Assignment 3

CompSci 1DM3

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Q1) show that $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil = \lfloor \frac{n^2}{4} \rfloor$ for all $n \in \mathbb{Z}$

Case 1: n is even

$n = 2K$, $K \in \mathbb{Z}$

$\frac{n}{2} = K$ and because $K \in \mathbb{Z}$

$\lfloor \frac{n}{2} \rfloor = K$, $\lceil \frac{n}{2} \rceil = K$

therefore $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil = K^2$

and as $K = \frac{n}{2}$ so $K^2 = \frac{n^2}{4}$

and we know that if K is an integer, then K^2 is also an integer so $\lfloor \frac{n^2}{4} \rfloor = \frac{n^2}{4}$

The left and right sides are equal so it's proved.

Case 2: n is odd

$n = 2K+1$, $K \in \mathbb{Z}$

$\frac{n}{2} = \frac{2K+1}{2} = K + \frac{1}{2}$

therefore:

$\lfloor \frac{n}{2} \rfloor = \lfloor K + \frac{1}{2} \rfloor = K$

$\lceil \frac{n}{2} \rceil = K+1$ so $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil = K(K+1)$

we know that $n = 2K+1$

So $n^2 = (2K+1)^2 = 4K^2 + 4K + 1$

$\frac{n^2}{4} = \frac{4K^2 + 4K + 1}{4} = K^2 + K + \frac{1}{4}$

and we know that K^2 and K are integers so $\lfloor \frac{n^2}{4} \rfloor = \lfloor K^2 + K + \frac{1}{4} \rfloor = K^2 + K$

$= K(K+1)$ So the left and right sides are equal.

Therefore, we can conclude that

$\lfloor \frac{n^2}{4} \rfloor \lceil \frac{n^2}{4} \rceil = \lfloor \frac{n^2}{4} \rfloor$ for all integers n.

Q2)

$a_n = 8a_{n-1} - 16a_{n-2}$

a) $a_n = 1$

$a_{n-1} = 1$ and $a_{n-2} = 1$

and we know that

$a_n = 8a_{n-1} - 16a_{n-2}$

So $a_3 = 8a_2 - 16a_1$

$a_3 = 8(1) - 16(1)$

$a_3 = -8$ so $a_n = 1$

is not a solution!

b) $a_n = 2^n$

$a_{n-1} = 2^{n-1}$, $a_{n-2} = 2^{n-2}$

we already know that

$a_n = 8a_{n-1} - 16a_{n-2}$

$a_n = 8 \times 2^{n-1} - 16 \times 2^{n-2}$

$2^n = 2^3 \times 2^{n-1} - 2^4 \times 2^{n-2}$

$2^n = 2^{n+2} - 2^{n+2}$

$2^n = 0$ wrong!

So $a_n = 2^n$ is not a solution!

Q3

a) $a_n = a_{n-1} - n$ with $a_0 = 4$ $\quad a_n = 4 - \frac{n(n+1)}{2}$

b) $a_n = 2a_{n-1} - 3$ with $a_0 = -1$ $\quad a_n = -4 \cdot 2^n + 3$

c) $a_n = (n+1)a_{n-1}$ with $a_0 = 2$ $\quad a_n = 2(n+1)!$

d) $a_n = 2na_{n-1}$ with $a_0 = 3$ $\quad a_n = 3 \times 2^n \times n!$

Q4)

a) $\sum_{j=10}^{20} j^2(j-3)$

$$\sum_{j=10}^{20} j^2(j-3) = \sum_{j=10}^{20}(j^3-3j^2) \text{ (expand)}$$

$$\sum_{j=10}^{20} j^3-3j^2 = \sum_{j=10}^{20} j^3 - 3\sum_{j=10}^{20} j^2$$

$$\sum_{j=1}^{n} j^3 = \left(\frac{n(n+1)}{2}\right)^2 \quad \text{(row 4)}$$

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{(row 3)}$$

$$\sum_{j=10}^{20} j^3 = \sum_{j=1}^{20} j^3 - \sum_{j=1}^{9} j^3 \text{ and } \sum_{j=10}^{20} j^2 = \sum_{j=1}^{20} j^2 - \sum_{j=1}^{9} j^3$$

Cubes:

$$\sum_{j=1}^{20} j^3 = \left(\frac{20\times 21}{2}\right)^2 = 210^2 = 44100$$

$$\sum_{j=1}^{9} j^3 = \left(\frac{9\times 10}{2}\right)^2 = 45^2 = 2025$$

$$\sum_{j=10}^{20} j^3 = 44100-2025 = 42075$$

Squares:

$$\sum_{j=1}^{20} j^2 = \frac{20\times 21\times 41}{6} = 2870$$

$$\sum_{j=1}^{9} j^2 = \frac{9\times 10\times 19}{6} = 285$$

$$\sum_{j=10}^{20} j^2 = 2870-285 = 2585$$

$$\sum_{j=10}^{20} j^3 - 3\sum_{j=10}^{20} j^2 = 42075-3(2585) = 34320$$

b) $\sum_{j=10}^{20}(j-1)(2j^2+1)$

$$\sum_{j=10}^{20}(j-1)(2j^2+1)=\sum_{j=10}^{20}(2j^3-2j^2+j-1)$$

$$\sum_{j=10}^{20}(2j^3-2j^2+j-1)=2\sum_{j=10}^{20}j^3-2\sum_{j=10}^{20}j^2+\sum_{j=10}^{20}j-\sum_{j=10}^{20}1$$ split into separte sums

$$\sum_{j=1}^{20}j^3=\left(\frac{j(j+1)}{2}\right)^2$$ row 4 and $$\sum_{j=1}^{20}j^2=\frac{j(j+1)(2j+1)}{6}$$ row 3

$$\sum_{j=1}^{20}j=\frac{j(j+1)}{2}$$ row 2 and $$\sum_{j=1}^{20}c=20c$$

Sum of Cubes: $\sum_{j=10}^{20}j^3=\sum_{j=1}^{20}j^3-\sum_{j=1}^{9}j^3=44100-2025=42075$

Sum of squares: $\sum_{j=10}^{20}j^2=\sum_{j=1}^{20}j^2-\sum_{j=1}^{9}j^2=2870-285=2585$

Sum of j: $\sum_{j=10}^{20}j=\sum_{j=1}^{20}j-\sum_{j=1}^{9}j=210-45=165$

Sum of constants: $\sum_{j=10}^{20}1=\sum_{j=1}^{20}1-\sum_{j=1}^{9}1=20-9=11$

Final answer: $2\sum_{j=10}^{20}j^3-2\sum_{j=10}^{20}j^2+\sum_{j=10}^{20}j-\sum_{j=10}^{20}1$

$=2(42075)-2(2585)+165-11$

$=79134$

Q5) $\log\left(1+\frac{1}{K}\right)=\log\left(\frac{K+1}{K}\right)$

So we have $\sum_{K=1}^{n}\log\left(\frac{K+1}{K}\right)$

$$\sum_{K=1}^{n}\log\left(\frac{K+1}{K}\right)=\sum_{K=1}^{n}\left(\log(K+1)-\log K\right)$$

So we have:

$$\left(\cancel{\log 2}-\log 1\right)+\left(\cancel{\log 3}-\cancel{\log 2}\right)+\left(\cancel{\log 4}-\cancel{\log 3}\right)+\cdots+\left(\log(n+1)-\cancel{\log n}\right)$$

$= -\log 1+\log(n+1)$

$= 0+\log(n+1) = \log(n+1)$

Q6) $A=\begin{bmatrix}3 & 5\\ 2 & 4\end{bmatrix}$ $C=\begin{bmatrix}2 & 1\\ 0 & 6\end{bmatrix}$

find B such that $AB=C$

$AB=C$

$A^{-1}AB=A^{-1}C$

$IB=A^{-1}C$

$B=A^{-1}C \longrightarrow B=\begin{bmatrix}4 & -13\\ -2 & 8\end{bmatrix}$

Q7) a) if $AB=AC$ then $B=C$ False

Counter example: A is the zero matrix but $B\neq C$

So we have $\left.\begin{matrix}AC=0\\ AB=0\end{matrix}\right\}\longrightarrow AB=AC$ but $B\neq C$

b) if $A=\begin{bmatrix}1 & 3\\ -5 & 2\end{bmatrix}$ then $A^2=\begin{bmatrix}1 & 9\\ 25 & 4\end{bmatrix}$ False

$A^2 = A \times A \longrightarrow \begin{bmatrix}1 & 3\\ -5 & 2\end{bmatrix}\begin{bmatrix}1 & 3\\ -5 & 2\end{bmatrix} = \begin{bmatrix}-14 & 9\\ -15 & -11\end{bmatrix}$

c) if A is 6×4 matrix and B is 4×5 matrix then

AB has 16 entries False

we can only multiply matricis A and B if the number of the columns of A is equal to number of rows of B. So if A is $m \times n$, B has to be $n \times p$ and the AB matrix would be a $m \times p$ matrix. So in this case as A is 6×4 and B is 4×5, the AB matris would be 6×5 and has 30 entries.

Q8)

```
def lorgest_prod (arr, N:real)      arr={a1,a2,...,an}
        t = 0
        for i=1 to n-1 :
            for j= i+1 to n :
                if ai*aj < N :
                    t= max(t, ai*aj)

        return t
```

the big-O of this algorithm is $O(n^2)$ as there are two nested for loops inside of each other.

Q9) prove that $\frac{6n^2+4n^5-4}{7n^2-3}$ is $O(n^3)$

we can break $\frac{6n^2+4n^5-4}{7n^2-3}$ into separate terms:

$$\frac{6n^2+4n^5-4}{7n^2-3} = \frac{6n^2}{7n^2-3} + \frac{4n^5}{7n^2-3} + \frac{-4}{7n^2-3}$$

As the term $\frac{4n^5}{7n^2-3}$ has the highest degree and grows the fastest, the big-o of the entire function is approaximately equal to the big-O of the term $\frac{4n^5}{7n^2-3}$.

also when grows, the -3 in the dinominator doesn't affect the big-O as much as n does, so we can get rid of the -3. therefore:

$$O\left(\frac{4n^5}{7n^2-3}\right) \approx O\left(\frac{4n^5}{7n^2}\right)$$

we already know that:

$|f(n)| \leq C\,|g(n)|$, $x > K$ $f(n)$ is $O(g(n))$

So we can say: $\frac{4}{7}n^3 \leq 1.n^3$ $n > 0$ $C=1$ $K=0$

we cancel out n^3 $\longrightarrow$ $\frac{4}{7} \leq 1$ ✓

So we conclude that $\frac{6n^2+4n^5-4}{7n^2-3}$ is $O(n^3)$

Q10)

a) $f(n) = 1+2+3+\dots+(n^2-1)+n^2$ $O(n^4)$

b) $f(n) = 3n^4$ $O(n^4)$

Q11)

a) prove that $\dfrac{x^3+7x^2+3}{2x+1}$ is $\Theta(x^2)$

$$\frac{x^3+7x^2+3}{2x+1} = \frac{x^3}{2x+1} + \frac{7x^2}{2x+1} + \frac{3}{2x+1}$$

this is the biggest term and grows the fastest, so it determines the big-O and big-Ω of the entire function

$$\frac{x^3}{2x+1} \leq \frac{x^3}{2x}$$

we decrease the dinominator, so the fraction increases.

$$\frac{x^3}{2x+1} \leq \frac{1}{2} \cdot x^2$$ where $x > 0$ $C = \frac{1}{2}, K = 0$

So $\dfrac{x^3}{2x+1}$ is $O(x^2)$ and therefore $\dfrac{x^3+7x^2+3}{2x+1}$ is $O(x^2)$

$$\frac{x^3}{2x+1} \geq \frac{x^3}{2x+x}$$

we increase the dinominator so the fraction decreases

$$\frac{x^3}{2x+1} \geqslant \frac{x^3}{3x}$$

$$\frac{x^3}{2x+1} \geqslant \frac{x^2}{3} \quad x>0 \quad C=\frac{1}{3}, K=0$$

So we conclude that $\frac{x^3}{2x+1}$ is $\Omega(x^2)$

therefore $\frac{x^3+7x^2+3}{2x+1}$ is $\Omega(x^2)$

As $\frac{x^3+7x+3}{2x+1}$ is both $O(x^2)$ and $\Omega(x^2)$,

it is $\Theta(x^2)$

Q12)

$$10000 < \log n^4 < 3n < n^2 \log n < n^3 + 88n^2 + 3 < n.2^n$$