

SYDE252 - lecture notes

09/01/18

Presented by: John Zelek
Systems Design Engineering
note: some material (figures) borrowed from various sources



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6. Fourier

09/11/18

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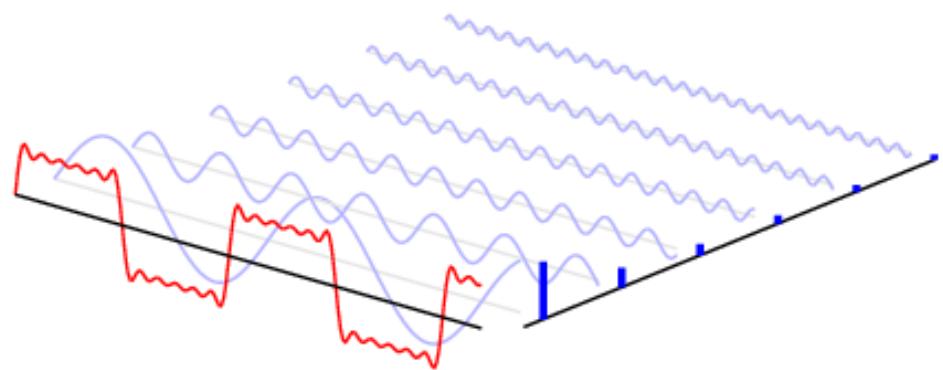
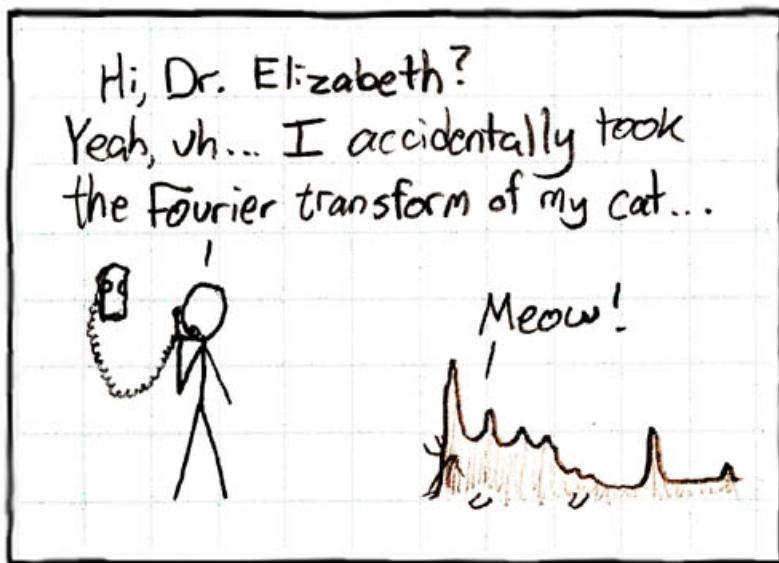


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inspiration

- “=Despots prefer the friendship of the dog, who, unjustly mistreated and debased, still loves and serves the man who wronged him. The method of doubt must be applied to civilization; we must doubt its necessity, its excellence, and its permanence”
— Charles Fourier



<http://math.sfsu.edu/beck/quotes.html>

- <http://pgfplots.net/tikz/examples/fourier-transform/>



Fourier Analysis: basics - Phasors

- phasor associated with the signal

$$v(t) = V_0 \cos(\omega t + \phi)$$

- is the complex number

$$\mathbf{V} = V_0 e^{j\phi}$$

- $v(t) = V_0 \cos(\omega t + \phi)$ \leftrightarrow $\mathbf{V} = V_0 e^{j\phi}$



Fourier Analysis: basics - Phasors

$x(t)$		\mathbf{X}
$A \cos \omega t$	\leftrightarrow	A
$A \cos(\omega t + \phi)$	\leftrightarrow	$Ae^{j\phi}$
$-A \cos(\omega t + \phi)$	\leftrightarrow	$Ae^{j(\phi \pm \pi)}$
$A \sin \omega t$	\leftrightarrow	$Ae^{-j\pi/2} = -jA$
$A \sin(\omega t + \phi)$	\leftrightarrow	$Ae^{j(\phi - \pi/2)}$
$-A \sin(\omega t + \phi)$	\leftrightarrow	$Ae^{j(\phi + \pi/2)}$
$\frac{d}{dt}[A \cos(\omega t + \phi)]$	\leftrightarrow	$j\omega A e^{j\phi}$
$\int A \cos(\omega t' + \phi) dt'$	\leftrightarrow	$\frac{1}{j\omega} A e^{j\phi}$



Fourier Analysis: basics - Phasors

- Use phasor analysis to solve the differential equation
($a=300$ and $b=50000$)

$$\frac{d^2y}{dt^2} + a \frac{dy}{dt} + by = 10 \sin(100t + 60^\circ)$$

- The phasor associated with the input is $X = 10e^{-j30^\circ}$
- The differential equation becomes algebraic
- The phasor associated with the output is then found as
-



Fourier Analysis: Fourier series

A spectrum of a signal is a way of describing a signal as a sum of sinusoids.

A signal can be described in 4 different ways:

1. a mathematical definition $x(t) = A_0 + \sum_{k=1}^N A_k \cos(2\pi f_k t + \phi_k)$
2. a plot of $x(t)$ versus time
3. a list of amplitudes, phases and frequencies such as $(A_0, (A_1, \phi_1, f_1), \dots (A_N, \phi_N, f_N))$
4. as a plot of the amplitudes and phases against frequencies.



Fourier Analysis: Fourier series

A Fourier series is a way of breaking of the signal into a weighted sum of a family of signals. We want to identify a family of signals such that $\{x_k(t)\}$ such that

1. every signal in the family passes through LTI system with only a scale change: $x_k(t) \rightarrow \lambda_k x_k(t)$
2. any signal can be represented as a linear combination of signals in their family:
 $x(t) = \sum_{k=-\infty}^{\infty} a_k x_k(t)$ and therefore $x(t) \rightarrow \sum_{k=-\infty}^{\infty} a_k x_k(t)$



Fourier Analysis: Fourier series

Definition: for an LTI system, if the output is a scaled version of the input, then the input function is called an eigenfunction of the system.

suppose: $x(t) = e^{st}$, see example: to see that this function is an eigenfunction.

lets consider $x(t) = a_1e^{s_1t} + a_2e^{s_2t} + a_3e^{s_3t}$ and recall that $e^{s_i t} \rightarrow H(s_i)e^{s_i t}$, therefore

$$y(t) = a_1H(s_1)e^{s_1t} + a_2H(s_2)e^{s_2t} + a_3H(s_3)e^{s_3t}$$

if the input is a linear combination of exponentials, so is the output!

Generally if $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{s_k t}$ (an infinite sum of complex exponentials), so is the output
 $y(t) = \sum_{k=-\infty}^{\infty} a_k H(s_k) e^{s_k t}$



Fourier Analysis: Fourier series of periodic signals

- Let $x(t)$ be **periodic**:
- Then $x(t)$ can be expressed as a linear combination of sinusoids at **harmonic** frequencies:

$$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)]$$

(sine/cosine representation)

$$= c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \phi_n)$$

(amplitude/phase representation)



Fourier Analysis: Fourier series of periodic signals

- Let $x(t)$ be **periodic**:
- Now $x(t)$ need not be real-valued
- Then $x(t)$ can be expressed as a linear combination of complex exponentials at **harmonic frequencies**:

$$x(t) = \sum_{n=-\infty}^{\infty} \mathbf{x}_n e^{jn\omega_0 t},$$

(exponential representation)



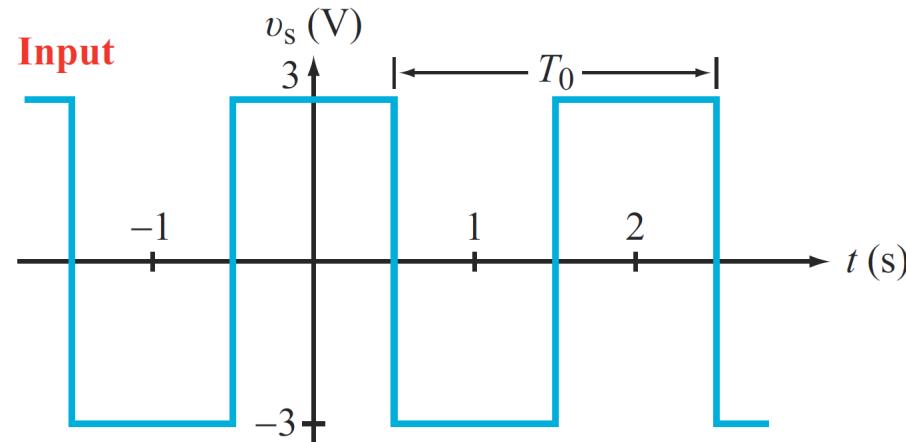
Fourier Analysis: Fourier series of periodic signals

- Fundamental angular frequency: $\omega_0 = 2\pi/T_0$
- dc or average term: a_0 or c_0
- Fundamental term: $c_1 \cos(\omega_0 t + \phi_1)$
- Harmonics: $c_n \cos(n\omega_0 t + \phi_n)$
- Fundamental has same period as $x(t)$
- In music: harmonics are known as overtones
- $\{a_n, b_n, c_n, x_n\}$ are Fourier coefficients



Fourier Analysis: Fourier series example

- The Fourier series expansion of the periodic signal



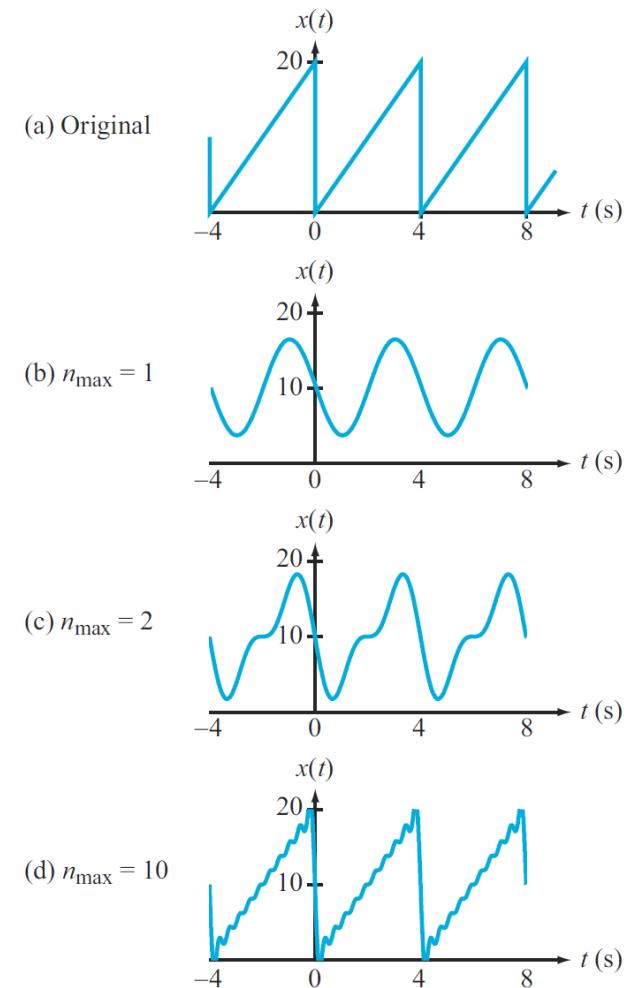
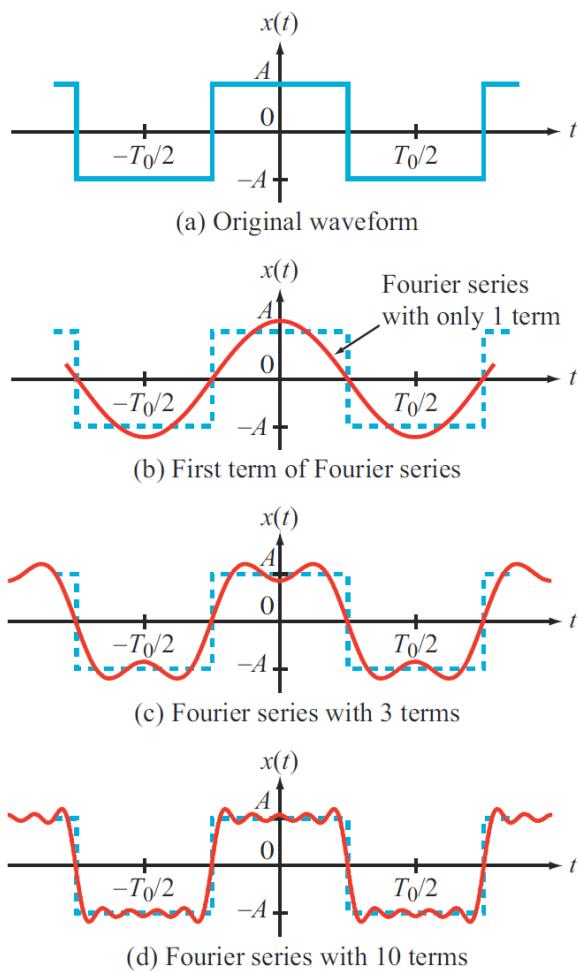
- is the infinite series

$$v_s(t) = \frac{12}{\pi} \left(\cos \omega_0 t - \frac{1}{3} \cos 3\omega_0 t + \frac{1}{5} \cos 5\omega_0 t - \dots \right)$$



Fourier Analysis: Fourier series more examples

- Adding more terms makes the Fourier series resemble more closely the original signal:

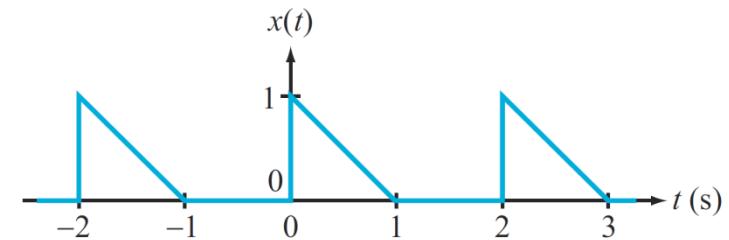
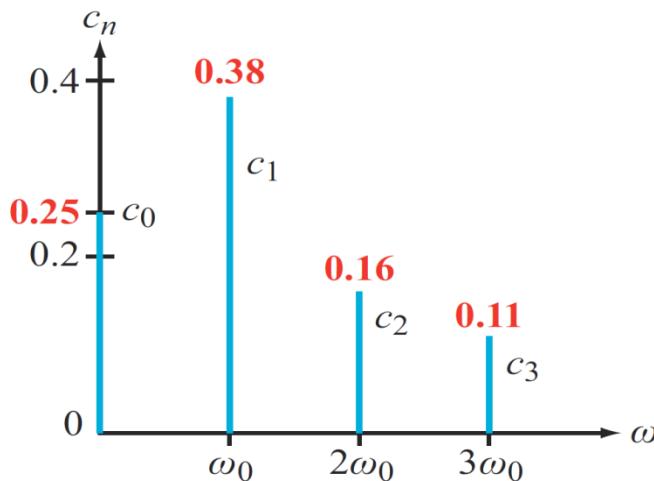


Fourier Analysis: Fourier series - one sided line spectrum

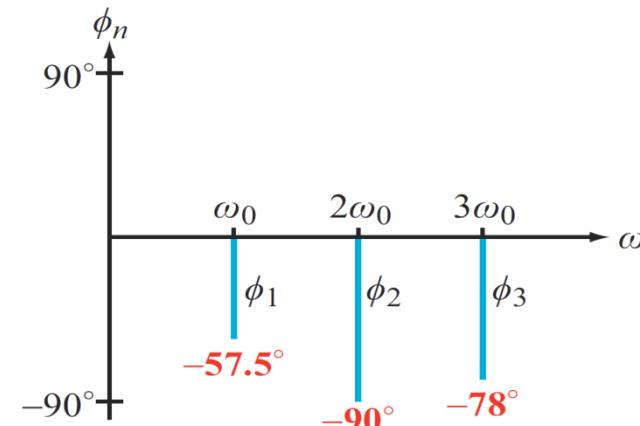
- Plot of amplitudes and phases vs. frequency .
- Example: $x(t)$ has period=2.

$$c_n = \begin{cases} \left(\frac{4}{n^4\pi^4} + \frac{1}{n^2\pi^2} \right)^{1/2} & \text{for } n = \text{odd}, \\ \frac{1}{n\pi} & \text{for } n = \text{even}. \end{cases}$$

- Amplitude spectrum:



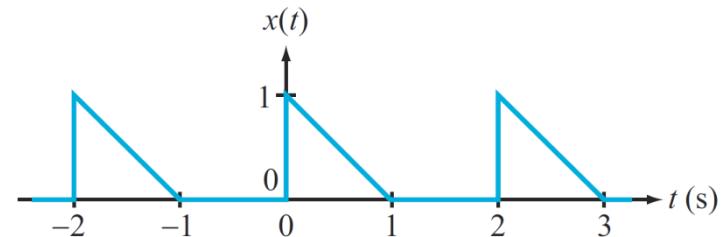
- Phase spectrum:



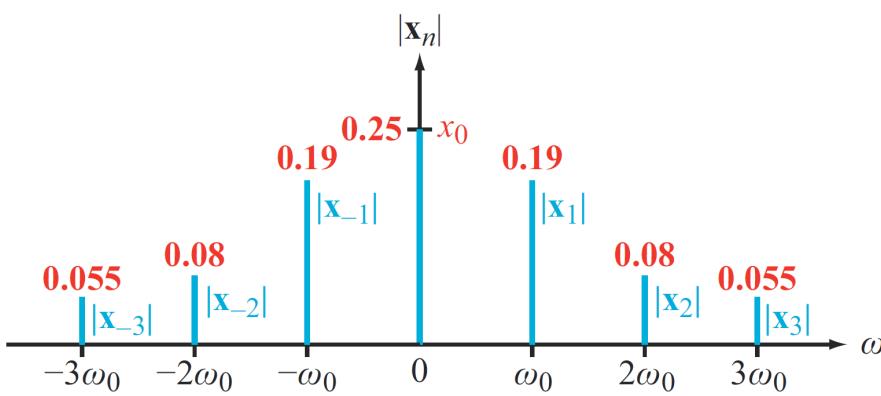
Fourier Analysis: Fourier series - two sided line spectrum

- Plot of magnitudes and phases vs. frequency .
- Example: $x(t)$ has period=2.

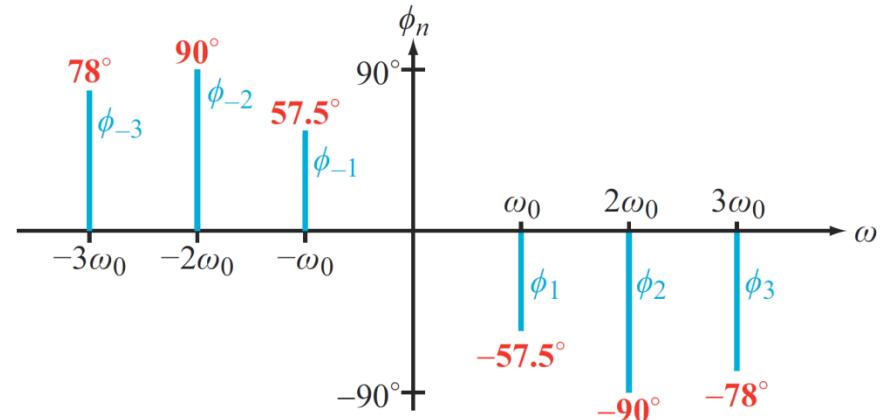
$$c_n = \begin{cases} \left(\frac{4}{n^4\pi^4} + \frac{1}{n^2\pi^2} \right)^{1/2} & \text{for } n = \text{odd}, \\ \frac{1}{n\pi} & \text{for } n = \text{even} \end{cases}$$



- Magnitude spectrum:



- Phase spectrum:



Fourier Analysis: Fourier series examples for some periodic functions

Fourier series expressions for a select set of periodic waveforms.

	Waveform	Fourier Series
1. Square Wave		$x(t) = \sum_{n=1}^{\infty} \frac{4A}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos\left(\frac{2n\pi t}{T_0}\right)$
2. Time-Shifted Square Wave		$x(t) = \sum_{n=\text{odd}}^{\infty} \frac{4A}{n\pi} \sin\left(\frac{2n\pi t}{T_0}\right)$
3. Pulse Train		$x(t) = \frac{A\tau}{T_0} + \sum_{n=1}^{\infty} \frac{2A}{n\pi} \sin\left(\frac{n\pi\tau}{T_0}\right) \cos\left(\frac{2n\pi t}{T_0}\right)$
4. Triangular Wave		$x(t) = \sum_{n=\text{odd}}^{\infty} \frac{8A}{n^2\pi^2} \cos\left(\frac{2n\pi t}{T_0}\right)$



Fourier Analysis: 3 forms of Fourier series expansion

$$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)].$$

(sine/cosine representation)

$$x(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \phi_n),$$

(amplitude/phase representation)

$$x(t) = \sum_{n=-\infty}^{\infty} \mathbf{x}_n e^{jn\omega_0 t},$$

(exponential representation)



Fourier Series: computing coefficients for sine/cosine form

$$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)].$$

(sine/cosine representation)

Compute coefficients from a single period of $x(t)$ using the formulae

$$a_0 = \frac{1}{T_0} \int_0^{T_0} x(t) dt,$$

$$a_n = \frac{2}{T_0} \int_0^{T_0} x(t) \cos(n\omega_0 t) dt,$$

$$\text{and } b_n = \frac{2}{T_0} \int_0^{T_0} x(t) \sin(n\omega_0 t) dt.$$



Fourier Series: computing coefficients for amplitude/phase form

$$x(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \phi_n),$$

(amplitude/phase representation)

Compute coefficients from a single period of $x(t)$ using the formula

$$c_n = \sqrt{a_n^2 + b_n^2}$$

and

$$\phi_n = \begin{cases} -\tan^{-1}\left(\frac{b_n}{a_n}\right), & a_n > 0 \\ \pi - \tan^{-1}\left(\frac{b_n}{a_n}\right), & a_n < 0 \end{cases}$$



Fourier Series: computing coefficients for exponential form

$$x(t) = \sum_{n=-\infty}^{\infty} \mathbf{x}_n e^{jn\omega_0 t},$$

(exponential representation)

Compute coefficients from a single period of $x(t)$ using the formula

$$\mathbf{x}_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-jn\omega_0 t} dt.$$

Note this is much simpler than using the other two representations.



Fourier Series: 3 representations summary

Cosine/Sine	Amplitude/Phase	Complex Exponential
$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)]$	$x(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \phi_n)$	$x(t) = \sum_{n=-\infty}^{\infty} \mathbf{x}_n e^{jn\omega_0 t}$
$a_0 = \frac{1}{T_0} \int_0^{T_0} x(t) dt$	$c_n e^{j\phi_n} = a_n - jb_n$	$\mathbf{x}_n = \mathbf{x}_n e^{j\phi_n}; \mathbf{x}_{-n} = \mathbf{x}_n^*; \phi_{-n} = -\phi_n$
$a_n = \frac{2}{T_0} \int_0^{T_0} x(t) \cos n\omega_0 t dt$	$c_n = \sqrt{a_n^2 + b_n^2}$	$ \mathbf{x}_n = c_n/2; x_0 = c_0$
$b_n = \frac{2}{T_0} \int_0^{T_0} x(t) \sin n\omega_0 t dt$	$\phi_n = \begin{cases} -\tan^{-1}(b_n/a_n), & a_n > 0 \\ \pi - \tan^{-1}(b_n/a_n), & a_n < 0 \end{cases}$	$\mathbf{x}_n = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-jn\omega_0 t} dt$
$a_0 = c_0 = x_0; a_n = c_n \cos \phi_n; b_n = -c_n \sin \phi_n; \mathbf{x}_n = \frac{1}{2}(a_n - jb_n)$		



Fourier Series: 3 representations summary - symmetry simplifies

Even Symmetry: $x(t) = x(-t)$

$$a_0 = \frac{2}{T_0} \int_0^{T_0/2} x(t) dt$$

$$a_n = \frac{4}{T_0} \int_0^{T_0/2} x(t) \cos(n\omega_0 t) dt$$

$$b_n = 0$$

$$c_n = |a_n|, \quad \phi_n = \begin{cases} 0 & \text{if } a_n > 0 \\ 180^\circ & \text{if } a_n < 0 \end{cases}$$

Odd Symmetry: $x(t) = -x(-t)$

$$a_0 = 0 \quad a_n = 0$$

$$b_n = \frac{4}{T_0} \int_0^{T_0/2} x(t) \sin(n\omega_0 t) dt$$

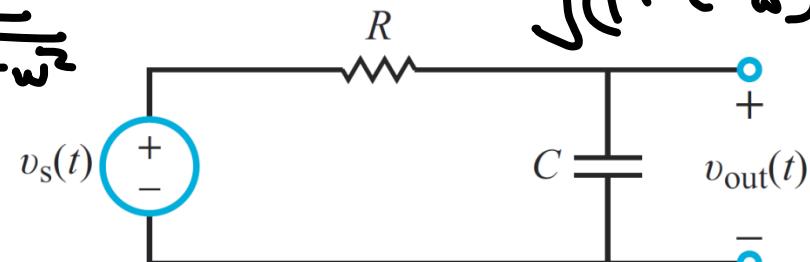
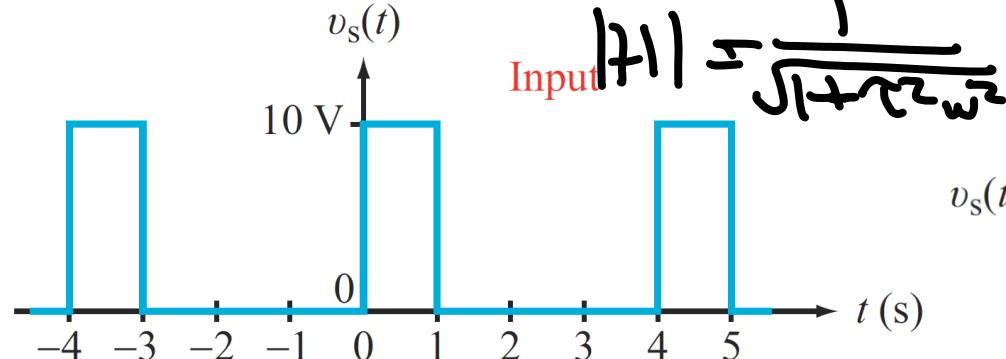
$$c_n = |b_n| \quad \phi_n = \begin{cases} -90^\circ & \text{if } b_n > 0 \\ 90^\circ & \text{if } b_n < 0 \end{cases}$$



Fourier Series: circuit analysis

- Compute the voltage response of the RC circuit shown to the input square wave shown, using Fourier series. The time constant RC of the circuit is $RC=2$ s.

$$H(j\omega) = \frac{1 - \tau j\omega}{(1 + \tau j\omega)(1 - \tau j\omega)} = \frac{1 - \tau j\omega}{1 + \tau^2 \omega^2} \quad |H| = \sqrt{\frac{1 + \tau^2 \omega^2}{(1 + \tau^2 \omega^2)^2}}$$



$$H(s) = \frac{1}{1 + \tau s} \quad s = j\omega$$

$$H(j\omega) = \frac{1}{1 + \tau j\omega}$$



Fourier Series: circuit analysis

- The input waveform has the Fourier series expansion

$$v_s(t) = a_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \phi_n)$$

$$c_n = a_n - jb_n = \frac{10}{n\pi} \left[\sin \frac{n\pi}{2} - j(1 - \cos \frac{n\pi}{2}) \right]$$

- The RC circuit has the transfer function ($RC=2$ s)

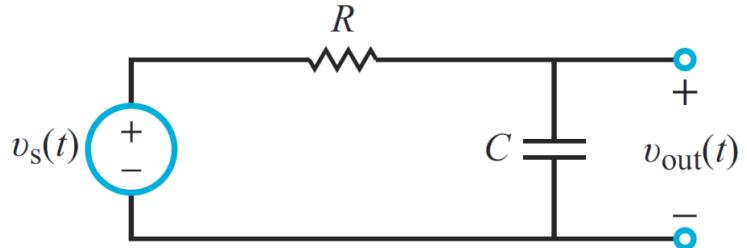
$$H(\omega) = \frac{1}{\sqrt{1+4\omega^2}} e^{-j\tan^{-1}(2\omega)}$$

$\angle H(j\omega)$

$|H(j\omega)| \angle H(j\omega)$



Fourier Series: circuit analysis

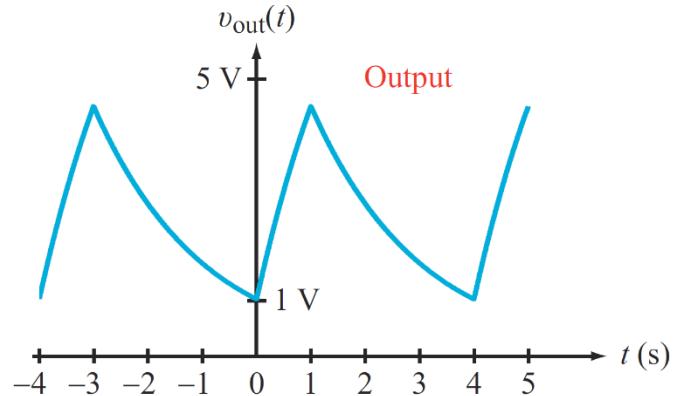
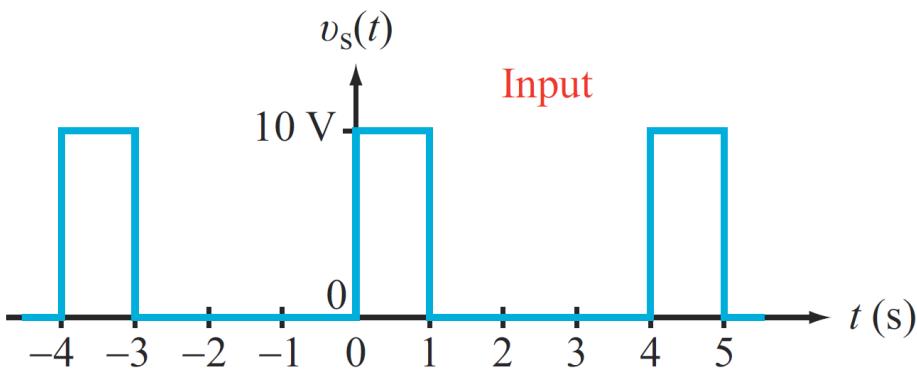


- The output waveform has the Fourier series expansion

$$v_{\text{out}}(t) = \underbrace{2.5}_{C_0} + \sum_{n=1}^{\infty} \text{Re} \left[c_n \frac{e^{j\omega_0 t + \phi_n}}{\sqrt{1+4n^2\omega_0^2}} \right]$$

j $\omega_0 t + \phi_n$

- Plot of the output waveform



Fourier Series: basic harmonics

$$T_0 = ?$$

② e.g., $x(t) = 1 + \frac{1}{2} \cos 2\pi t + \sin 3\pi t$

$$T=2$$

$$x(t) = 1 + \frac{1}{4} \left[e^{j2\pi t} + e^{-j2\pi t} \right] + \frac{1}{2j} \left[e^{j3\pi t} - e^{-j3\pi t} \right]$$

$$a_0 = 1, a_1 = 0, a_2 = \frac{1}{4}$$

$= a_{-1}$ $= a_{-2}$

$$a_3 = \frac{1}{2j}, a_{-3} = -\frac{1}{2j}$$



Fourier Series: basic harmonics

③ e.g., a) $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$ impulse train



Fourier Series: basic harmonics

e.g., $x(t) = 7\cos(\frac{1}{2}t + \theta_1) + 3\cos(\frac{2}{3}t + \theta_2)$
 $+ 5\cos(\frac{7}{6}t + \theta_3)$

find harmonic freq~



Fourier Series: basic harmonics

e.g., c) $x(t) = \cos(2t + \theta_1) + 5\sin(7t + \theta_2)$

e.g., d) $x(t) = 3\sin(3\sqrt{2}t + \theta_1) + 7\cos(6\sqrt{2}t + \theta_2)$



Fourier Series: fundamental frequency

Fundamental Frequency

The fundamental frequency ω_0 can be determined knowing the harmonics

- a sinusoid is periodic
- a periodic signal can be written in terms of a sum of sinusoids

Note: every frequency of a periodic signal is an integer multiple of the fundamental frequency

- ratio of any two frequencies must be of the form $\frac{m}{a}$ where m and a are integers
- ratio of any 2 frequencies must be a rational number
- frequencies must be harmonically related
- fundamental frequency is the GCF (greatest common factor) of all sinusoid frequencies



Fourier Series: representation requirements

Not every signal $x(t)$ can be decomposed as a linear combination of complex exponentials. One class of signals that decomposes are periodic signals. $x(t + T) = x(t)$.

The conditions that need to be satisfied are:

1. over any period, $x(t)$ must be absolutely integrable
$$\int_T |x(t)| dt < \infty$$
2. in any finite interval of time, $x(t)$ is of bounded variation: single period has finite amount of minima and maxima
3. in any finite interval of time, there are a finite number of discontinuities

for this class of signals

$$x(t) = \sum_{-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

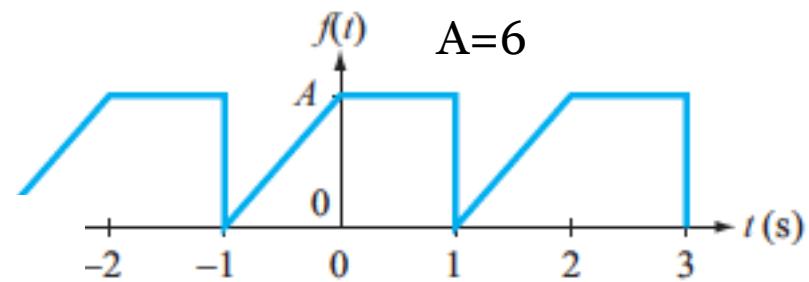
ω_0 is the fundamental frequency and recall that $\omega_0 = \frac{2\pi}{T}$



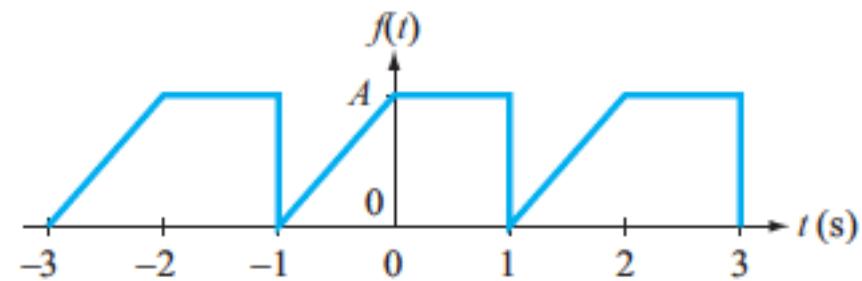
Fourier Series: example

Solution:

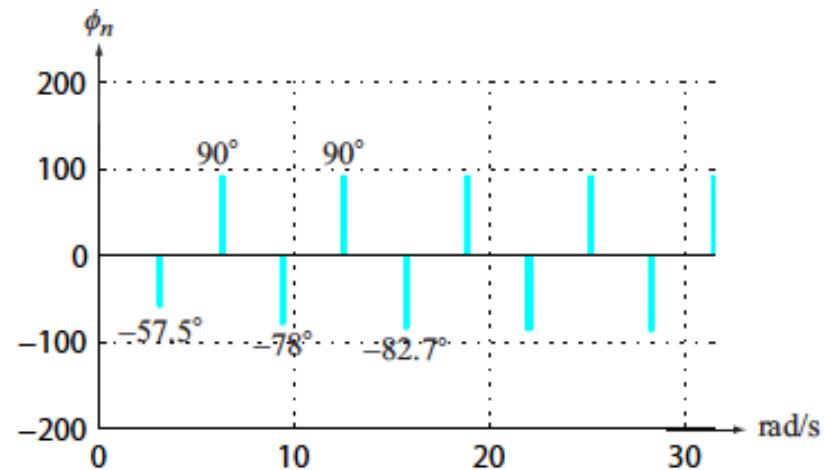
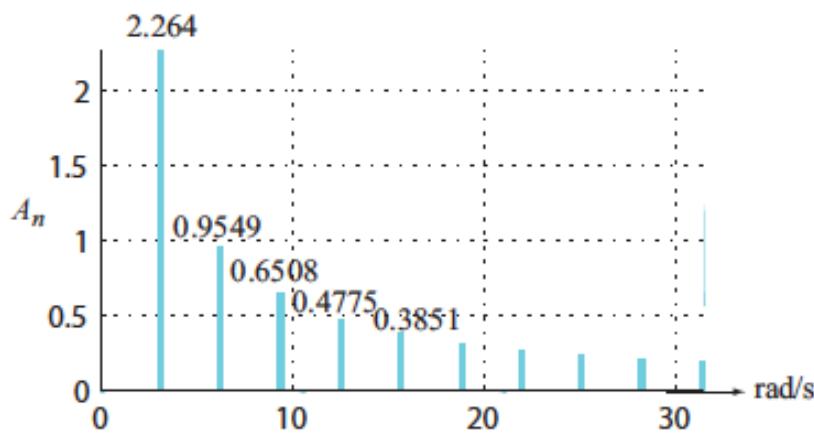
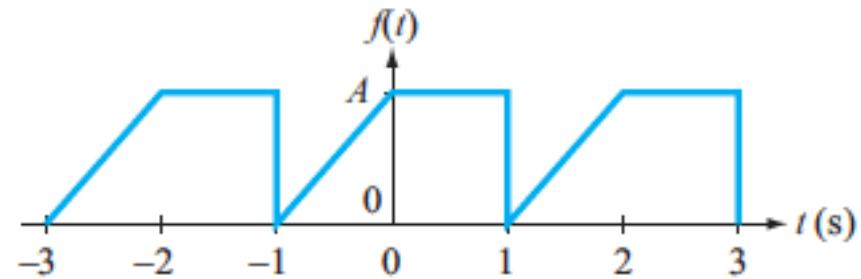
- (a) No symmetry
- (b) period $T = 2$



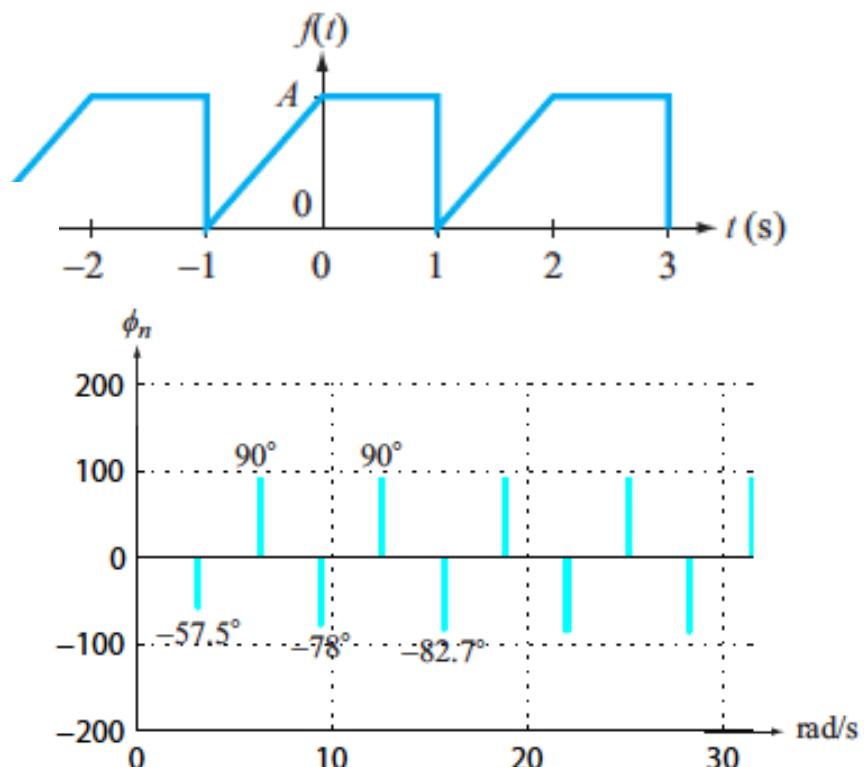
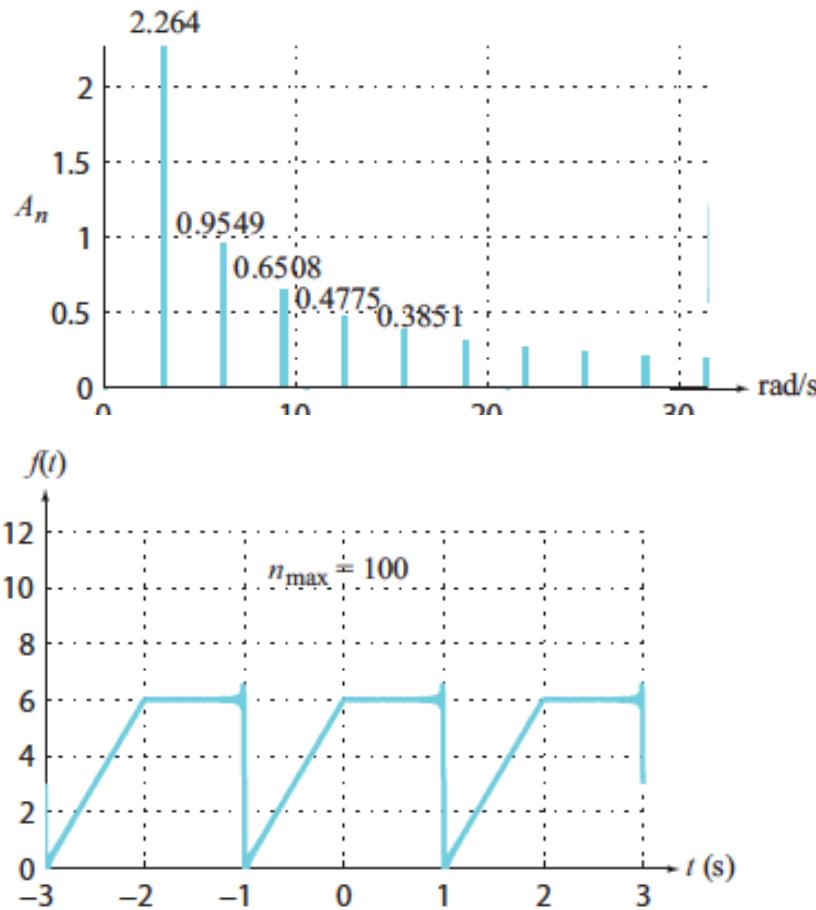
Fourier Series: example



Fourier Series: example



Fourier Series: example



Fourier Series: Parseval's theorem

Average Power is the same whether computed in the time domain or frequency domain

$$P_x = \frac{1}{T_0} \int_0^{T_0} |x(t)|^2 dt = a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)/2,$$

$$P_x = \frac{1}{T_0} \int_0^{T_0} |x(t)|^2 dt = c_0^2 + \sum_{n=1}^{\infty} c_n^2/2,$$

$$P_x = \frac{1}{T_0} \int_0^{T_0} |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |\mathbf{x}_n|^2.$$

Recall that average power of $c_n \cos(n\omega_0 t + \phi_n)$ is $c_n^2/2$



Fourier transform

- For non-periodic signals
- Obtain from Fourier series as period $\rightarrow \infty$
- Compute forward Fourier transform from $x(t)$:

$$\mathbf{X}(\omega) = \mathcal{F}[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

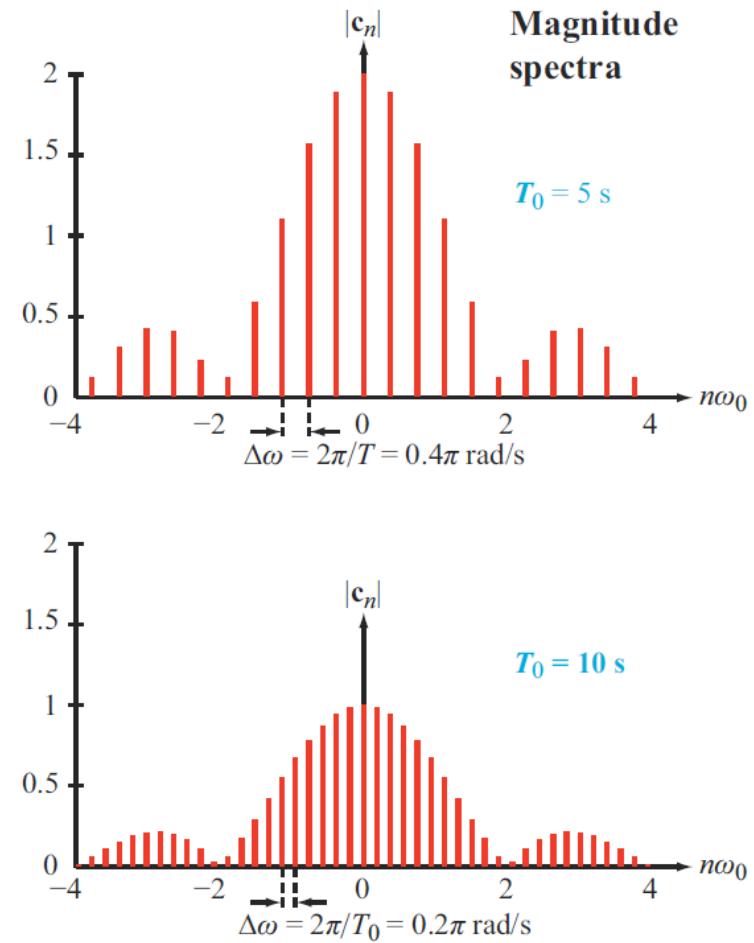
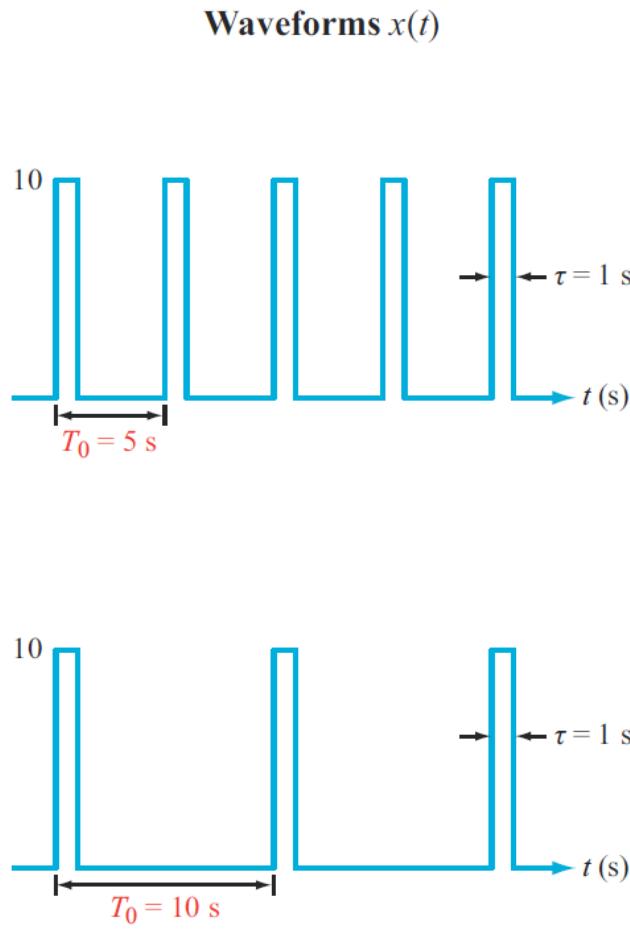
- Compute inverse Fourier transform: recover $x(t)$:

$$x(t) = \mathcal{F}^{-1}[\mathbf{X}(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{X}(\omega) e^{j\omega t} d\omega$$



Fourier series -> Fourier transform

- Consider a periodic pulse train. Let its period $\rightarrow \infty$. Then its two-sided magnitude line spectrum looks like:



Fourier series -> Fourier transform

- Exponential Fourier series:
- Coefficients computed using:
- Spacing between harmonics:
- Insert coefficient formula into exponential series:

$$x(t) = \sum_{n=-\infty}^{\infty} \mathbf{x}_n e^{jn\omega_0 t}$$

$$\mathbf{x}_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t') e^{-jn\omega_0 t'} dt'$$

$$\Delta\omega = (n + 1)\omega_0 - n\omega_0 = \omega_0 = \frac{2\pi}{T_0}$$

$$x(t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-T_0/2}^{T_0/2} x(t') e^{-jn\omega_0 t'} dt' \right] e^{jn\omega_0 t} \Delta\omega.$$

- Let the period $\rightarrow \infty$. Bracketed quantity is $\hat{X}(\omega)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(t') e^{-j\omega t'} dt' \right] e^{j\omega t} d\omega.$$



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Fourier transform - important transfer pairs

- Impulses in time and frequency:

$$\delta(t) \quad \leftrightarrow \quad 1. \qquad \qquad 1 \quad \leftrightarrow \quad 2\pi \delta(\omega).$$

- Causal exponential signals:

$$Ae^{-at} u(t) \quad \leftrightarrow \quad \frac{A}{a + j\omega} \quad \text{for } a > 0.$$

- Eternal sinusoidal signals:

$$\cos \omega_0 t \quad \leftrightarrow \quad \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)],$$

$$\sin \omega_0 t \quad \leftrightarrow \quad j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)].$$



Fourier transform - pairs (1)

No.	$x(t)$	$X(\omega)$	
1	$e^{-at}u(t)$	$\frac{1}{a + j\omega}$	$a > 0$
2	$e^{at}u(-t)$	$\frac{1}{a - j\omega}$	$a > 0$
3	$e^{-a t }$	$\frac{2a}{a^2 + \omega^2}$	$a > 0$
4	$te^{-at}u(t)$	$\frac{1}{(a + j\omega)^2}$	$a > 0$
5	$t^n e^{-at}u(t)$	$\frac{n!}{(a + j\omega)^{n+1}}$	$a > 0$
6	$\delta(t)$	1	
7	1	$2\pi\delta(\omega)$	
8	$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	
9	$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	
10	$\sin \omega_0 t$	$j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$	
11	$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$	



Fourier transform - pairs (1)

12	$\operatorname{sgn} t$	$\frac{2}{j\omega}$
13	$\cos \omega_0 t u(t)$	$\frac{\pi}{2} [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \frac{j\omega}{\omega_0^2 - \omega^2}$
14	$\sin \omega_0 t u(t)$	$\frac{\pi}{2j} [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] + \frac{\omega_0}{\omega_0^2 - \omega^2}$
15	$e^{-at} \sin \omega_0 t u(t)$	$\frac{\omega_0}{(a + j\omega)^2 + \omega_0^2}$
16	$e^{-at} \cos \omega_0 t u(t)$	$\frac{a + j\omega}{(a + j\omega)^2 + \omega_0^2}$
17	$\operatorname{rect}\left(\frac{t}{\tau}\right)$	$\tau \operatorname{sinc}\left(\frac{\omega\tau}{2}\right)$
18	$\frac{W}{\pi} \operatorname{sinc}(Wt)$	$\operatorname{rect}\left(\frac{\omega}{2W}\right)$
19	$\Delta\left(\frac{t}{\tau}\right)$	$\frac{\tau}{2} \operatorname{sinc}^2\left(\frac{\omega\tau}{4}\right)$
20	$\frac{W}{2\pi} \operatorname{sinc}^2\left(\frac{Wt}{2}\right)$	$\Delta\left(\frac{\omega}{2W}\right)$
21	$\sum_{n=-\infty}^{\infty} \delta(t - nT)$	$\omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi} e^{-\sigma^2\omega^2/2}$



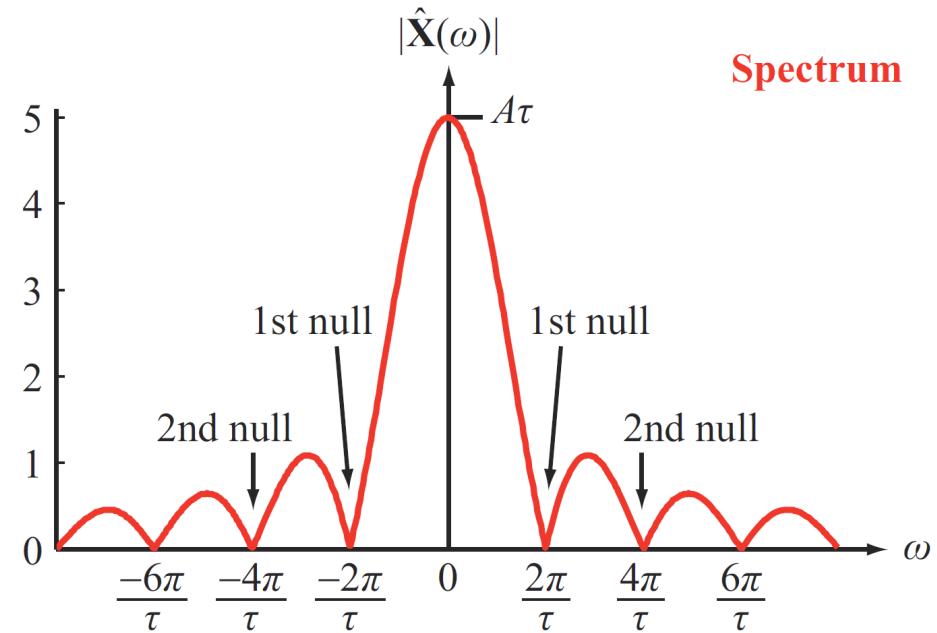
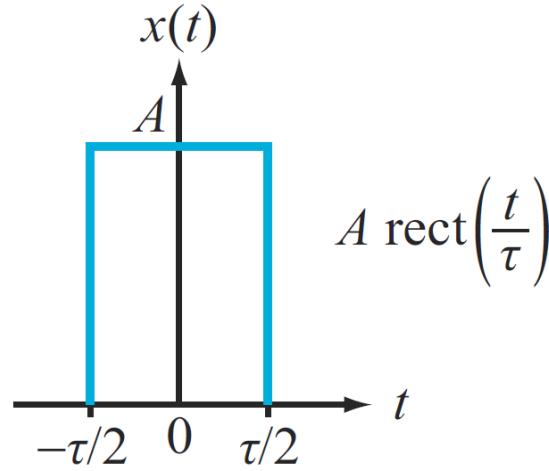
Fourier transform of a pulse

$$x(t) = A \operatorname{rect}(t/\tau) \text{ leads to } \hat{X}(\omega) = A\tau \frac{\sin(\omega\tau/2)}{(\omega\tau/2)} = A\tau \operatorname{sinc}\left(\frac{\omega\tau}{2}\right)$$

The sinc function is defined as

$$\operatorname{sinc}(\theta) = \frac{\sin \theta}{\theta} \quad \operatorname{sinc}(0) = \frac{\sin(\theta)}{\theta} \Big|_{\theta=0} = 1$$

Signal



Fourier transform - important properties

$$K_1 x_1(t) + K_2 x_2(t) \quad \leftrightarrow \quad K_1 \mathbf{X}_1(\omega) + K_2 \mathbf{X}_2(\omega),$$

(linearity property) (5.90)

$$x'(t) \quad \leftrightarrow \quad j\omega \mathbf{X}(\omega).$$

(derivative property)

$$\mathbf{X}(-\omega) = \mathbf{X}^*(\omega).$$

(reversal property)

$$x(at) \quad \leftrightarrow \quad \frac{1}{|a|} \mathbf{X}\left(\frac{\omega}{a}\right), \quad \text{for any } a.$$

(scaling property)

$$x(t) \cos(\omega_0 t) \quad \leftrightarrow \quad \frac{1}{2} [\mathbf{X}(\omega - \omega_0) + \mathbf{X}(\omega + \omega_0)].$$

(modulation property) (5.109)



Fourier transform - properties

Operation	$x(t)$	$X(\omega)$
Scalar multiplication	$kx(t)$	$kX(\omega)$
Addition	$x_1(t) + x_2(t)$	$X_1(\omega) + X_2(\omega)$
Conjugation	$x^*(t)$	$X^*(-\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Scaling (a real)	$x(at)$	$\frac{1}{ a } X\left(\frac{\omega}{a}\right)$
Time shifting	$x(t - t_0)$	$X(\omega)e^{-j\omega t_0}$
Frequency shifting (ω_0 real)	$x(t)e^{j\omega_0 t}$	$X(\omega - \omega_0)$
Time convolution	$x_1(t) * x_2(t)$	$X_1(\omega)X_2(\omega)$
Frequency convolution	$x_1(t)x_2(t)$	$\frac{1}{2\pi} X_1(\omega) * X_2(\omega)$
Time differentiation	$\frac{d^n x}{dt^n}$	$(j\omega)^n X(\omega)$
Time integration	$\int_{-\infty}^t x(u) du$	$\frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$



Fourier transform - DUAL properties

$$e^{j\omega_0 t} x(t) \leftrightarrow \mathbf{X}(\omega - \omega_0),$$

(frequency-shift property)

$$x(t - t_0) \leftrightarrow e^{-j\omega t_0} \mathbf{X}(\omega).$$

(time-shift property)



Fourier transform - Parseval's theorem

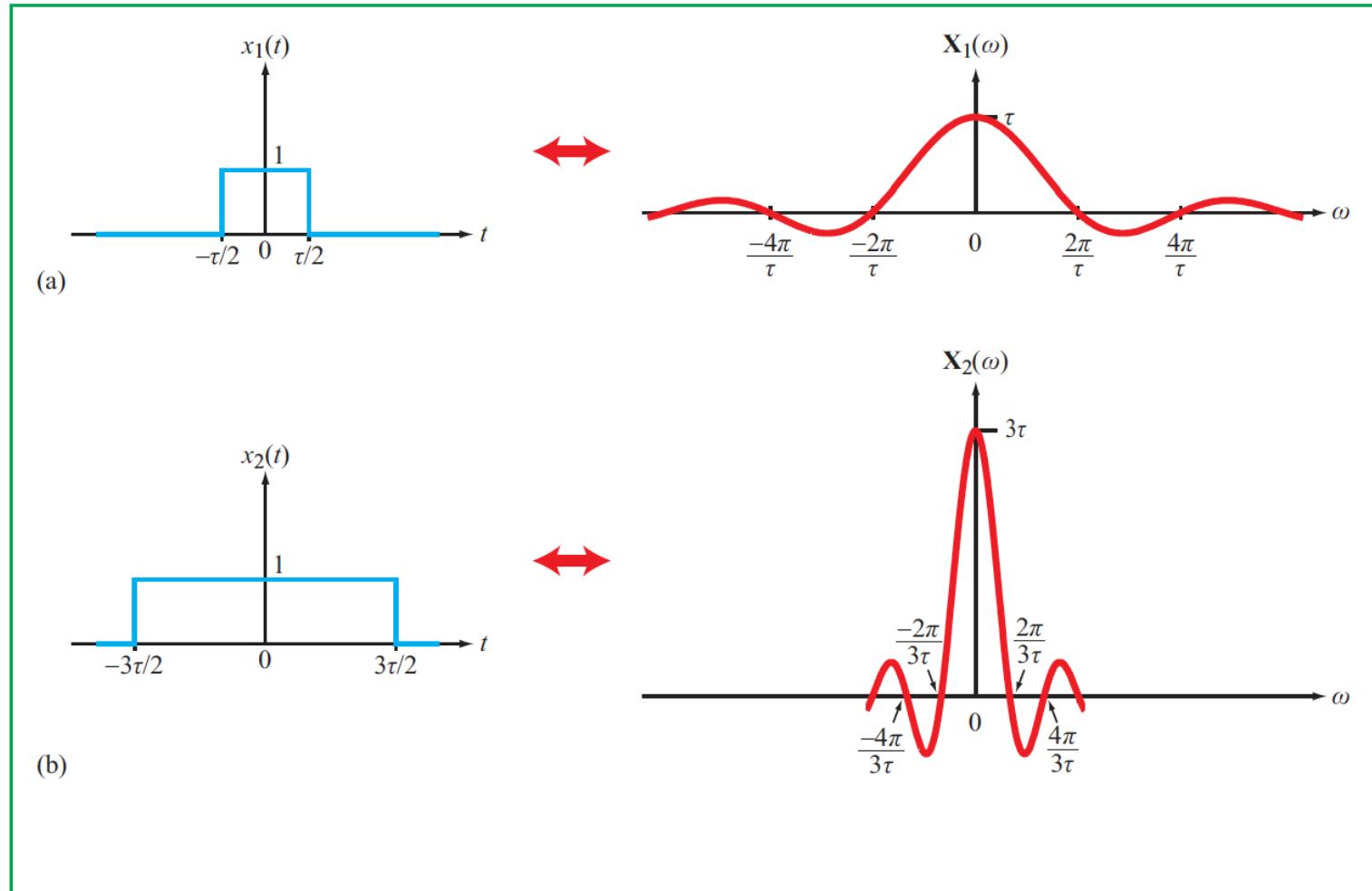
$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\mathbf{X}(\omega)|^2 d\omega.$$

(Parseval's theorem)

same in time domain or frequency domain!



Fourier transform - time scaling property



Fourier transform -Parseval example

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{X}(\omega)|^2 d\omega$$

$$Ae^{-at} u(t) \quad \leftrightarrow \quad \frac{A}{a + j\omega} \quad \text{for } a > 0$$

$$x(t) = e^{-at} u(t)$$

Energy in time domain:

$$\int_0^{\infty} |e^{-at}|^2 dt = \int_0^{\infty} e^{-2at} dt = \frac{1}{2a} .$$

Energy in frequency domain:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{1}{a + j\omega} \right|^2 d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{a^2 + \omega^2} d\omega = \frac{1}{2a}$$



Fourier transform - periodic signal

$$x(t) = \sum_{n=-\infty}^{\infty} \mathbf{x}_n e^{jn\omega_0 t}$$
$$\mathcal{F} \left\{ e^{j\omega_0 t} \right\} = 2\pi \delta(\omega - \omega_0)$$

$$\mathcal{F} \{x(t)\} = \sum_{n=-\infty}^{\infty} \mathbf{x}_n 2\pi \delta(\omega - n\omega_0)$$



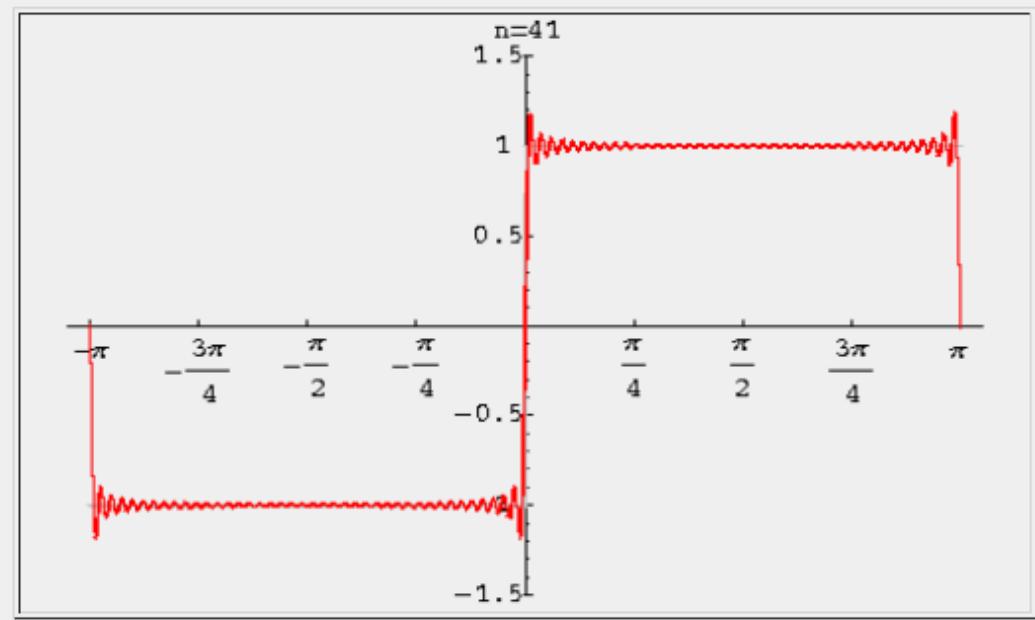
Fourier transform - Gibb's phenomenon

<http://www.sosmath.com/fourier/fourier3/gibbs.html>

$$f(x) = \begin{cases} 1 & 0 \leq x \leq \pi \\ -1 & -\pi \leq x < 0. \end{cases}$$

$$b_n = \frac{2}{\pi} \frac{(1 - (-1)^n)}{n}, \text{ for } n \geq 1.$$

$$f_{2n-1}(x) = \frac{4}{\pi} \left(\sin(x) + \frac{\sin(3x)}{3} + \dots + \frac{\sin((2n-1)x)}{(2n-1)} \right)$$



this is because of the discontinuity in $x(t)$





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