2020 Oct 28 | ROB 101

Summary • Let $f: \mathbb{R}^m \to \mathbb{R}^n$ be a function. Then $x^* \in \mathbb{R}^m$ is a root if $f(x^*) = 0_{n\times 1}$.

• $f(x) = \text{const vector} \in \mathbb{R}^n \iff \underbrace{f(x) - \text{const. vector}}_{\bar{f}(x)} = 0_{n\times 1}$

$\bar{f}(x) = 0_{n\times 1}$

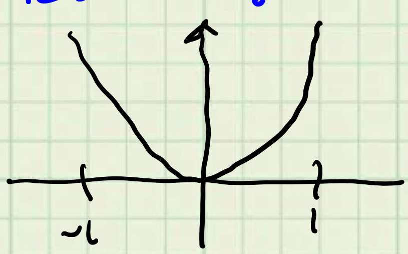
• Let $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2 + 1$. Then $f(x) = 0 \iff x^2+1 = 0$ does not have a root with our definition because roots are real numbers or real vectors.

Today $f: A \to B$ is a rule that maps each element $a \in A$ to an element $b \in B$ that we denote by $f(a)$.

Examples: $A = [-1, 1] \subset \mathbb{R}$, $B = \mathbb{R}$

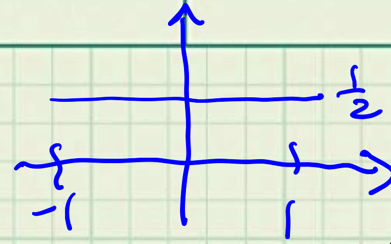
$f: A \to B$ is a function?

1) $f(a) = a^2$



Yes

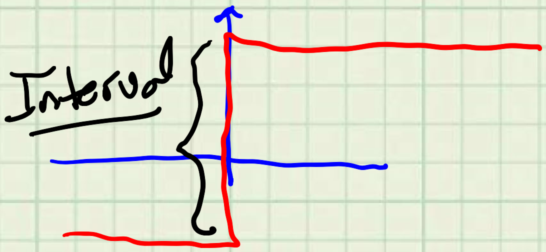
2) $f(a) = \frac{1}{2}$



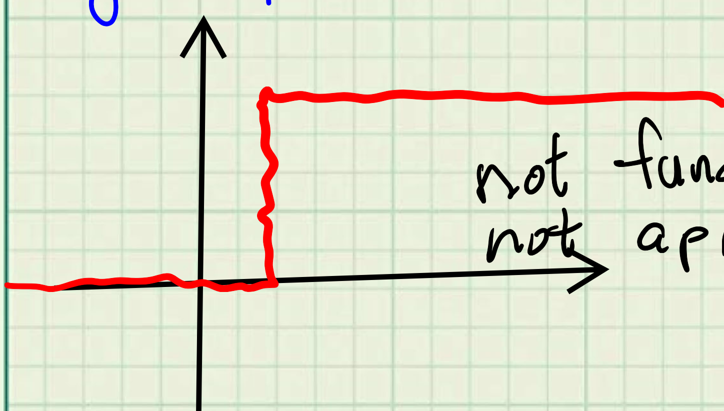
Yes

3) $f(a) = \begin{cases} -1 & -1 \le a < 0 \\ 2 & 0 < a \le 1 \\ [-1, 2] & a = 0 \end{cases}$

NOT a function



Def. (Tongue in Cheek) A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if you can draw its graph without lifting your pencil from the page!!



not function, so above does not apply!!!!

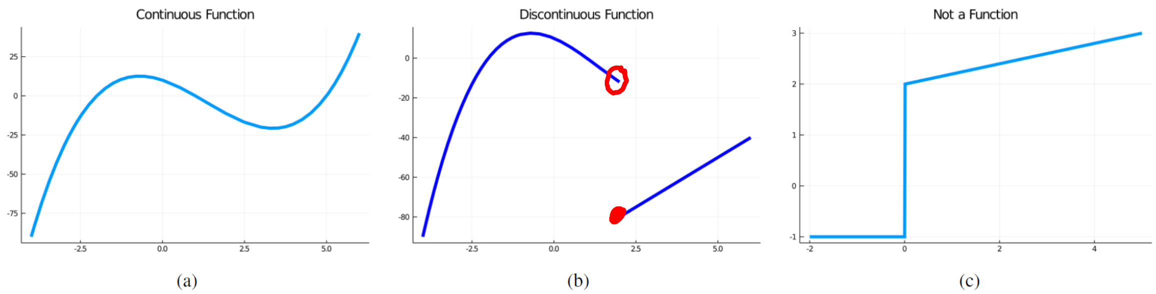


Figure 1: Examples of a continuous function, a discontinuous function, and a graph that is not a function. Yes, in (c), the point $x = 0$ is mapped to the interval $[-1, 2]$. To be a function, each point in the domain can only map to a single point in the range.

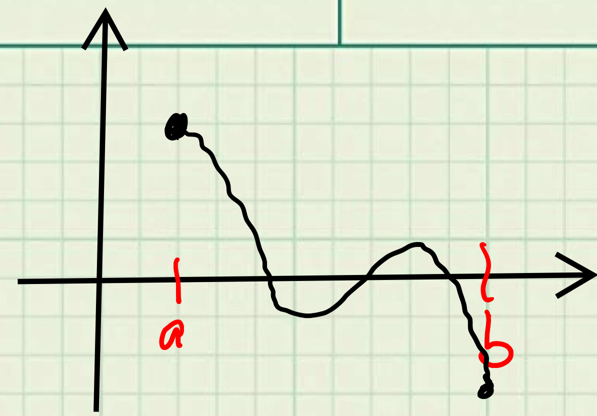
Bisection Algorithm: Needs Continuity.

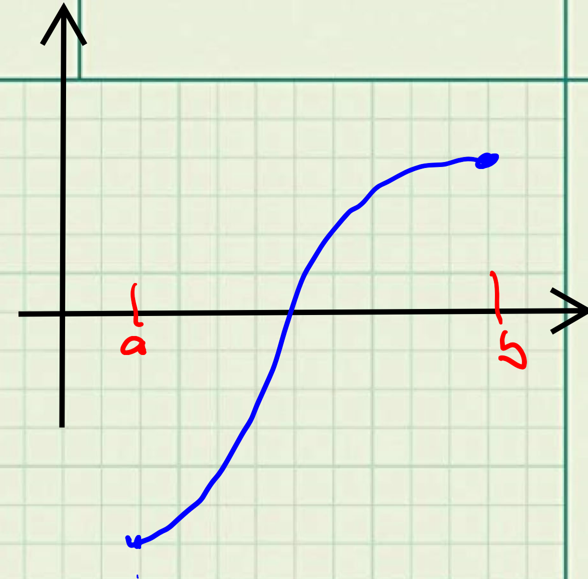
A practical and intuitive method to find roots of $f: \mathbb{R} \to \mathbb{R}$ [no vectors]

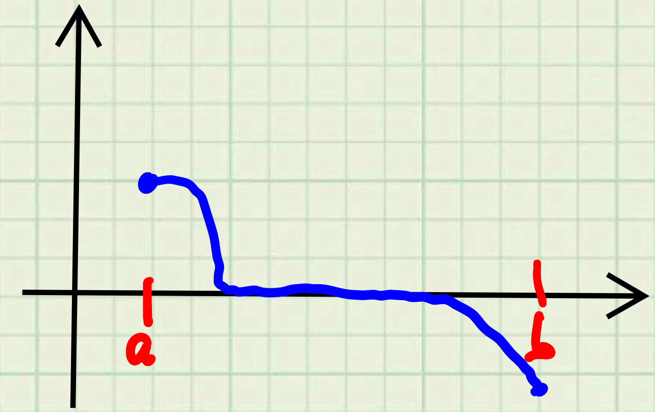
Intermediate Value Theorem:

Suppose that $f: \mathbb{R} \to \mathbb{R}$ is continuous and you know two real numbers $a < b$ such that $f(a) \cdot f(b) < 0$. Then, there exist a number $c \in \mathbb{R}$ such that

- $a < c < b$ (strictly between a & b)
- $f(c) = 0$ (c is a root)







Pseudo Code (Bisection Alg)

$$\boxed{\text{Initialize: Give } a<b \text{ such that } f(a)\cdot f(b)<0.}$$

Start Compute $c = \frac{a+b}{2}$

Exactly two things are possible:

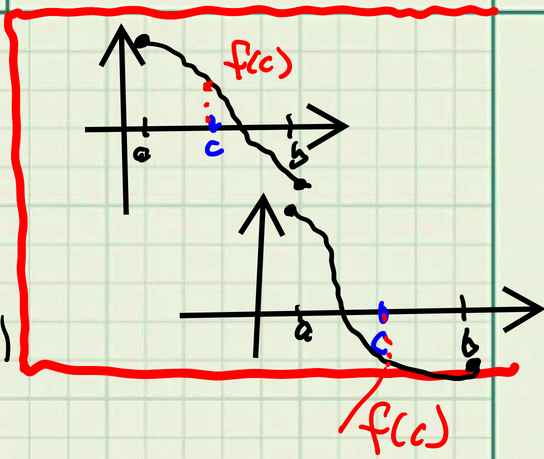
(1) $f(c) = 0$ We are done $x^* = c$!

(2) $f(c) \neq 0$ In this case, two things are possible

Either $f(a)\cdot f(c) < 0$

keep a, set $b = c$

new $f(a)\cdot f(b) < 0$



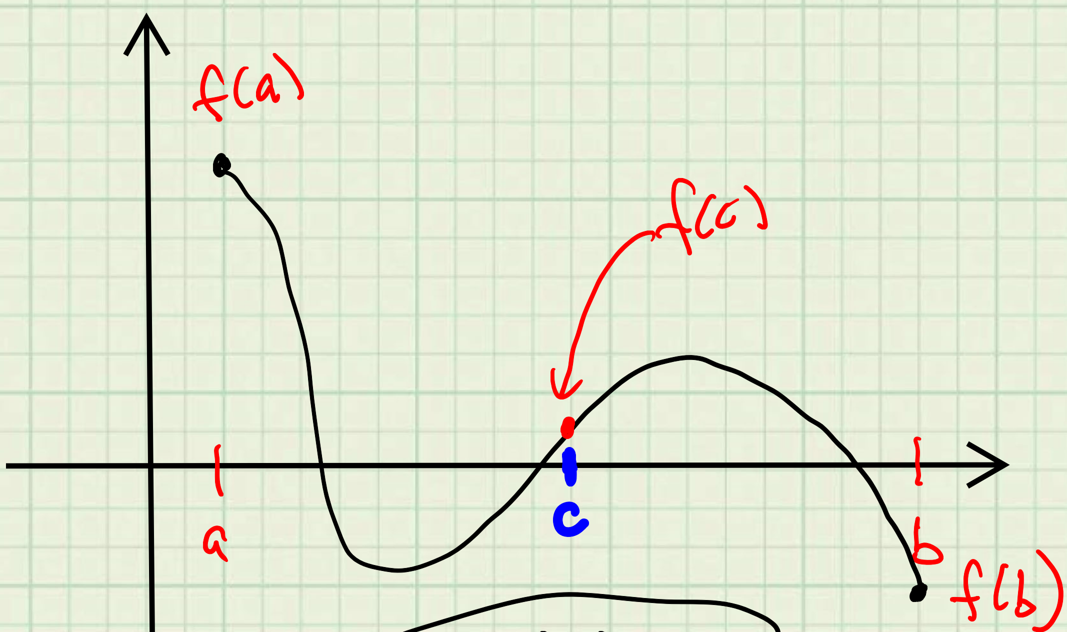
OR $f(a)\cdot f(c) > 0 \Leftrightarrow f(b)\cdot f(c) < 0$

keep b, set $a = c$

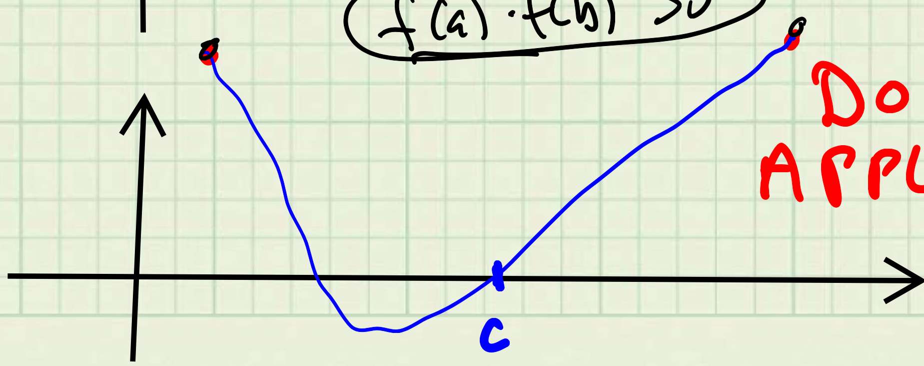
new $f(a)\cdot f(b) < 0$

END ALL IFS

LOOP back to start (wash rinse and repeat!)



$f(a)\cdot f(b) > 0$



DO NOT APPLY BISECTION

$0.2x^5 + x^3 + 3x + 1 = 0$

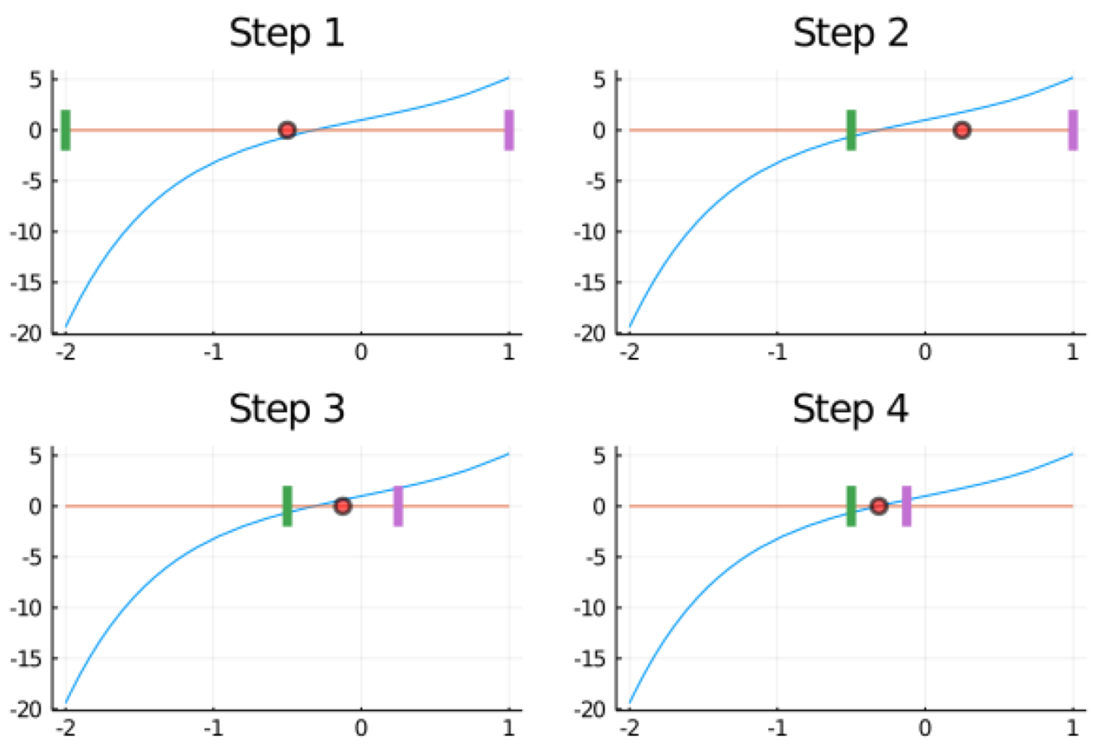


Figure 3: Evolution of the bracketing points a and b as well as the midpoint c in the first four steps of the Bisection Algorithm for finding a root of $0.2x^5 + x^3 + 3x + 1 = 0$. It is very clear that the algorithm hones in on a root!

Cool! For poly's of degree 5 and higher, it's been proven that closed form solutions for roots do not exist.

Does this "algorithm" terminate?

Vocabulary: a and b are said to bracket the root.

If a = rational, b = rational then $c = \frac{a+b}{2}$ = rational $\Longrightarrow$ at every step of the bisection algorithm c_k would be rational.

$f(x) = x^2 - 2$ know $x^* = \sqrt{2}$ is the root.

$a = 0$, $b = 2$ as the bracketing points. They are rational $\Longrightarrow$ c_k is always rational $\Longrightarrow (c_k)^2 - 2 \neq 0$

Algorithm never stops.

Must add termination criterion

If $|f(c)| < \text{tolerance}$, break.

Could aslo add $k \geq K_{max}$, terminate.

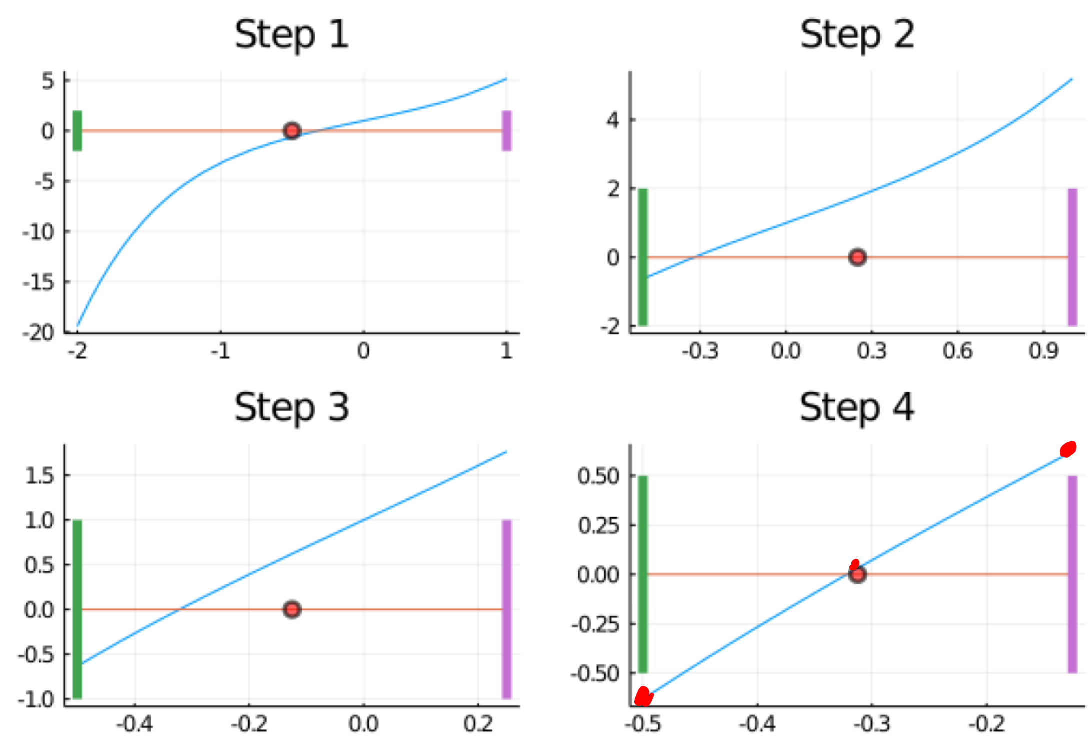


Figure 4: Zooms of the first four steps of the Bisection Algorithm for finding a root of $0.2x^5 + x^3 + 3x + 1 = 0$ that lies between -1 and 2. Observe that as we zoom into the function at a point, it looks more and more like a straight line!