

## CMPS 101

### Algorithms and Abstract Data Types Some Common Functions

We present several common functions and estimates which occur frequently in the analysis of algorithms.

#### Floors and Ceilings

Given  $x \in \mathbf{R}$ , we denote by  $\lfloor x \rfloor$  and  $\lceil x \rceil$  the *floor* of  $x$  and the *ceiling* of  $x$ , respectively. These are defined to be the unique integers satisfying

$$x-1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x+1$$

Equivalently, if  $x \in \mathbf{R}$  and  $N \in \mathbf{Z}$  then

- (1)  $N = \lfloor x \rfloor$  if and only if  $N \leq x < N+1$ , and
- (2)  $N = \lceil x \rceil$  if and only if  $N-1 < x \leq N$ .

In other words:

- (1)  $\lfloor x \rfloor$  is the *greatest integer less than or equal to  $x$* , and
- (2)  $\lceil x \rceil$  is the *least integer greater than or equal to  $x$* .

**Lemma 1:** Let  $x \in \mathbf{R}$  and  $a, b \in \mathbf{Z}$ . Then

- (1)  $a \leq x < b$  if and only if  $a \leq \lfloor x \rfloor < b$ , and
- (2)  $a < x \leq b$  if and only if  $a < \lceil x \rceil \leq b$ .

**Proof of (1):**

- (i)  $a \leq x$  implies  $a \leq \lfloor x \rfloor$ , since among all integers that are less than or equal to  $x$ ,  $\lfloor x \rfloor$  is the greatest.
- (ii)  $x < b$  implies  $\lfloor x \rfloor < b$ , since  $\lfloor x \rfloor \leq x$ .
- (iii)  $a \leq \lfloor x \rfloor$  implies  $a \leq x$ , since  $\lfloor x \rfloor \leq x$ .
- (iv)  $\lfloor x \rfloor < b$  implies  $x < b$ , since  $b \leq x$  implies  $b \leq \lfloor x \rfloor$ , by (i). ///

**Exercise:** prove part (2).

**Lemma 2:** Let  $x \in \mathbf{R}$  and  $m \in \mathbf{Z}^+$ . Then

- (1)  $\left\lfloor \frac{\lfloor x \rfloor}{m} \right\rfloor = \left\lfloor \frac{x}{m} \right\rfloor$ , and
- (2)  $\left\lceil \frac{\lceil x \rceil}{m} \right\rceil = \left\lceil \frac{x}{m} \right\rceil$ .

**Proof of (1):** Let  $N = \lfloor \lfloor x \rfloor / m \rfloor$ . Then

$$\begin{aligned}
N &\leq \frac{\lfloor x \rfloor}{m} < N+1 \\
\Rightarrow mN &\leq \lfloor x \rfloor < m(N+1) \\
\Rightarrow mN &\leq x < m(N+1) & \text{(by lemma 1)} \\
\Rightarrow N &\leq x/m < N+1 \\
\Rightarrow N &= \lfloor x/m \rfloor,
\end{aligned}$$

and therefore  $\lfloor \lfloor x \rfloor / m \rfloor = N = \lfloor x/m \rfloor$ . ///

**Exercise:** prove part (2).

**Lemma 3:** Let  $a, b, n \in \mathbf{Z}^+$ . Then

$$\begin{aligned}
(1) \quad \left\lfloor \frac{\lfloor n/a \rfloor}{b} \right\rfloor &= \left\lfloor \frac{n}{ab} \right\rfloor, \text{ and} \\
(2) \quad \left\lceil \frac{\lceil n/a \rceil}{b} \right\rceil &= \left\lceil \frac{n}{ab} \right\rceil.
\end{aligned}$$

**Proof:** Set  $x = n/a$  and  $m = b$  in lemma 2. ///

**Exercise**

Let  $n \in \mathbf{Z}$ . Show that (a)  $\left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil = n$ , (b)  $\left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{n+1}{2} \right\rfloor$ , and (c)  $\left\lfloor \frac{n}{2} \right\rfloor = \left\lceil \frac{n-1}{2} \right\rceil$ .

## Logarithms

Let  $x, a, b \in \mathbf{R}$  where  $x > 0$ ,  $a > 1$ , and  $b > 1$ . Then  $\log_a(x)$  denotes the exponent on  $a$  which gives  $x$ . In other words,  $\log_a(x)$  is the inverse function of  $a^x$ , which means  $a^{\log_a(x)} = x$  and  $\log_a(a^x) = x$ . Thus

$$x = a^{\log_a(x)} = \left(b^{\log_b(a)}\right)^{\log_a(x)} = b^{\log_b(a) \cdot \log_a(x)}$$

Taking  $\log_b$  of both sides of this equation yields

$$(*) \quad \log_b(x) = \log_b(a) \cdot \log_a(x),$$

which says in particular  $\log_b(x) = \text{constant} \cdot \log_a(x)$ , i.e. any two log functions differ by a constant multiple. It follows that  $\log_b(n) = \Theta(\log_a(n))$ , so speaking in terms of asymptotic growth rates, there is really only one log function. Equation (\*) implies

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

which shows how to convert from one log function to another. In particular  $\lg(x) = \frac{\ln(x)}{\ln(2)}$ . Here we use the standard notation  $\lg() = \log_2()$ , and  $\ln() = \log_e()$ , where  $e = 2.71828$ . Equation (\*) also implies  $a^{\lg_b(x)} = a^{\lg_a(x) \cdot \lg_b(a)} = \left(a^{\lg_a(x)}\right)^{\lg_b(a)} = x^{\lg_b(a)}$ , which gives us the useful formula

$$a^{\lg_b(x)} = x^{\lg_b(a)}.$$

## Stirling's Formula

Let  $n \in \mathbf{Z}^+$ . Then  $n! = \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n \cdot \left(1 + \Theta\left(\frac{1}{n}\right)\right)$ .

Stirling's formula gives a simple way to determine asymptotic (upper, lower, and tight) bounds on functions involving  $n!$ . An elementary proof can be found at

<http://www.sosmath.com/calculus/sequence/stirling/stirling.html>

### Corollary:

- (1)  $n! = o(n^n)$
- (2)  $n! = \omega(b^n)$  for any  $b > 0$
- (3)  $\log(n!) = \Theta(n \log(n))$

### **Proof of (1):**

$$\frac{n!}{n^n} = \frac{\sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n \cdot \left(1 + \Theta\left(\frac{1}{n}\right)\right)}{n^n} = \frac{\sqrt{2\pi n} \cdot \left(1 + \Theta\left(\frac{1}{n}\right)\right)}{e^n} \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ showing that } n! = o(n^n). \quad ///$$

**Proof of (3):** Taking log (any base) of both sides of Stirling's formula, we get

$$\begin{aligned} \log(n!) &= \log \sqrt{2\pi n} + \log \left(\frac{n}{e}\right)^n + \log \left(1 + \Theta\left(\frac{1}{n}\right)\right) \\ &= \frac{1}{2} \log(2\pi) + \frac{1}{2} \log(n) + n \log(n) - n \log(e) + \log \left(1 + \Theta\left(\frac{1}{n}\right)\right). \end{aligned}$$

Therefore

$$\frac{\log(n!)}{n \log(n)} = 1 + (\text{stuff that } \rightarrow 0 \text{ as } n \rightarrow \infty),$$

hence  $\lim_{n \rightarrow \infty} \left(\frac{\log(n!)}{n \log(n)}\right) = 1$ , proving that  $\log(n!) = \Theta(n \log(n))$ . ///

**Exercise:** Prove part (2) of the corollary.

**Exercise:** Prove that  $\binom{2n}{n} = \Theta\left(\frac{4^n}{\sqrt{n}}\right)$ , where  $\binom{m}{k}$  denotes the binomial coefficient  $\binom{m}{k} = \frac{m!}{k!(m-k)!}$ , for  $0 \leq k \leq m$ .

**Exercise:** Determine a number  $a > 0$  such that  $\binom{3n}{n} = \Theta\left(\frac{a^n}{\sqrt{n}}\right)$ .

**Exercise:** For each  $k \geq 2$ , determine a number  $a_k > 0$  such that  $\binom{kn}{n} = \Theta\left(\frac{a_k^n}{\sqrt{n}}\right)$ .