NAME: SOLUTIONS

STUDENT NUMBER: ____________________

MATH 1B03 (LINEAR ALGEBRA I) FINAL EXAM (VERSION 1)

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DAY CLASS

DURATION OF EXAM: 2.5 hours

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This examination paper includes 20 pages and 38 multiple choice questions printed on both sides of the page. The questions are on pages 2 through 20. Scrap paper is available for rough work. You are responsible for ensuring that your copy of the paper is complete. Bring any discrepancies to the attention of the invigilator.

SPECIAL INSTRUCTIONS

1. Select the one correct answer to each question and ENTER THAT ANSWER INTO THE SCAN CARD PROVIDED USING AN HB PENCIL.
2. You are required to submit this booklet along with your answer sheet. HOWEVER, NO MARKS WILL BE GIVEN FOR THE WORK IN THIS BOOKLET. Only the answers on the scan card count for credit. Each question is worth 1 mark. The final is graded out of 38. There is no penalty for incorrect answers.
3. **NO CALCULATORS ALLOWED.**

COMPUTER CARD INSTRUCTIONS:

IT IS YOUR RESPONSIBILITY TO ENSURE THAT THE ANSWER SHEET IS PROPERLY COMPLETED. YOUR TEST RESULTS DEPEND UPON PROPER ATTENTION TO THESE INSTRUCTIONS.

- Print your name, student number, course name, and the date in the space provided at the top of Side 1 (red side) of the form. Then the sheet **MUST** be signed in the space marked SIGNATURE.
- You are writing **VERSION 1**; indicate this in the version column.
- Mark your student number in the space provided on the sheet on Side 1 **and fill the corresponding bubbles underneath**. If you have a student number that is 7 digits, begin your student number with 00 (two zeroes).
- Mark only **ONE** choice (A, B, C, D, E) for each question.
- Begin answering questions using the first set of bubbles, marked "1".

For the next five questions, identify the correct term for the blank.

1. Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are ________________ if $\mathbf{u} \cdot \mathbf{v} = 0$.

   (a) invertible (b) a cross product $\boxed{\text{(c) orthogonal}}$ (d) unit vectors
   (e) a standard basis

2. A scalar λ is ________________ of an $n \times n$ matrix A if there is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$.

   (a) a projection $\boxed{\text{(b) an eigenvalue}}$ (c) a basis (d) an eigenvector
   (e) an echelon form

3. A nonzero vector is ________________ of an $n \times n$ matrix A if there is a scalar λ such that $A\mathbf{x} = \lambda\mathbf{x}$.

   (a) a projection (b) an eigenvalue (c) a basis $\boxed{\text{(d) an eigenvector}}$
   (e) an echelon form

4. An $n \times n$ matrix A is ________________ if there is an $n \times n$ matrix B such that $AB = BA = I_n$.

   $\boxed{\text{(a) an invertible matrix}}$ (b) a triangular matrix (c) a diagonal matrix
   (d) a symmetric matrix (e) a row reduced matrix

5. The set of vectors $\{(1,0,0), (0,1,0), (0,0,1)\}$ in $\mathbb{R}^3$ are ________________.

   (a) unit vectors (b) pairwise orthogonal (c) linearly independent
   (d) a standard basis for $\mathbb{R}^3$ $\boxed{\text{(e) all of the above}}$

6. Suppose that the augmented matrix of a system of linear equations has been placed into the following reduced row echelon form:

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & 1 & 0 & -1 & -10 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

← Free variables x_5 Set $x_5 = t$

The set of solutions for this systems is described by:

(a) $\begin{aligned} x_1 &= 10 \\ x_2 &= 10 \\ x_3 &= 10 \\ x_4 &= 30 \\ x_5 &= 20 \end{aligned}$

(b) $\boxed{\begin{aligned} x_1 &= t-10 \\ x_2 &= -t+30 \\ x_3 &= t-10 \\ x_4 &= t+10 \\ x_5 &= t \end{aligned}}$

(c) $\begin{aligned} x_1 &= t+s-10 \\ x_2 &= s \\ x_3 &= t-10 \\ x_4 &= t+10 \\ x_5 &= t \end{aligned}$

(d) $\begin{aligned} x_1 &= t+10 \\ x_2 &= -t-30 \\ x_3 &= t+10 \\ x_4 &= t-10 \\ x_5 &= t \end{aligned}$

$\left.\begin{aligned} x_1 &= -10 + x_5 \\ x_2 &= 30 - x_5 \\ x_3 &= -10 + x_5 \\ x_4 &= 10 + x_5 \\ x_5 &= x_5 \end{aligned}\right\}$ $\begin{aligned} x_1 &= t-10 \\ x_2 &= -t+30 \\ x_3 &= t-10 \\ x_4 &= t+10 \\ x_5 &= t \end{aligned}$

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7. Consider the following system of linear equations:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 1 \\ x_1 + 2x_2 + 3x_3 + 4x_4 &= 2016 \\ 2x_1 + 3x_2 + 4x_3 + 5x_4 &= 2017. \end{aligned}$$

How many solutions does it have?

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 2016 \\ 2 & 3 & 4 & 5 & 2017 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 2016 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

↑ infinite # of sol^ns since this will have a free variable

(a) 0

(b) 1

(c) 4

(d) 2017

(e) Infinitely many

8. Let

$$A = \begin{bmatrix} c & c & c \\ 2 & c & c \\ 2 & 2 & c \end{bmatrix}.$$

For what values of c is A invertible?

(a) All values of c make A invertible.

$\boxed{\text{(b) All } c \text{ except } c = 0 \text{ and } c = 2.}$

(c) Only $c = 0$ and $c = -1$.

(d) Only $c = 0$ and $c = 2$.

(e) No value of c makes A invertible.

A invertible $\Leftrightarrow \det(A) \neq 0$

$\det(A) = c^3 + 2c^2 + 4c - 2c^2 - 2c^2 - 2c^2$

$= c^3 - 4c^2 + 4c$

$= c(c^2 - 4c + 4)$

$= c(c-2)^2$

So $\det(A) = 0 \Leftrightarrow c = 0, 2$

Thus $\det(A) \neq 0 \Leftrightarrow c \neq 0, 2$

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9. Compute A if

$$(A^T + 5I_2)^{-1} = \begin{bmatrix} -1 & 3 \\ 2 & 3 \end{bmatrix}.$$

(a) $\frac{1}{9}\begin{bmatrix} -48 & 3 \\ 2 & -44 \end{bmatrix}$ (b) $\begin{bmatrix} 2017 & 1 \\ 2017 & 1 \end{bmatrix}$ (c) $-\frac{1}{9}\begin{bmatrix} -48 & 2 \\ 3 & -44 \end{bmatrix}$

$\boxed{\text{(d) } \frac{1}{9}\begin{bmatrix} -48 & 2 \\ 3 & -44 \end{bmatrix}}$ (e) $\frac{1}{9}\begin{bmatrix} 42 & 2 \\ 3 & 46 \end{bmatrix}$

$(A^T + 5I_2) = \begin{bmatrix} -1 & 3 \\ 2 & 3 \end{bmatrix}^{-1} = \frac{1}{-9}\begin{bmatrix} 3 & -3 \\ -2 & -1 \end{bmatrix}$

So $A^T = \frac{1}{-9}\begin{bmatrix} 3 & -3 \\ -2 & -1 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} -3/9 - 45/9 & +3/9 \\ +2/9 & 1/9 - 45/9 \end{bmatrix} = \begin{bmatrix} -48/9 & 3/9 \\ 2/9 & -44/9 \end{bmatrix}$

Thus $A = (A^T)^T = \begin{bmatrix} -48/9 & 2/9 \\ 3/9 & -44/9 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} -48 & 2 \\ 3 & -44 \end{bmatrix}$

10. Find the determinant of the following matrix:

$$\begin{bmatrix} 2 & 3 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 & -2 \\ 0 & 0 & -1 & -9 & 8 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}$$

(a) -14

$\boxed{\text{(b) } 0}$

(c) 10

(d) 12

* (e) ~~2016~~ 2017

Because the matrix is triangular,

$\det(A) = 2 \cdot 1 \cdot (-1) \cdot 0 \cdot 7 = 0$

$\begin{matrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 1 \end{matrix}$ $\begin{matrix} 2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{matrix}$ -4

$4 \times 2 = 8$

$B = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & -2 & 1 \end{bmatrix} A$

11. Given the matrices:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2a & 2b & 2c \\ g-3a & h-3b & i-3c \\ g-2d & h-2e & i-2f \end{bmatrix}$$

det B = det A.

and given that det(A)=2, compute det(B).

(a) 16

(b) 1

(c) -16

(d) -6

$\boxed{\text{(e) } 8}$

$$2\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} 2a & 2b & 2c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$-2\begin{vmatrix} 2a & 2b & 2c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} 2a & 2b & 2c \\ -2d & -2e & -2f \\ g & h & i \end{vmatrix}$$

$$= \begin{vmatrix} 2a & 2b & 2c \\ g-2d & h-2e & i-2f \\ g-3a & h-3b & i-3c \end{vmatrix} \begin{matrix} \leftarrow \text{row 2 + row 3} \\ \leftarrow \text{row 3} - \frac{3}{2}\text{ row 1} \end{matrix}$$

$$= -\begin{vmatrix} 2a & 2b & 2c \\ g-3a & h-3b & i-3c \\ g-2d & h-2e & i-2f \end{vmatrix}$$

So

$$\begin{vmatrix} 2a & 2b & 2c \\ g-3a & h-3b & i-3c \\ g-2d & h-2e & i-2f \end{vmatrix} = (-1)(-2)(2)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \boxed{8}$$

12. The matrix A is a 3×3 matrix, and the equation

$$A\mathbf{x} = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$$

has an infinite number of solutions. Which of the following may be a characterситic polynomial for A?

(a) $-\lambda^3 + \lambda^2 + \lambda - 36$

(b) $-\lambda^3 + \lambda^2 + \lambda + 6$

(c) $-\lambda^3 + \lambda^2 + \lambda + 1$

(d) $\boxed{-6\lambda^3 + 3\lambda^2 + \lambda}$

(e) Not enough information given.

$A\vec{x} = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$ has an infinite # of sol'^s

$\Longleftrightarrow \det(A) = 0$

$\Longleftrightarrow \lambda = 0$ is an eigenvalue

$\lambda = 0$ is a root only of (d)

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13. Given the 3×3 matrices A, B, C such that $\det(A) = 4$, $\det(B) = 3$, and $\det(C) = -2017$, evaluate $\det(3CA^TB^{-1}C^{-1})$.

(a) -36

(b) -4

(c) 4

(d) $\boxed{36}$

(e) Cannot be determined.

$\det(3CA^TB^{-1}C^{-1})$

$= 3^3 \det(C)\det(A^T)\det(B^{-1})\det(C^{-1})$

$= 27 \det(A) \cdot \frac{1}{\det(B)} = 27 \cdot 4 \cdot \frac{1}{3} = 9 \cdot 4$

$= \boxed{36}$

$9\ \cancel{27}^{3} \times \cancel{(-2017)} \times 4 \times \frac{1}{\cancel{3}} \times \frac{1}{\cancel{-2017}}$

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14. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}.$$

$\lambda I_2 - A = \begin{bmatrix} \lambda-2 & 12 \\ -1 & \lambda+5 \end{bmatrix}$

(a) $\boxed{-2, -1}$
(b) $-2, 1$
(c) -2 with algebraic multiplicty 2
(d) $2, -1$
(e) $2, 1$

$\begin{vmatrix} 2-\lambda & -12 \\ 1 & -5-\lambda \end{vmatrix} = 0$

$\det(\lambda I_2 - A) = (\lambda-2)(\lambda+5)+12$
$= \lambda^2+3\lambda-10+12$
$= \lambda^2+3\lambda+2$
$= (\lambda+2)(\lambda+1)$
roots $\lambda_1=-2, \lambda_2=-1$

$(2-\lambda)(-5-\lambda)+12=0 \quad 12-10-2\lambda+5\lambda+\lambda^2=0 \quad \lambda^2+3\lambda+2=0$
$(\lambda+1)(\lambda+2)=0$
$\lambda=-1$ or -2

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15. For the 3×3 matrix A, you are given the following results concerning eigenvectors

$$A\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad A\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix} \quad A\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \ (*)$$

1 $\quad$ -1 $\quad$ 0

If the matrix P is

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

what is $P^{-1}A^4P$ equal to? (do NOT try to solve for A or the inverse of P)

The $(*)$ implies $\lambda_1=1, \lambda_2=-1, \lambda_3=0$ are eigen values. So, can diagonalize A as

$A = P\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}P^{-1}$. (the order is given by the columns of P)

(a) $-A$
(b) A^4
(c) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & -64 & 0 \\ 1 & -64 & 0 \end{bmatrix}$
(d) $\boxed{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$
(e) Not enough information given.

Now $A^4 = P\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^4 P^{-1}$

So $P^{-1}A^4P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

16. Let A be a 2017×2017 upper triangular matrix. All the diagonal entries of A are $\frac{2}{i}$. What is the determinant of A?

(a) $i + 2017^2$ (b) i^{2017} (c) 2^{2017} $\boxed{\text{(d) } -2^{2017}i}$ (e) $2017 + 2^{2017}i$

$\det(A) = \left(\frac{2}{i}\right)^{2017}$. Now $\frac{1}{i} = -i$.

So $\det(A) = 2(-i)^{2017} = 2^{2017}(-i)^{2017} = -2^{2017}i$

17. Suppose $z_1 = 4\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$ and $z_2 = 2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$. Compute $z_1\overline{z_2}$.

(a) $8(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right))$
(b) $4(\cos\left(\frac{4\pi}{6}\right) + i\sin\left(\frac{4\pi}{6}\right))$
(c) $2(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right))$
(d) $\cos(\pi) + i\sin(\pi)$
$\boxed{\text{(e) } 8(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right))}$

If $z_2 = 2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$, then

$\overline{z_2} = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$

Thus $z_1\overline{z_2} = 2\cdot 4\left(\cos\left(\frac{\pi}{3} - \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{3} - \frac{\pi}{6}\right)\right)$
$= 8\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$

18. In any given year a person may or may not be infected with the dreaded "math-itis", the inability to answer linear algebra questions! Past records show that if a person has "math-itis" one year, then there is a 75% chance that they will not get the "math-itis" in the following year. If they don't have "math-itis" in a given year, then there is a 30% chance that they will get "math-itis" the following year. In the long run, what proportion of years does a person not have the "math-itis"?

(a) $\frac{1}{7}$

(b) $\frac{2}{7}$

(c) $\frac{3}{4}$

$\boxed{\text{(d) } \frac{5}{7}}$

(e) $\frac{30}{105}$

Transition matrix

$$P = \begin{array}{c c} & \begin{array}{cc} M & N \end{array} \\ \begin{array}{c} M \\ N \end{array} & \begin{bmatrix} 1/4 & 3/10 \\ 3/4 & 7/10 \end{bmatrix} \end{array}$$

M = mathitis

N = non-mathitis

Find prob-vector that is an eigenvector of $\lambda = 1$

$$1\lambda I_2 - P = \begin{bmatrix} 3/4 & -3/10 \\ -3/4 & 3/10 \end{bmatrix} \sim \begin{bmatrix} 3/4 & -3/10 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 15 & -6 \\ 0 & 0 \end{bmatrix}$$

So eigen space $E_{\lambda=1} = \left\{ t \begin{bmatrix} 6 \\ 15 \end{bmatrix} \mid t \in \mathbb{R} \right\}$

Prob-vector in $E_{\lambda=1}$ satisfies $6t + 15t = 1 \Rightarrow t = 1/21$

So $\begin{bmatrix} 6/21 \\ 15/21 \end{bmatrix} = \begin{bmatrix} 2/7 \\ 5/7 \end{bmatrix}$ is the steady state vector.

Since we want non-mathitis, look @ second entry.

19. The angle between $(2,0,1,1)$ and $(1,1,0,2)$ in $\mathbb{R}^4$ is

(a) $\frac{2}{3}$ $\boxed{\text{(b) } \cos^{-1}\left(\frac{2}{3}\right)}$ (c) $\cos\left(\frac{2}{3}\right)$ (d) $\frac{1}{2}$ (e) $\sin\left(\frac{1}{2}\right)$

$\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta \Rightarrow \theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}\right)$

$\vec{u}\cdot\vec{v} = 4$, $\|\vec{u}\| = \sqrt{6} = \|\vec{v}\|$. So $\theta = \cos^{-1}\left(\frac{4}{\sqrt{6}\sqrt{6}}\right) = \cos^{-1}\left(\frac{2}{3}\right)$

20. Find the area of the triangle with the given vertices $A(1,1)$, $B(2,2)$, and $C(3,-3)$.

(a) 1 $\boxed{\text{(b) } 3}$ (c) 6 (d) 7 (e) 14

Vector $\overrightarrow{AB} = (2-1, 2-1) = (1,1)$ and $\overrightarrow{AC} = (3-1, -3-1) = (2,-4)$

So Area of triangle $= \frac{1}{2}\left|\begin{matrix} 1 & 1 \\ 2 & -4 \end{matrix}\right| = \frac{1}{2}|-4-2| = \frac{1}{2}|-6| = \boxed{3}$

21. The projection of $\mathbf{u}$ onto $\mathbf{v}$ in $\mathbb{R}^5$ where $\mathbf{u} = (1,0,1,0,2)$ and $\mathbf{v} = (0,1,1,1,1)$ is

(a) $\frac{1}{2}(1,0,1,0,2)$
(b) $\frac{3}{4}(1,0,1,0,2)$
(c) $\frac{1}{2}(0,1,1,1,1)$
$\boxed{\text{(d) } \frac{3}{4}(0,1,1,1,1)}$
(e) None of these

$\text{Proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}$

$= \frac{1+2}{4}\vec{v} = \frac{3}{4}\vec{v}$

22. Two of the axioms of a vector space V are given below:

    Axiom 5. For each $\mathbf{u}$ in V, there is an object $-\mathbf{u}$ in V, called the negative of $\mathbf{u}$, such that $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$.

    Axiom 8. $(k+m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$ for any scalars k and m and vectors $\mathbf{u} \in V$.

    Consider the set $\mathbb{R}^3$ with the standard scalar multiplication, but with addition given by

    $$(u_1, u_2, u_3) + (v_1, v_2, v_3) = (u_1 + v_1, u_1 + u_2 + v_1 + v_2, u_1 + u_2 + u_3 + v_1 + v_2 + v_3).$$

    Then for this set with these two operations we know

    (a) Both Axiom 5 and Axiom 8 are true.

    (b) $\boxed{\text{Axiom 5 is true, but Axiom 8 fails.}}$

    (c) Axiom 5 fails, but Axiom 8 is true.

    (d) Both Axiom 5 and Axiom 8 fail.

There is a negative, namely $(-u_1, -u_2, -u_3)$ since

$(u_1, u_2, u_3) + (-u_1, -u_2, -u_3) = (u_1 - u_1, u_1 + u_2 - u_1 - u_2, u_1 + u_2 + u_3 - u_1 - u_2 - u_3)$
$= (0, 0, 0)$

So Axiom 5 true

Axiom 8 is false. E.g. Let $k = m = 1$, and $\vec{u} = (1,1,1)$

$(k+m)\vec{u} = (1+1)\vec{u} = 2\vec{u} = (2,2,2)$

$k\vec{u} + m\vec{u} = \vec{u} + \vec{u} = (1+1, 1+1+1+1, 1+1+1+1+1+1) = (2,4,6)$

So $(k+m)\vec{u} \neq k\vec{u} + m\vec{u}$

23. Consider the following subsets of $M_{2,2}(\mathbb{R})$, the vector space of 2×2 matrices:

W_1 is the set of 2×2 diagonal matrices.

W_2 is the set of 2×2 invertible matrices.

W_3 is the set of 2×2 matrices with integer entries.

Which subsets are subspaces of $M_{2,2}(\mathbb{R})$?

$\boxed{\text{(a) } W_1}$

(b) W_1 and W_2

(c) W_2 and W_3

(d) W_3

(e) None of them.

- W_2 is not a subspace since $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin W_2$

- W_3 is not a subspace since it is not closed under scalar multiplication. For example, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \in W_3$, but $\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \notin W_3$

- W_1 is a subspace since
  1. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W_1$ b/c this is a diagonal matrix
  2. If $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} \in W_1$, then so is .

     $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a+b & 0 \\ 0 & a+b \end{bmatrix} \in W_1$
  3. If $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \in W_1$ and $c \in \mathbb{R}$, $c\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} ca & 0 \\ 0 & ca \end{bmatrix} \in W_1$

24. Consider the following sets of vectors in $\mathbb{R}^3$

(1) $S = \left\{ \begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}$

(2) $T = \left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ -2 \end{bmatrix} \right\}$

Which of above sets are linearly independent?

(a) Neither set is linearly independent.

(b) Only S is linearly independent.

(c) Only T is linearly independent.

(d) Both sets are linearly independent.

• T cannot be linearly independent since we have 4 vectors in $\mathbb{R}^3$

S is linearly independent since

$$\det \begin{bmatrix} -3 & 5 & 1 \\ 0 & -1 & 1 \\ 4 & 2 & 3 \end{bmatrix} = 9 + 20 + 0 + 4 + 6 + 0 \neq 0$$

So system $\begin{bmatrix} -3 & 5 & 1 \\ 0 & -1 & 1 \\ 4 & 2 & 3 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ only has trivial $\underline{\text{sol}^n}$.

25. Consider the following three polynomials of $\mathbb{P}_2$:

$$\mathbf{p}_1(t) = 1 + 2t + 3t^2, \quad \mathbf{p}_2(t) = t + 2t^2, \quad \mathbf{p}_3(t) = -2 + t^2.$$

Which of the following statements is true about the set $S = \{\mathbf{p}_1(t), \mathbf{p}_2(t), \mathbf{p}_3(t)\}$?

(a) S is not linearly independent set and $\text{span}(S) \neq \mathbb{P}_2$.

(b) S is a linearly independent set and $\text{span}(S) \neq \mathbb{P}_2$

(c) S is not linearly independent and $\text{span}(S) = \mathbb{P}_2$.

(d) S is linearly independent and $\text{span}(S) = \mathbb{P}_2$.

We find all solns to

$$k_1(1+2t+3t^2) + k_2(t+2t^2) + k_3(-2+t^2) = 0 + 0x + 0x^2$$

$$\Leftrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

we compute the determinant: $1 - 8 + 0 + 6 + 0 + 0 = \boxed{-1}$

So, only trivial sol^{n}. So $\vec{p}_1$, $\vec{p}_2$ and $\vec{p}_3$ are linearly independent.

So $\text{span}\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ is 3-dimensional.

Since $\mathbb{P}_2$ is also 3-dimensional, these 3-vectors are a basis for $\mathbb{P}_2$.

i.e $\text{span}\{\vec{p}_1, \vec{p}_2, \vec{p}_3\} = \mathbb{P}_2$

26. The three vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ form a basis for $\mathbb{R}^3$. It is known that

$$\mathbf{v}_1 = \begin{bmatrix}1\\2\\1\end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix}2\\9\\0\end{bmatrix}.$$

Also, it is known that the coordinate vector of $\mathbf{v} = \begin{bmatrix}5\\-1\\9\end{bmatrix}$ relative to S is

$$(\mathbf{v})_S = (1, -1, 2)$$

What is the vector $\mathbf{v}_3$?

(a) $\begin{bmatrix}1\\-1\\2\end{bmatrix}$ $\boxed{\text{(b)} \begin{bmatrix}3\\3\\4\end{bmatrix}}$ (c) $\begin{bmatrix}7\\10\\10\end{bmatrix}$ (d) $\begin{bmatrix}3\\11\\1\end{bmatrix}$ (e) $\begin{bmatrix}6\\-2\\11\end{bmatrix}$

The vector $\vec{v}_3$ satisfies $1 \cdot \vec{v}_1 - \vec{v}_2 + 2\vec{v}_3 = \begin{bmatrix}5\\-1\\9\end{bmatrix}$

So, if $\vec{v}_3 = \begin{bmatrix}a\\b\\c\end{bmatrix}$, then

$$\begin{bmatrix}1\\2\\1\end{bmatrix} - \begin{bmatrix}2\\9\\0\end{bmatrix} + 2\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}5\\-1\\9\end{bmatrix} \iff \begin{bmatrix}-1+2a\\-7+2b\\1+2c\end{bmatrix} = \begin{bmatrix}5\\-1\\9\end{bmatrix}.$$

Solving for a, b, and c gives $a=3, b=3, c=4$.

So soln is $\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}3\\3\\4\end{bmatrix}$

27. Let $W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for $\mathbb{R}^3$ where

$$\mathbf{v}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix}1\\2\\0\end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix}0\\1\\2\end{bmatrix}.$$

The Gram-Schmidt process can find an orthogonal basis $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ of W. What is $\mathbf{u}_3$?

(a) $\begin{bmatrix}0\\1\\2\end{bmatrix}$ (b) $\begin{bmatrix}1\\1\\0\end{bmatrix}$ (c) $\begin{bmatrix}2017\\2017\\2017\end{bmatrix}$ (d) $\begin{bmatrix}-\frac{1}{2}\\\frac{1}{2}\\0\end{bmatrix}$ (e) $\boxed{\begin{bmatrix}0\\0\\2\end{bmatrix}}$

$\vec{u}_1 = \vec{v}_1$

$\vec{u}_2 = \vec{v}_2 - \dfrac{\vec{u}_1 \cdot \vec{v}_1}{\|\vec{u}_1\|^2}\vec{u}_1 = \begin{bmatrix}1\\2\\0\end{bmatrix} - \dfrac{(3)}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}-1/2\\1/2\\0\end{bmatrix}$

Note that we can now find $\vec{u}_3$ using the formula

$\vec{u}_3 = \vec{v}_3 - \dfrac{\vec{u}_1 \cdot \vec{v}_3}{\|\vec{u}_1\|^2}\vec{u}_1 - \dfrac{\vec{u}_2 \cdot \vec{v}_3}{\|\vec{u}_2\|^2}\vec{u}_2$.

However, we also know $\vec{u}_3$ will be orthogonal to $\vec{u}_1$ & $\vec{u}_2$. The only vector in the above list with this property is (e)

28. Which of the following are an orthonormal basis for $\mathbb{R}^3$?

(a) $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}$ (b) $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\-1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix}$

(c) $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\-1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix}$ (d) $\frac{1}{\sqrt{3}}\begin{bmatrix}1\\0\\1\end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\-1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix}$

* MISTAKE IN THIS PROBLEM

No orthonormal set!!

Gave everyone a bonus pt

~~For the other sets, there is at least one vector with norm ≠ 1.~~

WRONG. This set is not orthogonal.

29. The matrix A is row equivalent to the matrix B:

$$A = \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -3 & 0 & 5 & -7 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

What is a basis for Row(A), the row space of A.

(a) $\begin{bmatrix} 1 & -3 & 4 & -1 & 9 \end{bmatrix}, \begin{bmatrix} -2 & 6 & -6 & -1 & -10 \end{bmatrix}, \begin{bmatrix} -3 & 9 & -6 & -6 & -3 \end{bmatrix}, \begin{bmatrix} 3 & -9 & 3 & -9 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & -9 & 4 & 9 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -3 & 0 & 5 & -7 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 2 & -3 & 8 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 5 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & -3 & 0 & 5 & 7 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 5 \end{bmatrix}$

(e) Not enough information

The basis is the rows with leading ones.

30. Using A and B as in the Question 29, what is a basis for Col(A), the column space.

(a) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 8 \\ 5 \\ 0 \end{bmatrix}$
(b) $\begin{bmatrix} 1 \\ -2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} 9 \\ -10 \\ -3 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$
(d) $\begin{bmatrix} -3 \\ 6 \\ 9 \\ -9 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -6 \\ 9 \end{bmatrix}$

The basis of Col(A) are the columns of A corresponding to the columns with leading ones.

31. A $n \times n$ matrix A whose entries can be complex numbers is called *Hermitian* if $A = \overline{A^T}$, i.e., A is equal to taking the complex conjugate of each entry of the transpose of A. Consider the following statements about Hermitian matrices:

(1) The entries on the main diagonal of A are all real numbers. TRUE

(2) The sum of two Hermitian matrices is a Hermitian matrix. TRUE

(3) The product of two Hermitian matrices is a Hermitian matrix. FALSE

Which statements are true?

(a) (1) and (2) only

(b) (1) and (3) only

(c) (2) and (3) only

(d) All are true.

(e) None are true.

(2) true since $A^T + B^T = (A+B)^T$

(3) Is false. Ex. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$ are Hermitian

But $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} = \begin{bmatrix} 1-i & i+1 \\ 1-i & i+1 \end{bmatrix}$ is not Hermitian. (Fails (1))

---

32. The following commands are entered into Matlab:

```
b = @(i,j) 2*i+j
B = zeros(2,2)
B(1,1)=b(1,1)
B(1,2)=b(2,1)
B(2,1)=b(1,2)
B(2,2)=b(2,2)
det(tril(B))
```

$$B = \begin{bmatrix} 2\cdot 1+1 & 2\cdot 2+1 \\ 2\cdot 1+2 & 2\cdot 2+2 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$$

$$\text{tril}(B) = \begin{bmatrix} 3 & 0 \\ 4 & 6 \end{bmatrix}$$

What is the output of the last command?

(a) -18

(b) -2

(c) 18

(d) 0

(e) 2

$\det(\text{tril}(B)) = 18$

33. Which of the following statements are true?

(1) For any vector $\mathbf{v} \in \mathbb{R}^n$, $\|\mathbf{v}\| = \mathbf{v} \cdot \mathbf{v}$. False. $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$

(2) If $\mathbf{u}, \mathbf{v}$ are two distinct non-zero vectors in $\mathbb{R}^3$, then $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$. TRUE

(a) (1) is false and (2) is false.

(b) (1) is true and (2) is false.

(c) (1) is false and (2) is true.

(d) (1) is true and (2) is true.

---

34. Which of the following statements are true? Should be $\leq$

(1) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, then $\|\mathbf{u} + \mathbf{v}\| \geq \|\mathbf{u}\| + \|\mathbf{v}\|$. FALSE

(2) The vectors $\text{proj}_{\mathbf{a}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2} \mathbf{a}$ and $\mathbf{a}$ are orthogonal to each other. FALSE

(a) (1) is false and (2) is false. They are parallel to each other

(b) (1) is true and (2) is false.

(c) (1) is false and (2) is true.

(d) (1) is true and (2) is true.

---

35. Which of the following statements are true?

(1) There is a vector space with exactly one element. TRUE: $W = \{\vec{0}\}$

(2) Any subset W of a vector space V is a subspace. FALSE.

(a) (1) is false and (2) is false.

(b) (1) is true and (2) is false.

(c) (1) is false and (2) is true.

(d) (1) is true and (2) is true.

$V = \mathbb{R}$ is a vector space
$\{1\}$ is a subset of V but not a subspace

36. Which of the following statements are true?

    (1) If $S = \{v_1, \dots, v_n\}$ is a set of linearly independent elements in a vector space V, then S is a basis for V. FALSE. Only true if $\dim V = n$

    (2) If $S = \{v_1, \dots, v_n\}$ is a basis for a vector space V, then S spans V. TRUE

    (a) (1) is false and (2) is false.

    (b) (1) is true and (2) is false.

    (c) (1) is false and (2) is true.

    (d) (1) is true and (2) is true.

---

37. Which of the following statements are true?

    (1) If λ is an eigenvalue of an $n \times n$ matrix A, then the geometric multiplicity of λ equals the dimension of the null space of the matrix $(\lambda I_n - A)$. TRUE

    (2) If A is a 2017×7102 matrix, then $\operatorname{rank}(A) \leq 2017$. TRUE

    (a) (1) is false and (2) is false.

    (b) (1) is true and (2) is false.

    (c) (1) is false and (2) is true.

    (d) (1) is true and (2) is true.

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38. Who was your favourite Math 1B03 instructor? [Hint: Answer (e)]

    (a) Adam

    (b) Adam Van Tuyl

    (c) Dr. Adam

    (d) Professor Van Tuyl

    (e) All of the above

END OF TEST PAPER