AM5600: Computational Methods in Mechanics (July-Nov. 2019)

Assignment #6

Due: At the beginning of class on Oct. 22, 2019

1. Use Euler's method to approximate the solution $y(t) = t^{3/2}$ of the IVP
$$y' = 1.5y^{1/3},\ y(0) = 0,\ t \geq 0$$
Explain your results with the help of Lipschitz theorem. Furthermore, can you utilize Taylor's method to approximate the solution. Give appropriate justification.

2. Derive the general third-order Runge-Kutta method:
$$y_{n+1} = y_n + w_1k_1 + w_2k_2 + w_3k_3$$
Show that one such method is given by:
$$y_{n+1} = y_n + \frac{(k_1+4k_2+k_3)}{6},\ k_1 = hf(t_n, y_n)$$
$$k_2 = hf\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right),\ k_3 = hf(t_n + h, y_n - k_1h + 2k_2h)$$

3. Solve the differential equation below
$$y' = 2 - 2\mathrm{t} + 4t^2 - 4t^3 - 4t^4,\ y(0) = 0,\ t \geq 0$$
The exact solution is $y(t) = 1 + 2t - t^2 + \frac{4}{3}t^3 - t^4 - \frac{4}{5}t^5,\ t \geq 0$
   a) using Euler's method from $t = 0$ to 1 with step sizes of 0.2 and 0.1.
   b) Repeat (a) using Huen's method.
   c) Repeat (a) using RK4 method.

   Compare your findings with the exact solutions and discuss which method performs better with appropriate justification.

4. Solve the differential equation below:
$$y'' + 4y = 0,\ 0 \leq\ t \leq 2\pi,\ y(0) = 2\sqrt{2} \text{ and } y'(0) = -\sqrt{2}$$
The exact solution is $y(t) = \left(2/\sqrt{2}\right)(\cos 2t - sin2t)$. Compute the error at $t = 2\pi$ for $h = \left[\frac{2\pi}{16}, \frac{2\pi}{32}\right]$ using RK4 method.

AM5801/AM5810: Computational Lab (optional for students crediting AM5600)

Due: At the end of lab on Oct. 23, 2019

I. Solve the system of equations:

$$\frac{dx}{dt} = 10(y - x)$$
$$\frac{dy}{dt} = 25x - y - xz$$
$$\frac{dz}{dt} = xy - \frac{8}{3}z$$

Use the initial conditions as $x(0) = 1$, $y(0) = 1$, $z(0) = 1$. Develop a code for RK4 to solve this system of equations. Compare your results with in-built functions such as *ode45* (RK45) and *ode113* by using *tic-toc* to find which is the fastest method for computing the solution for $t \in [0,100]$. Set absolute and relative error tolerance at 10^{-6}. Use *plot3* to plot your solution in 3D. The trajectory of your solution should have a butterfly shape (known as Lorenz attractor, https://en.wikipedia.org/wiki/Lorenz_system). How accurate are your solution at $t = 100$ (select a tighter tolerance for comparison)?

II. Consider the differential equation

$$y' = \sin(t)(y^2 - \cos^2(t) - 1),\ y(0) = 1,\ t \geq 0$$

The exact solution to this non-linear ODE is $y(t) = \cos(t)$.

a) Using RK4 method solve the equation for $t \in [0,50]$. Plot the error vs. h for several values of h (use loglog). Does the error decay as $O(h^4)$?

b) Compare your results with Euler's method and RK45 (use built in *ode45)*. Solve the problem for several values of h.

III. Compute the solution to the systems of equation for the given initial conditions using RK4. Plot your results in the phase space $\left(y(t)\ vs.\ x(t)\right)$

Using initial conditions with $x(0)^2 + y(0)^2$ both inside and outside a circle of radius 2, solve:

$$x' = -4y + x(1 - x^2 - y^2),\ \ y' = 4x + y(1 - x^2 - y^2)$$

for $t \in [0,10]$. What is the final state of the system? Justify your solution by plotting the trajectories in the XY plane for different conditions.