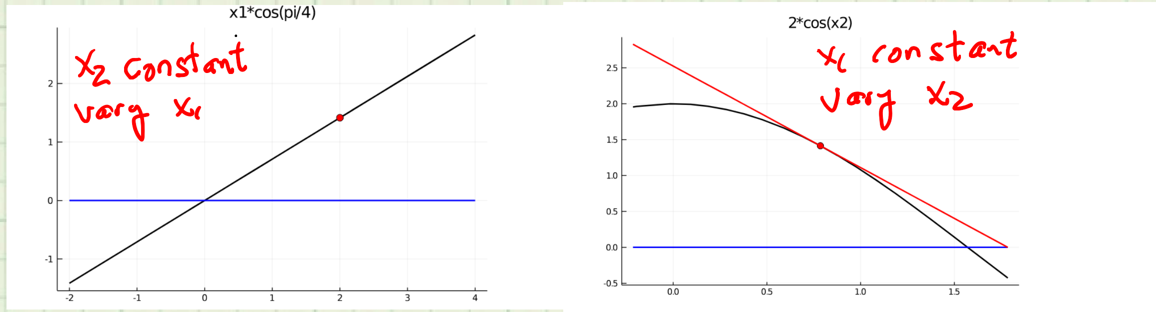
Where we are headed: roots of $f: \mathbb{R}^n \to \mathbb{R}^n$, that is n nonlinear equations in n unknowns! $f(x) \approx f(x_0) + A(x - x_0)$

$$0 = f(x_{k+1}) \approx f(x_k) + A(x_{k+1} - x_k) \Rightarrow x_{k+1} = x_k - A^{-1} f(x_k)$$

What is A?

Review: $f: \mathbb{R}^m \to \mathbb{R}$ "Partial derivatives"



$$\boxed{f(x_1, x_2) = x_1 \cos(x_2)}, \quad x_0 = \begin{bmatrix} 2 \\ \pi/4 \end{bmatrix}$$

$\frac{\partial}{\partial x_i}$ = partial derivative with respect to x_i = hold all x_j, $j \neq i$, constant and vary only x_i to compute a "slope" at a point, say $x_0 = [x_{01}; x_{02}; \cdots; x_{0m}]$

$$\frac{\partial f}{\partial x_i}(x_0) \approx \frac{f(x_0 + h e_i) - f(x_0)}{h} \quad \text{forward diff.}$$

$$\frac{\partial f}{\partial x_i}(x_0) \approx \frac{f(x_0 + h e_i) - f(x_0 - h e_i)}{2h} \quad \text{symmetric diff.}$$

Suppose $f: \mathbb{R}^3 \to \mathbb{R}$, so $f(x) = f(x_1, x_2, x_3)$, then

$$\frac{\partial f}{\partial x_2}(x_{01}, x_{02}, x_{03}) \approx \frac{f(x_{01}, x_{02} + h, x_{03}) - f(x_{01}, x_{02}, x_{03})}{h}$$

$$x_0 + he_2 = \begin{bmatrix} x01 \\ x02 \\ x03 \end{bmatrix} + h \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x01 \\ x02 + h \\ x03 \end{bmatrix}$$

Above underlines what the vector notation "$x_0 + he_2$" means in terms of components "$\begin{bmatrix} x01 \\ x02+h \\ x03 \end{bmatrix}$".

In a for loop, $e_i = Id[:,i]$, if you define **Id = zeros(n,n) + I** in Julia.

"reserved variable"

Linear Approx at a point x_0:

$$f(x) \approx f(x_0) + [a_1 \ a_2 \cdots a_m] \begin{bmatrix} x_1 - x01 \\ x_2 - x02 \\ \vdots \\ x_m - x0m \end{bmatrix}$$

$$\Updownarrow$$

$$a_i = \frac{\partial f(x_0)}{\partial x_i}$$

TODAY

Def. The gradient of $f: \mathbb{R}^m \to \mathbb{R}$ at $x_0 \in \mathbb{R}^m$ is

$$\nabla f(x_0) := \left[\frac{\partial f}{\partial x_1}(x_0) \;\; \frac{\partial f}{\partial x_2}(x_0) \; \cdots \; \frac{\partial f}{\partial x_m}(x_0)\right]_{1\times m}$$

∇ is pronounced "grad" and one says grad of f or grad-f for ∇f

Linear Approx of f at x_0

$$f(x) \approx \underbrace{f(x_0)}_{1\times 1} + \underbrace{\nabla f(x_0)}_{1\times m} \cdot \underbrace{(x - x_0)}_{m\times 1} \quad \in \mathbb{R}$$

$$= f(x_0) + \left[\frac{\partial f}{\partial x_1}(x_0) \;\; \frac{\partial f}{\partial x_2}(x_0) \cdots \frac{\partial f}{\partial x_m}(x_0)\right] \begin{bmatrix} x_1 - x_{01} \\ x_2 - x_{02} \\ \vdots \\ x_m - x_{0m} \end{bmatrix}$$

Compute a linear approx. of $f(x) = x_1 \cos(x_2)$ at the point $x_0 = \begin{bmatrix} 2 \\ \pi/4 \end{bmatrix}$

$$f(x) \approx f(x_0) + \nabla f(x_0)\,(x - x_0)$$

$$= f(x_0) + \left[\frac{\partial f}{\partial x_1}(x_0) \;\; \frac{\partial f}{\partial x_2}(x_0)\right] \begin{bmatrix} x_1 - x_{01} \\ x_2 - x_{02} \end{bmatrix}$$

$$= 2\cos(\pi/4) + \begin{bmatrix} \frac{\sqrt{2}}{2} & -\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 - 2 \\ x_2 - \pi/4 \end{bmatrix}$$

$$\boxed{Z = C_1 X_1 + C_2 X_2 + C_0}$$

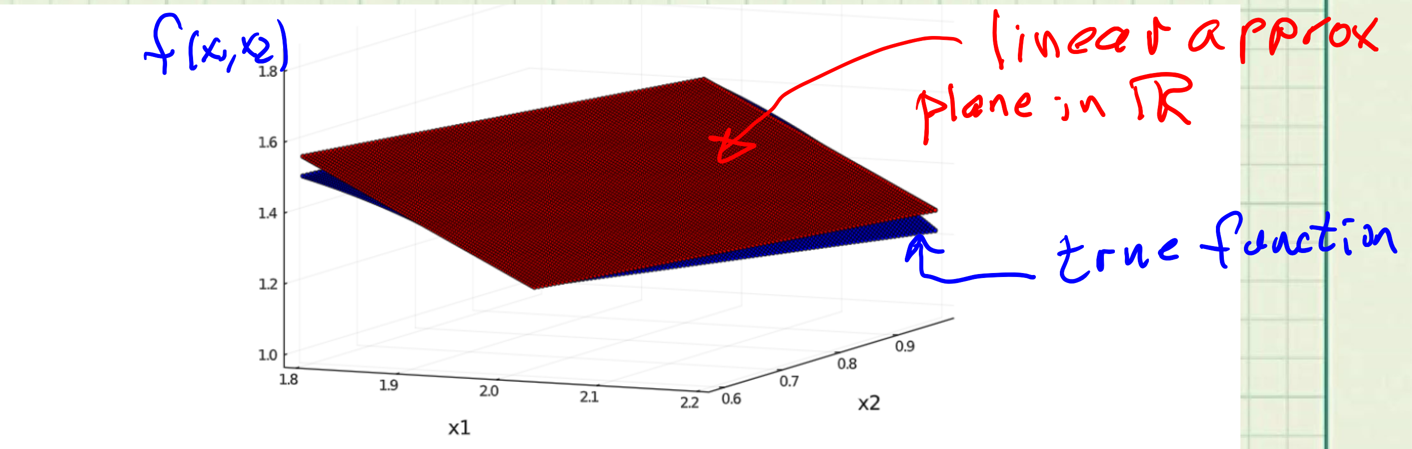


Figure 9: The function $f(x_1, x_2) = x_1 \cos(x_2)$ is plotted in blue, while in red is shown its linear approximation about the point $x_0 = [2 \;\; \pi/4]^\top$, that is, $f_{\text{lin}}(x) := f(x_0) + \nabla f(x_0)(x - x_0)$. This approximation can be done for any point x_0 at which the partial derivatives exists. In Calculus, the red plane is also called the **tangent plane at** $\mathbf{x_0}$. These linear approximations accurately represent the nonlinear function in a small enough region about a given point, allowing us to use our Linear Algebra skills.

```
f(x1,x2)=x1*cos(x2)
x0=[2;pi/4]
f1(x1)=f(x1,x0[2])
f2(x2)=f(x0[1],x2)
yzero(x)=0*f1(x)
titre="x1*cos(pi/4)"
p1=plot(f1,-2,4,linewidth=2,color=:black,legend=false,title=titre)
plot!(yzero,-2,4,linewidth=2,color=:blue)
scatter!([2],[f1(2)] , markersize=5, color=:red)
display(p1)
titre="2*cos(x2)"
h=0.01
dfdx2=(f2(pi/4+h)-f2(pi/4-h) )/(2*h)
linapprox(x2)=f2(pi/4)+dfdx2*(x2-pi/4)
p2=plot(f2,pi/4-1,pi/4+1,linewidth=2,color=:black,legend=false,title=titre)
plot!(yzero,pi/4-1,pi/4+1,linewidth=2,color=:blue)
scatter!([pi/4],[f2(pi/4)] , markersize=5, color=:red)
plot!(linapprox,pi/4-1,pi/4+1,linewidth=2,color=:red)
display(p2)
pi/4
```

General functions $f:\mathbb{R}^m \to \mathbb{R}^n$ and the Jacobian !!!

$$f:\mathbb{R}^m \to \mathbb{R}^n \iff \begin{bmatrix} f_1(x_1, \ldots, x_m) \\ f_2(x_1, \ldots, x_m) \\ \vdots \\ f_n(x_1, \ldots, x_m) \end{bmatrix}$$

We seek a linear approx at a point $x_0 \in \mathbb{R}^m$:

$$f(x) = \underbrace{f(x_0)}_{n\times 1} + \underbrace{A}_{n\times m} \underbrace{(x - x_0)}_{m\times 1} \in \mathbb{R}^n$$

$$= f(x_0) + \left[a_1^{col} \; a_2^{col} \cdots a_m^{col}\right](x - x_0)$$

How should we define a_i^{col} so that we have a linear approximation to f at the point x_0?

Just as we did for $f: \mathbb{R}^m \to \mathbb{R}$, we let $x = x_0 + he_i$. Hence

$x - x_0 = he_i$

Let's write $e_i[j] = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

and use our sum over columns times rows definition of matrix multiplication:

$$\left[a_1^{col} \; a_2^{col} \; \cdots \; a_m^{col}\right] he_i = h a_i^{col}$$

$$\boxed{f(x_0 + he_i) \approx f(x_0) + h a_i^{col}}$$

$$a_i^{col} \approx \frac{f(x_0 + he_i) - f(x_0)}{h} \xrightarrow[h \to 0]{} \frac{\partial f(x_0)}{\partial x_i}$$

Def. Let $f: \mathbb{R}^m \to \mathbb{R}^n$. The Jacobian of f at x_0 is

$$\frac{\partial f}{\partial x}(x_0) := \underbrace{\begin{bmatrix} \frac{\partial f}{\partial x_1}(x_0) & \frac{\partial f}{\partial x_2}(x_0) & \cdots & \frac{\partial f}{\partial x_m}(x_0) \end{bmatrix}}_{n \times m}$$

where $$\frac{\partial f}{\partial x_i}(x_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_i}(x_0) \\ \frac{\partial f_2}{\partial x_i}(x_0) \\ \vdots \\ \frac{\partial f_n}{\partial x_i}(x_0) \end{bmatrix}$$

HW#9 (Short Julia) introduces you to symbolic mathematics where a computer can actually manipulate variables and compute derivatives.

Linear Approx $f : \mathbb{R}^m \to \mathbb{R}^n$ at x_0 is

$$\boxed{\underbrace{f(x)}_{n\times 1} \approx f(x_0) + \underbrace{\frac{\partial f}{\partial x}(x_0)}_{n \times m} \underbrace{[x - x_0]}_{m \times 1}}$$

Compute the Linear Approx

of $$f(x_1, x_2, x_3) = \begin{bmatrix} x_1 x_2 x_3 \\ \log(2 + \cos(x_1)) + (x_2)^{x_1} \\ \dfrac{x_1 x_3}{1 + (x_2)^2} \end{bmatrix}$$

at the point $x_0 = \begin{bmatrix} \pi \\ 1 \\ 2 \end{bmatrix}$.