Symmetry groups: a powerful tool for solving differential equations

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Math 72.02: Lie Groups

1 Introduction

Differential equations are everywhere in mathematics and science, appearing in a wide range of applications such as physics, engineering, and finance. The study of differential equations has been an important topic in mathematics for centuries, and many techniques have been developed to solve them, and Lie group theory has risen to the forefront of this study.

In this paper, we will explore the uses and applications of Lie Groups to the analysis and solution of differential equations. We will start with a motivating example, and then move on to make some important definitions. We will then develop the tools we need to ascertain information about the symmetry groups of differential equations, and then discuss the applications of lie theory.

2 Motivating Example

We start by considering the following motivating example: the differential equation $\frac{dy}{dx} = f(x)$. In this section, we will take a look into this equation and the application of symmetry groups to the understanding and simplification of the equation at hand, an equation that helps represents a wide arrange of phenomena in various fields, such as population growth in biology.

Before discussing symmetry equations, let us first start by understanding the basic properties of $\frac{dy}{dx} = f(x)$. This is a separable ordinary differential equation, which means it can be written as

$$dy = f(x)dx$$

Thus, the general solution for this equation is:

$$y(x) = \int f(x)dx + C$$

where C is an arbitrary constant. Now, this is one way in which we can obtain a solution. However, it does not give us insight into the symmetries and structure of the equation. To gain this insight, we turn to Lie groups.

Let us now solve the same equation as before, but instead using methods involving symmetry groups. The first step in applying these methods is to find symmetry groups of $\frac{dy}{dx} = f(x)$, which is a group of transformations that leave the equation invariant. In other words, we are looking for transformations of the following form:

$$(x, y) \mapsto (x + \varepsilon\xi(x), y + \varepsilon\phi(x))$$

For some arbitrarily small ε. Now, one can easily check that the group of translations on the y axis $(x, y) \mapsto (x, y+\varepsilon)$ leaves the differential equation unchanged, since $\frac{d(y+\varepsilon)}{dx} = \frac{dy}{dx} + \frac{d\varepsilon}{dx} = \frac{dy}{dx} = f(x)$. Thus, letting $\xi(x) = 0,\ \phi(x) = 0$ we have the one parameter symmetry group of the equation, or in other words, the Lie group of the equation, since in this (general) case no other transformations leave the equation invariant. Then, we have that if y is a solution to the equation, every $\hat{y} = y + \varepsilon$ also is a solution to the equation, so the most general solution we can get when given a solution $y = g(x)$ is $g(x) + \varepsilon$. This means that the our one parameter symmetry group will map any solution of the equation to another solution.

Now, using the separation method from before we can always get the general solution of the equation, but using Lie groups we can find a type of transformation that maps one solution to another. This means if we have a solution y_1 and we have some coordinates of a solution y_2, we can find y_2 taking into account that we know the type of transformation that would map y_1 to y_2. Thus, Lie groups help understand the symmetries of a differential equation, so that if we have one particular solution of the equation, we will then have all of them. Furthermore, as we will discuss later, in more complicated examples we will be able to simplify a differential equation through Lie groups in order to solve the original equation in an easier fashion.

3 Differential Equations

Now that we have given some insight into why it is worth it to study the applications of Lie groups to differential equations, we will give some insight into the basic differential equation terminology that we will be using during this paper. Let us start by defining what an ODE is:

Definition 1. An ordinary differential equation, or ODE, is a differential equation in which only ordinary derivatives take place, as opposed to partial derivatives. The solutions to these equations are functions dependent only on one variable.

Example. Examples of ODEs are:

- $\frac{dy}{dx} = 2x$
- $\frac{dx}{dt} = t^2 + 2t$
- $\frac{du}{dx} = u + 2x$
- $\frac{dy}{dx} + \frac{dz}{dx} = 2x + zx$

Definition 2. A partial differential equation, or PDE, is a differential equation in which not all derivatives are ordinary derivative: there are, in fact, some partial derivatives. The solutions to these equations have at least one function with multiple parameters in them.

Example. Examples of PDEs are:

- $\frac{\partial f(x,y)}{\partial x} = F(x, y)$

- $\frac{\partial f(x,y)}{\partial y} = F(x,y)$
- $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ (Heat Equation).
- $\frac{\partial^2 \psi}{\partial^2 x^2} = \frac{1}{v^2}\frac{\partial \psi}{\partial t}$ (Wave Equation).

In some cases we might express the partial derivative of some function $u(x, y)$ as u_x. Furthermore, the second derivative will be u_{xx} (or u_{xy}), the third will be u_{xxx}, and so on. Lastly, let us define the notion of the order of a differential equation:

Definition 3. The order of a differential equation is the highest order of all its derivatives.

Example. To understand the notion of order, let us take a look at these examples:

- $\frac{dy}{dx} = f(x)$ is a differential equation of order 1.
- $\frac{d^2y}{dx^2} + \frac{dy}{dx} = f(x)$ is a differential equation of order 2.

4 Tools

Throughout this section we will build the vocabulary and tools needed to apply the theory of Lie groups to differential equations. Once we have the requisite tools available, we apply them to yield useful results in the classification of differential equations. Unless otherwise stated, G will be used to always refer to a Lie group.

4.1 Invariance

The key notion in the theory is invariance. Throughout, we will use this to characterize sets of points and functions. The following definitions capture this notion.

Definition 4. Let G be a group of transformations acting on a manifold M. A subset $S \subset M$ is said to be G**-invariant** if

$$\forall g \in G, x \in S : g \cdot x \in S.$$

Equivalently, $G = \mathrm{Symm}(X)$ (i.e. G is the symmetry group of X).

A similar notion also exists for functions.

Definition 5. Let $\gamma : M \to N$ be a map between manifolds, and let G act on M. Then γ is a G**-invariant function** if

$$\forall g \in G, x \in M : \gamma(g \cdot x) = \gamma(x).$$

If $N = \mathbb{R}$, γ is called an **invariant** of G.

Example. Let $M = \mathbb{R}^2$ and let $G = \{[(x, y) \mapsto (x + \varepsilon, y + \varepsilon)] : \varepsilon \in \mathbb{R}\}$. Then the set

$$X = \{(x, x + d) : x \in \mathbb{R}\} \subseteq M$$

(given by the line $y = x + d$) for any $d \in \mathbb{R}$ is G**-invariant**. Notice that G-invariant sets are precisely the orbits of G acting on M. Similarly, the function $f : (x, y) \to x - y$ is an invariant of G, since

$$f(x + \varepsilon, y + \varepsilon) = (x + \varepsilon) - (y + \varepsilon) = f(x, y).$$

These definitions are related by the following theorem.

Theorem 6. Suppose G acts on a manifold M and $F : M \to R^k$ is smooth. Then F is an G-invariant function if and only if each set $\{x : F(x) = c\}$ is G-invariant, for all $c \in M$.

Proof. Suppose that F is G-invariant. For arbitrary c, let $S_c = \{x : F(x) = c\}$, and let $s \in S_c$ be arbitrary. Then, for any $g \in G$, $F(g \cdot s) = F(s) = c$, so $g \cdot s \in S_c$. Since this holds for any s and c, it follows that any set S_c is G-invariant. Conversely, if all level sets of F are G-invariant, then $g \cdot x \in \{y : F(y) = F(x)\}$ for any $g \in G$. But then $F(g \cdot x) = F(x)$, so F is a G-invariant function. □

Our eventual goal will be to determine groups G for which the set of solutions to a differential equation are invariant. These criteria, however, are quite difficult to verify in practice, so in the context of a Lie group, we may replace them with a more obvious condition. First we recall an important definition[1]:

Definition 7. Let G be a Lie group, with associated Lie algebra $\mathfrak{g}$. Then the **infinitesimal generators** of G are the basis elements of the Lie algebra $\mathfrak{g}$.

Now we state the condition for invariance.

Theorem 8. Let G be a Lie group acting on a manifold M, and let V be the set of infinitesimal generators of G. Then a smooth function $\gamma : M \to \mathbb{R}$ is an invariant of G if and only if for any $x \in M$ and $v \in V$,

$$v \cdot \gamma(x) = 0.$$

Proof. Suppose γ is an invariant of G. Then[2]

$$\frac{d}{dt}\left[\gamma(e^{tv} \cdot x)\right] = v \cdot \gamma(x)\left[e^{tv} \cdot x\right] = 0;$$

and hence $\gamma(e^{tv} \cdot x)$ is constant. Since we may write g as the the exponentiation of some linear combination of vectors in V, it follows that $\gamma(g \cdot x) = \gamma(x)$ for all $x \in M, g \in G$, and so γ is an invariant of G.

Conversely, suppose $v \cdot \gamma(x) = 0$ for any $v \in V$ and $x \in M$. Observe that

$$\frac{d}{dt}\left[\gamma(e^{tv} \cdot x)\right] = v \cdot \gamma(x)\left[e^{tv} \cdot x\right] = 0;$$

hence $\gamma(e^{tv} \cdot x)$ is constant. Recall that for any $g \in G$, there exists some $A_g \in \mathfrak{g}$ such that $g = e^{tA_g}$. In particular, since A_g is a linear combination of vectors in V, $\gamma(e^{tA_g} \cdot x)$ is constant over t. So so

$$\gamma(g \cdot x) = \gamma(e^{tA_g} \cdot x) = \gamma(x),$$

[1] This omits a couple of steps, but is a useful definition nonetheless

[2] See [Olv86], equation 1.17

and γ is an invariant of G as claimed. $\square$

4.2 Differential Equations, Jet Spaces, and Prolongation

With these preliminary definitions out of the way, we begin to consider the interplay between Lie groups and differential equations. We start by exploring how a Lie group G acts on a function $f : X \to U$. We first identify f with its curve (or "graph") in $X \times U$:

$$\Gamma_f = \{(x, f(x) : x \in X\} \subseteq X \times U.$$

Say G acts on $X \times U$ by the following (general) transformation:

$$g \cdot (x, u) = (\alpha_g(x, u), \beta_g(x, u))$$

where α_g, β_g are smooth.

We want to determine what the graph of $g \cdot f$ is. Indeed, we first observe that Γ_f is parametrized by $(\mathbb{1} \times f)(x)$ (where $\times$ is the Cartesian product of functions). Then, simplifying, we find that that

$$g \cdot f(x) = [\beta \circ (\mathbb{1} \times f)] \circ [\alpha \circ (\mathbb{1} \times f)]^{-1} (x).$$

Example. Consider $G = \mathrm{SO}(2)$, and let $f(x) = 2x$. We may identify any $g \in G$ with its rotation angle θ_g. Then, g acts on $\mathbb{R} \times \mathbb{R}$ as follows:

$$g \cdot (x, u) = (x \cos(\theta_g) - u \sin(\theta_g), x \sin(\theta_g) + u \cos(\theta_g)).$$

Indeed, with $\alpha(x, u) = x \cos(\theta_g) - u \sin(\theta_g)$ and $\beta(x, u) = x \sin(\theta_g) + u \cos(\theta_g)$, it follows that

$$[\beta \circ (1 \times f)] (x) = (2 \cos(\theta_g) + \sin(\theta_g))x$$

and

$$[\alpha \circ (1 \times f)]^{-1} (x) = \frac{1}{\cos(\theta_g) - 2 \sin(\theta_g)} x.$$

So we have that

$$g \cdot f(x) = \frac{2 \cos(\theta_g) + \sin(\theta_g)}{\cos(\theta_g) - 2 \sin(\theta_g)} x.$$

Indeed, this corresponds to the rotation of the graph of $f(x) = 2x$ by θ_g degrees (as long as $\cot(\theta_g) \neq 2$).

We can finally define the symmetry group of a differential equation.

Definition 9. Suppose σ is a system of differential equations, and let G be a group of transformations acting on $X \times U$. Then G is a **symmetry group** for σ if whenever $u = f(x)$ is a solution to σ, then $u = g \cdot f(x)$ is also a solution whenever $g \cdot f$ is well-defined.

Example. Consider the first-order differential equation $\frac{d^2u}{dx^2} = 0$. The solutions are of the form $u = cx$ for some $c \in \mathbb{R}$. By a simple extension of the previous example, it is easy to see that $G = \mathrm{SO}(2)$ is a symmetry group for the differential equation.

An immediate issue is the following: without solving the system of differential equation, how can we determine if a group is a symmetry group? A useful approach is to define the differential equation as a geometric object in some ambient space, and to define a left group action of G on that space. To that end, we first define the ambient space, which we will call the jet space:

Definition 10. The n**th order jet space** of $X \times U$ is the space $X \times U \times U_1 \times \cdots \times U_n$, where U_k is the space of all kth order partial derivatives of U. Let $U^{(n)} = U \times U_1 \times \cdots \times U_n$; then the nth order jet space can be written as $X \times U^{(n)}$.

Under this definition, observe that any nth order differential equation can be represented as the solution to some system of equations $\Delta(x, u^{(n)}) = 0$ where $u^{(n)} \in U^{(n)}$.

We then define the nth "prolongation" of f to be the set of points f induces in the nth order jet space. In particular, we denote it as $\mathsf{pr}^n \circ f$ and $\mathsf{pr}^n \circ f(x)$ evaluates to a tuple of partial derivatives. A useful way to think about the prolongation is by the following equivalence:

$$\mathsf{pr}^{(n)} f \simeq f \times Df \times D^2 f \times \cdots \times D^n f$$

where D is the total differential operator. What remains is to suitably define the prolongation of a group action, such that the prolongation agrees with the action on the original function. To articulate this notion formally, we state the following definition:

Definition 11. Suppose G acts on $X \times U$. The n**th prolongation of** G, denoted $\mathsf{pr}^n G$, defines the action of G on the nth-order jet space $X \times U^{(n)}$ is defined such that, for $g \in G$,

$$(\mathsf{pr}^n g) \cdot (\mathsf{pr}^n f) = \mathsf{pr}^n (g \cdot f).$$

This definition leads to an immediate, but important, result which is central to our analysis of differential equations.

Theorem 12. Suppose $\Delta(x, u^{(n)}) = 0$ is a system of differential equations, which is satisfied by some set $S(\Delta) \subseteq X \times U^{(n)}$. Suppose $S(\Delta)$ is $(\mathsf{pr}^n G)$-invariant (as in [4]). Then G is a symmetry group of $\Delta(x, u^{(n)}) = 0$ (as in [9]).

A corollary of this theorem, along with [8], is the following:

Corollary 13. Suppose $\Delta(x, u^{(n)} = 0$ is a system of differential equations, and G is a group of transformations acting on $X \times U$ with infinitesimal generators V. Then if

$$\forall v \in V : (\mathsf{pr}^n v) \left[\Delta(x, u^{(n)}) \right] = 0$$

whenever

$$\Delta(x, u^{(n)}) = 0,$$

then G is a symmetry group for the system $\Delta(x, u^{(n)}) = 0$.

This gives us a rather straightforward criterion for determining whether a group of transformations is indeed a symmetry group for a differential equation.

Before we continue, we provide a definition, which we use for ease of representing infinitesimal generators, since in later sections we use this notation extensively.

Definition 14. Let G be a Lie group with Lie algebra $\mathfrak{g}$. For $g \in G$, such that G acts on $X \times U$, we define the corresponding element $v \in \mathfrak{g}$ such that $g : (x, u) \mapsto (x, u) + \Theta(x, u)$ where $v = \Theta_x \partial_x + \Theta_u \partial_u$.

Remark. One might consider v to be the total differential of a "flow" through (x, u), which comes from the construction of the Lie algebra.

4.3 Computing symmetry groups of differential equations

Though we have a method for verifying whether a group G is indeed a symmetry group to a differential equation, actually computing symmetry groups is nontrivial. In some cases, it is easy to extract some symmetry groups, especially if we have knowledge of the solutions to the differential equation or properties of the system generating the differenttial equations. However, it is often useful to know whether we truly have extracted all symmetries of the differential equation. So, we present some results that enable the computation of symmetry groups.

The following theorem enables us to compute symmetry groups, though the method is somewhat involved.

Theorem 15. Let $v = \sum_i \xi^{(i)}(x, u)\partial_{x^{(i)}} + \sum_i \phi^{(i)}(x, u)\partial_{u^{(i)}}$ be an infinitesimal generator of some symmetry group G, acting on $X \times U$. Then, for $n \geq 1$,

$$\mathsf{pr}^n(v) = v + \sum_i \sum_J \phi_J^\alpha(x, u^{(n)})\partial_{u_J^{(i)}},$$

where the inner summation is over all unordered multi-indices $J = \{j_1, ..., j_k\}$ with $1 \leq k \leq n$ and $1 \leq j_i \leq j_p$ (where $p = \dim X$), and ϕ_J^α are coefficient functions.

Remark. A couple of points become apparent here. First, the coefficient functions have an explicit expression, but their form is complicated and of little importance. Second, the formula is generally not useful, especially when in specific cases of X, U it is not too difficult to compute. Finally, we spare the reader the details of the proof, as it is arduous and not particularly useful to understanding the results. An interested reader may consult [Olv86], Theorem 2.36. We will, however, use specific corollaries of this result, so referring to the result is useful.

We illustrate this method through examples, as the specific techniques are dependent on the specific differential equation. The examples are motivated from [Olv10].

Example. Let $X, U \simeq \mathbb{R}$, and let $\Delta(x, u, u^{(1)}, u^{(2)}) = u_{xx} = 0$. Let's say G is a one-parameter symmetry group of Δ with infinitesimal generator v. Our goal is to compute all possible values this generator can take.

So, we write $v = \xi(x, u)\partial_x + \phi(x, u)\partial_u$.

It can be shown (by inspecting the action of v on the derivatives u_x and u_{xx}, though it may be simpler to refer to the general formula in [Olv86]) that the prolonged action $\mathsf{pr}^2(v)$ of v on $X \times U^{(2)}$ is given by

$$\mathsf{pr}^2(v) = v + \phi_1\partial_{u_x} + \phi_2\partial_{u_{xx}}$$

where

$$\phi_1 = \phi_x + (\phi_u - \xi_x)u_x - \xi_u u_x^2$$
$$\phi_2 = \phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x + (\phi_{uu} - 2\xi_{xu})u_x^2 - \xi_{uu}u_x^3 + (\phi_u - 2\xi_x)u_{xx} - 3\xi_u u_x u_{xx}.$$

Indeed, for $\mathrm{pr}^2(v)$ to leave solutions of Δ invariant, by applying [12] it must hold that ϕ_2 must equal zero whenever $u_{xx} = 0$. This implies the following method: set u_{xx} to zero, and equate ϕ_2 to zero over varying x, u, u_x. This then implies a set of relationships between ϕ and ξ, which we then solve. So,

$$\phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x + (\phi_{uu} - 2\xi_{xu})u_x^2 - \xi_{uu}u_x^3 = 0.$$

The following equations follow:

$$\phi_{xx} = 0$$
$$2\phi_{xu} = \xi_{xx}$$
$$\phi_{uu} = 2\xi_{xu}$$
$$\xi_{uu} = 0.$$

We then solve this system of **linear differential equations**. The details of solving it are omitted; it can be seen that the general solution is given by

$$\xi(x, u) = c_1x^2 + c_2xu + c_3x + c_4u + c_5.$$
$$\phi(x, u) = c_1xu + c_2u^2 + c_6x + c_7u + c_8.$$

These equations provide us with the description of every possible infinitesimal generator for the symmetry group. We can recover, for example, the generator $v = \partial_x$ corresponds to the translational invariance of the system under the action $(x, u) \to (x + \varepsilon, u)$.

But we can do more. In particular, the fact that our expressions for ξ and ϕ are the most general possible allow us to construct a Lie algebra $\mathfrak{g}$ for the symmetry group G of Δ: indeed, it has the following basis (using infinitesimal notation):

$$\mathfrak{g} =_{\mathbb{R}} \left\{\partial_x, \partial_u, x\partial_x, x\partial_u, u\partial_x, u\partial_u, x^2\partial_x + xu\partial_u, xu\partial_x + u^2\partial_u\right\}$$

whose symmetry group is the set of all transformations that preserve collinearity of points– the affine group!

We next consider an example in which $X \simeq \mathbb{R}^2, U \simeq \mathbb{R}$, which is of physical significance.

Example. Consider the equation $u_t = u_{xx}$ (the heat equation). Let $v = \xi(x,t,u)\partial_x + \eta(x,t,u)\partial_t + \phi(x,t,u)\partial_u$. An observation from the previous example is that it suffices to consider the action of the prolongation on the infinitesimals whose corresponding spaces are involved in the differential equation. In particular, it suffices to compute the coefficient functions for ∂_{u_t} and $\partial_{u_{xx}}$; define ϕ_{u_t} and $\phi_{u_{xx}}$ to be these functions, respectively. By the general prolongation formula, it can be shown that

$$\phi_{u_t} = \phi_t - \xi_t u_x + (\phi_u - \eta_t)u_t - \xi_u u_x u_t - \eta_u u_t^2$$

and

$$\begin{aligned}\phi_{u_{xx}} &= \phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x - \eta_{xx}u_t + (\phi_{uu} - 2\xi_{xu})u_x^2 - 2\tau_{xu}u_xu_t - \xi_{uu}u_x^3 - \eta_{uu}u_x^2u_t \\ &\quad + (\phi_u - 2\xi_x)u_{xx} - 2\eta u_{xt} - 3\xi_u u_x u_{xx} - \eta_u u_t u_{xx} - 2\eta_u u_x u_{xt}.\end{aligned}$$

We apply the same method as the previous example: set $u_t = u_{xx}$, and compute the conditions under which $\phi_{u_t} = \phi_{u_{xx}}$.

Again, we apply the same simplifications, yielding the following set of "determining equations":

$$\begin{aligned}\eta_u &= 0 \\ \eta_x &= 0 \\ \eta_{tt} &= 0 \\ \xi_u &= -\eta_{xu} \\ 2\xi_x + \eta_{xx} &= \phi_u \\ \xi_{uu} &= 0 \\ \phi_{uu} &= 2\xi_{xu} \\ -\xi_t &= 2\phi_{xu} - \xi_{xx} \\ \phi_t = \phi_{xx}.\end{aligned}$$

Again, observe this is a linear system of differential equations, so we may solve it, giving:

$$\begin{aligned}\xi &= c_1 + c_4x + 2c_5t + 4c_6xt \\ \eta &= c_2 + 2c_4t + 4c_6xt \\ \phi &= (c_3 - c_5x - 2c_6t - c_6x^2)u + \alpha(x,t)\end{aligned}$$

where $\alpha(x,t)$ is a function satisfying satisfies $\alpha_t = \alpha_{xx}$.

We can thus compute the Lie algebra $\mathfrak{g}$ for the symmetry group G:

$$\mathfrak{g} =_{\mathbb{R}} \left\{\partial_x, \partial_t, u\partial_u, x\partial_x + 2t\partial_t, 2t\partial_x - xu\partial_u, 4xt\partial_x + 4t^2\partial_t - (x^2+2t)\partial_u, \alpha(x,t)\partial_u\right\}$$

where α is as above. Observe that in contrast to the first example, this Lie algebra is infinite-dimensional, owing to the fact that there are infinite functions $\alpha(x,t)$ satisfying $\alpha_t = \alpha_{xx}$. Interestingly, many basis element of the Lie algebra tell us something meaningful about the nature of the heat equation: solutions are invariant under spatial and temporal transition, scaling. We shall see later that this is useful in constructing new solutions to the heat equation, from relatively trivial ones.

We end this section with some general remarks. Though the

4.4 Acknowledgements

Many of the definitions and theorems in this section are referenced, with modification and elaboration, from [Olv86].

5 Methods

The natural question that arises next is: what can one do with knowledge of the symmetry groups of a differential equation? We will explore several techniques to obtain solutions to differential equations assuming knowledge of the symmetry groups.

5.1 Constructing solutions

Constructing new solutions to a differential equation is the most immediate applications that comes to mind with knowledge of the symmetry group. The essence of this is captured by the following result, which follows from [12].

Corollary 16. Suppose $u = f(x)$ is a solution to the differential equation $\Delta(x, u^{(n)}) = 0$. Suppose further that G is a symmetry group for the differential equation. Then $u = g \cdot f(x)$ is also a solution to the differential equation for any $g \in G$.

Example. Consider the differential equation $u_{xx} = 0$. It can be shown, using the methods in the previous section, that SO(2) is a symmetry group for the differential equation. Also, trivially, $u = 0$ is a solution for the differential equation. As before, for any $g \in G$ we identify g with a rotation angle θ_g, and

$$g \cdot f(x) = \frac{\sin(\theta_g)}{\cos(\theta_g)} x = \tan(\theta_g) x.$$

Indeed, for any $\theta_g \in [0, 2\pi)$, $u = \tan(\theta_g)x$ satisfies the differential equation. We could also consider all transformations in the affine group (which we previously showed was a symmetry group of $u_{xx} = 0$), giving a similar result.

Example. Using our work from computing the symmetry algebra of the heat equation, we can show some useful results. For example, it is trivial to see that $u = 1$ is a solution to the heat equation. From this we may derive that, for example,

$$u(x,t) = \frac{1}{\sqrt{1 + 4\varepsilon t}} \exp\left(\frac{-\varepsilon x^2}{1 + 4\varepsilon t}\right)$$

is also a solution to the DE. One can derive a "general solution" by combining solutions using the symmetries of the Lie algebra, which is very useful in solving real-world systems.

As we can see, knowledge of the symmetry groups of a differential equation provides an approach for constructing solutions to the equation from a given solution. In particular, if we determine a nontrivial symmetry group of the differential equation, we may extend even trivial solutions (e.g. constant functions) of the differential equation to a meaningful solution set.

5.2 Integration of first-order ODEs

Given an arbitrary first-order ordinary differential equation $\frac{du}{dx} = F(x,u)$ one might hope to find a function $u = f(x)$ satisfying the equation. In fact, this is possible with some knowledge of the symmetry group.

Theorem 17. Consider the differential equation $\frac{du}{dx} = F(x,u)$, and suppose that G is a one-parameter symmetry group for the differential equation. If the infinitesimal generator of G does not vanish at (x_0, u_0), then there exists a solution to the differential equation obtainable by integration.

Proof. We start by first showing a useful result.

Proposition 18. Suppose $\mathbf{v}$ is a vector field not vanishing at a point $x_0 \in M$, i.e. $\mathbf{v}\big|_{x_0} \neq 0$. Then there is a local coordinate chart $y = (y^1, \cdots, y^m)$ at x_0 such that in terms of these coordinates, $\mathbf{v} = \partial_{y^1}$.

Proof. Let $\mathbf{v} = \xi(x,u)\partial_x + \eta(x,u)\partial_u$ be the infinitesimal generator of G. Then, first let us linearly change the coordinates so that $x_0 = 0$ and $\mathbf{v}\big|_{x_0} = \partial_{x^1}$. Then, by continuity the coefficient $\xi(x)$ is positive in a neighbourhood of x_0, since $\xi(x_0) = 1$. Since $\xi^1(x) = 0$, the integral curves of $\mathbf{v}$ cross the hyperplane $\{0, x^2, \cdots, x^m\}$ transversally, so in a neighbourhood of x_0, each point $(x^1, \cdots, x^m)$ can be defined as the flow of some $(0, y^2, \cdots, y^m)$ on this hyperplane. Consequently, we have

$$x = e^{y^1\mathbf{v}}(0, y^2, \cdots, y^m)$$

for some y^1 near 0, gives a diffeomorphism from $(x^1, \cdots, x^m)$ to $(y^1, \cdots, y^m)$ which defines the y coordinates. In terms of the y coordinates, we have that for small ε

$$e^{\varepsilon\mathbf{v}}(y^1, y^2, \cdots, y^m) = (y^1 + \varepsilon, y^2, \cdots, y^m)$$

so the flow is just the translation on the y^1 direction. Thus, a non vanishing vector field will always locally equivalent to the infinitesimal generator of a group of translations in one direction. □

Having proved the proposition above, let us now come back to the main proof. Let $\mathbf{v} = \xi(x,u)\partial_x + \phi\partial_x$ be the infinitesimal generator of G, and suppose $\mathbf{v}$ does not vanish at (x_0, u_0), that is, $\mathbf{v}\big|_{(x_0,u_0)} \neq 0$. Then, according to our proposition, we can introduce new coordinates

$$y = \eta(x,u),\ w = \zeta(x,u)$$

near (x_0, u_0) such that in terms of (y, w) coordinates, $\mathbf{v} = \partial_w$, with first prolongation

$$\mathsf{pr}^{(1)}\mathbf{v} = \mathbf{v} = \partial_w$$

Then in the new coordinate system, we have that our previous differential equation is equal to

$$\frac{dw}{dy} = H(y)$$

for some function H, since in order to be invariant the differential equation must be independent of w. Now, this new equation can be easily solved by integration: $w(y) = \int H(y)dy + C$. Then, a simple change of coordinates will give us a solution $u(x)$ to our original equation, so we have shown that there exists a solution to our differential equation that is also obtainable by integration. □

5.3 Reduction of order using symmetry

We will now show one of the most useful results of using Lie groups to solve differential equations. It is a generalization of 17, by using proposition 18. The result is as follows:

Theorem 19. Let $\Delta(x, u^{(n)}) = 0$ be an n-th order differential equation. If G is a one-parameter symmetry group of this equation, then we can reduce the order of the equation by one.

Proof. We start by construcing a change of coordinates $y = \eta(x, u), w = \zeta(x, u)$ much like in Theorem 17, such that the equation's group of symmetries gets changed into a group of translations with infinitesimal generator $\mathbf{v} = \partial_w$. Then, by the derivative chain rule we can express every derivative $\frac{d^k u}{dx^k}$ in terms of y, w, and derivatives of w with respect to y:

$$\frac{d^k u}{dx^k} = \delta_k(y, w, \frac{dw}{dy}, \cdots, \frac{d^k w}{dy^k})$$

for certain functions δ_k. Substituting these expressions into our equation, we get the following equivalent n-th order differential equation:

$$\tilde{\Delta}(y, w^{(n)}) = 0$$

in terms of y and w. Since G is a one-parameter group of the original equation, so does the equivalent equation. Then, the prolongation of the infinitesimal generator is:

$$\mathsf{pr}^{(n)}\mathbf{v} = \mathbf{v} = \partial_w$$

Thus, we have that by Theorem 12:

$$\mathsf{pr}^{(n)}\mathbf{v}(\tilde{\Delta}) = \frac{\partial \tilde{\Delta}}{\partial w} = 0 \text{ whenever } \tilde{\Delta}(y, w^{(n)}) = 0$$

Since the prolongation for this case is just the translations on the w axis, we can then set $w = 0$ by changing y appropriately to get an equivalent differential equation

$$\hat{\Delta}(y, \frac{dw}{dy}, \cdots, \frac{d^n w}{dy^n})$$

Whenever $\tilde{\Delta}(y, w^{(n)}) = 0$, i.e. whenever w is a solution for our original equation. Then, setting $z = w_y$ we have that our equation is equivalent to $\hat{\Delta}(y, z, \cdots, \frac{d^{n-1}z}{dz^{n-1}}) = 0$, which is an $n-1$-order differential equation. We have thus proven our claim.

□

5.4 Solvability by integration: an analogue to Galois theory

There is an incredible result in algebra which shows a polynomial is solvable by radicals iff its Galois group (i.e. the group of automorphisms that permutes roots of the polynomial) is solvable (i.e. there exists a chain of normal subgroups such that each quotient group is abelian).

Surprisingly, a similar result exists for differential equations! The following is the statement of the theorem, though the proof is outside the scope of the content of this paper.

Theorem 20. Let $\Delta(x, u^{(n)}) = 0$ be a system of differential equations. Suppose G is its (maximal) symmetry group. If G is solvable, then the differential equation is solvable by integration.

An interested reader might consult [Olv86] (Theorem 2.64) for the proof, though some background reading on differential invariants is likely needed.

6 Conclusions

There is a wealth of content in the subject of symmetry groups; this is a mere exposition into some of the key concepts and intuitions behind the theory of symmetry groups. With that said, symmetry groups are a powerful tool to analyzing differential equations, and in some sense provide a continuous analogue to Galois theory, one of the most celebrated results in mathematics. We hope to further explore these techniques moving forward, especially in considering the connections between the algebraic structure of symmetry groups and their role in computing symmetries.

References

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