Maxima and Minima

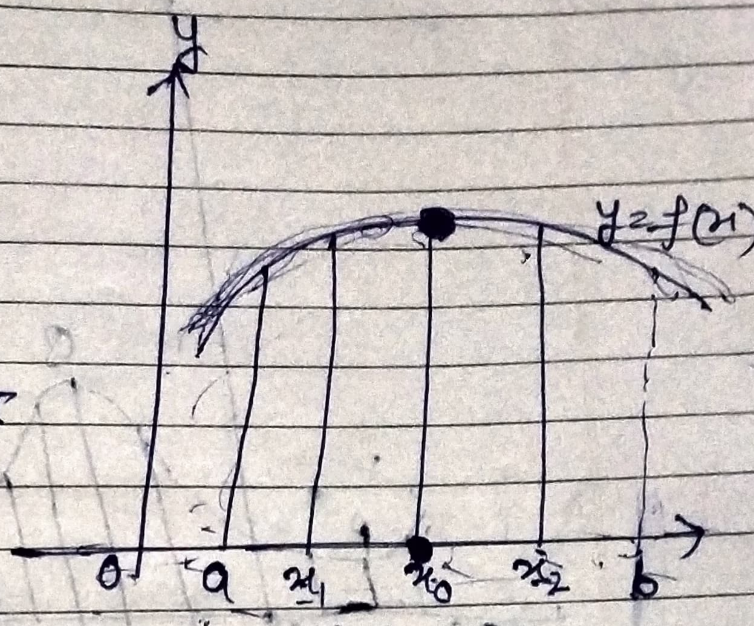
$y = f(x)$ $\qquad I$

$f(x_0) \geq f(x)$

for $x \in I$

$f(x_0)$

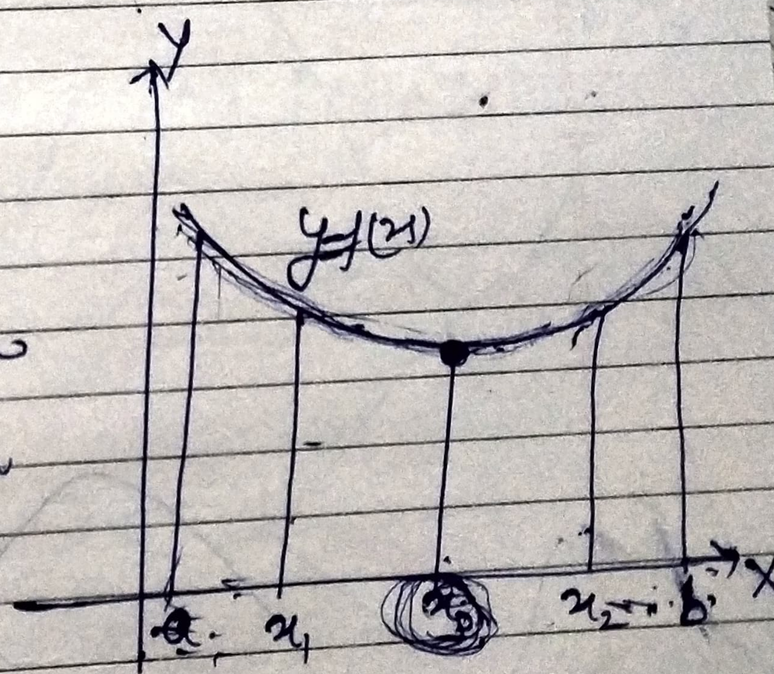
x_0 — Point of Maxima

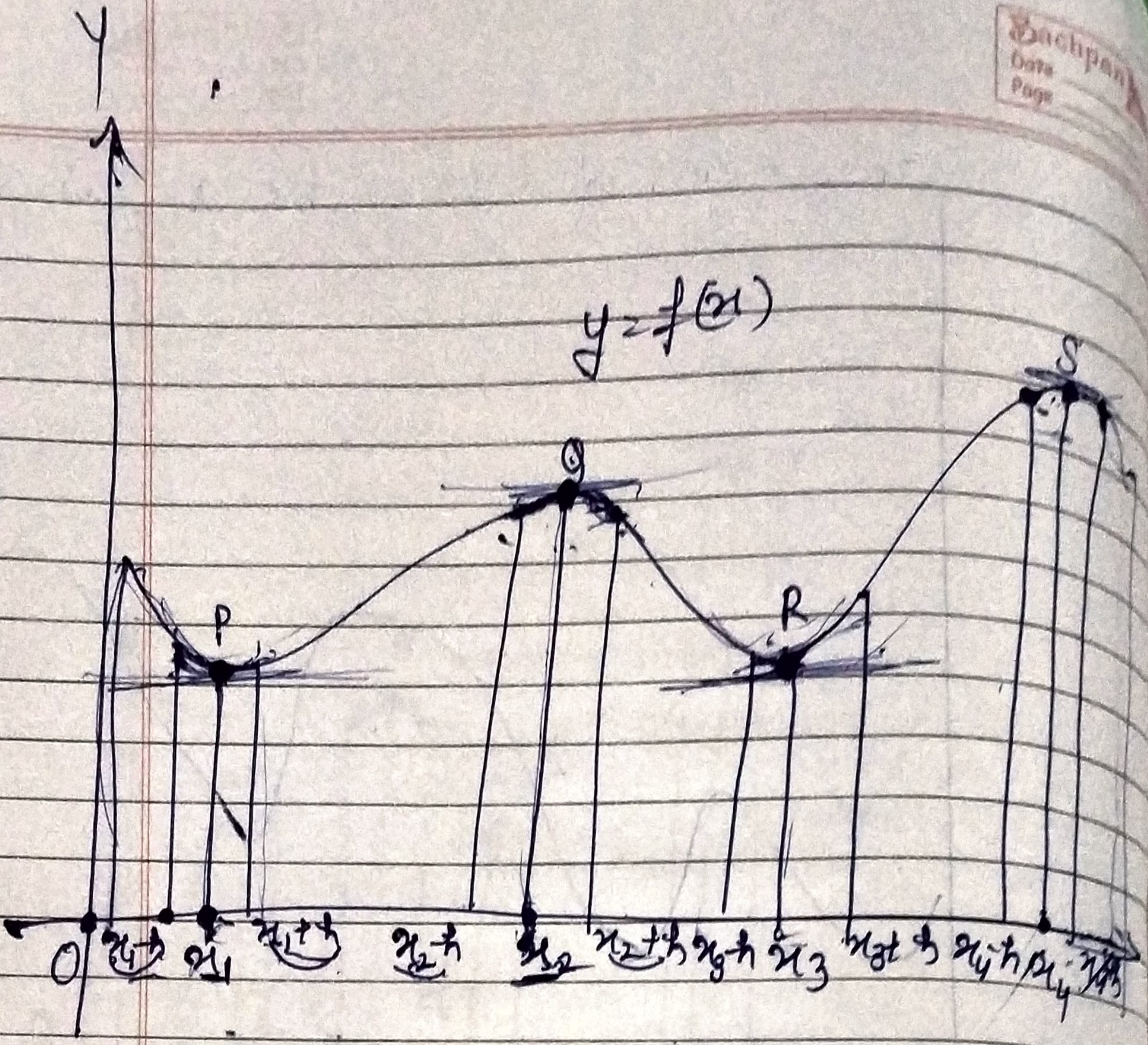


$f(x_0) \leq f(x)$

x_0 – Point of minima

$f(x_0)$ – Minimum





① first derivative test

$f(x)$

$f'(x) = 0$ $\qquad (-\frac{3}{2})$

$x_1, x_2, \ldots$

$3(-\frac{3}{2})^2 - 3$

$\frac{27}{4} - 3$

$x = -1$

when x is slightly less than -1

$f'(x) = +ve$ $\qquad 3(-\frac{1}{2})^2 - 3$

$\frac{3}{4} - 3$

when x is slightly greater than -1

$f'(x) = -ve$

$f'(x)$ change sign from +ve to −ve

So $f(x)$ is a local ~~minim~~ maxima at $x = -1$

(1) first derivative, (2) Higher order derivatives

$f(x)$

$f'(x)$

for critical value

$f'(x) = 0$

$x = a, b, c \dots$

when x is slightly less than a, $f'(x) = +ve$

when x is slightly greater than a, $f'(x) = -ve$

rule

1. If $f'(x)$ change sign from +ve to -ve
   $f(x)$ is a local Maxima at $x = a$
2. If $f'(x)$ change sign from -ve to +ve
   $f(x)$ is a local minima at $x = a$

(3) if $f'(x)$ does not change sign then $f(x)$ is neither maxima nor minima

point of inflexion

$f(x) = x^3 - 3x$

$f'(x) = 3x^2 - 3$

for maxima or minima

$f'(x) = 3x^2 - 3 = 0$

$3x^2 = 3$

$x^2 = 1$

$x = \pm 1$

$f(-1) = (-1)^3 - 3(-1)$

$= -1 + 3 = 2$

② Higher order derivative test.

$f(x)$

① $f'(x)$

② $f'(x) = 0$

$x = \underline{a}, b, \ldots$

$f''(x) = ?$

$f''(a) = +ve$, $f(x)$ is local minima at $x=a$

or $= -ve$, —— local maxima at $x=a$

$= 0$, then we check 3rd der

$f'''(x) = ?$

$f'''(a) = +ve, -ve$, point of inflexion at $x=a$

$f'''(a) = 0$, then we check 4th der

$f^{iv}(x) = ?$

$f^{iv}(a) = -ve$, local maxima

$= +ve$ —— minima

$= 0$ —— 5th deri

$f^{v}(a) = +ve, -ve$, point of inflexion

$= 0$, 6th

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if even derivative is +ve, then it is local minima

" —— -ve, local maxima

odd derivative +ve, -ve, point of inflexion

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$f(x) = (x+1)^6$

$f'(x) = 6(x+1)^5$

for critical value

$f'(x) = 0$

$6(x+1)^5 = 0$

$(x+1)^5 = 0$

$x = -1$

$f''(x) = 30(x+1)^4$

$f''(-1) = 30(-1+1)^4 = 0$

$f'''(x) = 120(x+1)^3$

$f'''(-1) = 120(-1+1)^3 = 0$

$f^{iv}(x) = 120 \times 3 (x+1)^2$

$f^{iv}(-1) = 360(-1+1)^2 = 0$

$f^{v}(x) = 720(x+1)$

$f^{v}(-1) = 720(-1+1)$

$f^{vi}(x) = 720$

$f^{vi}(-1) = 720 = +ve$

$f(x)$ is a local minima at $x = -1$

local minimum value

$f(-1) = (-1+1)^6 = 0$

when x is slightly less than -1

$f'(x) = -ve$

when x is slightly greater than -1

$f'(x) = +ve$

so $f'(x)$ changes sign from $-ve$ to $+ve$

so $f(x)$ is a local minimum at $x = -1$

local minimum value

$= 0$

Exercise 8.4

1. (i)

$f(x) = 3x^2 - 4x + 2$

$f'(x) = 6x - 4$

for maxima or minima

$f'(x) = 0$

$6x - 4 = 0$

$6x = 4$

$x = 4/6 = \frac{2}{3}$

$f''(x) = 6$

$f''\left(\frac{2}{3}\right) = 6$

$f''\left(\frac{2}{3}\right) = +ve$

So $f(x)$ is a local minima $x = 2/3$
and local minimum value.

$f\left(\frac{2}{3}\right) = 3\left(\frac{2}{3}\right)^2 - 4\left(\frac{2}{3}\right) + 2$

$= 3 \times \frac{4}{9} - \frac{8}{3} + 2$

$= -\frac{4}{3} + 2$

$= \frac{-4+6}{3} = \frac{2}{3}$

(iv)

$f(x) = x^5 - 5x^4 + 5x^3 - 1$

$f'(x) = 5x^4 - 20x^3 + 15x^2$

for maxima or minima

$f'(x) = 0$

$5x^4 - 20x^3 + 15x^2 = 0$

$5x^2(x^2 - 4x + 3) = 0$

$5x^2(x-3)(x-1) = 0$

$x = 0, 1, 3$

$f''(x) = 20x^3 - 60x^2 + 30x$

$f''(0) = 0$

$f''(1) = 20(1)^3 - 60(1)^2 + 30(1)$

$= 20 - 60 + 30 = -10 = -ve$

$f''(3) = 20(3)^3 - 60(3)^2 + 30(3)$

$= 540 - 540 + 90 = +90 = +ve$

$f(x)$ is local maxima at $x = 1$

$f(x)$ is ——— minima — $x = 3$

$f'''(x) = 60x^2 - 120x + 30$

$f'''(0) = 60(0)^2 - 120(0) + 30 = 30$

$f(x)$ is a point of inflexion at $x = 0$

$f(1) = 1^5 - 5(1)^4 + 5(1)^3 - 1$

$= 1 - 5 + 5 - 1 = 0$

local max value $= 0$ at $x = 1$

$f(3) = 3^5 - 5(3)^4 + 5(3)^3 - 5$

$= 243 - 405 + 135 - 5 = -28$ at $x = 3$

2(i) $f(x) = (x+1)^6$

(iii) $f(x) = (x+3)^3(x-4)^3$

$f'(x) = 3(x+3)^3(x-4)^2 + 3(x-4)^3(x+3)^2$

$f'(x) = 3(x+3)^2(x-4)^3[x+3+x-4)$

$f'(x) = 3(x+3)^2(x-4)^3(2x-1)$

for maxima or minima

$f'(x) = 0$

$3(x+3)^2(x-4)^3(2x-1) = 0$

$x = \frac{1}{2}, -3, 4$

at $x = \frac{1}{2}$

when x is slightly less than $\frac{1}{2}$

$f'(x) = (+)(-)(-) = +ve$

when x is slightly great than $\frac{1}{2}$

$f'(x) = (+)(-)(+) = -ve$

$f'(x)$ change sign from +ve to −ve

So $f(x)$ is local minima at $x = \frac{1}{2}$

local minimum value

$f(\frac{1}{2}) = (\frac{1}{2}+3)^3(\frac{1}{2}-4)^3$

$= (\frac{7}{2})^3(-\frac{7}{2})^3 = -$

A

(v) $f(x) = \dfrac{x}{(x-1)(x-4)}, \quad 1 < x < 4$

$f'(x) = \dfrac{(x-1)(x-4)\cdot 1 - x[x-1+x-4]}{[(x-1)(x-4)]^2}$

$= \dfrac{\cancel{x^2} - x - 4x + 4 - \cancel{x^2} - x - x^2 + 4x}{[(x-1)(x-4)]^2}$

$f'(x) = \dfrac{-x^2 - 10x + 4}{(x-1)^2(x-4)^2}$

$f'(x) = \dfrac{-[x^2 + 7x - 4)}{(x-1)^2(x-4)^2}$

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2 (v) $f(x) = \dfrac{x}{(x-1)(x-4)}, \quad 1 < x < 4$

$f(x) = \dfrac{x}{x^2 - 5x + 4}$

$f'(x) = \dfrac{x^2 - 5x + 4 - x(2x-5)}{(x^2-5x+4)^2}$

$f'(x) = \dfrac{x^2 - \cancel{5x} + 4 - 2x^2 + \cancel{5x}}{(x^2-5x+4)^2}$

$f'(x) = \dfrac{-x^2 + 4}{(x^2-5x+4)^2}$

$f'(x) = \dfrac{-(x^2-4)}{(x^2-5x+4)^2} \Rightarrow f'(x) = \dfrac{-(x-2)(x+2)}{(x^2-5x+4)^2}$

For maxima or minima

$f'(x) = 0$

$$\frac{-(x-2)(x+2)}{(x^2-5x+4)^2} = 0$$

$-(x-2)(x+2) = 0$

at $x=-2$ $x = 2, -2$ $1 \leq x < 4$

when x is slightly < -2

$$f'(x) = \frac{(-)(-)(-)}{(+)} = -ve$$

when x is slightly > -2

$f'(x) = (-)(-)(+) = +ve$

$f'(x)$ change sign from $-ve$ to $+ve$

So $f(x)$ is local minima at $x = -2$

local minimum $\Rightarrow f(-2) = \frac{-2}{(-2-1)(-2-4)}$

$$= \frac{-2}{-3 \times -6}$$

$$= -\frac{\cancel{2}}{\cancel{18}\,9} = \frac{-1}{9}$$

at $x = 2$

when x is slightly < 2

$f'(x) = (-)(-)(+) = +ve$

when x is slightly > 2

$f'(x) = (-)(+)(+) = -ve$

$f'(x)$ change sign from $+ve$ to $-ve$

So $f(x)$ is local maxima at $x=2$

Local Maximum value

$$f(2) = \frac{2}{(2-1)(2-4)} = \frac{2}{1(-2)} = -1$$

at $x = 2$

10) $f(x) = \frac{1}{x^2+2}$

$f'(x) = \frac{-2x}{(x^2+2)^2}$

for maxima or minima

$f'(x) = 0$

$\frac{-2x}{(x^2+2)^2} = 0$

$x = 0$

when x is slightly less than 0

$f'(x) = (-)(-) = +ve$

when x is slightly > 0, $f'(x) = -ve$

$f(x)$ is local maxima at $x = 0$

local maximum $= \frac{1}{2}$ at $x = 0$

11) $f(x) = -(x-1)^3(x+1)^2$

$f'(x) = -(x-1)^3 \times 2(x+1) - 3(x+1)^2(x-1)^2$

$= (x-1)^2(x+1)[-2(x-1) - 3(x+1)]$

$= (x-1)^2(x+1)(-2x+2-3x-3)$