

Supplementary Material

1 SUPPLEMENTARY MATHEMATICS

1.1 The derivation of equation

$$|Q_L| = 2^{L-2}(2^{L-1} + 1) \quad (S1)$$

Proof: For $L \geq 2$, there are 4^L IBD (identical by descent) probabilities $Q(i_1, i_2, \dots, i_L)$ since $i_l = 1, 2, 3$ or 4 and furthermore they add up to 1. A number of these probabilities are equal because of two symmetries: (1) the two homologous chromosomes in each individual play identical roles, and (2) the siblings play identical roles (assuming no sex-dependence of meiosis, so that for instance the recombination rates $r_{l,l'}$ are sex-independent). It is thus appropriate to use only one representative of each symmetry equivalence class, so that for instance one may impose this representative to have its first index, i_1 , equal to zero. In fact one can identify exactly one element in each class by imposing that the indices of the representative Q 's have either

1. $i_l \in \{0, 1\} \forall l \in \{2, \dots, L\}$, or
2. $i_l \in \{0, 1\} \forall l \in \{2, \dots, K-1\}$, $i_K = 2$ and $i_l \in \{0, 1, 2, 3\} \forall l \in \{K+1, \dots, L\}$

The number of equivalence classes and thus of Q 's to consider is then

$$|Q_L| = 2^{L-1} + \sum_{l=2}^L 2^{l-2} 4^{L-l} = 2^{L-1} + 2^{2L-2} \sum_{l=2}^L 2^{-l} \quad (S2)$$

Given that $\sum_{l=2}^L 2^{-l}$ is a geometric progression of common ratio 2^{-1} from 2 to L , the sum of its terms can be expressed as:

$$\sum_{l=2}^L 2^{-l} = \frac{2^{-2} - 2^{-(L-1)}}{1 - 2^{-1}} = 2^{-1} - 2^{-L} \quad (S3)$$

Substituting S3 in S2, we get

$$|Q_L| = 2^{L-1} + 2^{2L-2}(2^{-1} - 2^{-L}) = 2^{L-1} + 2^{2L-3} - 2^{L-2} \quad (S4)$$

Factorizing with respect to 2^{L-2} and after simplification, this gives

$$|Q_L| = 2^{L-2}(1 + 2^{L-1}). \quad (S5)$$

2 THE INHOMOGENEOUS EQUATION

$$4Q(0,0,0) + 4Q(0,0,1) + 8Q(0,0,2) + 4Q(0,1,0) + 4Q(0,1,1) + 8Q(1,1,2) + 8Q(0,2,0) + 8Q(0,2,1) + 8Q(0,2,2) + 8Q(0,2,3) = 1 \quad (\text{S6})$$

From equation S6 and Figures S9, S18, S27, S36, S45, S54, S54, S63, S72, S81, and S90 we get this system

$$\begin{bmatrix} 4 & 4 & 8 & 4 & 4 & 8 & 8 & 8 & 8 \\ \bar{r}_{12}\bar{r}_{23} - 1 & 0 & \frac{\bar{r}_{12}}{2} & 0 & 0 & 0 & \frac{\bar{r}_{13}}{2} & 0 & \frac{\bar{r}_{23}}{2} \\ r_{23}\bar{r}_{12} & -1 & \frac{\bar{r}_{12}}{2} & 0 & 0 & 0 & \frac{r_{13}}{2} & 0 & \frac{r_{23}}{2} \\ 0 & \frac{\bar{r}_{12}}{2} & \frac{\bar{r}_{12}-2}{2} & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ r_{12}r_{23} & 0 & \frac{r_{12}}{2} & -1 & 0 & 0 & \frac{\bar{r}_{13}}{2} & 0 & \frac{r_{23}}{2} \\ r_{12}\bar{r}_{23} & 0 & \frac{r_{12}}{2} & 0 & -1 & 0 & \frac{r_{13}}{2} & 0 & \frac{\bar{r}_{23}}{2} \\ 0 & \frac{r_{12}}{2} & \frac{r_{12}}{2} & 0 & 0 & -1 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{\bar{r}_{13}}{2} & 0 & \frac{1}{4} & \frac{\bar{r}_{13}-2}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{r_{13}}{2} & 0 & \frac{1}{4} & \frac{r_{13}}{2} & -1 & 0 \\ 0 & 0 & 0 & 0 & \frac{\bar{r}_{23}}{2} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{\bar{r}_{23}-2}{2} \end{bmatrix} \begin{bmatrix} Q(0,0,0) \\ Q(0,0,1) \\ Q(0,0,2) \\ Q(0,1,0) \\ Q(0,1,1) \\ Q(0,1,2) \\ Q(0,2,0) \\ Q(0,2,1) \\ Q(0,2,2) \\ Q(0,2,3) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{S7})$$

where $\bar{r} = 1 - r$; the complement value of the recombination rate.

3 THE SCHP EQUATIONS

3.1

See Figure S9

$$((1 - r_{12})(1 - r_{23}) - 1)Q(0, 0, 0) + \frac{1}{2}(1 - r_{12})Q(0, 0, 2) + \frac{1}{2}(1 - r_{13})Q(0, 2, 0) + \frac{1}{2}(1 - r_{23})Q(0, 2, 2) = 0 \quad (\text{S8})$$

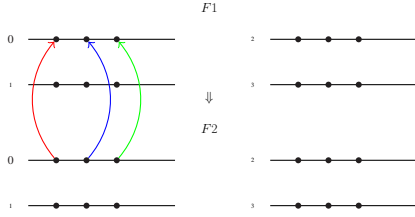


Figure S1. $Q(0, 0, 0) : \frac{1}{2} \times (1 - r_{12}) \times (1 - r_{23}) \times Q(0, 0, 0)$

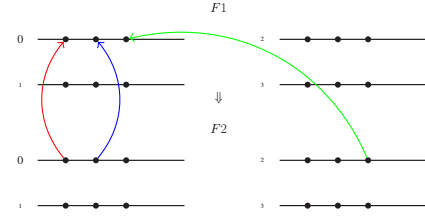
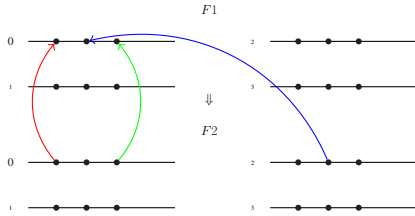


Figure S2. $Q(0, 0, 0) : \frac{1}{2} \times (1 - r_{12}) \times \frac{1}{2} \times Q(0, 0, 2)$

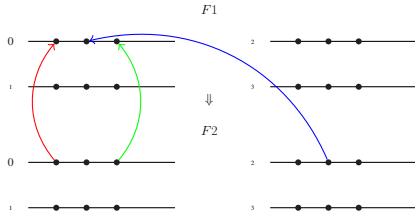


Figure S3. $Q(0, 0, 0) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{13}) \times Q(0, 2, 0)$

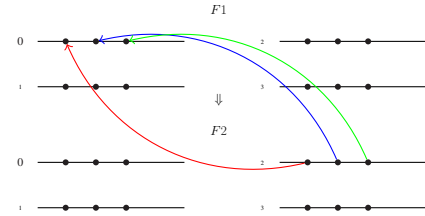
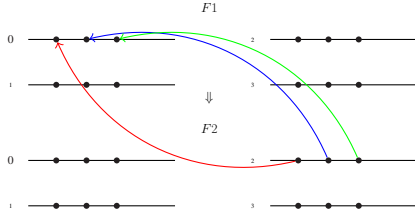


Figure S4. $Q(0, 0, 0) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{23}) \times Q(0, 2, 2)$

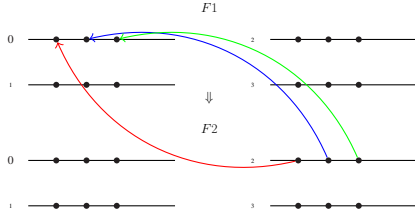


Figure S5. $Q(0, 0, 0) : \frac{1}{2} \times (1 - r_{12})(1 - r_{23}) \times Q(2, 2, 2)$

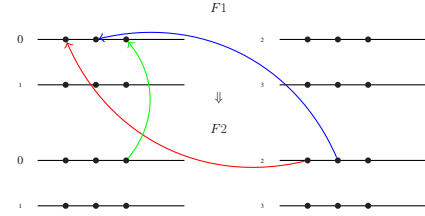
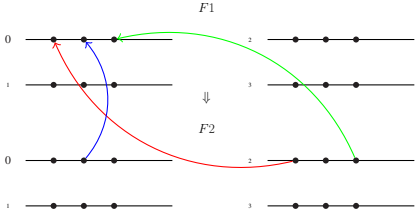


Figure S6. $Q(0, 0, 0) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{12}) \times Q(2, 2, 0)$

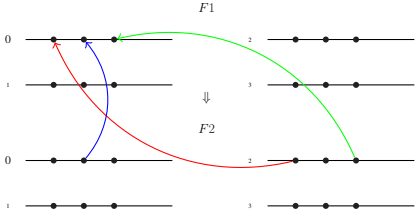


Figure S7. $Q(0, 0, 0) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{13}) \times Q(2, 0, 2)$

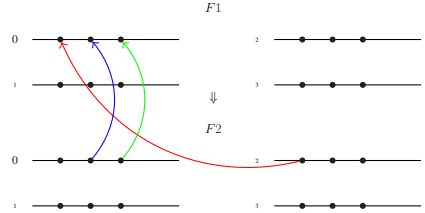


Figure S8. $Q(0, 0, 0) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{23}) \times Q(2, 0, 0)$

Figure S9. $Q(0, 0, 0)$

3.2

See Figure S18

$$(1 - r_{12})r_{23}Q(0, 0, 0) - Q(0, 0, 1) + \frac{1}{2}(1 - r_{12})Q(0, 0, 2) + \frac{1}{2}r_{13}Q(0, 2, 0) + \frac{1}{2}r_{23}Q(0, 2, 2) = 0 \quad (\text{S9})$$

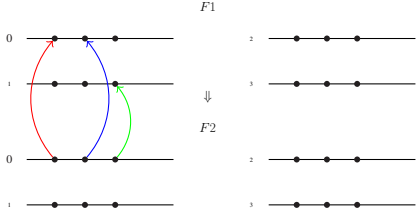


Figure S10. $Q(0, 0, 1) : \frac{1}{2} \times (1 - r_{12}) \times r_{23} \times Q(0, 0, 0)$

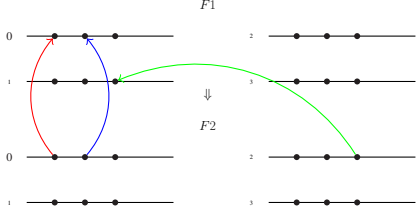


Figure S12. $Q(0, 0, 1) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{12}) \times Q(0, 0, 2)$

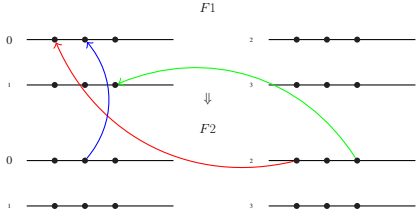


Figure S14. $Q(0, 0, 1) : \frac{1}{2} \times \frac{1}{2} \times r_{13} \times Q(2, 0, 2)$

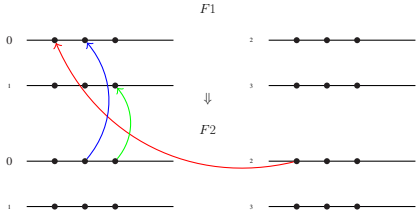


Figure S16. $Q(0, 0, 1) : \frac{1}{2} \times \frac{1}{2} \times r_{23} \times Q(2, 0, 0)$

Figure S18. $Q(0, 0, 1)$

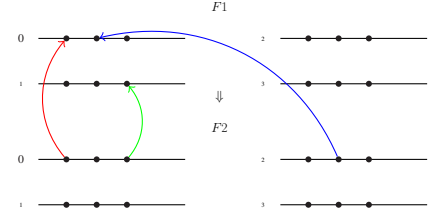


Figure S11. $Q(0, 0, 1) : \frac{1}{2} \times \frac{1}{2} \times r_{13} \times Q(0, 2, 0)$

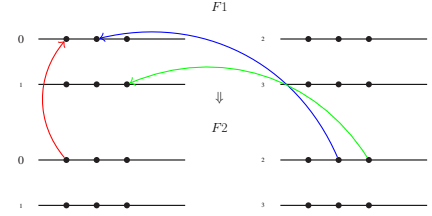


Figure S13. $Q(0, 0, 1) : \frac{1}{2} \times \frac{1}{2} \times r_{23} \times Q(0, 2, 2)$

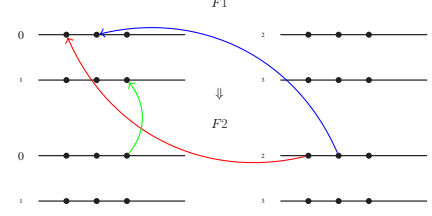


Figure S15. $Q(0, 0, 1) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{12}) \times Q(2, 2, 0)$

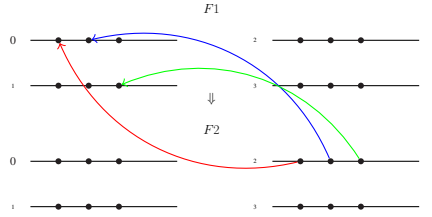


Figure S17. $Q(0, 0, 1) : \frac{1}{2} \times (1 - r_{12}) \times r_{23} \times Q(2, 2, 2)$

3.3

See Figure S27.

$$\frac{1}{2}(1 - r_{12})Q(0, 0, 1) + \left(\frac{1}{2}(1 - r_{12}) - 1\right)Q(0, 0, 2) + \frac{1}{4}Q(0, 2, 1) + \frac{1}{4}Q(0, 2, 3) = 0 \quad (\text{S10})$$

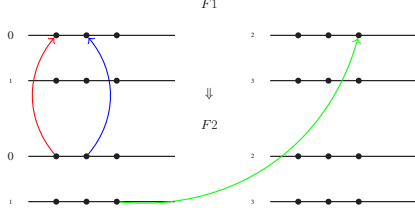


Figure S19. $Q(0, 0, 2) : \frac{1}{2} \times (1 - r_{12}) \times \frac{1}{2} \times Q(0, 0, 1)$

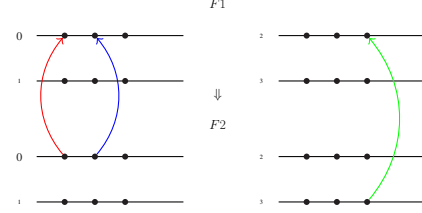


Figure S20. $Q(0, 0, 2) : \frac{1}{2} \times (1 - r_{12}) \times \frac{1}{2} \times Q(0, 0, 3)$

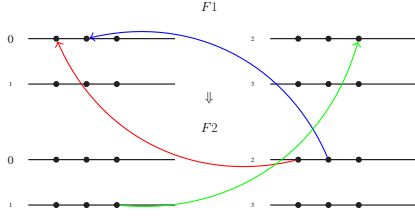


Figure S21. $Q(0, 0, 2) : \frac{1}{2} \times (1 - r_{12}) \times \frac{1}{2} \times Q(2, 2, 1)$

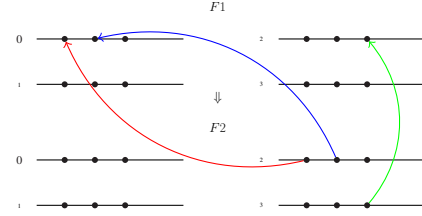


Figure S22. $Q(0, 0, 2) : \frac{1}{2} \times (1 - r_{12}) \times \frac{1}{2} \times Q(2, 2, 3)$

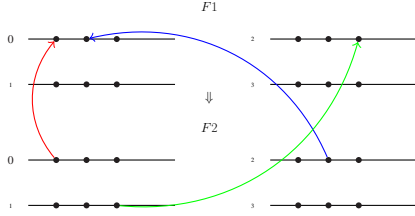


Figure S23. $Q(0, 0, 2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(0, 2, 1)$

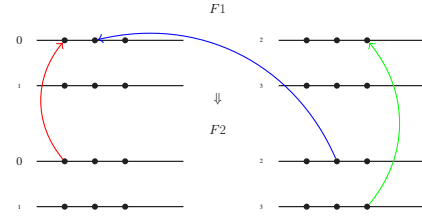


Figure S24. $Q(0, 0, 2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(0, 2, 3)$

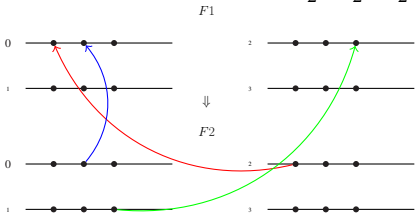


Figure S25. $Q(0, 0, 2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(2, 0, 1)$

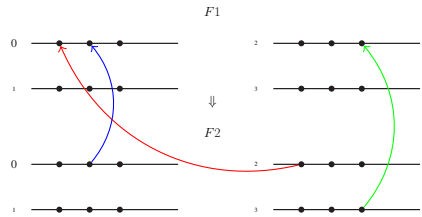


Figure S26. $Q(0, 0, 2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(2, 0, 3)$

Figure S27. $Q(0, 0, 2)$

3.4

See Figure S36.

$$r_{12}r_{23}Q(0,0,0) + \frac{1}{2}r_{12}Q(0,0,2) - Q(0,1,0) + \frac{1}{2}(1-r_{13})Q(0,2,0) + \frac{1}{2}r_{23}Q(0,2,2) \quad (\text{S11})$$

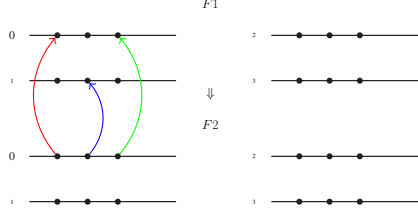


Figure S28. $Q(0,1,0) : \frac{1}{2} \times r_{12} \times (1 - r_{23}) \times Q(0,0,0)$

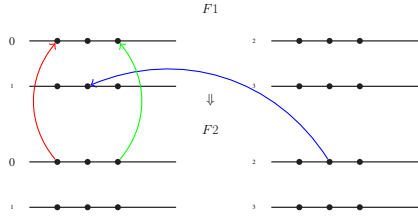


Figure S30. $Q(0,1,0) : \frac{1}{2} \times \frac{1}{2} \times r_{13} \times Q(0,2,0)$

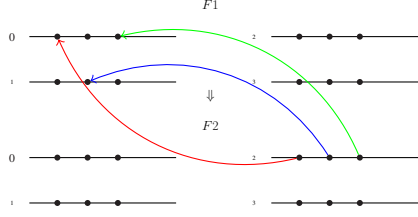


Figure S32. $Q(0,1,0) : \frac{1}{2} \times r_{12}r_{23} \times Q(2,2,2)$

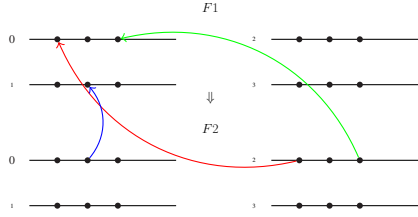


Figure S34. $Q(0,1,0) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{13}) \times Q(2,0,2)$

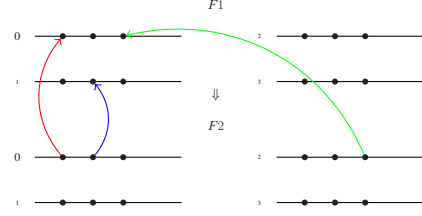


Figure S29. $Q(0,1,0) : \frac{1}{2} \times r_{12} \times \frac{1}{2} \times Q(0,0,2)$

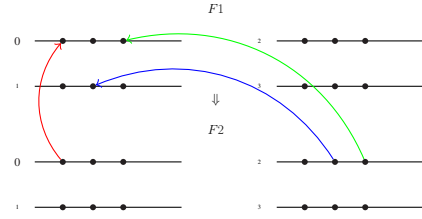


Figure S31. $Q(0,1,0) : \frac{1}{2} \times \frac{1}{2} \times r_{23} \times Q(0,2,2)$

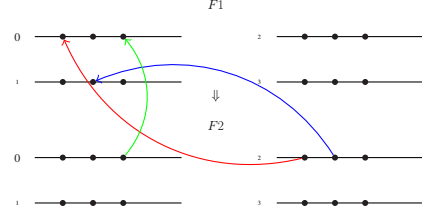


Figure S33. $Q(0,1,0) : \frac{1}{2} \times \frac{1}{2} \times r_{12} \times Q(2,2,0)$

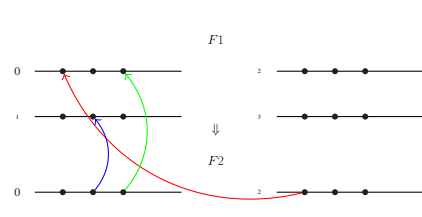


Figure S35. $Q(0,1,0) : \frac{1}{2} \times r_{23} \times \frac{1}{2} \times Q(2,0,0)$

Figure S36. $Q(0,1,0)$

3.5

See Figure S45

$$r_{12}(1 - r_{23})Q(0, 0, 0) + \frac{1}{2}r_{12}Q(0, 0, 2) - Q(0, 1, 1) + \frac{1}{2}r_{13}Q(0, 2, 0) + \frac{1}{2}(1 - r_{23})Q(0, 2, 2) \quad (\text{S12})$$

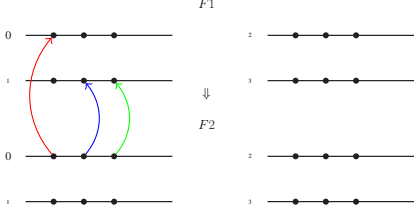


Figure S37. $Q(0, 1, 1) : \frac{1}{2} \times r_{12} \times (1 - r_{23}) \times Q(0, 0, 0)$

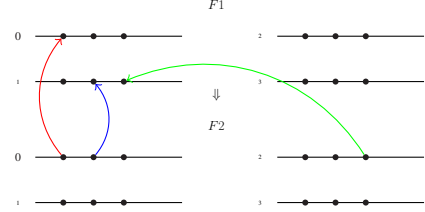


Figure S38. $Q(0, 1, 1) : \frac{1}{2} \times \frac{1}{2} \times r_{12} \times Q(0, 0, 2)$

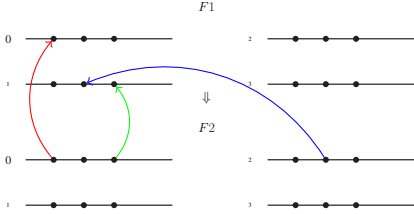


Figure S39. $Q(0, 1, 1) : \frac{1}{2} \times \frac{1}{2} \times r_{13} \times Q(0, 2, 0)$

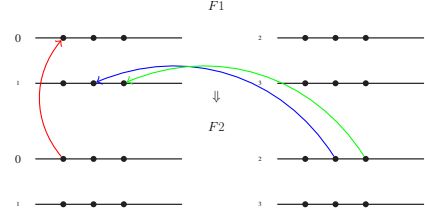


Figure S40. $Q(0, 1, 1) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{23}) \times Q(0, 2, 2)$

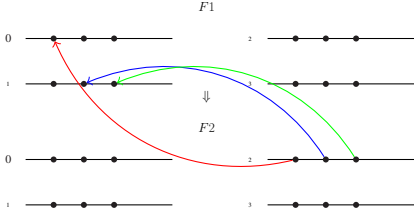


Figure S41. $Q(0, 1, 1) : \frac{1}{2} \times r_{12} \times (1 - r_{23}) \times Q(2, 2, 2)$

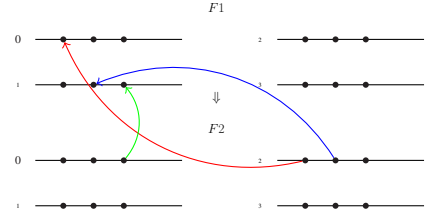


Figure S42. $Q(0, 1, 1) : \frac{1}{2} \times r_{12} \times \frac{1}{2} \times Q(2, 2, 0)$

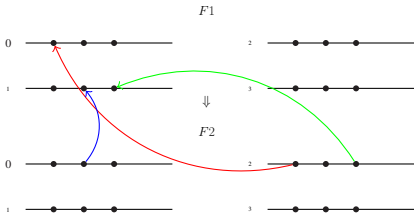


Figure S43. $Q(0, 1, 1) : \frac{1}{2} \times r_{13} \times \frac{1}{2} \times Q(2, 0, 2)$

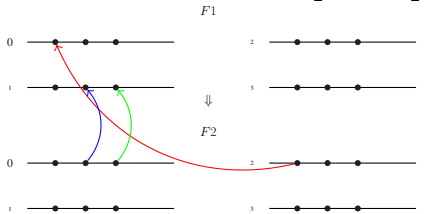


Figure S44. $Q(0, 1, 1) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{23}) \times Q(2, 0, 0)$

Figure S45. $Q(0, 1, 1)$

3.6

See Figure S54.

$$\frac{1}{2}r_{12}Q(0,0,1) + \frac{1}{2}r_{12}Q(0,0,2) - Q(0,1,2) + \frac{1}{4}Q(0,2,1) + \frac{1}{4}Q(0,2,3) \quad (\text{S13})$$

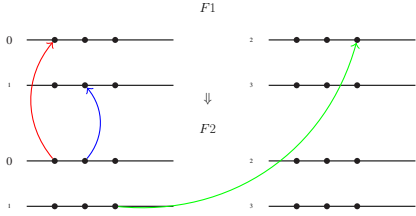


Figure S46. $Q(0,1,2) : \frac{1}{2} \times \frac{1}{2} \times r_{12} \times Q(0,0,1)$

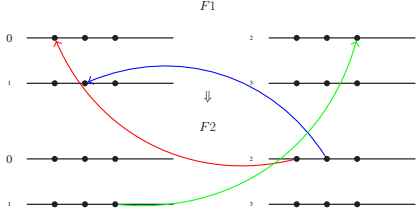


Figure S48. $Q(0,1,2) : \frac{1}{2} \times \frac{1}{2} \times r_{12} \times Q(2,2,1)$

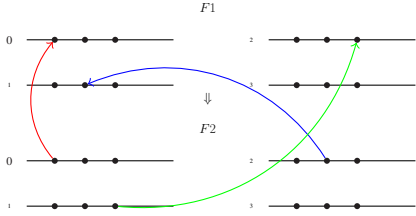


Figure S50. $Q(0,1,2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(0,2,1)$

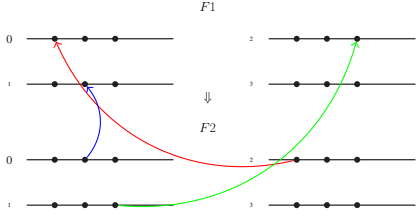


Figure S52. $Q(0,1,2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(2,0,1)$

Figure S54. $Q(0,1,2)$

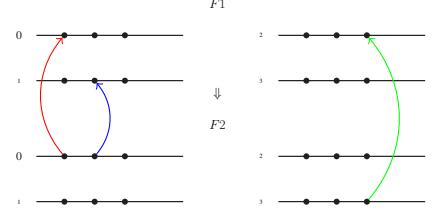


Figure S47. $Q(0,1,2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times r_{12} \times Q(0,0,3)$

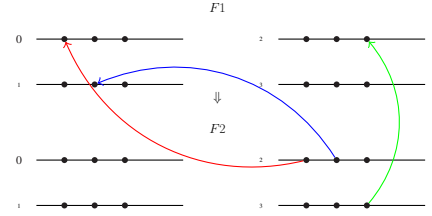


Figure S49. $Q(0,1,2) : \frac{1}{2} \times \frac{1}{2} \times r_{12} \times Q(2,2,3)$

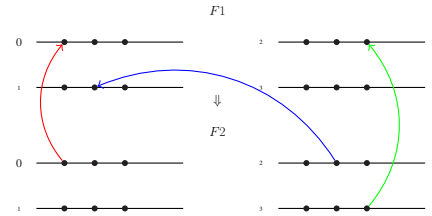


Figure S51. $Q(0,1,2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(0,2,3)$

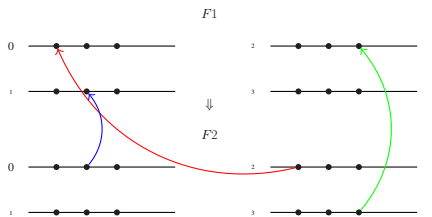


Figure S53. $Q(0,1,2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(2,0,3)$

3.7

See Figure S63.

$$\frac{1}{2}(1 - r_{13})Q(0, 1, 0) + \frac{1}{4}Q(0, 1, 2) + (\frac{1}{2}(1 - r_{13}) - 1)Q(0, 2, 0) + \frac{1}{4}Q(0, 1, 2) \quad (\text{S14})$$

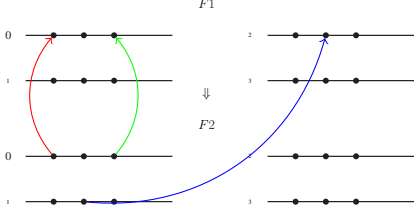


Figure S55. $Q(0, 2, 0) : \frac{1}{2} \times (1 - r_{13}) \times \frac{1}{2} \times Q(0, 1, 0)$

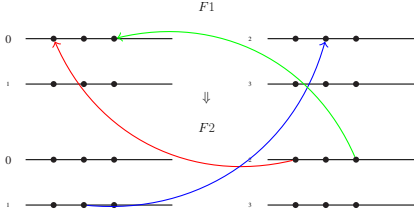


Figure S57. $Q(0, 2, 0) : \frac{1}{2} \times (1 - r_{13}) \times \frac{1}{2} \times Q(2, 1, 2)$

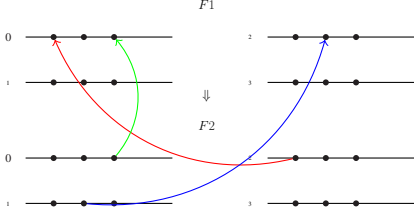


Figure S59. $Q(0, 2, 0) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(2, 1, 0)$

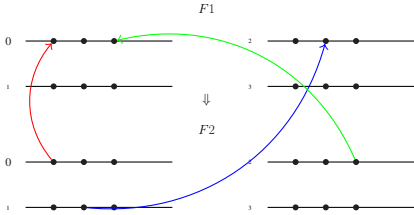


Figure S61. $Q(0, 2, 0) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(0, 1, 2)$

Figure S63. $Q(0, 2, 0)$

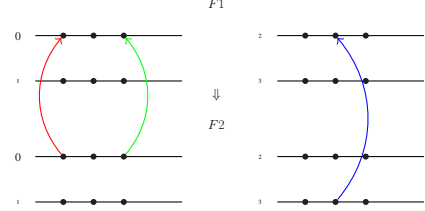


Figure S56. $Q(0, 2, 0) : \frac{1}{2} \times (1 - r_{13}) \times \frac{1}{2} \times Q(0, 3, 0)$

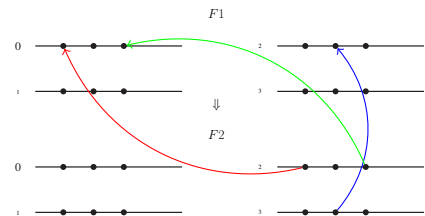


Figure S58. $Q(0, 2, 0) : \frac{1}{2} \times (1 - r_{13}) \times \frac{1}{2} \times Q(2, 3, 2)$

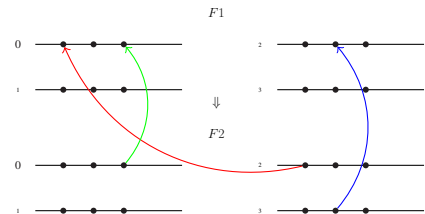


Figure S60. $Q(0, 2, 0) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(2, 3, 0)$

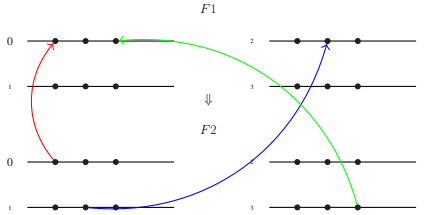


Figure S62. $Q(0, 2, 0) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(0, 1, 3)$

3.8

See Figure S72.

$$\frac{1}{2}r_{13}Q(0,1,0) + \frac{1}{4}Q(0,1,2) + \frac{1}{2}r_{13}Q(0,2,0) - Q(0,2,1) + \frac{1}{4}Q(0,2,3) \quad (\text{S15})$$

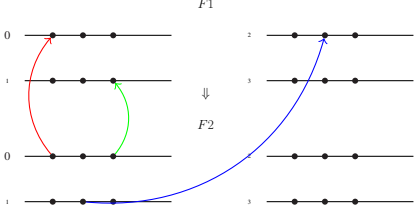


Figure S64. $Q(0,2,1) : \frac{1}{2} \times r_{13} \times \frac{1}{2} \times Q(0,1,0)$

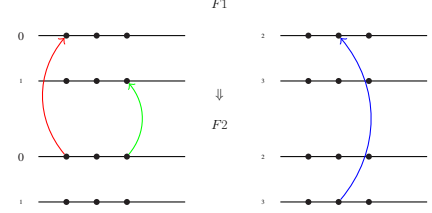
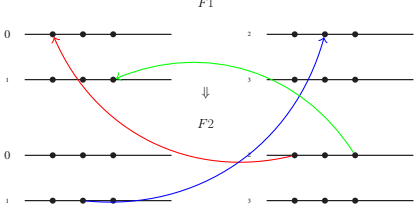


Figure S65. $Q(0,2,1) : \frac{1}{2} \times r_{13} \times \frac{1}{2} \times Q(0,3,0)$

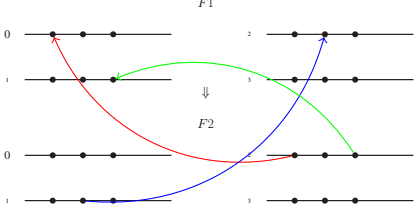


Figure S66. $Q(0,2,1) : \frac{1}{2} \times r_{13} \times \frac{1}{2} \times Q(2,1,2)$

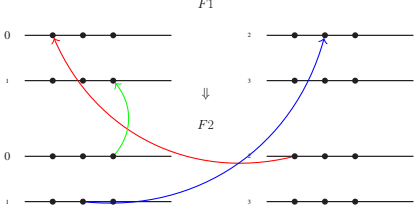


Figure S67. $Q(0,2,1) : \frac{1}{2} \times r_{13} \times \frac{1}{2} \times Q(2,3,2)$

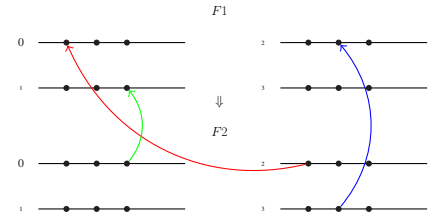


Figure S68. $Q(0,2,1) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(2,1,0)$

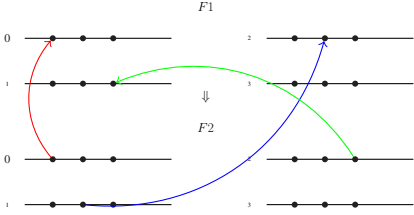


Figure S69. $Q(0,2,1) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(2,3,0)$

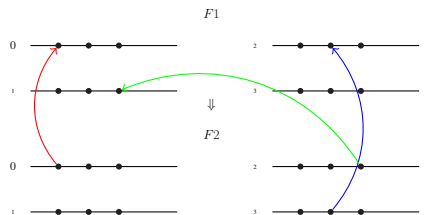


Figure S70. $Q(0,2,1) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(0,1,2)$

Figure S71. $Q(0,2,1) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(0,3,2)$

Figure S72. $Q(0,2,1)$

3.9

See Figure S81.

$$\frac{1}{2}(1 - r_{23})Q(0, 1, 1) + \frac{1}{4}Q(0, 1, 2) + \frac{1}{4}Q(0, 2, 1) + (\frac{1}{2}(1 - r_{23}) - 1)Q(0, 2, 2) \quad (\text{S16})$$

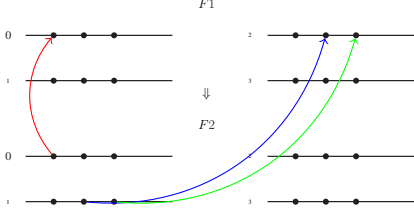


Figure S73. $Q(0, 2, 2) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{23}) \times Q(0, 1, 1)$

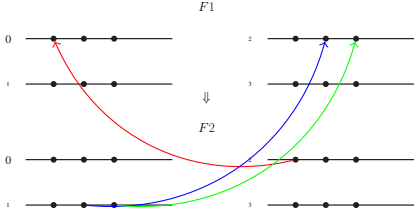


Figure S75. $Q(0, 2, 2) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{23}) \times Q(2, 1, 1)$

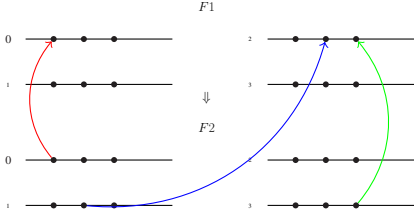


Figure S77. $Q(0, 2, 2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(0, 1, 3)$

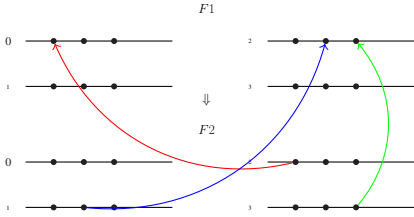


Figure S79. $Q(0, 2, 2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(2, 1, 3)$

Figure S81. $Q(0, 2, 2)$

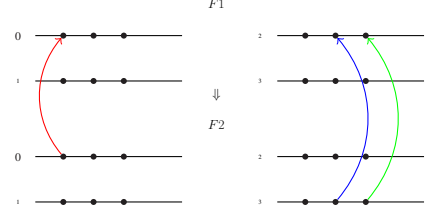


Figure S74. $Q(0, 2, 2) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{23}) \times Q(0, 3, 3)$

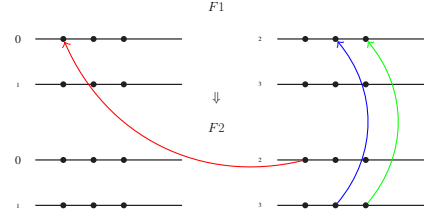


Figure S76. $Q(0, 2, 2) : \frac{1}{2} \times \frac{1}{2} \times (1 - r_{23}) \times Q(2, 3, 3)$

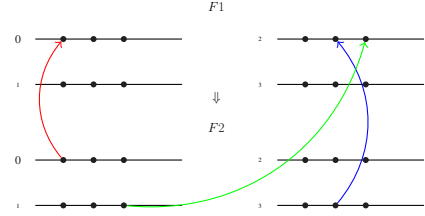


Figure S78. $Q(0, 2, 2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(031)$

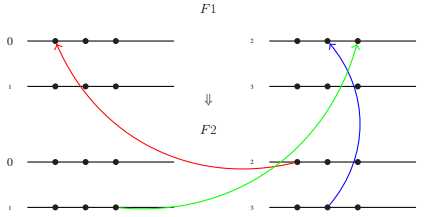


Figure S80. $Q(0, 2, 2) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(2, 3, 1)$

3.10

See Figure S90.

$$\frac{1}{2}r_{23}Q(0,1,1) + \frac{1}{4}Q(0,1,2) + \frac{1}{4}Q(0,2,1) + \frac{1}{2}r_{23}Q(0,2,2) - Q(0,2,3)$$

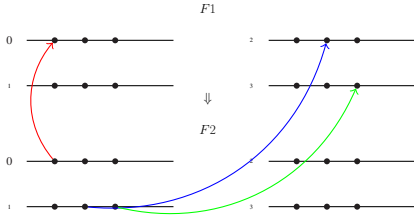


Figure S82. $Q(0,2,3) : \frac{1}{2} \times \frac{1}{2} \times r_{23} \times Q(0,1,1)$

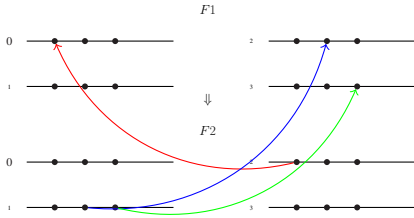


Figure S84. $Q(0,2,3) : \frac{1}{2} \times \frac{1}{2} \times r_{23} \times Q(2,1,1)$

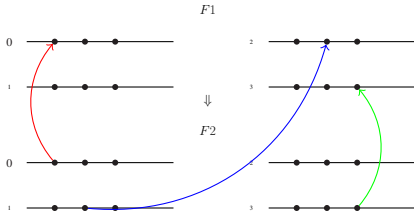


Figure S86. $Q(0,2,3) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(0,1,3)$

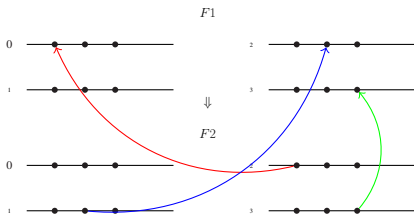


Figure S88. $Q(0,2,3) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(2,1,3)$

Figure S90. $Q(0,2,3)$

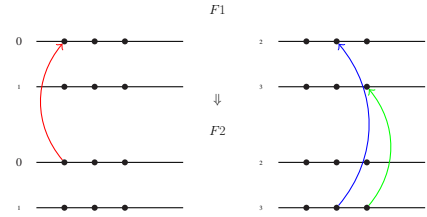


Figure S83. $Q(0,2,3) : \frac{1}{2} \times \frac{1}{2} \times r_{23} \times Q(0,3,3)$

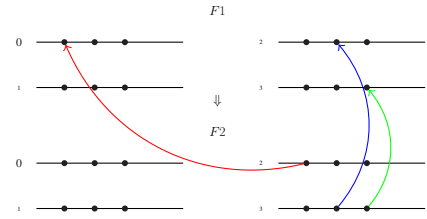


Figure S85. $Q(0,2,3) : \frac{1}{2} \times \frac{1}{2} \times r_{23} \times Q(2,3,3)$

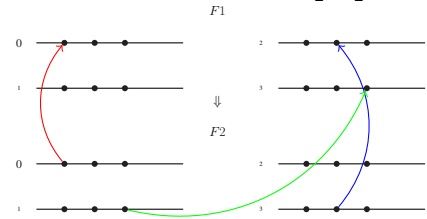


Figure S87. $Q(0,2,3) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(031)$

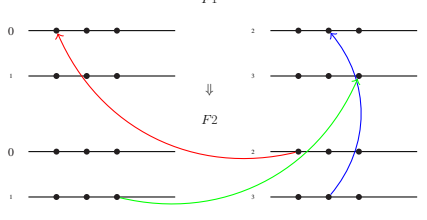


Figure S89. $Q(0,2,3) : \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times Q(2,3,1)$