MTE 203 – Advanced Calculus

Homework 7 (Solutions)

Directional Derivatives

Problem 1: [12.8, Prob. 11]

Find the rate of change of the function with respect to distance travelled along the curve $y = x^2 - 1$, $z = -2x$ in the direction of increasing x

$$f(x, y, z) = xy + z^2 \text{ at } (1,0,-2)$$

Solution:

Since parametric equations for the curve are $x = t$, $y = t^2 - 1$, $z = -2t$, a tangent vector to the curve is $\mathbf{T} = (1, 2t, -2)$. At the point $(1, 0, -2)$, a tangent vector is $(1, 2, -2)$, and

$$D_{\mathbf{T}} f = \nabla f_{|(1,0,-2)} \cdot \hat{\mathbf{T}} = (y, x, 2z)_{|(1,0,-2)} \cdot \frac{(1, 2, -2)}{\sqrt{1+4+4}} = (0, 1, -4) \cdot \frac{(1, 2, -2)}{3} = \frac{10}{3}.$$

Problem 2: [12.8, Prob. 15]

Find the direction in which the function increases most rapidly at the point. What is the rate of change in that direction?

$$f(x, y, z) = {}^{1}\!\Big/\!\sqrt{x^2 + y^2 + z^2} \text{ at } (1, -3, 2)$$

Solution:

The function increases most rapidly in the direction

$$\nabla f_{|(1,-3,2)} = \left[\frac{1}{(x^2 + y^2 + z^2)^{3/2}}(-x, -y, -z)\right]_{|(1,-3,2)} = \frac{1}{14\sqrt{14}}(-1, 3, -2),$$

or, $(-1, 3, -2)$. The rate of change in this direction is $\dfrac{1}{14\sqrt{14}}\sqrt{1+9+4} = \dfrac{1}{14}$.

Problem 3: [12.8, Prob. 25]

Find points on the curve $\boldsymbol{C}: x = t, y = 1 - 2t, z = t$ at which the rate of change of $f(x, y, z) = x^2 + xyz$ with respect to the distance travelled along the curve vanishes.

Solution:

Since $\mathbf{T} = (1, -2, 1)$ is a vector along the line, the rate of change of $f(x, y, z)$ with respect to distance travelled along the curve vanishes if

$$0 = D_{\mathbf{T}} f = \nabla f \cdot \hat{\mathbf{T}} = (2x + yz, xz, xy) \cdot \frac{(1, -2, 1)}{\sqrt{6}} \quad \Longrightarrow \quad 0 = 2x + yz - 2xz + xy.$$

If we substitute the parametric equations of the line into this equation,

$$0 = 2t + (1 - 2t)(t) - 2t(t) + t(1 - 2t) = -6t^2 + 4t = 2t(2 - 3t) \quad \Longrightarrow \quad t = 0 \text{ or } t = 2/3.$$

The required points are therfore $(0, 1, 0)$ and $(2/3, -1/3, 2/3)$.

Problem 4: [12.8, Prob. 31]

Rates of change of a function $f(x, y, z)$ at a point (x_0, y_0, z_0)in directions $\hat{\imath} + \hat{\jmath}$, $\widehat{2\imath} + \hat{k}$, and $\hat{\imath} - \hat{\jmath} + \hat{k}$ are $1, 2$ and -3, respectively. What is its partial derivative with respect to z at the point?

Solution:

Let (a, b, c) be the gradient of $f(x, y)$ at the point (x_0, y_0, z_0). Then,

$$\begin{aligned}
1 &= D_{\hat{\mathbf{i}}+\hat{\mathbf{j}}} f_{|(x_0,y_0,z_0)} = \nabla f_{|(x_0,y_0,z_0)} \cdot \frac{(1,1,0)}{\sqrt{2}} = (a,b,c) \cdot \frac{(1,1,0)}{\sqrt{2}} = \frac{a+b}{\sqrt{2}}, \\
2 &= D_{2\hat{\mathbf{i}}-\hat{\mathbf{k}}} f_{|(x_0,y_0,z_0)} = \nabla f_{|(x_0,y_0,z_0)} \cdot \frac{(2,0,-1)}{\sqrt{5}} = (a,b,c) \cdot \frac{(2,0,-1)}{\sqrt{5}} = \frac{2a-c}{\sqrt{5}}, \\
-3 &= D_{\hat{\mathbf{i}}-\hat{\mathbf{j}}+\hat{\mathbf{k}}} f_{(x_0,y_0,z_0)} = \nabla f_{|(x_0,y_0,z_0)} \cdot \frac{(1,-1,1)}{\sqrt{3}} = (a,b,c) \cdot \frac{(1,-1,1)}{\sqrt{3}} = \frac{a-b+c}{\sqrt{3}}.
\end{aligned}$$

These imply that $c = (\sqrt{2} - 3\sqrt{3} - 2\sqrt{5})/2$ and this is $\partial f / \partial z$ at the point.

<u>Tangent Lines and Tangent Planes</u>

Problem 5: [12.9, Prob. 19]

Find equations for the tangent line to the curve at the point $(1,1,\sqrt{2})$

$$x = t^2, y = t, z = \sqrt{t + t^4}$$

Solution:

A tangent vector at the point $(1, 1, \sqrt{2})$ is $\frac{d\mathbf{r}}{dt}_{|t=1} = (2t, 1, (1+4t^3)/(2\sqrt{t+t^4}))_{|t=1} = (2, 1, 5/(2\sqrt{2}))$. Since $(8, 4, 5\sqrt{2})$ is also a tangent vector, parametric equations for the tangent line are $x = 1 + 8u$, $y = 1 + 4u$, $z = \sqrt{2} + 5\sqrt{2}u$.

Problem 6: [12.9, Prob. 25]

Find an equation for the tangent plane to the surface at the point $(-1, -1, 1)$

$$x = y\sin(\frac{\pi z}{2})$$

Solution:

Since a normal to the tangent plane is

$$\nabla(y\sin(\pi z/2) - x)_{|(-1,-1,1)} = (-1, \sin(\pi z/2), (\pi y/2)\cos(\pi z/2))_{|(-1,-1,1)} = (-1, 1, 0),$$

as is $(1, -1, 0)$, the equation of the tangent plane is $0 = (1, -1, 0) \cdot (x+1, y+1, z-1) = x - y$.

Problem 7: [12.9, Prob. 31]

Find the derivative for the function $f(x, y, z) = xyz + xy + xz + yz$ at $(1, -2, 5)$ perpendicular to the surface $z = x^2 + y^2$

Solution:

Since a vector perpendicular to the surface at the point $(1, -2, 5)$ is

$$\mathbf{n} = \nabla(x^2 + y^2 - z)_{|(1,-2,5)} = (2x, 2y, -1)_{|(1,-2,5)} = (2, -4, -1),$$

the required derivative is

$$\pm D_{\hat{\mathbf{n}}} f = \pm \nabla f_{|(1,-2,5)} \cdot \hat{\mathbf{n}} = \pm(yz + y + z, xz + x + z, xy + x + y)_{|(1,-2,5)} \cdot \frac{(2, -4, -1)}{\sqrt{21}}$$
$$= \pm(-7, 11, -3) \cdot \frac{(2, -4, -1)}{\sqrt{21}} = \frac{\pm 55}{\sqrt{21}}.$$

Problem 8: [12.9, Prob. 39]

Show that the sum of the intercepts on the $x-$, $y-$, and $z-$ axes of the tangent plane to the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{a}$ at any point is a.

Solution:

Since a normal vector to the surface is $\nabla(\sqrt{x}+\sqrt{y}+\sqrt{z}-\sqrt{a}) = \left(\frac{1}{2\sqrt{x}}, \frac{1}{2\sqrt{y}}, \frac{1}{2\sqrt{z}}\right)$, the equation of the tangent plane at any point (x_0, y_0, z_0) is

$$0 = \left(\frac{1}{\sqrt{x_0}}, \frac{1}{\sqrt{y_0}}, \frac{1}{\sqrt{z_0}}\right) \cdot (x-x_0, y-y_0, z-z_0) = \frac{1}{\sqrt{x_0}}(x-x_0) + \frac{1}{\sqrt{y_0}}(y-y_0) + \frac{1}{\sqrt{z_0}}(z-z_0).$$

The x-intercept of this plane is given by

$$0 = \frac{1}{\sqrt{x_0}}(x-x_0) - \sqrt{y_0} - \sqrt{z_0} \quad \Longrightarrow \quad x = x_0 + \sqrt{x_0}(\sqrt{y_0}+\sqrt{z_0}) = \sqrt{x_0}(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}).$$

With similar expressions for the y- and z-intercepts, their sum is

$$\begin{aligned}\sqrt{x_0}(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}) + \sqrt{y_0}(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}) + \sqrt{z_0}(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}) \\ = (\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0})^2 = a.\end{aligned}$$

Solutions to the following problems can be found at the back of your textbook.

Warm-Up Problems

1. S. 12.8, Probs. 2, 4, 12, 14, 18
2. S. 12.9, Probs. 2, 6, 10, 16, 22

Extra Practice Problems

1. S. 12.8, Probs. 20, 26, 30
2. S. 12.9, Probs. 28, 34, 36

Extra Challenging Problems

1. S. 12.8, Probs. 34
2. S. 12.9, Probs. 38