Supplementary Material

1 SUPPLEMENTARY MATHEMATICS

1.1 The derivation of equation

$$|Q_L| = 2^{L-2}(2^{L-1}+1) \tag{S1}$$

Proof: For $L \geq 2$, there are 4^L IBD (identical by descent) probabilities $Q(i_1, i_2, \ldots i_L)$ since $i_l = 1, 2, 3$ or 4 and furthermore they add up to 1. A number of these probabilities are equal because of two symmetries: (1) the two homologous chromosomes in each individual play identical roles, and (2) the siblings play identical roles (assuming no sex-dependence of meiosis, so that for instance the recombination rates $r_{l,l'}$ are sex-independent. It is thus appropriate to use only one representative of each symmetry equivalence class, so that for instance one may impose this representative to have its first index, i_1, equal to zero. In fact one can identify exactly one element in each class by imposing that the indices of the representative Q's have either

1. $i_l \in \{0, 1\}\ \forall l \in \{2, .., L\}$, *or*
2. $i_l \in \{0, 1\}\ \forall l \in \{2, .., K-1\}$, $i_K = 2$ and $i_l \in \{0, 1, 2, 3\}\ \forall l \in \{K+1, .., L\}$

The number of equivalence classes and thus of Q's to consider is then

$$|Q_L| = 2^{L-1} + \sum_{l=2}^{L} 2^{l-2}4^{L-l} = 2^{L-1} + 2^{2L-2}\sum_{l=2}^{L} 2^{-l} \tag{S2}$$

Given that $\sum_{l=2}^{L} 2^{-l}$ is a geometric progression of common ratio 2^{-1} from 2 to L, the sum of its terms can be expressed as:

$$\sum_{l=2}^{L} 2^{-l} = \frac{2^{-2} - 2^{-(L-1)}}{1 - 2^{-1}} = 2^{-1} - 2^{-L} \tag{S3}$$

Substituting S3 in S2, we get

$$|Q_L| = 2^{L-1} + 2^{2L-2}(2^{-1} - 2^{-L}) = 2^{L-1} + 2^{2L-3} - 2^{L-2} \tag{S4}$$

Factorizing with respect to 2^{L-2} and after simplification, this gives

$$|Q_L| = 2^{L-2}(1 + 2^{L-1}). \tag{S5}$$

2 THREE LOCI 2-WAY RIL

2.1 The SCHP equations

2.1.1

See Figure S9

$$((1-r_{12})(1-r_{23})-1)Q(0,0,0)+\frac{1}{2}(1-r_{12})Q(0,0,2)+\frac{1}{2}(1-r_{13})Q(0,2,0)+\frac{1}{2}(1-r_{23})Q(0,2,2)=0 \tag{S6}$$

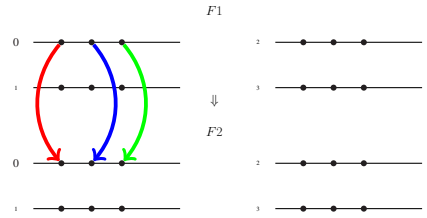

Figure S1. $Q(0,0,0) : \frac{1}{2} \times (1-r_{12}) \times (1-r_{23}) \times Q(0,0,0)$

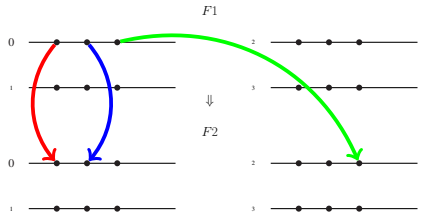

Figure S2. $Q(0,0,0) : \frac{1}{2} \times (1-r_{12}) \times \frac{1}{2} \times Q(0,0,2)$

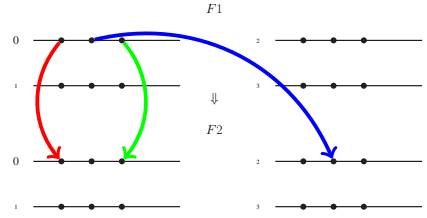

Figure S3. $Q(0,0,0) : \frac{1}{2} \times \frac{1}{2} \times (1-r_{13}) \times Q(0,2,0)$

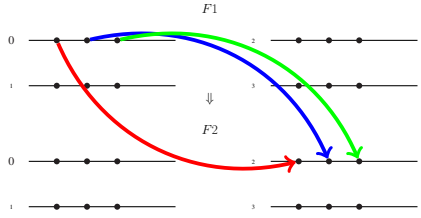

Figure S4. $Q(0,0,0) : \frac{1}{2} \times \frac{1}{2} \times (1-r_{23}) \times Q(0,2,2)$

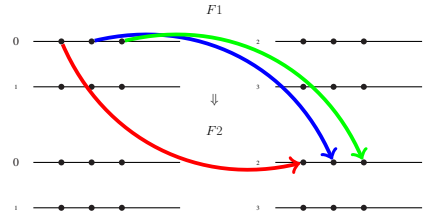

Figure S5. $Q(0,0,0) : \frac{1}{2} \times (1-r_{12})(1-r_{23}) \times Q(2,2,2)$

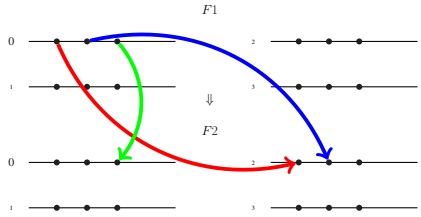

Figure S6. $Q(0,0,0) : \frac{1}{2} \times \frac{1}{2} \times (1-r_{12}) \times Q(2,2,0)$

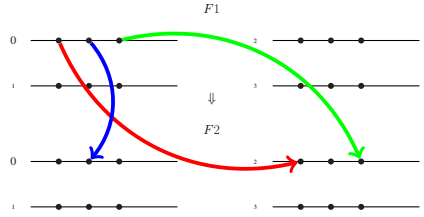

Figure S7. $Q(0,0,0) : \frac{1}{2} \times \frac{1}{2} \times (1-r_{13}) \times Q(2,0,2)$

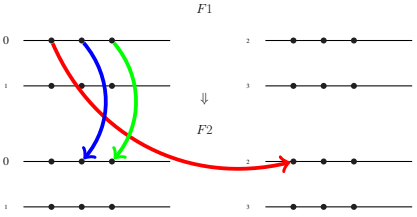

Figure S8. $Q(0,0,0) : \frac{1}{2} \times \frac{1}{2} \times (1-r_{23}) \times Q(2,0,0)$

Figure S9. $Q(0,0,0)$

2.1.2

See Figure S18

$$(1-r_{12})r_{23}Q(0,0,0) - Q(0,0,1) + \frac{1}{2}(1-r_{12})Q(0,0,2) + \frac{1}{2}r_{13}Q(0,2,0) + \frac{1}{2}r_{23}Q(0,2,2) = 0 \tag{S7}$$

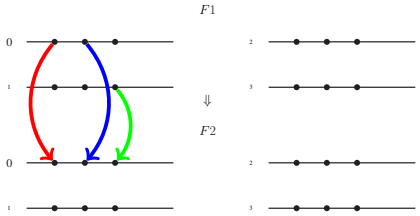

Figure S10. $Q(0,0,1) : \frac{1}{2} \times (1-r_{12}) \times r_{23} \times Q(0,0,0)$

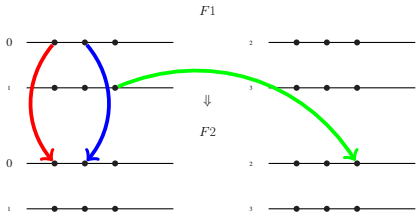

Figure S12. $Q(0,0,1) : \frac{1}{2} \times \frac{1}{2} \times (1-r_{12}) \times Q(0,0,2)$

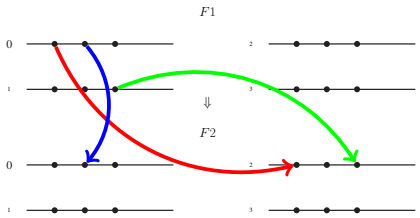

Figure S14. $Q(0,0,1) : \frac{1}{2} \times \frac{1}{2} \times r_{13} \times Q(2,0,2)$

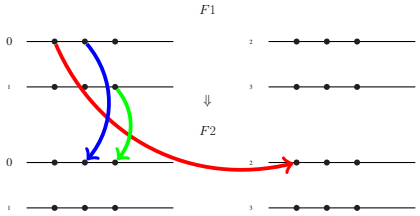

Figure S16. $Q(0,0,1) : \frac{1}{2} \times \frac{1}{2} \times r_{23} \times Q(2,0,0)$

Figure S18. $Q(0,0,1)$

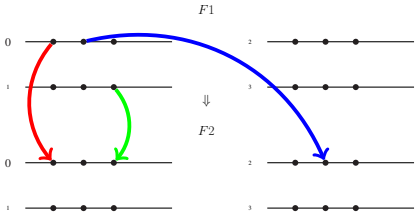

Figure S11. $Q(0,0,1) : \frac{1}{2} \times \frac{1}{2} \times r_{13} \times Q(0,2,0)$

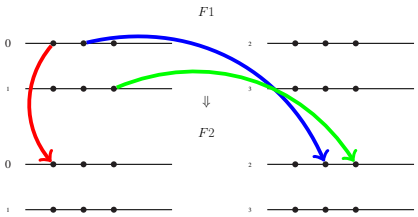

Figure S13. $Q(0,0,1) : \frac{1}{2} \times \frac{1}{2} \times r_{23} \times Q(0,2,2)$

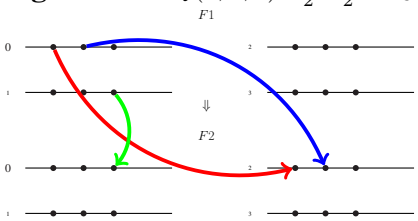

Figure S15. $Q(0,0,1) : \frac{1}{2} \times \frac{1}{2} \times (1-r_{12}) \times Q(2,2,0)$

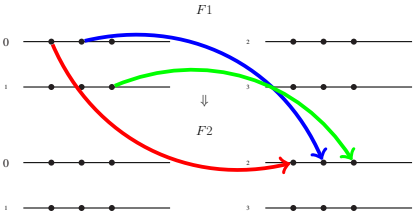

Figure S17. $Q(0,0,1) : \frac{1}{2} \times (1-r_{12}) \times r_{23} \times Q(2,2,2)$

2.1.3

See Figure S27.

$$\frac{1}{2}(1-r_{12})Q(0,0,1)+(\frac{1}{2}(1-r_{12})-1)Q(0,0,2)+\frac{1}{4}Q(0,2,1)+\frac{1}{4}Q(0,2,3)=0 \tag{S8}$$

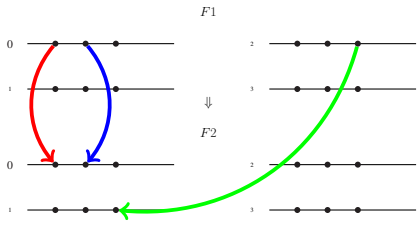

Figure S19. $Q(0,0,2):\frac{1}{2}\times(1-r_{12})\times\frac{1}{2}\times Q(0,0,1)$

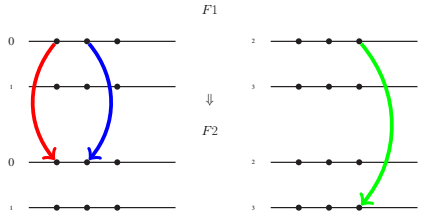

Figure S20. $Q(0,0,2):\frac{1}{2}\times(1-r_{12})\times\frac{1}{2}\times Q(0,0,3)$

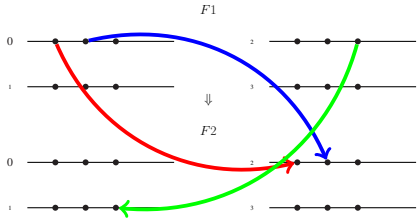

Figure S21. $Q(0,0,2):\frac{1}{2}\times(1-r_{12})\times\frac{1}{2}\times Q(2,2,1)$

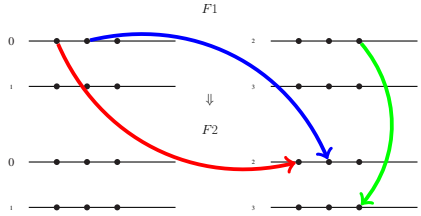

Figure S22. $Q(0,0,2):\frac{1}{2}\times(1-r_{12})\times\frac{1}{2}\times Q(2,2,3)$

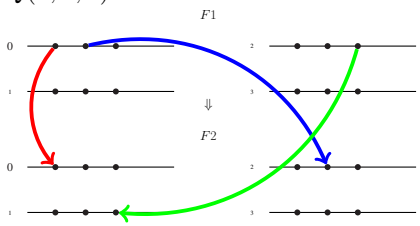

Figure S23. $Q(0,0,2):\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times Q(0,2,1)$

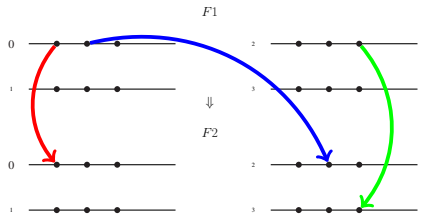

Figure S24. $Q(0,0,2):\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times Q(0,2,3)$

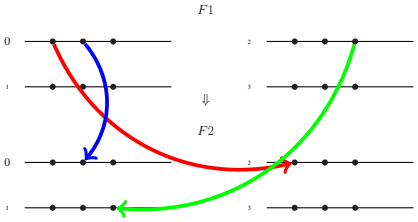

Figure S25. $Q(0,0,2):\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times Q(2,0,1)$

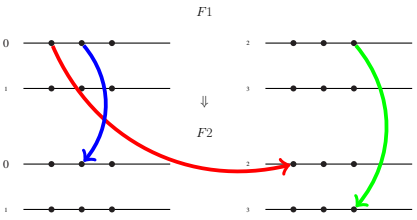

Figure S26. $Q(0,0,2):\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times Q(2,0,3)$

Figure S27. $Q(0,0,2)$

2.1.4

See Figure S36.

$$r_{12}r_{23}Q(0,0,0) + \frac{1}{2}r_{12}Q(0,0,2) - Q(0,1,0) + \frac{1}{2}(1-r_{13})Q(0,2,0) + \frac{1}{2}r_{23}Q(0,2,2) = 0 \qquad \text{(S9)}$$

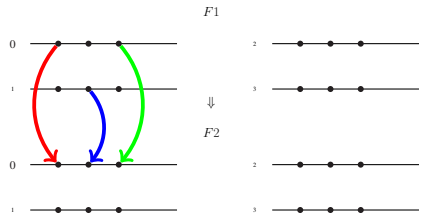

Figure S28. $Q(0,1,0) : \frac{1}{2} \times r_{12} \times (1-r_{23}) \times Q(0,0,0)$

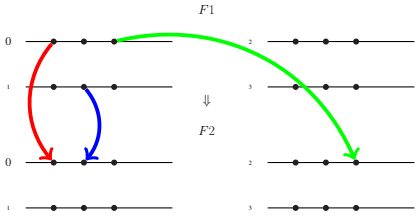

Figure S29. $Q(0,1,0) : \frac{1}{2} \times r_{12} \times \frac{1}{2} \times Q(0,0,2)$

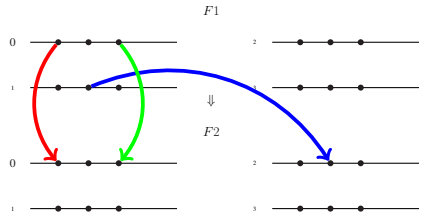

Figure S30. $Q(0,1,0) : \frac{1}{2} \times \frac{1}{2} \times r_{13} \times Q(0,2,0)$

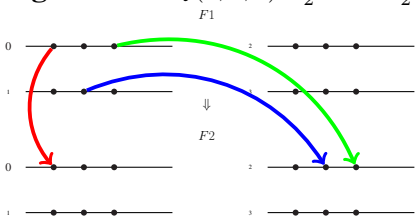

Figure S31. $Q(0,1,0) : \frac{1}{2} \times \frac{1}{2} \times r_{23} \times Q(0,2,2)$

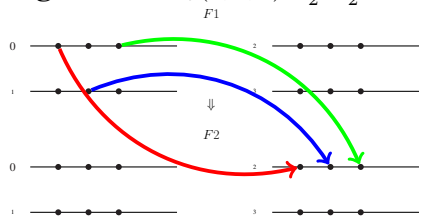

Figure S32. $Q(0,1,0) : \frac{1}{2} \times r_{12}r_{23} \times Q(2,2,2)$

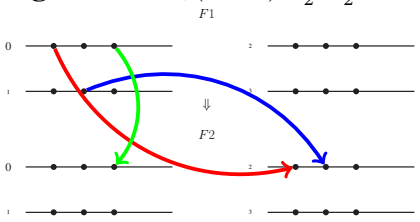

Figure S33. $Q(0,1,0) : \frac{1}{2} \times \frac{1}{2} \times r_{12} \times Q(2,2,0)$

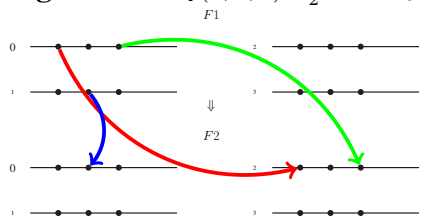

Figure S34. $Q(0,1,0) : \frac{1}{2} \times \frac{1}{2} \times (1-r_{13}) \times Q(2,0,2)$

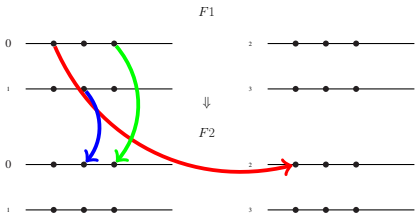

Figure S35. $Q(0,1,0) : \frac{1}{2} \times r_{23} \times \frac{1}{2} \times Q(2,0,0)$

Figure S36. $Q(0,1,0)$

2.1.5

See Figure S45

$$r_{12}(1-r_{23})Q(0,0,0)+\frac{1}{2}r_{12}Q(0,0,2)-Q(0,1,1)+\frac{1}{2}r_{13}Q(0,2,0)+\frac{1}{2}(1-r_{23})Q(0,2,2)=0 \tag{S10}$$

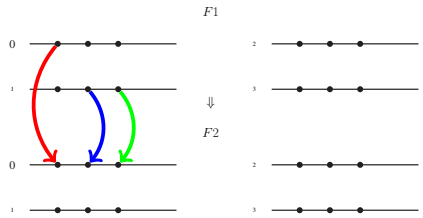

Figure S37. $Q(0,1,1):\frac{1}{2}\times r_{12}\times(1-r_{23})\times Q(0,0,0)$

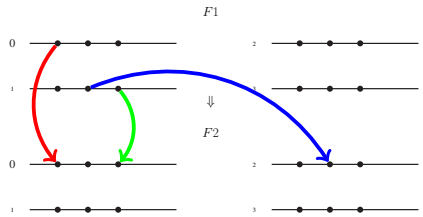

Figure S39. $Q(0,1,1):\frac{1}{2}\times\frac{1}{2}\times r_{13}\times Q(0,2,0)$

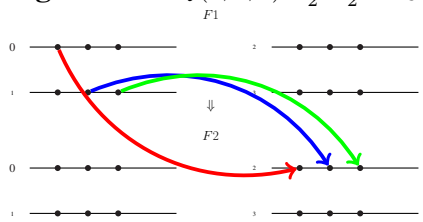

Figure S41. $Q(0,1,1):\frac{1}{2}\times r_{12}\times(1-r_{23})\times Q(2,2,2)$

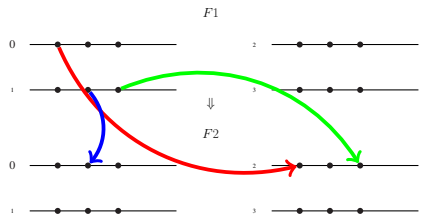

Figure S43. $Q(0,1,1):\frac{1}{2}\times r_{13}\times\frac{1}{2}\times Q(2,0,2)$

Figure S45. $Q(0,1,1)$

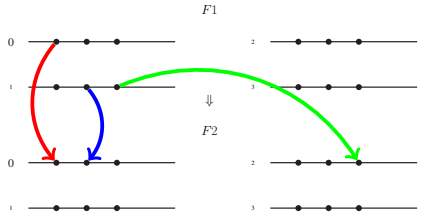

Figure S38. $Q(0,1,1):\frac{1}{2}\times\frac{1}{2}\times r_{12}\times Q(0,0,2)$

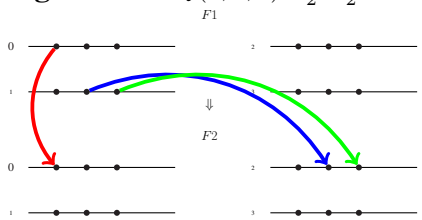

Figure S40. $Q(0,1,1):\frac{1}{2}\times\frac{1}{2}\times(1-r_{23})\times Q(0,2,2)$

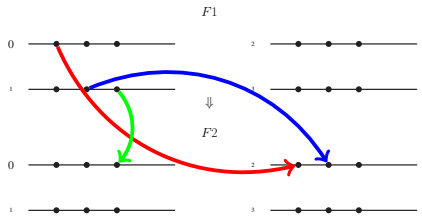

Figure S42. $Q(0,1,1):\frac{1}{2}\times r_{12}\times\frac{1}{2}\times Q(2,2,0)$

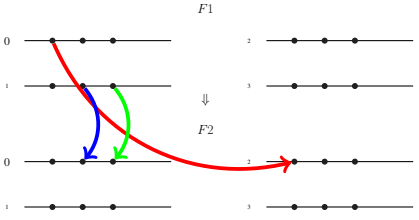

Figure S44. $Q(0,1,1):\frac{1}{2}\times\frac{1}{2}\times(1-r_{23})\times Q(2,0,0)$

2.1.6

See Figure S54.

$$\frac{1}{2}r_{12}Q(0,0,1)+\frac{1}{2}r_{12}Q(0,0,2)-Q(0,1,2)+\frac{1}{4}Q(0,2,1)+\frac{1}{4}Q(0,2,3)=0 \tag{S11}$$

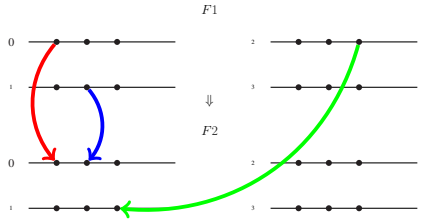

Figure S46. $Q(0,1,2):\frac{1}{2}\times\frac{1}{2}\times r_{12}\times Q(0,0,1)$

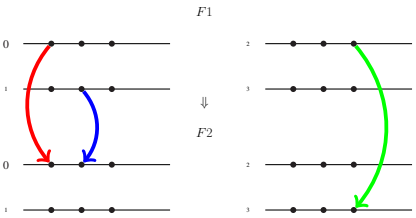

Figure S47. $Q(0,1,2):\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times r_{12}\times Q(0,0,3)$

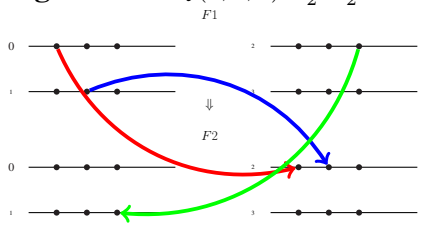

Figure S48. $Q(0,1,2):\frac{1}{2}\times\frac{1}{2}\times\times r_{12}\times Q(2,2,1)$

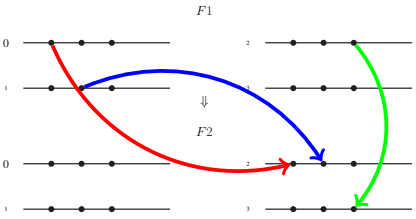

Figure S49. $Q(0,1,2):\frac{1}{2}\times\frac{1}{2}\times r_{12}\times Q(2,2,3)$

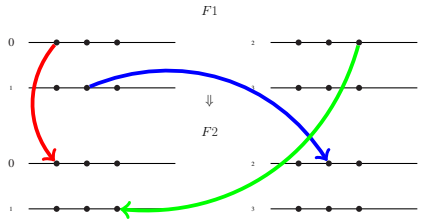

Figure S50. $Q(0,1,2):\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times Q(0,2,1)$

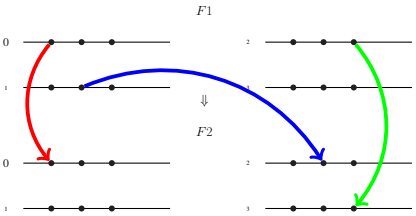

Figure S51. $Q(0,1,2):\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times Q(0,2,3)$

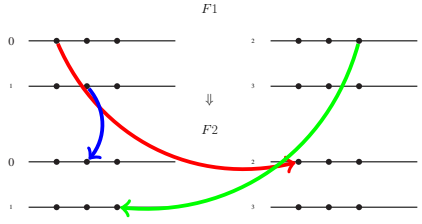

Figure S52. $Q(0,1,2):\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times Q(2,0,1)$

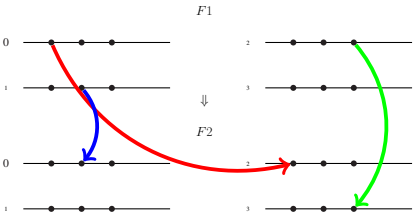

Figure S53. $Q(0,1,2):\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times Q(2,0,3)$

Figure S54. $Q(0,1,2)$

2.1.7

See Figure S63.

$$\frac{1}{2}(1-r_{13})Q(0,1,0)+\frac{1}{4}Q(0,1,2)+(\frac{1}{2}(1-r_{13})-1)Q(0,2,0)+\frac{1}{4}Q(0,1,2)=0 \tag{S12}$$

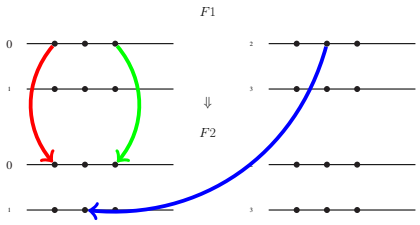

Figure S55. $Q(0,2,0):\frac{1}{2}\times(1-r_{13})\times\frac{1}{2}\times Q(0,1,0)$

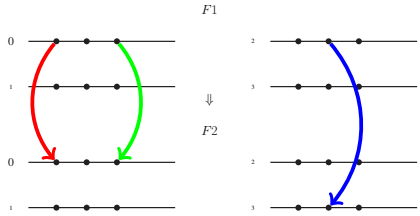

Figure S56. $Q(0,2,0):\frac{1}{2}\times(1-r_{13})\times\frac{1}{2}\times Q(0,3,0)$

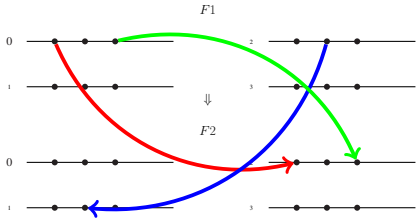

Figure S57. $Q(0,2,0):\frac{1}{2}\times(1-r_{13})\times\frac{1}{2}\times Q(2,1,2)$

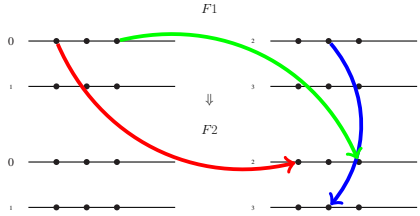

Figure S58. $Q(0,2,0):\frac{1}{2}\times(1-r_{13})\times\frac{1}{2}\times Q(2,3,2)$

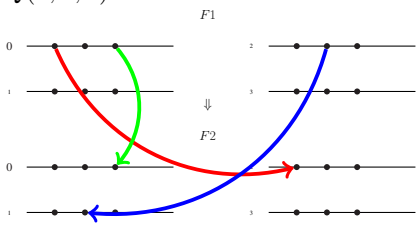

Figure S59. $Q(0,2,0):\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times Q(2,1,0)$

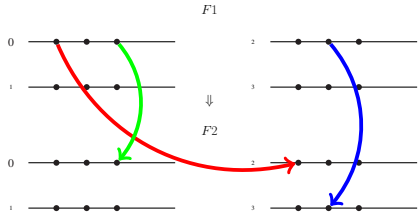

Figure S60. $Q(0,2,0):\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times Q(2,3,0)$

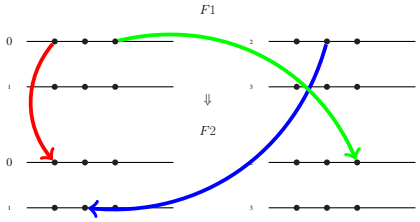

Figure S61. $Q(0,2,0):\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times Q(0,1,2)$

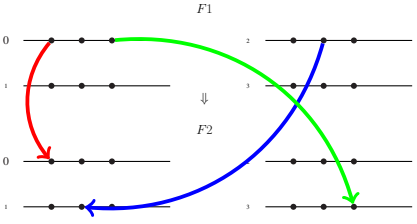

Figure S62. $Q(0,2,0):\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times Q(0,1,3)$

Figure S63. $Q(0,2,0)$

2.1.8

See Figure S72.

$$\frac{1}{2}r_{13}Q(0,1,0)+\frac{1}{4}Q(0,1,2)+\frac{1}{2}r_{13}Q(0,2,0)-Q(0,2,1)+\frac{1}{4}Q(0,2,3)=0 \tag{S13}$$

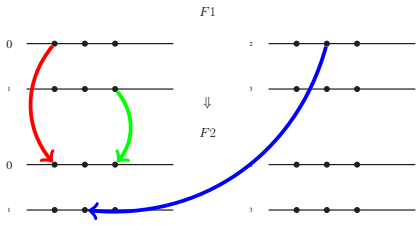

Figure S64. $Q(0,2,1):\frac{1}{2}\times r_{13}\times\frac{1}{2}\times Q(0,1,0)$

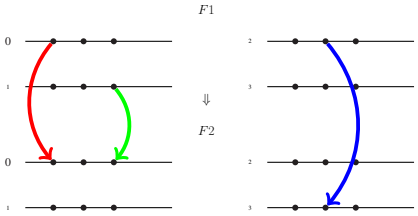

Figure S65. $Q(0,2,1):\frac{1}{2}\times r_{13}\times\frac{1}{2}\times Q(0,3,0)$

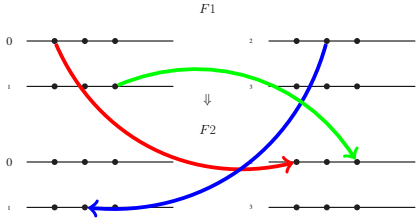

Figure S66. $Q(0,2,1):\frac{1}{2}\times r_{13}\times\frac{1}{2}\times Q(2,1,2)$

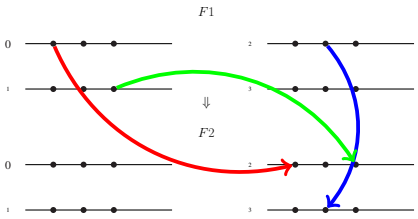

Figure S67. $Q(0,2,1):\frac{1}{2}\times r_{13}\times\frac{1}{2}\times Q(2,3,2)$

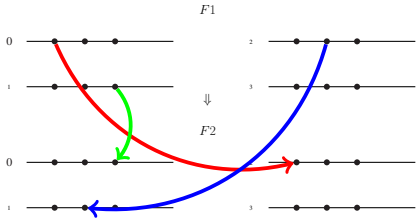

Figure S68. $Q(0,2,1):\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times Q(2,1,0)$

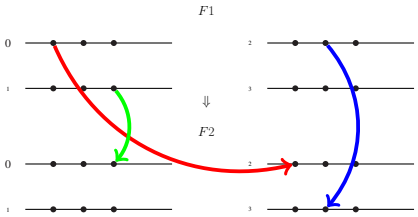

Figure S69. $Q(0,2,1):\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times Q(2,3,0)$

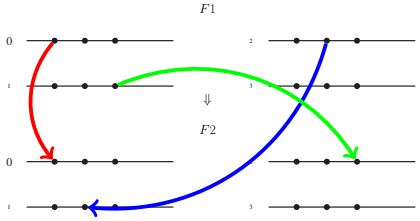

Figure S70. $Q(0,2,1):\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times Q(0,1,2)$

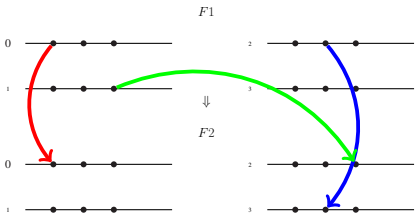

Figure S71. $Q(0,2,1):\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times Q(0,3,2)$

Figure S72. $Q(0,2,1)$

2.1.9

See Figure S81.

$$\frac{1}{2}(1-r_{23})Q(0,1,1)+\frac{1}{4}Q(0,1,2)+\frac{1}{4}Q(0,2,1)+(\frac{1}{2}(1-r_{23})-1)Q(0,2,2)=0 \tag{S14}$$

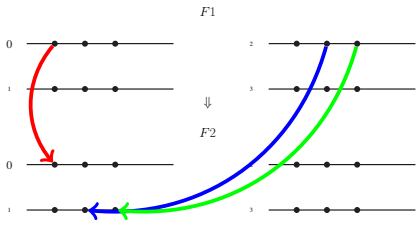

Figure S73. $Q(0,2,2):\frac{1}{2}\times\frac{1}{2}\times(1-r_{23})\times Q(0,1,1)$

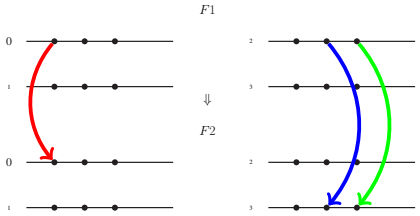

Figure S74. $Q(0,2,2):\frac{1}{2}\times\frac{1}{2}\times(1-r_{23})\times Q(0,3,3)$

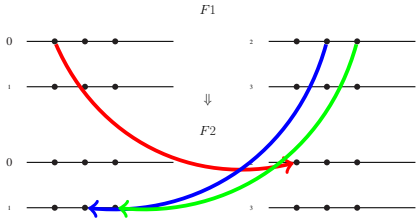

Figure S75. $Q(0,2,2):\frac{1}{2}\times\frac{1}{2}\times(1-r_{23})\times Q(2,1,1)$

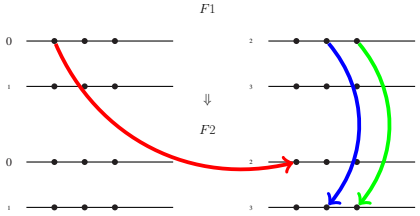

Figure S76. $Q(0,2,2):\frac{1}{2}\times\frac{1}{2}\times(1-r_{23})\times Q(2,3,3)$

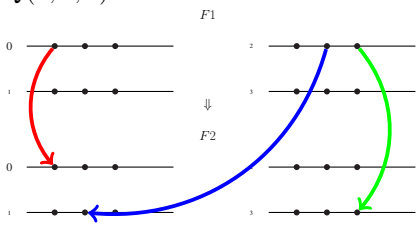

Figure S77. $Q(0,2,2):\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times Q(0,1,3)$

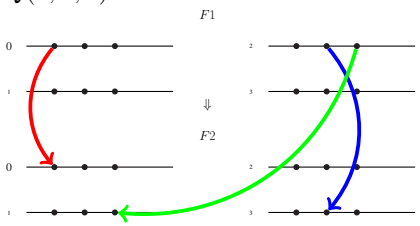

Figure S78. $Q(0,2,2):\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times Q(031)$

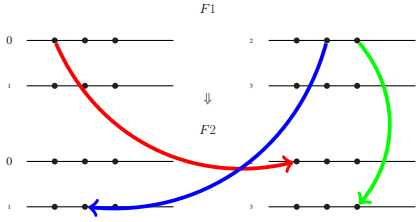

Figure S79. $Q(0,2,2):\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times Q(2,1,3)$

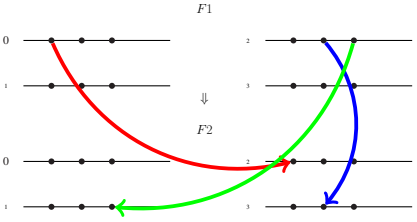

Figure S80. $Q(0,2,2):\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times Q(2,3,1)$

Figure S81. $Q(0,2,2)$

2.1.10

See Figure S90.

$$\frac{1}{2}r_{23}Q(0,1,1)+\frac{1}{4}Q(0,1,2)+\frac{1}{4}Q(0,2,1)+\frac{1}{2}r_{23}Q(0,2,2)-Q(0,2,3)=0$$

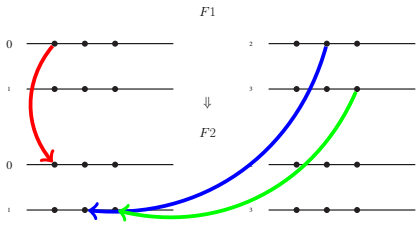

Figure S82. $Q(0,2,3):\frac{1}{2}\times\frac{1}{2}\times r_{23}\times Q(0,1,1)$

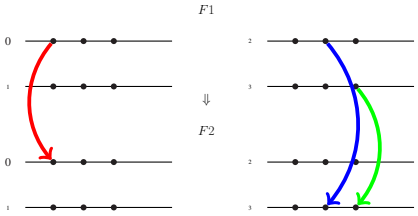

Figure S83. $Q(0,2,3):\frac{1}{2}\times\frac{1}{2}\times r_{23}\times Q(0,3,3)$

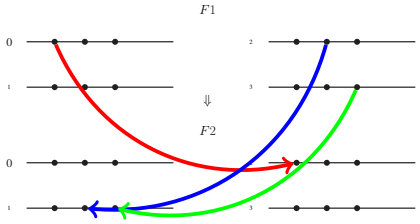

Figure S84. $Q(0,2,3):\frac{1}{2}\times\frac{1}{2}\times r_{23}\times Q(2,1,1)$

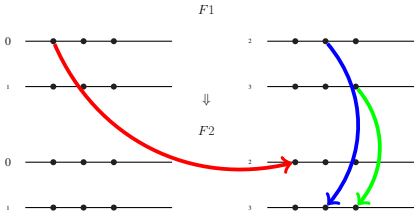

Figure S85. $Q(0,2,3):\frac{1}{2}\times\frac{1}{2}\times r_{23}\times Q(2,3,3)$

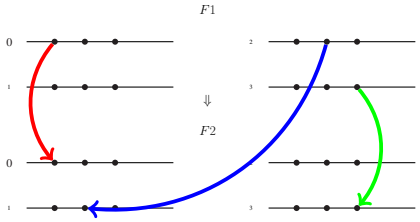

Figure S86. $Q(0,2,3):\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times Q(0,1,3)$

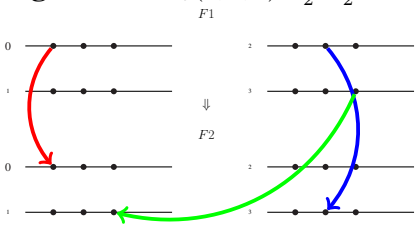

Figure S87. $Q(0,2,3):\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times Q(031)$

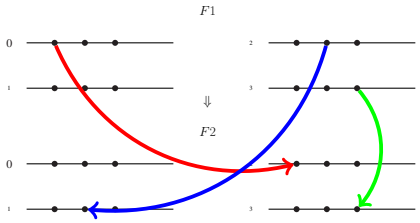

Figure S88. $Q(0,2,3):\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times Q(2,1,3)$

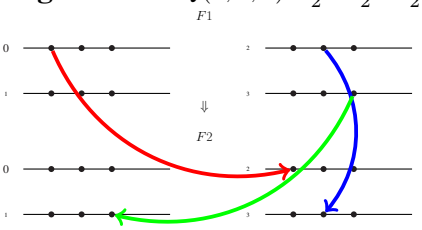

Figure S89. $Q(0,2,3):\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times Q(2,3,1)$

Figure S90. $Q(0,2,3)$