Assignment n.3 Group n.2

Hanna Kamil, Nuttini Elena, Rigamonti Matteo, Zhou Fude

March 2024

Contents

1 Bootstrap

We are provided with a dataset containing Interbank Market data as of February 15th, 2008 at 11:15 C.E.T. The dataset includes bid and ask spot rates for deposits with different maturities, bid and ask prices for STIR futures contracts and bid and ask quotes for plain vanilla Interest Rate Swaps (IRS) against 3-month Euribor for maturities ranging from 1 year to 50 years. The goal is to develop a Python algorithm that selects the most suitable product from the provided dataset and utilizes it to construct the spot discount factor curve and subsequently derive the zero rates curve.

In order to properly tackle this rates bootstrap problem, it is of paramount importance to take into account some fundamental concepts:

- **Settlement day**: usually $T+2$ (two business days after the trade date T). This is the day when the buyer must pay for the securities and the seller must deliver them. The settlement day is the day when the transaction is considered complete and the security officially changes hands.
- **Year fraction** (conventions): there exist different conventions to calculate the year fraction between two dates according to the particular contract. Some of them are: 30/360, ACT/360, ACT/365, etc. It's important to know the convention used in the contract to calculate the year fraction correctly.

In particular, in this problem:

- T: 2008-02-15 is a Friday.
- $T + 2$: 2008-02-19 is a Tuesday. This is the reference t_0.

The problem starts at 11:15 a.m. on T (2008-02-15), so Euribor rates are fixed and available.

1.1 IB Deposits

In the proposed dataset, 8 Inter Bank (IB) deposits are given, with maturities from overnight to 1 year. The Euribor rates for these maturities are given.
At the beginning of the resolution process we computed the spot discount factors $B(t_0, t_i)$. To address this, we firstly computed the following information about IB deposits:

- $\delta(t_0, t_i)$: year fraction between t_0 and t_i, computed according to the ACT/360 convention.
- $L(t_0, t_i)$: the Euribor rate for the period $[t_0, t_i]$, mid BID-ASK value.

The spot discount factors are computed as:

$$B(t_0, t_i) = \frac{1}{1 + \delta(t_0, t_i)L(t_0, t_i)}$$

Remark: only the first three are considered in the bootstrap process.

1.2 STIR Futures

9 Short Term Interest Rate (STIR) Futures are given. STIR Futures are settled at International Money Market (IMM) dates, which are the third Wednesday of March, June, September, and December and they are 3 months fixed duration. The price of each STIR Future is given by (price at expiry date t_{i+1}):

$$P(t_{i+1}) = 100 - L(t_i; t_i, t_{i+1})$$

where $L(t_i; t_i, t_{i+1})$ is the Euribor rate for the period $[t_i, t_{i+1}]$. BID and ASK prices are these prices plus/minus a spread.

For each STIR Future, t_i is settlement date (wednesday) and t_{i+1} is the corresponding expiry date. Unless $i = 0$, the quantity $L(t_i; t_i, t_{i+1})$ is not known at t_0, hence a proper approximation known at t_0 is used:

$$L(t_i; t_i, t_{i+1}) \approx L(t_0; t_i, t_{i+1})$$

So the price at expiry date t_{i+1} is:

$$P(t_{i+1}) \approx 100 - L(t_0; t_i, t_{i+1})$$

In STIR Future constracts, Forward Rate Agreements (FRA) are used to hedge the risk of interest rate changes. The FRA is a contract between two parties to exchange a fixed interest rate for a floating interest rate. The FRA is settled at the beginning of the period, so the FRA rate is known at t_0.

For each STIR Future, the FRA rate is given by:

$$FRA(t_i, t_{i+1}) = L(t_0; t_i, t_{i+1})$$

Along with the FRAs, the Forward discounts are defined as:

$$B(t_0; t_i, t_{i+1}) = \frac{1}{1 + \delta(t_i, t_{i+1})L(t_0; t_i, t_{i+1})} = \frac{B(t_0, t_{i+1})}{B(t_0, t_i)}$$

In order to compute the spot discount factors $B(t_0, t_{i+1})$, the following information can be retrieved:

- $\delta(t_i, t_{i+1})$: year fraction between t_i and t_{i+1}, computed according to the ACT/360 convention.
- $L(t_0; t_i, t_{i+1})$: the FRA for the period $[t_i, t_{i+1}]$.
- $B(t_0; t_i, t_{i+1})$: the forward discount for the period $[t_i, t_{i+1}]$.

Then, it is necessary to have a the first value of $B(t_0, t_1)$ to compute the rest of the spot discount factors iteratively by exploiting the FRA/forward discount relation. This first value is often interpolated, but in this specific case the last IB deposit expires the exact day of the first STIR Future settlement date, 2008-03-19, so we exploited this!

Note that the interpolation is done for each STIR Future, except for the first one, in order to compute each $B(t_0, t_i)$ iteratively, so you can compute all the spot discount factors $B(t_0, t_{i+1})$. The interpolation function used is the custom *getRatesLinInterpDiscount()* function, with ACT/365 convention.

Remark: both interpolation and extrapolation are performed according to the dates.
Remark: only the first seven are considered in the bootstrap process.

1.3 IR Swaps

Eventually, 50 Interest Rate (IR) Swaps are given (NO forward start, so vanilla IR Swaps). An IR Swap is a contract between two parties to exchange a fixed interest rate $S(t_0, t_i)$ for a floating interest rate. The IR Swap is settled at the beginning of the period, so the fixed rate is known at t_0. The floating rate is the Euribor rate for the period $[t_i, t_{i+1}]$.

For each IR Swap, the fixed rate is given by:

$$S(t_0, t_i) = \frac{1 - B(t_0, t_i)}{BPV_0(t_i)}$$

BID and ASK prices are these rates plus/minus a spread.
$BPV_0(t_i)$ is the Basis Point Value at t_0 for the period $[t_0, t_i]$.

The Basis Point Value is computed as:

$$\begin{aligned}
BPV_0(t_i) &= \sum_{n=1}^{i} \delta(t_{n-1}, t_n) B(t_0, t_n) \\
&= \sum_{n=1}^{i-1} \delta(t_{n-1}, t_n) B(t_0, t_n) + \delta(t_{i-1}, t_i) B(t_0, t_i) \\
&= BPV_0(t_{i-1}) + \delta(t_{i-1}, t_i) B(t_0, t_i)
\end{aligned}$$

In order to compute the spot discount factors $B(t_0, t_i)$, the following information can be retrieved:

- $\delta(t_{i-1}, t_i)$: year fraction between t_{i-1} and t_i, computed according to the 30/360 convention from the fixed leg of the swap.
- $S(t_0, t_i)$: the fixed rate is given, mid BID-ASK value.

Then, it is necessary to have the first value of $B(t_0, t_1)$ to compute the rest of the spot discount factors iteratively by exploiting the Basis Point Value formula. As for each $i = 2, \dots, 50$:

$$\begin{aligned}
S(t_0, t_i) &= \frac{1 - B(t_0, t_i)}{BPV_0(t_i)} \\
&= \frac{1 - B(t_0, t_i)}{BPV_0(t_{i-1}) + \delta(t_{i-1}, t_i) B(t_0, t_i)}
\end{aligned}$$

each $B(t_0, t_i)$ can be computed iteratively as:

$$B(t_0, t_i) = \frac{1 - S(t_0, t_i) BPV_0(t_{i-1})}{1 + S(t_0, t_i) \delta(t_{i-1}, t_i)}$$

In order to initiate, you need to compute the first value of $B(t_0, t_1)$ by linear interpolation using the discounts computed previously. The first Swap starts on 2009-02-18, so before the last considered STIR Future. The interpolation function used is the custom *getRatesLinInterpDiscount()* function, with ATC/365 convention.

Remark: first swap is used only as a reference to compute the first value of $B(t_0, t_1)$, then the spot discount factors are computed iteratively for the second swap and so on. The second swap is after the last considered STIR Future.

1.4 Bootstrap

The bootstrap process is performed by considering financial contracts from the most liquid to the least liquid. The process is iterative and the spot discount factors are computed by exploiting the information of the previous contracts. The liquidity of the contracts is determined by the volume of contracts traded in the market. The most liquid contracts are the ones with the highest trading volume. Furthermore, liquidity can be also measured by the bid-ask spread, the narrower the spread, the more liquid the contract.

Once we compute all the spot discount factors, the zero rates can be computed as:

$$z(t_0, t_i) = -\frac{\log B(t_0, t_i)}{\delta(t_0, t_i)}$$

time intervals for the zero rates are computed according to the ACT/365 convention. The zero rates give are the rate at which the following holds:

$$e^{z(t_0,t_i)\delta(t_0,t_i)} = \frac{1}{B(t_0, t_i)}$$

Meaning that in order to achieve the discounted value at t_0 equal to the cash flow at t_i (face value at t_i), one must invest at the rate $z(t_0, t_i)$ in t_0.
This is due to :

$$\text{Present Value} = \text{Future Value} \times B(t_0, t_i) = \text{Future Value} \times e^{-z(t_0,t_i)\delta(t_0,t_i)}$$

so in order to get the same Present Value = Present Value$_0$:

$$\text{Future Value} = \text{Present Value}_0 \times \frac{1}{B(t_0, t_i)} = \text{Present Value}_0 \times e^{z(t_0,t_i)\delta(t_0,t_i)}$$

$$\begin{aligned}
\text{Present Value} &= \text{Future Value} \times B(t_0, t_i) \\
&= \text{Present Value}_0 \times \frac{1}{B(t_0, t_i)} \times B(t_0, t_i) \\
&= \text{Future Value} \times e^{-z(t_0,t_i)\delta(t_0,t_i)} \\
&= \text{Present Value}_0 \times e^{z(t_0,t_i)\delta(t_0,t_i)} \times e^{-z(t_0,t_i)\delta(t_0,t_i)}
\end{aligned}$$

This holds true under the assumption that compounding is continuous.

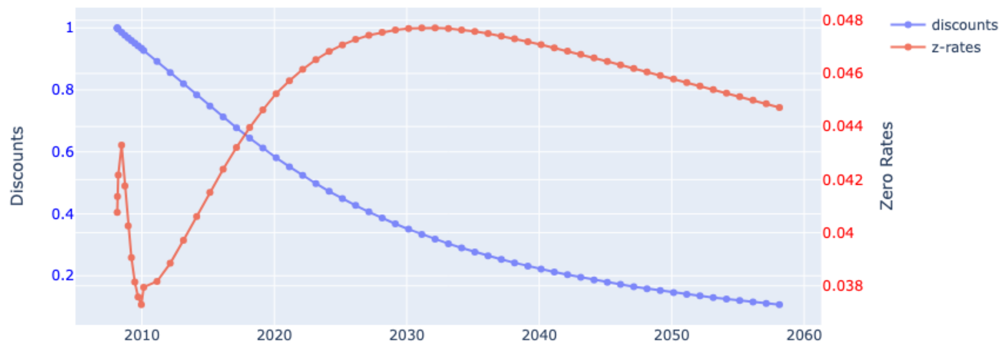

Figure 1: Bootstrap yield curve (Initial discounts & zero rates)

Let's give some interpretation to the curves we got.

The behaviour of the discounts' curve is intuitive and consistent with the common and the financial sense and it follows the rule of the *time value of money*: one unit of currency today is worth more than the same one unit tomorrow, therefore the curve has a monotonically decreasing behaviour.

As of the zero rates' curve, the first thing to say is that it's behaviour is strictly related to the particular financial situation you are bootstrapping these rates from. In this case, the 2008 financial crisis is depicted. Furthermore, you can spot the fact that the whole curve is actually a piece-wise curve and the behaviour of each piece depends on the particular security you are considering. We observed an upward slope in the zero rates curve during the initial section, which means that as the maturity of IB deposits increases, the credit risk increases as well. Subsequently, the zero rates curve exhibits a downward steepening, driven by an incentive to purchase futures. This phenomenon is commonplace following central bank decisions to lower interest rates to encourage consumption. In the final section, the curve slopes upwards over time. This pattern is attributed to the regime behavior of zero rates retrieved from IRSs up to 50 years maturity, the further in time you are, the more credit risk you have, therefore, the higher the risk-free rate will be.

1.5 Inconsistencies

From the data and the code that we were given we were able to spot some inconsistencies:

- The bid and ask value of a deposit were inverted
- The bid value of a swap wasn't correctly reported and had no decimal values
- The year fraction conventions were wrong for most products: futures need to be Act/360, zero rates Act/365.

2 Extra - Macaulay Duration

2.1 Quick Theory Recap

To validate the colleague's proposal and calculate the Macaulay Duration of the portfolio, we first need to determine whether the two bonds are traded at par. A bond is traded at par when its market price is equal to its face value.

To calculate the price of a bond, we can use the formula:

$$P = \sum_{i=1}^{n} \frac{C}{(1+r)^i} + \frac{F}{(1+r)^n}$$

where:

P is the price of the bond,
C is the coupon payment,
r is the yield to maturity (YTM) or market rate,
F is the face value of the bond,
n is the number of periods.

Once we have the prices of both bonds, we can calculate the weighted average Macaulay Duration of the portfolio using the formula:

$$\text{MacD}(t_0) = \frac{\sum_{i=1}^{n} C_i \cdot (t_i - t_0) \cdot B(t_0, t_i)}{\sum_{i=1}^{n} C_i \cdot B(t_0, t_i)}$$

where:

C_i is the cash flow (coupon payment) at time t_i,
t_0 is the current time,
$B(t_0, t_i)$ is the discount from time t_0 to t_i.

2.2 Results from Notebook

From the notebook, assuming a face value of 1000, we calculate the bond prices with the given formula and find that both bonds are traded above par.

```
Is the 5Y bond traded at par? False
5Y bond has price 1080.9077481742243 and face value 1000
Is the 10Y bond traded at par? False
10Y bond has price 1176.027476615829 and face value 1000
```

Figure 2: Bond Prices: Traded Above Par

Next, we compute the Macaulay Duration for both bonds:

```
5-Year Bond:
Macaulay Duration: 4.642657855365286
Modified Duration: 4.537833892449698

10-Year Bond:
Macaulay Duration: 8.444165334814443
Modified Duration: 8.243840022273204
```

Figure 3: Macaulay Duration for Bonds

Finally, we calculate the weighted Macaulay duration for the portfolio:

```
Weighted average Macaulay Duration of the portfolio: 6.543411595089864
```

Figure 4: Weighted Macaulay Duration for Portfolio

Given the current market rates, the colleague's portfolio proposal does not achieve the desired Macaulay Duration of 8.

To address this, we can implement a function that optimizes the portfolio's Macaulay duration to approach 8 by adjusting the bond weights. In the notebook we used *scipy.optimize.minimize* to achieve this.

```
Optimized Weights: [0.11683927 0.88316073]
Weighted Macaulay Duration: 7.999999994270893
```

Figure 5: Optimized Macaulay Duration

And we can see that we found the ideal weights of the portfolio to achieve a total Macaulay Duration of 8.