MTE 203 – Advanced Calculus
Homework 4 - Solutions

Normal Vectors, Curvature, and Radius of Curvature

Problem 1 [11.12, Prob. 9] :

Find $\hat{N}$ and $\hat{B}$ at the point $(5,2,10)$ on the curve: $x = y^2 + 1$, $z = x + 5$, directed so that y increases along the curve.

Solution:

With parametric equations $x = t^2 + 1$, $y = t$, $z = t^2 + 6$, a unit tangent vector to the curve is $\hat{\mathbf{T}} = \dfrac{(2t, 1, 2t)}{\sqrt{8t^2+1}}$. A vector in the direction of $\hat{\mathbf{N}}$ is

$$\mathbf{N} = \frac{d\hat{\mathbf{T}}}{dt} = \frac{-8t}{(1+8t^2)^{3/2}}(2t, 1, 2t) + \frac{(2,0,2)}{\sqrt{1+8t^2}}.$$

At $(5, 2, 10)$, $t = 2$, in which case

$$\mathbf{N}(2) = \frac{-16}{33\sqrt{33}}(4, 1, 4) + \frac{(2,0,2)}{\sqrt{33}} = \frac{2(1,-8,1)}{33\sqrt{33}}.$$

Hence, the principal normal at $(5, 2, 10)$ is $\hat{\mathbf{N}} = \dfrac{(1,-8,1)}{\sqrt{66}}$. Since a tangent vector at the point is $\mathbf{T}(2) = (4, 1, 4)$, the direction of the binormal at the point is

$$\mathbf{B}(2) = (4, 1, 4) \times (1, -8, 1) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 4 & 1 & 4 \\ 1 & -8 & 1 \end{vmatrix} = (33, 0, -33).$$

Thus, $\hat{\mathbf{B}}(2) = (1, 0, -1)/\sqrt{2}$.

Problem 2 [11.12, Probs. 15, 17]:

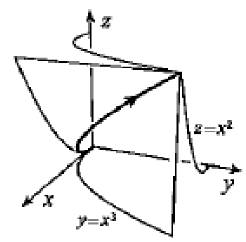
Find the curvature and the radius of curvature of the curve (if they exist). Draw each curve.

a. $x = t, y = t^3, z = t^2, \quad t \geq 0$
b. $x = t + 1, y = t^2 - 1, z = t + 1, \quad -\infty < t < \infty$

Solution:

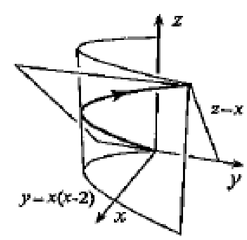
(a)

$$\kappa(t) = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} = \frac{|(1, 3t^2, 2t) \times (0, 6t, 2)|}{|(1, 3t^2, 2t)|^3}$$
$$= \frac{1}{(1+4t^2+9t^4)^{3/2}} \left\| \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 3t^2 & 2t \\ 0 & 6t & 2 \end{vmatrix} \right\|$$
$$= \frac{|(-6t^2, -2, 6t)|}{(1+4t^2+9t^4)^{3/2}} = \frac{2\sqrt{1+9t^2+9t^4}}{(1+4t^2+9t^4)^{3/2}}$$
and $\rho(t) = \frac{1}{\kappa} = \frac{(1+4t^2+9t^4)^{3/2}}{2\sqrt{1+9t^2+9t^4}}$.



(b)

$$\kappa(t) = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} = \frac{|(1, 2t, 1) \times (0, 2, 0)|}{|(1, 2t, 1)|^3} = \frac{1}{(2+4t^2)^{3/2}} \left\| \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2t & 1 \\ 0 & 2 & 0 \end{vmatrix} \right\|$$
$$= \frac{|(-2, 0, 2)|}{(2+4t^2)^{3/2}} = \frac{1}{(1+2t^2)^{3/2}}$$
and $\rho(t) = \frac{1}{\kappa} = (1+2t^2)^{3/2}$.



Problem 3: [11.12, Prob. 23]

Let C be the curve $x = t, y = t^2$ in the xy-plane.

a. At each point on C calculate the unit tangent $\hat{T}$ vector and the principal normal $\hat{N}$. What is $\hat{B}$?
b. $\vec{F} = t^2\,\hat{\imath} + t^4\,\hat{\jmath}$ is a vector that is defined at each point P on C. Denote by F_T and F_N the components of $\vec{F}$ in the directions $\hat{T}$ and $\hat{N}$ respectively. Find F_T and F_N as functions of t.
c. Express F in terms of $\hat{T}$ and $\hat{N}$.

Solution:

Example 11.47:

A smooth curve $C: x = x(t), y = y(t), \alpha \le t \le \beta$ in the xy –plane, the principle normal is

$$\hat{N} = \frac{sgn\left(\frac{dy}{dt}\frac{d^2x}{dt^2} - \frac{dx}{dt}\frac{d^2y}{dt^2}\right)\left(\frac{dy}{dt}, -\frac{dx}{dt}\right)}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}$$

Where:

$$sgn\ (u) = \begin{cases} 1 & if\ u > 0 \\ 0 & if\ u = 0 \\ -1 & if\ u < 0 \end{cases}$$

(a) $\hat{\mathbf{T}} = \dfrac{(1, 2t)}{\sqrt{1+4t^2}}$ According to Example 11.47,

$$\hat{\mathbf{N}} = \operatorname{sgn}\left[(2t)(0) - (1)(2)\right] \frac{(2t, -1)}{\sqrt{1+4t^2}} = \frac{(-2t, 1)}{\sqrt{1+4t^2}}.$$

The direction of the binormal is $\mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2t & 0 \\ -2t & 1 & 0 \end{vmatrix} = (0, 0, 1+4t^2)$. Thus, $\hat{\mathbf{B}} = (0, 0, 1)$.

(b) $F_T = \mathbf{F} \cdot \hat{\mathbf{T}} = (t^2, t^4) \cdot \dfrac{(1, 2t)}{\sqrt{1+4t^2}} = \dfrac{t^2 + 2t^5}{\sqrt{1+4t^2}}$; $F_N = \mathbf{F} \cdot \hat{\mathbf{N}} = (t^2, t^4) \cdot \dfrac{(-2t, 1)}{\sqrt{1+4t^2}} = \dfrac{t^4 - 2t^3}{\sqrt{1+4t^2}}$

(c) $\mathbf{F} = F_T \hat{\mathbf{T}} + F_N \hat{\mathbf{N}} = \dfrac{t^2 + 2t^5}{\sqrt{1+4t^2}} \hat{\mathbf{T}} + \dfrac{t^4 - 2t^3}{\sqrt{1+4t^2}} \hat{\mathbf{N}}$

Tangential and Normal Components of Acceleration

Problem 4: [11.13, Prob. 25]

Calculate the normal component a_N of the acceleration of a particle using equations 11.112 and 11.113 if its position is given by $x = t^2 + 1, y = 2t^2 - 1, z = t^2 + 5t\ , t \geq 0,$ (t being time).

Note:

Equation 11.112 (Trim Book):

$$a_T = a.\hat{T} = \frac{d}{dt}|v|, a_N = a.\hat{N} = |v| \left|\frac{d\hat{T}}{dt}\right|$$

Equation11.113 (Trim Book):

$$a_N = \sqrt{|a|^2 - a_T^2}$$

Solution:

Since $\hat{\mathbf{T}} = \dfrac{(2t, 4t, 2t+5)}{\sqrt{4t^2 + 16t^2 + (2t+5)^2}} = \dfrac{(2t, 4t, 2t+5)}{\sqrt{24t^2 + 20t + 25}}$,

$$
\begin{aligned}
\frac{d\hat{\mathbf{T}}}{dt} &= \frac{-(24t+10)}{(24t^2+20t+25)^{3/2}}(2t, 4t, 2t+5) + \frac{(2,4,2)}{\sqrt{24t^2+20t+5}} \\
&= \frac{1}{(24t^2+20t+25)^{3/2}}[-(24t+10)(2t, 4t, 2t+5) + (24t^2+20t+25)(2,4,2)] \\
&= \frac{10(2t+5, 4t+10, -10t)}{(24t^2+20t+25)^{3/2}}.
\end{aligned}
$$

According to 11.112b,

$$
\begin{aligned}
a_N &= |(2t, 4t, 2t+5)| \left| \frac{10(2t+5, 4t+10, -10t)}{(24t^2+20t+25)^{3/2}} \right| \\
&= \sqrt{24t^2+20t+25} \left[\frac{10\sqrt{(2t+5)^2 + (4t+10)^2 + 100t^2}}{(24t^2+20t+25)^{3/2}} \right] \\
&= \frac{10\sqrt{5}}{\sqrt{24t^2+20t+25}}.
\end{aligned}
$$

Since $\mathbf{v} = (2t, 4t, 2t+5)$ and $\mathbf{a} = (2,4,2)$,

$$
a_T = \frac{d}{dt}|\mathbf{v}| = \frac{d}{dt}\sqrt{24t^2+20t+25} = \frac{24t+10}{\sqrt{24t^2+20t+25}}.
$$

Equation 11.113 gives $a_N = \sqrt{24 - \dfrac{(24t+10)^2}{24t^2+20t+25}} = \dfrac{10\sqrt{5}}{\sqrt{24t^2+20t+25}}$.

Extra Practice Problems

Solutions to these problems can be found at the back of your textbook

1. S. 11.12, Probs. 18, 24, 26
2. S. 11.13, Probs. 14, 18, 24, 38