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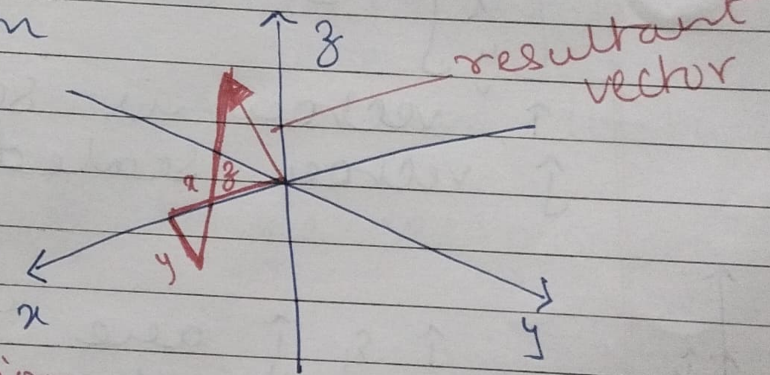
Linear Algebra

- In physics vector can move freely in a region of space
- In a coordinate system vector is almost always having its tail at origin
- Every pair of numbers gives you one vector only and vice versa.

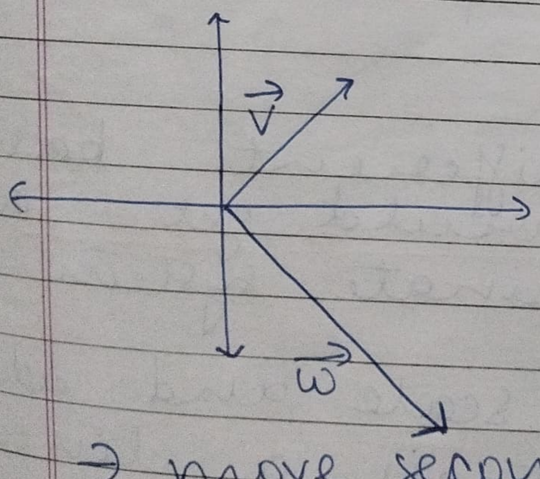
Eg $\begin{bmatrix} x \\ y \end{bmatrix}$ — x: how much distance along x axis; y: how much along y axis

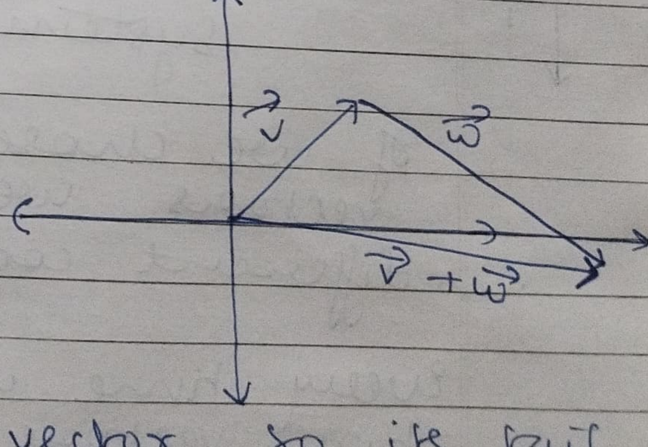
of 3D system

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$



Vector Addition





- move second vector so its tail is at head of 1st vector
- The sum is the vector whose tail is at tail of first and head at head of second vector.

Multiplying a vector — streching, compressing or flipping direction called scaling a number. Hence this no is called a scalar

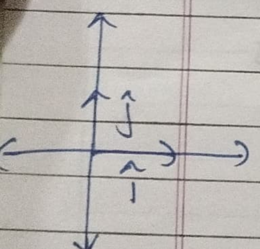
→ Multiplying by scalar is multiplying each compo-nent by that scalar.

\# ~~Spea~~ Consider a vector as a sum of 2 scaled vectors

$$\begin{bmatrix} x \\ y \end{bmatrix} = (x)\hat{i} + (y)\hat{j}$$

$\hat{i}$ vector is scaled by scalar x

$\hat{j}$ vector scaled by scalar y



$\hat{i}$ & $\hat{j}$ are "basis vectors" of xy coordinate system

If we chose different basis vectors we would get a different coordinate system

Every time we scale and add two vectors it is a linear combination of the two vectors

$$a\vec{v} + b\vec{w}$$

set of all vectors u can reach with a linear combination is called span of the vector

span of $\vec{w}$ & $\vec{v}$ is a set of all their linear combinations

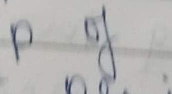
$a\vec{v} + b\vec{w}$

a & b vary over all linear numbers

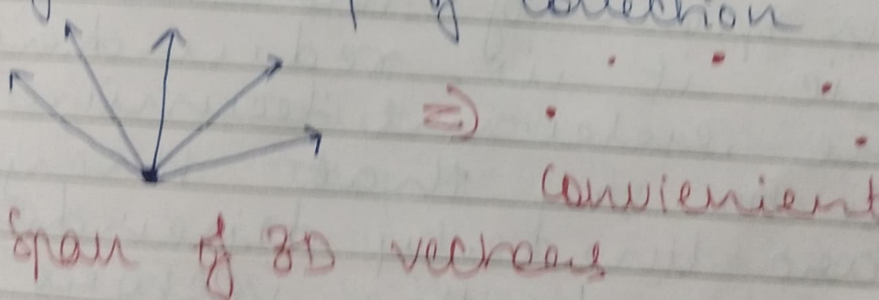
Vectors vs Points

when dealing with collection of vectors its easier to represent as a point in space. Point lies at tip of vector when its tail is at origin

→ if thinking of single vector



→ if thinking of collection



convienient as points

Span of 3D vectors

→ span of 3D vector is all possible results of linear combination of the 2 vectors

→ The tip of resultant of all linear combinations will trace out a flat sheet

This flat sheet is the span of 2 vectors

For span of 3 vectors is defined in same way as done for 2 vectors :-

linear combination of $\vec{v}$, $\vec{w}$ & $\vec{u}$

$$a\vec{v} + b\vec{w} + c\vec{u}$$

For span let these constants vary

→ If 3rd vector sits on span of 1st two then we get the same flat sheet

→ If its pointing in a seperate direction it unlocks access to every possible 3-D vector

→ Whenever one vector lies on span of the other 2 ie it is redundant the relevant terminology is to say that they are linearly dependent ie one of the vectors is a linear combination of other 2

$$\boxed{\vec{u} = a\vec{v} + b\vec{w}}$$

→ If one of the vectors adds another dimension to the span it is linearly independent

$\vec{u} \neq a\vec{v} + b\vec{w}$

The basis of a vector space is a set of linearly independent vectors that span the full space

Matrices as linear $\underbrace{\text{transformation}}_{\text{function}}$

→ ~~all lines must remain lines~~

→ Origin must remain fixed in space

→ linear transformation keeps grid lines parallel and evenly spaced

If $\bar{v} = x\hat{i} + y\hat{j}$

then transformed $\bar{v}$ = x (transformed $\hat{i}$) + y(transformed $\hat{j}$)

→ This allows us to predict where any vector lands by just knowing the initial position of $\hat{i}$ & $\hat{j}$ without needing to watch the transformation itself

eg If $\hat{i}$ transformed to $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

& $\hat{j}$ transformed to $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow x\begin{bmatrix} 1 \\ -2 \end{bmatrix} + y\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1x + 3y \\ -2x + 0y \end{bmatrix}$$

Hence a linear 2D transfor-mation is just described by 4 numbers — 2 coordinates for where $\hat{i}$ lands and 2 coordinates where $\hat{j}$ lands which are shown as a 2x2 Matrix

way of packaging the info

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{bmatrix} x \\ y \end{bmatrix}$$

place where $\hat{i}$ lands | place where $\hat{j}$ lands

vector to which the transfor-mation is applied

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{x \begin{bmatrix} a \\ c \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix}}_{\text{where the intuition is}}$$

$$= \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

Linear transformations are a way to move around space such that grid lines remain ‖ & evenly spaced and origin remain fixed.

A Linear transformation follows :-

Additivity $L(\vec{v} + \vec{w}) = L(\vec{v}) + L(\vec{w})$

Scaling $L(c\vec{v}) = cL(\vec{v})$

Linear trans. are functions ~~bg~~ with vectors as inputs and outputs

shear → when $\hat{i}$ not changed
and $\hat{j}$ → $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\underset{\text{shear}}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}} \underset{\text{rotation}}{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}} = \underset{\text{composition}}{\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}}$$

←
read Right to left

$f(g(x))$

$$\text{eg} \quad \overset{M_1}{\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}} \overset{M_2}{\begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix}} = \begin{bmatrix} ? \end{bmatrix}$$

$\hat{i}$ first here

$\hat{j}$ first here

after 2nd transformation

$$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & ? \\ 1 & ? \end{bmatrix}$$

makes up first col. of composition

$$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix}$$

makes up 2nd col of composition

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

$$\hat{i} \rightarrow \begin{bmatrix} e \\ g \end{bmatrix} \rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e \\ g \end{bmatrix} \rightarrow \begin{bmatrix} ae + bg \\ ce + dg \end{bmatrix}$$

$$\hat{j} \rightarrow \begin{bmatrix} f \\ h \end{bmatrix} \rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} f \\ h \end{bmatrix} \begin{bmatrix} af + bh \\ cf + dh \end{bmatrix}$$

Order of transformations matter

$M_1 M_2 \neq M_2 M_1$

however

$A(BC) = (AB)C$ Associativity

For 3D

$$\underbrace{\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}}_{\text{transformation}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow$$

→ Also read from ~~left~~ to right to left.

The Determinant

→ Related to change area of square in the grid

→ Scaling factor for the area is the determinant

→ Determinants allow -ve values
if its negative it means
orientation of space is -ve
its absolute value still shows
factor by which area is
scaled

For 3-D determinant shows
how much volume gets
scaled
→ Instead of unit square
we start with unit cube
which on transformation
gives a parallelopiped

$$\underbrace{\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}}_{\text{linearly dependent}} = 0$$

cube becomes a sheet or a point

If we can represent the 3
axes by right hand rule
first and then after trans
-formation by left hand rule
it means that orientation
is flipped and determinant
is -ve

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right)$$

$$= a \det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - b\det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right)$$

$$+ c \det\left(\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right)$$

→ A^{-1} is a unique transformation property such that if u apply transformation A then A^{-1} u end up back where u started (as long as det ≠ 0)

→ When determinant is 0 and is squished to a lower dimension u cannot have an inverse

→ When output of transformation is a line it is Rank 1

→ When output of transformation is a plane it is Rank 2

→ ∴ Rank is no of dimensions in the output of a transformation

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Span of columns of matrix $\Rightarrow$ column space

When rank = no of coloumns
$\Rightarrow$ max rank and matrix is called full rank

$\Rightarrow$ If 3D squished to 2D whole plane of vectors lie on origin
$\Rightarrow$ If 2D squished to a line whole line of vectors lie on origin after squishing
This plane or line is the Null space or kernel $\rightarrow$ it is the space of all vectors that become null

Non square matrices

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$$

where $\hat{i}$ lands, where $\hat{j}$ lands

$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$ If a 3x2 matrix $\rightarrow$ 2 basis vectors and 3 landing coordinates (mappind 2D to 3D)
If a 2x3 matrix $\rightarrow$ 3 basis
$\rightarrow$ 2 landing coordinates

DOT PRODUCT

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix} = a \cdot d + b \cdot e + c \cdot f$$

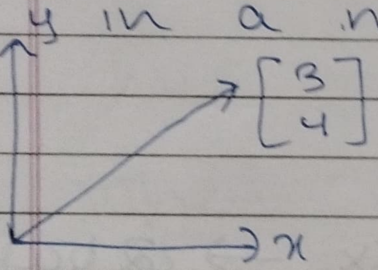
→ Indicates projection of one vector onto the other
→ If in similar directions dot product is +ve
→ If $\perp$ then dot product is 0
→ If generally opposite directions dot product is −ve
→ Order does not matter

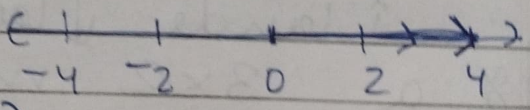
$\vec{v}\cdot\vec{w}$ → projecting $\vec{w}$ onto $\vec{v}$ and multiplying length of $\vec{v}$ by projection of $\vec{w}$

$$\underbrace{\begin{bmatrix} a & b \end{bmatrix}}_{\text{transform}} \begin{bmatrix} c \\ d \end{bmatrix}$$

vector

→ vector can be transformed to a ~~2x1~~ 1x2 matrix in a numberline



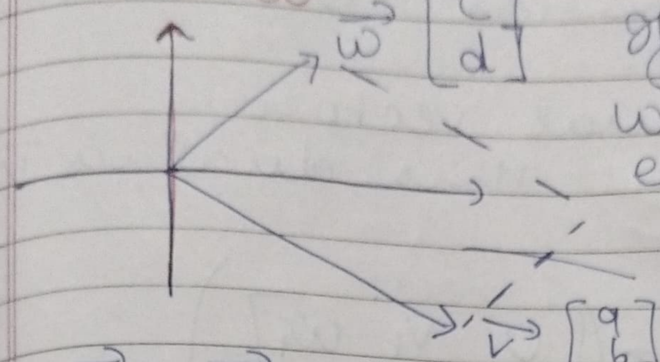


(Duality)

$$\underbrace{\begin{bmatrix} a \\ b \end{bmatrix} \cdot \overset{\vec{v}}{\begin{bmatrix} x \\ y \end{bmatrix}}}_{\text{Dot product}} = \overbrace{\begin{bmatrix} a & b \end{bmatrix}}^{\text{transform}} \overset{\vec{v}}{\begin{bmatrix} x \\ y \end{bmatrix}}$$

Cross product

$\vec{v} \times \vec{w}$



If v on right of w product is +ve else it is −ve

area of llgm = $\bar{v} \times \bar{w}$

$$\vec{v} \times \vec{w} = \det\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right)$$

llgm evolved from unit square
∴ $\vec{v} \times \vec{w}$ given by determinant with coordinates as columns
If v at left of w orientation was flipped ∴ negative

→ Cross product combines 2 vectors to give a new vector
→ The new vector will have length equal to area of llgm and will be ⊥ to llgm
→ cross product can be given by right hand rule

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \det\left(\begin{bmatrix} \hat{i} & v_1 & w_1 \\ \hat{j} & v_2 & w_2 \\ \hat{k} & v_3 & w_3 \end{bmatrix}\right)$$

$$\hat{i}\,(v_2 w_3 - v_3 w_2) + \hat{j}\,(v_3 w_1 - v_1 w_3) + \hat{k}\,(v_1 w_2 - v_2 w_1)$$

→ Define a 3D → 1D linear transformation in terms of $\vec{v}$ & $\vec{w}$

→ Find its dual vector

→ Show that this dual is $\vec{v} \times \vec{w}$

$$\vec{u} \times \vec{v} \times \vec{w} = \det\left(\begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}\right)$$

vol ~~area~~ of parallelopiped formed by the 3 vectors

$$\begin{bmatrix} ? & ? & ? \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \det\left(\begin{bmatrix} x & v_1 & w_1 \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{bmatrix}\right)$$

1 x 3 matrix

(This is ~~noamd~~ encoding 3D matrix to 1D linear multiplication transformation ust dot Product)

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \det\begin{bmatrix} x & v_1 & w_1 \\ y & v_2 & w_2 \\ z & v_3 & w_3 \end{bmatrix}$$

$$p_1 = v_2 w_3 - v_3 v_2$$
$$p_2 = v_3 w_1 - w_3 v_1$$
$$p_3 = v_1 w_2 - v_2 w_1$$

$= \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \cdot$ ~~(Area of ||gm)~~ (component of $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ ⊥ to $\vec{v}$ & $\vec{w}$ with length = area of ||gm)

$\vec{v}\times\vec{w}$ ∴ cross product is a vector ⊥ to $\vec{v}$ & $\vec{w}$ with length equal to area of llgm formed by the vectors

Change of basis

→ $\hat{i}$ & $\hat{j}$ are basis vectors in our standard coordinate system

→ Different basis make a different grid but origin will be same

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} p \\ q \end{bmatrix}$$

new basis

$\begin{bmatrix} p \\ q \end{bmatrix}$ is how

$\begin{bmatrix} x \\ y \end{bmatrix}$ is represented in standard coordinate system

our grid → new grid

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

our language ← new language

If we want to convert from standard grid to new one we use $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$

$$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

When transformations are to be done in a new basis

→ Step 1 → convert to standard

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$$

→ apply transformation in standard

$$\begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} r \\ s \end{bmatrix}$$

→ back to the new basis

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} r \\ s \end{bmatrix} = \underbrace{\begin{bmatrix} A \\ B \end{bmatrix}}_{\text{final ans}}$$

Eigenvectors and Eigen-values

→ On transformation vectors get knocked off their span

→ However few remain on their own span (such as for those on x, y axes diagonal lines)

→ These special vectors are called eigenvectors and the factor by which they are streched ~~and~~ or squished is called eigenvalue

→ Eigenvalue for only rotations is one

transformation

Eigenvalue

$$\vec{A}\,\vec{v} = \lambda\,\vec{v}$$

matrix multiplic.

eigenvector

scalar multiplication

scaling by λ = matrix multiplic by $\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$

$$\underbrace{(A - \lambda I)}\,\vec{v} = 0$$

This matrix looks like

$$\begin{bmatrix} a-\lambda & b & c \\ d & e-\lambda & f \\ g & h & i-\lambda \end{bmatrix}$$

(goal is to find λ that will make its determinant 0

This matrix squishes into one line

→ ~~A transformation~~ doesn't always ~~have eigenvalues~~
~~(Eg rotation by 90°)~~

→ It is possible to have 1 eigenvalue with more than 1 line of eigenvectors

→ If eigenvectors are basis vectors then ~~the~~ transformation matrix will be diagonal

→ U can change coordinate system so that eigenvectors become basis vectors take vectors u want to use as basis and as columns of a matrix

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

basis vectors

original transformation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

= diagonal matrix

basis vectors which are eigenvec. ~~basis~~ are "eigenbasis"

transformation frm basis vector perspective

$$\begin{bmatrix} p & 0 \\ 0 & s \end{bmatrix}$$

eigenbasis are used to make matrix computation easy

→ Abstract vector spaces

→ functions are like vectors in the sense that

1) They can be added to get a new func

2) They can be multiplied by a scalar $(2f)(x)$

3) functions can also be transformed by linear operators (Eg — derivative)

Formal definition of linearity

Additivity — $L(\vec{v}+\vec{w}) = L(\vec{v}) + L(\vec{w})$

Scaling — $L(c\vec{v}) = cL(\vec{v})$

Grid lines ∥ and evenly spaced is a result of these 2 properties

Derivative is linear

Eg $\frac{d}{dx}(x^3+x^2) = \frac{d}{dx}(x^3) + \frac{d}{dx}(x^2)$

$\frac{d}{dx}(4x^3) = 4\frac{d}{dx}(x^3)$

Eg Current space: All polynomials

$\frac{d}{dx}(x^3+5x^2+4x+5)$

$= 3x^2 + 10x + 4$

Basis functions

$b_0(x) = 1$

$b_1(x) = x$

$b_2(x) = x^2$

$b_3(x) = x^3$

$\vdots$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 3 & \cdots \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 5 \\ 1 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1\cdot 4 \\ 2\cdot 5 \\ 3\cdot 1 \\ 0 \\ \vdots \end{bmatrix}$$

Any vector space must satisfy the 8 axioms

1) $\vec{U} + (\vec{V}+\vec{w}) = (\vec{U}+\vec{V}) + (\vec{w})$

2) $\vec{V} + \vec{w} = \vec{w} + \vec{V}$

3) There is a vector 0 such that $0 + \vec{V} = \vec{V}$ for all $\vec{V}$

4) for every vector $\vec{v}$ there is vector $-\vec{V}$ so that $\vec{V} + (-\vec{V}) = 0$

5) $a(b\vec{V}) = (ab)\vec{V}$

6) $1\vec{V} = \vec{V}$

7 $a(\vec{v}+\vec{w}) = a\vec{v} + a\vec{w}$

8 $(a+b)\vec{v} = a\vec{v} + b\vec{v}$