THE UNIVERSITY OF ILLINOIS
Department of Statistics
STATISTICS 556
Advanced Time Series Analysis
Fall 2024

Homework 3 (Due: Oct 9th, 6pm)

1. For the standard Brownian motion $W(t), t \in [0,1]$, derive (a) $cov(W(s), W(t))$; (b) $cov(W(s)^2, W(t)^2)$ and (c) $cov(W(s)^2, W(t))$ for any $0 \le s < t \le 1$.

2. Suppose that η_t is a strictly stationary time series with finite fourth moment. Let $S_T(r) = T^{-1/2}\eta_{\lfloor Tr \rfloor}$ Show that $\sup_{r\in[0,1]} |S_T(r)| \to_p 0$.

3. Let $u_t \sim iid(0,1)$ and $E(|u_t|^{2+\delta}) < \infty$ for some $\delta > 0$. Define $X_T(r) = \frac{1}{T}\sum_{t=1}^{[Tr]} u_t$, and
$$X_T^*(r) = \frac{1}{T}\left(\sum_{t=1}^{\lfloor Tr \rfloor} u_t + (Tr - \lfloor Tr \rfloor)u_{\lfloor Tr \rfloor + 1}\right), \quad r \in [0,1].$$
Show that $\sup_{r\in[0,1]} \sqrt{T}|X_T(r) - X_T^*(r)| = o_p(1)$.

4. The limiting null distribution for the Dickey-Fuller test statistic $T(\hat{\rho}_T - 1)$ is
$$G = \frac{(1/2)\{W(1)^2 - 1\}}{\int_0^1 W(r)^2 dr},$$
where $W(r), r \in [0,1]$ is standard Brownian motion. To understand whether the numerator and denominator is independent, derive (a) $cov(W(1), \int_0^1 W(r)^2 dr)$. Furthermore, derive (b), $cov(W(1), \int_0^1 W(r)dr)$ and (c), $cov(W(1), \int_0^1 \tilde{W}(r)^2 dr)$, where $\tilde{W}(r) = W(r) - rW(1)$.

5. Simulating the distribution of G and H mentioned in class. (H is the limiting null distribution of t statistic for unit root testing). As mentioned in class, we can use the standardized partial sum of iid N(0,1) variables to approximate the standard Brownian motion. We can set time series length $T = 10^4$ and do $M = 10^4$ replications to get $G^{(i)},\ i = 1, \cdots, M$, where $G^{(i)}$ is approximately distributed as G. Plot the histogram based on $[G^{(i)}]_{i=1}^M$. What is the 5% quantile of G based on your simulation? Do the same for H.

6. AoFTS 2.11 (Please conduct ADF test to test for unit root. You can assume there is no time trend or intercept in ADF test, i.e., $C_t = 0$.)

7. AoFTS 2.12

1. For the standard Brownian motion $W(t), t \in [0,1]$, derive (a) $cov(W(s), W(t))$; (b) $cov(W(s)^2, W(t)^2)$ and (c) $cov(W(s)^2, W(t))$ for any $0 \le s < t \le 1$.

a. $cov(W(s), W(t))$ $\quad (E[W(s)] = 0)$

$$= E[W(s) \times (W(s) + (W(t) - W(s)))]$$

(by definition of brownian motion)

$$= E[W^2(s) + \cancel{W(s)(W(t) - W(s))}] \to 0$$

$$= s$$

b. $cov(W^2(s), W^2(t))$ $\quad (E[W^2(s)] = s)$

$$= E[(W^2(s) - s)(W^2(t) - t)]$$

$$= E[W^2(s) W^2(t)] - t\cancel{E[W^2(s)]}^{\to s} \cancel{- sE[W^2(t)]}^{\to t}$$

$$\cancel{+ st}$$

$$= E[W^2(s) W^2(t)] - st$$

$$= E\left[W^2(s)\left(W(s) + (W(t) - W(s))\right)^2\right] - st$$

$$= E\left[W^2(s)\left(W(s)^2 + 2W(s)(W(t) - W(s)) + (W(t) - W(s))^2\right)\right] - st$$

$$= E[W^4(s)] + 2E\left[W^3(s)(W(t) - W(s))\right]$$

$$+ E\left[W^2(s)(W(t) - W(s))^2\right] - st$$

$$\because \quad E[W^4(s)] = E[\sigma^4_{W(s)} X^4] \; ; \; X \sim N(0,1)$$

$$= s^2 \cancelto{3}{E[X^4]} = 3s^2$$

$$\therefore \; Cov(W^2(s), W^2(t)) = 3s^2 + 2\overbrace{\cancelto{0}{E[W^3(s)] \times E[W(t) - W(s)]}}^{\text{independent}}$$

$$+ \overbrace{\cancelto{s}{E[W^2(s)]}\; \cancelto{(t-s)}{E[(W(t) - W(s))^2]}}^{\text{independent}}$$

$$- st$$

$$= 3s^2 + s(\cancel{t}-s) - \cancel{st} = \boxed{2s^2}.$$

$$\boxed{\mathrm{Cov}(W^2(s), W(t))} = E[(W^2(s) - s)\, W(t)]$$

$$= E[W^2(s) W(t)] - s\,\cancel{E[W(t)]} \to 0.$$

$$= E[W^2(s)(W(s) + (W(t) - W(s)))]$$

$$= \cancel{E[W^3(s)]} \to 0 + E[W^2(s)(W(t) - W(s))]$$

independent.

$$= E[W^2(s)] \times \cancel{E[\underbrace{W(t) - W(s)}_{\sim N(0, (t-s)^2)}]} \to 0.$$

$$\boxed{= 0.}$$

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2. Suppose that η_t is a strictly stationary time series with finite fourth moment. Let $S_T(r) = T^{-1/2}\eta_{\lfloor Tr \rfloor}$ Show that $\sup_{r\in[0,1]} |S_T(r)| \to_p 0$.

According to Markov's inequality,

$$P\Big(\sup_{r\in[0,1]} |S_T(r)| \geq \varepsilon\Big) \leq \frac{E[\max[|S_T(r)|]]}{\varepsilon}$$ for a small ε.

$$= E\left[\max\left(\frac{|\eta_{\lfloor Tr \rfloor}|}{\sqrt{T}\,\varepsilon}\right)\right]$$

$$= \frac{E[\max(|\eta_{\lfloor Tr \rfloor}|)]}{\sqrt{T}\,\varepsilon}.$$

finite

As $T \to \infty$ then $P(|S_T(r)| \geq \varepsilon) = 0$

3. Let $u_t \sim iid(0,1)$ and $E(|u_t|^{2+\delta}) < \infty$ for some $\delta > 0$. Define $X_T(r) = \frac{1}{T}\sum_{t=1}^{[Tr]} u_t$, and

$$X_T^*(r) = \frac{1}{T}\left(\sum_{t=1}^{\lfloor Tr \rfloor} u_t + (Tr - \lfloor Tr \rfloor)u_{\lfloor Tr \rfloor + 1}\right), \quad r \in [0,1].$$

Show that $\sup_{r\in[0,1]} \sqrt{T}|X_T(r) - X_T^*(r)| = o_p(1)$.

$$\because \left| X_T(r) - X_T^*(r) \right|$$

$$= \left| \frac{(Tr - \lfloor Tr \rfloor)}{T} u_{\lfloor Tr \rfloor + 1} \right|$$

$$= \left(\frac{Tr - \lfloor Tr \rfloor}{T}\right) \left| u_{\lfloor Tr \rfloor + 1} \right|$$

As $Tr - \lfloor Tr \rfloor < 1$,

$$\sup_{r\in[0,1]} \left[\sqrt{T} \left| X_T^*(r) - X_T(r) \right| \right] \le \frac{1}{\sqrt{T}} \sup_{t=1,2,\ldots,T} |u_t| .$$

; ε is a small number.

Let $A_k = \left\{ \sqrt{T} \left| x_T^*(r) - x_T(r) \right| \geq \varepsilon \right\}$; $k=1, 2, \ldots, n$

$\therefore P\left(\sup_{k=0}^{N} \left(\sqrt{T} \left| x_T^*(r_k) - x_T(r_k) \right| \right) > \varepsilon \right)$

$$\leq \sum_{k=0}^{N} P(A_k)$$

$\therefore$ From Markov's inequality,

$$P(A_k) = P\left(\sqrt{T} \left| x_T(r_k) - x_T^*(r_k) \right| \geq \varepsilon \right)$$

$$\leq \frac{E[|u_t|^p]}{\varepsilon^p T^{p/2}}$$

finite

; $p \geq 2$

$\therefore \sum_{k=0}^{N} P(A_k) \leq \frac{N}{T^{(p/2)}}$ → constant.

So, as $T \to \infty$ then $\sum_{k=0}^{N} P(A_k) = O(1)$

$\therefore P\left(\sup_{r\in[0,1]} \left(\sqrt{T}\,|X_T(r) - X_T^{*}(r)|\right) \geq \varepsilon \right) \to 0$ as $T \to \infty$

$\left(\sup_{r\in[0,1]} \left(\sqrt{T}\,|X_T(r) - X_T^{*}(r)|\right) = O_p(1) \right)$ ✗R

4. The limiting null distribution for the Dickey-Fuller test statistic $T(\hat{\rho}_T - 1)$ is

$$G = \frac{(1/2)\{W(1)^2 - 1\}}{\int_0^1 W(r)^2 dr},$$

where $W(r), r \in [0,1]$ is standard Brownian motion. To understand whether the numerator and denominator is independent, derive (a) $cov(W(1), \int_0^1 W(r)^2 dr)$. Furthermore, derive (b), $cov(W(1), \int_0^1 W(r)dr)$ and (c), $cov(W(1), \int_0^1 \tilde{W}(r)^2 dr)$, where $\tilde{W}(r) = W(r) - rW(1)$.

a. $cov\left(W(1), \int_0^1 W^2(r)\,dr\right)$

$$= E\left[W(1)\int_0^1 W^2(r)\,dr\right] - \cancel{E[W(1)]}^{\,0} \times E\left[\int_0^1 W^2(r)\,dr\right]$$

$$= E\left[\int_0^1 W^2(r) \times W(1)\,dr\right]$$

$$= \int_0^1 E\left[W^2(r)\,W(1)\right] dr$$

$$= \int_0^1 E\left[W^2(r)\left(W(r) + (W(1) - W(r))\right)\right] dr$$

$$= \int_0^1 \cancel{E[W^2(r)]}^{0} + E[\underbrace{W^2(r)(W(1)-W(r))}_{\text{independent}}]\,dr$$

$$= \int_0^1 E[W^2(r)] \times \cancel{E[W(1)-W(r)]}^{0}\,dr$$

$$= 0$$

b. $\mathrm{Cov}\left(W(1), \int_0^1 W(r)\,dr\right)$

$$= E\left[W(1)\int_0^1 W(r)\,dr\right] - \cancel{E[W(1)]}^{0}\, E\left[\int_0^1 W(r)\,dr\right]$$

$$= \int_0^1 E[W(1)W(r)]\,dr$$

$$= \int_0^1 E[W(r)(W(r)+(W(1)-W(r)))]\,dr$$

(from 9.1)

$$= \int_0^1 E[w^2(r)] + \cancel{E[w(r)(w(1) - w(r))]}^{\,0}\, dr$$

$$= \int_0^1 r\,dr = \frac{r^2}{2}\Big|_0^1 = \frac{1}{2}$$

c. $cov\left(w(1), \int_0^1 (w(r) - r w(1))^2 dr\right)$

$$= E\left[w(1) \int_0^1 (w(r) - r(w(1)))^2 dr\right] - 0$$ (similar to the previous question)

$$= E\left[\int_0^1 (w^2(r) - 2r w(r) w(1) + w^2(1)) w(1)\, dr\right]$$ $(E[w^3(1)] = 0)$

0 (from 9.4.a) 0

$$= E\left[\int_0^1 (w(1) w^2(r) - 2r w(r) w^2(1) + w^3(1))\, dr\right]$$

$$= -\int_0^1 E\left[2rW(r)\left(W(r) + \left(W(1) - W(r)\right)\right)^2\right] dr$$

$$= -2\int_0^1 r E\left[\cancel{W^3(r)} + \cancel{2W^2(r)\left(W(1) - W(r)\right)} + \cancel{W(r)\left(W(1) - W(r)\right)^2}\right] dr$$

$0 \quad (E[W^3(r)] = 0)$

$0 \quad (E[W(1) - W(r)] = 0)$

$0 \quad (E[W(r)] = 0)$.

$$= 0.$$

5. Simulating the distribution of G and H mentioned in class. (H is the limiting null distribution of t statistic for unit root testing). As mentioned in class, we can use the standardized partial sum of iid N(0,1) variables to approximate the standard Brownian motion. We can set time series length $T = 10^4$ and do $M = 10^4$ replications to get $G^{(i)}$, $i = 1, \cdots, M$, where $G^{(i)}$ is approximately distributed as G. Plot the histogram based on $[G^{(i)}]_{i=1}^{M}$. What is the 5% quantile of G based on your simulation? Do the same for H.

$$G = \frac{(1/2)\{W(1)^2 - 1\}}{\int_0^1 W(r)^2 dr},$$

$$H = \frac{(1/2)\{W(1)^2 - 1\}}{\sqrt{\int_0^1 W(r)^2 dr}},$$

See Q5_Monte_Carlo_Simulation file.

6. AoFTS 2.11 (Please conduct ADF test to test for unit root. You can assume there is no time trend or intercept in ADF test, i.e., $C_t = 0$.)

2.11. Consider the monthly Aaa bond yields of the prior problem. Build a time series model for the series.

See Q6_ADF_TEST file.

2.12. Again, consider the two bond yield series, that is, Aaa and Baa. What is the relationship between the two series? To answer this question, build a time series model using yields of Aaa bonds as the dependent variable and yields of Baa bonds as independent variable.

Q7_ JOINT_ESTIMATION_AND_

REGRESSION. file.