8 Holt-Winters' Exponential Smoothing with Seasonality

8.1 So far on Exponential Smoothing and Holt's Linear model

In exponential smoothing, we've so far met:

Simple exponential smoothing : This model is good for non-seasonal data that is fairly level (no trend). It follows the equation

$$F_{t+1} = F_t + \alpha\,(y_t - F_t) = \alpha\, y_t + (1-\alpha)\, F_t,$$

for $0 \le \alpha \le 1$.

For a longer range forecasts, it is assumed that the forecast function is flat:

$$\boxed{F_{t+k} = F_{t+1}}$$

Holt's linear method/Double exponential smoothing : This model is good for non-seasonal data with a trend. It follows the equations

$$\begin{aligned}
\text{level } L_t &= \alpha y_t + (1-\alpha)(L_{t-1} + b_{t-1});\\
\text{trend } b_t &= \beta(L_t - L_{t-1}) + (1-\beta)b_{t-1},\\
\text{forecast } F_{t+1} &= L_t + b_t
\end{aligned}$$

for $0 \le \alpha \le 1$ and $0 \le \beta \le 1$.

For a longer range forecasts, it is assumed that the forecast function follows the trend:

$$\boxed{F_{t+k} = F_{t+k} = L_t + k\, b_t}$$

8.2 Holt Winters' Additive Model

What about an exponential smoothing for data with a trend and seasonal behaviour? Winters generalised Holt's linear method to come up with such a technique, now called Holt Winters. A seasonal equation is added to Holt's linear method equations:

$$\begin{aligned}
\text{level } L_t &= \alpha(y_t - S_{t-s}) + (1-\alpha)(L_{t-1} + b_{t-1});\\
\text{trend } b_t &= \beta(L_t - L_{t-1}) + (1-\beta)b_{t-1},\\
\text{seasonal } S_t &= \gamma(y_t - L_t) + (1-\gamma)S_{t-s}\\
\text{forecast } F_{t+k} &= L_t + kb_t + S_{t+k-s},
\end{aligned}$$

where s is the length of the seasonal cycle, for $0 \le \alpha \le 1$, $0 \le \beta \le 1$ and $0 \le \gamma \le 1$. The seasonal equation picks up differences between the current level and the data at that time in the seasonal cycle. This is then added to a forecast at the same point in the cycle.

We have to pick the values of α, β and γ. As with the other methods, we can use RMSE or MAPE to choose the best values.

8.1 Definition This particular model is known as the *additive model* (seasonality factor is added to forecast).

EXERCISE: what do these updating equations say?

8.2.1 Getting Initial Values

To get started, we need initial values of the level, trend and seasonality. To initialise the seasonality, *at least one complete seasonal cycle* $y_1, y_2, \ldots, y_s$ *is needed.* We cannot make forecasts until one complete cycle has been observed. We therefore initialise L_s, b_s and $S_1, \ldots, S_s$, then use the formulae above to make forecasts from time s onwards. It is in fact best to initialise the trend with 2 complete cycles. Sensible starting values are

$$
\begin{aligned}
L_s &= \frac{1}{s}\sum_{i=1}^{s} y_i; \\
b_s &= \frac{1}{s}\left[\frac{y_{s+1}-y_1}{s} + \frac{y_{s+2}-y_2}{s} + \cdots + \frac{y_{2s}-y_s}{s}\right]; \\
S_i &= y_i - L_s,\ i = 1, \ldots, s.
\end{aligned}
$$

8.2.2 Holt Winter's Additive Model Algorithm

Init:

$L_s = \frac{1}{s}\sum_{i=1}^{s} y_i$

$b_s = \frac{1}{s}\left[\frac{y_{s+1}-y_1}{s} + \frac{y_{s+2}-y_2}{s} + \cdots + \frac{y_{2s}-y_s}{s}\right]$

$S_i = y_i - L_s,\ i = 1, \ldots, s$

and choose $0 < \alpha < 1$ and $0 < \beta < 1$

Compute:

level	$L_t = \alpha\,(y_t - S_{t-s}) + (1-\alpha)\,(L_{t-1} + b_{t-1})$
trend	$b_t = \beta\,(L_t - L_{t-1}) + (1-\beta)\,b_{t-1}$,
seasonal	$S_t = \gamma\,(y_t - L_t) + (1-\gamma)\,S_{t-s}$
forecast	$F_{t+1} = L_t + b_t + S_{t+1-s}$

Until no more observation y_t are available

and subsequent forecasts: $F_{t+k} = L_t + k\,b_t + S_{t+k-s}$

Table 3: Holt Winter's Additive Model Algorithm.

8.3 Holt Winters' Multiplicative Model

An alternative Holt Winters' model multiplies the forecast by a seasonal factor. Its equations are:

$$\begin{array}{llcl} \text{level} & L_t & = & \alpha \dfrac{y_t}{S_{t-s}} + (1-\alpha)(L_{t-1}+b_{t-1}); \\ \text{trend} & b_t & = & \beta(L_t - L_{t-1}) + (1-\beta)b_{t-1}, \\ \text{seasonal} & S_t & = & \gamma \dfrac{y_t}{L_t} + (1-\gamma)S_{t-s} \\ \text{forecast} & F_{t+k} & = & (L_t + kb_t)S_{t+k-s}, \end{array}$$

where, as before, s is the length of the seasonal cycle, for $0 \le \alpha \le 1$, $0 \le \beta \le 1$ and $0 \le \gamma \le 1$. Initial values for L_s and b_s are the same as in the additive case, but the initial seasonal estimates are: $S_1 = y_1/L_s$, $S_2 = y_2/L_s, \ldots, S_s = y_s/L_s$.

8.4 Example

In the table on the next page are the first 14 months beer production data. Since the data have a 12 month seasonal cycle, we initialise L_{12}, b_{12} and $S_1, \ldots, S_{12}$. Use the additive model formulae to calculate month 13 and 14's level, trend and seasonality, and make a 1-step ahead forecast for months 13, 14 and 15. Use $\alpha = 0.5, \beta = 0.3$ and $\gamma = 0.9$.

Month No.	Production	Level L_t	Trend b_t	Seasonal S_t	Forecast F_t
1	164	–	–	5.75	–
2	148	–	–	−10.25	–
3	152	–	–	−6.25	–
4	144	–	–	−14.25	–
5	155	–	–	−3.25	–
6	125	–	–	−33.25	–
7	153	–	–	−5.25	–
8	148	–	–	−12.25	–
9	138	–	–	−20.25	–
10	190	–	–	31.75	–
11	192	–	–	33.75	–
12	192	158.25	−0.65	33.75	–
13	147				
14	133				
15	163				

Figures 18.9 show the one-step ahead forecast for the complete data set. The RMSE is 13.6.

8.5 Conclusion

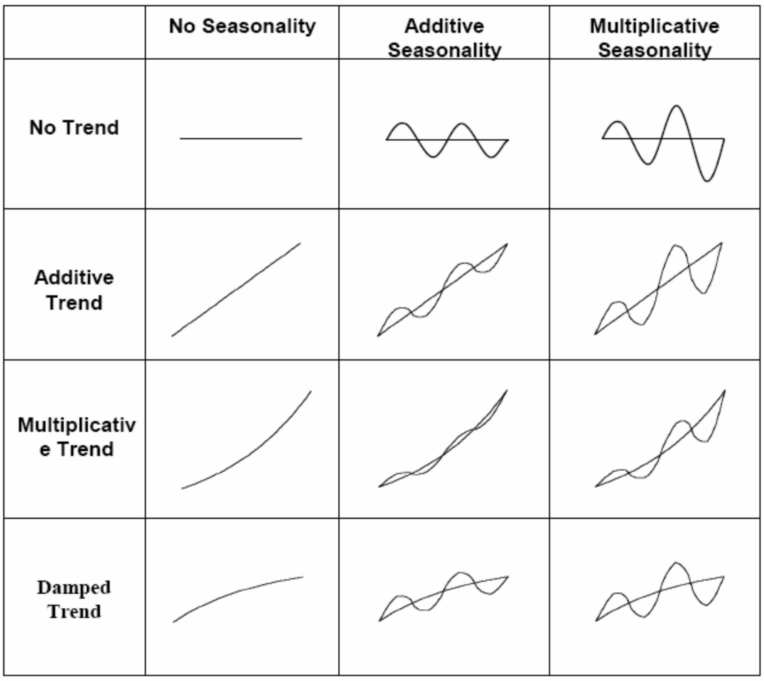

Figure 1: Trend and seasonality.

18.8 Comparing Forecasting Methods

18.9 Holt-Winters' Exponential Smoothing with Seasonality

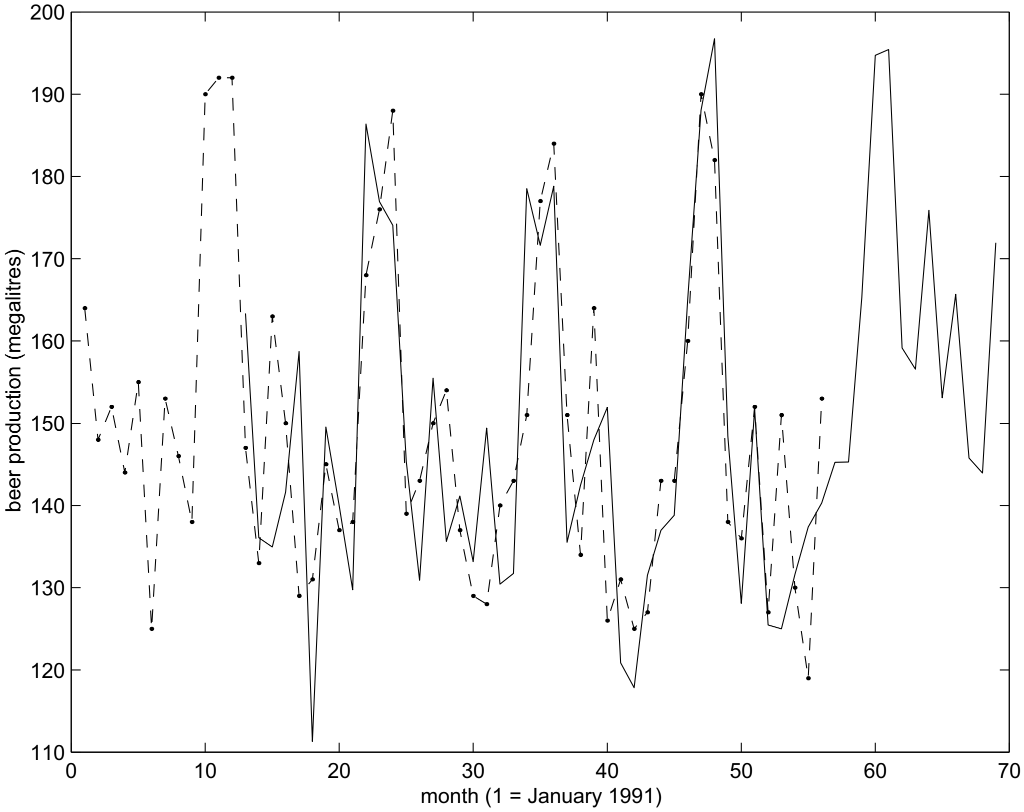

Figure 28: Monthly Australian beer production (dashed line) and forecasts by Holt Winters' additive model (one step ahead for observed values), with $\alpha = 0.5, \beta = 0.3$ and $\gamma = 0.9$.

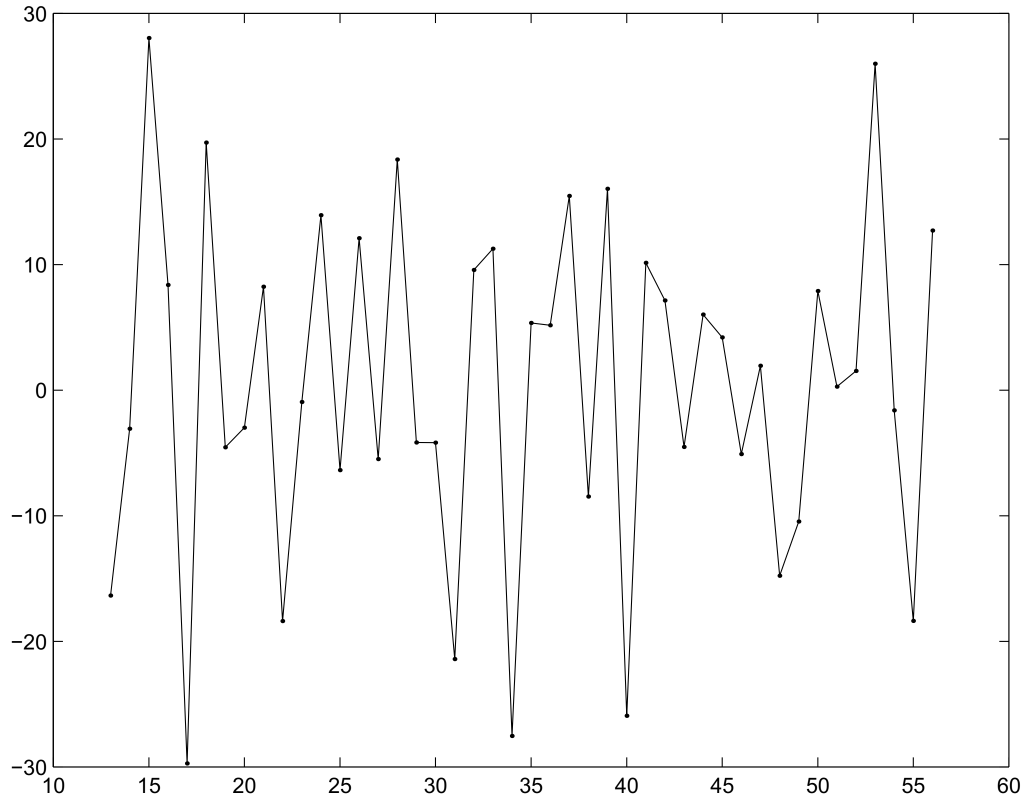

Figure 29: Timeplot of residuals in fitting by Holt Winters'.