- Lemma 1:

  If T is a tree with n vertices and m edges, then m=n-1

  i. Proof:

  ii. Base case:if m = 0, then T, being connected has only 1 vertice.

  iii. $\forall m > 0: P(0), ...P(m-1) \Rightarrow P(m)$

  iv. let m>0, assume for any tree T' with $|E(T')|$<m that

  v. $|E(T')| = |V(T')| - 1$

  vi. must show: if T has m edges and n vertice, then m=n-1

  vii. Pick any edge eεE(T) and remove it. This results in two subtrees: $T_1$and T_2each with fewer than m edges:

  viii. $|E(T_i)| < m$ for $i = 1, 2$

  ix. By the induction hypothesis, we have $|E(T_i)| = |V(T_i)| - 1$ for $i = 1, 2$

  x. since no vertices were removed, $|V(T_1)| + |V(T_2)| = n$

  xi. Therefore, $m = |E(T_1)| + |E(T_2)| + 1 = |V(T_1)| - 1 + |V(T_2)| - 1 + 1$

  xii. $= |V(T_1)| + |V(T_2)| - 1$

  xiii. =n-1 as required

- Lemma 2:

  If G is acylic with n vertices, m edges, and k connected componentsd, then m=n-k

  i. Proof:

  ii. Let T_1, T_2, T_3..........T_k denote then connected components of G, which are necessarily treees.

  iii. Let m_i, n_i denote #vertices and #edges in T_i $(1 \leq i \leq k)$

  iv. By Lemma 1: $|E(T_i)| = |V(T_i)| - 1 \quad (1 \leq i \leq k)$ ie m=n-1

  v. SO, $m = \sum_{i=1}^{k} m_i = \sum_{i=1}^{k} (n_i - 1)$

  vi. $= \sum_{i-1}^{k} (n_i) - \sum_{i=1}^{k} (1) = n - k$

- Lemma 3:

  If G is connected with n vertices and m edges, then $m \geq n-1$

  i. Proof

  ii. if m=0, G being connected, can have only 1 vertice, so n=1

     m=0≥0=n-1

iii. $\forall m > 0$: $P(0), P(1), ...P(m-1) \Rightarrow P(m)$

iv. let m>0 be chosen arbitarily

v. Assume for any connected Graph G' that $|E(G')| \geq |V(G')| - 1$

vi. Must show: if G is connected with m edges, n vertices, then m≥n-1

vii. Pick any eεE(G) and remove it. We have 2 cases:

a) Case 1: G-e is connected

Note G-e has n vertices and m-1 edges

By the induction hypothesis: $m-1 \geq n-1$

Therefore, $m \geq n \geq n-1$

$m \geq n-1$

b) Case 2:G-e is NOT connected

Claim: G-e consists of two connected componenets: $G_1 \,\text{and}\, G_2$

Note:$|E(G_i)| < m \quad i = 1, 2$

So, by Induction Hypothesis, we have $|E(G_i)| \geq |V(G_i)| - 1 \quad i = 1, 2$

also $|V(G_1)| + |V(G_2)| = n \qquad \text{since no vertices were removed}$

Thus, $m = |E(G_1)| + |E(G_2)| + 1 \geq |V(G_1)| - 1 + |V(G_2)| - 1$ by ind hyp

=$|V(G_1)| + |V(G_2)| - 1$

=n-1

=m≥n-1

in Each case m≥n-1

- Lemma 4:

  If G is a graph with n vertices, m edges, and k connected components, then m≥n-k

i. Proof:

ii. Let $G_1, G_2......., G_k \,\text{be the connected componenets of}\, G$

iii. By Lemma 3: Let $m_i, n_i \,\text{be \#edges and \#vertices in}\, G_i \; (1 \leq i \leq k)$

iv. $m_i \geq n-1$

v. then: $m = \sum_{i=1}^{k}(m_i) \geq \sum_{i=1}^{k}(n_i - 1) = \sum_{i=1}^{k}(n_i) - \sum_{i=1}^{k}(1)$

vi. =n-k

vii. therefore m≥n-k