

Supplementary Material

1 SUPPLEMENTARY MATHEMATICS

Here we prove the equation

$$|Q_L| = 2^{L-2}(2^{L-1} + 1) \quad (\text{S1})$$

Proof: For $L \geq 2$, there are 4^L IBD (identical by descent) probabilities $Q(i_1, i_2, \dots, i_L)$ since $i_l = 1, 2, 3$ or 4 and furthermore they add up to 1. A number of these probabilities are equal because of two symmetries: (1) the two homologous chromosomes in each individual play identical roles, and (2) the siblings play identical roles (assuming no sex-dependence of meiosis, so that for instance the recombination rates $r_{l,l'}$ are sex-independent. It is thus appropriate to use only one representative of each symmetry equivalence class, so that for instance one may impose this representative to have its first index, i_1 , equal to zero. In fact one can identify exactly one element in each class by imposing that the indices of the representative Q 's have either

1. $i_l \in \{0, 1\} \forall l \in \{2, \dots, L\}$, or
2. $i_l \in \{0, 1\} \forall l \in \{2, \dots, K-1\}$, $i_K = 2$ and $i_l \in \{0, 1, 2, 3\} \forall l \in \{K+1, \dots, L\}$

The number of equivalence classes and thus of Q 's to consider is then

$$|Q_L| = 2^{L-1} + \sum_{l=2}^L 2^{l-2} 4^{L-l} = 2^{L-1} + 2^{2L-2} \sum_{l=2}^L 2^{-l} \quad (\text{S2})$$

Given that $\sum_{l=2}^L 2^{-l}$ is a geometric progression of common ratio 2^{-1} from 2 to L , the sum of its terms can be expressed as:

$$\sum_{l=2}^L 2^{-l} = \frac{2^{-2} - 2^{-(L-1)}}{1 - 2^{-1}} = 2^{-1} - 2^{-L} \quad (\text{S3})$$

Substituting S3 in S2, we get

$$|Q_L| = 2^{L-1} + 2^{2L-2}(2^{-1} - 2^{-L}) = 2^{L-1} + 2^{2L-3} - 2^{L-2} \quad (\text{S4})$$

Factorizing with respect to 2^{L-2} and after simplification, this gives

$$|Q_L| = 2^{L-2}(1 + 2^{L-1}). \quad (\text{S5})$$

2 THE SELF-CONSISTENT EQUATIONS FOR THREE LOCI

Here we provide the coefficients entering each of the self-consistent equations.

2.1 The self consistent equation for $Q(0, 0, 0)$

In Figure S9 we show the graphical representation for each term entering this self-consistent equation.

$$((1 - r_{12})(1 - r_{23}) - 1)Q(0, 0, 0) + \frac{1}{2}(1 - r_{12})Q(0, 0, 2) + \frac{1}{2}(1 - r_{13})Q(0, 2, 0) + \frac{1}{2}(1 - r_{23})Q(0, 2, 2) = 0 \quad (\text{S6})$$

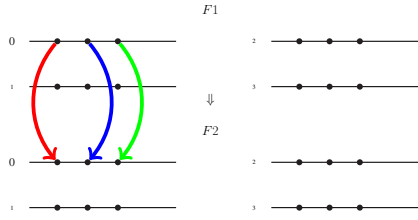


Figure S1. $Q(0, 0, 0) : \frac{1}{2} \times (1 - r_{12}) \times (1 - r_{23}) \times Q(0, 0, 0)$

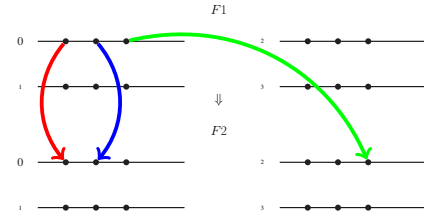
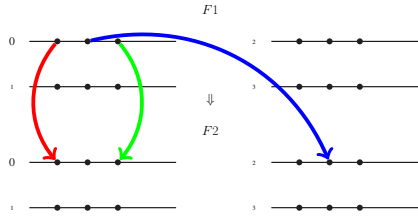


Figure S2. $Q(0, 0, 0) : \frac{1}{2} \times (1 - r_{12}) \times \frac{1}{2} \times Q(0, 0, 2)$

Figure S3. $Q(0, 0, 0) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{13}) \times Q(0, 2, 0)$

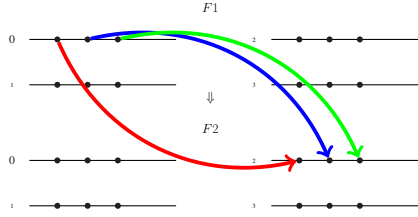


Figure S4. $Q(0, 0, 0) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{23}) \times Q(0, 2, 2)$

Figure S5. $Q(0, 0, 0) : \frac{1}{2} \times (1 - r_{12})(1 - r_{23}) \times Q(2, 2, 2)$

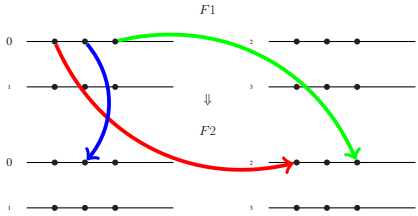


Figure S6. $Q(0, 0, 0) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{12}) \times Q(2, 2, 0)$

Figure S7. $Q(0, 0, 0) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{13}) \times Q(2, 0, 2)$

Figure S8. $Q(0, 0, 0) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{23}) \times Q(2, 0, 0)$

Figure S9. $Q(0, 0, 0)$

2.2 The self consistent equation for $Q(0, 0, 1)$

In Figure S18 we show the graphical representation for each term entering this self-consistent equation.

$$(1 - r_{12})r_{23}Q(0, 0, 0) - Q(0, 0, 1) + \frac{1}{2}(1 - r_{12})Q(0, 0, 2) + \frac{1}{2}r_{13}Q(0, 2, 0) + \frac{1}{2}r_{23}Q(0, 2, 2) = 0 \quad (\text{S7})$$

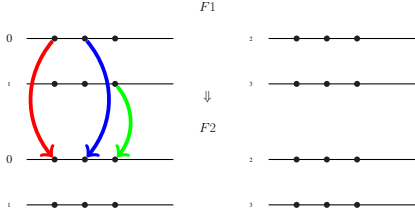


Figure S10. $Q(0, 0, 1) : \frac{1}{2} \times (1 - r_{12}) \times r_{23} \times Q(0, 0, 0)$

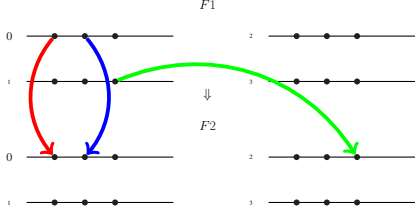


Figure S12. $Q(0, 0, 1) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{12}) \times Q(0, 0, 2)$

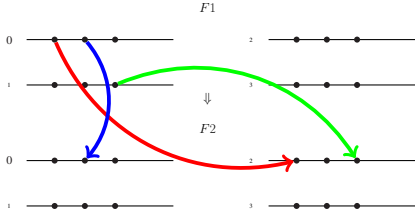


Figure S14. $Q(0, 0, 1) : \frac{1}{2} \times \frac{1}{2} \times r_{13} \times Q(2, 0, 2)$

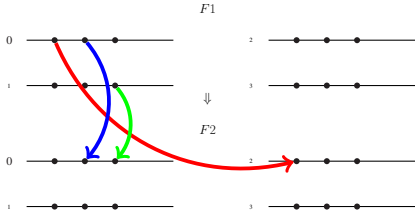


Figure S16. $Q(0, 0, 1) : \frac{1}{2} \times \frac{1}{2} \times r_{23} \times Q(2, 0, 0)$

Figure S18. $Q(0, 0, 1)$

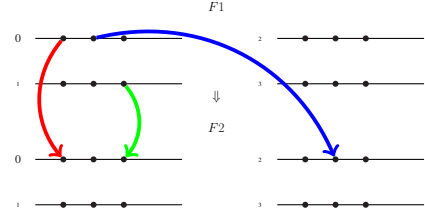


Figure S11. $Q(0, 0, 1) : \frac{1}{2} \times \frac{1}{2} \times r_{13} \times Q(0, 2, 0)$

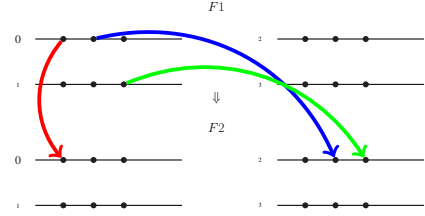


Figure S13. $Q(0, 0, 1) : \frac{1}{2} \times \frac{1}{2} \times r_{23} \times Q(0, 2, 2)$

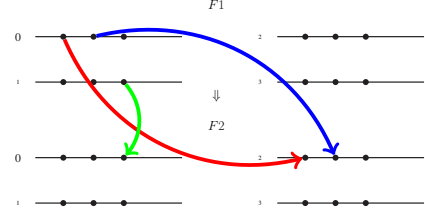


Figure S15. $Q(0, 0, 1) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{12}) \times Q(2, 2, 0)$

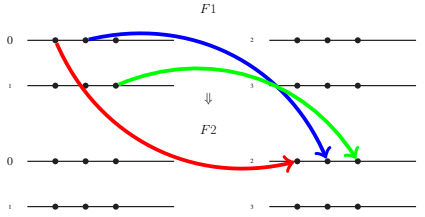


Figure S17. $Q(0, 0, 1) : \frac{1}{2} \times (1 - r_{12}) \times r_{23} \times Q(2, 2, 2)$

2.3 The self consistent equation for $Q(0, 0, 2)$

In Figure S27 we show the graphical representation for each term entering this self-consistent equation.

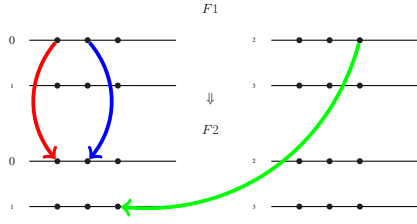


Figure S19. $Q(0, 0, 2) : \frac{1}{2} \times (1 - r_{12}) \times \frac{1}{2} \times Q(0, 0, 1)$

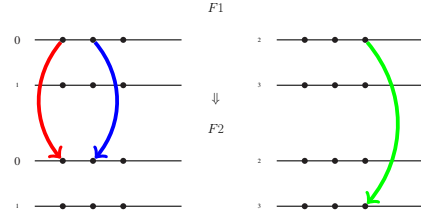


Figure S20. $Q(0, 0, 2) : \frac{1}{2} \times (1 - r_{12}) \times \frac{1}{2} \times Q(0, 0, 3)$

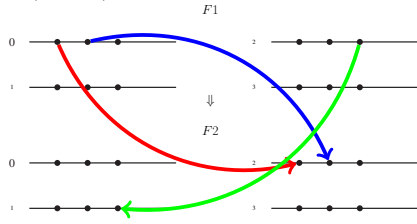


Figure S21. $Q(0, 0, 2) : \frac{1}{2} \times (1 - r_{12}) \times \frac{1}{2} \times Q(2, 2, 1)$

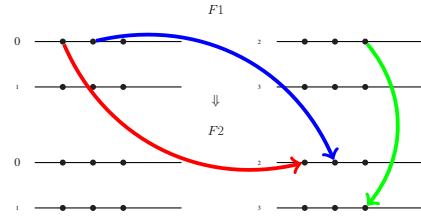


Figure S22. $Q(0, 0, 2) : \frac{1}{2} \times (1 - r_{12}) \times \frac{1}{2} \times Q(2, 2, 3)$

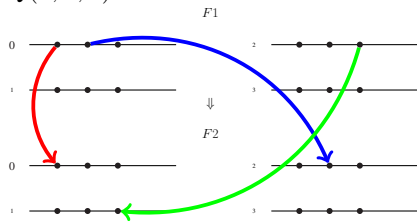


Figure S23. $Q(0, 0, 2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(0, 2, 1)$

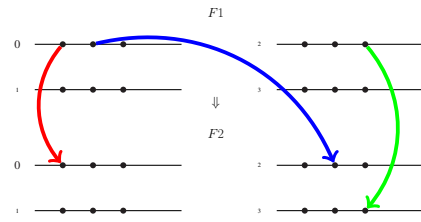


Figure S24. $Q(0, 0, 2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(0, 2, 3)$

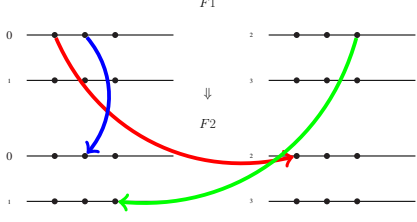


Figure S25. $Q(0, 0, 2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(2, 0, 1)$

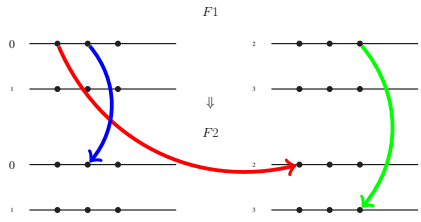


Figure S26. $Q(0, 0, 2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(2, 0, 3)$

Figure S27. $Q(0, 0, 2)$

2.4 The self consistent equation for $Q(0, 1, 0)$

In Figure S36 we show the graphical representation for each term entering this self-consistent equation.

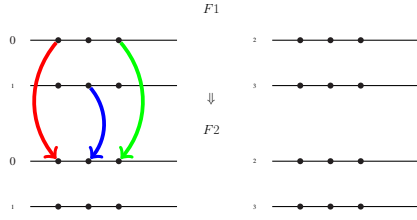


Figure S28. $Q(0, 1, 0) : \frac{1}{2} \times r_{12} \times (1 - r_{23}) \times Q(0, 0, 0)$

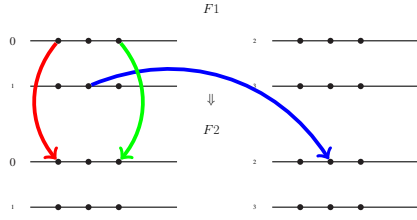


Figure S29. $Q(0, 1, 0) : \frac{1}{2} \times r_{12} \times \frac{1}{2} \times Q(0, 0, 2)$

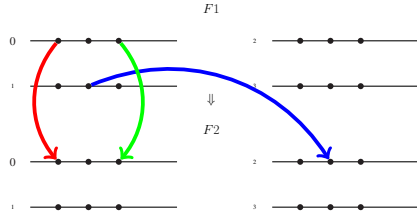


Figure S30. $Q(0, 1, 0) : \frac{1}{2} \times \frac{1}{2} \times r_{13} \times Q(0, 2, 0)$

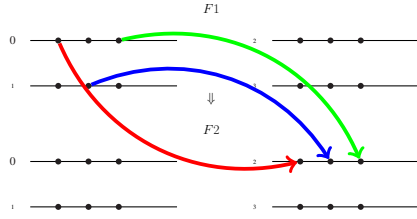


Figure S31. $Q(0, 1, 0) : \frac{1}{2} \times \frac{1}{2} \times r_{23} \times Q(0, 2, 2)$

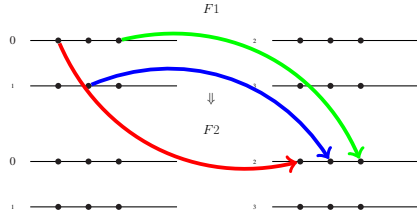


Figure S32. $Q(0, 1, 0) : \frac{1}{2} \times r_{12} r_{23} \times Q(2, 2, 2)$

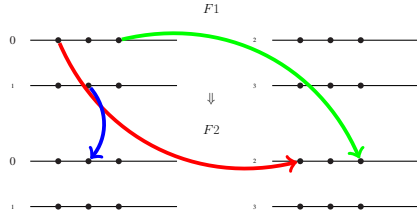


Figure S33. $Q(0, 1, 0) : \frac{1}{2} \times \frac{1}{2} \times r_{12} \times Q(2, 2, 0)$

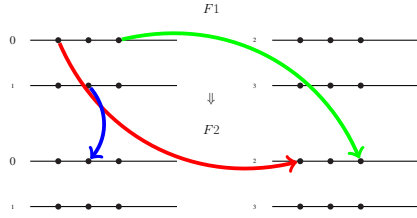


Figure S34. $Q(0, 1, 0) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{13}) \times Q(2, 0, 2)$

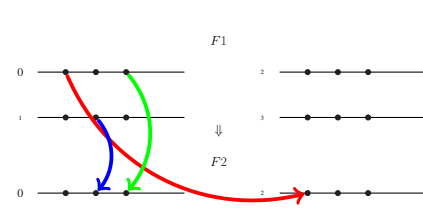


Figure S35. $Q(0, 1, 0) : \frac{1}{2} \times r_{23} \times \frac{1}{2} \times Q(2, 0, 0)$

Figure S36. $Q(0, 1, 0)$

2.5 The self consistent equation for $Q(0, 1, 1)$

In Figure S45 we show the graphical representation for each term entering this self-consistent equation.

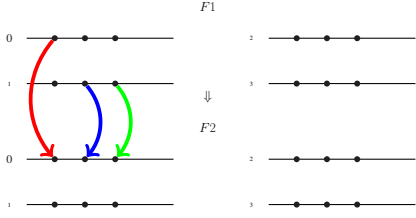


Figure S37. $Q(0, 1, 1) : \frac{1}{2} \times r_{12} \times (1 - r_{23}) \times Q(0, 0, 0)$

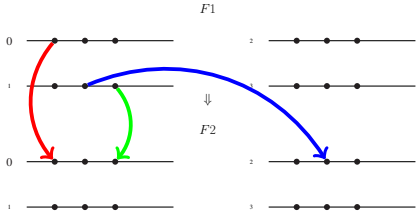


Figure S39. $Q(0, 1, 1) : \frac{1}{2} \times \frac{1}{2} \times r_{13} \times Q(0, 2, 0)$

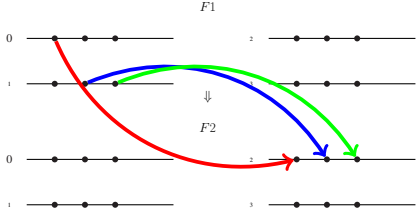


Figure S41. $Q(0, 1, 1) : \frac{1}{2} \times r_{12} \times (1 - r_{23}) \times Q(2, 2, 2)$

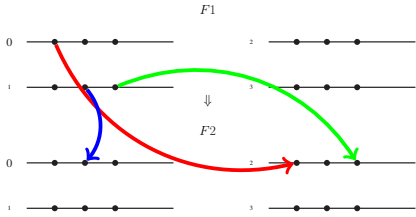


Figure S43. $Q(0, 1, 1) : \frac{1}{2} \times r_{13} \times \frac{1}{2} \times Q(2, 0, 2)$

Figure S45. $Q(0, 1, 1)$

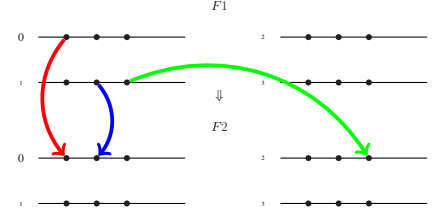


Figure S38. $Q(0, 1, 1) : \frac{1}{2} \times \frac{1}{2} \times r_{12} \times Q(0, 0, 2)$

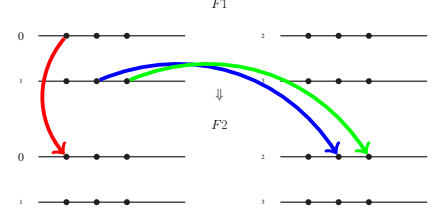


Figure S40. $Q(0, 1, 1) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{23}) \times Q(0, 2, 2)$

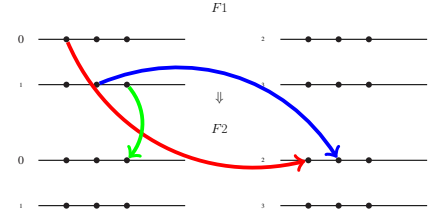


Figure S42. $Q(0, 1, 1) : \frac{1}{2} \times r_{12} \times \frac{1}{2} \times Q(2, 2, 0)$

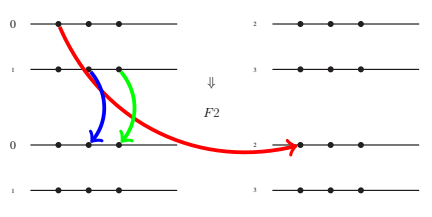


Figure S44. $Q(0, 1, 1) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{23}) \times Q(2, 0, 0)$

2.6 The self consistent equation for $Q(0, 1, 2)$

In Figure S54 we show the graphical representation for each term entering this self-consistent equation.

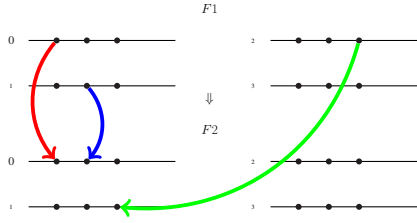


Figure S46. $Q(0, 1, 2) : \frac{1}{2} \times \frac{1}{2} \times r_{12} \times Q(0, 0, 1)$

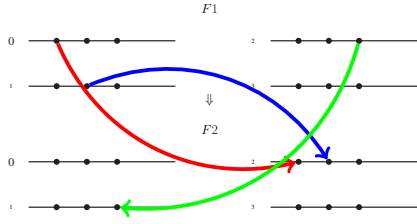


Figure S48. $Q(0, 1, 2) : \frac{1}{2} \times \frac{1}{2} \times r_{12} \times Q(2, 2, 1)$

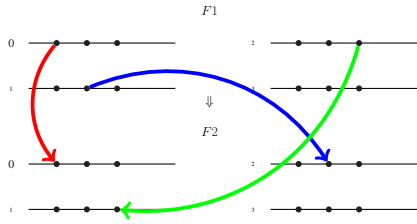


Figure S50. $Q(0, 1, 2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(0, 2, 1)$

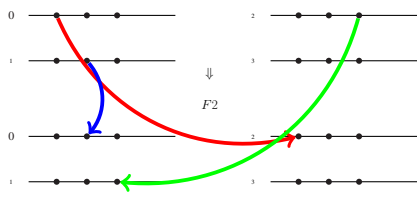


Figure S52. $Q(0, 1, 2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(2, 0, 1)$

Figure S54. $Q(0, 1, 2)$

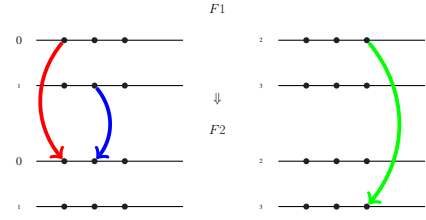


Figure S47. $Q(0, 1, 2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times r_{12} \times Q(0, 0, 3)$

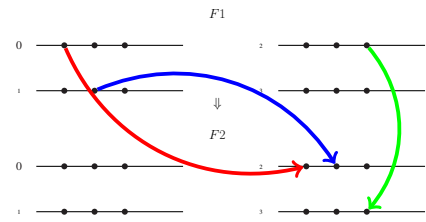


Figure S49. $Q(0, 1, 2) : \frac{1}{2} \times \frac{1}{2} \times r_{12} \times Q(2, 2, 3)$

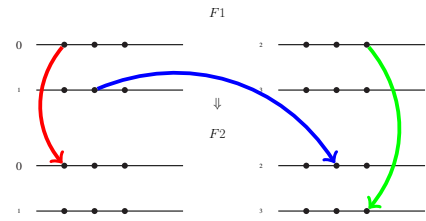


Figure S51. $Q(0, 1, 2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(0, 2, 3)$

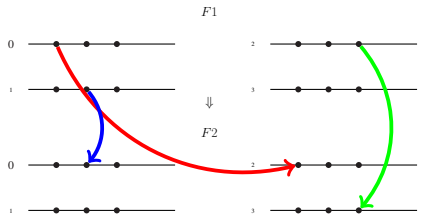


Figure S53. $Q(0, 1, 2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(2, 0, 3)$

2.7 The self consistent equation for $Q(0, 2, 0)$

In Figure S63 we show the graphical representation for each term entering this self-consistent equation.

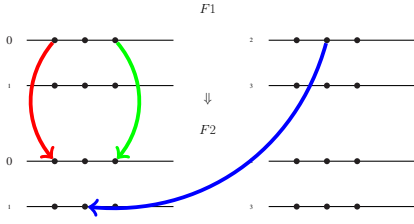


Figure S55. $Q(0, 2, 0) : \frac{1}{2} \times (1 - r_{13}) \times \frac{1}{2} \times Q(0, 1, 0)$

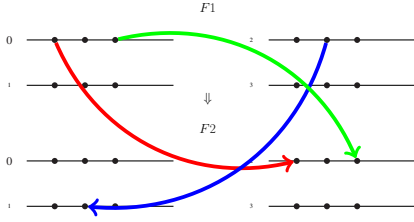


Figure S57. $Q(0, 2, 0) : \frac{1}{2} \times (1 - r_{13}) \times \frac{1}{2} \times Q(2, 1, 2)$

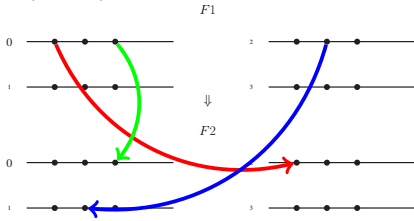


Figure S59. $Q(0, 2, 0) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(2, 1, 0)$

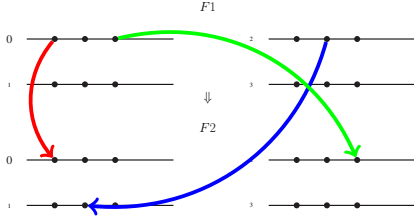


Figure S61. $Q(0, 2, 0) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(0, 1, 2)$

Figure S63. $Q(0, 2, 0)$

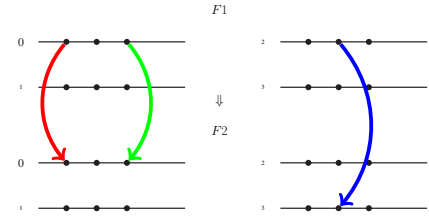


Figure S56. $Q(0, 2, 0) : \frac{1}{2} \times (1 - r_{13}) \times \frac{1}{2} \times Q(0, 3, 0)$

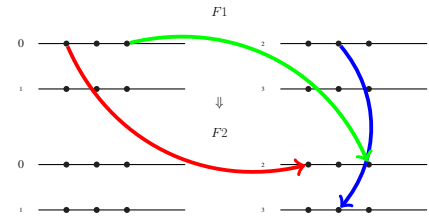


Figure S58. $Q(0, 2, 0) : \frac{1}{2} \times (1 - r_{13}) \times \frac{1}{2} \times Q(2, 3, 2)$

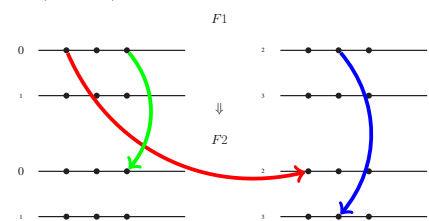


Figure S60. $Q(0, 2, 0) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(2, 3, 0)$

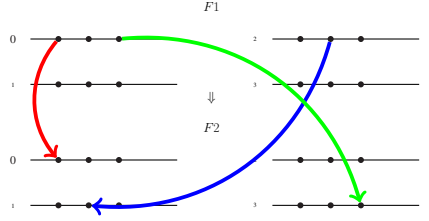


Figure S62. $Q(0, 2, 0) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(0, 1, 3)$

2.8 The self consistent equation for $Q(0, 2, 1)$

In Figure S72 we show the graphical representation for each term entering this self-consistent equation.

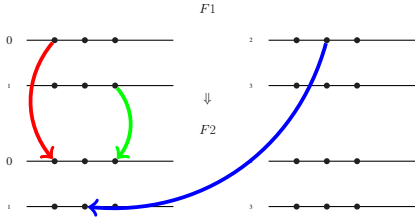


Figure S64. $Q(0, 2, 1) : \frac{1}{2} \times r_{13} \times \frac{1}{2} \times Q(0, 1, 0)$

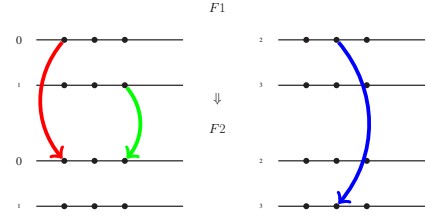
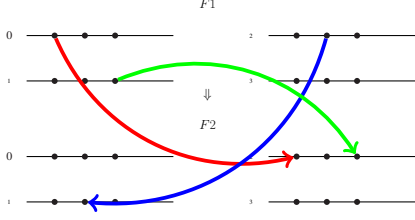


Figure S65. $Q(0, 2, 1) : \frac{1}{2} \times r_{13} \times \frac{1}{2} \times Q(0, 3, 0)$

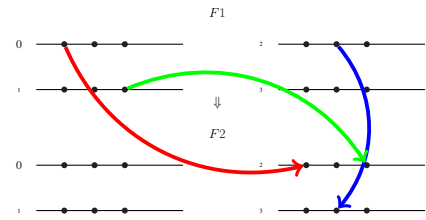
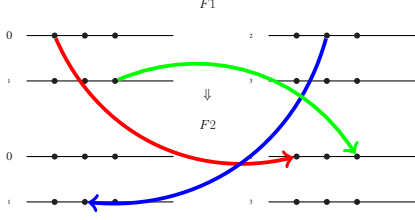


Figure S66. $Q(0, 2, 1) : \frac{1}{2} \times r_{13} \times \frac{1}{2} \times Q(2, 1, 2)$

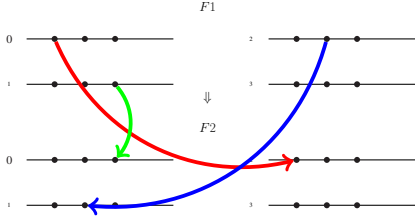


Figure S67. $Q(0, 2, 1) : \frac{1}{2} \times r_{13} \times \frac{1}{2} \times Q(2, 3, 2)$

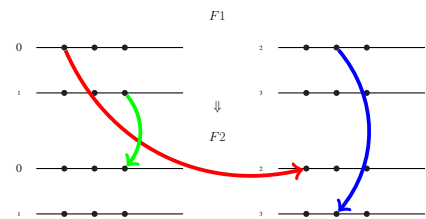
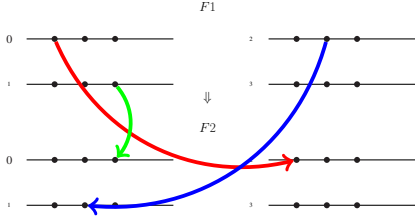


Figure S68. $Q(0, 2, 1) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(2, 1, 0)$

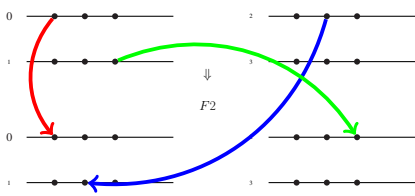


Figure S69. $Q(0, 2, 1) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(2, 3, 0)$

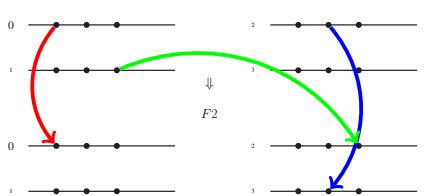
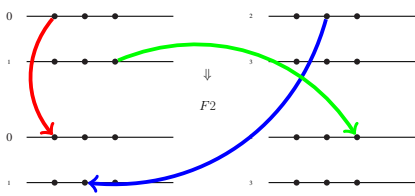


Figure S70. $Q(0, 2, 1) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(0, 1, 2)$

Figure S71. $Q(0, 2, 1) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(0, 3, 2)$

Figure S72. $Q(0, 2, 1)$

2.9 The self consistent equation for $Q(0, 2, 2)$

In Figure S81 we show the graphical representation for each term entering this self-consistent equation.

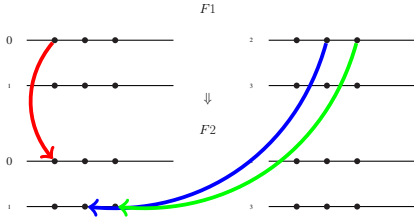


Figure S73. $Q(0, 2, 2) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{23}) \times Q(0, 1, 1)$

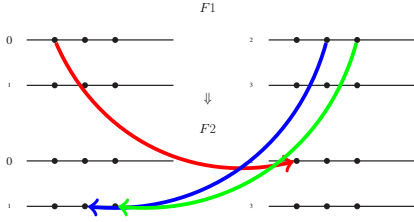


Figure S75. $Q(0, 2, 2) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{23}) \times Q(2, 1, 1)$

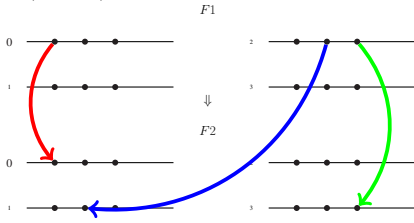


Figure S77. $Q(0, 2, 2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(0, 1, 3)$

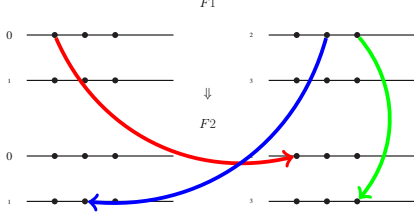


Figure S79. $Q(0, 2, 2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(2, 1, 3)$

Figure S81. $Q(0, 2, 2)$

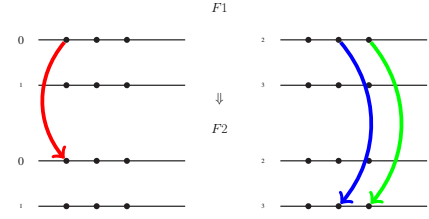


Figure S74. $Q(0, 2, 2) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{23}) \times Q(0, 3, 3)$

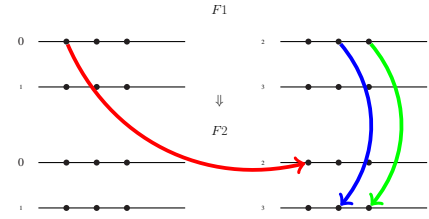


Figure S76. $Q(0, 2, 2) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{23}) \times Q(2, 3, 3)$

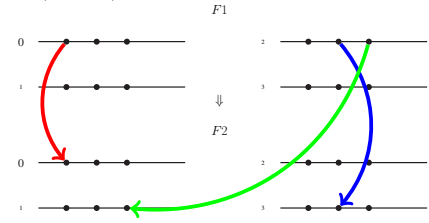


Figure S78. $Q(0, 2, 2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(031)$

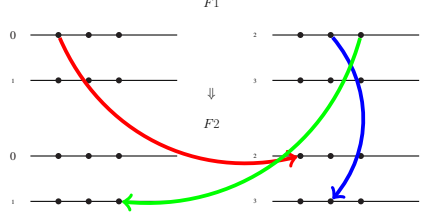


Figure S80. $Q(0, 2, 2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(2, 3, 1)$

2.10 The self consistent equation for $Q(0, 2, 3)$

In Figure S90 we show the graphical representation for each term entering this self-consistent equation.

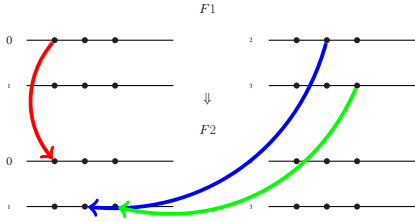


Figure S82. $Q(0, 2, 3) : \frac{1}{2} \times \frac{1}{2} \times r_{23} \times Q(0, 1, 1)$

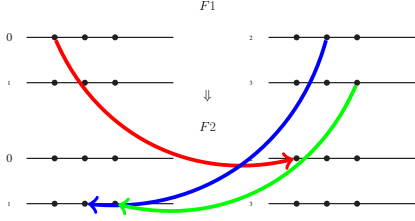


Figure S84. $Q(0, 2, 3) : \frac{1}{2} \times \frac{1}{2} \times r_{23} \times Q(2, 1, 1)$

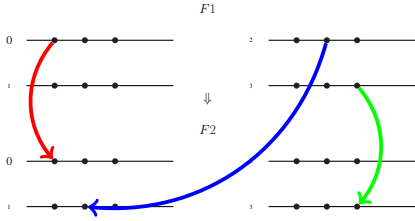


Figure S86. $Q(0, 2, 3) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(0, 1, 3)$

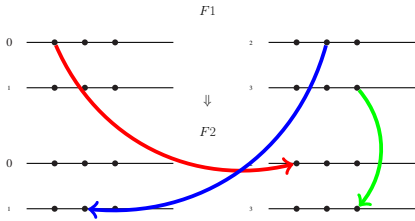


Figure S88. $Q(0, 2, 3) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(2, 1, 3)$

Figure S90. $Q(0, 2, 3)$

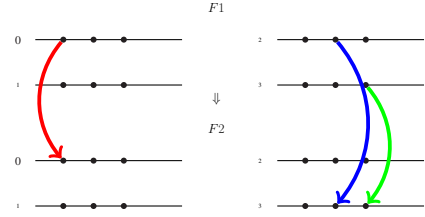


Figure S83. $Q(0, 2, 3) : \frac{1}{2} \times \frac{1}{2} \times r_{23} \times Q(0, 3, 3)$

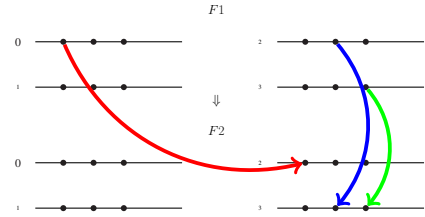


Figure S85. $Q(0, 2, 3) : \frac{1}{2} \times \frac{1}{2} \times r_{23} \times Q(2, 3, 3)$

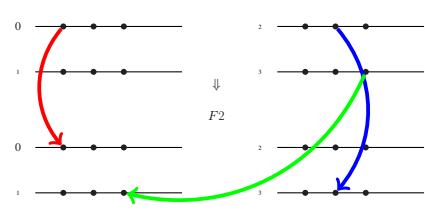


Figure S87. $Q(0, 2, 3) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(031)$

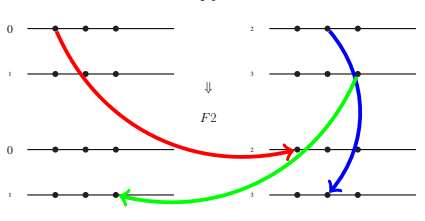


Figure S89. $Q(0, 2, 3) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(2, 3, 1)$