CMPS 101
Homework Assignment 3
Solutions

1. The last exercise in the handout entitled *Some Common Functions*.

   Use Stirling's formula to prove that $\binom{2n}{n} = \Theta\left(\frac{4^n}{\sqrt{n}}\right)$.

   Proof: By Stirling's formula

$$\binom{2n}{n} = \frac{(2n)!}{n!(2n-n)!} = \frac{(2n)!}{(n!)^2} = \frac{\sqrt{2\pi \cdot 2n} \cdot \left(\frac{2n}{e}\right)^{2n} \cdot \left(1+\Theta(1/2n)\right)}{\left(\sqrt{2\pi\, n} \cdot \left(\frac{n}{e}\right)^n \cdot \left(1+\Theta(1/n)\right)\right)^2}$$

$$= \frac{2^{2n}}{\sqrt{\pi n}} \cdot \frac{1+\Theta(1/2n)}{\left(1+\Theta(1/n)\right)^2} = \frac{1}{\sqrt{\pi}} \cdot \frac{4^n}{\sqrt{n}} \cdot \frac{1+\Theta(1/2n)}{\left(1+\Theta(1/n)\right)^2}$$

   so that

$$\frac{\binom{2n}{n}}{\frac{4^n}{\sqrt{n}}} = \frac{1}{\sqrt{\pi}} \cdot \frac{1+\Theta(1/2n)}{\left(1+\Theta(1/n)\right)^2} \to \frac{1}{\sqrt{\pi}} \quad \text{as} \quad n \to \infty$$

   The result now follows since $0 < \frac{1}{\sqrt{\pi}} < \infty$. ///

2. Let $f(n)$ be a positive, increasing function that satisfies $f(n/2) = \Theta(f(n))$. Show that

$$\sum_{i=1}^{n} f(i) = \Theta(nf(n))$$

   (Hint: follow the **Example** on page 4 of the handout on asymptotic growth rates in which it is proved that $\sum_{i=1}^{n} i^k = \Theta(n^{k+1})$ for any positive integer *k*.)

   Proof: Since $f(n)$ is increasing we have $\sum_{i=1}^{n} f(i) \le \sum_{i=1}^{n} f(n) = nf(n) = O(nf(n))$. Note also that

$$\begin{aligned}
\sum_{i=1}^{n} f(i) &\ge \sum_{i=\lceil n/2 \rceil}^{n} f(i) && \text{by discarding some positive terms} \\
&\ge \sum_{i=\lceil n/2 \rceil}^{n} f(\lceil n/2 \rceil) && \text{since } f(n) \text{ is increasing} \\
&= (n - \lceil n/2 \rceil + 1)\, f(\lceil n/2 \rceil) && \text{by counting terms} \\
&= (\lfloor n/2 \rfloor + 1)\, f(\lceil n/2 \rceil) && \text{since } n = \lfloor n/2 \rfloor + \lceil n/2 \rceil \\
&> ((n/2) - 1 + 1)\, f(n/2) && \text{since } f(n) \text{ is increasing, } \lceil x \rceil \ge x \text{, and } \lfloor x \rfloor > x-1 \\
&= (n/2)\, f(n/2) && \\
&= \Omega(nf(n)) && \text{since } f(n/2) = \Theta(f(n)) \text{, whence } f(n/2) = \Omega(f(n))
\end{aligned}$$

It follows from an exercise in the handout on Asymptotic Growth rates that $\sum_{i=1}^{n} f(i) = \Theta(nf(n))$, as claimed. ///

3. Use the result of the preceding problem to give an alternate proof of $\log(n!) = \Theta(n\log(n))$ that does not use Stirling's formula.

Proof:
Observe that $\log(n/2) = \log(n) - \log(2) = \Theta(\log(n))$. We may therefore apply the result of the preceding problem with $f(n) = \log(n)$, and properties of logarithms to get

$$\log(n!) = \sum_{i=1}^{n} \log(i) = \Theta(n\log(n))$$

as claimed. ///

4. Let $g(n)$ be an asymptotically non-negative function. Prove that $o(g(n)) \cap \Omega(g(n)) = \varnothing$.

Proof:
Assume to get a contradiction that $f(n) \in o(g(n)) \cap \Omega(g(n))$. Then since $f(n) = \Omega(g(n))$ we have

(1) $\exists c_1 > 0,\ \exists n_1 > 0,\ \forall n \geq n_1\colon \quad 0 \leq c_1 g(n) \leq f(n)$

Also, since $f(n) = o(g(n))$ we have

(2) $\forall c_2 > 0,\ \exists n_2 > 0,\ \forall n \geq n_2\colon \quad 0 \leq f(n) < c_2 g(n)$

Let $c_2 = c_1$. Then $c_2 > 0$, and by (2) there exists $n_2 > 0$ such that $0 \leq f(n) < c_1 g(n)$ for all $n \geq n_2$. Pick any $m \geq \max(n_1, n_2)$. Then by (1) we have $0 \leq c_1 g(m) \leq f(m) < c_1 g(m)$, and hence $c_1 g(m) < c_1 g(m)$, a contradiction. Our assumption was therefore false, and no such function $f(n)$ can exist. We conclude that $o(g(n)) \cap \Omega(g(n)) = \varnothing$. ///

5. Exercise 1 from the induction handout.

Prove that for all $n \geq 1$: $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$. Do this twice:

a. Using form IIa of the induction step.
b. Using form IIb of the induction step.

Proof: Let $P(n)$ be the equation $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$.

I. Observe that $\sum_{i=1}^{1} i^3 = 1^3 = 1^2 = \left(\frac{1\cdot(1+1)}{1}\right)^2$, whence $P(1)$ is true.

IIa. Let $n \geq 1$ and assume $P(n)$ is true, i.e. for this n, we assume that $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$. We must show that $P(n+1)$ holds: $\sum_{i=1}^{n+1} i^3 = \left(\frac{(n+1)((n+1)+1)}{2}\right)^2$. Thus

$$\begin{aligned}
\sum_{i=1}^{n+1} i^3 &= \sum_{i=1}^{n} i^3 + (n+1)^3 \\
&= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \quad \text{(by the induction hypothesis)} \\
&= \frac{n^2(n+1)^2 + 4(n+1)^3}{4} = \frac{(n+1)^2\left[n^2 + 4n + 4\right]}{4} \\
&= \frac{(n+1)^2(n+2)^2}{4} = \left(\frac{(n+1)(n+2)}{2}\right)^2
\end{aligned}$$

showing that $P(n+1)$ is true. ///

IIb. Let $n > 1$ and assume $P(n-1)$ is true, i.e. for this n, we assume that $\sum_{i=1}^{n-1} i^3 = \left(\frac{(n-1)n}{2}\right)^2$. We must show that $P(n)$ holds: $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$. Thus

$$\begin{aligned}
\sum_{i=1}^{n} i^3 &= \sum_{i=1}^{n-1} i^3 + n^3 \\
&= \left(\frac{(n-1)n}{2}\right)^2 + n^3 \quad \text{(by the induction hypothesis)} \\
&= \frac{(n-1)^2 n^2 + 4n^3}{4} = \frac{n^2\left[(n^2 - 2n + 1) + 4n\right]}{4} \\
&= \frac{n^2\left[n^2 + 2n + 1\right]}{4} = \frac{n^2(n+1)^2}{4} = \left(\frac{n(n+1)}{2}\right)^2
\end{aligned}$$

showing that $P(n)$ is true. ///

6. Exercise 2 from the induction handout.
Define $S(n)$ for $n \in Z^+$ by the recurrence:

$$S(n) = \begin{cases} 0 & \text{if } n = 1 \\ S(\lceil n/2 \rceil) + 1 & \text{if } n \geq 2 \end{cases}$$

Prove that $S(n) \geq \lg(n)$ for all $n \geq 1$, and hence $S(n) = \Omega(\lg n)$.

Proof: Let $P(n)$ be the inequality $S(n) \geq \lg(n)$.

I. The inequality $S(1) \geq \lg(1)$ reduces to $0 \geq 0$, which is obviously true, so $P(1)$ holds.

IId. Let $n > 1$ and assume for all k in the range $1 \leq k < n$ that $S(k) \geq \lg(k)$. Then

$$S(n) = S(\lceil n/2 \rceil) + 1 \qquad \text{(by the definition of } S(n)\text{)}$$

$$
\begin{aligned}
&\geq \lg\lceil n/2 \rceil + 1 && \text{(by the induction hypothesis with } k = \lceil n/2 \rceil\text{)} \\
&\geq \lg(n/2) + 1 && \text{(since } \lceil x \rceil \geq x \text{ for any } x\text{)} \\
&= \lg(n) - \lg(2) + 1 \\
&= \lg(n)
\end{aligned}
$$

showing that $P(n)$ holds. Therefore $S(n) \geq \lg(n)$ for all $n \geq 1$, as claimed. ///

7. Let $T(n)$ be defined by the recurrence formula:

$$
T(n) = \begin{cases} 1 & n = 1 \\ T(\lfloor n/2 \rfloor) + n^2 & n \geq 2 \end{cases}
$$

Show that $\forall n \geq 1$: $T(n) \leq \frac{4}{3}n^2$, and hence $T(n) = O(n^2)$. (Hint: follow Example 3 on page 3 of the induction handout.)

Proof:
Let $P(n)$ be the statement $T(n) \leq (4/3)n^2$. Then $P(1)$ is true, since $T(1) = 1 \leq 4/3 = (4/3)\cdot 1^2$, and the base case is satisfied.

Let $n > 1$ be chosen arbitrarily, and suppose for all k in the range $1 \leq k < n$ that $T(k) \leq (4/3)k^2$. We must show as a consequence that $T(n) \leq (4/3)n^2$. Observe

$$
\begin{aligned}
T(n) &= T(\lfloor n/2 \rfloor) + n^2 && \text{by the recurrence formula for } T(n) \\
&\leq (4/3)\lfloor n/2 \rfloor^2 + n^2 && \text{by the induction hypothesis with } k = \lfloor n/2 \rfloor \\
&\leq (4/3)(n/2)^2 + n^2 && \text{since } \lfloor x \rfloor \leq x \text{ for any } x \\
&= n^2/3 + n^2 \\
&= (4/3)n^2,
\end{aligned}
$$

as required. ///

8. Let $T(n)$ be defined by the recurrence formula:

$$
T(n) = \begin{cases} 2 & n = 1, 2 \\ 9T(\lfloor n/3 \rfloor) + 1 & n \geq 3 \end{cases}
$$

Show that $\forall n \geq 1$: $T(n) \leq 3n^2 - 1$, and hence $T(n) = O(n^2)$. (Hint: emulate Example 4 on page 4 of the induction handout. I. Base: check the two cases $n = 1$, and $n = 2$. II. Induction step: show that for all $n \geq 3$, if for any k in the range $1 \leq k < n$ we have $T(k) \leq 3k^2 - 1$, then $T(n) \leq 3n^2 - 1$.)

Proof:

Let $P(n)$ be the statement $T(n) \le 3n^2 - 1$. $P(1)$ is true since $T(1) = 2 = 3 \cdot 1^2 - 1$, and $P(2)$ is true because $T(2) = 2 \le 11 = 3 \cdot 2^2 - 1$.

Let $n > 2$ be arbitrary, and assume for all k in the range $1 \le k < n$ that $T(k) \le 3k^2 - 1$. Note that in particular $1 \le \lfloor n/3 \rfloor < n$ (since $n \ge 3 \Rightarrow n/3 \ge 1 \Rightarrow \lfloor n/3 \rfloor \ge 1$) and hence $T(\lfloor n/3 \rfloor) \le 3\lfloor n/3 \rfloor^2 - 1$. We must show as a consequence that $T(n) \le 3n^2 - 1$.

$$
\begin{aligned}
T(n) &= 9T(\lfloor n/3 \rfloor) + 1 && \text{by the recurrence formula for } T(n) \\
&\le 9\left(3\lfloor n/3 \rfloor^2 - 1\right) + 1 && \text{by the induction hypothesis} \\
&= 9 \cdot 3\lfloor n/3 \rfloor^2 - 9 + 1 && \\
&\le 9 \cdot 3(n/3)^2 - 9 + 1 && \text{since } \lfloor x \rfloor \le x \text{ for any } x \\
&= 9 \cdot 3(n^2/3^2) - 9 + 1 && \\
&= 3n^2 - 8 && \\
&\le 3n^2 - 1 && \text{since } -8 \le -1
\end{aligned}
$$

and therefore $T(n) \le 3n^2 - 1$, as required. ///