# Assignment 3

## General Instructions

- Solutions due by Submission due by $17^{th}$ September 2019 for Batch I (Dr. Tiwari's class); $18^{th}$ September 2019 for Batch II (Dr. Pallab's class).
- Hand calculations can be done as homework and should be submitted as a report. Write your name and roll-number on your report.

## Hand calculations

1. Consider the matrix B as,

$$B = \begin{bmatrix} 1+s & -s \\ s & 1-s \end{bmatrix}$$

Calculate $p$ and $q$ such that $[B]^n = p[B] + q[I]$. Determine $e^{[B]}$.

2. We will examine the round-off errors in numerically solving linear systems using different pivoting techniques. The relative error between the exact and the numerical solution (using floating point arithmetic) is related to the *Growth factor*, defined as $\dfrac{\max_{i,j} |u_{ij}|}{\max_{i,j} |a_{ij}|}$. The smaller the growth factor, lesser will the number of digits you lose in the solution. Similarly, if the growth factor is larger, you will lose more digits in the solution, leading to higher error.

   (a) Consider the matrix:

$$\begin{pmatrix} \epsilon & 1 \\ 1 & 1 \end{pmatrix} \tag{1}$$

   where $\epsilon$ is a very small number. Determine the growth factor using Gaussian Elimination (i) without pivoting and (ii) with partial pivoting; Compute the growth factor as $\epsilon \to 0$.

   (b) In partial pivoting technique, while eliminating the $k^{th}$ column, row swaps (with row index $i \geq k$) are performed to get the largest element in magnitude within the $k^{th}$ column as the pivot. Whereas, in *complete pivoting* technique, while eliminating the $k^{th}$ column, rows with index $j \geq k$ and columns with index $j \geq k$ are swapped to get the largest element in magnitude as the pivot.
   Consider the matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{pmatrix} \tag{2}$$

   Find the growth factor using Gaussian Elimination with pivoting. What are your estimates of growth factors for a similar matrix with size $N \times N$?

3. Calculate the 1-, $\infty$-, and Frobenius norms and the corresponding condition numbers for the following matrices

$$A = \begin{bmatrix} 2 & -2 \\ 1 & -3 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 5 & 6 \\ 2 & -1 & 3 \end{bmatrix}$$

$$C = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. Calculate the condition number using Frobenius norm of the coefficient matrix of the system $Ax = b$, with,

$$A = \begin{bmatrix} 1 & 10^6 \\ 2 & 3 \end{bmatrix}$$

and $b = [1 \quad 3]^T$. Is this system well-conditioned? If not, multiply one of the equations through by a suitably chosen constant so as to make the system better conditioned. Calculate the condition number of the coefficient matrix in your new system of equations.

5. Consider the matrix,

$$A(\alpha) = \begin{bmatrix} 0.1\alpha & 0.1\alpha \\ 1.0 & 1.5 \end{bmatrix}$$

Determine $\alpha$ such that the $\text{cond}(A(\alpha))$ is minimized. Use the maximum norm.

# Programming

1. Use your Gauss elimination program to solve the following system,

$$2x_2 - x_3 = 5 \tag{3}$$
$$x_1 - 4x_2 + 4x_3 + 7x_4 = 4 \tag{4}$$
$$2x_1 + x_2 + x_3 + 4x_4 = 2 \tag{5}$$
$$2x_1 - 3x_2 + 2x_3 - 5x_4 = 9 \tag{6}$$

2. Using LU decomposition program, solve the $60 \times 60$ systems of equations $Ax = b$ with $A$ having the form given below. Consider $b = [1, 1, 1...1]^T$.

$$\begin{pmatrix} 1 & & & & 1 \\ -1 & 1 & & & 1 \\ -1 & -1 & 1 & & 1 \\ -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 \end{pmatrix} \tag{7}$$

Calculate the growth factor.

3. (a) Consider the following system,

$$\begin{pmatrix} 1 & 2 & 0 & . & . & 0 \\ 1 & 4 & 1 & 0 & . & 0 \\ 0 & 1 & 4 & 1 & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & 1 & 4 & 1 \\ 0 & . & . & 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ x_2 \\ . \\ . \\ . \\ y_{N-1} \\ y_N \end{pmatrix} = \begin{pmatrix} \frac{N}{3}\left(-\frac{5}{2}f(x_0) + 2f(x_1) + \frac{1}{2}f(x_2)\right) \\ N\left(f(x_2) - f(x_0)\right) \\ N\left(f(x_3) - f(x_1)\right) \\ . \\ . \\ . \\ N\left(f(x_n) - f(x_{N-2})\right) \\ \frac{N}{3}\left(\frac{5}{2}f(x_N) - 2f(x_{N-1}) - \frac{1}{2}f(x_{N-2})\right) \end{pmatrix}$$

where $f(x_j) = \sin(5x_j)$ with $x_j = \frac{3j}{N}, \ \ j = 0, 1, ..., N$. Compute the solution $y_j$ and plot $y_j$ versus $x_j$ for $N = 15, 25$ and 50 using LU decomposition algorithm (Doolittle algorithm).

(b) The matrix in the previous problem is called a scalar-tridiagonal system. It can be solved using the so-called Thomas algorithm, which is essentially a simplified form of Gaussian elimination. Convert the Gaussian elimination program (written for Assignment 1) to Thomas algorithm and calculate the solution for the scalar-tridiagonal system.