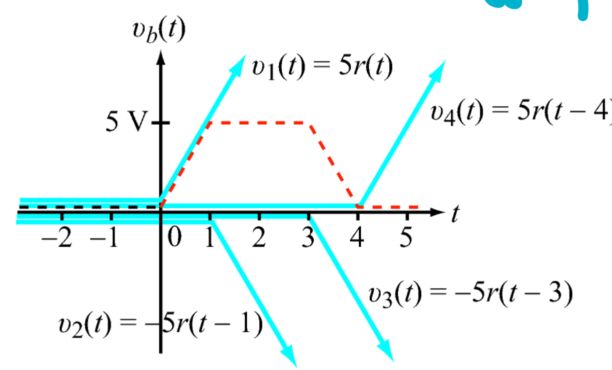
(a) Rectangular pulse



(b) Trapezoidal pulse



(c) $v_a(t) = 4u(t-2) - 4u(t-4)$



(d) $v_b(t) = v_1(t) + v_2(t) + v_3(t) + v_4(t)$

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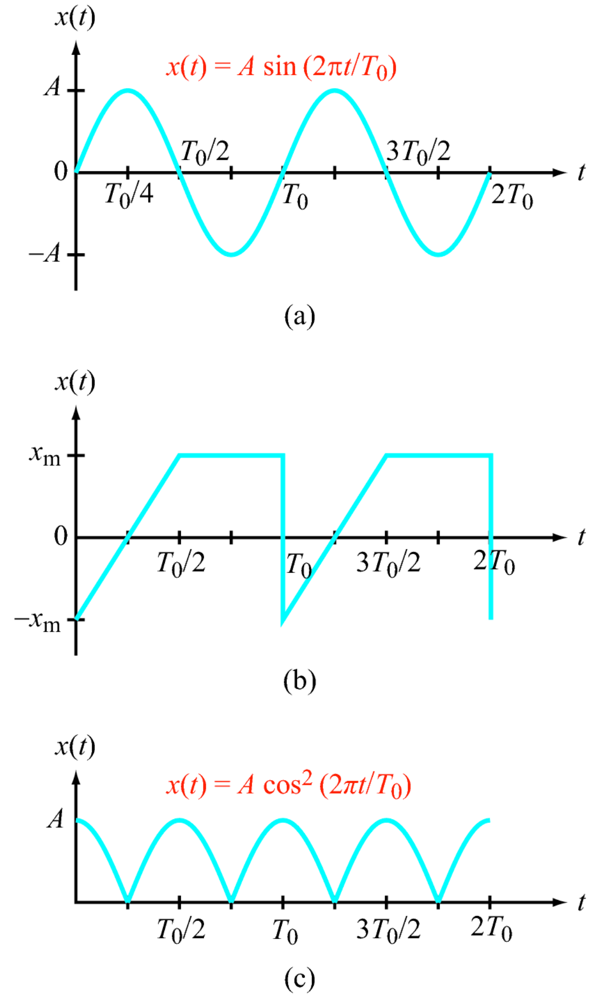
signals - useful functions - periodic waveforms

- a periodic signal x(t) of period T satisfies the periodicity property:
- x(t) = x(t +nT)
- for all integer values of n and all times t



signals - useful functions - sinusoids

sinusoidal function: $x(t) = A\cos(w_o t + \theta)$; A is the amplitude of the signal, w_0 is the radian frequency in radians per second units and θ is the phase angle in radians; recall that $w_0 = 2\pi f_0$

- if $\theta = 0$, the cosine waveform has a peak at the origin $t_0 = 0$, and the positive or negative peaks of the waveform would occur at time instants t_k solution to $2\pi f_0 t_k = k\pi$ where k is an integer and $t_k = \frac{k}{2f_0}$
- with a different value of θ, first peak is shifted to $t_0 = \frac{-\theta}{2\pi f_0}$ where $t_k = \frac{k\pi - \theta}{2\pi f_0}$ which are solutions to $2\pi f_0 t_k + \theta = k\pi$ where k is an integer

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signals - useful functions - impulse function

unit impulse function $\delta(t) = 0, t \neq 0$ and $\int_{-\infty}^{\infty} \delta(t)dt = 1$;; some properties are: Figure 5

$\times\ \delta(t) = 1$
$= \phi(0)$

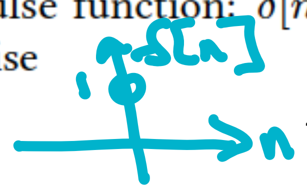
- multiplication of a function by an impulse
  $\phi(t)\delta(t) = \phi(0)\delta(t)$
  $\phi(t)\delta(t-T) = \phi(T)\delta(t-T)$

- sampling property
  $\int_{-\infty}^{\infty} \phi(t)\delta(t)dt = \int_{-\infty}^{\infty} \phi(0)\delta(t)dt = \phi(0)\int_{-\infty}^{\infty} \delta(t)dt = \phi(0)$

- the area under the curve obtained by the product of the unit impulse function shifted by T and is the value of the function $\phi(t)$ for $t = T$
  $\int_{-\infty}^{\infty} \phi(t)\delta(t-T)dt = \phi(T)$

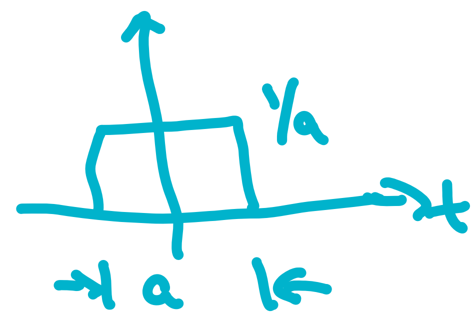
- unit step is the integral of the unit impulse function
  $\frac{du(t)}{dt} = \delta(t); \int_{-\infty}^{t} \delta(t)dt = u(t)$

- discrete time impulse function: $\delta[n] = 1$ if $n = 0$ or 0 otherwise

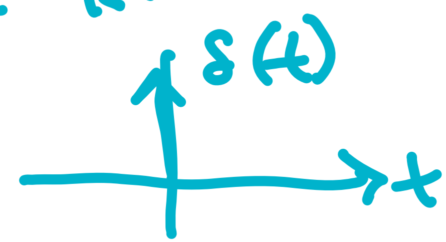


Time scaling property of Dirac

$$\delta(at) = \frac{1}{|a|}\delta(t).$$



let $a \to 0$

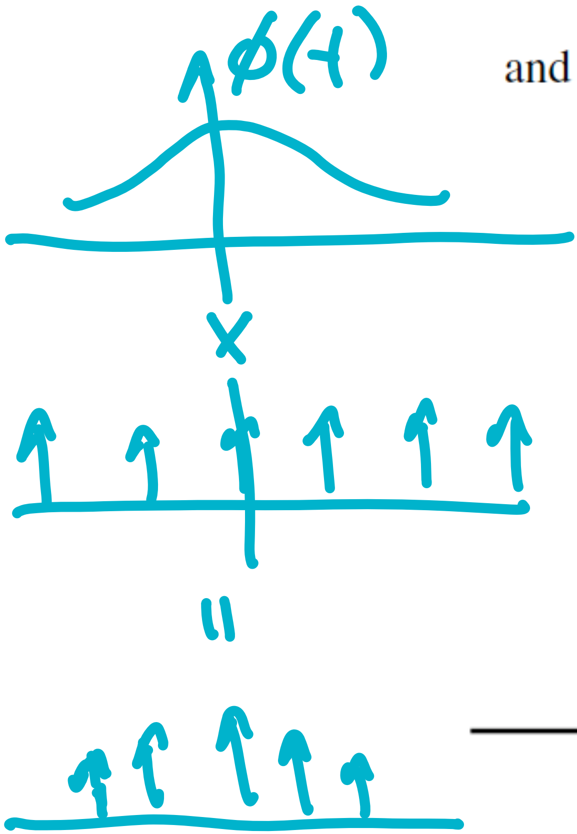


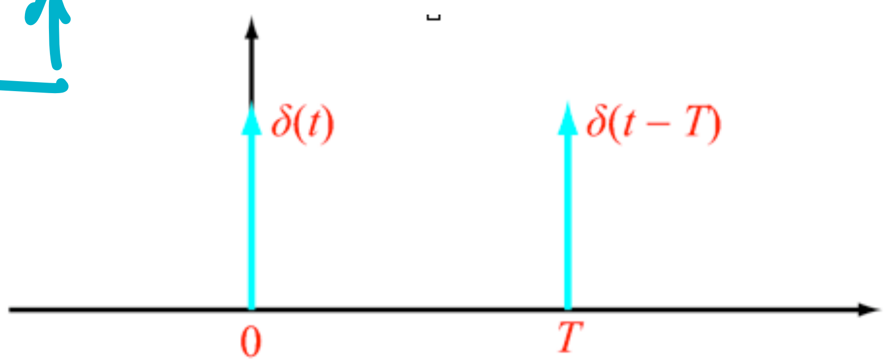
signals - useful functions - impulse function

$$\delta(t - T) = 0 \qquad \text{for } t \neq T$$

and

$$\int_{-\infty}^{\infty} \delta(t - T)\, dt = 1.$$



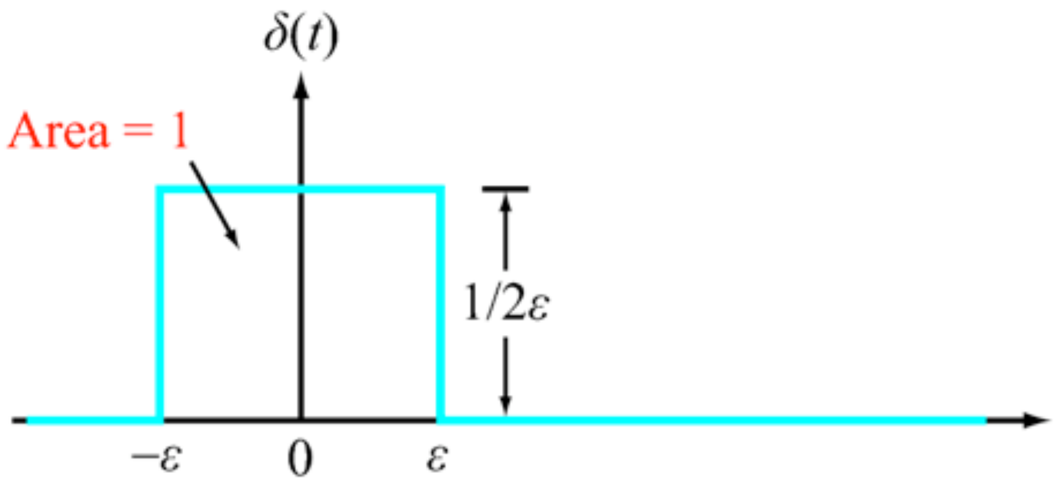


(a) $\delta(t)$ and $\delta(t - T)$

signals - useful functions - impulse function

$$\frac{d}{dt}[u(t-T)] = \delta(t-T),$$

$$u(t-T) = \int_{-\infty}^{t} \delta(\tau - T)\, d\tau.$$



(b) Rectangle model for $\delta(t)$

signals - useful functions - sampling property of dirac (delta)

$$\int_{-\infty}^{\infty} x(t)\, \delta(t - T)\, dt = x(T).$$

(sampling property)

$$\int_{-\infty}^{\infty} x(t) \boxed{\sum_{n=-\infty}^{n=\infty} \delta(t-n)} dt$$ - impulse train

signals - useful functions - example

Evaluate $\int_1^2 t^2\, \delta(2t-3)\, dt$.

Solution:

$\delta(2t-3) = \delta\left(2\left(t-\frac{3}{2}\right)\right)$

$= \frac{1}{2}\delta\left(t-\frac{3}{2}\right)$

be

$\int_1^2 t^2 \delta(2t-3)dt = \frac{1}{2}\int_1^2 t^2 \delta\left(t-\frac{3}{2}\right)dt$

$= \frac{1}{2}\left(\frac{3}{2}\right)^2$

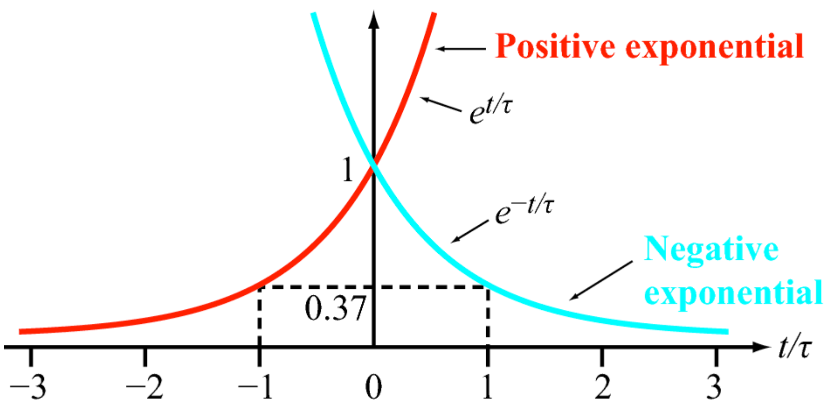
$= \frac{9}{8}$

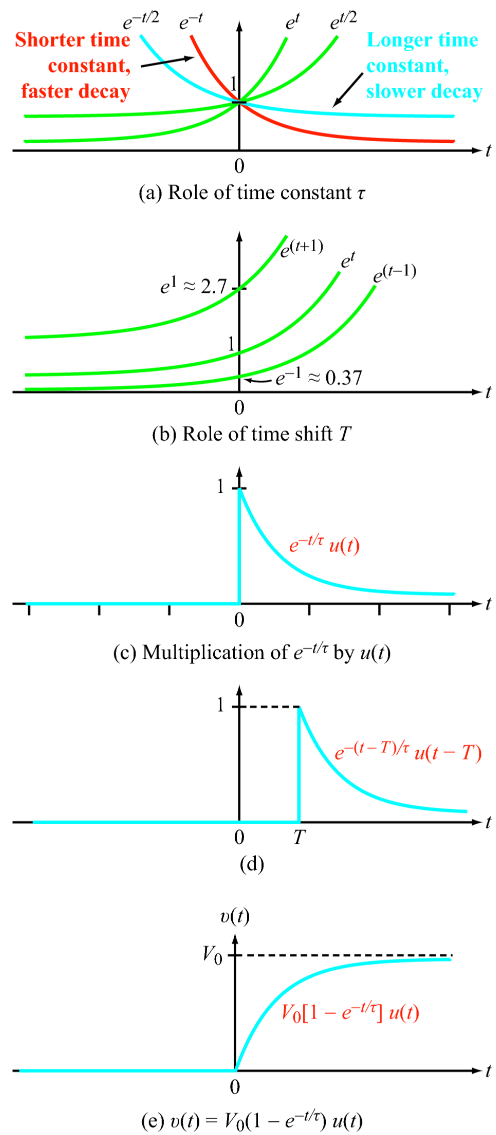
$\int f(t)\delta(t-a)dt$

$= f(a)$

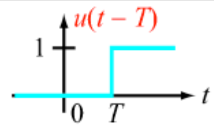
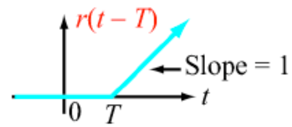
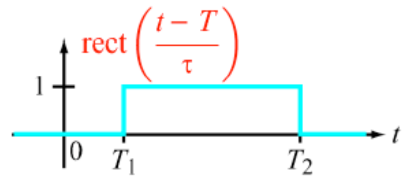
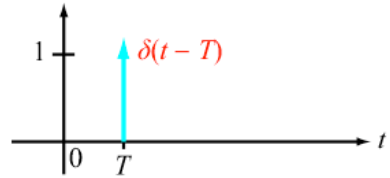
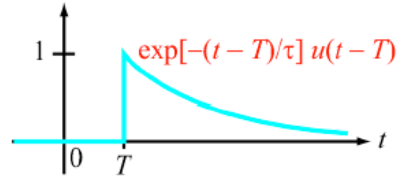
signals - useful functions - exponential

- exponential waveform





signals - useful functions - summary

Function	Expression	General Shape
Step	$u(t-T) = \begin{cases} 0 & \text{for } t < T \\ 1 & \text{for } t > T \end{cases}$	
Ramp	$r(t-T) = (t-T)\,u(t-T)$	
Rectangle	$\text{rect}\left(\frac{t-T}{\tau}\right) = u(t-T_1) - u(t-T_2)$ $T_1 = T - \frac{\tau}{2}; \quad T_2 = T + \frac{\tau}{2}$	
Impulse	$\delta(t-T)$	
Exponential	$\exp[-(t-T)/\tau]\,u(t-T)$	

signals - useful functions - complex exponential

- complex exponential waveform
- review complex numbers

$f(t) = Ae^{jwt}$

- Eulers relations
  $Ae^{jwt} = A\cos(wt) + jA\sin(wt)$

  $\cos(wt) = \frac{e^{jwt}+e^{-jwt}}{2}$
  $\sin(wt) = \frac{e^{jwt}-e^{-jwt}}{2}$
  $e^{jwt} = \cos(wt) + j\sin(wt)$

- discrete time complex exponential
  $f[n] = Be^{snT}$
  $= Be^{jwnT}$
  where $s = jw$ and $k = nT$

- function generalization e^{st}
  let $s = \sigma + jw$
  therefore $e^{st} = e^{(\sigma+jw)t} = e^{\sigma t}e^{jwt}$
  $= e^{\sigma t}(\cos(wt) + j\sin(wt))$
  if $s^* = \sigma - jw$ that being the conjugate of s, then
  $e^{s^*t} = e^{(\sigma-jw)t} = e^{\sigma t}e^{-jwt}$
  $=e^{\sigma t}(\cos(wt) - j\sin(wt))$
- e^{st} can be used to model a large class of signals
  - if $s = 0$ then the signal is a constant k
  - if $w = 0$ and $s = \sigma$ then the signal is a monotonic exponential $e^{\sigma t}$
  - if $\sigma = 0$ and $s = \pm jw$ then the signal is a sinusoid $\cos(wt)$

  - if $s = \sigma \pm jw$ then the signal is an exponentially varying sinusoid $e^{\sigma t}\cos(wt)$

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signals - discrete time signals

- 3 ways to define
  - graphically
  - as a discrete function
  - as a list with a datum
- operations
  - arithmetic with constant
  - multiplication with constant
  - arithmetic 2 series
  - multiplication
  - time shift
  - time scale
  - time reversal

arithmetic with constant $g[n] = x[n] + A$

multiplication with constant $g[n] = Bx[n]$

arithmetic with 2 series $g[n] = x[n] + y[n]$, note: make sure Indices line up

mutliplication with 2 series $g[n] = x[n]y[n]$, same note, make sure indices line up

time shifting $g[n] = x[n-k]$, k is an integer

time scaling $g[n] = x[kn]$, k is an integer

time reversal $g[n] = x[-n]$

signals - discrete time signals

unit impulse

$$\begin{aligned}\delta[n] &= 1, n = 0 \\ &= 0, n \neq 0\end{aligned}$$

$$\begin{aligned}a\delta[n - n_1] &= a, n = n_1 \\ &= 0, n \neq n_1\end{aligned}$$

unit step function

$$\begin{aligned}u[n] &= 1, n \geq 0 \\ &= 0, n < 0\end{aligned}$$

unit ramp function

$$\begin{aligned}u[n] &= n, n \geq 0 \\ &= 0, n < 0\end{aligned}$$

signals - discrete time signals

sinusoidal signals
$x[n] = A\cos(\Omega_0 n + \theta)$, A is amplitude, Ω_0 is the angular frequency in radians and θ is the phase angle in radians

$$
\begin{aligned}
x_k[n] = x[k]\delta[n-k] &= x[k], n = k \\
&= 0, n \neq k
\end{aligned}
$$

real vs. complex
$x[n] = x_r[n] + x_i[n]$ or in polar form
$x[n] = |x[n]|e^{\angle x[n]}$

signals - discrete time signals

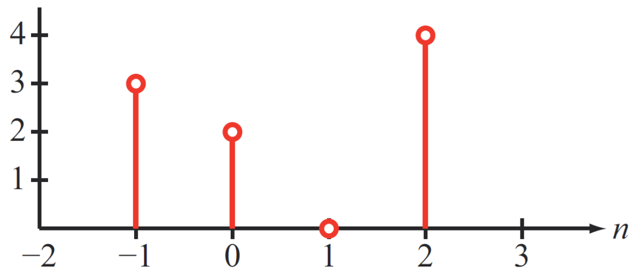
- discrete-time signals x[n] are defined only at integer times n
- continuous time signals x(t) are defined for all real numbers t
- discrete time signals can be represented in 3 different ways:

1. Enclosing all non-zero values in brackets: $x[n]=\{3,\underline{2},0,4\}$. The underline indicates x[n] at time n=0. Here, x[0]=2.

2. Listing all of its non-zero values:

$$x[n] = \begin{cases} 3 & \text{for } n = -1, \\ 2 & \text{for } n = 0, \\ 4 & \text{for } n = 2, \\ 0 & \text{for all other } n. \end{cases}$$

3. Using a stem plot:

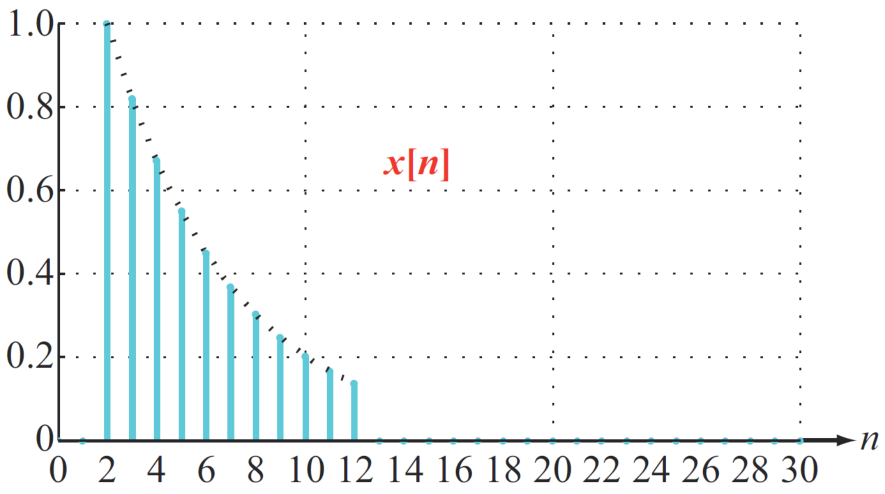


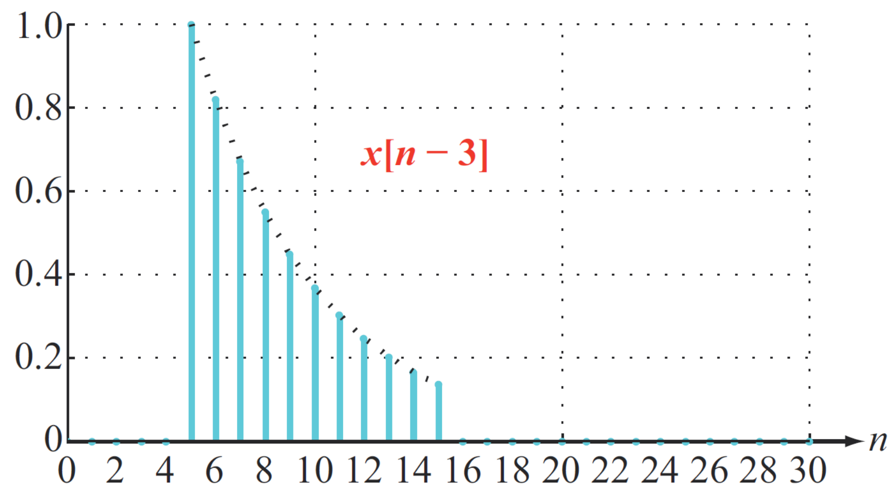
signals - discrete time signals - finite length

Let x[n]=0 for all n<a and all n>b. The length or duration of x[n] is b-a+1, not b-a.

Example: Let x[n]={3,2,0,4}=0 for n<-1 and n>2. The length of x[n] is 2-(-1)+1=4.

Time shifts work the same way in discrete time as time delays work in continuous time.



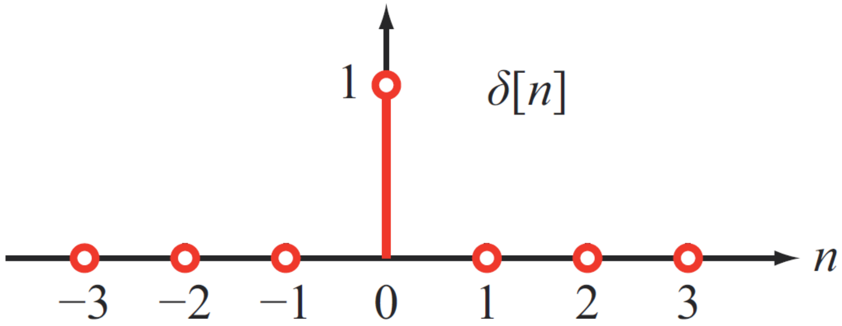


signals - discrete time signals - functions

discrete-time impulse, discrete-time step

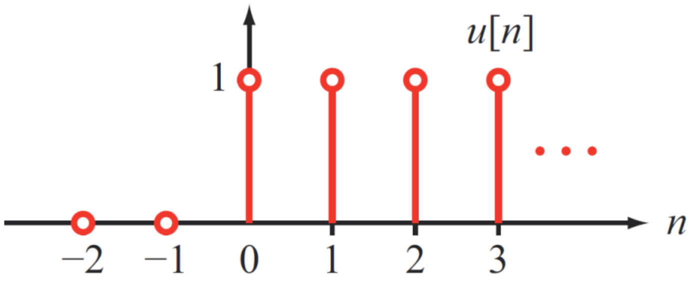
Discrete-time Impulse:

$$\delta[n] = \{\underline{1}\} = \begin{cases} 1 & \text{for } n = 0, \\ 0 & \text{for } n \neq 0. \end{cases}$$



Discrete-time Step:

$$u[n] = \{\underline{1}, 1, 1, \ldots\} = \begin{cases} 1 & \text{for } n \geq 0, \\ 0 & \text{for } n < 0, \end{cases}$$



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signals - discrete time signals - impulses & steps

discrete vs. cts. time : sampling, steps, impulses

Property:	Discrete Time:	Continuous Time:
Sampling:	$x[n] = \sum_{i=-\infty}^{\infty} x[i]\,\delta[i-n]$	$x(T) = \int_{-\infty}^{\infty} x(t)\,\delta(t-T)\,dt$
Relation between step and impulse	$u[n] = \sum_{i=-\infty}^{n} \delta[i]$	$u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau$
Value at time=0	u[0]=1 and (0)=1.	u(0) undefined (0) → ∞

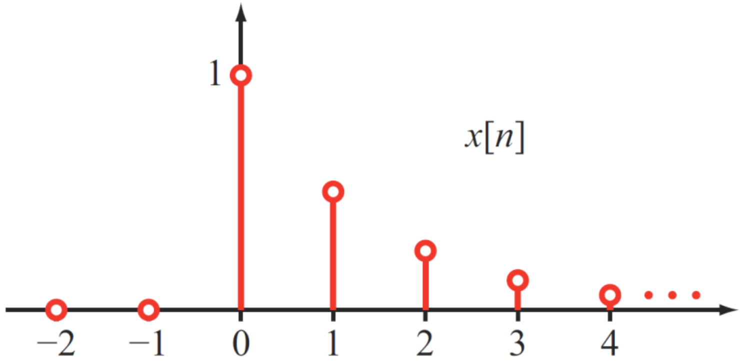
signals - discrete time signals - exponentials

discrete vs. cts. time : geometric signals, exponential signals

Discrete Time:

Geometric Signals

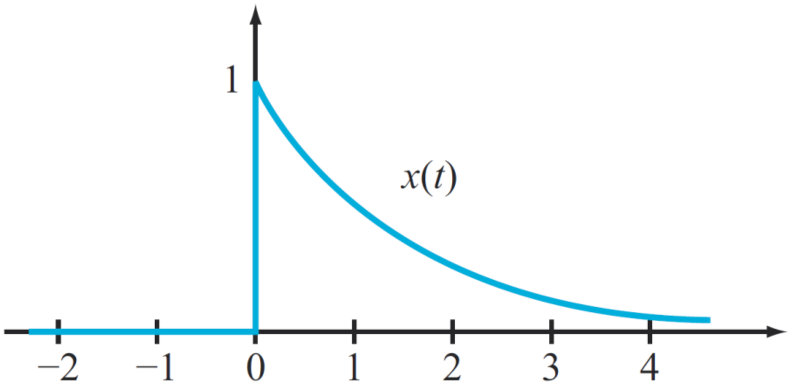
$$x[n] = \mathbf{p}^n \ u[n]$$



Continuous Time:

Exponential Signals

$$x_1(t) = e^{-at} \ u(t)$$



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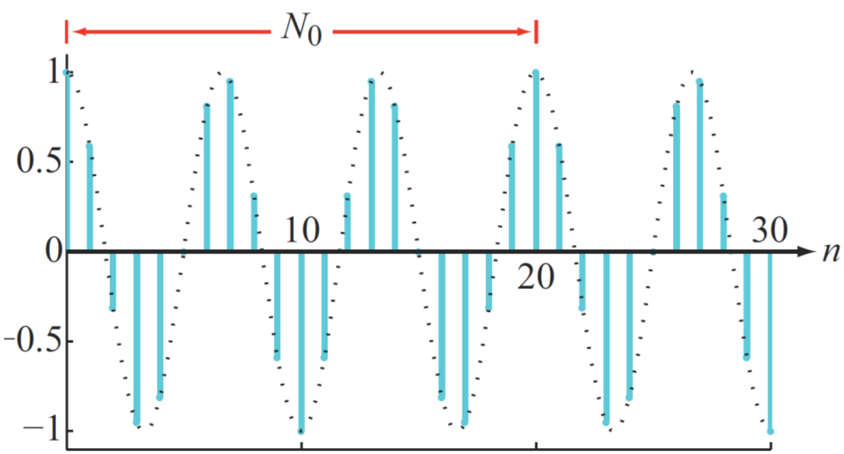
signals - discrete time signals - sinusoids

discrete vs. cts. time : sinusoids

periodic vs. non-periodic, if signal satisfies $x[n] = x[n+N]$ then the signal is periodic for an integer value n for $N \neq 0$

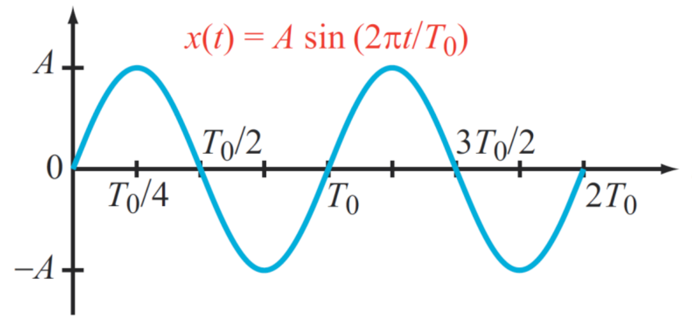
Discrete Time:

$$= A\cos(\Omega n + \theta)$$



Continuous Time:

$$x(t) = A\cos(\omega_0 t + \theta)$$



Period: $N_0 = \dfrac{2\pi k}{\Omega}$ if k exists that makes this an integer

signals - discrete time signals - example

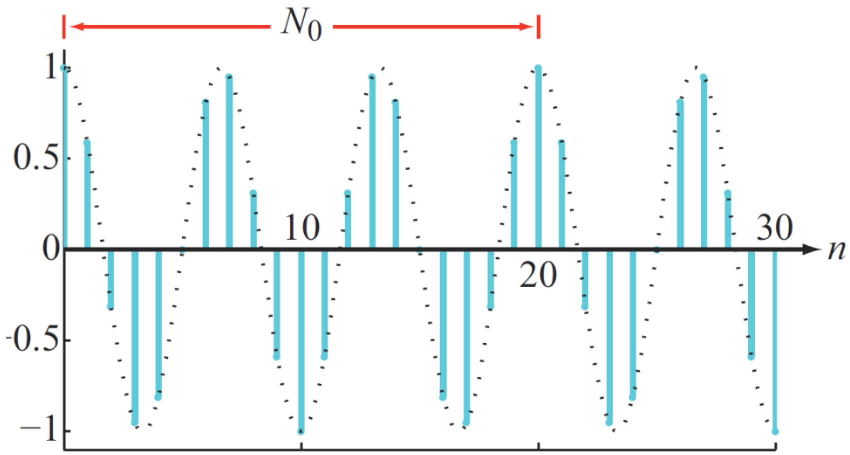
period of discrete time sinusoid

Goal: Determine the period of $x[n] = \cos(0.3\pi n)$

Solution: $N_0 = \frac{2\pi k}{\Omega} = \frac{2\pi k}{0.3\pi} = \frac{20k}{3}$ k=3→period=20.

x[n+20]=cos(0.3π(n+20))=cos(0.3πn+6π)=cos(0.3πn).

Note: Most discrete-time sinusoids are not periodic!



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signals - discrete time signals - example

To find the integer period $N > 0$ (i.e., ($N \in \mathbb{Z}^+$) of a general discrete-time sinusoid $x[n] = A\cos(\Omega n + \phi)$:

$$
\begin{aligned}
x[n] &= x[n + N] \\
A\cos(\Omega n + \phi) &= A\cos(\Omega(n + N) + \phi) \\
A\cos(\Omega n + \phi + 2\pi k) &= A\cos(\Omega n + \phi + \Omega N) \\
\therefore \quad 2\pi k &= \Omega N \\
N &= \frac{2\pi k}{\Omega}
\end{aligned}
$$

where $k \in \mathbb{Z}$.

<u>Note</u>: there may not exist a $k \in \mathbb{Z}$ such that $\frac{2\pi k}{\Omega}$ is an integer.

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signals - discrete time signals - example

Example i: $\Omega = \frac{37}{11}\pi$

Example ii: $\Omega = 2$

Example iii: $\Omega = \sqrt{2}\pi$

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signals - discrete time signals - example

$$
\begin{aligned}
N &= \frac{2\pi k}{\Omega} \\
\Omega &= \frac{2\pi k}{N} = 2\pi \frac{k}{N} = \pi \cdot \underbrace{\boxed{\frac{2k}{N}}}_{RATIONAL}
\end{aligned}
$$

Therefore, a discrete-time sinusoid is periodic if its radian frequency Ω is a rational multiple of π.

Otherwise, the discrete-time sinusoid is non-periodic.

signals - discrete time signals - example

Example 1: $\Omega = \pi/6 = \pi \cdot \boxed{\frac{1}{6}}$

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signals - discrete time signals - example

Example 2: $\Omega = 8\pi/31 = \pi \cdot \boxed{\frac{8}{31}}$

signals - discrete time signals - example

Example 3: $\Omega = 1/6 = \pi \cdot \boxed{\frac{1}{6\pi}}$

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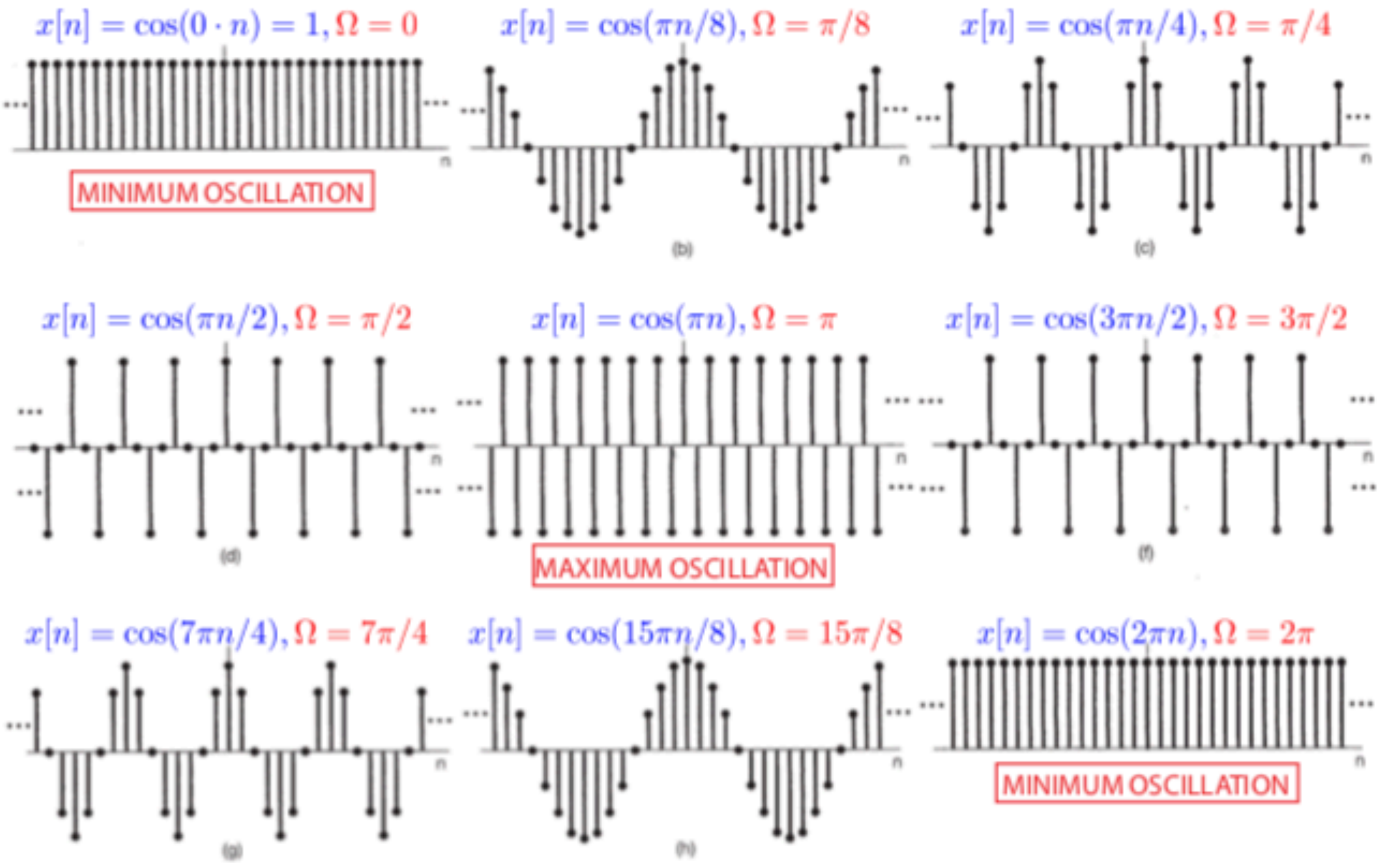
signals - discrete time signals - example



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signals - discrete time signals - properties

periodic vs. non-periodic, if signal satisfies $x[n] = x[n+N]$ then the signal is periodic for an integer value n for $N \neq 0$

energy of a signal
$E_s = \sum_{n=-\infty}^{n=\infty} x^2[n]$ and
$E_s = \sum_{n=-\infty}^{n=\infty} |x[n]|^2$ if complex

time averaging $< x[n] >= \frac{1}{N} \sum_{x=0}^{n=N-1} x[n]$

power of a signal $P_s =< x^2[n] >$

signals - discrete time signals - qualitative properties

general

- u decays if $u(t) \to 0$ as $t \to \infty$
- u converges if $u(t) \to a$ as $t \to \infty$ where a is some constant
- u is bounded if its peak is finite
- u is unbounded or blows up if its peak is infinite u is periodic if for some $T > 0, u(t+T) = u(t)$ holds for all t

- in practice we are interested in more specific quantitative questions such as (i) how fast does u decay or converge or (ii) how large is the peak of u

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signals - discrete time signals - qualitative properties

bounded

- a continuous signal bounded if there exists a M s.t. for all $t \in R$, $|x(t)| \leq M$
- a discrete signal is bounded if there exists an M s.t. for all $n \in Z$, $|x[n]| \leq M$
- a signal is unbounded if it is not bounded

stable

- a continuous signal is absolutely integratable if $\int_{-\infty}^{\infty} |x(t)| < \infty$
- a discrete time signal is absolutely summable if $\sum_{-\infty}^{\infty} |x[n]| < \infty$
- a CT (continuous time) or DT (discrete time) signal is stable if it is absolutely integratable or summable

signals - discrete time signals - qualitative properties

causal

- a CT signal is causal if for all $t < 0$, $x(t) = 0$

side-ness

- a CT signal is right-sided if there exists a T s.t. for all $t < T$, $x(t) = 0$
- a DT signal is right-sided if there exists a T s.t. for all $n < T$, $x[n] = 0$
- a CT signal is left-sided if there exists a T s.t. $t > T$, $x(t) = 0$
- a DT signal is left-sided if there exists a T s.t. for all $n > T$, $x[n] = 0$
- a CT or DT signal is said to be two-sided if it is not left-sided or right-sided


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