CMPS 101
Homework Assignment 2
Solutions

1. p.50: 3.1-1
   Let $f(n)$ and $g(n)$ be asymptotically non-negative functions. Using the basic definition of Θ-notation, prove that $f(n) + g(n) = \Theta(\max(f(n), g(n)))$.

   Proof:
   Since $f(n)$ and $g(n)$ are asymptotically non-negative, there exists a positive constant n_0 such that $f(n) \geq 0$ and $g(n) \geq 0$ for all $n \geq n_0$. For such n we have

   $$0 \leq \max(f(n), g(n))$$
   $$\leq \min(f(n), g(n)) + \max(f(n), g(n))$$
   $$\leq 2 \cdot \max(f(n), g(n)).$$

   But $f(n) + g(n) = \min(f(n), g(n)) + \max(f(n), g(n))$, so for all $n \geq n_0$ we have

   $$0 \leq 1 \cdot \max(f(n), g(n)) \leq f(n) + g(n) \leq 2 \cdot \max(f(n), g(n)).$$

   Thus $f(n) + g(n) = \Theta(\max(f(n), g(n)))$, as required.   ///

2. p.50: 3.1-3
   Explain why the statement "The running time of algorithm A is at least $O(n^2)$" is meaningless.

   Solution:
   This statement is true under all circumstances, hence it conveys no useful information, and is therefore meaningless. To illustrate, let $T(n)$ be the running time of algorithm A. To say that $T(n)$ is "at least $O(n^2)$" is to say that $T(n)$ is bounded below by a function that is bounded above (asymptotically) by n^2. If $T(n)$ in the class $O(n^2)$, then $T(n)$ is bounded below by itself, which is bounded above asymptotically by n^2, and hence the statement is true. If, on the other hand, $T(n)$ is in the class $\Omega(n^2)$, then $T(n)$ is bounded below by cn^2 (for sufficiently large n), which is bounded above asymptotically by n^2, and again the statement is true. Even if $T(n)$ is not comparable to n^2, $T(n)$ is bounded below by some positive constant, which is bounded above by n^2. (This is true since even if algorithm A performs no operations, it takes some time to start and stop, so that $T(n)$ is greater than some, possibly small, positive number.) In all cases the statement is true, and therefore devoid of meaning.   ///

3. p. 50: 3.1-4
   Determine whether the following statements are true or false.

   a. $2^{n+1} = O(2^n)$
      Solution: True since $2^{n+1} = 2 \cdot 2^n = \text{const} \cdot 2^n = O(2^n)$.
   b. $2^{2n} = O(2^n)$
      Solution: False since $2^{2n} = (2^2)^n = 4^n = \omega(2^n)$ and since $\omega(2^n) \cap O(2^n) = \varnothing$.

4. p.58: 3-2abcdef
   Indicate, for each pair of expressions (A, B) in the table below, whether A is O, o, Ω, ω, or Θ of B. Assume that $k \geq 1$, $\varepsilon > 0$, and $c > 1$ are constants. Place 'yes' or 'no' in each of the empty cells below, and justify your answers.

	A	B	O	o	Ω	ω	Θ
a.	$\lg^k n$	n^ε	yes	yes	no	no	no
b.	n^k	c^n	yes	yes	no	no	no
c.	$\sqrt{n}$	$n^{\sin n}$	no	no	no	no	no
d.	2^n	$2^{n/2}$	no	no	yes	yes	no
e.	$n^{\lg c}$	$c^{\lg n}$	yes	no	yes	no	yes
f.	$\lg(n!)$	$\lg(n^n)$	yes	no	yes	no	yes

Justifications:

a. Applying l'Hopital's rule $\lceil k \rceil$ times, we get $\lim\limits_{n \to \infty} \dfrac{(\lg n)^k}{n^\varepsilon} = 0$, whence $\lg^k n = o(n^\varepsilon)$.

b. Again, by $\lceil k \rceil$ applications of l'Hopital's rule, $\lim\limits_{n \to \infty} \dfrac{n^k}{c^n} = 0$, whence $n^k = o(c^n)$.

c. Let $c > 0$, and observe that $n^{\sin n} > c\sqrt{n}$ for $n = 2\pi k + \dfrac{\pi}{2}$, where k is a sufficiently large positive integer. Therefore the inequality $n^{\sin n} \leq c\sqrt{n}$ is false for arbitrarily large n, and hence $n^{\sin(n)} \neq O(\sqrt{n})$. Also note that $n^{\sin n} < c\sqrt{n}$ for $n = 2\pi k + \dfrac{3\pi}{2}$, where k is sufficiently large, and therefore the inequality $n^{\sin n} \geq c\sqrt{n}$ is false for arbitrarily large n, whence $n^{\sin(n)} \neq \Omega(\sqrt{n})$.

d. Observe $\lim\limits_{n \to \infty}(2^n / 2^{n/2}) = \lim\limits_{n \to \infty}(\sqrt{2})^n = \infty$, and therefore $2^n = \omega(2^{n/2})$.

e. Recall it was shown in class that $n^{\lg c} = c^{\lg n}$.

f. Note $\lg(n^n) = n\lg(n)$, and it was shown using Stirling's formula that $\lg(n!) = \Theta(n\lg(n))$.

5. p.58: 3-4cdeh
   Let $f(n)$ and $g(n)$ be asymptotically positive functions (i.e. $f(n) > 0$ and $g(n) > 0$ for sufficiently large n.) Prove or disprove the following statements.

c. Assume $\lg(g(n)) \geq 1$ and $f(n) \geq 1$ for all sufficiently large n. Then $f(n) = O(g(n))$ implies $\lg(f(n)) = O(\lg(g(n)))$. **True**
   Proof:
   Let $f(n) = O(g(n))$. Then there exists positive c and n_0 such that $f(n) \leq cg(n)$ for all $n \geq n_0$. The above hypotheses say we can take n_0 large enough to also guarantee $f(n) \geq 1$ and $\lg(g(n)) \geq 1$ for $n \geq n_0$. Thus $1 \leq f(n) \leq cg(n)$ for all $n \geq n_0$. Take $\log_2$ of this inequality to obtain

   (*) $\qquad 0 \leq \lg(f(n)) \leq \lg(c) + \lg(g(n)) = \left(\dfrac{\lg(c)}{\lg(g(n))} + 1 \right)\lg(g(n)) \leq (\lg(c) + 1)\lg(g(n))$.

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The last inequality is a consequence of $\lg(g(n)) \geq 1$, for this implies $1 \geq \dfrac{1}{\lg(g(n))}$, which implies

$\lg(c) \geq \dfrac{\lg(c)}{\lg(g(n))}$. Define the constant $b = \lg(c) + 1$. Inequality (*) gives $0 \leq \lg(f(n)) \leq b \lg(g(n))$

for all $n \geq n_0$, showing that $\lg(f(n)) = O(\lg(g(n)))$, as claimed.   ///

d. $f(n) = O(g(n))$ implies $2^{f(n)} = O(2^{g(n)})$. **False**

   Counter-Example:

   Let $f(n) = 2n$ and $g(n) = n$. Then $2^{g(n)} = 2^n$ and $2^{f(n)} = 2^{2n} = 4^n = \omega(2^n)$, so $2^{f(n)} = \omega(2^{g(n)})$,

   and therefore $2^{f(n)} \neq O(2^{g(n)})$.

e. $f(n) = O((f(n))^2)$. **False**

   Counter-Example:

   Let $f(n) = 1/n$. Then $f(n) = \omega((f(n))^2)$ since $\lim\limits_{n \to \infty} \dfrac{f(n)}{f(n)^2} = \lim\limits_{n \to \infty} \dfrac{1}{f(n)} = \lim\limits_{n \to \infty} n = \infty$, and therefore

   $f(n) \neq O((f(n))^2)$.   ///

h. $f(n) + o(f(n)) = \Theta(f(n))$. **True**

   Proof:

   In the above formula, $o(f(n))$ stands for some anonymous function $h(n)$ in the class $o(f(n))$,

   whence $\lim\limits_{n \to \infty} \dfrac{h(n)}{f(n)} = 0$. Thus $\lim\limits_{n \to \infty} \dfrac{f(n) + h(n)}{f(n)} = \lim\limits_{n \to \infty}\left(1 + \dfrac{h(n)}{f(n)}\right) = 1$, and $f(n) + h(n) = \Theta(f(n))$, as

   claimed.   ///

6. Let $f(n) = \Theta(n)$. Prove that $\displaystyle\sum_{i=1}^{n} f(i) = \Theta(n^2)$. (See the hint at bottom of p.4 of the handout on asymptotic growth rates.)

Proof:
Since $f(n) = \Theta(n)$ there exist positive constants c_1, c_2, and n_0 such that for all $n \geq n_0$ the inequality $0 \leq c_1 n \leq f(n) \leq c_2 n$ holds. In particular, given $n_0 \leq i \leq n$ we have $c_1 i \leq f(i) \leq c_2 i$. Upon summing these inequalities from $i = n_0$ to $i = n$ we obtain

$$\sum_{i=n_0}^{n} c_1 i \ \leq \ \sum_{i=n_0}^{n} f(i) \ \leq \ \sum_{i=n_0}^{n} c_2 i$$

for any $n \geq n_0$. Define $A = \displaystyle\sum_{i=1}^{n_0 - 1} f(i)$. Then the above inequality can be rewritten as

$$\sum_{i=1}^{n} c_1 i \ - \ \sum_{i=1}^{n_0 - 1} c_1 i \ \leq \ \sum_{i=1}^{n} f(i) \ - \ A \ \leq \ \sum_{i=1}^{n} c_2 i \ - \ \sum_{i=1}^{n_0 - 1} c_2 i$$

Adding the constant A to all terms, and using the well known formula $\sum_{i=1}^{n} i = \dfrac{n(n+1)}{2}$, we obtain

$$c_1\left(\frac{n(n+1)}{2}\right) - c_1\left(\frac{n_0(n_0-1)}{2}\right) + A \leq \sum_{i=1}^{n} f(i) \leq c_2\left(\frac{n(n+1)}{2}\right) - c_2\left(\frac{n_0(n_0-1)}{2}\right) + A$$

Now observe that since c_1, c_2, n_0, and A are constants, the left hand side is in the class $\Omega(n^2)$ and the right hand side is in $O(n^2)$, and this inequality holds for all $n \geq n_0$. In other words, the inequality $h_1(n) \leq \sum_{i=1}^{n} f(i) \leq h_2(n)$ holds for all sufficiently large n, where $h_1(n) = \Omega(n^2)$ and $h_2(n) = O(n^2)$. By an exercise in the handout on asymptotic growth of functions, we conclude that $\sum_{i=1}^{n} f(i) = \Theta(n^2)$, as required. $/\!/\!/$