Assignment 2

General Instructions

- Submission due by 27^{th} August 2019 for Batch I (Dr. Tiwari's class); 28^{th} August 2019 for Batch II (Dr. Pallab's class).
- Hand calculations can be done as homework and should be submitted as a report. Write your name and roll-number on your report.

Hand calculations

1. Solve the equations

$$x_1 + 2x_2 - x_3 = 2 \tag{1}$$
$$3x_1 + 6x_2 + x_3 = 1 \tag{2}$$
$$3x_1 + 3x_2 + 2x_3 = 3 \tag{3}$$

   using (i) Cramer's rule (ii) Inverse matrix method (iii) Gauss elimination method.

2. Find inverse of the matrix

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 2 \end{pmatrix} \tag{4}$$

   using LU decomposition method. Take $u_{11} = u_{22} = u_{33} = 1$.

3. Show that LU decomposition fails to solve the system of equations

$$x_1 + x_2 - x_3 = 2 \tag{5}$$
$$2x_1 + 2x_2 + 5x_3 = -3 \tag{6}$$
$$3x_1 + 2x_2 - 3x_3 = 6 \tag{7}$$

   The exact solution of the system is $x_1 = 1, x_2 = 0$ and $x_3 = -1$.

4. We learnt two ways to get $A = LU$ in class:
   (a) From Gaussian Elimination and pre-multiplication by E matrices
   (b) Using Doolittle's algorithm.

   Consider the matrix:

$$\begin{pmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{pmatrix} \tag{8}$$

   Get L and U matrices both ways keeping $l_{ii} = 1$ for $i = \{1, 2, 3, 4\}$. Are they the same?

5. Solve the system of equations,

$$2x_1 + x_2 + x_3 - 2x_4 = -10 \tag{9}$$
$$4x_1 + 2x_3 + x_4 = 8 \tag{10}$$
$$3x_1 + 2x_2 + 2x_3 = 7 \tag{11}$$
$$x_1 + 3x_2 + 2x_3 - x_4 = -5 \tag{12}$$

   using Gauss elimination method with partial pivoting.

6. Write a pseudocode for a specific test case that solves the system of equations shown below for various values of n.

$$\sum_{j=1}^{n}(1+i)^{j-1}x_j = \frac{1}{i}[(1+i)^n - 1] \quad for \quad (1 \leq i \leq n) \tag{13}$$

Programming

7. Write a program that performs Gaussian elimination for a square system $Ax = b$ of size 5×5. Elements in i^{th} row and j^{th} column of the coefficient matrix can be taken as i^j for $1 \leq i \leq 5$ and $1 \leq j \leq 5$. Also consider $b = [1, 1, ..., 1]^T$.

   Consider writing it as a modular program with separate functions or subroutines that perform forward-elimination and backward-substitution. Solve the system to obtain the solution vector x.

8. Generalize the program of the previous question for any n. To test the program, solve the system $Ax = b$ with $A = [a_{ij}]$ defined by

$$a_{ij} = \max(i, j).$$

   Also define $b = [1, 1, ..., 1]^T$. Solve the system to obtain the solution vector x, for $n = 32, 128, 512, 1024$.

9. Write a program of LU decomposition to find out the inverse of the matrix

$$\begin{pmatrix} 3 & 2 & 1 & 5 \\ 2 & 3 & 2 & 7 \\ 1 & 2 & 2 & 6 \\ 4 & 5 & 6 & 5 \end{pmatrix} \tag{14}$$

   Take $l_{11} = l_{22} = l_{33} = l_{44} = 1$.

10. Consider the recurrence $u_{n+2} = 3\,u_{n+1} - 2u_n$, where $n \geq 0$. Take $u_0 = u_1 = 2.9689$.
    (a) Determine u_n manually for $n = 2, .., 6$. What value does u_n take for any n?
    (b) Determine the values of u_n on a computer for $n = 2, \ldots, 64$. Do the values agree with (a)?
    (c) Repeat (b) when $u_0 = u_1 = 2.96875$. Do the values agree with (a)?
    (d) Explain your observations in (b) and (c)