Help Yourself: Create a Google Doc that summarizes the commands you have learned in Julia & material covered in Linear Algebra. Details posted on PIAZZA under **LECTURE**.

Not Required. Does not count for a grade. Will only be used if you ~~fall~~ fall on the borderline between two grades, such as B^+/A^-.

Review:

- $\underset{n\times m}{A}\,\alpha_{m\times 1} = 0_{n\times 1} \iff \underbrace{A^\top A}_{\text{square } m\times m}\alpha = 0_{m\times 1}$

- There exists $\alpha \neq 0_{m\times 1}$ such that $A\alpha = 0_{n\times 1}$

$\iff$ columns of A are linearly dependent

$\iff$ There exists $\alpha \neq 0_{m\times 1}$ such that $A^\top A \alpha = 0_{m\times 1}$

$\iff$ columns of $A^\top A$ are linearly dependent

$\iff \det(A^\top A) = 0_{1\times 1}$

Remarks (paraphrased from Office Hours)

What does uniqueness of solutions mean for $Ax=b$?

Ans.: If you compute a solution and I compute a solution (correctly!), then they must agree.

$$\underbrace{\begin{bmatrix} 1 & 2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{3}_{b}$$

$(x_1 + 2x_2 = 3)$

You: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

ME: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

NOT the same because columns of A are NOT linearly independent.

You should vote too!

What does linear combination mean for $Ax=b$?

Ans: Let's see!

$$\underbrace{\begin{bmatrix}1\\2\end{bmatrix}}_{A} x_1 = \underbrace{\begin{bmatrix}b_1\\b_2\end{bmatrix}}_{b}$$

If $b_1=1$ and $b_2=1$, for example, then we have

$$\left.\begin{aligned} x_1 &= 1 \\ 2x_1 &= 1 \end{aligned}\right\}$$ Inconsistent equations and hence, NO SOLUTION

If $b=\begin{bmatrix}b_1\\b_2\end{bmatrix}=\alpha\begin{bmatrix}1\\2\end{bmatrix}$ (b is a linear combination of the columns of A), then the equations are consistent.

$\begin{pmatrix}Ax=b \\ \text{consistent} \\ \text{equations}\end{pmatrix} \Leftrightarrow \begin{pmatrix}Ax=b \\ \text{has a} \\ \text{solution}\end{pmatrix} \Leftrightarrow \begin{pmatrix}b \text{ is a lin. comb.} \\ \text{of columns of } A\end{pmatrix}$

Today

Problem (juliahw5) If $\det(A^TA)=0$, how to actually solve for $\alpha \neq 0_{m\times 1}$ such that $A\alpha = 0_{n\times 1}$?

Solution:

$$A = \begin{array}{c} \\ \\ \\ \text{Error here} \\ \\ \end{array} \begin{array}{c} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 3 & 0 & 3 \\ \boxed{0} & 1 & 4 \end{bmatrix} \\ \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \end{array}$$

Check at end!

A is supposed to be

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 3 & 0 & 3 \\ 3 & 1 & 4 \end{bmatrix}$$

problem is reworked at end of notes

Know $Ax=0 \iff A^TAx=0$

Compute $P\cdot(A^TA) = L\cdot U$

$P\cdot(A^TA)x=0 \iff A^TAx=0$ because $\det(P)=\pm 1$

$P\cdot A^TAx=0 \iff L\cdot Ux=0$

L is uni-triangular $\Rightarrow \det(L)=1$ $\Rightarrow$ invertible

$$\boxed{\therefore\ A^TAx=0 \iff Ux=0}$$

$\uparrow$ Triangular

In our case

$$U = \begin{bmatrix} 24 & 6 & 30 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\det(U) = 0 \iff \det(A^TA) = 0 \iff$ columns of A are linearly dependent

$Ux = 0$. What is x?

$$\begin{bmatrix} 24 & 6 & 30 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$Ux = 0 \iff \begin{aligned} 24x_1 + 6x_2 + 30x_3 &= 0 \\ x_2 + x_3 &= 0 \\ 0 \cdot x_3 &= 0 \end{aligned}$$

x_3 is arbitrary, so we take say $x_3 = 1$

$\therefore \quad x_2 = -x_3 = -1$

$\therefore \quad 24x_1 = -6x_2 - 30x_3 = (-6)(-1) - 30(1)$

$= -24$

$\boxed{x_1 = 1}$

$$x = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 3 & 0 & 3 \\ 6 & 1 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \\ \\ \end{bmatrix}$$

Grizzle's numbers are wrong.

Method is correct! 

Chapter 8: Norms and Least Squared Error Solutions to Linear Equations

- Meta Message: In Engineering as in Life, the most interesting problems do not have exact answers!

Linear Regression: Super Powers of Linear Algebra!

Norm or "Length" of a vector

Def. For $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}_{n \times 1} \in \mathbb{R}^n$,

$$\|v\| := \sqrt{(v_1)^2 + (v_2)^2 + \cdots + (v_n)^2}$$

We can also write this as

$$\|v\| = \sqrt{v^T v} = \sqrt{\sum_{i=1}^{n} (v_i)^2}$$

$\boxed{\text{Remark:}}$ $\|v\|^2 = (v_1)^2 + (v_2)^2 + \cdots + (v_n)^2 = v^T \cdot v$

Examples $v = \begin{bmatrix} \sqrt{3} \\ -1 \\ \sqrt{5} \end{bmatrix}$

$$\|v\|^2 = (\sqrt{3})^2 + (-1)^2 + (\sqrt{5})^2 = 9$$

$$\|v\| = \sqrt{9} = 3$$

$$W = \begin{bmatrix} -2 \\ 7 \\ 6 \\ 4 \end{bmatrix}$$

$$\|w\|^2 = (-2)^2 + (7)^2 + (6)^2 + (4)^2$$
$$= 4 + 49 + 36 + 16 = 105$$

$$\|w\| = \sqrt{105} \approx 10.? \checkmark$$

Properties of the Norm

- $\|v\| = 0_{1\times 1} \iff v = 0_{n\times 1}$
- For any $\alpha \in \mathbb{R}$ and $v \in \mathbb{R}^n$

$$\|\alpha v\| = |\alpha| \cdot \|v\|$$

Why? $v \in \mathbb{R}^2$ $\quad \left\|\alpha \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right\|^2 = \left\|\begin{bmatrix} \alpha v_1 \\ \alpha v_2 \end{bmatrix}\right\|^2$

$$= (\alpha v_1)^2 + (\alpha v_2)^2$$
$$= (\alpha)^2 \left[(v_1)^2 + (v_2)^2\right]$$

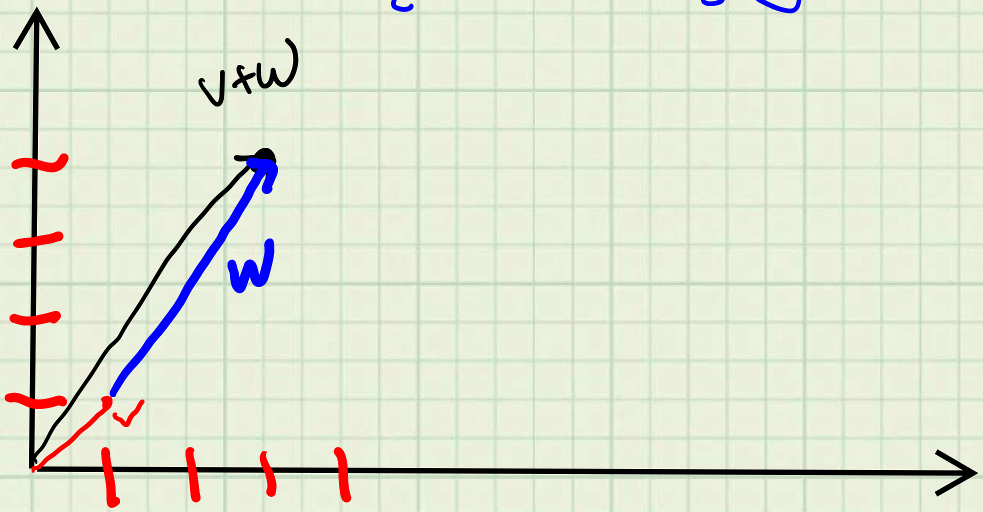
$$\left\|\alpha \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right\| = \sqrt{\alpha^2} \cdot \sqrt{(v_1)^2 + (v_2)^2}$$
$$= |\alpha| \cdot \|v\|$$

- For any pair of vectors $v \in \mathbb{R}^n$ and $w \in \mathbb{R}^n$,

$$\|v+w\| \leq \|v\| + \|w\|$$

This is called the triangle inequality.

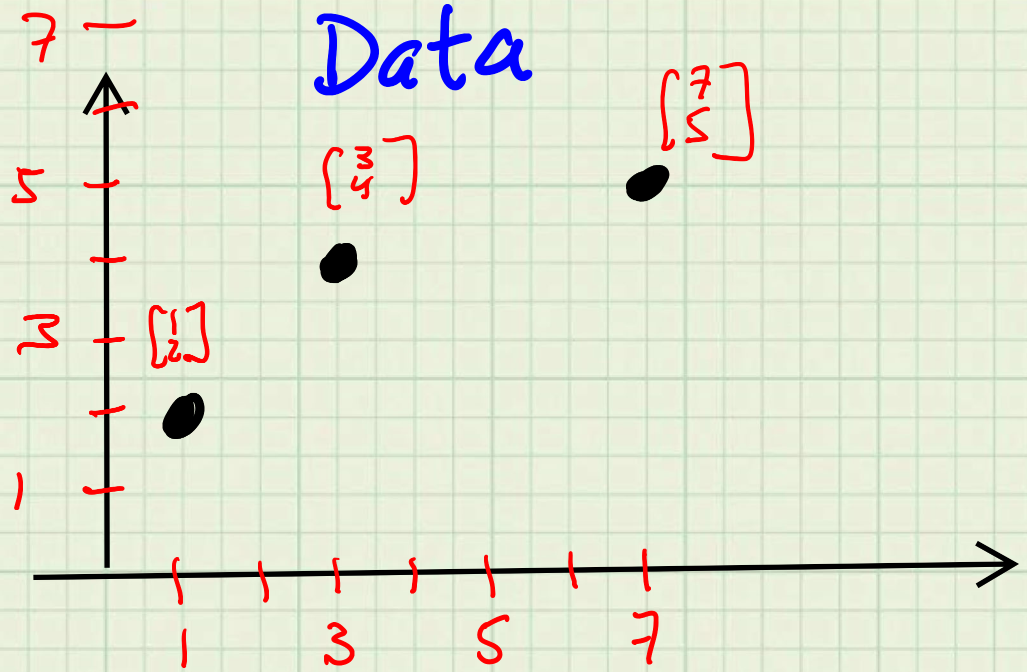
$v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $w = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $v+w = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$



Least Squared Error: Why Should we Care?

Revisit an example of fitting a line to data

Data



Seek a linear model that explains the data

$$y = mx + b$$

$\begin{bmatrix} x_i \\ y_i \end{bmatrix}$ 3 data points

$$y_i = m x_i + b$$

unknowns are $\begin{bmatrix} m \\ b \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} 1 & 1 \\ 3 & 1 \\ 7 & 1 \end{bmatrix}}_{\Phi} \underbrace{\begin{bmatrix} m \\ b \end{bmatrix}}_{\alpha}$$

$$\boxed{Y = \Phi \cdot \alpha}$$

Re-do of Example: Find α s.t. $A\alpha = 0$!

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 3 & 0 & 3 \\ 3 & 1 & 4 \end{bmatrix}$$

Know: $Ax = 0 \iff A^T A x = 0$

Compute $P \cdot (A^T A) = L \cdot U$ $\qquad \therefore PA^TAx = 0 \iff LUx = 0$

$A^TAx = 0 \iff P \cdot A^T A x = 0$ **because** $\det(P) = \pm 1 \Rightarrow$ P^{-1} exists.

$$\therefore PA^TAx = 0 \iff \underbrace{P^{-1}(P)}_{I} (A^TA)x = P^{-1} \cdot 0$$

$$\therefore A^TAx = 0.$$

$L \cdot Ux = 0 \iff Ux = 0$ because $\det(L) = 1$

In our case, $U = \begin{bmatrix} 24 & 6 & 30 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$$\therefore\ Ux = 0_{3\times 1} \iff \begin{aligned} 24x_1 + 6x_2 + 30x_3 &= 0 \\ x_2 + x_3 &= 0 \\ 0\cdot x_3 &= 0 \end{aligned}$$

$\therefore$ x_3 is arbitrary. We select $x_3 = 1$

$\therefore\ x_2 = -x_3 = -1$

$$\therefore\ 24x_1 = -6x_2 - 30x_3$$
$$= -6(-1) - 30(1)$$
$$= -24$$

$\therefore\ x_1 = -1$

$$\therefore\ x = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Let's check that $Ax = 0$.

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 3 & 0 & 3 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \checkmark$$