Part 3. Linear Algebraic Equations
Chapter 11. Special Matrices and Gauss-Seidel

Lecture 13 & 14

Special Matrices and Gauss Seidel

11.1, 11.2

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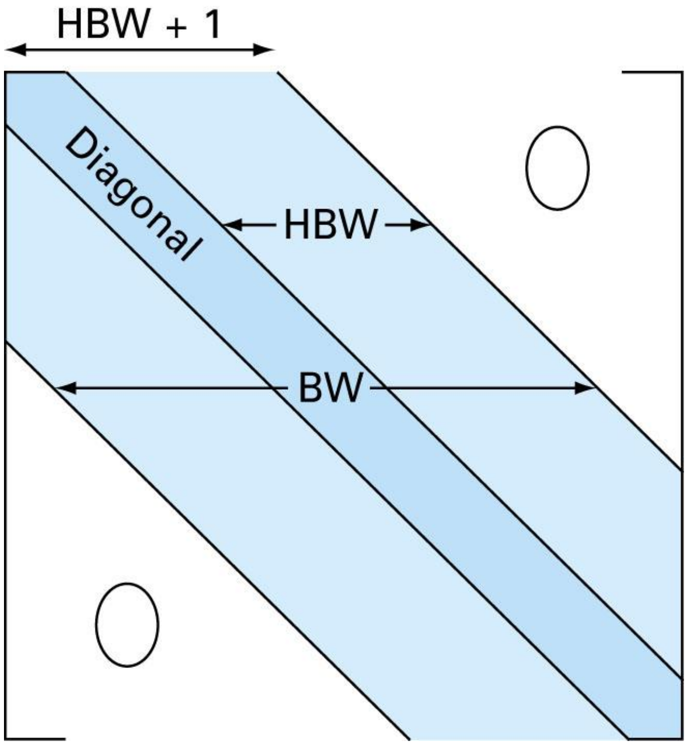
Objectives and Learning Outcomes

- Solve a set of equations using Gauss-Seidel method
- Determine convergence conditions of Gauss-Seidel method
- Recognize the pitfalls of the method and how to improve it

Special Matrices

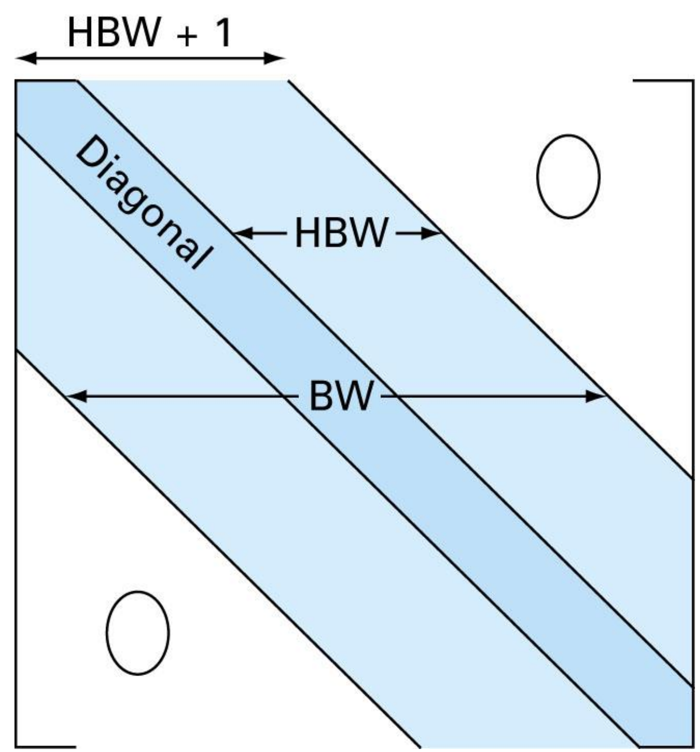
- Certain matrices have particular structures that can be exploited to develop efficient solution schemes.

Banded matrix is a square matrix that has all elements equal to zero, with the exception of a band centered on the main diagonal. These matrices typically occur in solution of differential equations.



Special Matrices

The dimensions of a banded system can be quantified by two parameters: the band width BW and half-bandwidth HBW. These two values are related by **BW=2HBW+1**.



- Gauss elimination or conventional LU decomposition methods are inefficient in solving banded equations because pivoting becomes unnecessary.

Gauss-Seidel: Features

- It is an iterative or approximate method
- Provides an alternative to the elimination methods
- One of the most commonly used iterative method.
- The system $[A]\{X\}=\{B\}$ is reshaped by solving the first equation for x_1, the second equation for x_2, and the third for x_3, …and n^{th} equation for x_n.

Gauss-Seidel Method-Iterative Procedure

1

- Solve each linear equation algebraically for x_i

2

- Assume an initial guess solution array

3

- Solve for each x_i and repeat

4

- Use absolute relative approximate error after each iteration and compare with a pre-specified tolerance.

Gauss-Seidel Method- Motivation

A

Error Control

- The Gauss-Seidel Method allows the user to control round-off error

B

Superiority

- Elimination methods such as Gaussian Elimination and LU Decomposition are prone to prone to round-off error.

C

Initial Guess Close to Solution

•If the physics of the problem are understood, a close initial guess can be made, decreasing the number of iterations needed.

Gauss-Seidel Method-Algorithm

n equations and *n* unknowns:

If: the diagonal elements are non-zero

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n &= b_n
\end{aligned}
$$

Rewrite each equation solving for the corresponding unknown

Gauss-Seidel Method-Algorithm

Rewrite each equation:

From equation *1*

$$x_1 = \frac{c_1 - a_{12}x_2 - a_{13}x_3 \ldots\ldots - a_{1n}x_n}{a_{11}}$$

From equation 2

$$x_2 = \frac{c_2 - a_{21}x_1 - a_{23}x_3 \ldots\ldots - a_{2n}x_n}{a_{22}}$$

.
.
.

$$\vdots \qquad \vdots \qquad \vdots$$

From equation *n-1*

$$x_{n-1} = \frac{c_{n-1} - a_{n-1,1}x_1 - a_{n-1,2}x_2 \ldots\ldots - a_{n-1,n-2}x_{n-2} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$

From equation *n*

$$x_n = \frac{c_n - a_{n1}x_1 - a_{n2}x_2 - \ldots\ldots - a_{n,n-1}x_{n-1}}{a_{nn}}$$

Gauss-Seidel Method-Algorithm

General form for x_i

$$x_1 = \frac{c_1 - \sum\limits_{\substack{j=1 \\ j\neq 1}}^{n} a_{1j} x_j}{a_{11}}$$

$$x_{n-1} = \frac{c_{n-1} - \sum\limits_{\substack{j=1 \\ j\neq n-1}}^{n} a_{n-1,j} x_j}{a_{n-1,n-1}}$$

$$x_2 = \frac{c_2 - \sum\limits_{\substack{j=1 \\ j\neq 2}}^{n} a_{2j} x_j}{a_{22}}$$

$$x_n = \frac{c_n - \sum\limits_{\substack{j=1 \\ j\neq n}}^{n} a_{nj} x_j}{a_{nn}}$$

Gauss-Seidel Method-Algorithm

First Iteration

$$x_1 = (c_1 - a_{12}x_2 - a_{13}x_3)/a_{11}$$

$$x_2 = (c_2 - a_{21}x_1 - a_{23}x_3)/a_{22}$$

$$x_3 = (c_3 - a_{31}x_1 - a_{32}x_2)/a_{33}$$

Second Iteration

$$x_1 = (c_1 - a_{12}x_2 - a_{13}x_3)/a_{11}$$

$$x_2 = (c_2 - a_{21}x_1 - a_{23}x_3)/a_{22}$$

$$x_3 = (c_3 - a_{31}x_1 - a_{32}x_2)/a_{33}$$

Gauss-Seidel Method-Algorithm

Requires an initial guess for $[X]$

$$[X] = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

Use rewritten equations to solve for each value of x_i.

Note: The x_i is the most recent value.

Apply values calculated to the calculations remaining in the <u>current</u> iteration.

Gauss-Seidel Method- Error Calculation

The Absolute Relative Approximate Error is:

$$\left|\in_a\right|_i = \left|\frac{x_i^{new} - x_i^{old}}{x_i^{new}}\right| \times 100$$

When the absolute relative approximate error is less than the defined stopping criterion, the iterations are stopped.

Example 1. Gauss Seidel Method. The upward velocity of a rocket is given at three different times. The velocity data is approximated by a polynomial as:

Time (s)	Velocity (m/s)
5	106.8
8	177.2
12	279.2

$$v(t) = a_1 t^2 + a_2 t + a_3 \,, 5 \le t \le 12.$$

$$\begin{bmatrix} t_1^2 & t_1 & 1 \\ t_2^2 & t_2 & 1 \\ t_3^2 & t_3 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Initial guess for *a* matrix:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

Example 1. Continued. Gauss Seidel Method. Rewrite each equation

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Multiplication of each row

From 1st row $\longrightarrow$ $$a_1 = \frac{106.8 - 5a_2 - a_3}{25}$$

From 2nd row $\longrightarrow$ $$a_2 = \frac{177.2 - 64a_1 - a_3}{8}$$

From 3rd row $\longrightarrow$ $$a_3 = \frac{279.2 - 144a_1 - 12a_2}{1}$$

Example 1. Continued. Gauss Seidel Method. Evaluate at initial guesses

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

Initial Guess

Iteration # 1

$$a_1 = \frac{106.8 - 5(2) - (5)}{25} = 3.6720$$

$$a_2 = \frac{177.2 - 64(3.6720) - (5)}{8} = -7.8510$$

$$a_3 = \frac{279.2 - 144(3.6720) - 12(-7.8510)}{1} = -155.36$$

Note: The a_i is the most recent value.

Example 1. Continued. Gauss Seidel Method. Find Error in 1st Iteration

At the end of 1st iteration:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 3.6720 \\ -7.8510 \\ -155.36 \end{bmatrix}$$

$$\left|\in_a\right|_i = \left|\frac{x_i^{new} - x_i^{old}}{x_i^{new}}\right| \times 100$$

$$\left|\in_a\right|_1 = \left|\frac{3.6720 - 1.0000}{3.6720}\right| \times 100 = \boxed{72.76\%}$$

$$\left|\in_a\right|_2 = \left|\frac{-7.8510 - 2.0000}{-7.8510}\right| \times 100 = \boxed{125.47\%}$$

← Maximum error

$$\left|\in_a\right|_3 = \left|\frac{-155.36 - 5.0000}{-155.36}\right| \times 100 = \boxed{103.22\%}$$

Example 1. Continued. Gauss Seidel Method. Evaluate solutions in the last iteration:

Iteration # 2

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 3.6720 \\ -7.8510 \\ -155.36 \end{bmatrix}$$ from iteration #1

$$a_1 = \frac{106.8 - 5(-7.8510) - 155.36}{25} = 12.056$$

$$a_2 = \frac{177.2 - 64(12.056) - 155.36}{8} = -54.882$$

$$a_3 = \frac{279.2 - 144(12.056) - 12(-54.882)}{1} = -798.34$$

Example 1. Continued. Gauss Seidel Method. Find Error in 2nd Iteration

At the end of 2nd iteration:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 12.056 \\ -54.882 \\ -798.54 \end{bmatrix}$$

$$\left|\in_a\right|_i = \left|\frac{x_i^{new} - x_i^{old}}{x_i^{new}}\right| \times 100$$

$$\left|\in_a\right|_1 = \left|\frac{12.056 - 3.6720}{12.056}\right| x100 = 69.543\%$$

$$\left|\in_a\right|_2 = \left|\frac{-54.882 - (-7.8510)}{-54.882}\right| x100 = 85.695\%$$

← Maximum error

$$\left|\in_a\right|_3 = \left|\frac{-798.34 - (-155.36)}{-798.34}\right| x100 = 80.540\%$$

Example 1. Continued. Gauss Seidel Method. Next Iterations

Iteration	a_1	$\left\|\in_a\right\|_1 \%$	a_2	$\left\|\in_a\right\|_2 \%$	a_3	$\left\|\in_a\right\|_3 \%$
1	3.6720	72.767	−7.8510	125.47	−155.36	103.22
2	12.056	69.543	−54.882	85.695	−798.34	80.540
3	47.182	74.447	−255.51	78.521	−3448.9	76.852
4	193.33	75.595	−1093.4	76.632	−14440	76.116
5	800.53	75.850	−4577.2	76.112	−60072	75.963
6	3322.6	75.906	−19049	75.972	−249580	75.931

The relative errors are not decreasing at any significant rate

not converging to the true solution of $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.29048 \\ 19.690 \\ 1.0857 \end{bmatrix}$

Pitfall of Gauss-Seidel Method

<u>Not all</u> systems of equations will converge

System of equations with

Diagonally Dominant Coefficient Matrix

↓

can always converge

Gauss Seidel: Diagonally Dominant Matrix

[A] in **[A] [X] = [B]** is diagonally dominant if:

$$\left|a_{\mathrm{ii}}\right| \geq \sum_{\substack{j=1 \\ j \neq \mathrm{i}}}^{n} \left|a_{ij}\right|$$

for all 'i'

$$\left|a_{ii}\right| > \sum_{\substack{j=1 \\ j \neq i}}^{n} \left|a_{ij}\right|$$

for at least one 'i'

The coefficient on the diagonal must be at least equal to the sum of the other coefficients in that row and at least one row with a diagonal coefficient greater than the sum of the other coefficients in that row.

Gauss Seidel: Diagonally Dominant Matrix

e.g. for a system of **3** equations, the coefficient matrix **[A]** is diagonally dominant if

$$|a_{11}| \geq |a_{12}| + |a_{13}|$$

$$|a_{22}| \geq |a_{21}| + |a_{23}|$$

$$|a_{33}| \geq |a_{31}| + |a_{32}|$$

Which coefficient matrix is diagonally dominant?

$$[A] = \begin{bmatrix} 2 & 5.81 & 34 \\ 45 & 43 & 1 \\ 123 & 16 & 1 \end{bmatrix} \qquad [B] = \begin{bmatrix} 124 & 34 & 56 \\ 23 & 53 & 5 \\ 96 & 34 & 129 \end{bmatrix}$$

Example 2. Gauss Seidel Method. Diagonally Dominant Matrix.

Check convergence and estimate solution after 2 iteration with the given initial guess.

$$12x_1 + 3x_2 - 5x_3 = 1$$
$$x_1 + 5x_2 + 3x_3 = 28$$
$$3x_1 + 7x_2 + 13x_3 = 76$$

$$[A] = \begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Initial guess

Check Convergence: Is [A] diagonally dominant?

$$|a_{11}| = |12| = 12 \geq |a_{12}| + |a_{13}| = |3| + |-5| = 8$$
$$|a_{22}| = |5| = 5 \geq |a_{21}| + |a_{23}| = |1| + |3| = 4$$
$$|a_{33}| = |13| = 13 \geq |a_{31}| + |a_{32}| = |3| + |7| = 10$$

The inequalities are all true → solution should converge using the Gauss-Seidel Method

Example 2. Gauss Seidel Method. Rewrite each equation

$$\begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 28 \\ 76 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Evaluate x_i at initial values

$$x_1 = \frac{1 - 3x_2 + 5x_3}{12}$$

$$x_2 = \frac{28 - x_1 - 3x_3}{5}$$

$$x_3 = \frac{76 - 3x_1 - 7x_2}{13}$$

$$x_1 = \frac{1 - 3(0) + 5(1)}{12} = 0.50000$$

$$x_2 = \frac{28 - (0.5) - 3(1)}{5} = 4.9000$$

$$x_3 = \frac{76 - 3(0.50000) - 7(4.9000)}{13} = 3.0923$$

Example 2. Gauss Seidel Method. Calculate error after 1st iteration

At the end of 1st iteration:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.5000 \\ 4.9000 \\ 3.0923 \end{bmatrix}$$

$$\left|\in_a\right|_i = \left|\frac{x_i^{new} - x_i^{old}}{x_i^{new}}\right| \times 100$$

$$\left|\in_a\right|_1 = \left|\frac{0.50000 - 1.0000}{0.50000}\right| \times 100 = 100.00\%$$

$$\left|\in_a\right|_2 = \left|\frac{4.9000 - 0}{4.9000}\right| \times 100 = 100.00\%$$

← Maximum error

$$\left|\in_a\right|_3 = \left|\frac{3.0923 - 1.0000}{3.0923}\right| \times 100 = 67.662\%$$

Example 2. Continued. Gauss Seidel Method. Evaluate solutions in the last iteration:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.5000 \\ 4.9000 \\ 3.0923 \end{bmatrix} \text{ from iteration \#1}$$

Iteration # 2

Evaluate x_i at recent values

$$x_1 = \frac{1 - 3(4.9000) + 5(3.0923)}{12} = 0.14679$$

$$x_2 = \frac{28 - (0.14679) - 3(3.0923)}{5} = 3.7153$$

$$x_3 = \frac{76 - 3(0.14679) - 7(4.900)}{13} = 3.8118$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.14679 \\ 3.7153 \\ 3.8118 \end{bmatrix}$$

Example 2. Gauss Seidel Method. Calculate error after 1st iteration

At the end of 2nd iteration:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.14679 \\ 3.7153 \\ 3.8118 \end{bmatrix}$$

$$\left|\in_a\right|_i = \left|\frac{x_i^{new} - x_i^{old}}{x_i^{new}}\right| \times 100$$

$$\left|\in_a\right|_1 = \left|\frac{0.14679 - 0.50000}{0.14679}\right| \times 100 = 240.61\%$$

← Maximum error

???

$$\left|\in_a\right|_2 = \left|\frac{3.7153 - 4.9000}{3.7153}\right| \times 100 = 31.889\%$$

$$\left|\in_a\right|_3 = \left|\frac{3.8118 - 3.0923}{3.8118}\right| \times 100 = 18.874\%$$

Example 2. Continued. Gauss Seidel Method. Next Iterations

Iteration	a_1	$\left\|\in_a\right\|_1 \%$	a_2	$\left\|\in_a\right\|_2 \%$	a_3	$\left\|\in_a\right\|_3 \%$
1	0.50000	100.00	4.9000	100.00	3.0923	67.662
2	0.14679	240.61	3.7153	31.889	3.8118	18.876
3	0.74275	80.236	3.1644	17.408	3.9708	4.0042
4	0.94675	21.546	3.0281	4.4996	3.9971	0.65772
5	0.99177	4.5391	3.0034	0.82499	4.0001	0.074383
6	0.99919	0.74307	3.0001	0.10856	4.0001	0.00101

Numerical Solution→ $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.99919 \\ 3.0001 \\ 4.0001 \end{bmatrix}$.

Exact Solution → $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$.

Example 3. Gauss Seidel Method. How about this system?

$$3x_1 + 7x_2 + 13x_3 = 76$$
$$x_1 + 5x_2 + 3x_3 = 28$$
$$12x_1 + 3x_2 - 5x_3 = 1$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Initial guess

$$x_1 = \frac{76 - 7x_2 - 13x_3}{3}$$

$$x_2 = \frac{28 - x_1 - 3x_3}{5}$$

$$x_3 = \frac{1 - 12x_1 - 3x_2}{-5}$$

What happens after 6 iterations?

Example 3. Continued. Gauss Seidel Method. Next Iterations

Iteration	a_1	$\lvert\in_a\rvert_1\,\%$	a_2	$\lvert\in_a\rvert_2\,\%$	a_3	$\lvert\in_a\rvert_3\,\%$
1	21.000	95.238	0.80000	100.00	50.680	98.027
2	−196.15	110.71	14.421	94.453	−462.30	110.96
3	−1995.0	109.83	−116.02	112.43	4718.1	109.80
4	−20149	109.90	1204.6	109.63	−47636	109.90
5	2.0364×10^5	109.89	−12140	109.92	4.8144×10^5	109.89
6	-2.0579×10^5	109.89	1.2272×10^5	109.89	-4.8653×10^6	109.89

Not converging.

Gauss-Seidel method works?

Example 3. Continued. Gauss Seidel Method. Conclusion

The Gauss-Seidel Method can still be used

The coefficient matrix is not diagonally dominant

$$[A] = \begin{bmatrix} 3 & 7 & 13 \\ 1 & 5 & 3 \\ 12 & 3 & -5 \end{bmatrix}$$

But this is the same set of equations used in example 2, which did converge.

$$[A] = \begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix}$$

Note. If [A]is not diagonally dominant:

Can rearranging the equations form a diagonally dominant matrix?