Part 6. Numerical Differentiation and Integration
Chapter 21. Newton-Cotes Integration Formulas

Lecture 19

Trapezoidal Rule

21.1

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Motivation

- Calculus is the mathematics of change.
- Engineers continuously deal with systems and processes that change.

- Standing in the heart of calculus are mathematical concepts of:

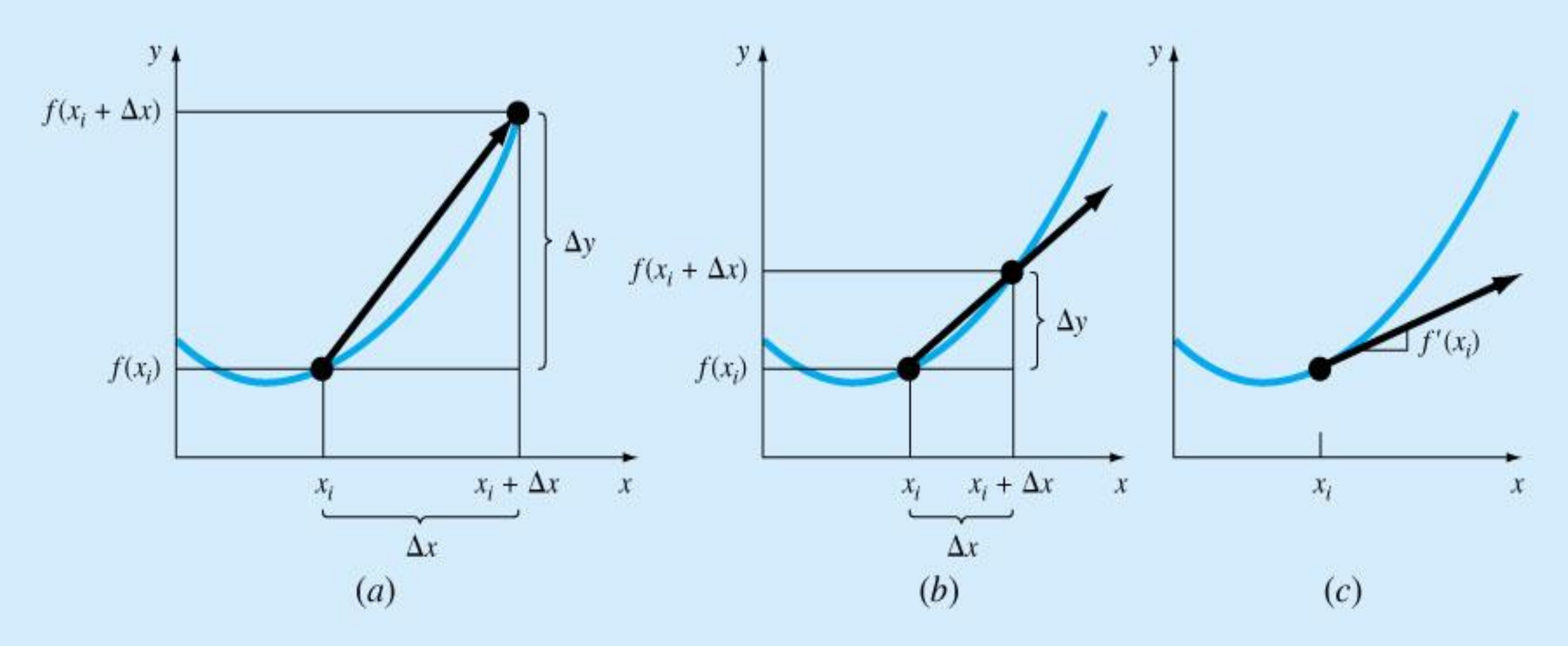
Differentiation

$$\left\{ \begin{aligned} \frac{\Delta y}{\Delta x} &= \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x} \\ \frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x} \end{aligned} \right.$$

Integration

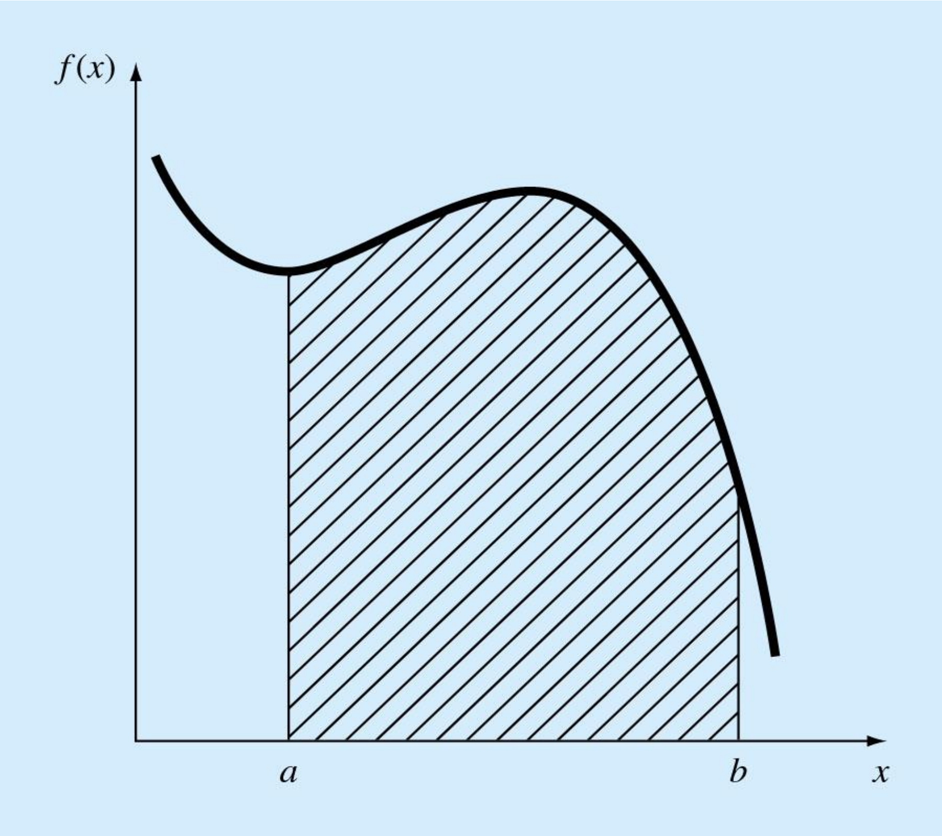
$$I = \int_a^b f(x)\,dx$$

Difference Approximation and Derivative



The graphical definition of a derivative: as Δx approaches zero in going from (*a*) to (*c*), the difference approximation becomes a derivative.

Integration



The integral of $f(x)$ between the limits $x = a$ to b is equivalent to the area under the curve.

Noncomputer Methods for Differentiation & Integration

- The function can typically be in one of these forms:

1 Simple, continuous function

- such as polynomial, an exponential, or a trigonometric function.

2 Complicated, continuous function

- that is difficult or impossible to differentiate or integrate directly.

3 Tabulated function

- where values of x and f(x) are given at a number of discrete points, as is often the case with experimental or field data.

Newton-Cotes Integration Formulas

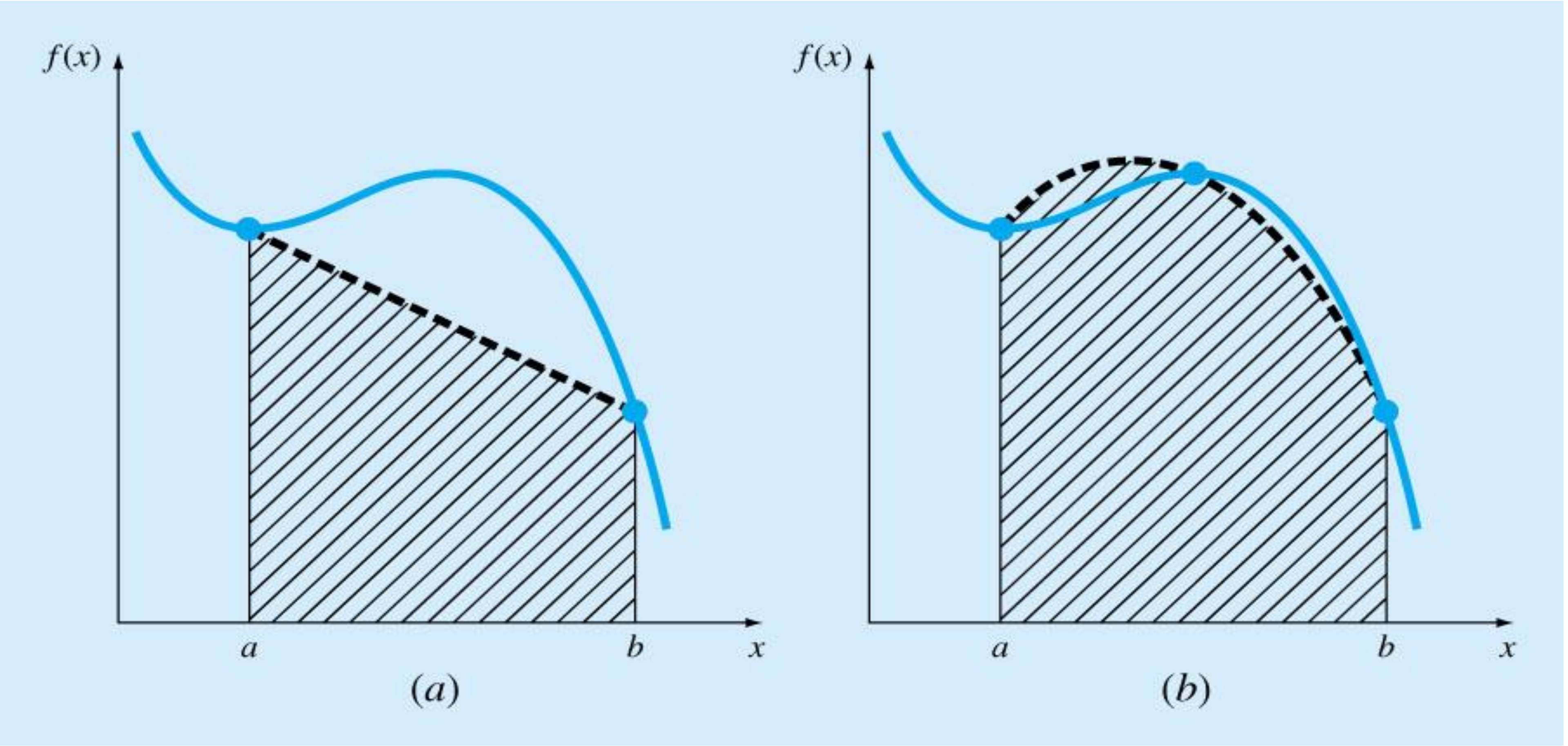
They are based on the strategy of replacing a complicated function or tabulated data with an approximating function that is easy to integrate:

$$I = \int_a^b f(x)dx \cong \int_a^b f_n(x)dx$$

$$f_n(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + a_n x^n$$

e.g. when ***n =1***, the area under the 1st-order polynomial or the line is calculated to approximate the value of integral

Integral Approximation

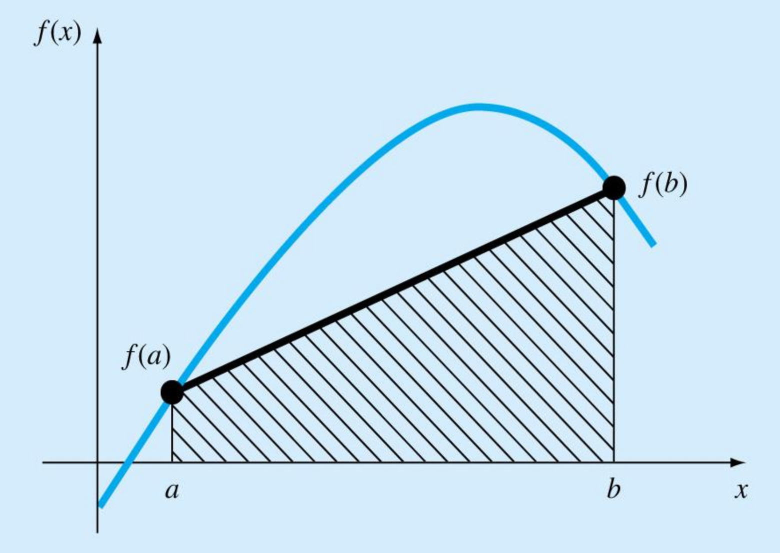


Using a straight line Using a parabola

The Trapezoidal Rule

- It is the first of the Newton-Cotes closed integration formulas, that the 1st – order polynomial is used to approximate the integral

$$I = \int_a^b f(x)\ dx \cong \int_a^b f_1(x)\ dx$$



$$\int_a^b f(x)dx \approx Area\ of\ trapezoid$$

$$= \frac{1}{2}(Sum\ of\ parallel\ sides)(height)$$

$$= \frac{1}{2}\left(f(b) + f(a)\right)(b - a)$$

$$\boxed{I = (b-a)\frac{f(a) + f(b)}{2}}$$

Example 1: The vertical distance of a rocket between 8 to 30 seconds is given by:

$$x = \int_{8}^{30} \left(2000 \ln \left[\frac{140000}{140000 - 2100\,t} \right] - 9.8\,t \right) dt$$

Use Trapezoidal rule to find a) the distance covered., b) the true error, E_t for part (a), and c) the absolute relative true error, $|\in_a|$ for part (a).

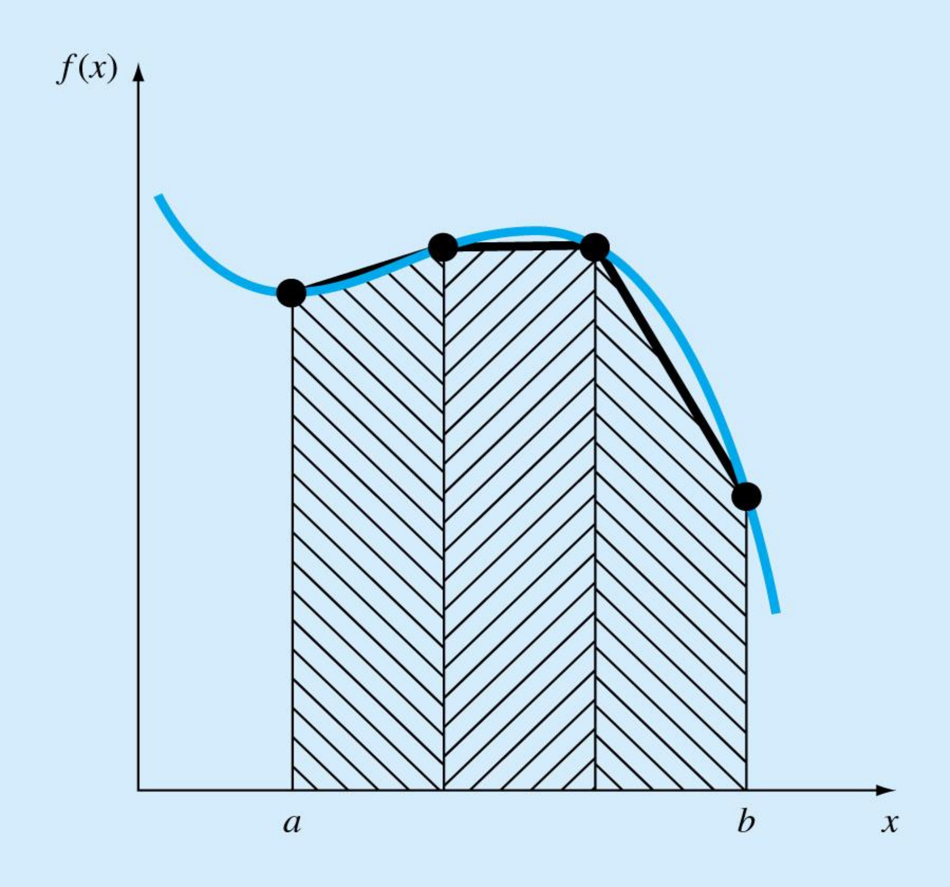
The Multiple-Section Trapezoidal Rule

- Divide into equal segments.
- Width of each segment is:

$$h = \frac{b-a}{n}$$

The integral I is:

$$I = \int_a^b f(x)dx$$



The Multiple-Section Trapezoidal Rule

The integral *I* can be broken into *h* integrals as:

$$\int_a^b f(x)dx = \int_a^{a+h} f(x)dx + \int_{a+h}^{a+2h} f(x)dx + \ldots + \int_{a+(n-2)h}^{a+(n-1)h} f(x)dx + \int_{a+(n-1)h}^{b} f(x)dx$$

Applying Trapezoidal rule on each segment gives:

$$\int_a^b f(x)dx = \frac{b-a}{2n}\left[f(a) + 2\left\{ \sum_{i=1}^{n-1} f(a+ih) \right\} + f(b) \right]$$

Example 2: The vertical distance of a rocket between 8 to 30 seconds is given by:

$$x = \int_{8}^{30} \left(2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

Use two- section Trapezoidal rule to find a) the distance covered., b) the true error, E_t for part (a), and c) the absolute relative true error, $\left| \in_a \right|$ for part (a).

Accuracy of Approximation using Trapezoidal Rule

$$I_{exact} = \int_a^b f(x)dx$$

$$I_{exact} = I_{trapezoidal} + E$$

Error from Taylor series

$$E = \frac{C}{n^2} + \boxed{\frac{D}{n^4} + \frac{F}{n^6} + \ldots\ldots\ldots}$$

For large n, neglect higher order terms

$$E \propto \frac{1}{n^2}$$

n = number of sections

e.g. if you double # of sections→ truncation error will be quartered

Group Problem Solving: Find the area under the following curve between 0 to 10.

$$f(x) = \frac{300x}{1+e^{x}}$$

Use two- section Trapezoidal rule to: a) estimate the integral b) the true error, E_t for part (a), and c) the absolute relative true error, $|\in_a|$ for part (a).