Summary: $v_k = u_k - \frac{u_k \bullet v_1}{v_1 \bullet v_1} v_1 - \frac{u_k \bullet v_2}{v_2 \bullet v_2} v_2 - \cdots \frac{u_k \bullet v_{k-1}}{v_{k-1} \bullet v_{k-1}} v_{k-1}$

- Gram-Schmidt: $\underbrace{\{u_1, .., u_m\}}_{\text{Linearly Indep}} \xrightarrow{\text{G.S.}} \underbrace{\{v_1, .., v_m\}}_{\text{orthogonal}}$

- $A = n \times m$, $\text{rank}(A) = m = \#$ columns

$$A = Q \cdot R, \quad Q^T \cdot Q = I_m$$

$R = m \times m$ upper triang.
$\det(R) \neq 0$

$A = [u_1 \; u_2 \; \cdots \; u_m]$

$$Q = \left[\frac{v_1}{\|v_1\|} \quad \frac{v_2}{\|v_2\|} \quad \cdots \quad \frac{v_m}{\|v_m\|}\right]$$

Columns of Q are ORTHONORMAL

- $Ax = b$ $\quad A = n \times m$, $\text{rank}(A) = m$, $A = Q \cdot R$

Case 1: $b \in \text{range}(A) = \text{colspan}(A)$

$$Ax = b \iff \boxed{Rx = Q^T b} \iff x = (R^{-1}) \cdot Q^T b$$

Case 2: $b \notin \text{range}(A)$

$$x^* = \underset{x \in \mathbb{R}}{\arg\min} \|Ax - b\|^2 \iff A^T A x^* = A^T b \iff \boxed{Rx^* = Q^T b}$$

Day 1 of ROB 101 (matrix version)

i) $\begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ no solution

ii) $\begin{bmatrix} 3 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$ unique solution

iii) $\begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ -6 \end{bmatrix}$ infinite number of solutions

How to handle $Ax=b$ when it has an infinite number of solutions?

$Ax=b$ $\quad A = n \times m \quad b \in \text{range}(A)$

$\iff$ **A solution exists**

Let $\bar{x}$ be a solution : $A\bar{x} = b$.

Define $\text{null}(A) = \{x \in \mathbb{R}^m \mid Ax = 0\}$

Solution of $Ax=b$ is unique $\iff$

$\text{null}(A) = \{0_{m\times 1}\} \iff \text{nullity}(A) = 0$

$\Leftrightarrow$ rank$(A) = m$ #columns

Note $\tilde{x} \in \text{null}(A)$, then $\bar{x} + \tilde{x}$ is still a solution of the equation $Ax = b$!!

$$A(\bar{x}+\tilde{x}) = A\bar{x} + \cancel{A\tilde{x}}^{\,0} = b$$

Question: If nullity$(A) > 0$, how to choose among all of the solutions to $Ax = b$?

Suggestion: $$x^* = \underset{Ax=b}{\arg\min} \|x\| = \underset{Ax=b}{\arg\min} \|x\|^2$$

Among all solutions, choose the one of least squared norm.

Constrained optimization problem:

minimize $\|x\|^2$ subject to $Ax = b$

$Ax = b$ is a constraint on $x \in \mathbb{R}^m$

Fact: Suppose the columns of A^T are linearly indep. (rows of A are linearly indep). Then

$$x^* = \underset{Ax=b}{\arg\min} \|x\|^2 \iff x^* = A^T \cdot (A \cdot A^T)^{-1} \cdot b$$

$$\iff x^* = A^T \cdot \alpha \quad \& \quad (A \cdot A^T)\alpha = b$$

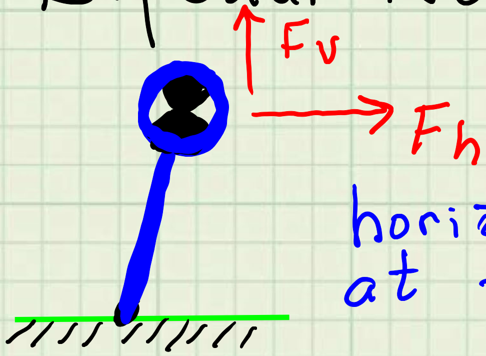
2 1

Book: Applies QR. Do that on your own!

$\boxed{\text{Insert Cassie}}$

Example $\quad x^* = \arg\min_{Ax=b} \|x\|^2$

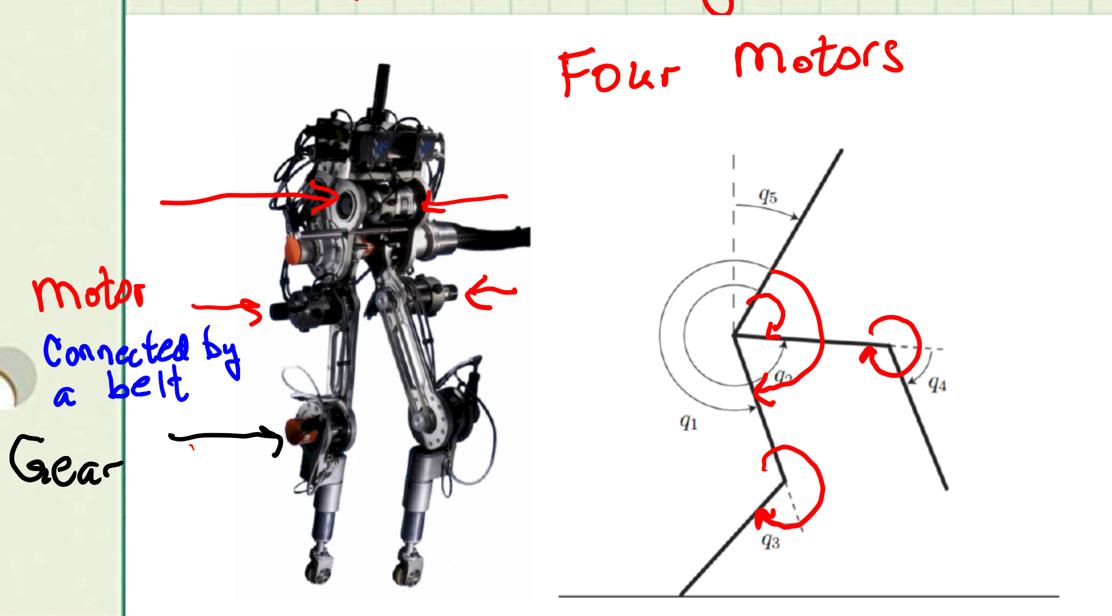
Bipedal Robot: Walking $\leftrightarrow$ {Regulating Forces Applied by Feet}



horizontal and vertical forces at the robot's center of mass

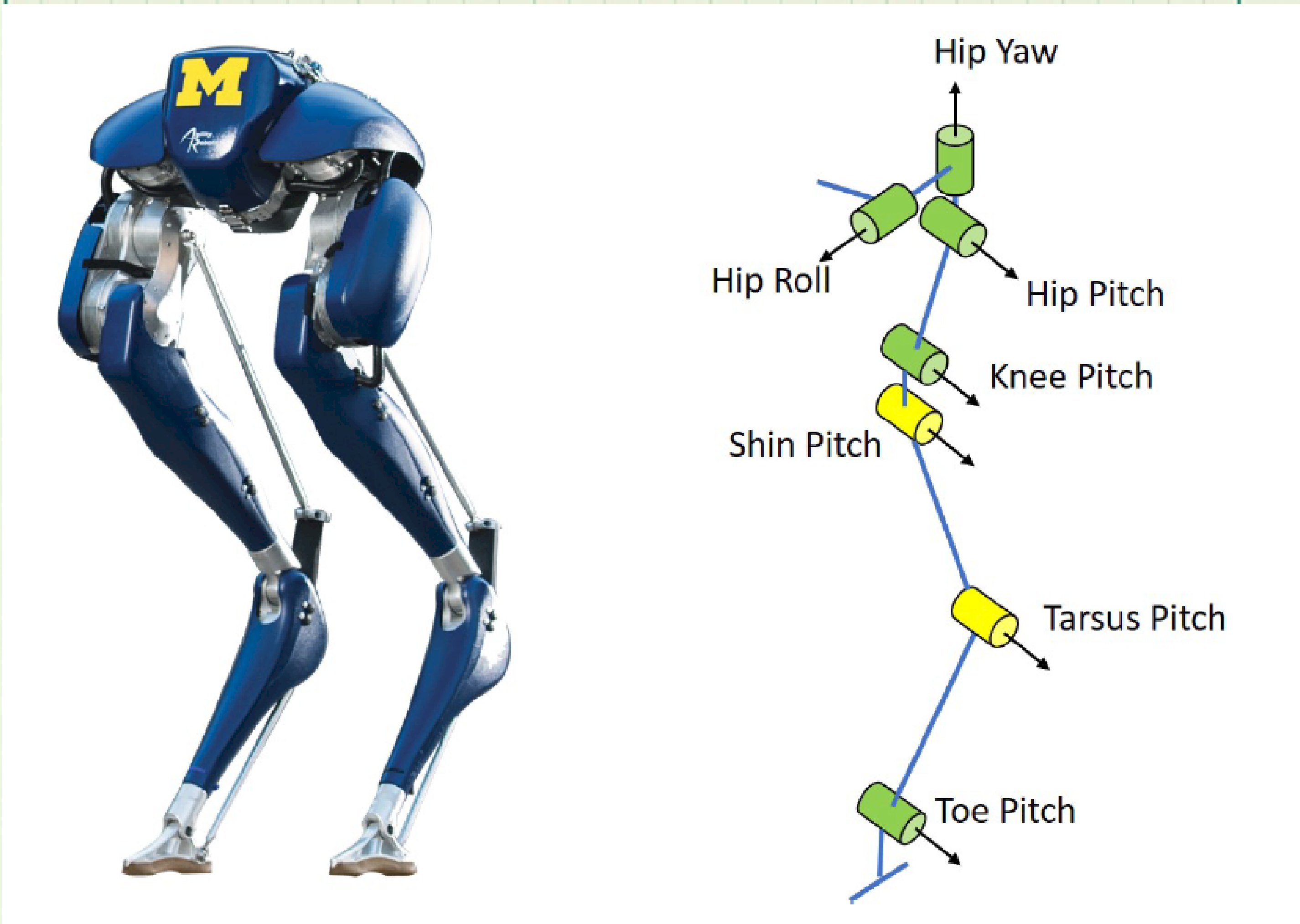
Physics: Equal & Opposite Forces at the Ground Contact Point

Generate forces by applying torques at joints via motors



Rabbit

TEN MOTORS



Each "green can" is a "motor"

Physics: $\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}_{foot} = A_q \cdot \begin{bmatrix} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_{10} \end{bmatrix}$

where $F_\bullet$ = Force component

τ_i = torque i

q = vector joint angles

A_q = Matrix relating motor torques to ground reaction forces

$$\boxed{\begin{aligned} \tau^* = \arg\min \ & \|\tau\|^2 \\ & A_q \tau = F \end{aligned}}$$

Physical Reality: $lb_i \leq \tau_i \leq ub_i$

lb = lower bound on torque ≈ -7 Nm
ub = upper bound on torque $\approx +7$ Nm

Real Problem

$$\left.\begin{aligned} \tau^* = \arg\min \ & \|\tau\|^2 \\ A_q \tau &= F \\ \tau &\leq ub \\ -\tau &\leq -lb \end{aligned}\right\}$$

Quadratic Program

OR

QP for short

Chapter 10: Solving Nonlinear Equations.

Learning Objectives

- Extend our horizons from linear equations to nonlinear equations.
- Appreciate the power of using algorithms to iteratively construct approximate solutions to a problem.
- Accomplish all of this without assuming a background in Calculus.

Outcomes

- Learn that a root is a solution of an equation of the form $f(x) = 0$.
- Learn two methods for finding roots of real-valued functions of a real variable, that is for $f : \mathbb{R} \to \mathbb{R}$, namely the Bisection Method and Newton's Method
- Become comfortable with the notion of a "local slope" of a function at a point and how to compute it numerically.
- Linear approximations of nonlinear functions.
- Extensions of these ideas to vector-valued functions of several variables, that is $f : \mathbb{R}^m \to \mathbb{R}^n$, with key notions being the gradient and Jacobian of a function and their use in the Newton-Raphson Algorithm.

Let $f: \mathbb{R}^m \to \mathbb{R}^n$ be a function.

Then $\boxed{f(x) = 0}$ is an equation.

equals sign

A solution to $f(x) = 0$ is called a root. x^* is a root $\iff f(x^*) = 0$.

Suppose $f(x) = c$ is the equation for $c \in \mathbb{R}^n$, a constant vector.

Is $f(x^*) = c$ a root of the equation?

NO. Technically not. It is a root of $\bar{f}(x) = 0$, where $\bar{f}(x) := f(x) - c$.

Most engineers could care less!!

Now, we are looking at $f: \mathbb{R}^m \to \mathbb{R}^n$, so we are working with real numbers.

What about $x^2 + 1 = 0$? $x \in \mathbb{R}$

$x^2 + 1 = 0 \Leftrightarrow x^2 = -1 \Leftrightarrow$ imaginary answers allowed.

$x^2 + 1$ does not have a root.