Assignment n.1 Group n.2

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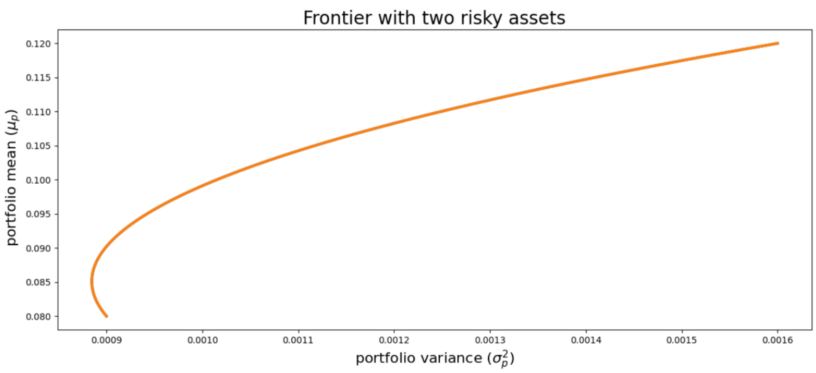
1 Minimum Variance Portfolio with 2 Risky Assets

In this scenario, a portfolio comprising two risky assets, denoted as X and Y, is considered. The expected returns and standard deviations of both assets are known. Asset X exhibits a higher return along with increased risk, while asset Y presents a lower return coupled with reduced risk.

Asset	μ	σ
X	12%	4%
Y	8%	3%

Table 1: Expected Returns (μ) and Standard Deviations (σ) for Assets X and Y with $\rho = 0.65$.

The variable w represents the weight assigned to one of the risky assets in the portfolio. To explore the efficient frontier, 2000 distinct values of w are simulated, plotted, and observed to align with the efficient frontier. The efficient frontier represents the set of portfolios that offers the maximum expected return for a given level of risk or the minimum risk for a given level of expected return.



To compute the optimal portfolio, specifically the one with the minimum variance, the variance is derived with respect to w. In the two-dimensional case, an analytical solution is obtainable by setting the derivative equal to zero. The resulting point on the curve with the minimum variance is identified through graphical representation.

$$w^* = \arg \min_{w \in [0,1]} \sigma^2(w)$$

$$\frac{\partial \sigma^2(w)}{\partial w} = w\left(\sigma_X^2 + \sigma_Y^2 - 2\rho\sigma_X\sigma_Y\right) - \sigma_Y^2 + \rho\sigma_X\sigma_Y = 0$$

$$w^* = \frac{\sigma_Y^2 - \rho\sigma_X\sigma_Y}{\sigma_X^2 + \sigma_Y^2 - 2\rho\sigma_X\sigma_Y}.$$

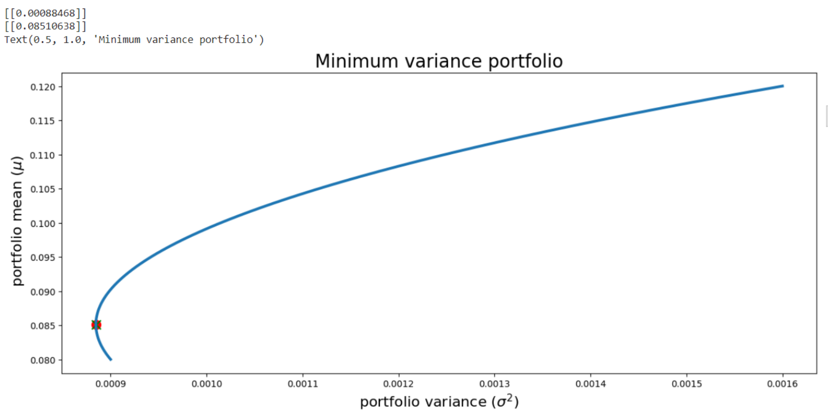
Additionally, Merton coefficients A, B, C, and D are computed. The analytical solution, expressed through the $first mutual fund theorem$, provides a benchmark for comparison with numerical approximations. The latter is obtained using the *scipy.optimize.minimize_scalar* function. The comparison reveals a high degree of alignment, though the numerical solution may exhibit slightly diminished precision. We can see that they coincide from the plot.

The analytical solutions are:

- $\sigma_p = 0.00088468$
- $\mu_p = 0.08510638$

And the numerical ones:

- $\sigma^* = 0.00088468085106382\,96$
- $\mu^* = 0.08510638305469787$



2 Maximum Utility Portfolio with N-Risky Assets

In this instance, a scenario involving N-risky assets is considered, with N specifically set to 4, encompassing stocks from Google, Amazon, Apple, and Tesla. After initial data preprocessing, expected returns and variances are computed exploiting logarithmic scaling.

Expected Returns

Asset	**Expected Return**
Google	0.014559
Amazon	0.018679
Tesla	0.032732
Apple	0.017859

Covariance Matrix

	Google	Amazon	Tesla	Apple
Google	0.004655	0.003165	0.003550	0.002350
Amazon	0.003165	0.007154	0.004711	0.003212
Tesla	0.003550	0.004711	0.026630	0.004951
Apple	0.002350	0.003212	0.004951	0.006125

To explore the efficient frontier, 25,000 potential portfolios are simulated by randomly assigning weight parameters $\mathbf{w} \in [0,1]^4$. These portfolios are visualized through a scatterplot.

The utility function is maximized by minimizing its negative counterpart using *scipy.optimize.minimize* with the Sequential Least Square Programming (*SLSQP*), subject to the constraint that its wights sum up to 1, that is: $\mathbf{1}^T\mathbf{w} = 1$.

$$\begin{aligned} &\max_{w} w^T\mu - \frac{\gamma}{2} w^T \Sigma w \\ &1^T w = 1 \end{aligned}$$

This numerical optimization method is sensitive to the initial guess, yielding slightltly different results even with slight variations to the initial guess for $\mathbf{w}$, up to the second decimal digit.

Analytical Solution

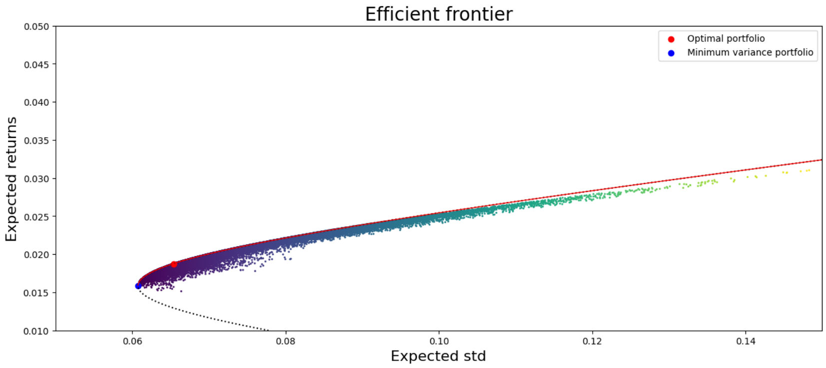
$$[0.3281523 \quad 0.2231252 \quad 0.12133784 \quad 0.32738466]$$

Numerical Solution

$$[0.32304746 \quad 0.22001489 \quad 0.12082691 \quad 0.33611074]$$

Analytical results are obtained using the first mutual fund theorem and coefficients A, B, C, and D are used to determine the minimum variance portfolio as well (that is when $D = 0$).

A comparison between the analytical efficient frontier and the simulated points illustrates that the latters indeed delineates the boundary of the cloud of points.



Additional computations and checks were performed in the attached notebook along with further comments. We found out consistent outcomes by exploiting the results of the first mutual fund theorem, both in the multi-dimensional case and in the two dimensional case, and by exploiting both analytical methods and numerical ones.