1993 Higher School Certificate Solutions

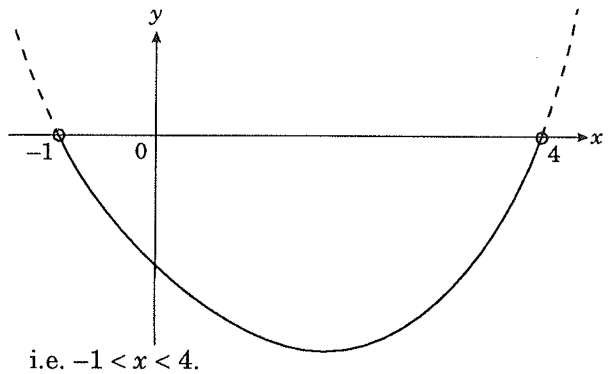
3/4 UNIT MATHEMATICS

QUESTION ONE

(a) $x^2-3x<4 \qquad x^2-3x-4<0$

$(x-4)(x+1)<0.$

Consider values of x for which $y=(x-4)(x+1)$ is *below* the x axis.



i.e. $-1<x<4$.

(b) $$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x\ dx = \left[\tan x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$
$$= \tan\frac{\pi}{3} - \tan\frac{\pi}{6}$$
$$= \sqrt{3} - \frac{1}{\sqrt{3}}$$
$$= \frac{2}{\sqrt{3}} \text{ or } \frac{2\sqrt{3}}{3}.$$

(c) $u=2t-1 \Rightarrow \dfrac{du}{dt}=2 \qquad$ (i.e. $du=2\,dt$).

$t=\frac{1}{2},\quad u=0.$

$t=1,\quad u=1.$

$u=2t-1 \Rightarrow t=\frac{1}{2}(u+1)$

$$\therefore \int_{\frac{1}{2}}^{1} 4t\left(2t-1\right)^5\ dt = \int_0^1 2(u+1)u^5.\frac{1}{2}\ du$$
$$= \int_0^1 (u^6+u^5)\ du$$
$$= \left[\frac{1}{7}u^7+\frac{1}{6}u^6\right]_0^1$$
$$= \left(\frac{1}{7}+\frac{1}{6}\right)-(0+0) = \frac{13}{42}.$$

(d) $y=\cos(\ln x)$.

(i) $$\frac{dy}{dx} = -\sin(\ln x)\times\frac{1}{x} = \frac{-\sin(\ln x)}{x}.$$

(ii) $$\frac{d^2y}{dx^2} = \frac{x\times\dfrac{-\cos(\ln x)}{x}+\sin(\ln x)}{x^2} = \frac{\sin(\ln x)-\cos(\ln x)}{x^2}.$$

(e) Number of ways $= {}^{10}C_3\times{}^{12}C_2$

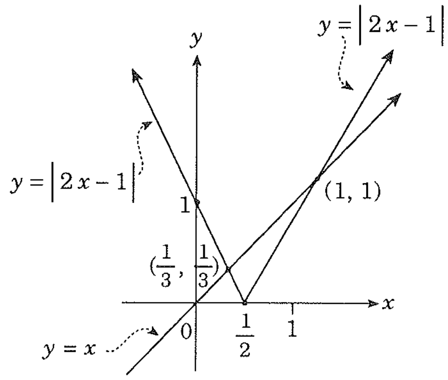
$= 120\times 66$

$= 7920$ ways.

QUESTION TWO

(a) LHS
$$= \frac{\sin A(\cos A-\sin A)+\sin A(\cos A+\sin A)}{(\cos A+\sin A)(\cos A-\sin A)}$$
$$= \frac{2\sin A\cos A}{\cos^2 A-\sin^2 A}$$
$$= \frac{\sin 2A}{\cos 2A}$$
$$= \tan 2A$$
$$= \text{RHS}.$$

(b) (i)



The lines meet at $(1, 1)$ and $\left(\frac{1}{3}, \frac{1}{3}\right)$,

[found by solving $y=x,\ y=-(2x-1)$].

(ii) Using (i):

$$y=\left|2x-1\right| \text{ and } y=x-\frac{1}{2}$$

would meet at $\left(\frac{1}{2},\ 0\right)$ only,

i.e. $c=-\frac{1}{2}$ gives only one solution,

$c<-\frac{1}{2}$ gives no solutions,

i.e. $c>-\frac{1}{2}$ gives exactly two solutions.

Or otherwise:

Using the fact that $\left|z\right|=+\sqrt{z^2}$:

$\left|2x-1\right|=x+c$

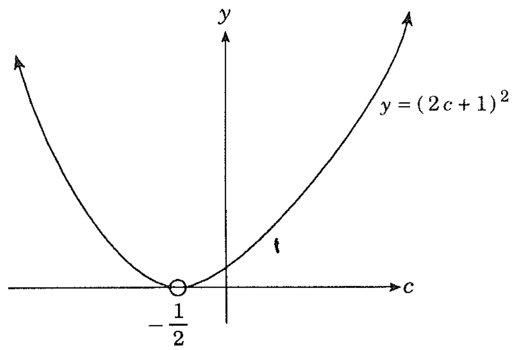
$$\Rightarrow \quad \sqrt{(2x-1)^2}=x+c$$
$$(2x-1)^2=(x+c)^2$$
$$4x^2-4x+1=x^2+2cx+c^2$$
$$3x^2-2(2+c)x+1-c^2=0.$$

This has exactly two solutions for $\Delta > 0$,

i.e. $4(2+c)^2-12(1-c^2)>0.$

i.e. $\quad 4c^2+4c+1>0$

$$(2c+1)^2>0.$$



From the graph it can be seen that $y=(2c+1)^2$ has positive values for y everywhere except at $-\frac{1}{2}$,

i.e. $\quad (2c+1)^2>0$

$$c>-\frac{1}{2}.$$

(c) $f(x)=x^3+ax^2+bx+c.$

(i) $f'(x)=3x^2+2ax+b.$

Relative max. at $x=\alpha$ means $f'(\alpha)=0$,

i.e. $\quad 3\alpha^2+2a\alpha+b=0.$ —①

Similarly, relative min. at $x=\beta$ means that $f'(\beta)=0$,

i.e. $\quad 3\beta^2+2a\beta+b=0.$ —②

① − ②:

$$3(\alpha^2-\beta^2)+2a(\alpha-\beta)=0$$
$$3(\alpha-\beta)(\alpha+\beta)+2a(\alpha-\beta)=0,$$

$\therefore$ $\alpha\neq\beta$ as a point cannot be both a max. and a min.

$\div\ (\alpha-\beta)$:

$$3(\alpha+\beta)=-2a$$

$$\therefore \quad \alpha+\beta=-\frac{2}{3}a.$$

(ii) $f''(x)=6x+2a.$

Because there is a max. at $x=\alpha$ and a min. at $x=\beta$, there *will* be a point of inflexion where $f''(x)=0$,

i.e. at $6x+2a=0$,

$\therefore \quad x=-\frac{a}{3}.$ —①

From (i), $\alpha+\beta=-\frac{2a}{3}$,

$\therefore \quad \frac{\alpha+\beta}{2}=-\frac{a}{3}.$ —②

Equating ① and ②: $x=\frac{\alpha+\beta}{2}.$

Also, since $x=\frac{\alpha+\beta}{2}=-\frac{a}{3}$,

if $\quad x=-\frac{a^-}{3},\quad f''(x)=6x+2a<0$

$\quad x=-\frac{a^+}{3},\quad f''(x)=6x+2a>0.$

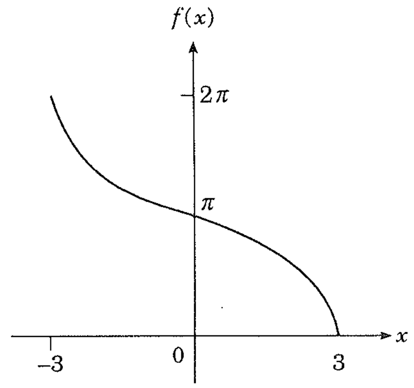
So there is a change of concavity.

QUESTION THREE

(a) (i) $f(0)=2\cos^{-1}\frac{0}{3}$

$$=2\times\frac{\pi}{2}$$
$$=\pi.$$

(ii)



(iii) Domain: $\left\{x:\quad -3\le x\le 3\right\}.$

Range: $\left\{f(x):\quad 0\le 2\cos^{-1}\frac{x}{3}\le 2\pi\right\}.$

(b) $P(x)=(x-1)(x+1)Q(x)+3x-1.$

$$\frac{P(x)}{x-1}=(x+1)Q(x)+\frac{3x-1}{x-1}$$
$$=(x+1)Q(x)+\frac{3(x-1)}{x-1}+\frac{2}{x-1}$$

$$= (x+1)Q(x) + \frac{2}{x-1},$$

$$\therefore \; P(x) = \left\{(x+1)Q(x)+3\right\}x - 1 + 2.$$

Remainder is 2.

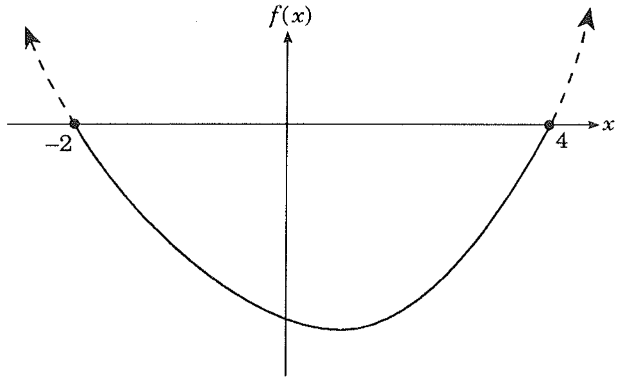
OR

$$P(x) = (x-1)(x+1)Q(x) + 3x - 1.$$

By Remainder Theorem, since $x - 1$ is a linear term, $P(1) = 2$, $\therefore$ remainder is 2.

(c) (i) $v^2 \geq 0$,

$$\therefore \; 8 + 2x - x^2 \geq 0$$
$$x^2 - 2x - 8 \leq 0$$
$$(x-4)(x+2) \leq 0.$$



i.e. $-2 \leq x \leq 4$.

$\therefore$ The particle oscillates between $x = -2$ and $x = 4$.

(ii) Distance between $x = -2$ and $x = 4$ is 6 units.

$$\therefore \text{ Amplitude } = \frac{1}{2} \times 6 \text{ units}$$
$$= 3 \text{ units.}$$

(iii)
$$\frac{1}{2}v^2 = 4 + x - \frac{1}{2}x^2.$$
$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 1 - x,$$

i.e. $\text{accel.} = 1 - x$
$$= \ddot{x}.$$

(iv) Observe that the point oscillates between -2 and 6, so $+1$ is the centre of the motion.

$$\ddot{x} = 1 - x$$
$$= -1(x-1).$$

This is in the form

$$\ddot{X} = -n^2 X \text{ where } X = x - 1,$$

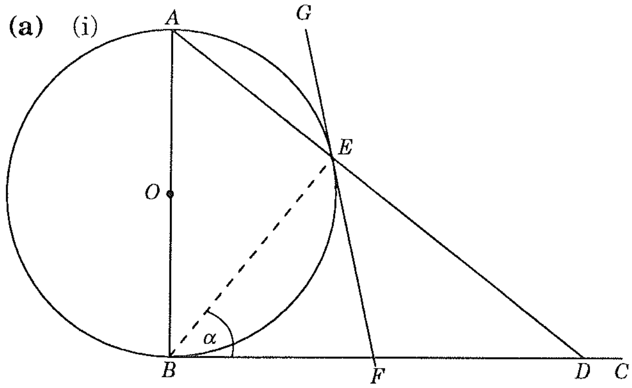
i.e. $n^2 = 1$

$\therefore$ $n = 1$ (since $n > 0$).

$$\therefore \text{ Period } = \frac{2\pi}{n}$$
$$= 2\pi.$$

QUESTION FOUR

(a) (i)



(ii) $\angle BAE = \alpha$ (alternate segment)

$\therefore \angle BEF = \alpha$ (angle in alternate segment equals angle between tangent GF and chord BE).

$\angle AEB = 90°$ (angle in semicircle)
$$= \frac{\pi}{2}$$

$\therefore \angle BED = \frac{\pi}{2}$ (AD a st. line)

$\therefore \angle DEF = \angle BED - \angle BEF$
$$= \frac{\pi}{2} - \alpha \quad \text{(adj. compl. } \angle\text{s).}$$

OR

$A\hat{E}B = \frac{\pi}{2}$ (angle in semicircle)

$\therefore \; B\hat{E}D = \frac{\pi}{2}$ (st. line)

$FE = FB$ (tangents from a point)

$B\hat{E}F = \alpha$ (isosceles ΔBEF)

$F\hat{E}D = B\hat{F}D - B\hat{E}F$
$$= \frac{\pi}{2} - \alpha.$$

(iii) ΔBEF is isosceles since

$\angle BEF = \angle EBF$ (from (i))
$$= \alpha$$

$\therefore BF = EF$ (sides oppos. equal angles).

In ΔABD,

$\angle ABD = \frac{\pi}{2}$ (radius $\perp$ tangent)

$\therefore \angle ADB = \frac{\pi}{2} - \alpha$ (angle sum Δ)

$\therefore \Delta EFD$ is isosceles

since $\angle FED = \angle EDF = \frac{\pi}{2} - \alpha$.

$\therefore$ $EF = FD$ (angles oppos. equal sides)

$\therefore$ $BF = FD$ (both equal EF).

OR

$BF = EF$ (tangents at a point) —①

$E\hat{F}D = \alpha + \alpha$ (external angle of ΔFEB)
$$= 2\alpha.$$

$$E\hat{F}D + F\hat{E}D + E\hat{D}F = \pi \quad (\text{angle sum } \Delta FED)$$

$$2\alpha + \frac{\pi}{2} - \alpha + E\hat{D}F = \pi$$

$\therefore$ $$E\hat{D}F = \frac{\pi}{2} - \alpha,$$

$\therefore$ ΔFED is isosceles

$\therefore$ $$FD = FE,$$

but $$FE = BF \ \left(\text{from (i)}\right)$$

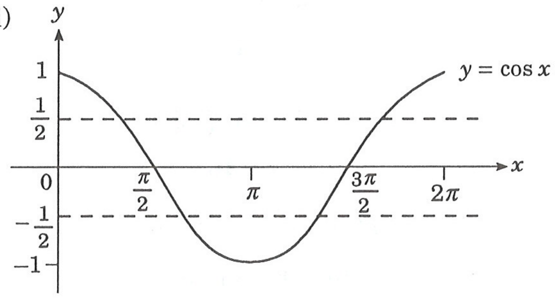
$\therefore$ $$FD = BF.$$

(b) (i) 5^5 ways

(ii) 5! ways

(iii) Husband and wife have a choice of 5 hotels.
$\therefore$ Other three can choose from *any* of 4 remaining hotels—that is, in 4^3 ways.
$\therefore$ There will be $\cancel{(5 \times 4^3)}$ different accommodation arrangements. $\times(5\times 3^4)$

(c) (i)



(ii) Limiting sum only exists if $|r| < 1$, where $r = 2\cos x$.

That is, $$-1 < 2\cos x < 1$$
$$-\frac{1}{2} < \cos x < \frac{1}{2}.$$

From graph in (i),
$$-\frac{1}{2} < \cos x < \frac{1}{2}, \text{ for}$$
$$\frac{\pi}{3} < x < \frac{2\pi}{3} \quad \text{and}$$
$$\frac{4\pi}{3} < x < \frac{5\pi}{3}.$$

QUESTION FIVE

(a) (i) For $n = 1$, $\text{LHS} = 1^2 = 1$.
$$\text{RHS} = \frac{1}{6} \times 1 \times 2 \times 3$$
$$= 1 \quad = \text{LHS}.$$

$\therefore$ Holds for $n = 1$.
Let S_k be true, where k is a positive integer. Hence prove, for $n = k + 1$, that S_{k+1} is true, i.e. prove:
$$1^2 + 2^2 + \cdots + k^2 + (k+1)^2$$
$$= \frac{1}{6}(k+1)(k+2)(2k+3).$$

$$\begin{aligned}\text{LHS} &= S_k + (k+1)^2 \\ &= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 \\ &= \frac{1}{6}(k+1)\left[k(2k+1) + 6(k+1)\right] \\ &= \frac{1}{6}(k+1)(2k^2 + 7k + 6) \\ &= \frac{1}{6}(k+1)(k+2)(2k+3) \\ &= \text{RHS}.\end{aligned}$$

$\therefore$ True for $n = k + 1$.
$\therefore$ True for $n = 1$, $n = 1 + 1 = 2$, $n = 2 + 1 = 3$, and so on for all positive integral values of n.

(ii) $$S_n = \frac{1}{6}n(n+1)(2n+1)$$
$$> \frac{1}{6} \cdot n \cdot n \cdot 2n$$

$\therefore$ $$S_n > \frac{1}{3}n^3.$$

For $$\frac{1}{3}n^3 = \left(3^{-\frac{1}{3}}n\right)^3.$$

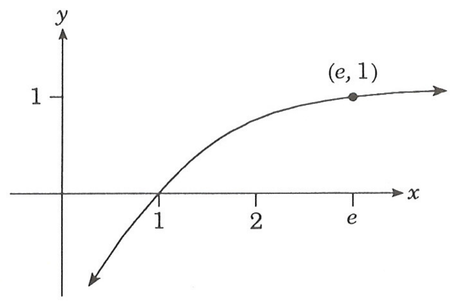
For $$\left(3^{-\frac{1}{3}}n\right)^3 = 10^9$$
$$= (10^3)^3$$
$$3^{-\frac{1}{3}} . n = 10^3$$
$$n = 3^{\frac{1}{3}} \times 10^3$$
$$\approx 1422 \cdot 2.$$

Estimate for n is 1422 (allowing 0·2 for bit extra which was dropped off).
Note For $n = 1422$,
$$S_n = 1 \cdot 000\,52 \times 10^9$$
which is very close.
$$n = 1441, \quad S_n = 9 \cdot 98.$$
Estimation works well!

(b) (i) $y = \ln x$, $x > 0$. $$\frac{dy}{dx} = \frac{1}{x}$$
$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}.$$

$x^2 > 0$ for all x $(x \neq 0)$ and $-\dfrac{1}{x^2}$ is thus always negative, i.e. $\dfrac{d^2y}{dx^2} < 0$,
i.e. $y = \ln x$ is concave down.

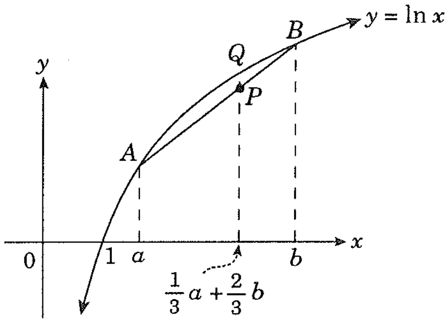
(ii)



(iii) $A(a,\ \ln a)$ and $B(b,\ \ln b)$.
Let P have coordinates (x, y).

$$\therefore\ (x,\ y)=\left(\frac{a+2b}{3},\ \frac{\ln a+2\ln b}{3}\right)$$
$$=\left(\frac{1}{3}a+\frac{2}{3}b,\ \frac{1}{3}\ln a+\frac{2}{3}\ln b\right).$$

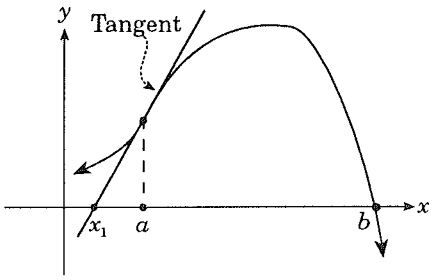
(iv)



From (iii), P, which lies on line segment AB, has a y coord. of $\frac{1}{3}\ln a+\frac{2}{3}\ln b$.
Because $y=\ln x$ is concave down, and increasing, this means that the y coord. of Q, the point on $y=\ln x$ with x coord. of $\frac{1}{3}a+\frac{2}{3}b$, will have a y coordinate of $\ln\left(\frac{1}{3}a+\frac{2}{3}b\right)$, which is above the y coord. of P,

i.e. $\ln\left(\frac{1}{3}a+\frac{2}{3}b\right)>\frac{1}{3}\ln a+\frac{2}{3}\ln b$.

(c)



If $x_0=a$, x_1 is as shown, Newton's method finds where the tangent at $x_0=a$ cuts the x axis. x_1 is this point. It is clearly not a better approximation.

In general, if there is a stationary point between the solution to $f(x)=0$ and the approximation x_0, then x_1 by Newton's method will not be a better approximation.

QUESTION SIX

(a) (i) $$T=A+Ce^{kt}, \qquad \text{—①}$$
$$\therefore\quad Ce^{kt}=T-A. \qquad \text{—②}$$

Differentiate T (in ①) w.r.t. t.

$$\therefore\quad \frac{dT}{dt}=k\times Ce^{kt}$$
$$=k(T-A) \qquad \text{from②},$$
$$\text{i.e.}\quad \frac{dT}{dt}\propto(T-A).$$

(ii) $A=5 \qquad t=0,\ T=20$

$$\therefore\quad 20=5+Ce^{0}$$
$$\therefore\quad C=15$$
$$\therefore\quad T=5+15e^{kt}. \qquad \text{—③}$$

$t=\frac{1}{2},\ T=17$

$$\therefore\quad 17=5+15e^{\frac{1}{2}k}$$
$$12=15e^{\frac{1}{2}k}$$
$$e^{\frac{1}{2}k}=0{\cdot}8$$
$$\frac{1}{2}k=\ln 0{\cdot}8 \qquad \left[\begin{array}{l} k=2\ln 0{\cdot}8 \\ \quad\approx -0{\cdot}4463 \end{array}\right]$$
$$k=2\ln 0{\cdot}8.$$

$T=10,\ t=?$

$$10=5+15e^{kt} \qquad \text{from ③},$$
$$5=15e^{kt}$$
$$\frac{1}{3}=e^{kt}$$
$$kt=-\ln 3$$
$$t=\frac{-\ln 3}{2\ln 0{\cdot}8}$$
$$t\approx 2{\cdot}46.$$

i.e. after approx. $2\frac{1}{2}$ h.
(Actually 2 h 28 min.)

(b) (i) $$\frac{AP}{AB}=\sec\theta,$$
$$\therefore\quad AP=d\sec\theta.$$

$\therefore$ Resistance to flow in AP is $c_2 d\sec\theta$, where c_2 is a constant of proportionality.

$$\frac{PB}{AB}=\tan\theta$$
$$\therefore\quad PB=d\tan\theta$$
$$\therefore\quad OP=OB-PB$$
$$=l-d\tan\theta.$$

$\therefore$ Resistance to flow in OP is $c_1(l-d\tan\theta)$, where c_1 is a constant of proportionality.

$$\therefore\ R=c_1\left(l-d\tan\theta\right)+c_2 d\sec\theta.$$

(ii) $$\frac{dR}{d\theta}=c_1(-d\sec^2\theta)+c_2 d\sec\theta\tan\theta$$
$$=0 \text{ for max. or min.}$$
$$\therefore\quad c_2 d\sec\theta\tan\theta-c_1 d\sec^2\theta=0$$
$$d\sec\theta(c_2\tan\theta-c_1\sec\theta)=0$$
$$\therefore\ \underbrace{\sec\theta=0}_{\Uparrow\ \text{impossible}} \text{ or } c_2\tan\theta-c_1\sec\theta=0$$
$$\therefore\ \frac{c_2}{c_1}\tan\theta-\sec\theta=0$$
$$\therefore\quad 2\tan\theta-\sec\theta=0$$
$$\left(\text{since } \frac{c_2}{c_1}=2\right).$$

$\therefore \quad \dfrac{2\sin\theta}{\cos\theta} - \dfrac{1}{\cos\theta} = 0,$

$\therefore \quad 2\sin\theta - 1 = 0 \quad (\text{since } \theta < \frac{\pi}{2})$

$\sin\theta = \dfrac{1}{2}$

$\theta = \dfrac{\pi}{6}.$

Check $\theta = \dfrac{\pi}{6}$ for max. or min.:

θ	$\frac{\pi}{6}^-$	$\frac{\pi}{6}$	$\frac{\pi}{6}^+$
$\frac{dR}{d\theta}$	$-$	0	$+$

i.e. $\dfrac{\pi}{6}$ minimizes R.

QUESTION SEVEN

(a) (i) $y = px - ap^2$ —①

$\therefore \quad y = qx - aq^2.$ —②

①$-$②: $\quad 0 = (p-q)x - ap^2 + aq^2,$

$\therefore \quad (p-q)x = a(p^2 - q^2)$

$(p-q)x = a(p-q)(p+q),$

$\therefore \quad x = a(p+q)$, where $p \neq q$.

Subst. for $x = a(p+q)$ in ①:

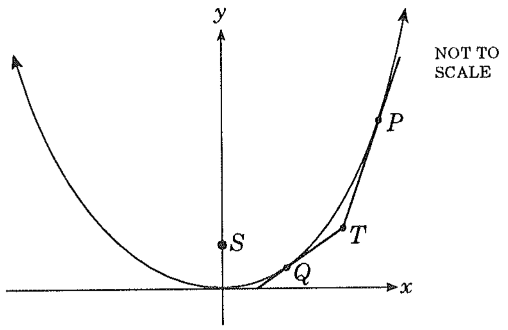
$\therefore \quad y = pa(p+q) - ap^2$

$= ap^2 + apq - ap^2,$

$\therefore \quad y = apq.$

$\therefore$ T has coords. $\big(a(p+q),\ apq\big)$.

(ii)



$S(0,\ a)$ and $P(2ap,\ ap^2)$,

$\therefore \quad SP = \sqrt{(2ap-0)^2 + (ap^2 - a)^2}$

$= \sqrt{4a^2p^2 + a^2p^4 - 2a^2p^2 + a^2}$

$= \sqrt{a^2p^4 + 2a^2p^2 + a^2}$

$= \sqrt{(ap^2 + a)^2}$

$= a(p^2 + 1).$

(iii) From (ii), $SP = a(p^2+1)$.

Similarly, $SQ = a(q^2+1)$.

$\therefore$ $SP + SQ = 4a$ means that

$a(p^2+1) + a(q^2+1) = 4a,$

i.e. $\quad p^2 + 1 + q^2 + 1 = 4,$

$\therefore \quad p^2 + q^2 = 2.$

From (i), T has coordinates given by:

$x = a(p+q)$ and $y = apq$.

$\therefore \quad \dfrac{y}{a} = pq$, and $\dfrac{x}{a} = p + q$.

$\therefore \quad \dfrac{x^2}{a^2} = (p+q)^2,$

$\therefore \quad \dfrac{x^2}{a^2} = p^2 + q^2 + 2pq.$

But $p^2 + q^2 = 2$ and $pq = \dfrac{y}{a}$.

$\therefore \quad \dfrac{x^2}{a^2} = 2 + \dfrac{2y}{a},$

$\therefore \quad x^2 = 2a^2 + 2ay$, which is a parabola.

(b) (i) $x = Vt\cos\theta \Rightarrow t = \dfrac{x}{V\cos\theta}$

Substitute for $t = \dfrac{x}{V\cos\theta}$ into expression for y.

$\therefore \quad y = -\dfrac{g}{2}\dfrac{x^2}{V^2\cos^2\theta} + V\sin\theta \times \dfrac{x}{V\cos\theta}$

$= -\dfrac{g}{2V^2}x^2(1 + \tan^2\theta) + x\tan\theta$

$= -\dfrac{1}{4h}x^2(1 + \tan^2\theta) + x\tan\theta$

where $\dfrac{V^2}{2g} = h.$

(ii) Need to look at the solutions to eqn. of flight of projectile for varying values of θ:

$y = x\tan\theta - \dfrac{1}{4h}x^2\left(1 + \tan^2\theta\right)$

$\Rightarrow \quad \dfrac{x^2}{4h}(1 + \tan^2\theta) - x\tan\theta + y = 0.$

$\left(\dfrac{x^2}{4h}\right)\tan^2\theta - (x)\tan\theta + \left(y + \dfrac{x^2}{4h}\right) = 0.$

Thus $\tan\theta$ (and then θ) will have two different values when $\Delta > 0$,

i.e. $\quad x^2 - 4 \times \dfrac{x^2}{4h}\left(y + \dfrac{x^2}{4h}\right) > 0,$

$\therefore \quad 4hx^2 - 4x^2\left(y + \dfrac{x^2}{4h}\right) > 0,$

$\therefore \quad h - \left(y + \dfrac{x^2}{4h}\right) > 0, \quad x \neq 0,$

$$h > y + \frac{x^2}{4h}$$
$$h - y > \frac{x^2}{4h}$$
$$4h(h-y) > x^2,$$

i.e. (X, Y) can be hit by firing at 2 different angles, θ_1 and θ_2, provided that

$$4h(h-y) > X^2.$$

(iii) From (i),

$$\frac{1}{4h}x^2(1+\tan^2\theta) - x\tan\theta + y = 0,$$

i.e. $x^2\tan^2\theta - 4hx\tan\theta + x^2 + 4hy = 0.$

If $\tan\theta_1$ and $\tan\theta_2$ are the roots of this quadratic equation, the PRODUCT of the roots is given by:

$$\tan\theta_1\tan\theta_2 = \frac{x^2 + 4hy}{x^2}$$
$$= 1 + \frac{4hy}{x^2}.$$

The projectile is ABOVE the x axis;
i.e. $y > 0$,
i.e. $\tan\theta_1\tan\theta_2 > 1$, since $\frac{4hy}{x^2} > 0$.

If their product is greater than 1, then both angles will be acute *but* one of the angles must be bigger than $\frac{\pi}{4}$, (since $\tan\frac{\pi}{4} = 1$).

OR

If $0 < \theta_1 < \frac{\pi}{4}$ and $\theta < \theta_2 < \frac{\pi}{4}$,
then $0 < \theta_1\theta_2 < \frac{\pi^2}{16}$.

But in eqn. (C),

$$\frac{x^2}{4h^2}\tan^2\theta - x\tan\theta + y + \frac{x^2}{4h} = 0,$$

product of roots θ_1, θ_2 is

$$\theta_1\theta_2 = \frac{y + \frac{x^2}{4h}}{\frac{x^2}{4h}}$$

$$\therefore \quad 0 < \frac{y + \frac{x^2}{4h}}{\frac{x^2}{4h}} < \frac{\pi^2}{16},$$

i.e. $0 < y + \frac{x^2}{4h} < \frac{\pi^2}{16}\frac{x^2}{4h}$ $\quad \left(h = \frac{v^2}{2g} > 0\right),$

i.e. $0 < y < \left(\frac{\pi^2}{16} - 1\right)\frac{x^2}{4h}.$ (D)

But $\pi < 4$
$\therefore$ $\pi^2 < 16$
$\therefore$ $\frac{\pi^2}{16} - 1 < 0.$

$\therefore$ Inequality (D) is impossible if $y > 0$.
$\therefore$ There is no point above the x axis.

COMMENTS ON THE 1993 3/4 UNIT PAPER

5. (b) This is a specific instance of the general result:

$$r\ln a+(1-r)\ln b \;\le\; \ln\big(ra+(1-r)b\big),$$

where $0\le r\le 1$. Consequently we have the geometric mean – arithmetic mean inequality $a^r b^{1-r} \le ra+(1-r)b$.

When $r=\frac{1}{2}$, we obtain $\sqrt{ab} \;\le\; \frac{a+b}{2}$.

This type of concavity argument is very powerful, but anything more sophisticated would be more appropriate for 4 Unit. Perhaps 4 Unit students could be asked to sketch $y=-x\ln x,\ 0<x\le 1$ and invited to rework the argument. Another common type of graphical argument is to start from an inequality such as $x-1 \ge \ln x$, for $x > 0$ and then deduce an inequality such as:

$$\sum y_k - \sum x_k \ge \sum \ln\left(\frac{y_k}{x_k}\right)^{x_k}.$$

If the LHS vanishes, we have:

$$1 \ge \left(\frac{y_1}{x_1}\right)^{x_1} \cdots \left(\frac{y_n}{x_n}\right)^{x_n}.$$

6. (a) What would happen if A were not constant here? 4 Unit students might appreciate a more general discussion. It might be possible to set a question on this more general situation in the 3 Unit Paper. Perhaps someone could write a short note for *Reflections*.

(b) Another instance of Fermat's principle. (A more standard example occurred as Question 10(b) on the 1991 2 Unit Paper.) There is a nice discussion of Fermat's principle, or 'the best path' in *Genius*, James Gleick's biography of Richard Feynman.

7. (a) (iii) Good students should be able to work out what part of the parabola is the locus of T.

(b) Students can easily check that the directrix of the parabola of (i) is independent of θ. Indeed it is also the directrix of the enveloping parabola of (ii). Given this, remember Einstein's paradigm that the geometric world and physical world are identical; we should be able to find some physical characterization of the directrix. Good students should be able to check that the speed of a projectile at any point on its parabolic path is equal to the speed under free fall to that point from rest on the directrix.

When answering (iii) some students may argue that the trajectories of projectiles fired at the two different angles θ_1 and θ_2 where $\theta_1<\frac{\pi}{4}$, $\theta_2<\frac{\pi}{4}$ can never intersect. In fact, all trajectories intersect (though the times at which they pass through their points of intersection are different). Good students should be able to verify this.

When teaching projectile motion it would be a pity if students were not made aware of some of the pioneering and fumbling ideas of people such as Galileo. Remember here that the science of dynamics played a significant role in the development of the Calculus.*) As we know with the benefit of hindsight, the velocity of a falling body is propor-tional to time. Galileo first guessed that it was proportional to distance travelled, but unexpectedly discovered that motion cannot take place at all under this assumption. Using the Calculus it is easy to verify this, but Galileo's original argument is more illuminating.

* Kepler's seminal work on wine barrels was also significant. One of his barrels appeared as 4(c) on the 1992 Paper.