

Supplementary Material

1 SUPPLEMENTARY MATHEMATICS

Here we prove the equation

$$|Q_L| = 2^{L-2}(2^{L-1} + 1) \quad (\text{S1})$$

Proof: For $L \geq 2$, there are 4^L IBD (identical by descent) probabilities $Q(i_1, i_2, \dots, i_L)$ since $i_l = 0, 1, 2$ or 3 and furthermore they add up to 1. A number of these probabilities are equal because of two symmetries: (1) the two homologous chromosomes in each individual play identical roles, and (2) the siblings play identical roles (assuming no sex-dependence of meiosis, so that the recombination rates $r_{l,l'}$ are sex-independent). It is thus appropriate to use only one representative of each symmetry equivalence class. A way to do this is to impose that this representative have its first index, i_1 , equal to zero. In fact we can identify exactly one element in each class by imposing that the indices of the representative Q 's have either

1. $i_l \in \{0, 1\} \forall l \in \{2, \dots, L\}$, or
2. $i_l \in \{0, 1\} \forall l \in \{2, \dots, K-1\}$, $i_K = 2$ and $i_l \in \{0, 1, 2, 3\} \forall l \in \{K+1, \dots, L\}$

The number of equivalence classes and thus of Q 's to consider is then

$$|Q_L| = 2^{L-1} + \sum_{l=2}^L 2^{l-2} 4^{L-l} = 2^{L-1} + 2^{2L-2} \sum_{l=2}^L 2^{-l} \quad (\text{S2})$$

Given that $\sum_{l=2}^L 2^{-l}$ is a geometric progression of common ratio 2^{-1} from 2 to L , the sum of its terms can be expressed as:

$$\sum_{l=2}^L 2^{-l} = \frac{2^{-2} - 2^{-(L-1)}}{1 - 2^{-1}} = 2^{-1} - 2^{-L} \quad (\text{S3})$$

Substituting S3 in S2, we get

$$|Q_L| = 2^{L-1} + 2^{2L-2}(2^{-1} - 2^{-L}) = 2^{L-1} + 2^{2L-3} - 2^{L-2} \quad (\text{S4})$$

Factorizing with respect to 2^{L-2} and after simplification, this gives

$$|Q_L| = 2^{L-2}(1 + 2^{L-1}). \quad (\text{S5})$$

2 THE SELF-CONSISTENT EQUATIONS FOR THREE LOCI

Here we provide the coefficients entering each of the $|Q_L| = 10$ self-consistent equations for $L = 3$.

2.1 The self consistent equation for $Q(0, 0, 0)$

Figure S1 displays the 8 factors in the self-consistent equation for $Q(0, 0, 0)$:

$$Q(0, 0, 0) = \frac{1}{2}(1 - r_{12})(1 - r_{23})[Q(0, 0, 0) + Q(2, 2, 2)] + \frac{1}{4}(1 - r_{12})[Q(0, 0, 2) + Q(2, 2, 0)] + \frac{1}{4}(1 - r_{13})[Q(0, 2, 0) + Q(2, 0, 2)] + \frac{1}{4}(1 - r_{23})[Q(0, 2, 2) + Q(2, 0, 0)] \quad (\text{S6})$$

After use of symmetry to keep only non-equivalent Q s, this leads to

$$Q(0, 0, 0) = (1 - r_{12})(1 - r_{23})Q(0, 0, 0) + \frac{1}{2}(1 - r_{12})Q(0, 0, 2) + \frac{1}{2}(1 - r_{13})Q(0, 2, 0) + \frac{1}{2}(1 - r_{23})Q(0, 2, 2)$$

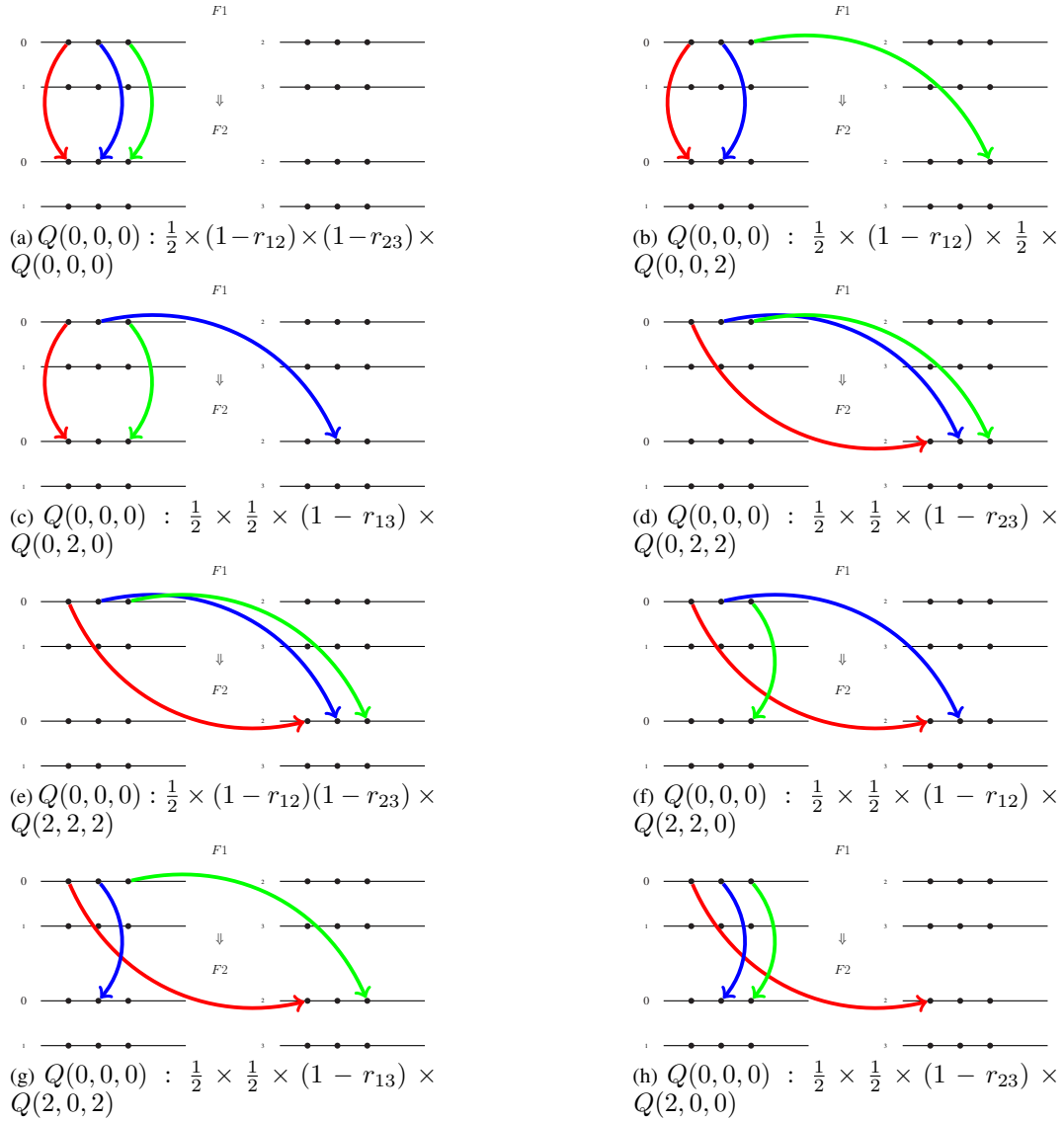


Figure S1: The graphical representation of the factors multiplying each Q on the right-hand side of Eq. S6 for $Q(0, 0, 0)$.

2.2 The self consistent equation for $Q(0, 0, 1)$

Figure S2 displays the 8 factors in the self-consistent equation for $Q(0, 0, 1)$:

$$Q(0, 0, 1) = \frac{1}{2}(1 - r_{12})r_{23}[Q(0, 0, 0) + Q(2, 2, 2)] + \frac{1}{4}(1 - r_{12})[Q(0, 0, 2) + Q(2, 2, 0)] + \frac{1}{4}r_{13}[Q(0, 2, 0)Q(2, 0, 2)] + \frac{1}{4}r_{23}[Q(0, 2, 2) + Q(2, 0, 0)] \quad (\text{S7})$$

After use of symmetry to keep only non-equivalent Q s, this leads to

$$Q(0, 0, 1) = (1 - r_{12})r_{23}Q(0, 0, 0) + \frac{1}{2}(1 - r_{12})Q(0, 0, 2) + \frac{1}{2}r_{13}Q(0, 2, 0) + \frac{1}{2}r_{23}Q(0, 2, 2)$$

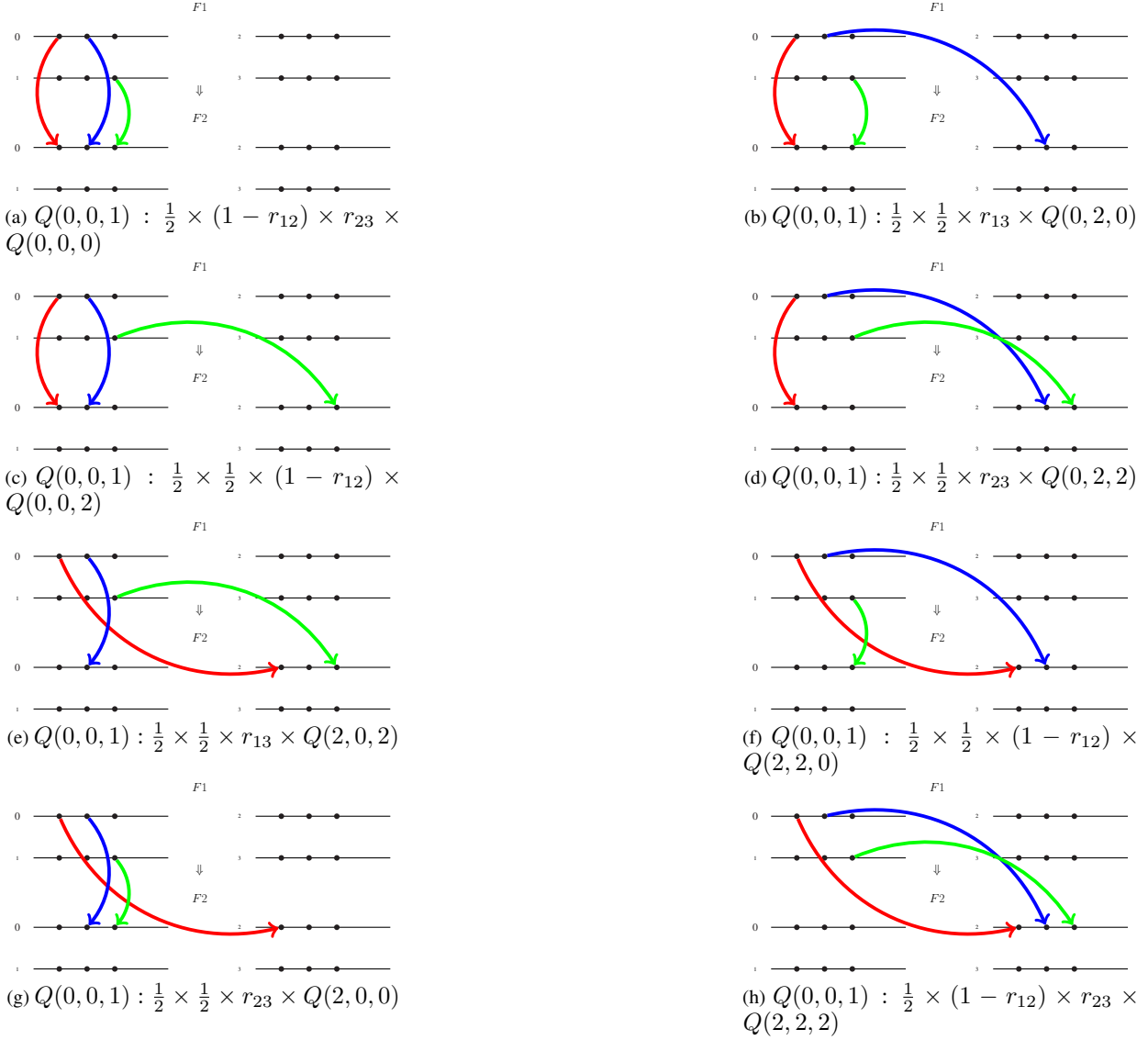


Figure S2: The graphical representation of the factors multiplying each Q on the right-hand side of Eq. S7 for $Q(0, 0, 1)$.

2.3 The self consistent equation for $Q(0, 0, 2)$

Figure S3 displays the 8 factors in the self-consistent equation for $Q(0, 0, 2)$:

$$Q(0, 0, 2) = \frac{1}{4}(1 - r_{12})[Q(0, 0, 1) + Q(2, 2, 3)] + \frac{1}{4}(1 - r_{12})[Q(0, 0, 3) + Q(2, 2, 1)] + \frac{1}{8}[Q(0, 2, 1) + Q(2, 0, 3)] + \frac{1}{8}[Q(0, 2, 3) + Q(2, 0, 1)] \quad (\text{S8})$$

After use of symmetry to keep only non-equivalent Q s, this leads to

$$Q(0, 0, 2) = \frac{1}{2}(1 - r_{12})Q(0, 0, 1) + \frac{1}{2}(1 - r_{12})Q(0, 0, 2) + \frac{1}{4}Q(0, 2, 1) + \frac{1}{4}Q(0, 2, 3)$$

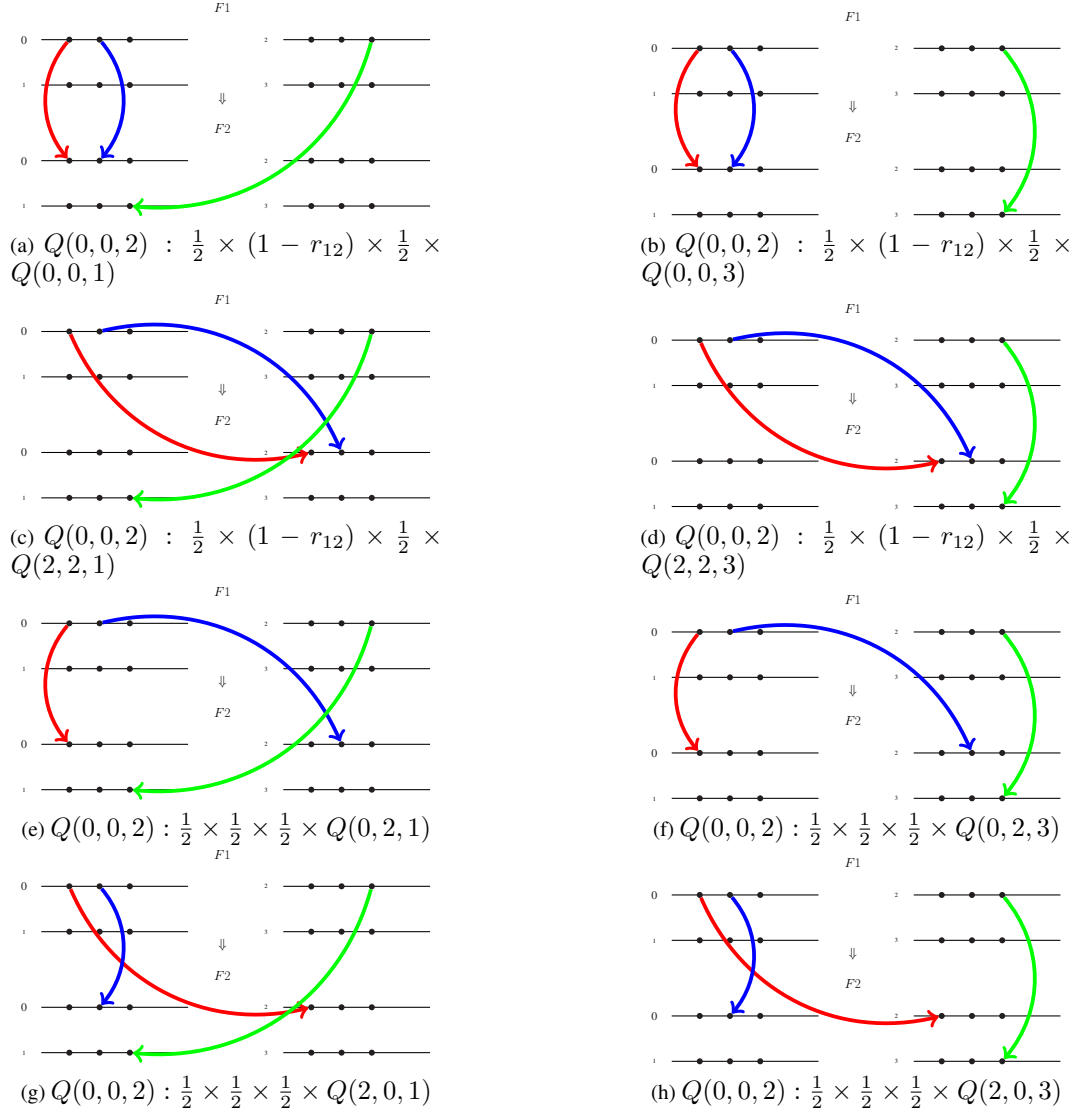


Figure S3: The graphical representation of the factors multiplying each Q on the right-hand side of Eq. S8 for $Q(0, 0, 2)$.

2.4 The self consistent equation for $Q(0, 1, 0)$

Figure S4 displays the 8 factors in the self-consistent equation for $Q(0, 1, 0)$:

$$Q(0, 1, 0) = \frac{1}{2}r_{12}r_{23}[Q(0, 0, 0) + Q(2, 2, 2)] + \frac{1}{4}r_{12}[Q(0, 0, 2) + Q(2, 2, 0)] + \frac{1}{4}(1 - r_{13})[Q(0, 2, 0) + Q(2, 0, 2)] + \frac{1}{4}r_{23}[Q(0, 2, 2) + Q(2, 0, 0)] \quad (\text{S9})$$

After use of symmetry to keep only non-equivalent Q s, this leads to

$$Q(0, 1, 0) = r_{12}r_{23}Q(0, 0, 0) + \frac{1}{2}r_{12}Q(0, 0, 2) + \frac{1}{2}(1 - r_{13})Q(0, 2, 0) + \frac{1}{2}r_{23}Q(0, 2, 2)$$

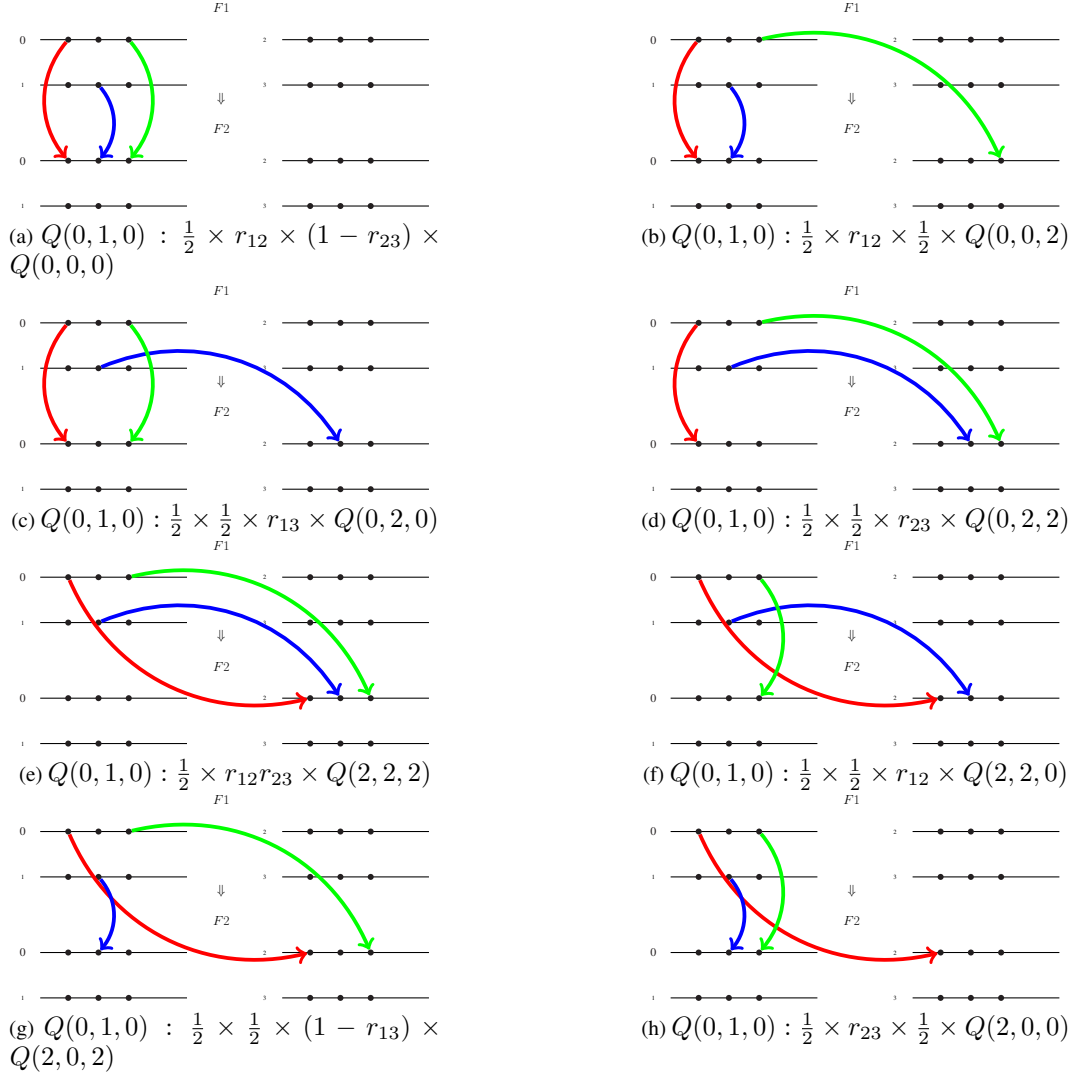


Figure S4: The graphical representation of the factors multiplying each Q on the right-hand side of Eq. S9 for $Q(0, 1, 0)$.

2.5 The self consistent equation for $Q(0, 1, 1)$

Figure S5 displays the 8 factors in the self-consistent equation for $Q(0, 1, 1)$:

$$Q(0, 1, 1) = \frac{1}{2}r_{12}(1 - r_{23})[Q(0, 0, 0) + Q(2, 2, 2)] + \frac{1}{4}r_{12}[Q(0, 0, 2) + Q(2, 2, 0)] + \frac{1}{4}r_{13}[Q(0, 2, 0) + Q(2, 0, 2)] + \frac{1}{4}(1 - r_{23})[Q(0, 2, 2) + Q(2, 0, 0)] \quad (\text{S10})$$

After use of symmetry to keep only non-equivalent Q s, this leads to

$$Q(0, 1, 1) = r_{12}(1 - r_{23})Q(0, 0, 0) + \frac{1}{2}r_{12}Q(0, 0, 2) + \frac{1}{2}r_{13}Q(0, 2, 0) + \frac{1}{2}(1 - r_{23})Q(0, 2, 2)$$

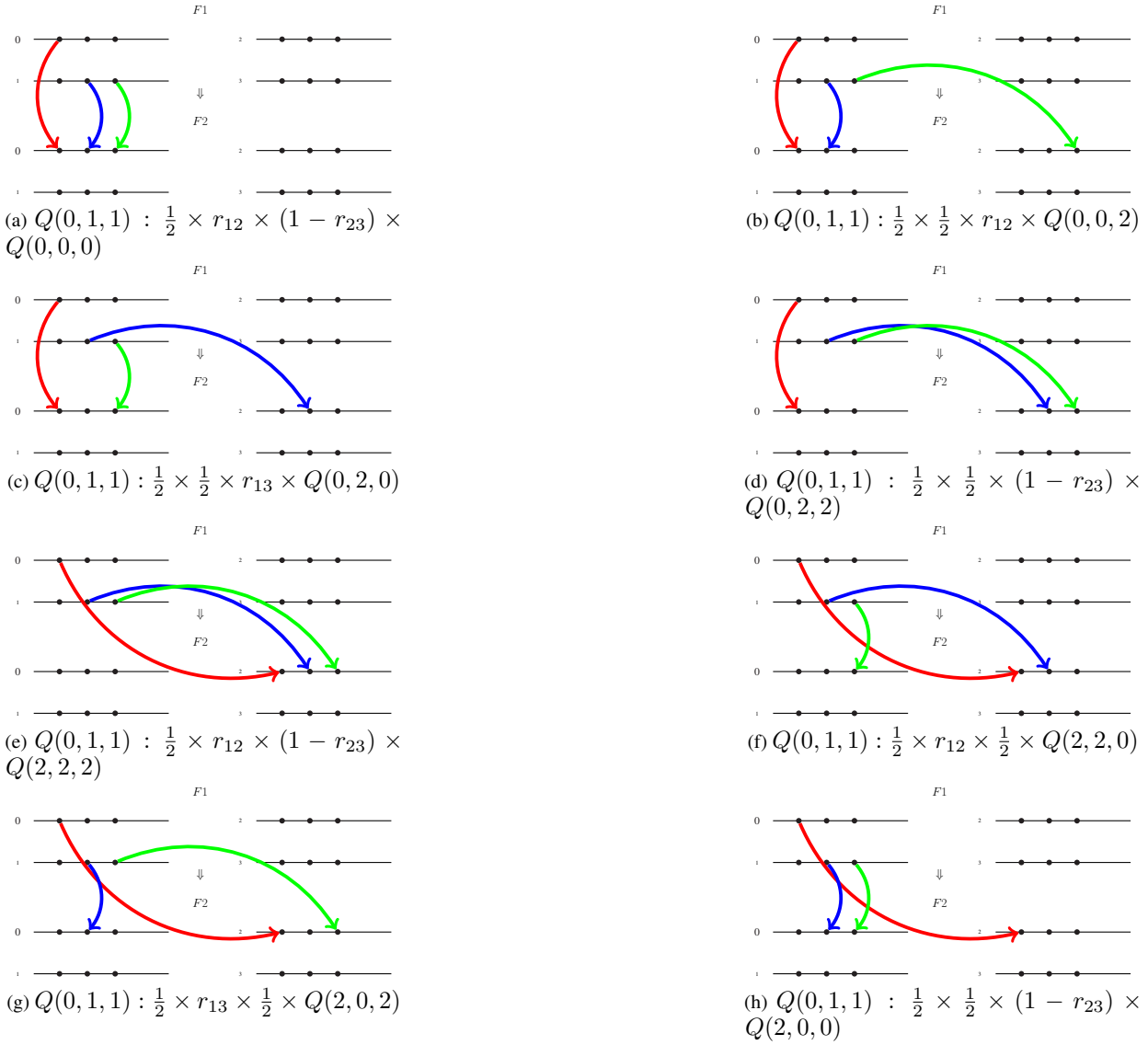


Figure S5: The graphical representation of the factors multiplying each Q on the right-hand side of Eq. S10 for $Q(0, 1, 1)$.

2.6 The self consistent equation for $Q(0, 1, 2)$

Figure S6 displays the 8 factors in the self-consistent equation for $Q(0, 1, 2)$:

$$Q(0, 1, 2) = \frac{1}{4}r_{12}[Q(0, 0, 1) + Q(2, 2, 3)] + \frac{1}{4}r_{12}[Q(0, 0, 3) + Q(2, 2, 1)] + \frac{1}{8}[Q(0, 2, 1) + Q(2, 0, 3)] + \frac{1}{8}[Q(0, 2, 3) + Q(2, 0, 1)] \quad (\text{S11})$$

After use of symmetry to keep only non-equivalent Q s, this leads to

$$Q(0, 1, 2) = \frac{1}{2}r_{12}Q(0, 0, 1) + \frac{1}{2}r_{12}Q(0, 0, 2) + \frac{1}{4}Q(0, 2, 1) + \frac{1}{4}Q(0, 2, 3)$$

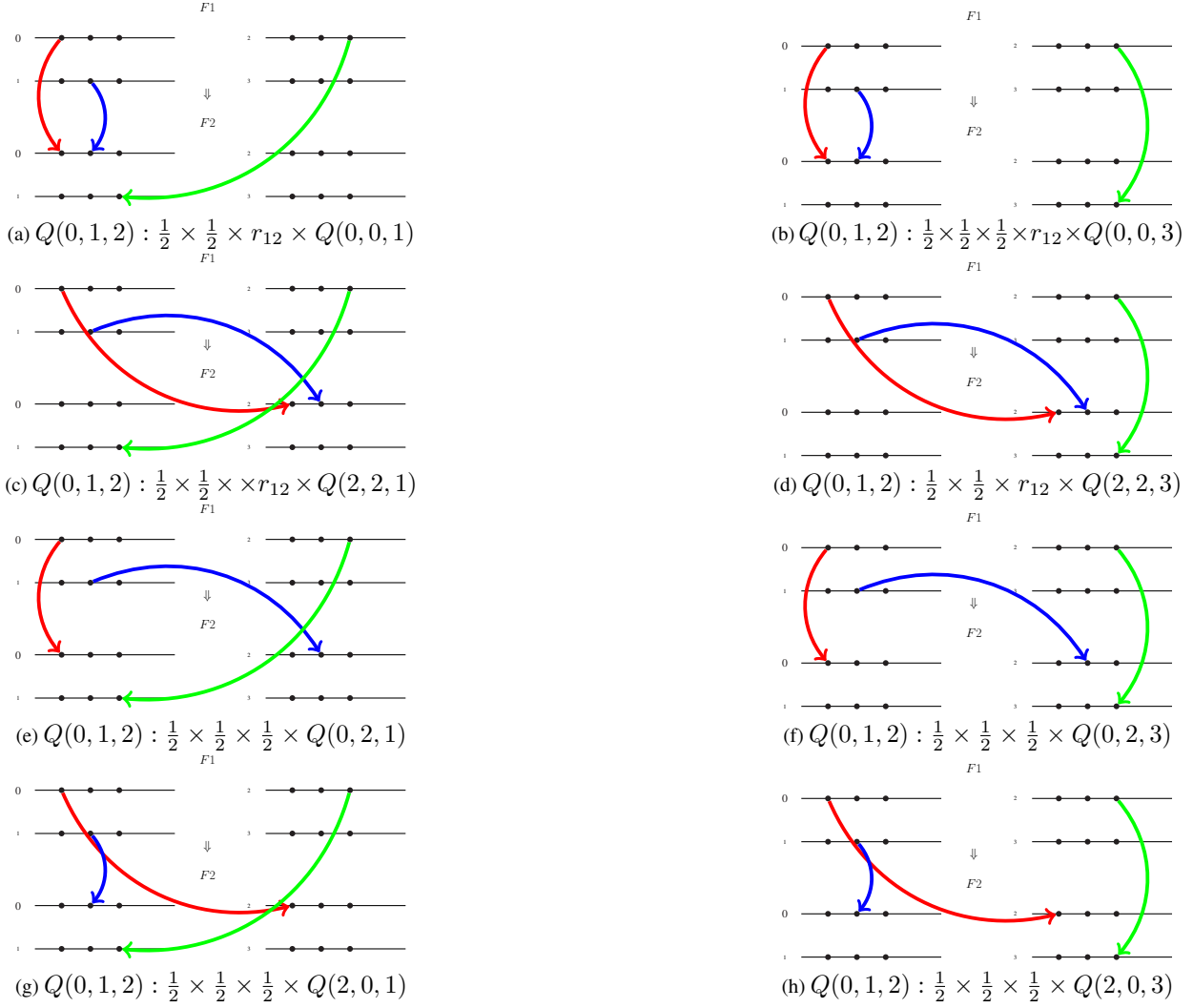


Figure S6: The graphical representation of the factors multiplying each Q on the right-hand side of Eq. S11 for $Q(0, 1, 2)$.

2.7 The self consistent equation for $Q(0, 2, 0)$

Figure S7 displays the 8 factors in the self-consistent equation for $Q(0, 2, 0)$:

$$Q(0, 2, 0) = \frac{1}{4}(1 - r_{13})[Q(0, 1, 0) + Q(2, 3, 2)] + \frac{1}{8}[Q(0, 1, 2) + Q(2, 3, 0)] + \frac{1}{4}(1 - r_{13})[Q(0, 3, 0) + Q(2, 1, 2)] + \frac{1}{8}[Q(0, 1, 3) + Q(2, 1, 0)] \quad (\text{S12})$$

After use of symmetry to keep only non-equivalent Q s, this leads to

$$Q(0, 2, 0) = \frac{1}{2}(1 - r_{13})Q(0, 1, 0) + \frac{1}{4}Q(0, 1, 2) + \frac{1}{2}(1 - r_{13})Q(0, 2, 0) + \frac{1}{4}Q(0, 1, 2)$$

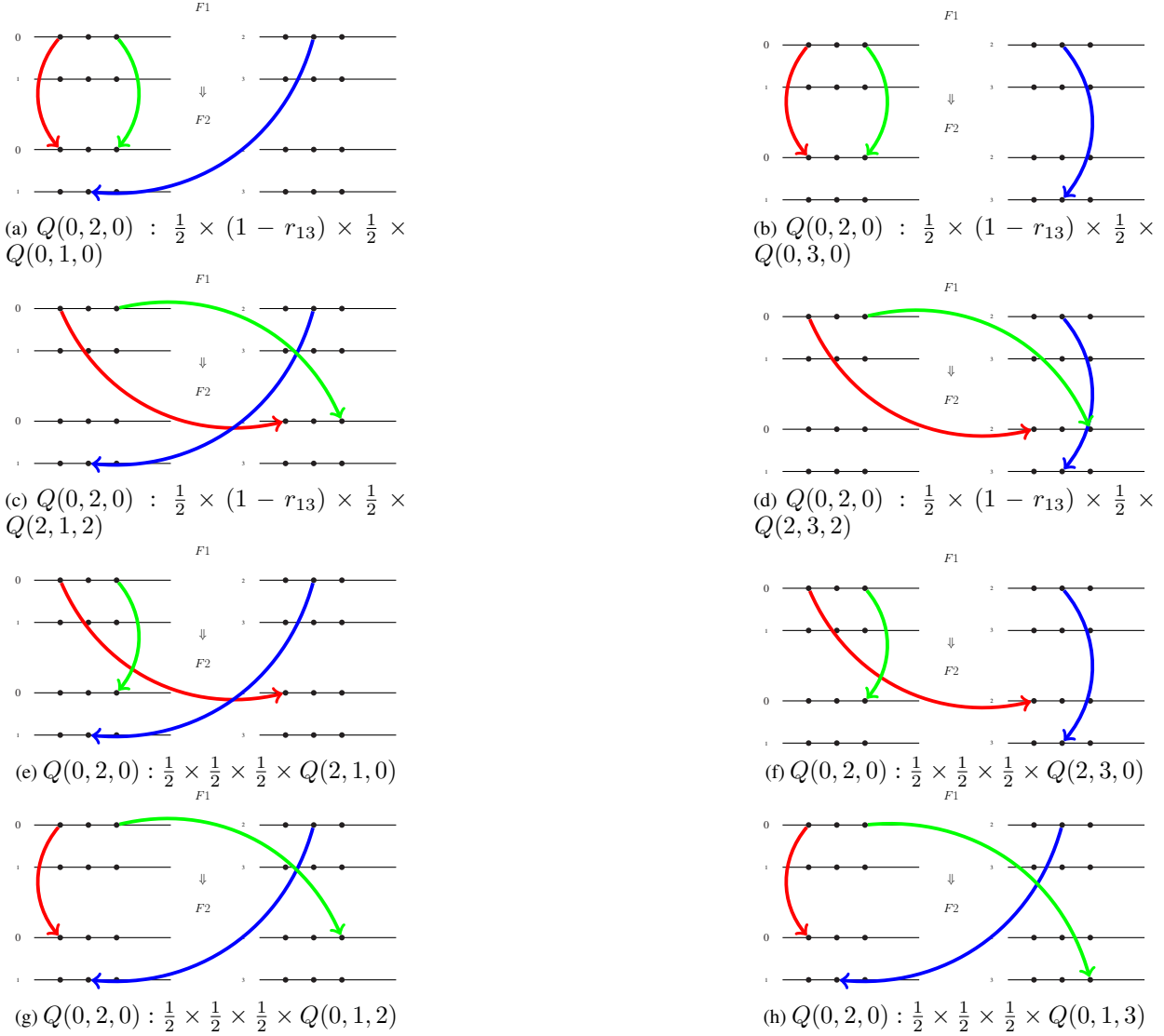


Figure S7: The graphical representation of the factors multiplying each Q on the right-hand side of Eq. S12 for $Q(0, 2, 0)$.

2.8 The self consistent equation for $Q(0, 2, 1)$

Figure S8 displays the 8 factors in the self-consistent equation for $Q(0, 2, 1)$:

$$Q(0, 2, 1) = \frac{1}{4}r_{13}[Q(0, 1, 0) + Q(2, 3, 2)] + \frac{1}{8}[Q(0, 1, 2) + Q(2, 3, 0)] + \frac{1}{4}r_{13}[Q(0, 3, 0) + Q(2, 1, 2)] + \frac{1}{8}[Q(0, 3, 2) + Q(2, 1, 0)] \quad (\text{S13})$$

After use of symmetry to keep only non-equivalent Q s, this leads to

$$Q(0, 2, 1) = \frac{1}{2}r_{13}Q(0, 1, 0) + \frac{1}{4}Q(0, 1, 2) + \frac{1}{2}r_{13}Q(0, 2, 0) + \frac{1}{4}Q(0, 2, 3)$$

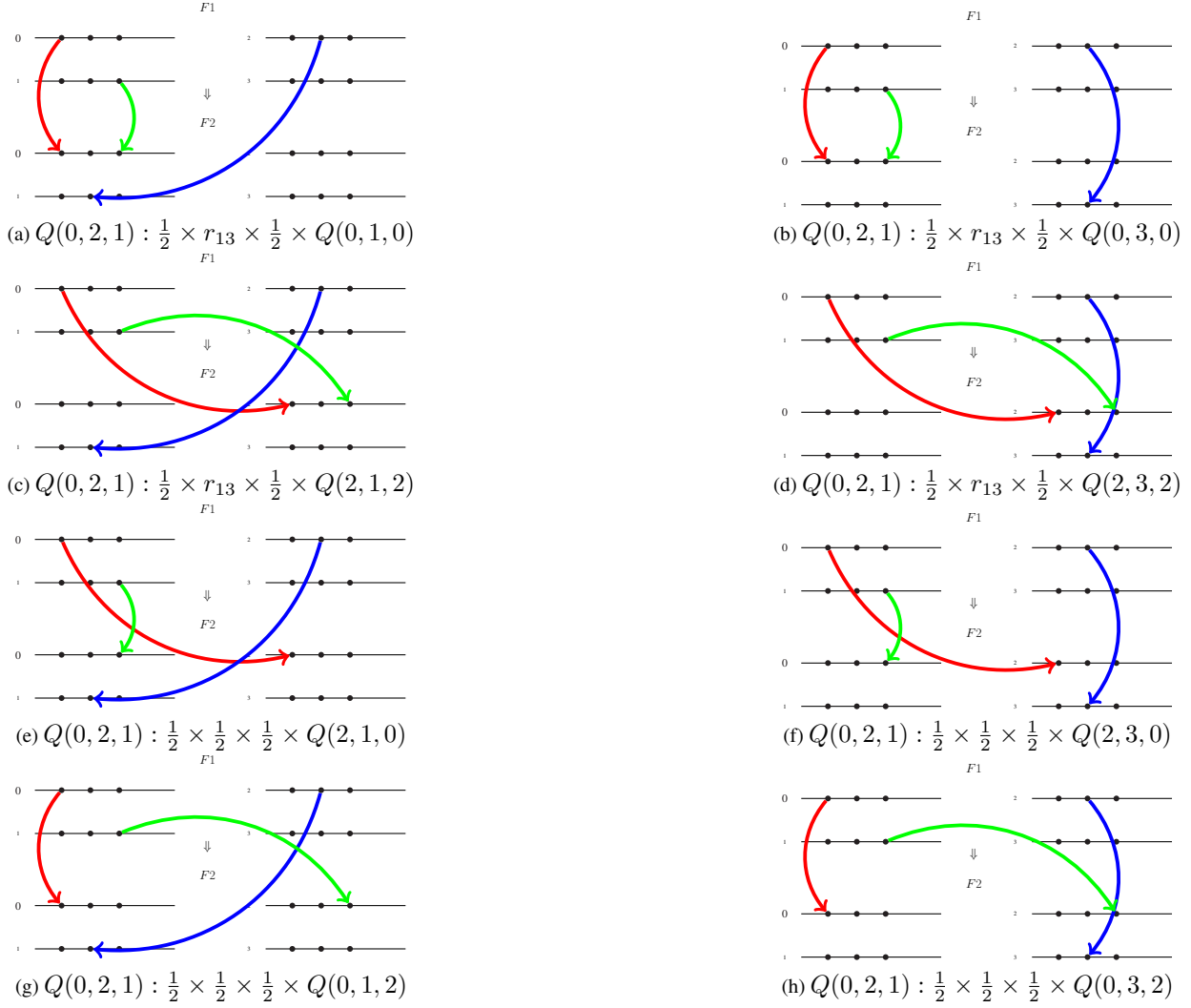


Figure S8: The graphical representation of the factors multiplying each Q on the right-hand side of Eq. S13 for $Q(0, 2, 1)$.

2.9 The self consistent equation for $Q(0, 2, 2)$

Figure S9 displays the 8 factors in the self-consistent equation for $Q(0, 2, 2)$:

$$Q(0, 2, 2) = \frac{1}{4}(1 - r_{23})[Q(0, 1, 1) + Q(2, 3, 3)] + \frac{1}{8}[Q(0, 1, 3) + Q(2, 3, 1)] + \frac{1}{8}[Q(0, 3, 1) + Q(2, 1, 3)] + \frac{1}{4}(1 - r_{23})[Q(0, 3, 3) + Q(2, 1, 1)] \quad (\text{S14})$$

After use of symmetry to keep only non-equivalent Q s, this leads to

$$Q(0, 2, 2) = \frac{1}{2}(1 - r_{23})Q(0, 1, 1) + \frac{1}{4}Q(0, 1, 2) + \frac{1}{4}Q(0, 2, 1) + \frac{1}{2}(1 - r_{23})Q(0, 2, 2)$$

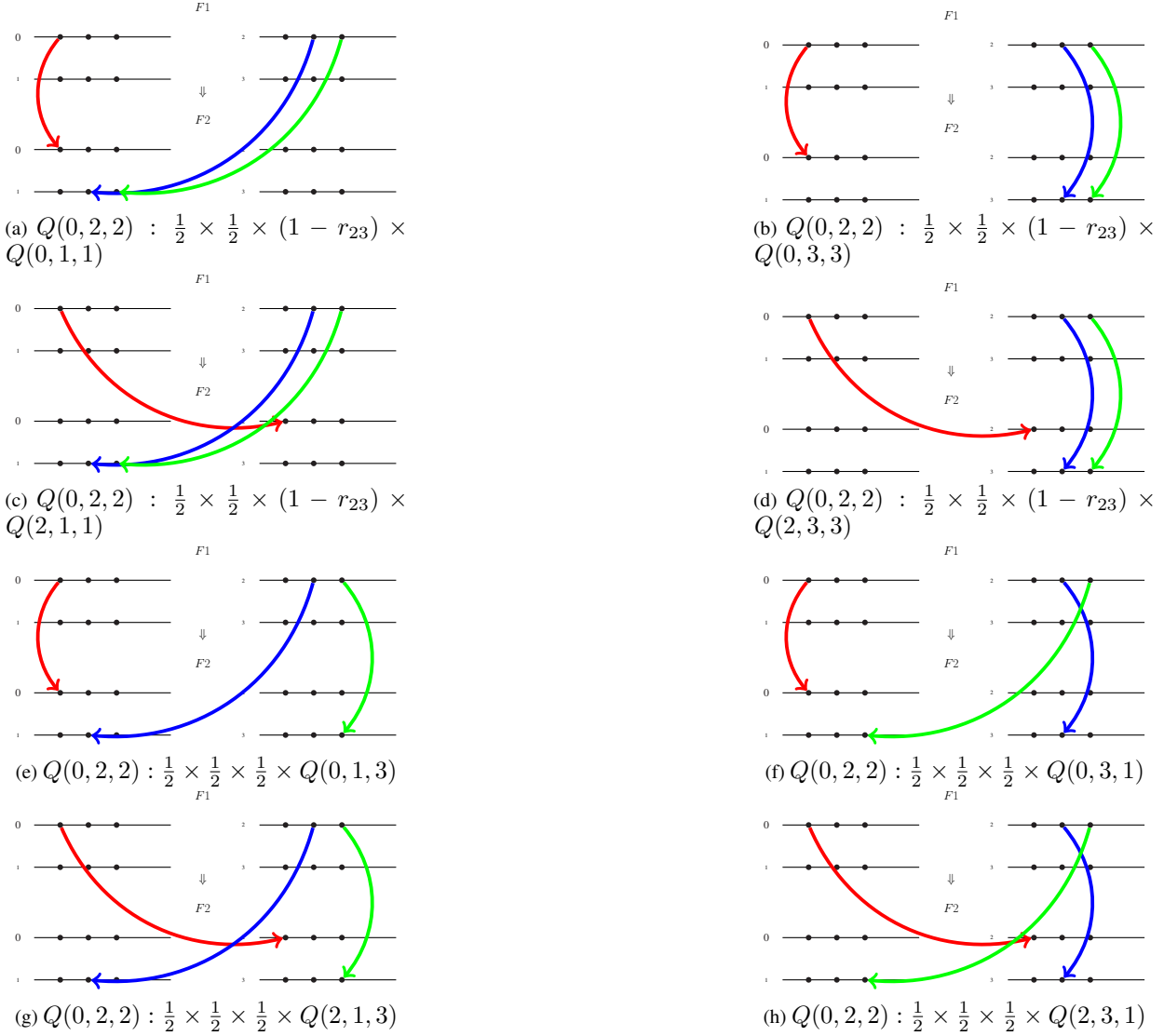


Figure S9: The graphical representation of the factors multiplying each Q on the right-hand side of Eq. S14 for $Q(0, 2, 2)$.

2.10 The self consistent equation for $Q(0, 2, 3)$

Figure S10 displays the 8 factors in the self-consistent equation for $Q(0, 2, 3)$:

$$Q(0, 2, 3) = \frac{1}{4}r_{23}[Q(0, 1, 1) + Q(2, 3, 3)] + \frac{1}{8}[Q(0, 1, 3) + Q(2, 3, 1)] + \frac{1}{8}[Q(0, 2, 1) + Q(2, 1, 3)] + \frac{1}{4}r_{23}[Q(0, 3, 3) + Q(2, 1, 1)] \quad (\text{S15})$$

After use of symmetry to keep only non-equivalent Q s, this leads to

$$Q(0, 2, 3) = \frac{1}{2}r_{23}Q(0, 1, 1) + \frac{1}{4}Q(0, 1, 2) + \frac{1}{4}Q(0, 2, 1) + \frac{1}{2}r_{23}Q(0, 2, 2)$$

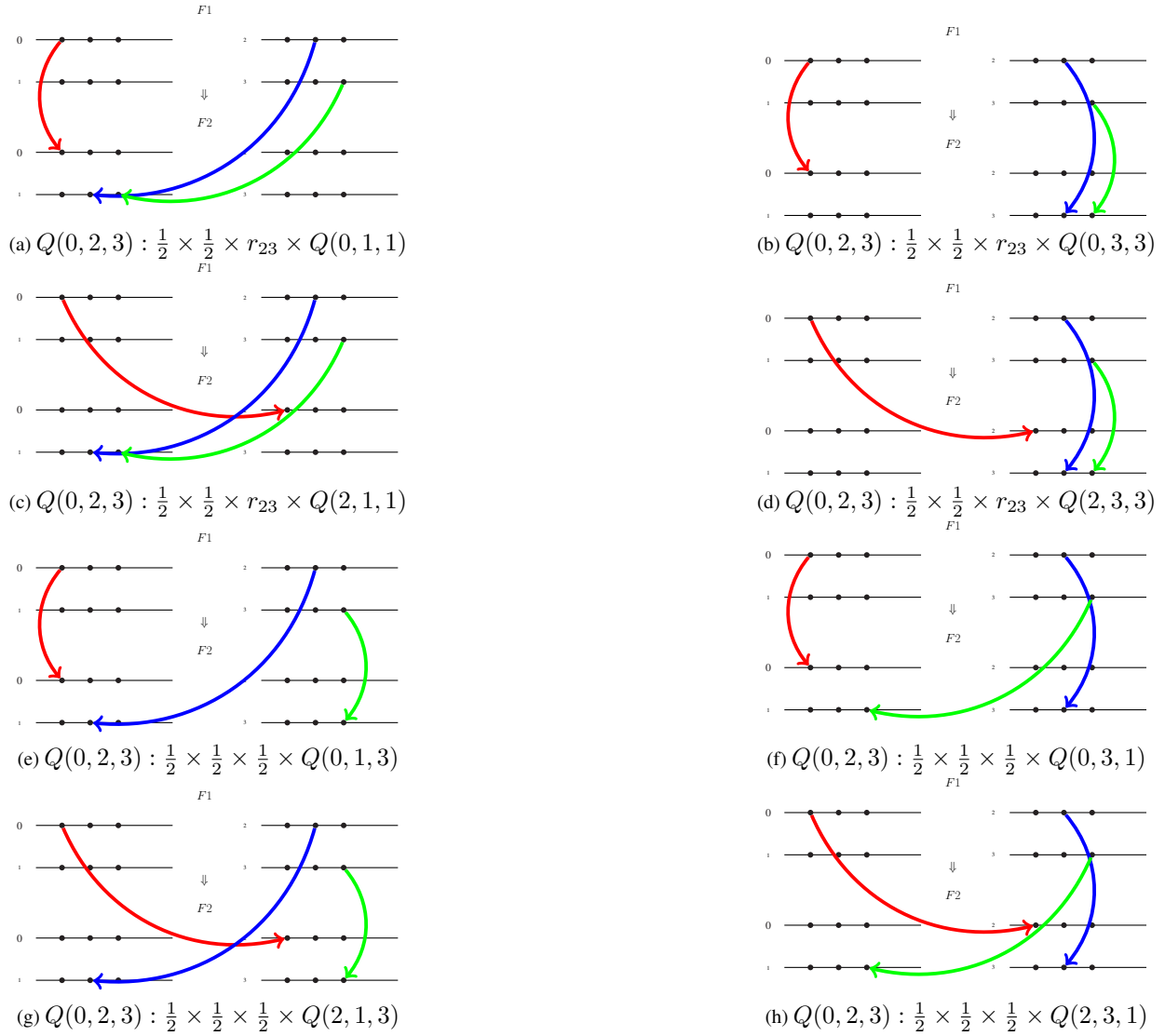


Figure S10: The graphical representation of the factors multiplying each Q on the right-hand side of Eq. S15 for $Q(0, 2, 3)$.