Final Project

Andreotti P., Hanna K., Nuttini E., Rigamonti M., Zhou F.

June 2024

Contents

1 Problem setting

In this final assignment we are required to tackle the pricing, sensitivity analysis and hedging of a so called *exotic derivative*, which is a **structured bond** issued by Bank XX, which from now one will be referred as Party A. In particular we are considering a contract between Party A and an Investment Bank, which from now on will be referred as Party B.

To make a long story short, the main difference between a plain-vanilla bond and a structured bond is the fact that the latter has a more complex structure for the coupon, which, in this particular case, follows the behavior of an *Asian option* with ENEL stock as underlying. Therefore, the exotic-ness of this derivative is due to its layered structure: you have the derivative itself, the structured bond, and within it you have embedded derivatives such as the Asian option and the **hedging swap** between Party A and Party B.

It's important to highlight the fact that when dealing with this kind of structured bond, you usually have three parties, and the third one is the (retail) investor, which is the Party C. The big picture is the following:

1. Party C enters in the proper structured bond contract with Party A, the flows that are exchanged are the standard ones for a bond:

   - Party C pays the structured bond;
   - Party A pays the coupons up to maturity along with the face value at maturity.

   This is the proper structured bond flows exchange;

2. at the same time, Party A wants to hedge its risk with party B, and they agree on a hedging swap contract. This is due to the fact that Party A (Bank XX) wants to receive a floater during the life time of the contract. The flows that are exchanged are the following:

   - Party B pays an initial upfront and the same aforementioned coupon, the floater;
   - Party A pays 3 month Euribor and a Spread over Libor.

   This is the the hedging swap flows exchange.

From now the view we are interested in is only the second one related to the hedging swap. The reason why Party A need to enter in this contract with Party B is for hedging purpose because the structured bond as it is has several sources of risk:

1. interest rates (*DV01*);
2. price of the underlying (Δ);
3. volatility of the underlying ($\mathcal{V}$).

Finally, as the underlying stock ENEL pays also a dividend, the proper model to use is no more Black & Scholes but **Garman & Kohlhagen**, which takes into account the dividend yield factor. As *B&S*, also *G&H* model has closed form solution whenever the derivative satisfies the model assumptions, but still numerical methods are required, in particular when dealing with the embedded Asian option.

2 Pricing the upfront $X\%$

Since a contract must be favorable for both parties, we price it by setting its total NPV equal to zero, meaning that $NPV_{PartyA} = NPV_{PartyB}$. By considering the inflows only, the NPVs are composed as follows:

- $NPV_{PartyA} = X\% + NPV_{Asian}$;
- $NPV_{PartyB} = NPV_{rate} + NPV_{Spol}$.

Furthermore, from now on, when we refer to the total NPV we mean the following:

$$\begin{aligned} NPV_{tot} &= NPV_{PartyA} - NPV_{PartyB} \\ &= X\% + NPV_{Asian} - NPV_{rate} - NPV_{Spol} \end{aligned}$$

In general, NPV_{tot} refers simply to the total inflows in Party A, minus the total outflows from Party A.

The reference t_0 of the problem is Tuesday 19th February 2008. Here we show the cash flows of the contract, in detail we have the upfront $X\%$ which is paid in t_0 that is the settlement date, we have the coupon $\bar{c}$ relative to the asian option, and in the lower part we have the Euribor rate 3m plus a spread, which, for historical reasons, we will call *spread over libor*.

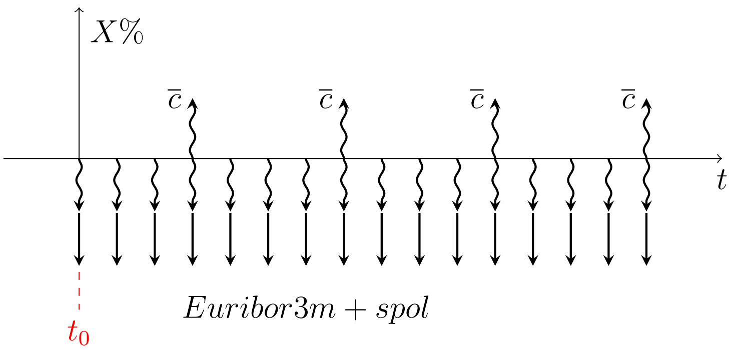

Figure 1: Cash Flows of the hedging swap

In the given contract, the NPV consists in many different parts:

- NPV_{Asian} includes all the coupon flows;
- NPV_{rate} includes all the 3 months Euribor flows;
- NPV_{Spol} includes all the spread over Libor flows;
- $X\%$ is computed as: $NPV_{rate} + NPV_{Spol} - NPV_{Asian}$

2.1 NPVs calculation

The first step is to construct the Interest Rates bootstrap curves. This is achieved by utilizing the most liquid products currently available in the market: the first three overnight deposits, the first seven futures, and the remaining swaps. By doing so, we establish precise reference points to compute risk-free rates and discount factors throughout our analysis.

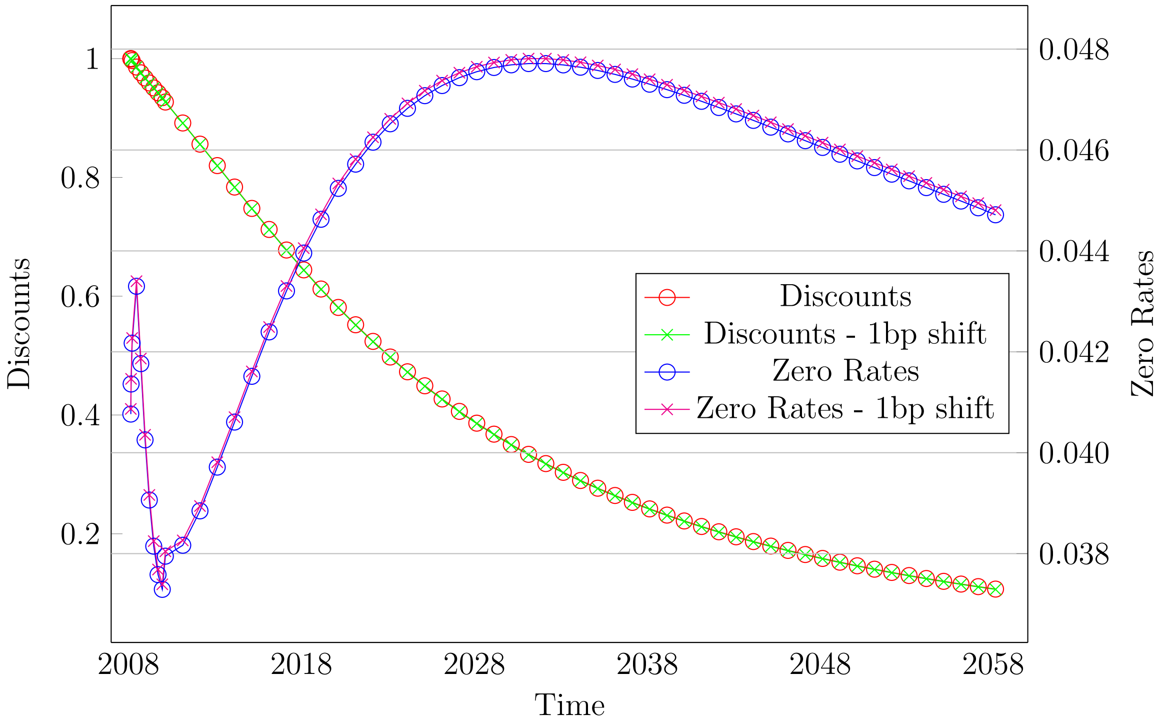

Figure 2: Original vs. Shifted bootstrap curves

2.1.1 NPV of Asian Option

Modeling the Underlying Dynamics In order to price our Asian option, we first need to model the underlying dynamics of ENEL stock. For simplicity, we assume a geometric Brownian motion for the underlying:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

Solving this stochastic differential equation (SDE) with the initial condition $S_{t_0} = S_0$, we obtain the analytical formula for the underlying asset's dynamic:

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)(t - t_0) + \sigma W_t\right)$$

This is equivalent in law to:

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)(t - t_0) + \sigma\sqrt{t - t_0}g\right)$$

where $g \overset{\text{iid}}{\sim} \mathcal{N}(0, 1)$.

To model the price of forwards, we use the *G&H* model, which is appropriate given the continuous dividend yield d. Given the interest rate r and $\mu = r - d$ the dynamic of the underlying is as follows:

$$S_t = S_0 \exp\left(\left(r - d - \frac{\sigma^2}{2}\right)(t - t_0) + \sigma\sqrt{t - t_0}g\right)$$

Given that we have four monitoring dates, we directly simulated the prices of the underlying at each date as follows:

$$S_{t_{i+1}} = S_{t_i} \exp\left(\left(z_{i,i+1} - d - \frac{\sigma^2}{2}\right)(t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i}g\right)$$

In this way at each iteration we keep the $i - th$ simulated price as starting price for the $(i + 1) - th$ simulation, with initial value the given S_{t_0}.
During this iterative process all the related time differences are computed according to the 30/360 European day count convention and the most important aspect to highlight is how the forward zero rate $z_{i,i+1}$ for the period (t_i, t_{i+1}) is retrieved. The computation of $z_{i,i+1}$ at each iteration is performed, by exploiting the already computer IR bootstrap curves, as follows:

1. interpolate the zero rates z_i and z_{i+1} from the zero rates curve;
2. compute the corresponding spot discounts factors as $B(t_0, t_i) = \exp(-z_i(t_i - t_0))$ and $B(t_0, t_{i+1}) = \exp(-z_{i+1}(t_{i+1} - t_0))$;
3. compute the corresponding forward discount $B(t_0; t_i, t_{i+1}) = \frac{B(t_0, t_{i+1})}{B(t_0, t_i)}$;
4. compute the forward zero rate as: $z_{i,i+1} = -\frac{\log(B(t_0; t_i, t_{i+1})}{t_{i+1} - t_i}$.

Remark: the actual S_{t_0} we are given is related to the closing price of ENEL on Friday 15th February 2008 but our simulation starts on the next Tuesday and we still used S_{t_0} as starting price by assuming that the effect of Monday in the price variation is negligible.

Asian Option Pricing For the Asian option pricing we used *Monte Carlo simulation* with *antithetic variates* for reducing the variability of the final Monte Carlo estimate.

Since the payment of the coupons follows the behavior of an Asian call option payoff, at each monitoring date, defined as:

$$P(t_i) = \alpha(E(t_i) - 1)^+$$

where:

$$E(t_i) = \frac{1}{i} \sum_{k=1}^{i} \frac{S(t_k)}{S(t_0)}$$

Basically, as coupons are paid annually, the magnitude of the payments depend on the average performance of ENEL up to the current monitoring date over its initial price. The NPV related to these flows are computed as follows:

$$NPV_{Asian} = \sum_{i=1}^{4} P(t_i) B(t_0, t_i)$$

2.1.2 What are the averaging steps involved during Monte Carlo?

During the Asian option pricing we need to work with a matrix M constructed as follows:

- on the rows we have all the different paths of Monte Carlo simulations;
- on the columns we have all the prices of the underlying on the monitoring dates.

When we pass the quantity $E(t_i)$ to the pay off function we are considering the proper *Asian option averaging*, meaning that we are computing the average performance for each path of Monte Carlo, so from the matrix M we obtain a column vector of payoffs.

When instead we discount back the payoffs, we are instead considering the *Monte Carlo averaging*, so from the vector of payoffs we get a scalar value.

2.1.3 NPV of Euribor rate

The NPV for the 3 months Euribor rate flows can be calculated by recognizing that receiving a floater in t_{i+1} equals receiving 1 in t_i and paying 1 in t_{i+1}. Therefore, we can use the *telescopic sum* to simplify the computation:

$$\begin{aligned} NPV_{rate} &= \sum_{i=0}^{n-1} B(t_0, t_i) - B(t_0, t_{i+1}) \\ &= B(t_0, t_0) - B(t_0, t_n) \\ &= 1 - B(t_0, t_n) \end{aligned}$$

In this case, the spot discount factor $B(t_0, t_n)$ is computed by interpolating the zero rate at t_n and considering the time difference with Act/360 convention.

2.1.4 NPV of spread over Libor

The NPV of the spread over Libor is nothing but the sum of the discounted cash flows:

$$\begin{aligned} NPV_{Spol} &= \sum_{i=1}^{n} S_{Spol} B(t_0, t_i) \\ &= S_{Spol} \sum_{i=1}^{n} B(t_0, t_i) \end{aligned}$$

Discount factors computed in the same way as for the one in NPV_{rate}.

2.2 Pricing X%

At this point we have all the needed quantities for pricing $X\%$ and to retrieve this value we just impose $NPV_{tot} = 0$:

$$NPV_{tot} = 0 \rightarrow X\% = NPV_{rate} + NPV_{Spol} - NPV_{Asian}$$

Hence, we can compute the total upfront payment from Party B as:

$$X\% \times \text{Principal Amount}$$

3 Computing the Sensitivities

Since we are not treating a derivative (exotic Asian option) that satisfies *G&H* model assumptions, we cannot compute the Greeks with any closed form solution, therefore we employed *finite differences numerical methods* in order to compute Δ and $\mathcal{V}$ of the embedded Asian option. As of *DV01*, its computation is performed by passing shifted IR bootstrap curves in the NPV_{shift} calculation.

3.1 Delta

The Delta of the portfolio is analytically $\Delta = 0$. This is because, given the nature of the payoff function, which considers $E(t_i)$ as defined above, the sensibility of the Asian option with respect to the underlying price is exactly 0 as the following quantity does not depend on $S_{t_0} = S_0$:

$$
\begin{aligned}
E(t_1) &= \frac{S(t_1)}{S(t_0)} \\
&= \frac{S_{t_0}}{S_{t_0}} \exp\left(\left(r - d - \frac{\sigma^2}{2}\right)(t_1 - t_0) + \sigma\sqrt{t_1 - t_0}g\right) \\
&= \exp\left(\left(r - d - \frac{\sigma^2}{2}\right)(t_1 - t_0) + \sigma\sqrt{t_1 - t_0}g\right)
\end{aligned}
$$

The same consideration can be made for the remaining values of $E(t_i)$.

3.1.1 Delta: a numerical perspective

During the numerical computation of Δ with finite differences methods we encountered a quite oscillatory behavior in the results when varying h.

In particular, with very small h the results of different finite differences were not converging at the same value and not even the sign of the derivative was consistent. The reason behind this was due to floating values precision, meaning that, due to numerical truncation or round-offs, the price function was highly oscillatory, so we decided to build a variational Asian option pricing function but with fixed precision to control the very subtle numerical differences. Here is a plot in the neighborhood of S_0.

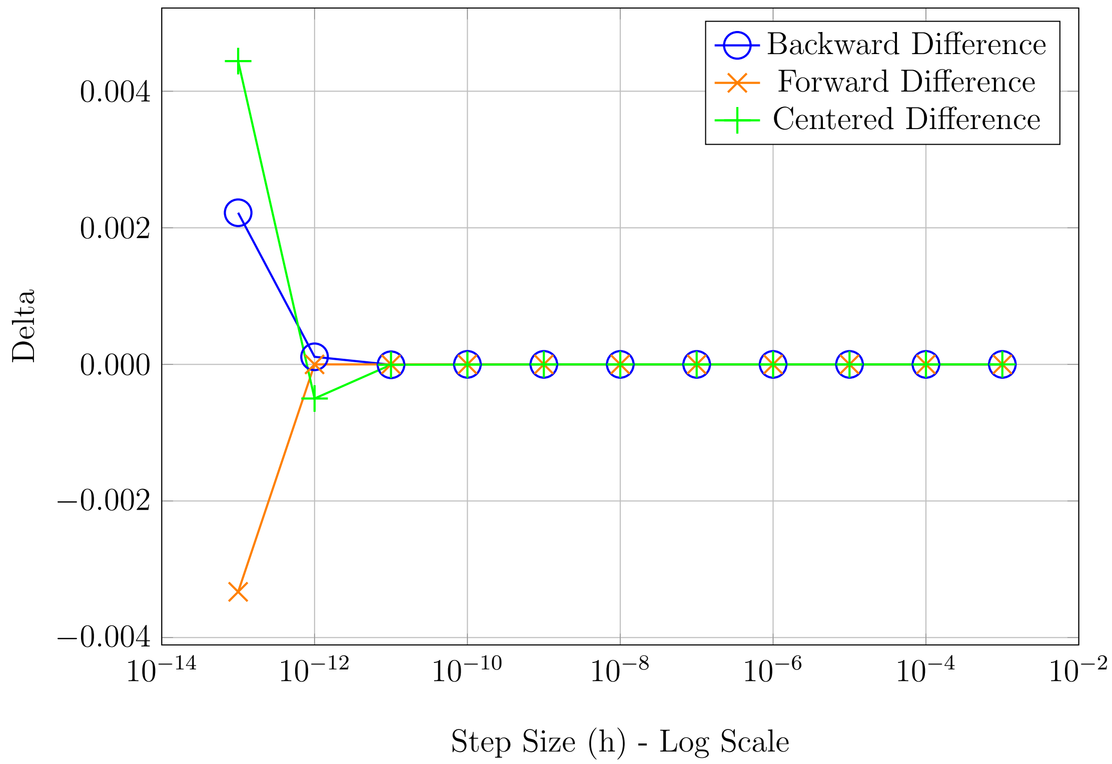

Figure 3: Oscillatory behavior of Delta

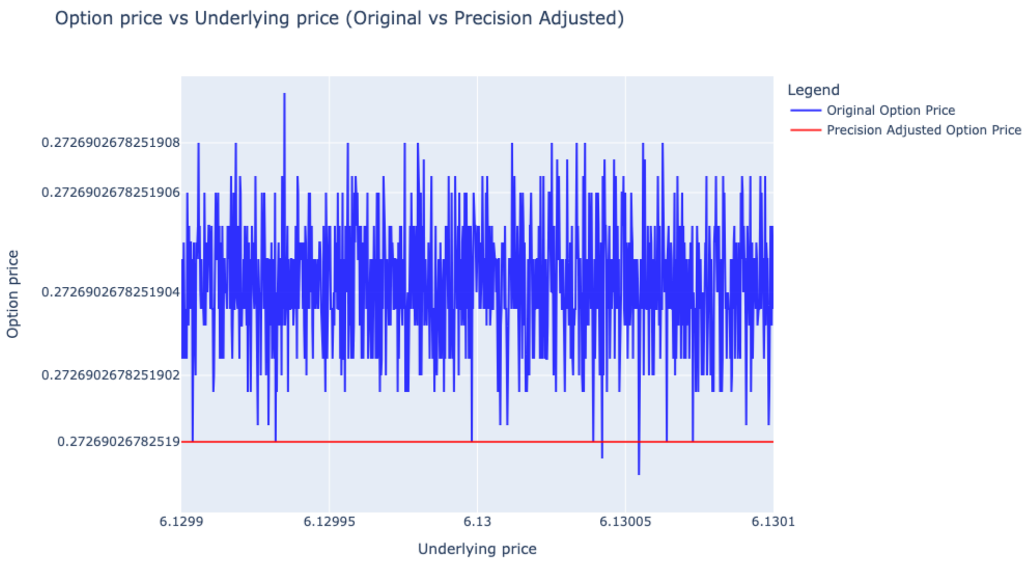

Figure 4: Original vs. Precision Adjusted Price function (price)

With the precision adjusted approach, all the Δs were numerically equal to 0.

3.2 Vega

The computation of $\mathcal{V}$ was straightforward as the price function with respect to the underlying volatility is smooth.

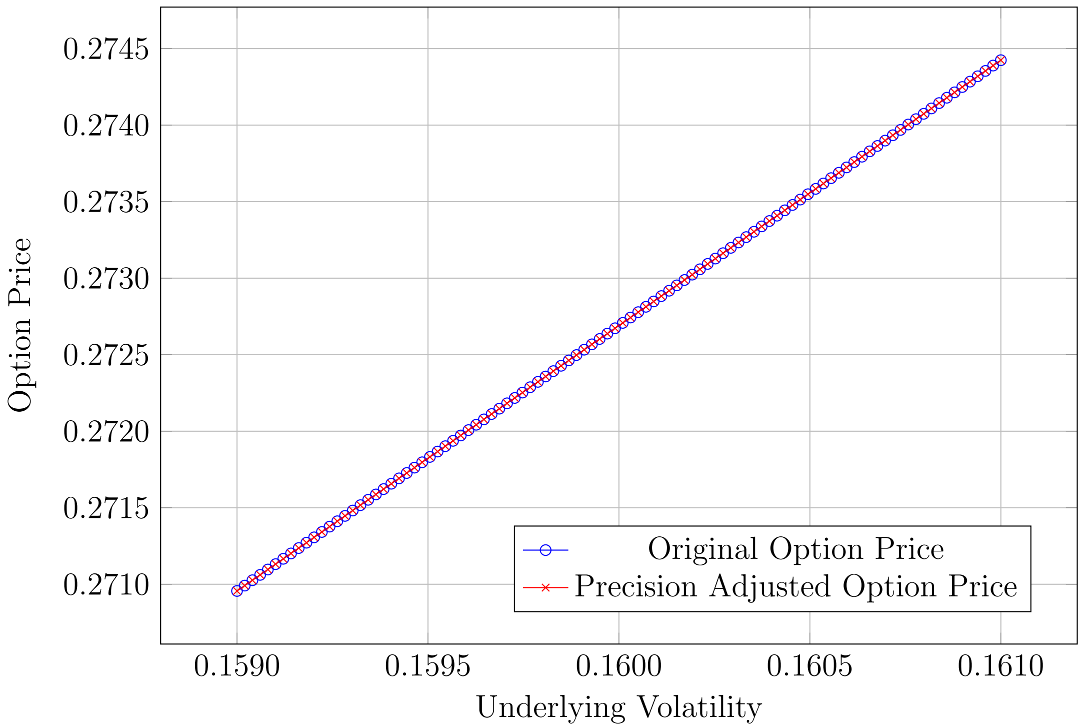

Figure 5: Original vs. Precision Adjusted Price function (volatility)

Therefore, numerical methods returned the very same value of $\mathcal{V}$ up to truncation or round-offs.

3.3 DV01

To compute the $DV01$ of the portfolio, we simply repeated the $NPVs$ computation process but using the shifted IR bootstrap curves by 1 Basis Point.

$$DV01 = NPV_{shift} - NPV$$

4 Hedging

During the hedging process it is of paramount importance to consider all the possible perturbations given by new instruments involved. In this case we are required to hedge in such a way that the resulting portfolio presents no sensitivities, that is: $\Delta = \mathcal{V} = \mathit{DV01} = 0$.

In order to achieve this, we need to use in principle at least three different instruments:

1. the underlying ENEL stock for Δ;
2. a 4 years ATM-Forward European Call on the same underlying for $\mathcal{V}$;
3. a 4 years IR swap vs. 3 months Euribor for *DV01*.

If we start by hedging Δ, we may need to readjust it after hedging $\mathcal{V}$, as hedging $\mathcal{V}$ involves entering in new option positions in our portfolio, thereby altering the total Δ. The same reasoning applies with interest rates hedging. Since we introduce a quantity of European call options in our portfolio, we must take them into account in the *DV01*.

Our hedging order is: $\mathcal{V}$, Δ and *DV01*

4.1 $\mathcal{V}$ Hedging

A 4 years ATM-Forward European Call is no more than a call option with strike price $K = F_0$, where $F_0 = S_0 \exp((r-d)T)$.

To hedge Vega, we sell European call options, which also reduces Delta. We must first compute the $\mathcal{V}_{European}$ of the European call option, and then sell an amount of European call options equal to $\frac{\mathcal{V}_{Asian}}{\mathcal{V}_{European}}$ so that the resulting portfolio has total $\mathcal{V}_{tot} = \mathcal{V}_{Asian} - \frac{\mathcal{V}_{Asian}}{\mathcal{V}_{European}} \times \mathcal{V}_{European}$ equal to zero.

In the *G&H* model, we derived the Vega of a European option as follows. Given the price of an European Call Option in presence of a dividend yield d:

$$C(0,T) = Se^{-dT}N(d_1) - Ke^{-rT}N(d_2)$$

where:

$$d_1 = \frac{\ln(S/K) + (r - d + 0.5\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

The Vega of the call option is the partial derivative of the option price with respect to the volatility σ:

$$\mathcal{V} = \frac{\partial C}{\partial \sigma}$$

Taking the derivative, we get:

$$\mathcal{V} = \frac{\partial}{\partial \sigma} \left(S e^{-dT} N(d_1) - K e^{-rT} N(d_2) \right)$$

Considering the dependence of d_1 and d_2 on σ:

$$\mathcal{V} = S e^{-dT} \frac{\partial N(d_1)}{\partial \sigma} - K e^{-rT} \frac{\partial N(d_2)}{\partial \sigma}$$

Using the chain rule again:

$$\frac{\partial N(d_1)}{\partial \sigma} = N'(d_1) \frac{\partial d_1}{\partial \sigma}$$

$$\frac{\partial N(d_2)}{\partial \sigma} = N'(d_2) \frac{\partial d_2}{\partial \sigma}$$

The derivatives of d_1 and d_2 with respect to σ are:

$$\frac{\partial d_1}{\partial \sigma} = \frac{\partial}{\partial \sigma} \left(\frac{\ln(S/K) + (r - d + 0.5\sigma^2)T}{\sigma \sqrt{T}} \right)$$

$$\frac{\partial d_2}{\partial \sigma} = \frac{\partial d_1}{\partial \sigma} - \sqrt{T}$$

Calculating $\frac{\partial d_1}{\partial \sigma}$:

$$\begin{aligned}
\frac{\partial d_1}{\partial \sigma} &= \frac{(r - d + 0.5\sigma^2)T}{\sigma \sqrt{T}} \cdot \frac{1}{\sigma} - \frac{\ln(S/K) + (r - d + 0.5\sigma^2)T}{\sigma^2 \sqrt{T}} \cdot \sigma \\
&= \frac{T(r - d + 0.5\sigma^2)}{\sigma \sqrt{T} \sigma} - \frac{\ln(S/K) + (r - d + 0.5\sigma^2)T}{\sigma^2 \sqrt{T}} \\
&= \frac{T(r - d + 0.5\sigma^2)}{\sigma^2 \sqrt{T}} - \frac{\ln(S/K) + (r - d + 0.5\sigma^2)T}{\sigma^2 \sqrt{T}} \\
&= \frac{T(r - d + 0.5\sigma^2) - (\ln(S/K) + (r - d + 0.5\sigma^2)T)}{\sigma^2 \sqrt{T}} \\
&= \frac{T(r - d + 0.5\sigma^2) - \ln(S/K) - T(r - d + 0.5\sigma^2)}{\sigma^2 \sqrt{T}}
\end{aligned}$$

$$= \frac{-\ln(S/K)}{\sigma^2\sqrt{T}}$$

Simplifying:

$$\mathcal{V} = Se^{-dT}N'(d_1)\sqrt{T}$$

Given that $N'(d_1) = \frac{1}{\sqrt{2\pi}}e^{-d_1^2/2}$:

$$\mathcal{V} = Se^{-dT}\sqrt{T}\frac{1}{\sqrt{2\pi}}e^{-d_1^2/2}$$

Therefore:

$$\mathcal{V}_{European} = Se^{-dT}\sqrt{T}N'(d_1)$$

We have a closed form solution for Vega!

4.2 Δ Hedging

Since our Asian option has $\Delta_{Asian} = 0$, we only have to hedge the effect of the European option, hence selling $\Delta_{tot} = \Delta_{Asian} - \frac{\mathcal{V}_{Asian}}{\mathcal{V}_{European}}\Delta_{European}$ ENEL shares. In the *G&H* framework, we compute delta as follows:

$$\Delta = \frac{\partial C}{\partial S}$$

Taking the derivative, we get:

$$\Delta = \frac{\partial}{\partial S}\left(Se^{-dT}N(d_1) - Ke^{-rT}N(d_2)\right)$$

Since both d_1 and d_2 depend on S, we need to use the chain rule:

$$\Delta = e^{-dT}N(d_1) + Se^{-dT}\frac{\partial N(d_1)}{\partial S} - Ke^{-rT}\frac{\partial N(d_2)}{\partial S}$$

Using the chain rule for the normal cumulative distribution function $N(\cdot)$, we get:

$$\frac{\partial N(d_1)}{\partial S} = N'(d_1)\frac{\partial d_1}{\partial S}$$
$$\frac{\partial N(d_2)}{\partial S} = N'(d_2)\frac{\partial d_2}{\partial S}$$

The derivative of d_1 with respect to S is:

$$\frac{\partial d_1}{\partial S} = \frac{1}{S\sigma\sqrt{T}}$$

And the derivative of d_2 with respect to S is:

$$\frac{\partial d_2}{\partial S} = \frac{\partial d_1}{\partial S} = \frac{1}{S\sigma\sqrt{T}}$$

Thus:

$$\Delta = e^{-dT}N(d_1) + Se^{-dT}N'(d_1)\frac{1}{S\sigma\sqrt{T}} - Ke^{-rT}N'(d_2)\frac{1}{S\sigma\sqrt{T}}$$

Simplifying, we get:

$$\Delta = e^{-dT}N(d_1) + e^{-dT}\frac{N'(d_1)}{\sigma\sqrt{T}} - Ke^{-rT}\frac{N'(d_2)}{\sigma\sqrt{T}}$$

Since $d_2 = d_1 - \sigma\sqrt{T}$ and $N'(d_1) = \frac{1}{\sqrt{2\pi}}e^{-d_1^2/2}$, we have:

$$\Delta = e^{-dT}N(d_1)$$

Therefore, the Delta of the European call option with dividend yield is:

$$\Delta_{European} = e^{-dT}N(d_1)$$

Again, we have closed form solution for Delta!

4.3 ***DV01*** Hedging

As previously stated, in order to hedge *DV01* we need to enter in a IR Swap contract. All the details related to how this swap contract is structured is on the notebook.

To effectively hedge *DV01*, we first compute the *DV01* of our hedging contract by taking into account also the European option that we introduced in our portfolio after Vega and Delta hedging. Then, we compute the *DV01* of the swap contract that we want to enter and we enter it with notional amount equal to $\frac{DV01}{DV01_{swap}} \times$ Principal Amount. This way we can effectively hedge against variations in the floating rates. Note that the notional amount of the swap is not exchanged in practice between the two parties, it's just used as a reference for the calculations of the payments.