Assignment 5

General Instructions

- Solutions due by Submission due by 15^{th} October 2019 for Batch I (Dr. Tiwari's class); 16^{th} October 2019 for Batch II (Dr. Pallab's class).
- Hand calculations can be done as homework and should be submitted as a report.

Hand calculations

1. Given the initial value problem $y\prime = -2ty^2$, $y(0) = 1$. Estimate $y(0.4)$ using

   (a) the mid-point method

   (b) the Heun method, and

   (c) the fourth order classical Runge-Kutta method

   with $h = 0.2$. Determine the percentage relative error for each method at $t = 0.4$ with respect to the exact solution.

2. Find single step methods for the differential equation $y\prime = f(t, y)$, which produce exact results for

   (a) $y(t) = a + be^{-t}$.

   (b) $y(t) = a + bcost + csint$.

3. The following scheme has been proposed for solving $y' = f(y)$:

$$y_{i+1} = y_i + \omega_1 k_1 + \omega_2 k_2,$$

   where

$$\begin{aligned} k_1 &= h\ f(y_i) \\ k_0 &= h\ f(y_i + \beta_0 k_1) \\ k_2 &= h\ f(y_i + \beta_1 k_0) \end{aligned}$$

   with h being the step size.
   (a) Determine the coefficients $\omega_1, \omega_2, \beta_0$ and β_1 that would maximize the order of accuracy of the method.
   (b) Applying this method to $y' = \alpha y$, what is the maximum step size h for α pure imaginary?
   (c) Applying this method to $y' = \alpha y$, what is the maximum step size h for α real negative?

4. Consider solving the following first-order initial value problem:

$$y'(x) = \frac{dy}{dx} = f(x, y) = x + y, \quad y(0) = 1, \quad 0 \le x \le 1,$$

   using a fourth-order accurate Taylor-series method given as follows:

$$y(x_i + h) = y(x_i) + h\ T_4(x_i, y_i, h)$$

   where

$$T_4(x_i, y_i, h) = y'(x_i) + y''(x_i)\frac{h}{2!} + y'''(x_i)\frac{h^2}{3!} + y^{iv}(x_i)\frac{h^3}{4!}.$$

   Starting with $x = 0$ using a step size of $h = 0.1$, calculate the solution at $x = 1$. Tabulate your results showing the computed solution, exact solution and error at each step. The exact analytical solution is given by $y(x) = 2e^x - x - 1$.

Programming

1. Consider the following initial value problem

$$\frac{dy}{dt} = -\lambda \left(y - e^{-t}\right) - e^{-t}, \quad \lambda > 0$$

with the intial condition $y(t = 0) = y_0$. Solve this initial value problem for $\lambda = 10$ and $y_0 = 10$ using explicit Euler, implicit Euler, Heun, fourth-order Runge-Kutta method. Use time step sizes $h = 0.1, 0.05, 0.025, 0.0125, 0.00625$ and integrate from $t = 0$ to 0.8. The exact analytical solution to this equation is

$$y = e^{-t} + (y_0 - 1)e^{-\lambda t}.$$

(a) For each method, plot the solutions obtained along with the exact solution from $t = 0$ to $t = 0.8$.
(b) Evaluate the error between the numerical and the exact solution at $t = 0.8$.