Math 543: Homework Set 1

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> 1. Let $G = \{e, a, b, c\}$ be a group of order 4, where e is the identity of G. Deduce multiplication tables for all the possible isomorphism classes of G. Argue from the denitions that you have found them all.

Solution. In the first, Say an element which is not e a. Then there are two cases which value a^2 could have. One thing is that $a^2 = e$. The other thing is that $a^2 \neq e$.

Now, consider first case. $G = \{e, a\}, a^2 = e$ already satisfies group condition. However, to satisfy 4 elements condition of G, we need another distinct element b. Since $a \neq e and b \neq e$, $ab \neq a, ab \neq b \to ab = c$. So $G = \{e, a, b, c\}, a^2 = e, ab = c$. To satisfy 4 element condition, $b^{-1} = b^2 = c^{-1} = c^2 = e$ and $bc = a$. Finally, we know that $G = \{e, a, b, c | a^2 = b^2 = c^2 = e, ab = c, bc = a, ac = a(ab) = b\}$

Consider second case. Since $a^2 \neq a (\because a^2 = a \to a = e), a^2$ should be some other distinct element. Say this distinct element b. Hence $a^2 = b$ Then G have $e, a, b(a^2 = b, a^2 \neq a)$. Now consider a^{-1}. Since $a^2 \neq e$, $a^{-1} \neq a$, so $a^{-1} = a^2$ or other distinct element c. If $a^{-1} = a^2$ then $a^3 = e$. It means $G = \{e, a, a^2\}$. But a condition $|G| = 4$ makes insert another element c into G. But in this case, ac could not be e, a, a^2. It means $|G| > 4$. Hence, $a^{-1} = c$. Moreover, since $|G| = 4$, $a * a^2 = a^3 = c(\because a^3 \neq e, a^3 \neq a, a^3 \neq a^2)$. Finally $G = \{e, a, b, c\} = \{e, a, a^2, a^3\} \to cyclic group$

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$*$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

2. Let $H < G$ and define $C_G(H) = \{g \in G : gh = hg \quad \forall h \in H\}$.

a. Show that $C_G(H)$ is a subgroup of G.

b. Let $G = \mathcal{D}_4, H = \langle F \rangle$. Find $C_G(H)$.

Solution.

a. To prove $C_G(H)$ is a subgroup of G, we only need to show that $ab^{-1} \in C_G(H)$ when $a \in C_G(H), b \in C_G(H)$.
$\forall h, ab^{-1}h = ahb^{-1} (\because b \in C_G(H) \to bh = hb \to b^{-1}bh = h = b^{-1}hb \to hb^{-1} = b^{-1}hbb^{-1} = b^{-1}h)$
$= hab^{-1} (\because a \in C_G(H) \to ah = ha)$
Therefore, $ab^{-1} \in C_G(H)$.
It means $C_G(H)$ is a subgroup of G.

b. $H =< F >= \{e, F\}$.
So, we only need to find elements X in $\mathcal{D}_4$ which satisfy $XF = FX$.
($\because \quad \forall x, xe = x = ex$ is trivial, we do not need to check.)
Trivially, $eF = F = Fe$. So $e \in C_G(< F >)$.
It just have 7 more elements. Let's check all cases.
$(R)F = F(R^3) \neq F(R)$. So $R \notin C_G(< F >)$.
$(R^2)F = F(R^2)$. So $R^2 \in C_G(< F >)$.
$(R^3)F = F(R) \neq F(R^3)$. So $R^3 \notin C_G(< F >)$.
$(F)F = F^2 = F(F)$. So $F \in C_G(< F >)$.
$(FR)F = FFR^3 = R^3 \neq F(FR) = R$. So $(FR) \notin C_G(< F >)$.
$(FR^2)F = R^2FF = R^2 = F(FR^2)$. So $(FR^2) \in C_G(< F >)$.
$(FR^3)F = RFF = R \neq F(FR^3) = R^3$. So $(FR^3 \notin C_G(< F >)$.
Hence, $C_G(< F >) = C_G(< F >) = \{e, R^2, F, FR^2\}$.

3. Suppose $\varphi : G_1 \to G_2$ is a non-trivial homomorphism of groups and that $|G_1| = p$, a prime. Show that φ is injective.

Solution. By the first Isomorphism theorem, $ker\varphi \lhd G_1$ and $G_1/ker\varphi \equiv \varphi(G_1)$.
So $\frac{|G_1|}{|ker\varphi|} = |\varphi(G_1)|$.
It means that $\frac{|G_1|}{|ker\varphi|}$ is an integer. Since $|G_1|$ is a prime p, $|ker\varphi|$ should be either 1 or p.
But φ is non-trivial homomorphism, $|ker\varphi|$ should be 1. So, $ker\varphi = \{e\}$. It means that φ is injective.