IB Extended Essay

Mathematics

<u>Discretization of First-Order Linear Differential Equations</u>

To what extent can the analytical solutions of first-order linear differential equations be approximated accurately through discretization using Euler's Method?

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Introduction

Differential equations are ubiquitous in the natural sciences. They serve as indispensable tools in mathematical modelling, allowing us to ascertain quantitative laws in many scenarios. Differential equations model rates of change, and are often used to model time-dependent variables. For example, in chemistry, the rate of the reaction whose mechanism is $A \rightarrow B + C$ is modelled as $\frac{d[A]}{dt} = (-k)[A]$, where $[A]$ is the concentration of compound A as a function of time. The solution of this differential equation models the concentration of A as a function of time. Upon solution, it can be found that the concentration of A follows an exponential decay function: $A(t) = A_0 e^{-kt}$, where $A_0 = A(0)$.

Similarly, in physics, the velocity of an object thrown straight up with the presence of air resistance can be modelled by the differential equation $\frac{dv}{dt} = (-g) - \frac{b}{m}v$, where g, b, and m are constants, and $v(t)$ is the velocity of the object as a function of time. The solution to this differential equation is $v(t) = (v_0 - \frac{mg}{b})e^{-\frac{b}{m}t} - \frac{mg}{b}$ where $v_0 = v(0)$. There are many more examples of the relevance of differential equations to the natural sciences; they are thus invaluable in these applications, allowing for accurate predictions of dynamic variables in the natural sciences.

In mathematics, differential equations have been studied intensively since they were first developed. Differential equations were first considered by Leibniz, following which they have been the subject of analysis in both pure and applied mathematics. Specifically, in pure mathematics, the focus has been on determining analytical solutions to all differential equations. This has given rise to two categories of differential equations: those which can be reduced to analytical solutions, hereafter referred to as ‘solvable’, and those which can only be solved through recursion following discretization, hereafter referred to as ‘unsolvable’.

In order to apply the powerful modelling tool of differential equations when the solution creates an ‘unsolvable’ equation, we use discrete approximations. One such method, Euler’s Method, will be the focus of this paper, and is discussed further in the section of this paper titled “Discretization, Recurrence Relations and Euler’s Method”. Other numerical methods also exist; however, they are far less elegant in their mathematics and they are far more computationally intensive. Thus, Euler’s Method is worthy of consideration, and is an extremely interesting topic in the field of pure mathematics as well.

The question “To what extent can the analytical solutions of first-order linear differential equations be approximated accurately through discretization using Euler’s Method?” is thus extremely relevant to the study of pure and applied mathematics. The insights derived from applying Euler’s Method to solvable differential equations could prove valuable in understanding the application of Euler’s Method to unsolvable differential equations as well, and could help improve numerical mathematical models in complex scenarios with high degrees of dynamism.

Differential Equations, First-Order Linear Differential Equations and Initial Value Problems

A differential equation is an equation which relates the dependent variable (*y*), independent variable (*x*) and the derivative $\frac{dy}{dx}$. In other words, a differential equation is of the form

$$\sum_{i=0}^{n} \frac{d^i(y)}{dx^i} \cdot f_i(x, y) = 0$$

where $f_i(x, y)$ is any function of the variables x and y, and n is the order of the differential equation. Consequently, the order of the differential equation is defined as the order of the highest derivative contained within the differential equation. For example, if the equation contains terms in which the highest-order derivative is $f_i(x, y) \cdot \frac{d^2y}{dx^2}$, the equation is considered second-order. However, the focus of this paper will be on first-order differential equations, or those of the form $\frac{dy}{dx} = f(x, y)$.

This paper will focus on a specific type of first-order differential equation: first-order linear differential equations (FOLDEs). A FOLDE is linear in the dependent variable (*y*); mathematically, a FOLDE has the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where *P(x)* and *Q(x)* are both functions of *x*. For example, the following differential equation is a FOLDE:

$$\frac{dy}{dx} + 2xy = \sin(x)$$

where $P(x) = 2x$ and $Q(x) = \sin(x)$.

All FOLDEs have a closed-form solution if both $P(x)$ and $e^{\int P(x)dx}Q(x)$ are integrable. While the solution method for FOLDEs had been alluded to indirectly in Leonhard Euler's *Institutionum Calculi Integralis*, as well as in works by Bernoulli, Leibniz and Newton, the first explicit mention of the use of the integrating factor $e^{\int P(x)dx}$ appeared in W. C. Ottley's *A Treatise on Differential Equations*. This method is almost universally employed in the solution of FOLDEs.

To determine a particular closed-form solution of a FOLDE, it is imperative that an initial condition be provided. A combination of a differential equation with an initial condition is known as an initial value problem (IVP), and is used to find particular solutions to differential equations. Thus, to determine the particular solution of a FOLDE, it can be transformed into an IVP by assigning the value $y(0) = \gamma$ where γ is an arbitrary constant.

For the IVP

$$\tfrac{dy}{dx} + P(x)y = Q(x),\ y(0) = \gamma \qquad \text{(Eq. 1)}$$

We first define an integrating factor $\mu(x)$ such that

$$\mu(x) = e^{\int P(x)dx} \qquad \text{(Eq. 2)}$$

Following which both sides of (Eq. 1) can be multiplied by $\mu(x)$ to yield

$$e^{\int P(x)dx}\tfrac{dy}{dx} + e^{\int P(x)dx}P(x)y = e^{\int P(x)dx}Q(x)\,.$$

In this equation, it can be easily shown that the left side can be rewritten as $\tfrac{d}{dx}(e^{\int P(x)dx}y)$. Thus, the equation can be solved as follows:

$$e^{\int P(x)dx} y + c = \int e^{\int P(x)dx} \cdot Q(x)dx \, .$$

Now, we can define $q(x)$ such that $q(x) = \int \mu(x) \cdot Q(x)dx$. Then,

$$\mu(x)y + c = q(x) \, . \qquad \text{(Eq. 3)}$$

Substituting the arbitrary initial condition $y(0) = \gamma$ into (Eq. 3) yields

$$c = q(0) - \mu(0)\gamma \, .$$

The value of c can finally be substituted into (Eq. 3) to yield the solution of the IVP:

$$y = \frac{q(x) - q(0) + \mu(0)\gamma}{\mu(x)} \, . \qquad \text{(Eq. 4)}$$

This method will be employed throughout this paper in determining the solutions to various FOLDEs (Ottley 15).

Discretization, Recurrence Relations and Euler's Method

The process of mathematical discretization involves taking a continuous function $y(x)$ and redefining it as a new function y_x over discrete values of x. Discretization is commonly used when a closed-form solution to a complex continuous function cannot be obtained. Often, the result of discretization is a recurrence relation. A recurrence relation is defined as an equation of the form

$$y_n = \sum_{i=0}^{n-1} f_i(i, y_i)$$

where $f(i, y_i)$ is any function, n is an integer, and y_i denotes discrete values of y which precede y_n. A first-order recurrence relation is an equation of the form

$$f(y_{n+1}) = g(y_n)$$

where both $f(x)$ and $g(x)$ are functions. A homogeneous first-order recurrence relation, which is a subset of all first-order recurrence relations, is one which can be written in the form

$$ay_{n+1} + by_n = 0$$

where a and b are constants. A linear first-order recurrence relation is linear in y_n; that is,

$$ay_{n+1} + by_n = c_n$$

where, once again, a and b are constants, and c_n is a known sequence varying with n.

One common example of a recurrence relation is Euler's Method, which is a recursive method used to approximate the solution to a differential equation. Specifically, it uses the value of $\frac{dy}{dx}$ at discrete intervals to create a set of linear approximations to the solution curve of a differential equation.

Euler's Method was first published in Euler's *Institutionum Calculi Integralis* in 1768, and is widely used to numerically solve differential equations whose analytic solutions cannot be obtained easily, or those whose analytical solutions are yet to be discovered. While Euler's Method is superseded by other numerical methods in terms of accuracy, it is still used due to its relative computational ease and mathematical elegance.

A mathematical representation of Euler's Method can be obtained from the discretization of the differential operator (Euler 747). Setting the interval between two successive values of x_n to be h, and defining y_n as the value of y after n intervals, the following discretization can be made for any first-order differential equation:

$$\frac{\Delta y}{\Delta x} = f(x,y)$$

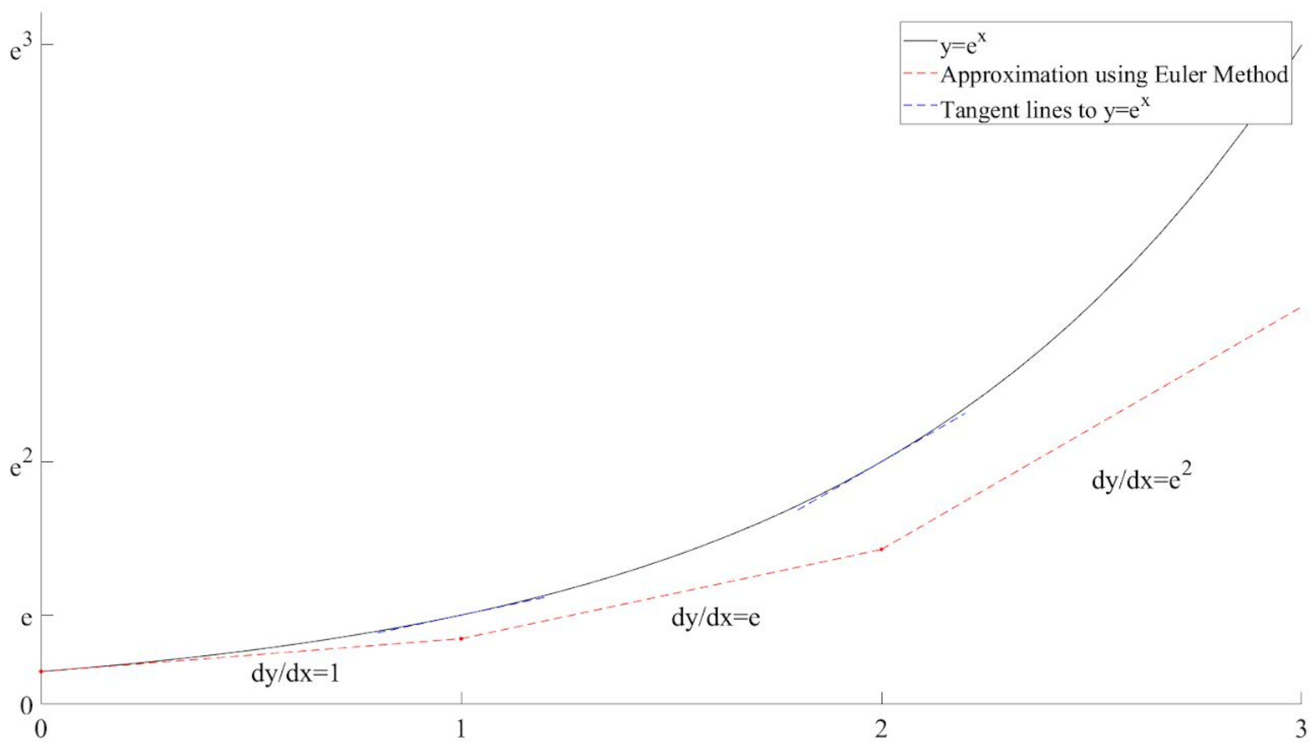
$$\frac{y_{n+1}-y_n}{x_{n+1}-x_n} = f(x_n, y_n)\,.$$

Given that $x_{n+1} - x_n = h$, we can arrive at a recursive definition of Euler's Method:

$$y_{n+1} = y_n + hf(x_n, y_n) \qquad \text{(Eq. 5)}$$

which will be employed throughout this paper. Graphically, Euler's method can be depicted as follows:

Graph 1: Euler's Method for $\frac{dy}{dx} = e^x,\ y(0) = 1$ *with step size h=1*



Here, we can see that there is an error induced when Euler's Method is used. The nature of this error is dependent on the function, the step size, and the interval across which the approximation is being carried out. This will be explored further below, in the section titled "Approximation Error".

Scope of Paper

In this paper, the goal is to find an explicit, closed-form expression for Euler's Method in initial value problems involving FOLDEs. Additionally, once expressions are derived, the accuracy of the linear approximation will be considered by computing the error of the discretized approximation compared to the analytical solution of the FOLDEs. The paper will also consider situations in which the accuracy of the linear approximation is computationally impractical to determine.

The mathematical analysis will consist of various 'cases' of increasing complexity. Eventually, the paper will attempt to generalize its findings to all FOLDEs, or establish circumstances in which the findings are applicable.

Analysis

To begin, we consider various examples of FOLDEs. Recalling (Eq. 1), we can express all FOLDEs in the following form:

$$\frac{dy}{dx} = Q(x) - P(x)y$$

The table below details the progression of cases employed in arriving at a generalization:

Case	$\frac{dy}{dx}$	$Q(x)$	$P(x)$	$y(0)$	Form of Recurrence Relation
1	$b - ky$	b	k	γ	$y_{n+1} = \alpha y_n + \beta$
2	$a(x) - ky$	$a(x)$	k	γ	$y_{n+1} = \alpha y_n + c_n$
3	$a(x) - b(x)y$	$a(x)$	$b(x)$	γ	$y_{n+1} = d_n y_n + c_n$

Table 1: the set of IVPs to be discretized, in order of complexity.

Case 1: $\frac{dy}{dx} = b - ky,\ y(0) = \gamma$

We begin by considering the simplest case. The analytical solution of this case can be obtained through separation of variables; however, it will be obtained using Ottley's integrating factor method described above for the sake of consistency. The differential equation above can be written in the form

$$\frac{dy}{dx} + ky = b\,,$$

and it immediately becomes apparent using (Eq. 2) that

$$\mu(x) = e^{kx}\,.$$

We can thus, after multiplying by the integrating factor, simplify the differential equation to

$$\tfrac{d}{dx}(e^{kx}y) = be^{kx}$$

which can be integrated to yield

$$e^{kx}y = \tfrac{b}{k}e^{kx} + C\,. \qquad \text{(Eq. 6)}$$

This equation can be solved for y, yielding

$$y = \tfrac{b}{k} + Ce^{-kx}\,.$$

The value of C can be determined by substituting the initial condition, $y(0) = \gamma$ into (Eq. 6):

$$C = \gamma - \tfrac{b}{k}\,.$$

Finally, the analytical solution of this IVP can be determined by substituting the value of C into (Eq. 10), yielding

$$y = \tfrac{b}{k} + (\gamma - \tfrac{b}{k})e^{-kx} \qquad \text{(Eq. 7)}$$

Employing the recurrence relation of Euler's Method as defined as in (Eq. 5), we can arrive at the recurrence relation

$$y_{n+1} = y_n + (b - ky_n) \cdot h\,,$$

which can be simplified to the linear recurrence relation

$$y_{n+1} = (1 - kh)y_n + bh\,. \qquad \text{(Eq. 8)}$$

This is a nonhomogeneous linear recurrence relation, and its solution method thus merits separate consideration. Based on the asymptotic form of the analytical solution, we first employ the ansatz $y_n = y^*$, where y^* is a constant that represents the 'steady state' of the recurrence relation. We can then solve for the value of y^* using the ansatz above:

$$y^* = (1 - kh)y^* + bh \qquad \text{(Eq. 9)}$$

which solves to yield

$$y^* = \tfrac{b}{k}\,.$$

To verify the ansatz, we confirm that, in (Eq. 8) when $y_n = y^*$, $y_{n+1} = y_n = \tfrac{b}{k}$, and hence we can show that the ansatz is valid.

We can then write a new equation by subtracting (Eq. 9) from (Eq. 8), yielding

$$(y_{n+1} - y^*) = (1 - kh)(y_n - y^*)$$

which, upon the substitution $x_n = y_n - y^*$ yields the homogeneous recurrence relation

$$x_{n+1} = (1 - kh)x_n\,.$$

Once the substitution has converted (Eq. 9) to a homogeneous form, it is easily solvable, resulting in the explicit expression

$$x_n = x_0(1 - kh)^n\,.$$

Finally, undoing the substitution $x_n = y_n - y^*$ leads to the equation

$$y_n - \tfrac{b}{k} = (y_0 - \tfrac{b}{k})(1 - kh)^n$$

which, upon substituting the initial condition $y_0 = \gamma$ and simplifying, yields

$$y_n = (\gamma - \tfrac{b}{k})(1 - kh)^n + \tfrac{b}{k} \qquad \text{(Eq. 10)}$$

Case 2: $\frac{dy}{dx} = a(x) - ky,\ y(0) = \gamma$

As in previous cases, we use Ottley's method to solve the differential equation exactly. The details of this process are fairly similar to the previous case; the only condition is that $\int e^{kx}\alpha(x)dx$ must be defined. We can solve this IVP using the exact same method as before, and thus, the solution alone is provided; see appendix A for the full solution method. For the IVP

$$\tfrac{dy}{dx} + ky = a(x),\ y(0) = \gamma\,,$$

the solution is

$$y = e^{-kx}(A(x) + \gamma - A(0))$$

where $A(x) = \int e^{kx} a(x)dx$. While this solution is not closed-form, it is still useful and can be employed for a variety of functions.

To discretize this IVP, we write

$$y_{n+1} = y_n + (a(x_n) - ky_n) \cdot h, y_0 = \gamma ,$$

which can be easily simplified to

$$y_{n+1} = ha(x_n) + (1 - kh)y_n . \quad \text{(Eq. 11)}$$

This recurrence relation has no closed-form solution, since the definition of $a(x_n)$ is the primary determinant of the solution form. To solve an equation of this form, we employ the method discussed in *Discrete Dynamical Models* (Salinelli and Tomarelli 50-60). In their paper, they discuss the Z-transform, defined as $Z\{x_n\} = x(z) = \sum_{k=0}^{+\infty} z^{-k} x_k$. The Z-transform can be used to solve linear recurrence relations when there is a nonconstant term involved, and is analogous to the use of Laplace transform in solving differential equations (Kanasewich).

While it is possible to obtain an analytical expression for many Z-transforms, it is often easier to use a table. Appendix B contains a table of Z-transforms and expressions taken from Salinelli and Tomarelli's *Discrete Dynamical Systems*.

It is easy to verify the linearity of the Z-transform operator (Salinelli and Tomarelli); it can be easily shown, using the definition of the Z-transform, that

$$Z\{ax_n + by_n\} = aZ\{x_n\} + bZ\{y_n\} .$$

In (Eq. 11), after making the substitutions $\alpha = (1 - kh)$ and $c_n = ha(x_n)$, we can apply the Z-transform to both sides, yielding

$$Z\{y_{n+1} - \alpha y_n\} = Z\{c_n\} \qquad \text{(Eq. 12)}$$

Given that the Z-transform operator is linear, we can rewrite (Eq. 12) as

$$Z\{y_{n+1}\} - \alpha Z\{y_n\} = Z\{c_n\}\,,$$

and upon applying the Z-transform operator on each term, using the table in appendix B, we arrive at the following:

$$z \cdot y(z) - z \cdot y_0 - \alpha \cdot y(z) = Z\{c_n\}\,.$$

This can be simplified to

$$y(z) = \frac{Z\{c_n\} - z\gamma}{z - \alpha}$$

and upon inversion of the Z-transform, we find that

$$y_n = Z^{-1}\{\tfrac{Z\{c_n\} - z\gamma}{z - \alpha}\} \qquad \text{(Eq. 13)}$$

where $Z^{-1}\{x\}$ is the operation such that $Z^{-1}\{Z\{x_n\}\} = x_n$. However, this result, although useful, is not closed-form. Thus, in order to demonstrate the usefulness of this method, we will use it to discretize a specific differential equation.

Case 2.1 - example

For the equation $\frac{dy}{dx} + ky = \frac{2^{\frac{x}{h}}}{h}$, we can use (Eq. 12) to arrive at a discrete form of the differential equation:

$$y_{n+1} = (1 - kh)y_n + 2^n$$

Here, given that $c_n = 2^n$, we can proceed to use the result in (Eq. 14):

$$y_n = Z^{-1}\{\tfrac{Z\{2^n\} - z\gamma}{z - \alpha}\} \qquad \text{(Eq. 14)}$$

and using the Z-transform table in appendix B, we find:

$$Z\{2^n\} = \frac{z}{z-2} .$$

We can thus substitute this into (Eq. 14) to yield

$$y_n = Z^{-1}\{\frac{\frac{z}{z-2} - z\gamma}{z-\alpha}\}$$

which can be simplified to

$$y(z) = \frac{z(1-(z-2)\gamma)}{(z-\alpha)(z-2)} .$$

To solve this, we can employ partial fraction decomposition to yield

$$\frac{y(z)}{z} = \frac{A}{z-\alpha} + \frac{B}{z-2}$$

and A and B can be solved to yield $A = \frac{1+(2-\alpha)\gamma}{\alpha-2}$, $B = \frac{-1}{\alpha-2}$. Now, we can invert the Z-transform to yield:

$$Z^{-1}\{y(z)\} = AZ^{-1}\{\frac{z}{z-\alpha}\} + BZ^{-1}\{\frac{z}{z-2}\} ,$$

and using the Z-transform table from appendix B, we find the expression for y_n :

$$y_n = \frac{1+(2-\alpha)\gamma}{\alpha-2}\alpha^n - \frac{1}{\alpha-2}2^n$$

where $\alpha = 1 - kh$. This can be rewritten as:

$$y_n = \frac{1}{1+kh}2^n - (\frac{1}{1+kh} + \gamma)(1-kh)^n \qquad \text{(Eq. 15)}$$

Through this method, it can be seen that nearly any differential equation of the form $\frac{dy}{dx} + ky = a(x)$ can be discretized. Only if the Z-transform ($Z\{x\}$) or inverse Z-transform ($Z^{-1}\{x\}$) is undefined will this method fail, in which case a recursive approach is the only way to numerically approximate the solution to the differential equation.

Case 3: $\frac{dy}{dx} = a(x) - b(x)y$

This case will consider the generalization of the previous results to any FOLDE. To solve the FOLDE, we use the result of (Eq. 4), defining $\mu(x) = e^{\int b(x)dx}$ and $A(x) = \int \mu(x)a(x)dx$, to arrive at the following solution (although not closed-form):

$$y = \frac{A(x) - A(0) + \mu(0)\gamma}{\mu(x)} .$$

To discretise this result, the same process can be employed as in previous cases:

$$y_{n+1} = y_n + h \cdot (a(x_n) - b(x_n)y_n)$$

$$y_{n+1} = (1 - b(x_n)h)y_n + h \cdot a(x_n)$$

In this case, the analytical methods used in previous examples fail to yield a result. Additionally, the solution method for this case varies based on the nature of the function $b(x)$, and thus is practically impossible. Using advanced methods such as the Casorati determinant could prove useful in discretizing some differential equations (Salinelli and Tomarelli); however, these methods are far outside the scope of this paper.

Approximation Error

In order to fully answer the research question, we must also consider the accuracy of the discretized solutions. Additionally, by deriving an expression for the error of Euler's Method in different cases, we can determine the minimum number of steps to reduce the error below a certain value. This would be helpful in developing algorithms for numerical IVP-solving computer programs.

To do this, we must develop a set of notation. First, we define a variable I that represents the variable finite interval across which the discretization is being conducted. We also define a variable n which represents the variable number of discrete steps within the interval, such that $n = \frac{I}{h}$ and n is an integer. Next, we define an operator $\Delta\{\frac{dy}{dx}\}(I, n)$ such that

$$\Delta\{\tfrac{dy}{dx}\}(I, n) = \left|f(I) - y_n\right|$$

where $f(x)$ represents the value of the solution to the differential equation upon which the operator is being applied, and y_n represents the discretized solution to the differential equation as explored above.

For Case 1, this metric can be algebraically evaluated. Using (Eq. 7) and (Eq. 10), the metric can be written as

$$\varepsilon = \left|\tfrac{b}{k} + (\gamma - \tfrac{b}{k})e^{-kI} - ((\gamma - \tfrac{b}{k})(1 - kh)^n + \tfrac{b}{k})\right|$$

where $\varepsilon = \Delta\{\frac{dy}{dx}\}(I, n)$.

Upon further simplifying this equation, we can see that

$$\varepsilon = \left|(\gamma - \tfrac{b}{k})(e^{-kI} - (1 - kh)^n)\right|$$

which is then equivalent to

$$\varepsilon = \left|(\gamma - \tfrac{b}{k})(e^{-kI} - (1 - k\tfrac{I}{n})^n)\right|.$$

With this expression, we can easily determine the minimum number of steps required to reduce the error of the discretized solution below a certain amount, by substituting the values of γ, b, k and ε into the above expression.

Interestingly, as the number of steps (n) tends to infinity, $(1 - \frac{kI}{n})^n$ tends to e^{-kI}. This can be proven easily - the proof is given below.

If we let $a = \lim_{n\to\infty}(1 - \frac{kI}{n})^n$, then:

$$\ln(a) = \lim_{n\to\infty} \ln((1 - \tfrac{kI}{n})^n)$$

$$= \lim_{n\to\infty}(\ln(n(1 - \tfrac{kI}{n}))$$

$$= \lim_{n\to\infty}(\frac{\ln(1-\frac{kI}{n})}{\frac{1}{n}})$$

As $n \to \infty$, it can be shown that the limit approaches the indeterminate form $\frac{0}{0}$; thus, using L'Hopital's rule, we can evaluate the limit by differentiating both the numerator and the denominator. This yields:

$$\ln(a) = \lim_{n\to\infty}(\frac{(\ln(1-\frac{kI}{n}))'}{(\frac{1}{n})'})$$

$$= \lim_{n\to\infty}(\frac{(1-\frac{kI}{n})^{-1}\cdot kIn^{-2}}{-n^{-2}})$$

$$= \lim_{n\to\infty}(\frac{-kI}{1-\frac{kI}{n}})$$

Now, evaluating the limit yields

$$\ln(a) = (-kI)$$

which can be rewritten as $\lim_{n\to\infty}(1 - \frac{kI}{n})^n = e^{-kI}$; the limit thus converges to the continuous solution to the differential equation.

Given that the definition of a derivative involves the infinite reduction in the step size of a discrete slope equation, the convergence of the discrete result to the continuous one verifies the solution presented earlier in the paper.

For Case 2, a problem arises. The lack of a closed-form solution to the case means that no closed-form expression can be derived for the error metric. It can, however, be shown that

$$\varepsilon = \left| e^{-k\frac{L}{n}}\left(A(\tfrac{L}{n}) + \gamma - A(\tfrac{L}{n})\right) - Z^{-1}\left\{\frac{Z\{c_n\} - z\gamma}{z-\alpha}\right\} \right|.$$

This result, despite the apparent complexity, is practically meaningless. Based on this result, however, we can ascertain a method by which we can reduce the error of specific FOLDEs below a certain level, by employing the appropriate substitutions into expression above.

Conclusion

In this paper, a method for discretizing FOLDEs has been explored, for various scenarios. However, given that a majority of FOLDEs can already be solved explicitly (i.e. without any numerical methods), this method becomes far more applicable with other types of differential equations (those that cannot be explicitly solved). However, given that the paper focuses on FOLDEs, one extension would be to use the same recursive definition of Euler's method with other differential equations. Another potential extension would be to approximate differential equations to FOLDEs based on the graphical form of their solutions. This method, while introducing additional inaccuracy, will allow for the explicit, discrete approximation of other differential equations, and for the approximate determination of the minimum number of steps required to achieve specific levels of accuracy in numerical approximations.

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Appendices

Appendix A: Solution of the IVP from Case 2

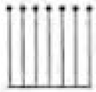
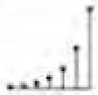
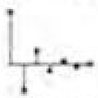
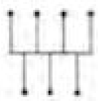
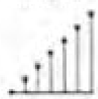
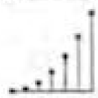
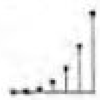
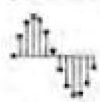
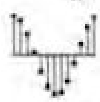
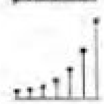
For the IVP $\frac{dy}{dx} + ky = a(x),\ y(0) = \gamma$, we must define an integrating factor $\mu(x) = e^{\int k dx}$. Then, after multiplying the differential equation by the integrating factor, the equation can be rewritten as $\frac{d}{dx}(e^{kx}y) = e^{kx}a(x)$. Upon integration, this yields:

$$e^{kx}y = A(x) + C,$$

where $A(x) = \int e^{kx}a(x)$. Substituting the initial condition, we find that $C = \gamma - A(0)$, and this leads to the final solution given:

$$y = e^{-kx}(A(x) + \gamma - A(0)).$$

Appendix B: A table of useful Z-transforms

	X_k	$\mathcal{Z}\{X\}(z) = x(z) = \sum_{k=0}^{+\infty} \frac{X_k}{z^k}$	
	1	$\frac{z}{z-1}$	
	a^k	$\frac{z}{z-a}$	$a > 1$
	a^k	$\frac{z}{z-a}$	$0 < a < 1$
	a^k	$\frac{z}{z-a}$	$-1 < a < 0$
	$(-1)^k$	$\frac{z}{z+1}$	
	k	$\frac{z}{(z-1)^2}$	
	k^2	$\frac{z(z+1)}{(z-1)^3}$	
	k^3	$\frac{-11z^3+4z^2+z}{(z-1)^4}$	
	$\sin(ak)$	$\frac{z\sin(a)}{z^2-2\cos(a)z+1}$	
	$\cos(ak)$	$\frac{z^2-z\cos(a)}{z^2-2\cos(a)z+1}$	
	$\sinh(ak)$	$\frac{z\sinh(a)}{z^2-2\cosh(a)z+1}$	
	$\cosh(ak)$	$\frac{z^2-z\cosh(a)}{z^2-2\cosh(a)z+1}$	

Note: this table is taken from Salinelli and Tomarelli's *Discrete Dynamical Systems*.