MTE 203 – Advanced Calculus
Homework 8

Relative and Absolute Maxima and Minima

Problem 1: [12.10, Prob. 3]

Find all the critical points for the given function and classify each as yielding a relative maximum, a relative minimum, a saddle point, or none of these.

$$f(x,y) = x^3 - 3x + y^2 + 2y$$

Solution:

For critical points we solve $0 = \dfrac{\partial f}{\partial x} = 3x^2 - 3$, $0 = \dfrac{\partial f}{\partial y} = 2y + 2$. Solutions are $(\pm 1, -1)$.

$$\frac{\partial^2 f}{\partial x^2} = 6x, \qquad \frac{\partial^2 f}{\partial x \partial y} = 0, \qquad \frac{\partial^2 f}{\partial y^2} = 2$$

At $(-1,-1)$, $B^2 - AC = 0 + 12$, and therefore $(-1,-1)$ yields a saddle point. At $(1,-1)$, $B^2 - AC = 0 - 12$ and $A = 6$. Thus, $(1,-1)$ gives a relative minimum.

Problem 2: [12.10, Prob. 13]

Find all the critical points for the given function and classify each as yielding a relative maximum, a relative minimum, a saddle point, or none of these.

$$f(x,y) = (1-x)(1-y)(x+y-1)$$

For critical points we solve

$$0 = \frac{\partial f}{\partial x} = -(1-y)(x+y-1) + (1-x)(1-y) = (1-y)(2-2x-y),$$
$$0 = \frac{\partial f}{\partial y} = -(1-x)(x+y-1) + (1-y)(1-x) = (1-x)(2-2y-x).$$

Solutions are $(1,1)$, $(0,1)$, $(1,0)$ and $(2/3, 2/3)$.

$$\frac{\partial^2 f}{\partial x^2} = 2(y-1), \qquad \frac{\partial^2 f}{\partial x \partial y} = -(2-2x-y) - (1-y) = -3 + 2x + 2y, \qquad \frac{\partial^2 f}{\partial y^2} = 2(x-1)$$

At $(1,1)$, $(0,1)$ and $(1,0)$, $B^2 - AC = 1$, so that each gives a saddle point. At $(2/3, 2/3)$, $B^2 - AC = (-1/3)^2 - 4(-1/3)(-1/3) = -1/3$ and $A = -2/3$. This critical point therefore yields a relative maximum.

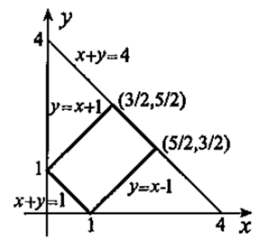
Problem 3: [12.11, Prob.3]

Find the maximum and minimum values of the function on the region:

$f(x,y) = 3x + 4y$ on region R bounded by the lines $x + y = 1$, $x + y = 4$, $1 + y = x$, and $y - 1 = x$,

Solution:

The function has no critical points inside C. When $f(x,y)$ is expressed in terms of one variable on each part of C, the resulting function is linear, and therefore has no critical points. It follows that maximum and minimum of the function must occur at the vertices of the rectangle. Since $f(1,0) = \boxed{3}$, $f(0,1) = \boxed{4}$, $f(3/2,5/2) = \boxed{29/2}$, and $f(5/2,3/2) = \boxed{27/2}$, maximum and minimum values are 29/2 and 3.



<u>**Constrained Maxima and Minima**</u>

Problem 4: [12.11, Prob. 15]

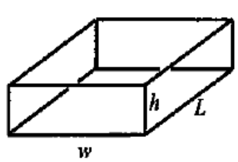
a. When a rectangular box is sent through the mail, the post office demands that the length of the box plus twice the sum of its height and width be no more than 250 cm. Find the dimensions of the box satisfying this requirement that encloses the largest possible volume.
b. **[Optional]** Verify your answer by solving the problem using Lagrange multipliers.

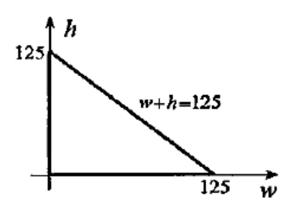
Solution part a:

The volume of the box is $V = Lwh$. Since $L + 2(w+h) \le 250$, we set $L = 250 - 2w - 2h$, in which case $V = wh(250 - 2w - 2h)$. This function must be maximized on the triangle in the right figure below. For critical points we solve

$$0 = \frac{\partial V}{\partial w} = 250h - 4wh - 2h^2, \qquad 0 = \frac{\partial V}{\partial h} = 250w - 2w^2 - 4wh.$$

The only solution inside the triangle is $w = h = 125/3$. Since $V = 0$ on the edges of the triangle, it follows that this critical point must yield a maximum volume. Dimensions of the box are therefore $w = h = 125/3$ cm and $L = 250/3$ cm.





Part b: Left for you

Problem 5: [12.11, Prob. 25]

Find the maximum and minimum values of $f(x,y) = |x - 2y|$ on the curve $|x| + |y| = 1$.

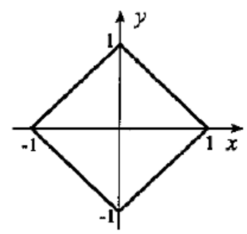
Solution:

On the top half of the square $y = 1 - |x|$, in which case we can write that

$$f(x,y) = F(x) = \big|x - 2 + 2|x|\big|, \quad -1 \le x \le 1.$$

For critical points of this function, we solve

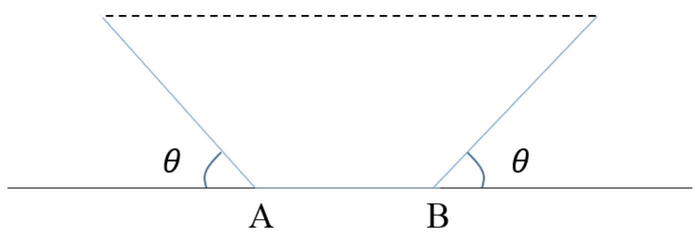
$$0 = F'(x) = \frac{|x - 2 + 2|x||}{x - 2 + 2|x|}\left(1 + \frac{2|x|}{x}\right).$$



There are no solutions of this equation, but the derivative does not exist at $x = 0$ and $x = 2/3$. We now calculate $F(-1) = 1$, $F(0) = 2$, $F(2/3) = 0$, and $F(1) = 1$. A similar analysis on the bottom half of the square leads to the same values. Hence, maximum and minimum values of $f(x,y)$ are 2 and 0.

Problem 6: [12.11, Prob.31]

A long piece of metal 1 m wide is bent at A and B, as shown in the figure below, to form a channel with three straight sides. If the bends are equidistant from the ends, where should they be made in order to obtain maximum possible flow of fluid along the channel?

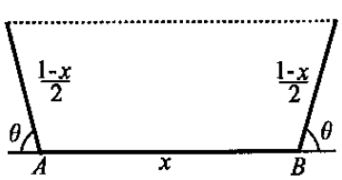


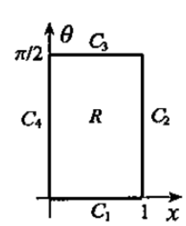
Solution:

If the length of AB is x, and the bends are at angle θ, then the area of the trapezoid is

$$F(x,\theta) = \frac{1}{2}\left(\frac{1-x}{2}\right)\sin\theta\left[2x + 2\left(\frac{1-x}{2}\right)\cos\theta\right] = \frac{1}{4}(1-x)\sin\theta[2x + (1-x)\cos\theta].$$

This function must be maximized for the region R of the $x\theta$-plane shown to the right below.





For critical points, we solve

$$0 = \frac{\partial F}{\partial x} = \frac{1}{4}\sin\theta[2 - 4x - 2(1-x)\cos\theta],$$
$$0 = \frac{\partial F}{\partial \theta} = \frac{1}{4}(1-x)[2x\cos\theta + (1-x)(\cos^2\theta - \sin^2\theta)].$$

Since $\sin\theta = 0$ and $1 - x = 0$ correspond to edges of R, we set

$$1 - 2x - (1-x)\cos\theta = 0, \qquad 2x\cos\theta + (1-x)(\cos^2\theta - \sin^2\theta) = 0.$$

The first equation implies that $\cos\theta = (1-2x)/(1-x)$, and when this is substituted into the second equation,

$$0 = 2x\left(\frac{1-2x}{1-x}\right) + (1-x)\left[2\left(\frac{1-2x}{1-x}\right)^2 - 1\right].$$

This simplifies to $0 = 3x^2 - 4x + 1 = (3x-1)(x-1)$. Thus, $x = 1/3$, and from this $\theta = \pi/3$. The area of the trapezoid so formed is

$$F = \frac{1}{4}\left(\frac{2}{3}\right)\left(\frac{\sqrt{3}}{2}\right)\left[\frac{2}{3} + \frac{2}{3}\left(\frac{1}{2}\right)\right] = \boxed{\frac{\sqrt{3}}{12}}.$$

For values of x and θ along edges C_1 and C_2 of R, the area of the trapezoid is $\boxed{0}$. Along C_3,

$$F = \frac{1}{4}(1-x)2x = \frac{x(1-x)}{2}, \quad 0 \le x \le 1.$$

For critical points, $0 = dF/dx = (1-2x)/2$. At the critical point $x = 1/2$, $F(1/2) = \boxed{1/8}$.

Along C_4,

$$F = \frac{1}{4}\sin\theta\,\cos\theta = \frac{1}{8}\sin 2\theta, \quad 0 \le \theta \le \pi/2.$$

For critical points, $0 = dF/d\theta = (1/4)\cos 2\theta$. At the critical point $\theta = \pi/4$, $F(\pi/4) = \boxed{1/8}$.

Finally, at the four vertices of the rectangle,

$$F(0,0) = \boxed{0}, \quad F(1,0) = \boxed{0}, \quad F(1,\pi/2) = \boxed{0}, \quad F(0,\pi/2) = \boxed{0}.$$

Thus, area is maximized when $\theta = \pi/3$ and $x = 1/3$ m.

Lagrange Multipliers

Problem 7: [12.12, Prob.3]

Use Lagrange multipliers to find the maximum and minimum values of the given function subject to the constraints. Also interpret the constraint geometrically.

Solution:

The constraint $(x-1)^2+y^2=1$ defines a **closed** curve (a circle). We define the Lagrangian

$$L(x,y,\lambda)=x+y+\lambda[(x-1)^2+y^2-1].$$

For critical points of L, we solve

$$0=\frac{\partial L}{\partial x}=1+2\lambda(x-1), \quad 0=\frac{\partial L}{\partial y}=1+2\lambda y, \quad 0=\frac{\partial L}{\partial \lambda}=(x-1)^2+y^2-1.$$

Critical points (x,y) are $(1\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$. Since $f(1\pm 1/\sqrt{2}, \pm 1/\sqrt{2})=1\pm\sqrt{2}$, these are maximum and minimum values of $f(x,y)$.

Problem 8: [12.12, Prob.27]

Find the maximum value of $f(x,y,z)=x^2yz-xzy^2$ subject to constraints $x^2+y^2=1$, $z=\sqrt{x^2+y^2}$

Solution:

Since z is always equal to 1 on the curve of intersection of the surfaces, we maximize the function $F(x,y)=x^2y-xy^2$ subject to the constraint $x^2+y^2=1$. Critical points of the Lagrangian $L(x,y,\lambda)=x^2y-xy^2+\lambda(x^2+y^2-1)$ are given by

$$0=\frac{\partial L}{\partial x}=2xy-y^2+2\lambda x, \quad 0=\frac{\partial L}{\partial y}=x^2-2xy+2\lambda y, \quad 0=\frac{\partial L}{\partial \lambda}=x^2+y^2-1.$$

Solutions for (x,y) are $(\pm 1/\sqrt{2}, \mp 1/\sqrt{2})$, $\left(\sqrt{(3\pm\sqrt{5})/6}, 1/[3\sqrt{3\pm\sqrt{5})/6}]\right)$, and $\left(-\sqrt{(3\pm\sqrt{5})/6}, -1/[3\sqrt{3\pm\sqrt{5})/6}]\right)$. When these are substituted into $F(x,y)$, the largest value is $1/\sqrt{2}$.

Warm-Up Problems

Solutions to these problems can be found at the back of your textbook

1. S. 12.10, Probs. 2, 6, 10, 18,
2. S. 12.11, Probs. 2, 4, 10
3. S. 12.12, Probs. 2, 4, 8

Extra Practice Problems

Solutions to these problems can be found at the back of your textbook

1. S. 12.10, Probs. 8, 12, 20
2. S. 12.11, Probs. 12, 20, 30, 32
3. S. 12.12, Probs. 18, 24, 26, 28

Extra Challenging Problems

Solutions to these problems can be found at the back of your textbook

1. S. 12.10, Probs. 22
2. S. 12.11, Probs. 28, 38
3. S. 12.12, Probs. 22, 34