

# Supplementary Material

## 1 SUPPLEMENTARY MATHEMATICS

### 1.1 The derivation of equation

$$|Q_L| = 2^{L-2}(2^{L-1} + 1) \quad (S1)$$

**Proof:** For  $L \geq 2$ , there are  $4^L$  IBD (identical by descent) probabilities  $Q(i_1, i_2, \dots, i_L)$  since  $i_l = 1, 2, 3$  or  $4$  and furthermore they add up to 1. A number of these probabilities are equal because of two symmetries: (1) the two homologous chromosomes in each individual play identical roles, and (2) the siblings play identical roles (assuming no sex-dependence of meiosis, so that for instance the recombination rates  $r_{l,l'}$  are sex-independent). It is thus appropriate to use only one representative of each symmetry equivalence class, so that for instance one may impose this representative to have its first index,  $i_1$ , equal to zero. In fact one can identify exactly one element in each class by imposing that the indices of the representative  $Q$ 's have either

1.  $i_l \in \{0, 1\} \forall l \in \{2, \dots, L\}$ , or
2.  $i_l \in \{0, 1\} \forall l \in \{2, \dots, K-1\}$ ,  $i_K = 2$  and  $i_l \in \{0, 1, 2, 3\} \forall l \in \{K+1, \dots, L\}$

The number of equivalence classes and thus of  $Q$ 's to consider is then

$$|Q_L| = 2^{L-1} + \sum_{l=2}^L 2^{l-2} 4^{L-l} = 2^{L-1} + 2^{2L-2} \sum_{l=2}^L 2^{-l} \quad (S2)$$

Given that  $\sum_{l=2}^L 2^{-l}$  is a geometric progression of common ratio  $2^{-1}$  from 2 to  $L$ , the sum of its terms can be expressed as:

$$\sum_{l=2}^L 2^{-l} = \frac{2^{-2} - 2^{-(L-1)}}{1 - 2^{-1}} = 2^{-1} - 2^{-L} \quad (S3)$$

Substituting S3 in S2, we get

$$|Q_L| = 2^{L-1} + 2^{2L-2}(2^{-1} - 2^{-L}) = 2^{L-1} + 2^{2L-3} - 2^{L-2} \quad (S4)$$

Factorizing with respect to  $2^{L-2}$  and after simplification, this gives

$$|Q_L| = 2^{L-2}(1 + 2^{L-1}). \quad (S5)$$

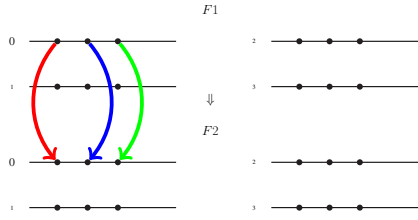
## 2 THREE LOCI 2-WAY RIL

### 2.1 The SCHP equations

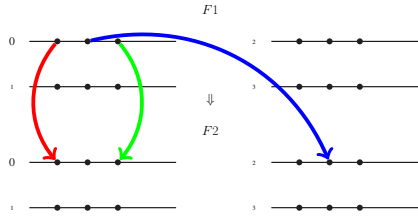
#### 2.1.1

See Figure S9

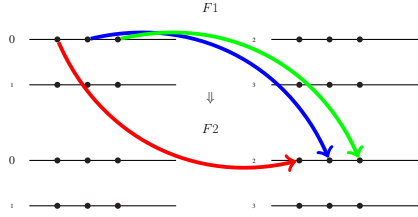
$$((1 - r_{12})(1 - r_{23}) - 1)Q(0, 0, 0) + \frac{1}{2}(1 - r_{12})Q(0, 0, 2) + \frac{1}{2}(1 - r_{13})Q(0, 2, 0) + \frac{1}{2}(1 - r_{23})Q(0, 2, 2) = 0 \quad (\text{S6})$$



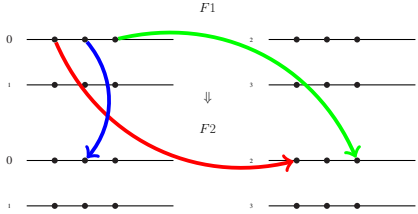
**Figure S1.**  $Q(0, 0, 0) : \frac{1}{2} \times (1 - r_{12}) \times (1 - r_{23}) \times Q(0, 0, 0)$



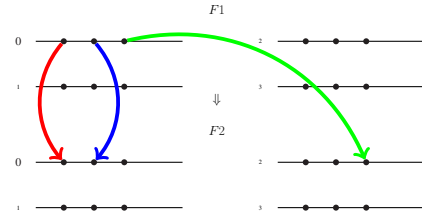
**Figure S3.**  $Q(0, 0, 0) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{13}) \times Q(0, 2, 0)$



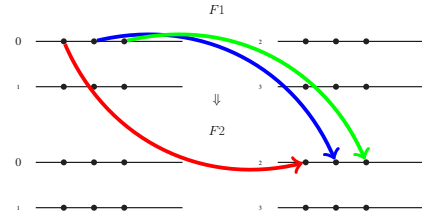
**Figure S5.**  $Q(0, 0, 0) : \frac{1}{2} \times (1 - r_{12})(1 - r_{23}) \times Q(2, 2, 2)$



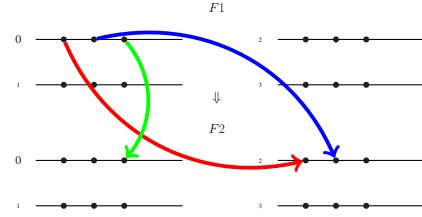
**Figure S7.**  $Q(0, 0, 0) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{13}) \times Q(2, 0, 2)$



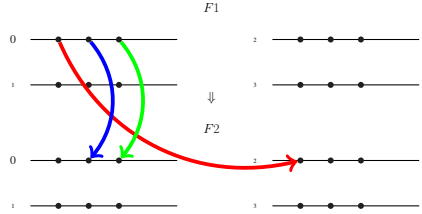
**Figure S2.**  $Q(0, 0, 0) : \frac{1}{2} \times (1 - r_{12}) \times \frac{1}{2} \times Q(0, 0, 2)$



**Figure S4.**  $Q(0, 0, 0) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{23}) \times Q(0, 2, 2)$



**Figure S6.**  $Q(0, 0, 0) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{12}) \times Q(2, 2, 0)$



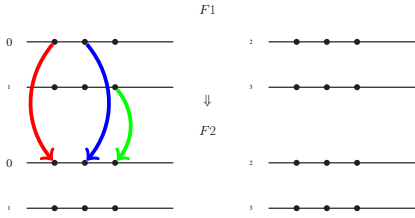
**Figure S8.**  $Q(0, 0, 0) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{23}) \times Q(2, 0, 0)$

**Figure S9.**  $Q(0, 0, 0)$

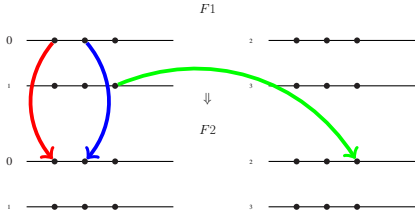
## 2.1.2

See Figure S18

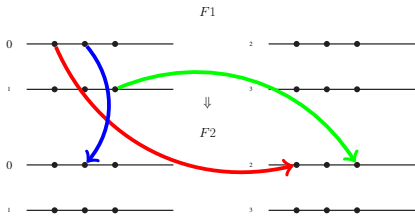
$$(1 - r_{12})r_{23}Q(0, 0, 0) - Q(0, 0, 1) + \frac{1}{2}(1 - r_{12})Q(0, 0, 2) + \frac{1}{2}r_{13}Q(0, 2, 0) + \frac{1}{2}r_{23}Q(0, 2, 2) = 0 \quad (\text{S7})$$



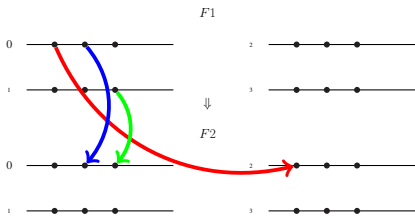
**Figure S10.**  $Q(0, 0, 1) : \frac{1}{2} \times (1 - r_{12}) \times r_{23} \times Q(0, 0, 0)$



**Figure S12.**  $Q(0, 0, 1) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{12}) \times Q(0, 0, 2)$

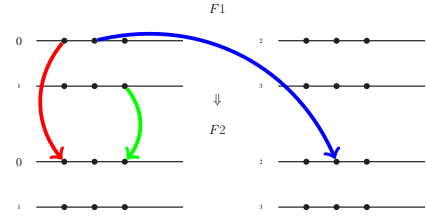


**Figure S14.**  $Q(0, 0, 1) : \frac{1}{2} \times \frac{1}{2} \times r_{13} \times Q(2, 0, 2)$

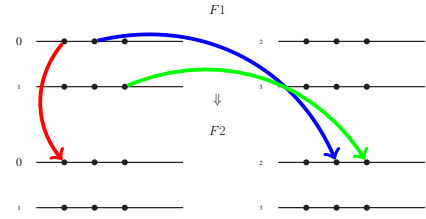


**Figure S16.**  $Q(0, 0, 1) : \frac{1}{2} \times \frac{1}{2} \times r_{23} \times Q(2, 0, 0)$

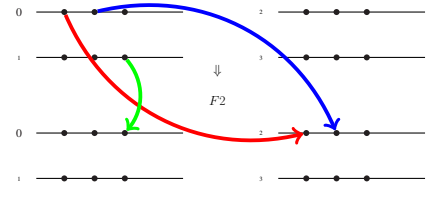
**Figure S18.**  $Q(0, 0, 1)$



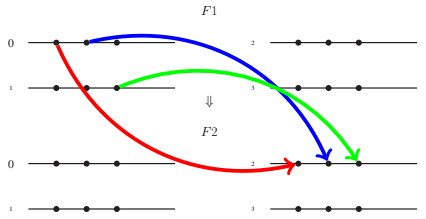
**Figure S11.**  $Q(0, 0, 1) : \frac{1}{2} \times \frac{1}{2} \times r_{13} \times Q(0, 2, 0)$



**Figure S13.**  $Q(0, 0, 1) : \frac{1}{2} \times \frac{1}{2} \times r_{23} \times Q(0, 2, 2)$



**Figure S15.**  $Q(0, 0, 1) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{12}) \times Q(2, 2, 0)$

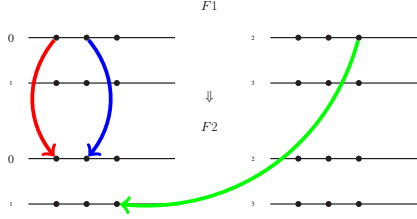


**Figure S17.**  $Q(0, 0, 1) : \frac{1}{2} \times (1 - r_{12}) \times r_{23} \times Q(2, 2, 2)$

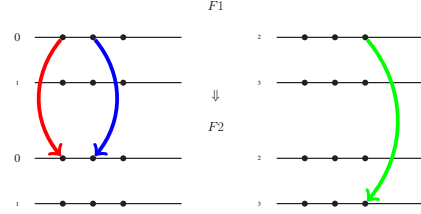
### 2.1.3

See Figure S27.

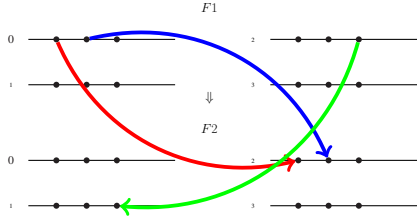
$$\frac{1}{2}(1 - r_{12})Q(0, 0, 1) + \left(\frac{1}{2}(1 - r_{12}) - 1\right)Q(0, 0, 2) + \frac{1}{4}Q(0, 2, 1) + \frac{1}{4}Q(0, 2, 3) = 0 \quad (\text{S8})$$



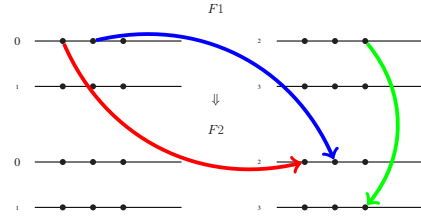
**Figure S19.**  $Q(0, 0, 2) : \frac{1}{2} \times (1 - r_{12}) \times \frac{1}{2} \times Q(0, 0, 1)$



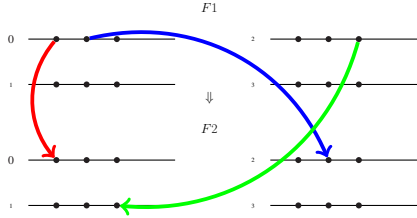
**Figure S20.**  $Q(0, 0, 2) : \frac{1}{2} \times (1 - r_{12}) \times \frac{1}{2} \times Q(0, 0, 3)$



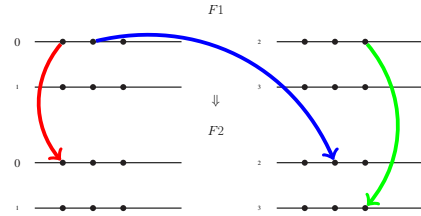
**Figure S21.**  $Q(0, 0, 2) : \frac{1}{2} \times (1 - r_{12}) \times \frac{1}{2} \times Q(2, 2, 1)$



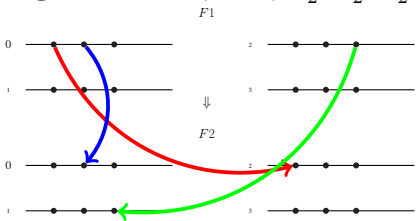
**Figure S22.**  $Q(0, 0, 2) : \frac{1}{2} \times (1 - r_{12}) \times \frac{1}{2} \times Q(2, 2, 3)$



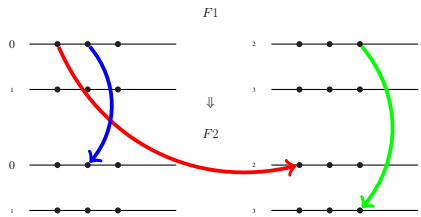
**Figure S23.**  $Q(0, 0, 2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(0, 2, 1)$



**Figure S24.**  $Q(0, 0, 2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(0, 2, 3)$



**Figure S25.**  $Q(0, 0, 2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(2, 0, 1)$



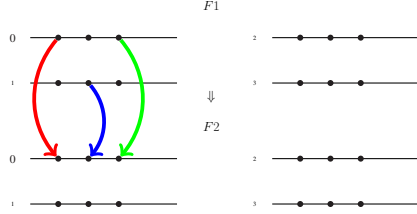
**Figure S26.**  $Q(0, 0, 2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(2, 0, 3)$

**Figure S27.**  $Q(0, 0, 2)$

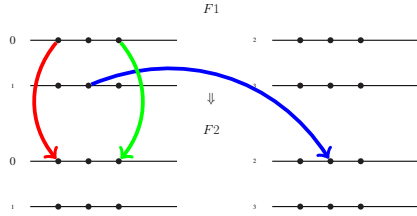
## 2.1.4

See Figure S36.

$$r_{12}r_{23}Q(0,0,0) + \frac{1}{2}r_{12}Q(0,0,2) - Q(0,1,0) + \frac{1}{2}(1-r_{13})Q(0,2,0) + \frac{1}{2}r_{23}Q(0,2,2) = 0 \quad (\text{S9})$$

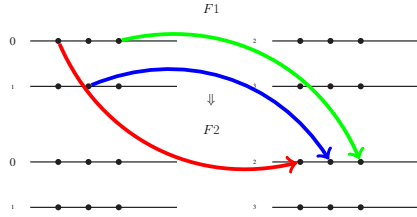


**Figure S28.**  $Q(0,1,0) : \frac{1}{2} \times r_{12} \times (1 - r_{23}) \times Q(0,0,0)$

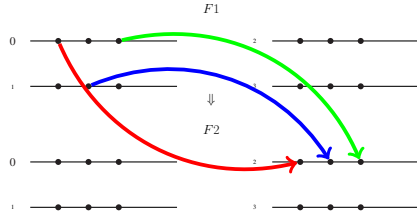


**Figure S29.**  $Q(0,1,0) : \frac{1}{2} \times r_{12} \times \frac{1}{2} \times Q(0,0,2)$

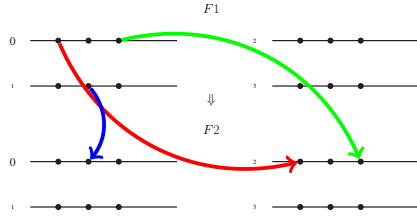
**Figure S30.**  $Q(0,1,0) : \frac{1}{2} \times \frac{1}{2} \times r_{13} \times Q(0,2,0)$



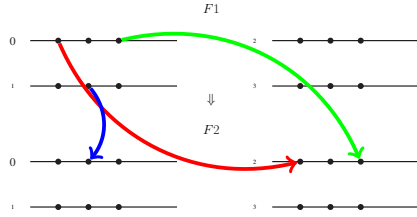
**Figure S31.**  $Q(0,1,0) : \frac{1}{2} \times \frac{1}{2} \times r_{23} \times Q(0,2,2)$



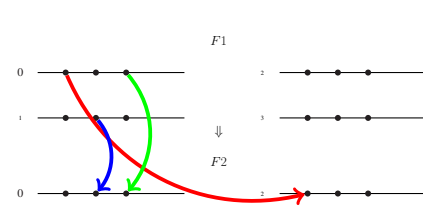
**Figure S32.**  $Q(0,1,0) : \frac{1}{2} \times r_{12}r_{23} \times Q(2,2,2)$



**Figure S33.**  $Q(0,1,0) : \frac{1}{2} \times \frac{1}{2} \times r_{12} \times Q(2,2,0)$



**Figure S34.**  $Q(0,1,0) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{13}) \times Q(2,0,2)$



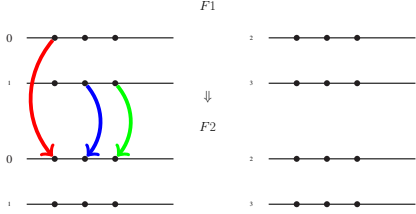
**Figure S35.**  $Q(0,1,0) : \frac{1}{2} \times r_{23} \times \frac{1}{2} \times Q(2,0,0)$

**Figure S36.**  $Q(0,1,0)$

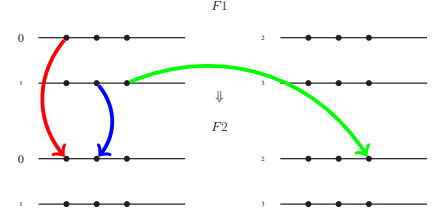
### 2.1.5

See Figure S45

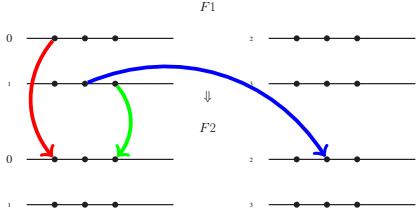
$$r_{12}(1 - r_{23})Q(0, 0, 0) + \frac{1}{2}r_{12}Q(0, 0, 2) - Q(0, 1, 1) + \frac{1}{2}r_{13}Q(0, 2, 0) + \frac{1}{2}(1 - r_{23})Q(0, 2, 2) = 0 \quad (\text{S10})$$



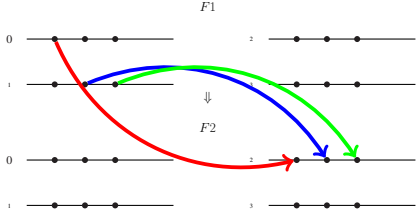
**Figure S37.**  $Q(0, 1, 1) : \frac{1}{2} \times r_{12} \times (1 - r_{23}) \times Q(0, 0, 0)$



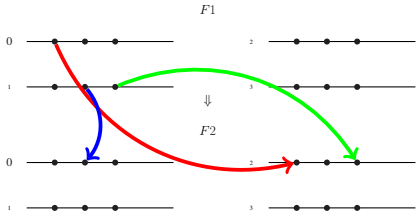
**Figure S38.**  $Q(0, 1, 1) : \frac{1}{2} \times \frac{1}{2} \times r_{12} \times Q(0, 0, 2)$



**Figure S39.**  $Q(0, 1, 1) : \frac{1}{2} \times \frac{1}{2} \times r_{13} \times Q(0, 2, 0)$

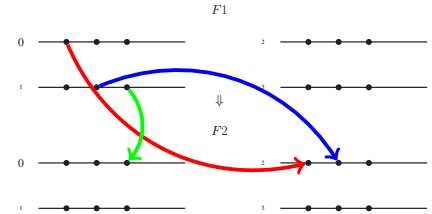


**Figure S41.**  $Q(0, 1, 1) : \frac{1}{2} \times r_{12} \times (1 - r_{23}) \times Q(2, 2, 2)$

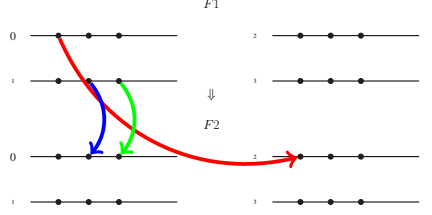


**Figure S43.**  $Q(0, 1, 1) : \frac{1}{2} \times r_{13} \times \frac{1}{2} \times Q(2, 0, 2)$

**Figure S45.**  $Q(0, 1, 1)$



**Figure S42.**  $Q(0, 1, 1) : \frac{1}{2} \times r_{12} \times \frac{1}{2} \times Q(2, 2, 0)$

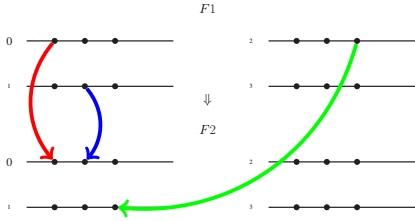


**Figure S44.**  $Q(0, 1, 1) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{23}) \times Q(2, 0, 0)$

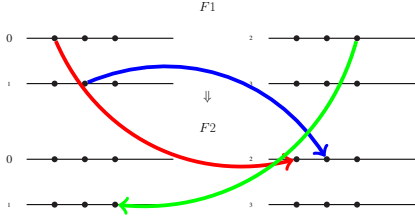
## 2.1.6

See Figure S54.

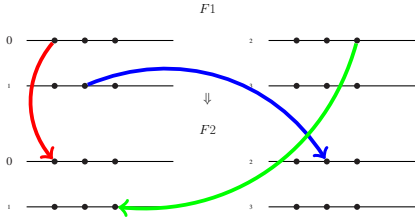
$$\frac{1}{2}r_{12}Q(0,0,1) + \frac{1}{2}r_{12}Q(0,0,2) - Q(0,1,2) + \frac{1}{4}Q(0,2,1) + \frac{1}{4}Q(0,2,3) = 0 \quad (\text{S11})$$



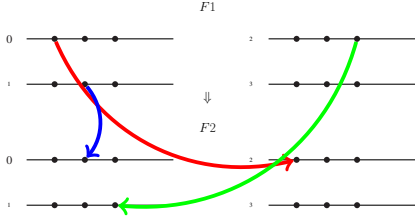
**Figure S46.**  $Q(0,1,2) : \frac{1}{2} \times \frac{1}{2} \times r_{12} \times Q(0,0,1)$



**Figure S48.**  $Q(0,1,2) : \frac{1}{2} \times \frac{1}{2} \times r_{12} \times Q(2,2,1)$

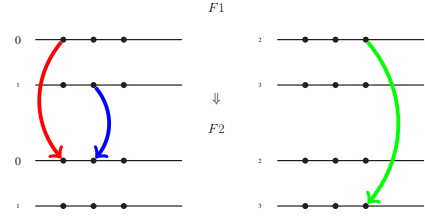


**Figure S50.**  $Q(0,1,2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(0,2,1)$

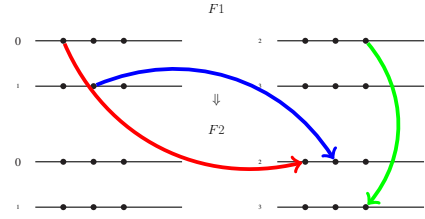


**Figure S52.**  $Q(0,1,2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(2,0,1)$

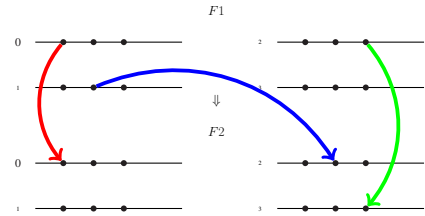
**Figure S54.**  $Q(0,1,2)$



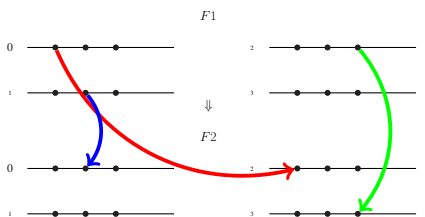
**Figure S47.**  $Q(0,1,2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times r_{12} \times Q(0,0,3)$



**Figure S49.**  $Q(0,1,2) : \frac{1}{2} \times \frac{1}{2} \times r_{12} \times Q(2,2,3)$



**Figure S51.**  $Q(0,1,2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(0,2,3)$

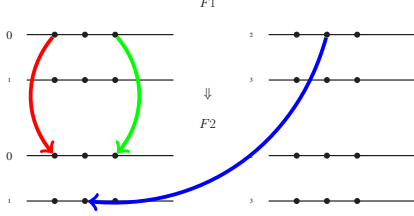


**Figure S53.**  $Q(0,1,2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(2,0,3)$

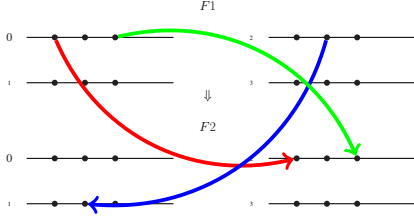
### 2.1.7

See Figure S63.

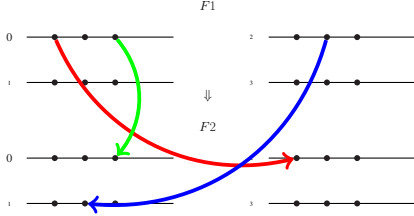
$$\frac{1}{2}(1 - r_{13})Q(0, 1, 0) + \frac{1}{4}Q(0, 1, 2) + (\frac{1}{2}(1 - r_{13}) - 1)Q(0, 2, 0) + \frac{1}{4}Q(0, 1, 2) = 0 \quad (\text{S12})$$



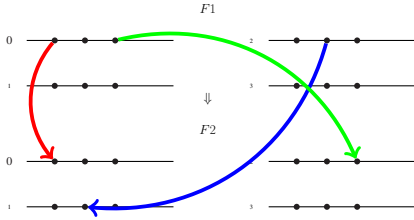
**Figure S55.**  $Q(0, 2, 0) : \frac{1}{2} \times (1 - r_{13}) \times \frac{1}{2} \times Q(0, 1, 0)$



**Figure S57.**  $Q(0, 2, 0) : \frac{1}{2} \times (1 - r_{13}) \times \frac{1}{2} \times Q(2, 1, 2)$

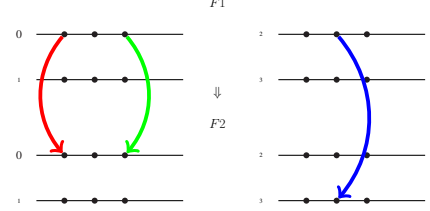


**Figure S59.**  $Q(0, 2, 0) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(2, 1, 0)$

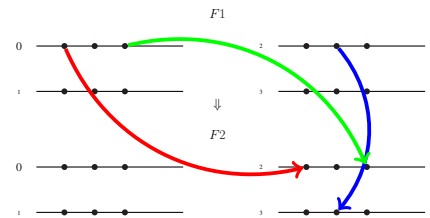


**Figure S61.**  $Q(0, 2, 0) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(0, 1, 2)$

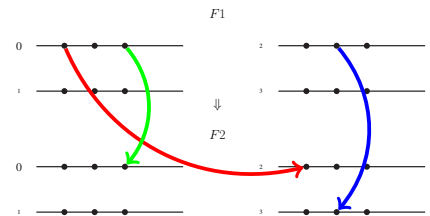
**Figure S63.**  $Q(0, 2, 0)$



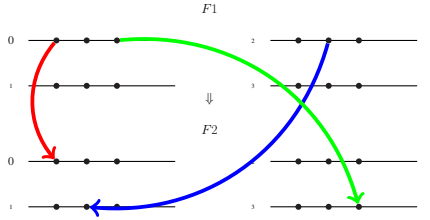
**Figure S56.**  $Q(0, 2, 0) : \frac{1}{2} \times (1 - r_{13}) \times \frac{1}{2} \times Q(0, 3, 0)$



**Figure S58.**  $Q(0, 2, 0) : \frac{1}{2} \times (1 - r_{13}) \times \frac{1}{2} \times Q(2, 3, 2)$



**Figure S60.**  $Q(0, 2, 0) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(2, 3, 0)$



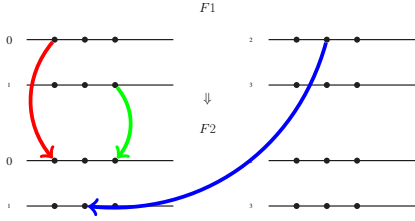
**Figure S62.**  $Q(0, 2, 0) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(0, 1, 3)$



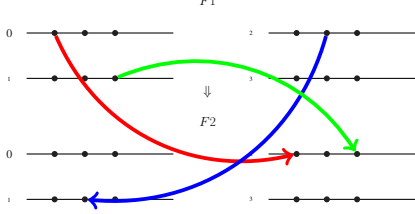
## 2.1.8

See Figure S72.

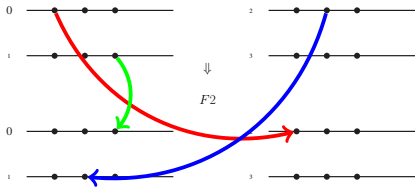
$$\frac{1}{2}r_{13}Q(0,1,0) + \frac{1}{4}Q(0,1,2) + \frac{1}{2}r_{13}Q(0,2,0) - Q(0,2,1) + \frac{1}{4}Q(0,2,3) = 0 \quad (\text{S13})$$



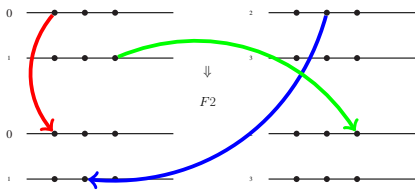
**Figure S64.**  $Q(0,2,1) : \frac{1}{2} \times r_{13} \times \frac{1}{2} \times Q(0,1,0)$



**Figure S66.**  $Q(0,2,1) : \frac{1}{2} \times r_{13} \times \frac{1}{2} \times Q(2,1,2)$

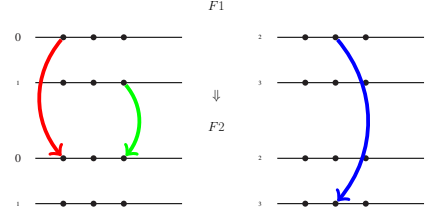


**Figure S68.**  $Q(0,2,1) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(2,1,0)$

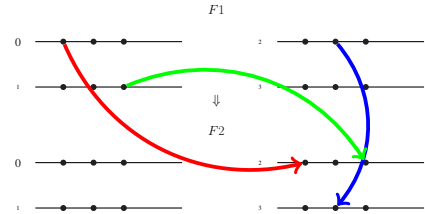


**Figure S70.**  $Q(0,2,1) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(0,1,2)$

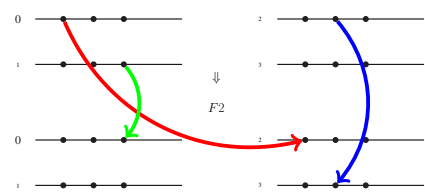
**Figure S72.**  $Q(0,2,1)$



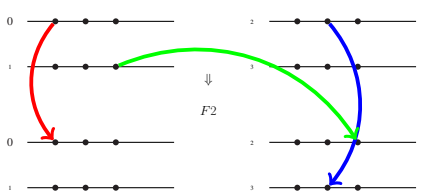
**Figure S65.**  $Q(0,2,1) : \frac{1}{2} \times r_{13} \times \frac{1}{2} \times Q(0,3,0)$



**Figure S67.**  $Q(0,2,1) : \frac{1}{2} \times r_{13} \times \frac{1}{2} \times Q(2,3,2)$



**Figure S69.**  $Q(0,2,1) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(2,3,0)$

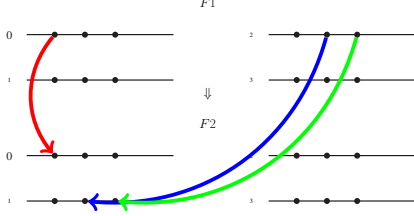


**Figure S71.**  $Q(0,2,1) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(0,3,2)$

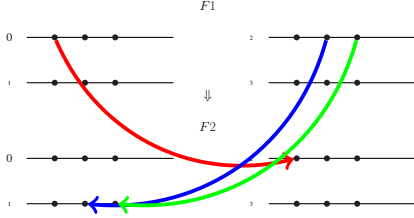
## 2.1.9

See Figure S81.

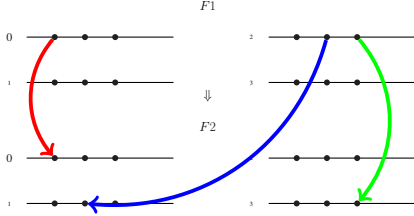
$$\frac{1}{2}(1 - r_{23})Q(0, 1, 1) + \frac{1}{4}Q(0, 1, 2) + \frac{1}{4}Q(0, 2, 1) + (\frac{1}{2}(1 - r_{23}) - 1)Q(0, 2, 2) = 0 \quad (\text{S14})$$



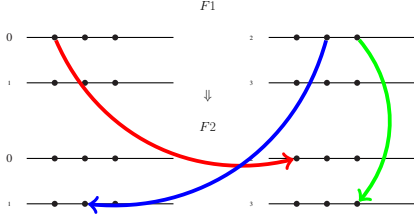
**Figure S73.**  $Q(0, 2, 2) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{23}) \times Q(0, 1, 1)$



**Figure S75.**  $Q(0, 2, 2) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{23}) \times Q(2, 1, 1)$

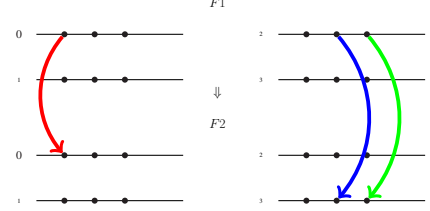


**Figure S77.**  $Q(0, 2, 2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(0, 1, 3)$

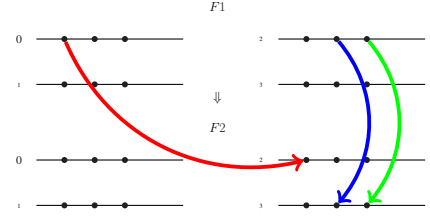


**Figure S79.**  $Q(0, 2, 2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(2, 1, 3)$

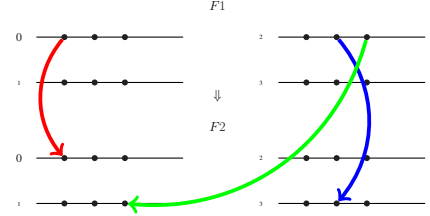
**Figure S81.**  $Q(0, 2, 2)$



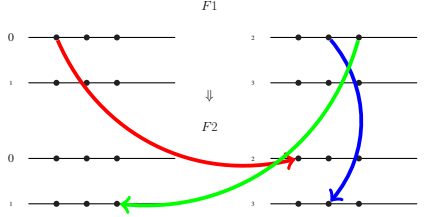
**Figure S74.**  $Q(0, 2, 2) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{23}) \times Q(0, 3, 3)$



**Figure S76.**  $Q(0, 2, 2) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{23}) \times Q(2, 3, 3)$



**Figure S78.**  $Q(0, 2, 2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(031)$

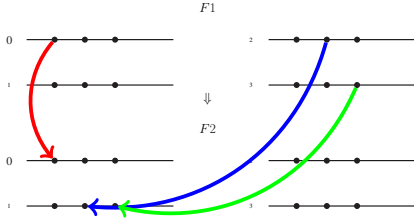


**Figure S80.**  $Q(0, 2, 2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(2, 3, 1)$

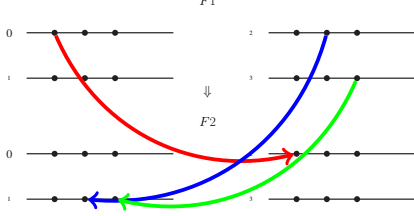
## 2.1.10

See Figure S90.

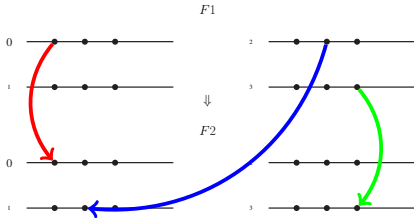
$$\frac{1}{2}r_{23}Q(0,1,1) + \frac{1}{4}Q(0,1,2) + \frac{1}{4}Q(0,2,1) + \frac{1}{2}r_{23}Q(0,2,2) - Q(0,2,3) = 0$$



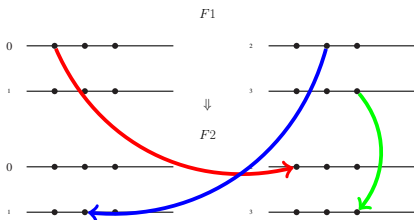
**Figure S82.**  $Q(0,2,3) : \frac{1}{2} \times \frac{1}{2} \times r_{23} \times Q(0,1,1)$



**Figure S84.**  $Q(0,2,3) : \frac{1}{2} \times \frac{1}{2} \times r_{23} \times Q(2,1,1)$

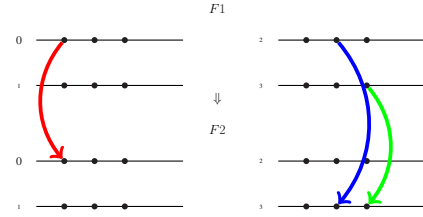


**Figure S86.**  $Q(0,2,3) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(0,1,3)$

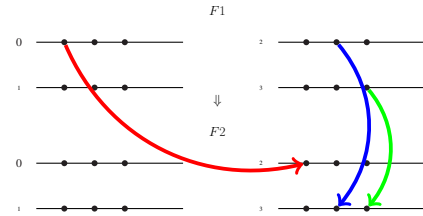


**Figure S88.**  $Q(0,2,3) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(2,1,3)$

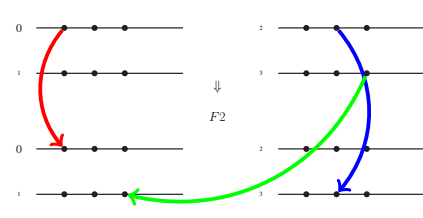
**Figure S90.**  $Q(0,2,3)$



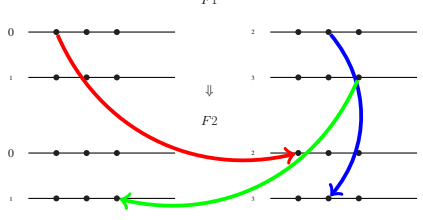
**Figure S83.**  $Q(0,2,3) : \frac{1}{2} \times \frac{1}{2} \times r_{23} \times Q(0,3,3)$



**Figure S85.**  $Q(0,2,3) : \frac{1}{2} \times \frac{1}{2} \times r_{23} \times Q(2,3,3)$



**Figure S87.**  $Q(0,2,3) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(031)$



**Figure S89.**  $Q(0,2,3) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(2,3,1)$