Problem Set #6 Solutions

21.6 Analytical solution:

$$\int_0^3 x^2 e^x \, dx = \left[(x^2 - 2x + 2)e^x \right]_0^3 = 98.42768$$

Trapezoidal rule ($n = 4$):

$$I = (3-0)\frac{0 + 2(1.190813 + 10.0838 + 48.03166) + 180.7698}{8} = 112.2684 \qquad \varepsilon_t = 14.062\%$$

Simpson's rule ($n = 4$):

$$I = (3-0)\frac{0 + 4(1.190813 + 48.03166) + 2(10.0838) + 180.7698}{12} = 99.45683 \qquad \varepsilon_t = 1.046\%$$

21.9 Analytical solution:

$$z(t) = \int_0^t \sqrt{\frac{gm}{c_d}} \tanh\left(\sqrt{\frac{gc_d}{m}}t\right) dt = \left[\frac{m}{c_d}\ln\left[\cosh\left(\sqrt{\frac{gc_d}{m}}t\right)\right]\right]_0^t$$

$$z(10) = \left[\frac{68.1}{0.25}\ln\left[\cosh\left(\sqrt{\frac{9.81(0.25)}{68.1}}(10)\right)\right]\right]_0^{10} = 334.1782$$

Thus, the result to 3 significant digits is 334. Here are results for various multiple-segment trapezoidal rules:

n	*I*
1	247.1068
2	314.6304
3	325.7253
4	329.4623
5	331.1708
6	332.0937
7	332.6485
8	333.0079
9	333.254
10	333.4298
11	333.5599

Thus, an 11-segment application gives the result to 3 significant digits.

21.11

(a) Trapezoidal rule ($n = 6$):

$$I = (10-(-2))\frac{35+2(5-10+2+5+3)+20}{12} = 65$$

(b) Simpson's rules ($n = 6$):

$$I = (10-(-2))\frac{35+4(5+2+3)+2(-10+5)+20}{18} = 56.66667$$

22.2 The integral can be evaluated analytically as,

$$I = \int_1^2 \left(x+\frac{1}{x}\right)^2 dx = \int_1^2 x^2+2+x^{-2}\ dx$$

$$I = \left[\frac{x^3}{3}+2x-\frac{1}{x}\right]_1^2 = \frac{2^3}{3}+2(2)-\frac{1}{2}-\frac{1^3}{3}-2(1)+\frac{1}{1} = 4.8333$$

The tableau depicting the implementation of Romberg integration to $\varepsilon_s = 0.5\%$ is

iteration→	**1**	**2**	**3**
$\varepsilon_t \to$	6.0345%	0.0958%	0.0028%
$\varepsilon_a \to$		1.4833%	0.0058%
1	5.12500000	4.83796296	4.83347014
2	4.90972222	4.83375094	
4	4.85274376		

Thus, the result is 4.83347014.

22.3

	1	**2**	**3**
n	$\varepsilon_a \to$	**7.9715%**	**0.0997%**
1	1.34376994	1.97282684	1.94183605
2	1.81556261	1.94377297	
4	1.91172038		

22.4 Change of variable:

$$x = \frac{2+1}{2} + \frac{2-1}{2} x_d = 1.5 + 0.5x_d$$

$$dx = \frac{2-1}{2} dx_d = 0.5dx_d$$

$$I = \int_{-1}^{1} \left(1.5 + 0.5x_d + \frac{1}{1.5 + 0.5x_d}\right)^2 0.5dx_d$$

Therefore, the transformed function is

$$f(x_d) = 0.5\left(1.5 + 0.5x_d + \frac{1}{1.5 + 0.5x_d}\right)^2$$

Two-point formula:

$$I = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = 2.074414 + 2.755961 = 4.830375$$

$$\varepsilon_t = \left|\frac{4.833333 - 4.830375}{4.833333}\right| \times 100\% = 0.0612\%$$

Three-point formula:

$$\begin{aligned} I &= 0.5555556 f(-0.7745967) + 0.8888889 f(0) + 0.5555556 f(0.7745967) \\ &= 0.5555556(2.022895) + 0.8888889(2.347222) + 0.5555556(2.921322) \\ &= 1.123831 + 2.08642 + 1.622957 = 4.833207 \end{aligned}$$

$$\varepsilon_t = \left|\frac{4.833333 - 4.833207}{4.833207}\right| \times 100\% = 0.0026\%$$

Four-point formula:

$$\begin{aligned} I &= 0.3478548 f(-0.861136312) + 0.6521452 f(-0.339981044) + 0.6521452 f(0.339981044) \\ &\qquad + 0.3478548 f(0.861136312) \\ &= 0.3478548(2.009026) + 0.6521452(2.16712) + 0.6521452(2.573718) + 0.3478548(2.997699) \\ &= 0.698849 + 1.413277 + 1.678438 + 1.042764 = 4.833328 \end{aligned}$$

$$\varepsilon_t = \left|\frac{4.833333 - 4.833328}{4.833333}\right| \times 100\% = 0.00010\%$$

22.14 Change of variable:

$$x = \frac{1.5+0}{2} + \frac{1.5-0}{2} x_d = 0.75 + 0.75 x_d$$

$$dx = \frac{1.5-0}{2} dx_d = 0.75 dx_d$$

$$I = \int_{-1}^{1} \frac{1.5}{\sqrt{\pi}} e^{-(0.75+0.75x_d)^2} \, dx_d$$

Therefore, the transformed function is

$$f(x_d) = \frac{1.5}{\sqrt{\pi}} e^{-(0.75+0.75x_d)^2}$$

Two-point formula:

$$I = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = 0.765382 + 0.208792 = 0.974173$$

$$\varepsilon_t = \left| \frac{0.966105 - 0.974173}{0.966105} \right| \times 100\% = 0.835\%$$