CMPS 101: Master Theorem

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I. The Master Method

Let A≥1, b>1, $f(n)$ asyomptic positive.

Let $T(n)$ be defined by $T(n) = \mathrm{aT}\left(\frac{n}{b}\right) + f(n)$

We have three cases

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some $\epsilon > 0$, then $T(n) = \theta(n^{\log_b a})$
2. If $f(n0 = \theta(n^{\log_b n})$, then $T(n) = \theta(n^{\log_b a} \log(n)) = \theta(f(n)\,\mathrm{logn})$
3. if $f(n) = \Omega(n^{\log_b a + \epsilon})$for some ξ>0 and if af$\left(\frac{n}{b}\right) \le$ cf(n) for some c in $0 < c < 1$

   all sufficiently large n, then $T(n) = \theta(f(n))$

Remember for all cases we compare f(n) to $n^{\log_b a}$

in each case the Winner (faster grouping function) determine assyoptic Solution, and the winner must win by a polynomial factor

EXAMPLE 1:

- $T(n) = T\left(\dfrac{n}{2}\right) + n^2$
- compare: n^3 to X$\mathrm{n}^{\log_2 X} = n^3$
- by case 2: $T(n) = \theta(n^3\mathrm{logn})$

EXAMPLE 2:

- $T(n) = 5T\left(\dfrac{n}{4}\right) + n$
- compare: n to $n^{\log_4 5}$
- note: 4<5 => 1<$\log_4 5$
- Let ϵ =$\log_4 5 - 1$ then ϵ>0 and $1 = \log_4 5 - \epsilon$, therefore n=$O(n)$
- by case 1: $T(n) = \theta(n^{\log_4 5})$

Example 3:

- $T(n) = 5T\left(\dfrac{n}{4}\right) + n^2$
- Compare n^2 to $n^{\log_4 5}$
- note 5<16 implies $\log_4 5 < 2$

- Let $\varepsilon = 2 - \log_4 5$ then $\varepsilon > 0$ and $2 = \log_4 5 + 2$, ≥ 0

- must also show $5\left(\frac{n}{4}\right)^2 \leq \text{cn}^2$

- $\frac{5}{16} n^2 \leq \text{cn}^2$

- IE $\frac{5}{16} \leq c$

- therefore Pick any c in $\frac{5}{16} \leq c < 1$

  This is possinle since $\frac{5}{16} < 1$

Example 4:

- $T(n) = 8T\left(\frac{n}{2}\right) + 10n^3 + 15n^2 - n\sqrt{n} + \text{nlogn} - 1$

- Simplify to $T(n) = 8T\left(\frac{n}{2}\right) + n^3$

- Answer by case 2 : $T(n) = \theta(n^3 \text{logn})$

Often: we write the recurrense as

$T(n) = \text{aT}\left(\frac{n}{b}\right) + \theta(f(n))$

Exercises:

- Prove that if f(n) is a polynomial, and if deg(f)$>\log_b a$, then case 3 holds, and regularity condition is necessarily satisfied

EXAMPLE 5:

- $T(n) = T\left(\lfloor\frac{n}{2}\rfloor\right) + 2T\left(\lceil\frac{n}{2}\rceil\right) + \log(n!)$

- simplify $T(n) = 3T\left(\frac{n}{2}\right) + \text{nlogn}$

- compare nlogn to $n^{\log_2 3}$

- Let $\epsilon = \frac{1}{2}(\log_2 3 - 1)$ since $2 < 3 \Rightarrow 1 < \log_2 3$, we have $\epsilon > 0$

- $2\varepsilon = \log_2 3 - 1$

- $1 + \varepsilon = \log_2 3 - \epsilon$

- NOTE: $\frac{\text{nlogn}}{n^{1+\varepsilon}} = \frac{\text{nlogn}}{n\, n^{\epsilon}} {=>} 0$ as n$\Rightarrow\infty$

- therefore nlongn$= o(n^{1+\varepsilon}) \subseteq O(n^{1+\epsilon}) = O(n^{\log_2 3 - \epsilon})$

- by case 1: $T(n) = \theta(n^{\log_2 3})$