Assignment 6

General Instructions

- Solutions due by Submission due by 11^{th} November 2019 for Batch I (Dr. Tiwari's class); 13^{th} November 2019 for Batch II (Dr. Pallab's class).
- Hand calculations can be done as homework and should be submitted as a report.

Hand calculations

1. Consider the numerical solution of $y' = i\omega y$, $y(0) = 1$ using the Explicit Euler method and the second order RK method. Suppose $\omega h = 0.1$ and the differential is integrated for $n = 1000$ time steps (*i.e.* $0 \le t \le nh/\omega$). What is the solution after n time steps for each time advancement scheme?

2. Use fourth order Adams P-C method to solve $y\prime = 4e^{0.8x} - 0.5y$ from $x = 0$ to $x = 4$ using a step size of 1. Start the method using RK-4. The expressions for AB-4 and AM-4 are

$$y_{n+1}^{(P)} = y_n + \frac{h}{24}\left(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}\right) \tag{1}$$

$$y_{n+1}^{(C)} = y_n + \frac{h}{24}\left(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}\right) \tag{2}$$

3. Derive a fourth-order accurate forward-difference formula for evaluating the second derivative on a uniform grid of size Δx.

4. Calculate the truncation error and examine the consistency and stability of Dufort-Frankel's method for the solution of one-dimensional transient heat conduction equation given by

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = \frac{u_{i+1}^n - u_i^{n+1} - u_i^{n-1} + u_{i-1}^n}{\Delta x^2} \tag{3}$$

5. Consider a non-uniform grid with the left and right-hand side grid sizes of point x_i are h_l and h_r respectively and the grid size ratio is a constant ($h_r/h_l = r$). Denoting the function values at the grid points $x_i, (x_i - h_l), (x_i + h_r)$ as $u(x_i), u(x_i - h_l)$ and $u(x_i + h_r)$ respectively. Derive second-order accurate central-difference formulae for the first and second derivatives at the grid point x_i. Express your answer in terms of $u(x_i), u(x_i - h_l), u(x_i + h_r), r$ and h_l.

6. Calculate the truncation error and examine the consistency and stability of the following discretizations for the solution of first-order wave equation give by

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \quad c > 0 \tag{4}$$

(a) Euler explicit method

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\,\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = 0 \tag{5}$$

(b) Lax method

$$\frac{u_i^{n+1} - \frac{1}{2}(u_{i+1}^n + u_{i-1}^n)}{\Delta t} + c\,\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = 0 \tag{6}$$

Programming

1. Consider the differential equation governing simple harmonic motion: $y(t)$

$$y'' + \omega^2 y = 0 \text{ for } t > 0$$
$$y(0) = y_0;\ y'(0) = 0.$$

Use $y_0 = 1$ and $\omega = 4$.

(a) If the general form of the exact solution is $y(t) = A\sin(\omega t) + B\cos(\omega t)$, obtain the exact solution to the initial value problem for the given conditions

(b) Re-write the above second order differential equation as a system of first order differential equations.

(c) Solve this system using the method: $y^{(n+1)} = y^{(n-1)} + 2hf(y^{(n)}, t^{(n)})$ with time step size $h = 0.1$s for $0 \leq t \leq 9$ s. Use Euler explicit method to start the calculation.

2. Consider the following equation that arises in the solution of transient heat conduction in a plane wall

$$u(x) = Bi - x\ \tan(x) \tag{7}$$

where Bi is the Biot number in the present case taken to be 7 and $x = [-1, 1]$. Write a program to compute the first derivative using first, second and fourth-order accuarate forward-difference formulae. Using grid sizes of $\Delta x = 0.1, 0.01, 0.001$ plot the magnitude of the truncation-error versus the grid sizes for all the three schemes on a log-log plot.

3. The heat equation with a source term is

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + S(x), \quad 0 \leq x \leq L \tag{8}$$

The initial and boundary conditions are $T(x, 0) = 0, T(0, t) = 0$ and $T(L, t) = T_{steady}(L)$. Taking $\alpha = 1, L = 15$ and $S(x) = -(x^2 - 4x + 2)e^{-x}$. The exact steady solution is

$$T_{steady}(x) = x^2 e^{-x}. \tag{9}$$

Solve the equation to steady state on a uniform grid with a grid spacing of $\Delta x = 1, 0.1$ and employing a time step of $\Delta t = 0.005$. For each of the following methods (i) plot the variation of error in $1-$norm with respect to Δx on a log-log plot for a $\Delta t = 0.005$ (ii) plot the exact and the numerical solutions for $\Delta x = 1$ and $\Delta t = 0.005$. Comment on the maximum time step you can use in each of the following methods and show that by employing a larger Δt you can take fewer iterations to arrive at the steady solution in case of implicit methods. (a) Explicit Euler time-advacement with second-order central difference scheme for spatial derivate (b) Implicit Euler time-advancement with second-order central difference scheme for spatial derivate (c) Crank-Nicolson method.