Part 3. Linear Algebraic Equations
Chapter 9. Gaussian Elimination

Lecture 8

Systems of Linear Equations: Introduction & Naïve Gaussian Elimination

PT3.1 – PT3.3 & 9.1

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General Overview

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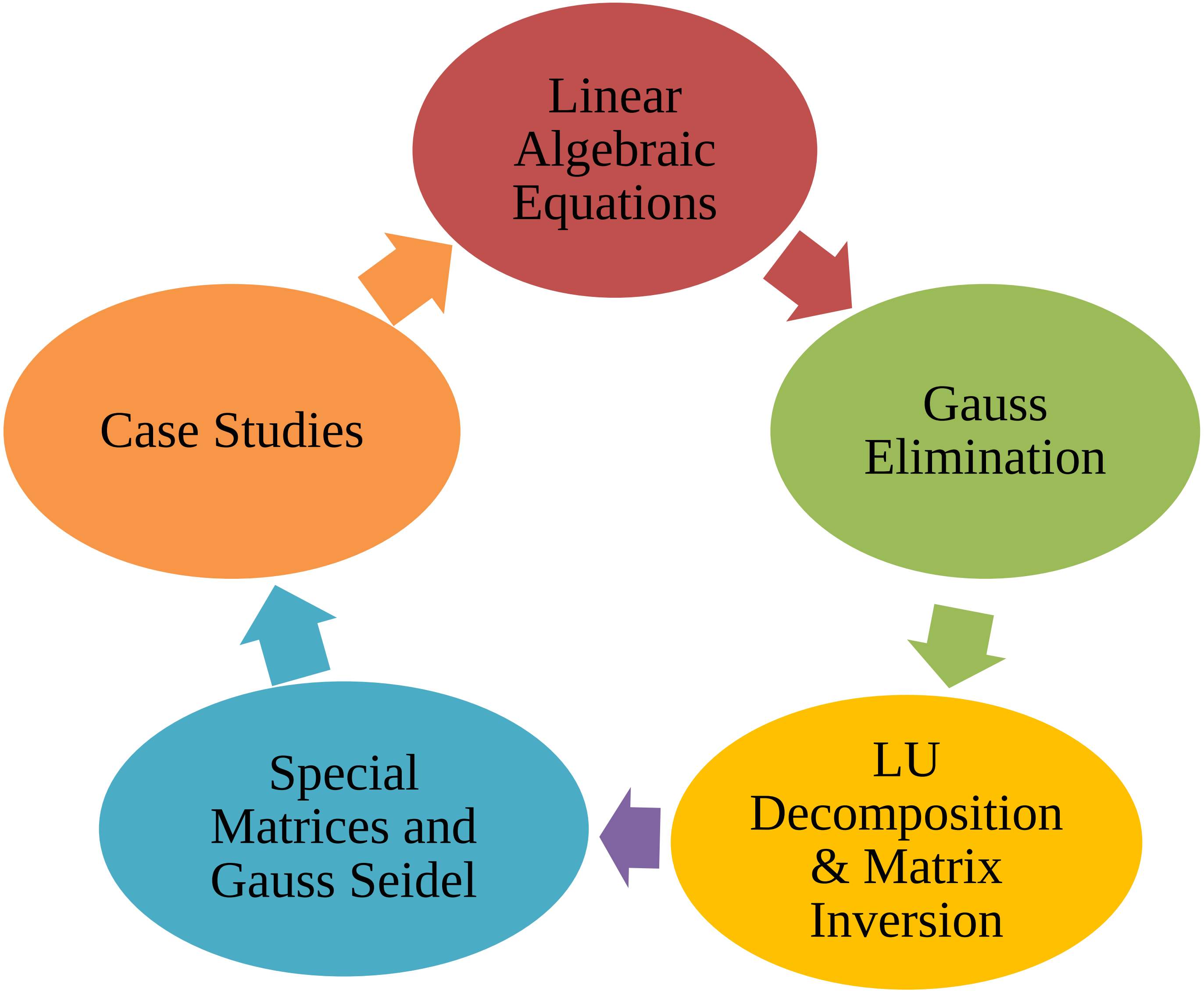
Roots of Equation

found value that satisfied a single equation

Now

find values $x_1 , x_2 ,\ldots , x_n$ that satisfy multiple equations

Orientation: Scope and Overview



Linear Algebraic Equations: Motivation

$ax + by + c = 0$ or $ax + by = -c$ (linear equation in x & y variables)

$ax + by + cz = d$ (linear equation in x, y & z variables)

$$\boxed{a_1x_1 + a_2x_2 + \ldots + a_nx_n = b}$$ (linear equation in n variables)

A solution of equation in n variables consists of real numbers:

$c_1, c_2, c_3, \ldots, c_n$

If # of linear equations, $n > 1$ → all equations are solved simultaneously

Noncomputer Methods for Solving Systems of Equations

If # of equations ($n \leq 3$) → Solved by "method of elimination"

Linear algebra provides the tools to solve such systems of linear equations.

Solution of large sets of linear algebraic equations → possible and practical with computers.

Gauss Elimination

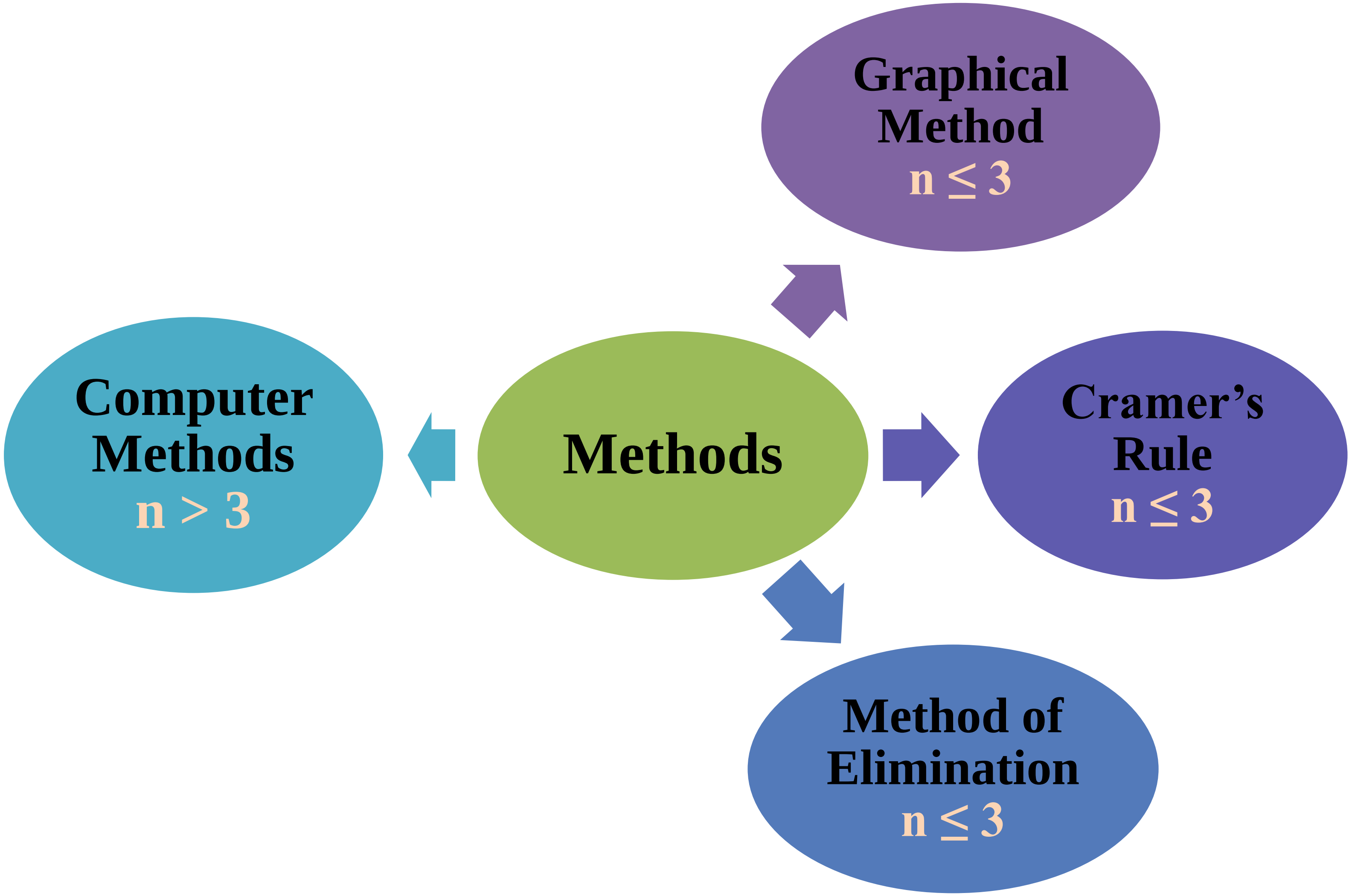
$$a_{11}\ x_1 + a_{12}\ x_2 + \ \dots\ + a_{1n}\ x_n = b_1$$

$$a_{21}\ x_1 + a_{22}\ x_2 + \ \dots\ + a_{2n}\ x_n = b_2$$

$$a_{n1}\ x_1 + a_{n2}\ x_2 + \ \dots\ + a_n\ x_n = b_n$$

- *a*'s constant coefficient and *b*'s constant
- It involves combining equations to eliminate unknowns
- Earliest and most popular for solving simultaneous equations

Solving Small Numbers of Equations



Solving Small Numbers of Equations: Graphical Solution

Graphical Method
n ≤ 3

For two linear equations:

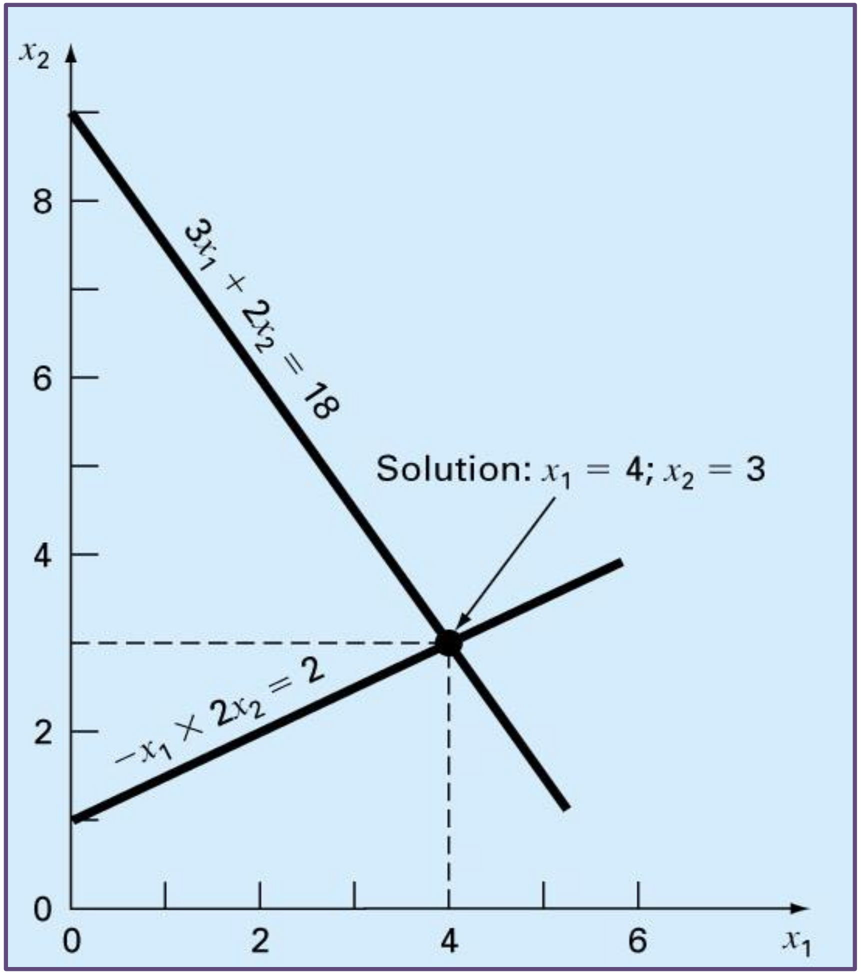
$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

Solve both equations for x_2:

$$\boxed{\begin{aligned} x_2 &= -\left(\frac{a_{11}}{a_{12}}\right)x_1 + \frac{b_1}{a_{12}} \\ x_2 &= -\left(\frac{a_{21}}{a_{22}}\right)x_1 + \frac{b_2}{a_{22}} \end{aligned}} \Rightarrow \quad x_2 = (\text{slope})x_1 + \text{intercept}$$

Plot both lines →

Solving Small Numbers of Equations: Graphical Solution

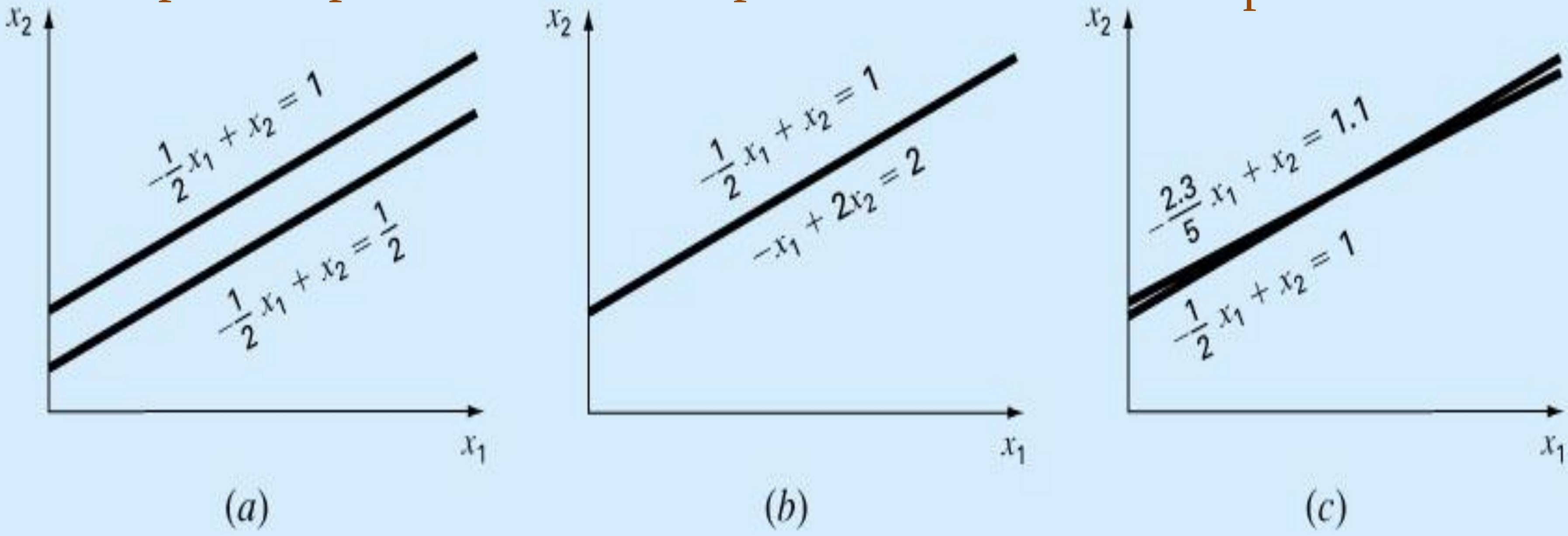


- Graphical solution of a set of two simultaneous linear algebraic equations
- The intersection of the lines present the solution.

Solving Small Numbers of Equations: Graphical Solution

Graphical depiction of singular and ill-conditioned systems

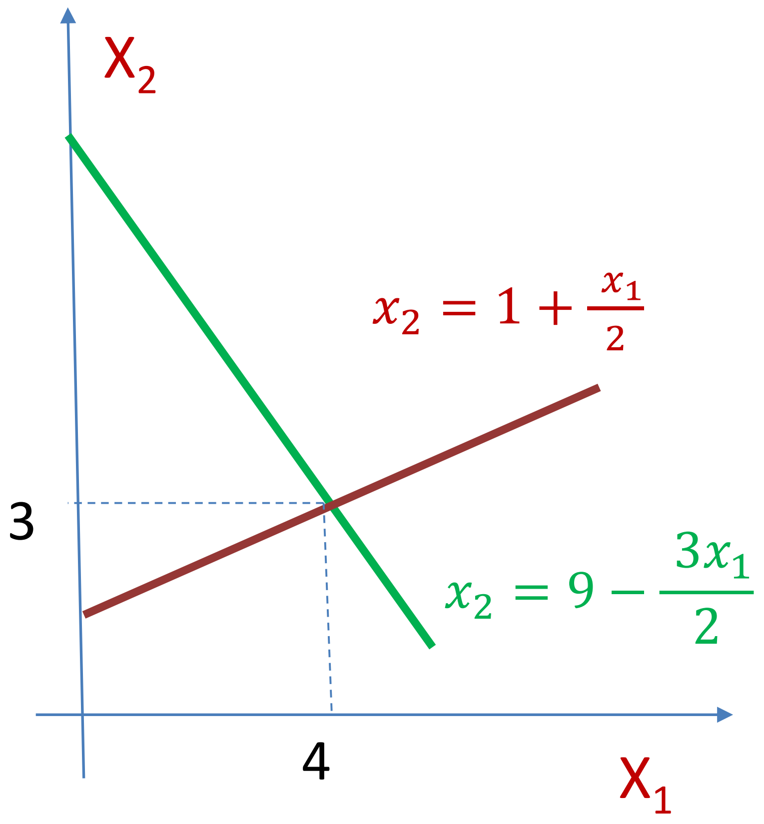
Slopes are parallel | Slopes coincide | Slopes are so close



No solution | Infinite solutions | Ill-conditioned system

Example 1. Find graphical solution for 2 simultaneous equations:

$$\begin{cases} 3x_1 + 2x_2 = 18 \\ -x_1 + 2x_2 = 2 \end{cases}$$



Solve both equations for x_2

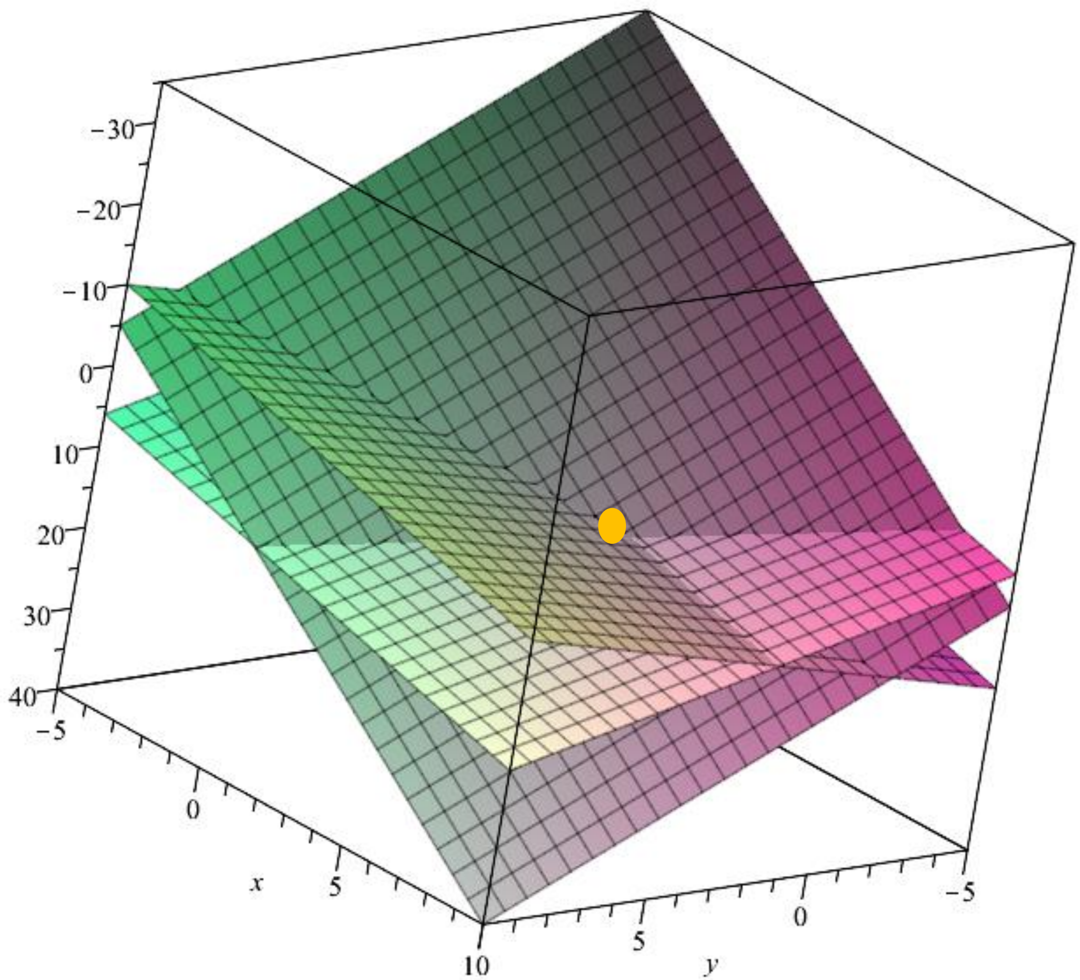
$$\begin{cases} x_2 = \dfrac{(18 - 3x_1)}{2} = 9 - \dfrac{3x_1}{2} \\ x_2 = 1 + \dfrac{x_1}{2} \end{cases}$$

$$1 + \frac{x_1}{2} = 9 - \frac{3x_1}{2}$$

Example 2. Find graphical solution for 3 simultaneous equations:

Plot3d ({*x-y+5*, *x+y+1*, *3*x+2*y-10*}, x = -5 .. 10, y = -5 .. 10)

Form $z = f(x, y)$



Point represents the intersection of 3 planes

Point is the solution of system of equations

Determinants and Cramer's Rule: Matrix Representation of Equations

A set of 3 equations, expressed in matrix form:

Cramer's Rule
n ≤ 3

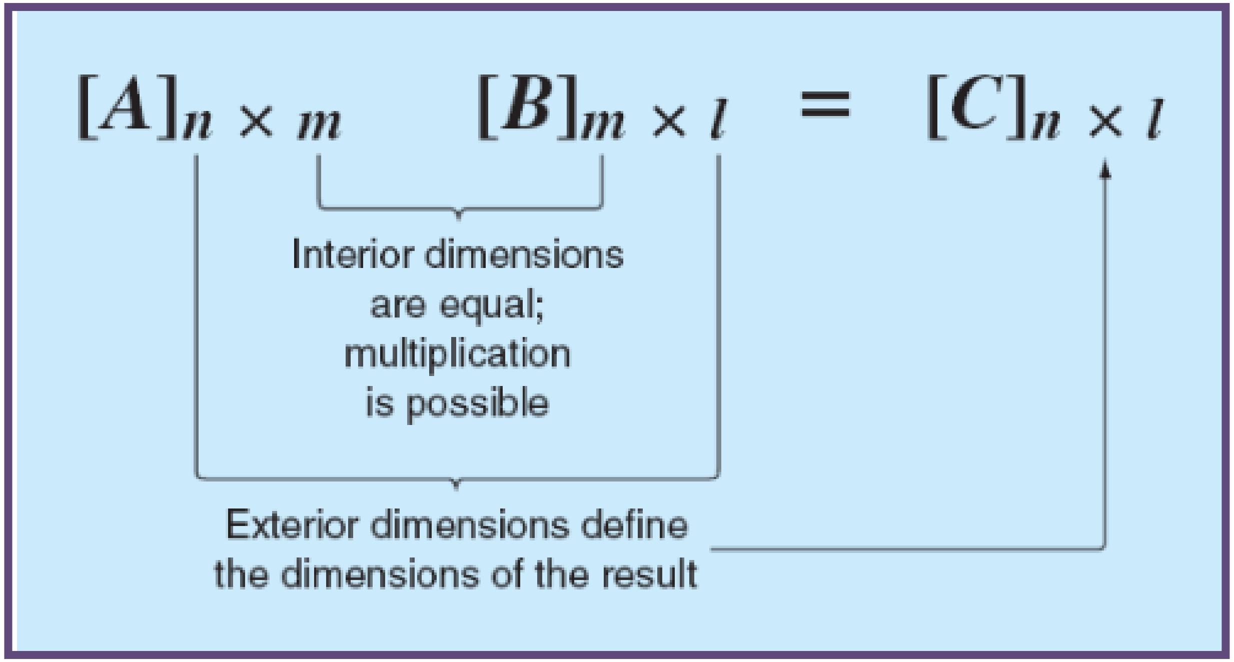
$$[A]\{x\} = \{B\}$$

[A]: coefficient matrix:

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- For calculating determinant → all matrices need to be square matrices
- Determinant of [A]: a number associated with each square matrix [A]
- For a square matrix of order 3, the *minor* of an element a_{ij} is the determinant of the matrix of order 2 by deleting row i and column j of [A]

Recall: Matrix Multiplication



Product of two matrices: [C] = [A] [B] where elements of [C] are defined as:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Determinants and Cramer's Rule: Matrix Representation of Equations

Example 3. Matrices of 3 Simultaneous Equations

What are [*A*], {*x*} and [*B*] matrices for this set of 3 linear algebraic eqs.?

$$\begin{cases} 0.3\ x_1 + 0.52\ x_2 + x_3 = -0.01 \\ 0.5\ x_1 + x_2 + 1.9\ x_3 = 0.67 \\ 0.1\ x_1 + 0.3\ x_2 + 0.5\ x_3 = -0.44 \end{cases}$$

$$\begin{bmatrix} 0.3 & 0.52 & 1 \\ 0.5 & 1 & 1.9 \\ 0.1 & 0.3 & 0.5 \end{bmatrix} \times \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} -0.01 \\ 0.67 \\ -0.44 \end{bmatrix}$$

$$[A] \times \{x\} = [B]$$

Coefficients Unknowns Constants

Determinants: Third- and Second-Order

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Third-order determinant of 3×3 coefficient matrix [A]

Second-order (2×2) determinant (minors) of the first row of coefficient matrix

$$D_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}\,a_{33} - a_{32}\,a_{23}$$

$$D_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}\,a_{33} - a_{31}\,a_{23}$$

$$D_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}\,a_{32} - a_{31}\,a_{22}$$

Determinants: Third- and Second Order

Third-order determinant of [A] in terms of the second-order determinants

$$D = a_{11} D_{11} - a_{12} D_{12} + a_{13} D_{13}$$

$$\boxed{D = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}$$

Cramer's Rule

- It expresses the solution of a systems of linear equations is in terms of ratios of determinants of the array of coefficients of the equations, e.g. → x_1 is computed as:

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{D}$$

← D_{x1} : the first column is replaced with the constant column

$x_1 = D_{x1} / D$ $\quad$ $x_2 = D_{x2} / D$ $\quad$ $x_3 = D_{x3} / D$

Example 3. Cramer's Rule

Find solution of the following system using Cramer's rule

$$\begin{cases} 0.3\ x_1 + 0.52\, x_2 + \quad x_3 = -0.01 \\ 0.5\ x_1 + \quad x_2 + 1.9\ x_3 = 0.67 \\ 0.1\ x_1 + 0.3\ x_2 + 0.5\ x_3 = -0.44 \end{cases}$$

Constants column

Strategy:

- ❑ Calculate determinant of coefficient matrix
  - Calculate the minors first
- ❑ Apply Cramer's equations: $x_1 = D_{x1} / D$
  $x_2 = D_{x2} / D$
  $x_3 = D_{x3} / D$

- for D_{x1} replace coefficient of 1st column with constants column
- for D_{x2} replace coefficient of 2nd column with constants column
- for D_{x3} replace coefficient 3rd column with constants column

Example 3. Cramer's Rule. Continued

$$D= \begin{vmatrix} 0.3 & 0.52 & 1 \\ 0.5 & 1 & 1.9 \\ 0.1 & 0.3 & 0.5 \end{vmatrix} = -0.0022$$

$$x_1 = \frac{\begin{vmatrix} & 0.52 & 1 \\ & 1 & 1.9 \\ & 0.3 & 0.5 \end{vmatrix}}{-0.0022}$$

$$x_2 = \frac{\begin{vmatrix} 0.3 & & 1 \\ 0.5 & & 1.9 \\ 0.1 & & 0.5 \end{vmatrix}}{-0.0022}$$

$$x_3 = \frac{\begin{vmatrix} 0.3 & 0.52 & \\ 0.5 & 1 & \\ 0.1 & 0.3 & \end{vmatrix}}{-0.0022}$$

$x_1 =$

$x_2 =$

$x_1 =$

Method of Elimination (of Unknowns)

- Solves one of the equations of the set for one of the unknowns and to eliminate that variable from the remaining equations by substitution.

Method of Elimination
$n \leq 3$

- Can be extended to systems with more than 2 or 3 equations → but would be extremely tedious to solve by hand.

Example 4. Method of Elimination of Unknowns

$$\begin{cases} 2x_1 + 3x_2 = 5 \\ 5x_1 - 6x_2 = 1 \end{cases}$$

Multiply equations by constants

Combine equations to remove one unknown

$$2\,(2x_1 + 3x_2) = 2\,(5)$$

Multiply top equation by 2

$$\begin{aligned} 4x_1 + 6x_2 &= 10 \\ + \quad 5x_1 - 6x_2 &= 1 \end{aligned}$$

Add equation to eliminate x_2

$$9x_1 = 11 \rightarrow x_1 = 11 / 9$$

Then x_2 can be easily calculated by substitution

Also works with 3, more equations but is complicated – requires computer for calculations

Part 3. Linear Algebraic Equations
Chapter 9. Gaussian Elimination

Lecture 9

Naive Gauss Elimination

9.2

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Naive Gauss Elimination

$$\begin{cases} a_{11}\, x_1 + a_{12}\, x_2 + \ \dots\ + a_{1n}\, x_n = c_1 \\ a_{21}\, x_1 + a_{22}\, x_2 + \ \dots\ + a_{2n}\, x_n = c_2 \\ \\ \\ a_{n1}\, x_1 + a_{n2}\, x_2 + \ \dots\ + a_{nn}\, x_n = c_n \end{cases}$$

General set of n equations

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots\dots & a_{1n} & c_1 \\ a_{21} & a_{22} & \dots & a_{2n} & c_2 \\ \\ a_{n1} & a_{n2} & --- & a_{nn} & c_n \end{array}\right]$$

Naive Gauss Elimination

- Extension of *method of elimination* to large sets of equations by developing a systematic scheme or algorithm to eliminate unknowns and to back substitute.

- The technique for n equations consists of two phases:

1
- Forward Elimination of Unknowns

2
- Back Substitution

Naive Gauss Elimination

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & c_1 \\ a_{21} & a_{22} & a_{23} & c_2 \\ a_{31} & a_{32} & a_{33} & c_3 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & c_1 \\ & a'_{22} & a'_{23} & c'_2 \\ & & a''_{33} & c''_3 \end{array}\right]$$

Forward elimination

$$\Downarrow$$

$$x_3 = c''_3 / a''_{33}$$

$$x_2 = (c'_2 - a'_{23} x_3) / a'_{22}$$

$$x_1 = (c_1 - a_{12} x_2 - a_{13} x_3) / a_{11}$$

Back substitution

Step 1. Forward Elimination of Unknowns

1

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

2

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \mathbf{0} & a'_{22} & a'_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b'_2 \\ b_3 \end{Bmatrix}$$

3

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \mathbf{0} & a'_{22} & a'_{23} \\ \mathbf{0} & a'_{32} & a'_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b'_2 \\ b'_3 \end{Bmatrix}$$

4

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \mathbf{0} & a'_{22} & a'_{23} \\ \mathbf{0} & \mathbf{0} & a''_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b'_2 \\ b''_3 \end{Bmatrix}$$

Now ready for Back Substitution

Example 5. Gaussian Elimination

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$
$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$
$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} \\ \\ \\ \end{Bmatrix}$$

Step 1

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \mathbf{0} & a'_{22} & a'_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b'_2 \\ b_3 \end{Bmatrix}$$

Calculate

$a'_{22} =$

$a'_{23} =$

$b'_2 =$

Example 5. continued. Gaussian Elimination

Step 2

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \mathbf{0} & a'_{22} & a'_{23} \\ \mathbf{0} & a'_{32} & a'_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b'_2 \\ b'_3 \end{Bmatrix}$$

Calculate:

$a'_{32} =$

$a'_{33} =$

$b'_3 =$

Example 5. continued. Gaussian Elimination

Step 3

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \mathbf{0} & a'_{22} & a'_{23} \\ \mathbf{0} & \mathbf{0} & a''_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b'_2 \\ b''_3 \end{Bmatrix}$$

Calculate:

$a''_{33} =$

$b''_3 =$

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.00333 & -0.29333 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.5617 \\ 71.4 \end{Bmatrix}$$

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.00333 & -0.29333 \\ 0 & -0.19 & 10.02 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.5617 \\ 70.615 \end{Bmatrix}$$

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0 & 7.00333 & -0.29333 \\ 0 & 0 & 10.012 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.5617 \\ 70.0843 \end{Bmatrix}$$

Ready for Back Substitution