CMPS 101
Homework Assignment 4 Solutions

1. Define $T(n)$ defined by the recurrence formula

$$T(n) = \begin{cases} 6 & 1 \le n < 3 \\ 2T(\lfloor n/3 \rfloor) + n & n \ge 3 \end{cases}$$

Use induction to show that $\forall n \ge 1 : T(n) \le 6n$, and hence $T(n) = O(n)$. (Hint use strong induction with two base cases: $n = 1$ and $n = 2$.)

Proof:

I. $T(1) = 6 \le 6 \cdot 1$ and $T(2) = 6 \le 12 = 6 \cdot 2$, so both base cases are satisfied.

II. Let $n > 1$ and assume for all k in the range that $1 \le k < n$ that $T(k) \le 6k$. We must show that $T(n) \le 6n$. Observe

$$\begin{aligned} T(n) &= 2T(\lfloor n/3 \rfloor) + n \\ &\le 2 \cdot 6\lfloor n/3 \rfloor + n && \text{by the induction hypothesis with } k = \lfloor n/2 \rfloor \\ &\le 12(n/3) + n && \text{since } \lfloor x \rfloor \le x \\ &= 4n + n \\ &= 5n \\ &\le 6n \end{aligned}$$

as required. ■

2. Let T be a tree with n vertices and m edges. Prove that $m = n - 1$ by induction on m.

Proof:
This result was proved in the handout on Induction Proofs by induction on n. We prove it here by induction on m.

I. If $m = 0$, then T can have only one vertex, since T is connected. Thus $n = 1$, establishing the base case.

II. Let $m > 0$ and assume that any tree T' with fewer than m edges satisfies $|E(T')| = |V(T')| - 1$. We must show that if a tree T has m edges, then $|E(T)| = |V(T)| - 1$. Pick an edge $e \in E(T)$ and remove it. The resulting graph consists of two trees T_1 and T_2, each having fewer than m edges. Suppose T_i has m_i edges and n_i vertices (for $i = 1, 2$). Then the induction hypothesis implies $m_i = n_i - 1$ (for $i = 1, 2$). Also $n_1 + n_2 = n$, since no vertices were removed. Therefore

$$m = m_1 + m_2 + 1 = (n_1 - 1) + (n_2 - 1) + 1 = (n_1 + n_2) - 1 = n - 1$$

as required. ■

3. Let G be an acyclic graph with n vertices, m edges and k connected components. Use the result of the preceding problem to prove that $m = n - k$. (Hint: apply the preceding result to each of the k trees composing G.)

Proof:
Let the connected components of G (which are necessarily trees) be $T_1, T_2, \ldots, T_k$. Suppose T_i has m_i edges and n_i vertices (for $1 \le i \le k$). By the result of the preceding problem we have $m_i = n_i - 1$ $(1 \le i < n)$. Therefore

$$m = \sum_{i=1}^{k} m_i = \sum_{i=1}^{k} (n_i - 1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1 = n - k$$

as claimed. ∎

4. Use the iteration method to find an exact solution to the recurrence:

$$T(n) = \begin{cases} 1 & 1 \le n < 3 \\ 2T(\lfloor n/3 \rfloor) + 5 & n \ge 3 \end{cases}$$

Solution:
Recurring down to the k^{th} level, we have

$$\begin{aligned} T(n) &= 5 + 2T(\lfloor n/3 \rfloor) \\ &= 5 + 2\left(5 + 2T\left(\left\lfloor \frac{\lfloor n/3 \rfloor}{3} \right\rfloor\right)\right) = 5 + 2 \cdot 5 + 2^2 T(\lfloor n/3^2 \rfloor) \\ &= 5 + 2 \cdot 5 + 2^2\left(5 + 2T\left(\left\lfloor \frac{\lfloor n/3^2 \rfloor}{3} \right\rfloor\right)\right) = 5 + 2 \cdot 5 + 2^2 \cdot 5 + 2^3 T(\lfloor n/3^3 \rfloor) \\ &\vdots \\ &= \sum_{i=0}^{k-1} 5 \cdot 2^i + 2^k T(\lfloor n/3^k \rfloor) \end{aligned}$$

The recursion terminates when the recursion depth k satisfies $1 \le \lfloor n/3^k \rfloor < 3$, which is equivalent to $k = \lfloor \log_3(n) \rfloor$. For this value of k we have $T(\lfloor n/3^k \rfloor) = 1$, and therefore

$$T(n) = \sum_{i=0}^{k-1} 5 \cdot 2^i + 2^k = 5\left(\frac{2^k - 1}{2 - 1}\right) + 2^k = 6 \cdot 2^k - 5$$

so the exact solution is $T(n) = 6 \cdot 2^{\lfloor \log_3(n) \rfloor} - 5$ ∎