# CS 101: Iteration Method

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I. Recurrence Relations:

- Example: Iteration method

  $T(n) = \begin{Bmatrix} 0 & n=1 \\ T\left(\frac{n}{3}\right)_{\text{floor}}+1 & n\geq 2 \end{Bmatrix}$

  1. $T(n) = 1 + T\left(\frac{n}{2}\right)_{\text{floor}}$

     $= 1 + 1 + T\left(\frac{\left(\frac{n}{2}\right)_{\text{floor}}}{2}\right)_{\text{floor}}$

     $= 2 + T\left(\frac{n}{2^2}\right)_{\text{floor}}$

     $= 2 + 1 + T\left(\frac{\frac{n}{2}}{2^2}\right)_{\text{floor}}$

     $= 3 + T\left(\frac{n}{2^3}\right)$

     $= k + T\left(\frac{n}{2^k}\right)_{\text{floor}}$

  2. Bottoms out recursion when $\left(\frac{n}{2^k}\right)_{\text{floor}} = 1$

     $\left(\frac{n}{2^k}\right)_{\text{floor}} = 1$

     $1 \leq \frac{n}{2^k} < 2$

     $2^k \leq n < 2^{k+1}$

     $k \leq \log(n) < k+1$

     k=(log(n))$_{\text{floor}}$

  3. Thus: $T(n) = (\log(n))_{\text{floor}}1$

     Exercise: Check by direct Substituion than $T(n) = \text{logn}_{\text{floor}}$ is the solution to the original recurrence

- Notice that $T(n) = \theta\text{logn}$

- Example 2:

  1. $T(n) = \begin{Bmatrix} c & 1\leq n< n_0 \\ T\left(\frac{n}{2}\right)_{\text{floor}}+d & n\geq n_0 \end{Bmatrix}$

  2. $T(n) = d + T\left(\frac{n}{2}\right)_{\text{floor}}$

$$=d+d+T\left(\frac{\frac{n}{2}}{2}\right)_{\text{floor}}$$

$$=2d+T\left(\frac{n}{2^2}\right)$$

$$=3d+T\left(\frac{n}{2^3}\right)_{\text{floor}}$$

$$=\text{kd}+T\left(\frac{n}{2^k}\right)_{\text{floor}}$$

3. We seek the smallest k such that $1 \leq \left(\frac{n}{2^k}\right)_{\text{floor}} < n_0$

   $\frac{n}{2^k} < n_0$

   $\frac{n}{n_0} < 2^k$

   $\log\frac{n}{n_0} < k$

   $k-1 \leq \log\frac{n}{n_0} < k$ Since k is least integer satisfying

   k-1$=(\log\frac{n}{n_0})_{\text{floor}}$

   $k=(\text{logn}-\text{logn}_0)+1$

   Thus: $T(n)=d((\text{logn}-\text{logn}_0)+1)+c$

   Therfore $T(n)=\theta(\text{logn})$

- Example 3:

1. $T(n)=\begin{cases} 1 & n=1 \\ T\left(\frac{n}{2}\right)_{\text{floor}}+n^2 & n\geq 2 \end{cases}$

2. $T(n)=n^2+T\left(\frac{n}{2}\right)_{\text{floor}}$

$$=n^2+\left(\frac{n}{2}\right)^2_{\text{floor}}+T\left(\frac{\frac{n}{2}}{2}\right)_{\text{floor}}$$

$$=n^2+\left(\frac{n}{2}\right)^2_{\text{floor}}+T\left(\frac{n}{2^2}\right)_{\text{floor}}$$

$$=n^2+\left(\frac{n}{2}\right)^2_{\text{floor}}+\left(\frac{n}{2}\right)^2_{\text{floor}}+T\left(\frac{n}{2^3}\right)_{\text{floor}}$$

$$=\sum_{i=o}^{k-1}\left(\frac{n}{2^i}\right)^2_{\text{floor}}+T\left(\frac{n}{2^k}\right)_{\text{floor}}$$

3. Solve $\left(\frac{n}{2^k}\right)_{\text{floor}}=1$ for k

   we get $k=(\text{logn})_{\text{floor}}$

4. Hence: $T(n)=\sum_{i=0}^{(\text{logn})_{\text{floor}}-1}\left(\frac{n}{2^i}\right)^2_{\text{foor}}+1$

5. Use this sum to get asymoptic solution

   i. $T(n)=\sum_{i=0}^{k-1}\left(\frac{n}{2^i}\right)^2_{\text{floor}}+1$

ii. $=\sum_{i=0}^{k-1}\left(\frac{n}{2^i}\right)^2+1$ since $x_{\text{floor}}\leq x$

iii. $=n^2\sum_{i=0}^{k-1}\left(\frac{1}{4}\right)^i+1$

iv. $\leq n^2\sum_{i=0}^{\infty}$

v. $=n^2\left(\frac{1}{1-\frac{1}{4}}\right)+1$

vi. $=\frac{4}{3}n^2+1=\mathrm{O}(\mathrm{n}^2)$

vii. Therefore $T(n)=O(n^2)$

viii. Also: $T(n)=\sum_{i=0}^{k-1}\left(\frac{n}{2^i}\right)_{\text{floor}}^2+1$

ix. $\geq\sum_{i=0}^{k—1}\left(\frac{n}{2^i}-1\right)^2+1$ Since $x_{\text{floor}}>x-1$

x. $=\sum_{i=0}^{k-1}\left(\frac{n^2}{4}-\frac{2n}{2^i}+1\right)+1$

xi. $=n^2\sum_{i=0}^{k-1}\left(\frac{1}{4}\right)^i-2n\sum_{i=0}^{k-1}\left(\frac{1}{2}\right)^i+k \quad +1$

xii. $\geq n^2-2n\left(\frac{1}{1-\frac{1}{2}}\right)+\mathrm{logn}_{\text{floor}}+1$

xiii. $=\Omega(n^2)$

xiv. there fore, $T(n)=\Omega(n^2)$

xv. and: $T(n)=\theta(n^2)$

II. Master Method:

- Determine Asymoption solution to

  $T(n)=\mathrm{aT}\left(\frac{n}{b}\right)+f(n)$

  Where a$\geq$1, b$\geq$1