Dr. Jozo Dujmović

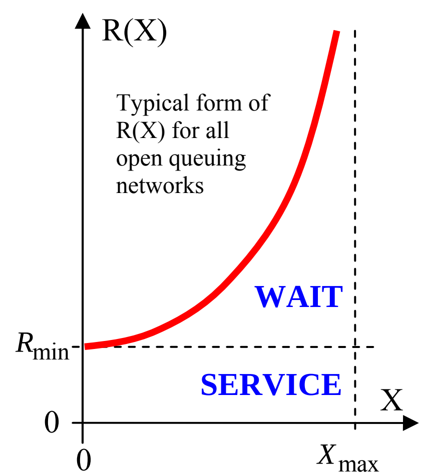
ANALYSIS OF OPEN QUEUING NETWORKS (a short summary)

Exponential single server models are based on the following M/M/1 formulas:

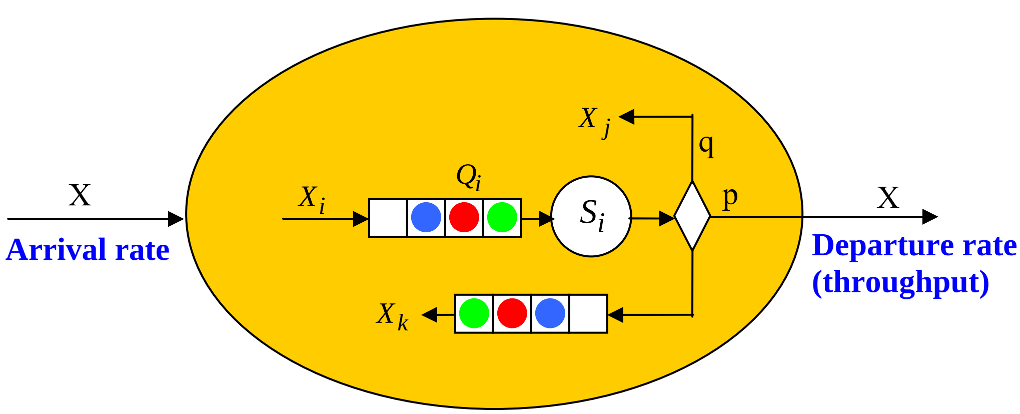
$$U = SX \qquad \text{(Utilization law)}$$
$$Q = RX \qquad \text{(Little's formula)}$$
$$R = QS + S = S(Q+1) \qquad \text{(Response time formula)}$$

The utilization formula $U=SX$ can be interpreted as Little's formula applied only to the server: the utilization of server is the average number of customers at the server. Using these formulas we can compute the average response time (R), the average queue length (Q), the average wait time (W), the average server utilization (U), and the average number of customers that are waiting in the queue before service (Q_w), as follows:

$$R = QS + S = RXS + S = RU + S$$
$$R(1-U) = S; \qquad R_{\min} = S$$
$$R = \frac{S}{1-U} = \frac{S}{1-SX} = \frac{1}{1/S - X} = \frac{1}{X_{\max} - X}$$
$$Q = RX = \frac{SX}{1-SX} = \frac{U}{1-U}$$
$$Q - QU = U$$
$$U = \frac{Q}{Q+1} = \frac{QS}{R}$$
$$W = R - S$$
$$Q_W = WX = RX - XS = Q - U = QU$$
$$Q = Q_W + U$$



M/M/1 formulas can be used to analyze open queuing networks with *n* service stations. Assumptions: (1) Poisson arrival process, (2) all servers are exponential, (3) all jobs are identical.



A general open queuing network with *n* resources

Step 1. Compute throughputs in all branches

In normal operation, the average arrival rate equals the average departure rate and starting from the output we have the following throughput in individual branches:

$$X = pX_i$$

$$X_i = X / p$$

$$X_j = qX_i = Xq / p \qquad (p \text{ and } q \text{ are given probabilities})$$

$$X_k = (1-p-q)X_i = X(1-p-q)/p$$

Step 2. Compute the average number of visits to each service center:

$$X_i = V_i X; \qquad V_i = 1/p$$

$$X_j = V_j X; \qquad V_j = q/p$$

$$X_k = V_k X; \qquad V_k = (1-p-q)/p$$

Step 3. Compute all server utilizations and demands:

$$U_i = S_i X_i = S_i V_i X = D_i X, \qquad i = 1,\ldots,n$$

$D_i = S_i V_i =$ demand for resource i (total service time at the i^{th} resource per job)

Step 4. Find the bottleneck resource, the maximum throughput, resource utilization ratios, and the maximum utilization of all resources:

$U_b = \max(U_1,\ldots,U_n)$ Bottleneck device is the device that has both the

$D_b = \max(D_1,\ldots,D_n)$ maximum demand and the maximum utilization

$U_{b\max} = D_b X_{\max} = 1;$ Maximum utilization of the bottleneck device is 1

$X_{\max} = 1/D_b = 1/S_b V_b$ Maximum system throughput

$\frac{U_i}{U_b} = \frac{D_i}{D_b} = \frac{S_i V_i}{S_b V_b}, \quad i = 1,\ldots,n$ Utilization ratios are constant for each resource

$U_{i\max} = \frac{D_i}{D_b}, \quad i = 1,\ldots,n$ Maximum resource utilizations

Step 5. Compute queue lengths and response times for all service centers:

$$Q_i = \frac{U_i}{1-U_i} = \frac{XD_i}{1-XD_i}, \qquad R_i = \frac{Q_i}{X_i} = \frac{S_i}{1-U_i}, \qquad i = 1,\ldots,n$$

Step 6. Compute the total number of jobs (customers) in the network:

$$Q = Q_1 + \ldots + Q_n$$

Step 7. Compute the min/average/max response time for the whole network:

$$Q = XR$$

$$R(X) = \frac{1}{X}\sum_{i=1}^{n} Q_i = \frac{1}{X}\sum_{i=1}^{n}\frac{U_i}{1-U_i} = \frac{1}{X}\sum_{i=1}^{n}\frac{S_i V_i X}{1-U_i} = \sum_{i=1}^{n}\frac{S_i V_i}{1-U_i} = \sum_{i=1}^{n} V_i R_i = \sum_{i=1}^{n}\frac{D_i}{1-U_i} = \sum_{i=1}^{n}\frac{D_i}{1-XD_i}$$

$$R_{\min} = R(0) = R\big|_{U_1=\ldots=U_n=0} = \sum_{i=1}^{n} S_i V_i = \sum_{i=1}^{n} D_i; \qquad R_{\max} = R(X_{\max}) = +\infty$$