Assignment n.2 Group n.2

Hanna Kamil, Nuttini Elena, Rigamonti Matteo, Zhou Fude

March 2024

Contents

1 About Q1

In this problem we are considering again the 2 risky asset only and minimum variance portfolio optimization.
You have two assets with the following characteristics:

- X, with expected return: $\mu_X = 0.5 \times 0.03 + 0.5 \times (-0.01) = 0.01$ and variance $\sigma_X^2 = 0.5 \times (0.03 - 0.01)^2 + 0.5 \times (-0.01 - 0.01)^2 = 0.0004$, so the standard deviation is $\sigma_X = 0.02$.
- Y, with expected return: $\mu_Y = 0.5 \times 0.05 + 0.5 \times (-0.03) = 0.01$ and variance $\sigma_Y^2 = 0.5 \times (0.05 - 0.01)^2 + 0.5 \times (-0.03 - 0.01)^2 = 0.0016$, so the standard deviation is $\sigma_Y = 0.04$.

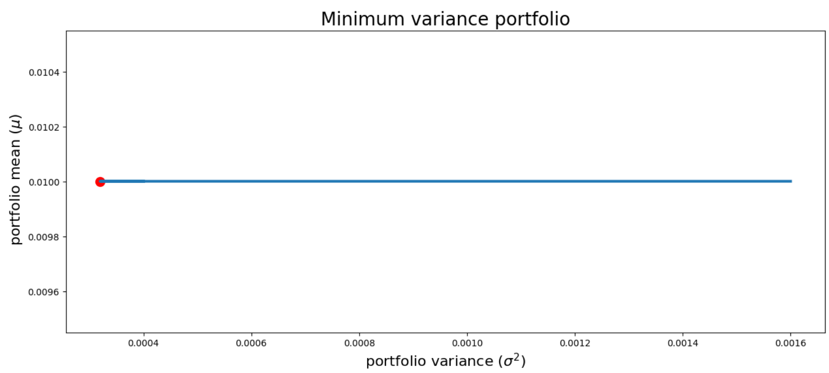
Asset	μ	σ
X	1%	2%
Y	1%	4%

Now assume that X and Y are independent random variables. The correlation between the two assets is zero.

We can solve this problem analytically given:

$$\mu(w) = w\mu_X + (1-w)\mu_Y,$$
$$\sigma(w)^2 = w^2\sigma_X^2 + (1-w)^2\sigma_Y^2 + 2w(1-w)\rho\sigma_X\sigma_Y.$$

Remark: you can already see that $\mu(w) = 0.01$ for all w. Here is the plot of the minimum variace portfolio: This is a weird shape for an efficient fron-



tier, but is consistent with the fact that, no matter the level of volatility, the combination of the two assets will always have the same expected return.

What is fundamental to highlight is that this result confirmed the importance of diversification, that is: the overall volatility of the any diversified portfolio is always lower than the least volatile asset. Indeed, the minimum variance portfolio has variance $\sigma_p^2 = 0.00032$ and $\sigma_p = 0.0179$

2 Proofs

Given the following results of the first mutual fund theorem (about Merton coefficients):

$$\begin{aligned} A &= \mathbf{1}^T \Sigma^{-1} \mu \\ B &= \mu^T \Sigma^{-1} \mu \geq 0 \\ C &= \mathbf{1}^T \Sigma^{-1} \mathbf{1} > 0 \\ D &= BC - A^2 \geq 0 \end{aligned}$$

Let's prove for B, C, and D.

First remark is that Σ is the covariance matrix, and under the assumption that no risk-free asset is involved, Σ is symmetric and positive definite.

Definition: given a matrix $S \in R^{n \times n}$ such that: $S = S^T$ and $\text{rank}(S) = n$, then S is positive definite. This means:

- all eigenvalues of S are strictly positive;
- eigenvectors of S form an orthogonal basis of R^n;
- the energy of S is strictly positive, meaning: $x^T S x > 0$ for all $x \in R^n \setminus \{\mathbf{0}\}$.

Proposition: given a spd matrix $S \in R^{n \times n}$, you can apply the **spectral theorem** to S and write it as $S = Q \Lambda Q^T$, where:

- Q is the orthogonal matrix composed of eigenvectors of S;
- Λ is the diagonal matrix of eigenvalues of S.

Remark: orthogonal matrix Q is such that $Q^T Q = I = Q Q^T$.
Proposition: given a spd matrix $S \in R^{n \times n}$, then S^{-1} is also spd.

Proof: apply the spectral theorem to S and write it as $S = Q \Lambda Q^T$. Then, $S^{-1} = (Q \Lambda Q^T)^{-1} = Q \Lambda^{-1} Q^T$. Since Λ is diagonal, then Λ^{-1} is also diagonal and has all elements strictly positive. Therefore, S^{-1} is spd.
This means that the inverse of the covariance matrix Σ^{-1} is also spd!

2.1 Proof for B

We need to prove that $B = \mu^T \Sigma^{-1} \mu \geq 0$.

As Σ^{-1} is spd, then the energy of $\mu^T \Sigma^{-1} \mu$ is strictly positive, meaning: $\mu^T \Sigma^{-1} \mu > 0$.
The only case where $\mu^T \Sigma^{-1} \mu = 0$ is when $\mu = \mathbf{0}$, meaning that the vector of expected returns is null.

2.2 Proof for C

We need to prove that $C = \mathbf{1}^T \Sigma^{-1} \mathbf{1} > 0$, but this is trivial since the energy of $\mathbf{1}^T \Sigma^{-1} \mathbf{1}$ is strictly positive.

2.3 Proof for D

In order to prove this, let's first recall the Cauchy-Schwarz inequality.

Proposition: The **Cauchy-Schwartz inequality** states that for any two vectors $\mathbf{v}$ and $\mathbf{w}$ in R^n, $|\mathbf{v}^T \mathbf{w}|^2 \leq \|\mathbf{v}\|^2 \|\mathbf{w}\|^2$.

Let's address explicitly the term $D = BC - A^2$:

$$\begin{aligned} D &= BC - A^2 \\ &= \mu^T \Sigma^{-1} \mu \cdot \mathbf{1}^T \Sigma^{-1} \mathbf{1} - (\mathbf{1}^T \Sigma^{-1} \mu)^2 \end{aligned}$$

Recall that Σ^{-1} is spd, so the Spectral Theorem applies and we can write $\Sigma^{-1} = Q \Lambda Q^T$, where Q is the orthogonal matrix composed by eigenvectors of Σ^{-1} and Λ is the diagonal matrix of eigenvalues of Σ^{-1}.
By exploiting the the aformentioned properties, you can write $\Sigma^{-1} = \Sigma^{-1/2} \Sigma^{-1/2}$, where $\Sigma^{-1/2} = Q \Lambda^{1/2} Q^T$.

Let's rewrite the term D:

$$\begin{aligned} D &= \mu^T \Sigma^{-1} \mu \cdot \mathbf{1}^T \Sigma^{-1} \mathbf{1} - (\mathbf{1}^T \Sigma^{-1} \mu)^2 \\ &= \mu^T \Sigma^{-1/2} \Sigma^{-1/2} \mu \cdot \mathbf{1}^T \Sigma^{-1/2} \Sigma^{-1/2} \mathbf{1} - (\mathbf{1}^T \Sigma^{-1/2} \Sigma^{-1/2} \mu)^2 \\ &= (\Sigma^{-1/2} \mu)^T (\Sigma^{-1/2} \mu) \cdot (\Sigma^{-1/2} \mathbf{1})^T (\Sigma^{-1/2} \mathbf{1}) - ((\Sigma^{-1/2} \mathbf{1})^T (\Sigma^{-1/2} \mu))^2 \end{aligned}$$

Let's define the following vectors:

- $\mathbf{v} = \Sigma^{-1/2} \mu$;

- $\mathbf{w} = \Sigma^{-1/2}\mathbf{1}$.

Then, the term D can be rewritten as:

$$D = \mathbf{v}^T\mathbf{v} \cdot \mathbf{w}^T\mathbf{w} - (\mathbf{w}^T\mathbf{v})^2 \geq 0$$

So:

$$\begin{aligned}(\mathbf{w}^T\mathbf{v})^2 &\leq \mathbf{v}^T\mathbf{v} \cdot \mathbf{w}^T\mathbf{w} \\ |\mathbf{w}^T\mathbf{v}|^2 &\leq \|\mathbf{v}\|^2\|\mathbf{w}\|^2\end{aligned}$$

2.4 Proof of Q3

Why $D = 0$ is equivalent to state that $\mu = \hat{\mu}\mathbf{1}$ with $\hat{\mu} \in R$ (homogeneous portfolio)?

This is a direct consequence of the Cauchy-Schwarz inequality proved in the previous section.

The term D can be rewritten as:

$$D = \mathbf{v}^T\mathbf{v} \cdot \mathbf{w}^T\mathbf{w} - (\mathbf{w}^T\mathbf{v})^2$$

If $D = 0$, then the Cauchy-Schwarz inequality becomes an equality, meaning that $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent. This means that $\mathbf{v} = \hat{\mu}\mathbf{w}$ with $\hat{\mu} \in R$. Recall that $\mathbf{v} = \Sigma^{-1/2}\mu$ and $\mathbf{w} = \Sigma^{-1/2}\mathbf{1}$. Then, $\mu = \hat{\mu}\mathbf{1}$ with $\hat{\mu} \in R$.

Let's define the new vectors in the following way:

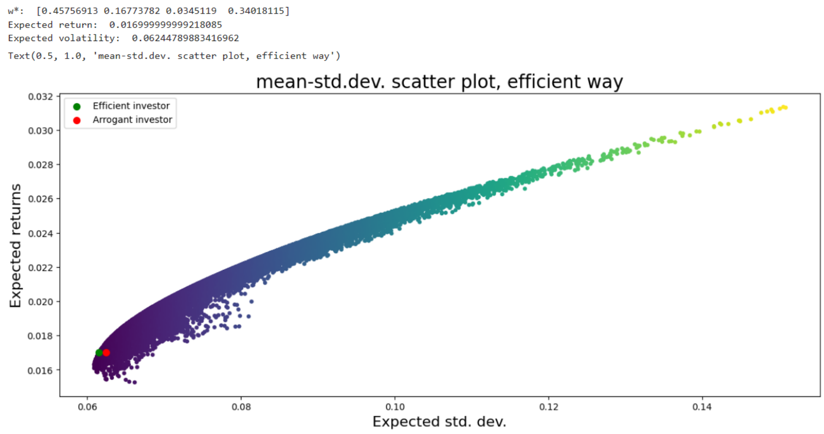
$$\begin{aligned}\mathbf{v} &= \Sigma^{-1/2}\mu = \Sigma^{-1/2}\hat{\mu}\mathbf{1} = \hat{\mu}\Sigma^{-1/2}\mathbf{1} \\ \mathbf{w} &= \Sigma^{-1/2}\mathbf{1}\end{aligned}$$

Then, the term D can be rewritten as:

$$\begin{aligned}D &= \mathbf{v}^T\mathbf{v} \cdot \mathbf{w}^T\mathbf{w} - (\mathbf{w}^T\mathbf{v})^2 \\ &= (\hat{\mu}\Sigma^{-1/2}\mathbf{1})^T(\hat{\mu}\Sigma^{-1/2}\mathbf{1}) \cdot (\Sigma^{-1/2}\mathbf{1})^T(\Sigma^{-1/2}\mathbf{1}) - ((\Sigma^{-1/2}\mathbf{1})^T(\hat{\mu}\Sigma^{-1/2}\mathbf{1}))^2 \\ &= \|\hat{\mu}\Sigma^{-1/2}\mathbf{1}\|^2 \cdot \|\Sigma^{-1/2}\mathbf{1}\|^2 - \hat{\mu}^2((\Sigma^{-1/2}\mathbf{1})^T(\Sigma^{-1/2}\mathbf{1}))^2 \\ &= \hat{\mu}^2\|\Sigma^{-1/2}\mathbf{1}\|^2 \cdot \|\Sigma^{-1/2}\mathbf{1}\|^2 - \hat{\mu}^2((\Sigma^{-1/2}\mathbf{1})^T(\Sigma^{-1/2}\mathbf{1}))^2 \\ &= \hat{\mu}^2\|\Sigma^{-1/2}\mathbf{1}\|^4 - \hat{\mu}^2\|\Sigma^{-1/2}\mathbf{1}\|^4 = 0\end{aligned}$$

3 Arrogant Investor

In the arrogant investor scenario we computed the suggested portfolio with the client constraints and we noticed that, due to these constraints, it was not lying on the efficient frontier. We decided to remove the constraints to explore some possible alternative solutions and we noticed that by removing only the constraint on the google stock we could reach a portfolio lying on the efficient frontier.



From this result we can say that it is a necessary, but not sufficient, condition for the weight vector to satisfy $\mathbf{w}^T\mathbf{1} = 1$ in order for the portfolio to lay on the efficient frontier.
Again, without the arrogant investor's constraint we were able to find an optimal portfolio which at the same level of expected return has lower volatility.

Additional plots and comments are on the notebook.

4 Efficient Frontier with risk-free asset

In order to tackle this problem, we can use the following steps:

1. Compute the efficient frontier for the risky assets only;
2. Compute the Market Portfolio (Sharpe ratio involved);
3. Compute the Capital Market Line;
4. Plot the efficient frontier and the Capital Market Line.

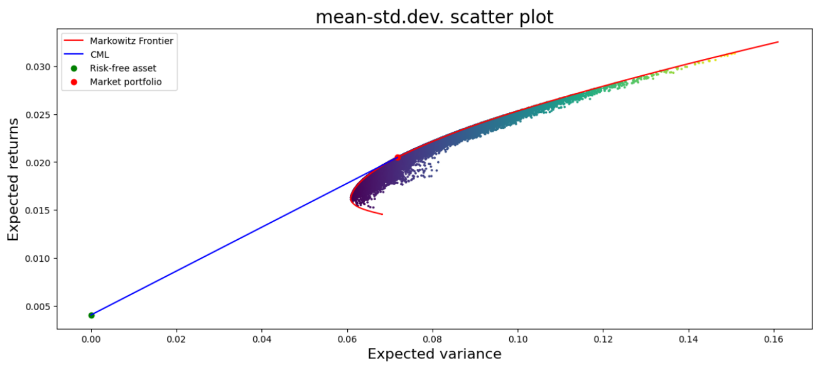
Definition: the Sharpe ratio is defined as:

$$\text{Sharpe ratio}_i = \frac{\mu_i - r_0}{\sigma_i}$$

where:

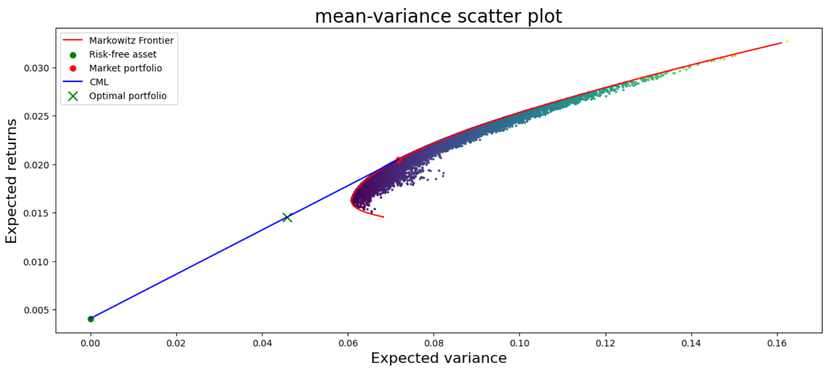
- μ_i is the expected return of the asset i;
- r_0 is the risk-free rate;
- σ_i is the standard deviation of the asset i.

In the scenario where you are given the possibility to hold also one (and only one) risk free asset, the efficient portfolios are no more all placed on the Markowitz frontier. In this frame work all the efficient portfolio have the same Sharpe ratio, that is when the Sharpe ratio achieved its maximum.
Once found the maximum Sharpe ratio, which has a unique Market portfolio on the Markowitz frontier, you can compute the line that links the risk free asset corresponding to zero volatility, and the Market portfolio. Here is the plot of Capital Market line: Now we need to find the optimal portfolio in



terms of maximum expected utility. We expected this portfolio to lay on the Capital Market line, and our result is consistent with what we were expecting.

Here is the plot:



mean-variance scatter plot
Markowitz Frontier
Risk-free asset
Market portfolio
CML
Optimal portfolio
Expected returns
Expected variance
0.030
0.025
0.020
0.015
0.010
0.005
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16

5 CAPM

In this simplified Capital Asset Pricing Model, without any risk-free asset involved, we are checking if the following holds for any asset with respect to the Market Portfolio:

$$\mu_j - r_0 = \beta_{jM} \left(\mu_M - r_0 \right).$$

The same relation can be written as:

$$\mu_j - r_0 = \alpha_j + \beta_{jM} \left(\mu_M - r_0 \right) + \epsilon$$

Under the hypothesis of normal log-returns we have a formula to estimate β which is given by:

$$\beta_i = \frac{Cov\left[X_i, X_M\right]}{Var\left[X_M\right]} = \frac{\sigma_{iM}}{\sigma_{MM}},$$

The goal here is to verify if the linear regression lines return the same β_i as those computed with the latter formula.

So, for each asset j we are fitting the following:

$$\mu_j = \alpha_j + \beta_{jM} \mu_M$$

where:

- α_j is the Jensen's alpha and the intercept of the regression line. This parameter is the measure of how much each asset, or portfolio, is outperforming the expected return of the market. "Big" alphas are what hedge funds are looking for;
- β_{jM} is the slope of the regression line, so its meaning is strictly related to how much the $j - th$ asset is changing along with a variation of the Market Portfolio.

In order to fit these regression lines we first addressed the problem with a "hand-made" linear algebra solution, that is by solving many systems of normal equations: $A^T A \mathbf{x_j} = A \mathbf{b_j}$, where A is the enhanced matrix composed by a column of ones and the log returns of the market portfolio, $\mathbf{b_j}$ is the vector containing the log returns of each single $j - th$ asset and $\mathbf{x_j}$ is the unknown vector for each pair of (α_j, β_j) we are looking for.
Then we tried also the build-in *OLS* method from *statsmodel* library.

Both ways lead to the same results are the ones computed with the formula.
More details and commentts are on the notebook