Experimental Statistics for Engineers I

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Problem 1: A mail-order computer business has six telephone lines. Let X denote the number of lines in use at a specified time. Suppose the pmf of X is as given (see Table 1)

Find the probability

(a) at most three lines are in use

$$P[X \leq 3] = P[X = 0] + P[X = 1] + P[X = 2] + P[X = 3] = 0.10 + 0.15 + 0.20 + 0.25 = 0.7$$

(b) Fewer than three lines are in use

$$P[X < 3] = P[X = 0] + P[X = 1] + P[X = 2] = 0.10 + 0.15 + 0.20 = 0.45$$

(c) At least three lines are in use

$$P[X \geq 3] = 1 - P[X < 3] = 1 - 0.45 = 0.55$$

(d) Between two and five lines, inclusive, are in use

$$P[2 \leq X \leq 5] = P[X = 2]+P[X = 3]+P[X = 4]+P[X = 5] = 0.20+0.25+0.20+0.06 = 0.71$$

(e) At least 4 lines are not in use

$$P[X \leq 2] = P[X = 0] + P[X = 1] + P[X = 2] = 0.10 + 0.15 + 0.20 = 0.45$$

(f) Compute E(X)

$$E[X] = 0 \cdot 0.10 + 1 \cdot 0.15 + 2 \cdot 0.20 + 3 \cdot 0.25 + 4 \cdot 0.20 + 5 \cdot 0.06 + 6 \cdot 0.04 = 2.64$$

x	0	1	2	3	4	5	6
p(x)	0.10	0.15	0.20	0.25	0.20	0.06	0.04

Table 1

(g) Compute Var(X)

$$Var[X] = \sum_D (x-\mu)^2 \cdot p(x) = (0-2.64)^2 \cdot 0.10 + (1-2.64)^2 \cdot 0.15$$
$$+(2-0.2.64)^2 \cdot 0.20 + (3-2.64)^2 \cdot 0.25 + (4-2.64)^2 \cdot 0.20$$
$$+(5-2.64)^2 \cdot 0.06 + (6-2.64)^2 \cdot 0.04 = 2.3704$$

Note that from the table one can verify that it is pmf because sum of all $p(x)$ values give 1 and also all $p(x) \geq 0$.

Problem 2: An investment firm offers its customers municipal bonds that mature after varying numbers of years. Given that the cumulative distribution function of T, the number of years to maturity for a randomly selected bond is

$$F(t) = \begin{cases} 0 & t < 1 \\ \frac{1}{4} & 1 \leq t < 3 \\ \frac{1}{2} & 3 \leq t < 5 \\ \frac{3}{4} & 5 \leq t < 7 \\ 1 & t \geq 7 \end{cases}$$

(a)
$$P[T=5] = P[T \leq 5] - P[T < 5] = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$$

(b)
$$P[T > 3] = 1 - F(3) = 1 - \frac{1}{2} = \frac{1}{2}$$

(c)
$$P[1.4 < T < 6] = F(6) - F(1.4) = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$$

(d)
$$P[T \leq 5 | T \geq 2] = \frac{P[T \leq 5 \cap T \geq 2]}{P[T \geq 2]} = \frac{P[2 \leq T \leq 5]}{P[T \geq 2]} = \frac{F(5) - F(2)}{1 - F(2)} = \frac{\frac{3}{4} - \frac{1}{4}}{1 - \frac{1}{4}} = \frac{2}{3}$$

Problem 3: Let W be a random variable giving the number of heads minus the number of tails in three tosses of a coin. List the elements of the sample space S for the three tosses of the coin and to each sample point assign a value w of W. Find the probability distribution of the random variable W assuming that the coin is biased so that a head is twice as likely to occur as a tail.

Let S be a sample space for the three tosses of the coin.

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Possible outcomes	w
HHH	3
HHT	1
HTH	1
HTT	-1
THH	1
THT	-1
TTH	-1
TTT	-3

Table 2

W	-3	-1	1	3	$\sum P(w)$
$P(w)$	$\frac{1}{27}$	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{8}{27}$	1

Table 3

Let W be a random variable giving the number of heads minus the number of tails in three tosses of a coin and w is assigned value of W. The sample space for W(see Table 2):

$$W = \{3, 1, -1, -3\}$$

$P(H) = 2P(T)$ and note that we also have $P(H) + P(T) = 1$. Thus, we get $2P(T) + P(T) = 1$, $3P(T) = 1$, $P(T) = \frac{1}{3}$. So, $P(H) = \frac{2}{3}$

When outcome is 0 heads and 3 tails, the probability is

$$P(W = -3) = \frac{1}{3}\frac{1}{3}\frac{2}{3} = \frac{1}{27}$$

When outcome is 3 heads and 0 tails, the probability is

$$P(W = 3) = \frac{2}{3}\frac{2}{3}\frac{2}{3} = \frac{8}{27}$$

When outcome is 1 heads and 2 tails, the probability is

$$P(W = -1) = 3\frac{1}{3}\frac{1}{3}\frac{2}{3} = \frac{2}{9}$$

When outcome is 2 heads and 1 tails, the probability is

$$P(W = 1) = 3\frac{2}{3}\frac{2}{3}\frac{1}{3} = \frac{4}{9}$$

Note that from the table one can verify that sum of all $P(w)$ values give 1 and also all $P(w) \geq 0$.

Problem 4: Consider a random variable Y with $p(y) = \frac{e^{-\mu}\mu^y}{y!}$

(a) Derive the MGF of Y given on the formula sheet.

$$M_Y(t) = E[e^{ty}] = \sum_{y=0}^{\infty} e^{ty} \frac{e^{-\mu}\mu^y}{y!} = e^{-\mu} \sum_{y=0}^{\infty} \frac{(e^t\mu)^y}{y!} = e^{-\mu} e^{e^t\mu} = e^{\mu(e^t-1)}$$

(b) Use the MGF to derive the mean of Y.

$$M_Y(t)' = \mu e^t e^{\mu(e^t-1)}$$

$$E[Y] = M_Y(0)' = \mu$$

Problem 5: Suppose there is a 1 in 50 chance of injury on a single skydiving attempt.

(a) If we assume that the outcomes of different jumps are **independent**, what is the probability that a skydiver is injured if she jumps twice?

Let A1 be event that the skydiver is injured on the first jump and A2 the event she is injured on the second.

$$P(A1 \cup A2) = P(A1) + P(A2) - P(A1 \cap A2) = P(A1) + P(A2) - P(A1) \cdot P(A2) =$$
$$= \frac{1}{50} + \frac{1}{50} - \frac{1}{50} \cdot \frac{1}{50} = 0.0396$$

Alternatively, we can solve it using binomial formula. If we say that A is event skydiver get injured. There are two possible scenarios, one is $X = 1$(either of jumps injuring happen) and another $X = 2$ (both attempts ends with injuring). Number of trials (n) is 2. Probability of injuring (p) is $\frac{1}{50}$. Thus,

$$P(X=1)+P(X=2) = b(1,2,\frac{1}{50})+b(2,2,\frac{1}{50}) = \binom{2}{1}(\frac{1}{50})^1(1-\frac{1}{50})^1+\binom{2}{2}(\frac{1}{50})^2(1-\frac{1}{50})^0 = 0.0396$$

(b) A friend claims if there is a 1 in 50 chance of injury on a single jump then there is a 100% chance at least one injury if a skydiver jumps 50 times. Is your friend correct? Give a probability argument to back up your choice.

No, friend's opinion is wrong. Let us prove it. Assume I is the the event that the skydiver is injured in one of the 50 jumps. Then $\overline{I}$ is the event that the skydiver makes 50 jumps without injury. Since the jumps are independent,

$$P(\bar{I}) = (1 - \frac{1}{50})^{50} \approx 0.36$$

Then,

$$P(I) = 1 - P(\bar{I}) \approx 0.64 \approx 64\%$$

which creates contradiction to statement given in the problem.
