Example. Predicting Thermal Expansion Coefficient at Specific Temperature for a Trunnion using Interpolation. (6 data points is given in a certain temperature range)

For the purpose of shrinking a trunnion into a hub, the reduction of diameter ΔD of a trunnion shaft by cooling it through a temperature change of ΔT is given by $\Delta D = D\alpha\Delta T$ where D is original diameter (in.) and α is coefficient of thermal expansion at average temperature $(\text{in/in/}\,^\circ\text{F})$. The trunnion is cooled from 80°F to -108°F, giving the average temperature as -14°F. The table of the coefficient of thermal expansion vs. temperature data is given in Table 1. If the coefficient of thermal expansion needs to be calculated at the average temperature of -14°F, determine the value of the coefficient of thermal expansion at $T = -14^\circ\text{F}$ using the direct method of interpolation and

1) a **first order** polynomial (linear interpolation),
2) a **second order** polynomial (quadratic interpolation),
3) a **third order** polynomial (cubic interpolation).

Table 1 Thermal expansion coefficient as a function of temperature.

Temperature, T $(^\circ\text{F})$	Thermal Expansion Coefficient, α $(\text{in/in/}\,^\circ\text{F})$
80	6.47×10^{-6}
0	6.00×10^{-6}
–60	5.58×10^{-6}
–160	4.72×10^{-6}
–260	3.58×10^{-6}
–340	2.45×10^{-6}

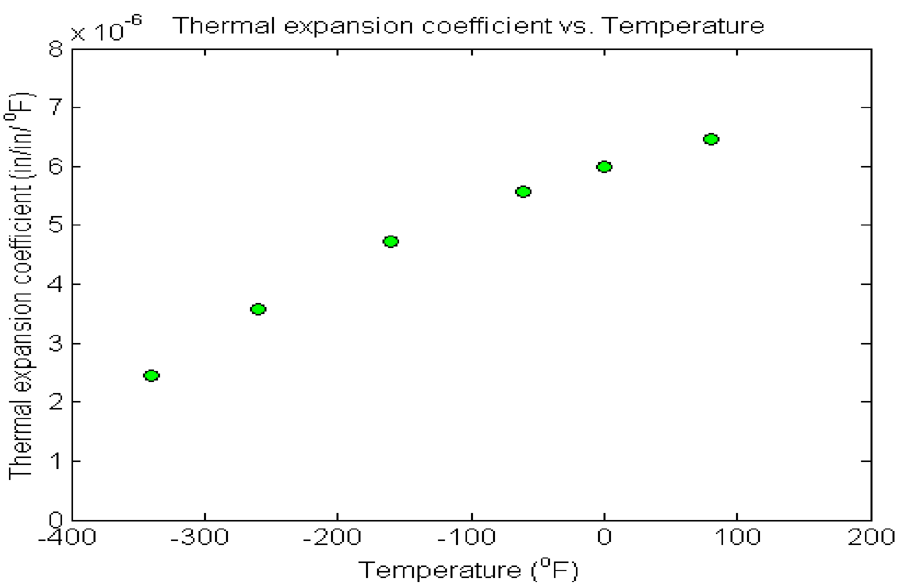
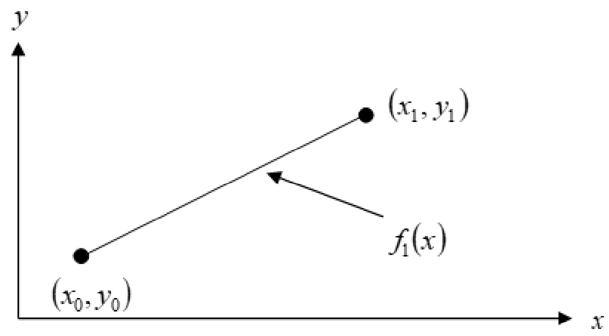


Figure 1 Thermal expansion coefficient vs. temperature.

Linear Interpolation: For first order polynomial interpolation (also called linear interpolation), we choose the coefficient of thermal expansion given by $\alpha(T) = a_0 + a_1 T$



Since we want to find the coefficient of thermal expansion at $T = -14\,°\text{F}$, and we are using a first order polynomial, we need to <u>choose the two data points</u> that are closest to $T = -14\,°\text{F}$ that also <u>bracket</u> $T = -14\,°\text{F}$ to evaluate it. The two points are $T_0 = 0\,°\text{F}$ and $T_1 = -60\,°\text{F}$.

Then: $T_0 = 0, \quad \alpha(T_0) = 6.00 \times 10^{-6}$

$T_1 = -60, \ \alpha(T_1) = 5.58 \times 10^{-6}$ Gives

$$\alpha(0) = a_0 + a_1(0) = 6.00 \times 10^{-6}$$

$$\alpha(-60) = a_0 + a_1(-60) = 5.58 \times 10^{-6}$$

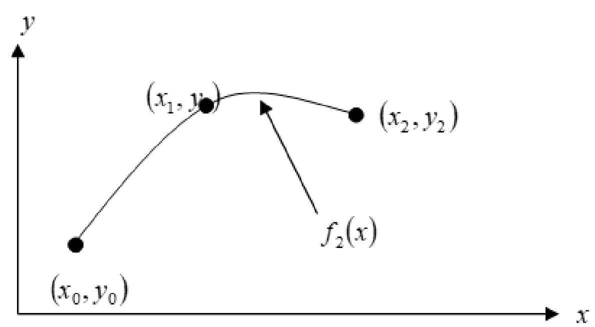
Writing the equations in matrix form, we have $\begin{bmatrix} 1 & 0 \\ 1 & -60 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 6.00 \times 10^{-6} \\ 5.58 \times 10^{-6} \end{bmatrix}$

Solving the above two equations gives $a_0 = 6.00 \times 10^{-6}$, $a_1 = 0.007 \times 10^{-6}$. Hence

$$\alpha(T) = a_0 + a_1 T = 6.00 \times 10^{-6} + 0.007 \times 10^{-6} T, \quad -60 \le T \le 0$$

At $T = -14\,°\text{F}$, $\alpha(-14) = 6.00 \times 10^{-6} + 0.007 \times 10^{-6}(-14) = 5.902 \times 10^{-6}$ in/in/°F

Quadratic Interpolation: For second order polynomial interpolation (also called quadratic interpolation), we choose the coefficient of thermal expansion given by $\alpha(T) = a_0 + a_1 T + a_2 T^2$. Since we want to find the coefficient of thermal expansion at $T = -14\,°\text{F}$, and we are using a second order polynomial, we need to <u>choose the three data points</u> that are closest to $T = -14\,°\text{F}$ that <u>also bracket</u> $T = -14\,°\text{F}$ to evaluate it.



These three points are $T_0 = 80\,°\text{F}$, $T_1 = 0\,°\text{F}$ and $T_2 = -60\,°\text{F}$. Then:

$T_0 = 80, \quad \alpha(T_0) = 6.47 \times 10^{-6}$

$$\alpha(80) = a_0 + a_1(80) + a_2(80)^2 = 6.47 \times 10^{-6}$$

$T_1 = 0, \quad \alpha(T_1) = 6.00 \times 10^{-6}$ gives $\alpha(0) = a_0 + a_1(0) + a_2(0)^2 = 6.00 \times 10^{-6}$

$T_2 = -60, \quad \alpha(T_2) = 5.58 \times 10^{-6}$ $\quad \alpha(-60) = a_0 + a_1(-60) + a_2(-60)^2 = 5.58 \times 10^{-6}$

Writing the three equations in matrix form, we have $\begin{bmatrix} 1 & 80 & 6400 \\ 1 & 0 & 0 \\ 1 & -60 & 3600 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 6.47 \times 10^{-6} \\ 6.00 \times 10^{-6} \\ 5.58 \times 10^{-6} \end{bmatrix}$. Solving the above three equations gives

$a_0 = 6.00 \times 10^{-6}$

$a_1 = 6.5179 \times 10^{-9}$

$a_2 = -8.0357 \times 10^{-12}$

Hence $\alpha(T) = 6.00 \times 10^{-6} + 6.5179 \times 10^{-9} T - 8.0357 \times 10^{-12} T^2, \; -60 \le T \le 80$

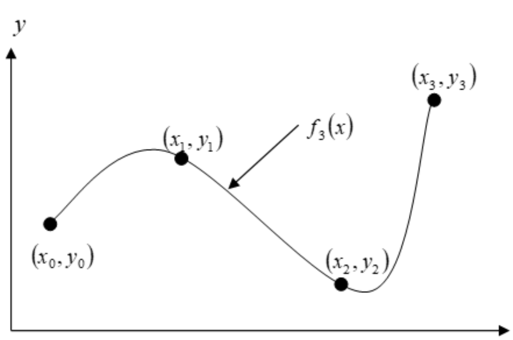
At $T = -14\,°\text{F}$,

$$\alpha(-14) = 6.00 \times 10^{-6} + 6.5179 \times 10^{-9}(-14) - 8.0357 \times 10^{-12}(-14)^2 = 5.9072 \times 10^{-6} \text{ in/in/°F}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$|\in_a| = \left| \frac{5.9072 \times 10^{-6} - 5.902 \times 10^{-6}}{5.9072 \times 10^{-6}} \right| \times 100 = 0.087605\%$$

Cubic (Spline) Interpolation: a) For third order polynomial interpolation (also called cubic interpolation), we choose the coefficient of thermal expansion given by $\alpha(T) = a_0 + a_1 T + a_2 T^2 + a_3 T^3$



We'd like to find: a) Find the absolute relative approximate error for the third order polynomial approximation, b) The actual reduction in diameter is given by

$$\Delta D = D \int_{T_r}^{T_f} \alpha \, dT \quad \text{where} \quad T_r = \text{ room temperature } (°\text{F})$$

$$T_f = \text{ temperature of cooling medium } (°\text{F})$$

Since $T_r = 80\,°\text{F}, \quad T_f = -108°\text{F} \quad \rightarrow \quad \Delta D = D \int_{80}^{-108} \alpha \, dT$

Find out the percentage difference in the reduction in the diameter by the above integral formula and the result using the thermal expansion coefficient from part (a).

Since we want to find the coefficient of thermal expansion at $T = -14\,°\mathrm{F}$, and we are using a third order polynomial, we need to <u>choose the four data points</u> closest to $T = -14\,°\mathrm{F}$ that <u>also bracket</u> $T = -14\,°\mathrm{F}$ to evaluate it. Then the four points are $\mathrm{T}_0 = 80°\mathrm{F}$, $\mathrm{T}_1 = 0\,°\mathrm{F}$, $T_2 = -60\,°\mathrm{F}$ and $T_3 = -160°\mathrm{F}$.

$T_0 = 80, \quad \alpha(T_0) = 6.47\times10^{-6} \qquad \alpha(80) = a_0 + a_1(80) + a_2(80)^2 + a_3(80)^3 = 6.47\times10^{-6}$

$T_1 = 0, \quad \alpha(T_1) = 6.00\times10^{-6} \qquad \alpha(0) = a_0 + a_1(0) + a_2(0)^2 + a_3(0)^3 = 6.00\times10^{-6}$

$T_2 = -60, \quad \alpha(T_2) = 5.58\times10^{-6} \qquad \alpha(-60) = a_0 + a_1(-60) + a_2(-60)^2 + a_3(-60)^3 = 5.58\times10^{-6}$

$T_3 = -160, \quad \alpha(T_3) = 4.72\times10^{-6}$

$\alpha(-160) = a_0 + a_1(-160) + a_2(-160)^2 + a_3(-160)^3 = 4.72\times10^{-6}$

Writing the four equations in matrix form, we have

$$\begin{bmatrix} 1 & 80 & 6400 & 5.12\times10^5 \\ 1 & 0 & 0 & 0 \\ 1 & -60 & 3600 & -2.16\times10^5 \\ 1 & -160 & 25600 & -4.096\times10^6 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 6.47\times10^{-6} \\ 6.00\times10^{-6} \\ 5.58\times10^{-6} \\ 4.72\times10^{-6} \end{bmatrix}$$

Solving the above four equations gives
$a_0 = 6.00\times10^{-6} \qquad a_1 = 6.4786\times10^{-9}$
$a_2 = -8.1994\times10^{-12} \qquad a_3 = 8.1845\times10^{-15}$

Hence

$$\begin{aligned}\alpha(T) &= a_0 + a_1 T + a_2 T^2 + a_3 T^3 \\ &= 6.00\times10^{-6} + 6.4786\times10^{-9}T - 8.1994\times10^{-12}T^2 + 8.1845\times10^{-15}T^3, \quad -160 \le T \le 80\end{aligned}$$

$$\begin{aligned}\alpha(-14) &= 6.00\times10^{-6} + 6.4786\times10^{-9}(-14) - 8.1994\times10^{-12}(-14)^2 + 8.1845\times10^{-15}(-14)^3 \\ &= 5.9077\times10^{-6}\ \text{in/in/°F}\end{aligned}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the second and third order polynomial is

$$|\in_a| = \left|\frac{5.9077\times10^{-6} - 5.9072\times10^{-6}}{5.9077\times10^{-6}}\right| \times 100$$
$$= 0.0083867\%$$

b) In finding the percentage difference in the reduction in diameter, we can rearrange the integral formula to

$\dfrac{\Delta D}{D} = \displaystyle\int_{T_r}^{T_f} \alpha\, dT$ and since we know from part (a) that

$\alpha(T) = 6.00\times10^{-6} + 6.4786\times10^{-9}T - 8.1994\times10^{-12}T^2 + 8.1845\times10^{-15}T^3, \quad -160 \le T \le 80$ we see that we can use the integral formula in the range from $T_f = -108°\mathrm{F}$ to $T_r = 80\,°\mathrm{F}$

Therefore,

$$\frac{\Delta D}{D} = \int_{T_r}^{T_f} \alpha\, dT = \int_{80}^{-108} \left(6.00\times10^{-6} + 6.4786\times10^{-9}T - 8.1994\times10^{-12}T^2 + 8.1845\times10^{-15}T^3\right) dT$$

$$= \left[6.00 \times 10^{-6} T + 6.4786 \times 10^{-9} \frac{T^2}{2} - 8.1994 \times 10^{-12} \frac{T^3}{3} + 8.1845 \times 10^{-15} \frac{T^4}{4} \right]_{80}^{-108}$$

$$= -1105.9 \times 10^{-6}$$

So $\frac{\Delta D}{D} = -1105.9 \times 10^{-6}$ in/in using the actual reduction in diameter integral formula. If we use the average value for the coefficient of thermal expansion from part (a), we get

$$\frac{\Delta D}{D} = \alpha \Delta T = \alpha \left(T_f - T_r \right) = 5.9077 \times 10^{-6} \left(-108 - 80 \right) = -1110.6 \times 10^{-6}$$

and $\frac{\Delta D}{D} = -1110.6 \times 10^{-6}$ in/in using the average value of the coefficient of thermal expansion using a third order polynomial. Considering the integral to be the more accurate calculation, the percentage difference would be

$$\left| \in_a \right| = \left| \frac{-1105.9 \times 10^{-6} - \left(-1110.6 \times 10^{-6} \right)}{-1105.9 \times 10^{-6}} \right| \times 100$$

$$= 0.42775\%$$