THE UNIVERSITY OF ILLINOIS
Department of Statistics
STATISTICS 556
Advanced Time Series Analysis
Fall 2024

Homework 1 (Due: Sept 11, 12pm noon)

1. Let $a_t \sim iid\ N(0,1)$ (that is, a sequence of independent and identically distributed variables with marginal distribution being standard normal). Let $e_t = a_t a_{t-1}$. Show that e_t is a white noise sequence but not iid.

2. Two random variables X and Y can be uncorrelated but dependent. (a) Let $X \sim N(0,1)$ and $Y = X^2 - 1$. Show that they are uncorrelated but dependent; (b) Give another example where X and Y are uncorrelated but dependent. Provide justification.

3. For a linear process $X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$, where $\sum_{j=0}^{\infty} \psi_j^2 < \infty$ and ε_t are iid$(0, \sigma^2)$. Find the form of the autocorrelation function $\rho_l, l \geq 0$ and show that $\rho_l \to 0$ as $l \to \infty$.

4. For the following AR(2) model: $X_t - (\sqrt{2}/2)X_{t-1} + (1/4)X_{t-2} = Z_t$, where $Z_t \sim$ WN$(0, \sigma^2)$, find a closed-form solution for its autocorrelation function $\rho(h)$, $h \in \mathbb{N}$. (Here WN$(0, \sigma^2)$ stands for white noise, i.e., a sequence of uncorrelated random variables with mean zero and marginal variance σ^2).

5. For the general causal AR(2) model: $X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = Z_t$, where $Z_t \sim$ WN$(0, \sigma^2)$, Denote by $\phi(z) = 1 - \phi_1 z - \phi_2 z^2$ the AR polynomial and its roots are $r_1 = r_2 = r$. In the case $r_1 = r_2$, $\rho(h) = r^{-h}(C_1 + C_2 h)$, where C_1 and C_2 are constants.

   a. Verify that the solution provided above indeed satisfy the homogeneous difference equation of order 2, i.e., $\rho(h) - \phi_1 \rho(h-1) - \phi_2 \rho(h-2) = 0$ for $h \geq 1$.

   b. Find the closed-from solution for its autocorrelation function $\rho(h)$ in term of ϕ_1 and ϕ_2. (Hint: So the key is to express C_1, C_2 and r in terms of ϕ_1, ϕ_2.).

   c. Show the autocorrelation function of the following particular model: $X_t - 1.6X_{t-1} + 0.64X_{t-2} = Z_t$.

6. AoFTS: Exericise 2.2

7. AoFTS: Excercise 2.13

8. AoFTS: Excercise 2.15

Notes:

Please submit your homework solution at Canvas course website by the deadline.

AoFTS stands for Tsay, R. (2010) Analysis of Financial Time Series, 3rd edition. (E-version Available at our library).

All the datasets mentioned in the exercises of AoFTS can be downloaded from https://faculty.chicagobooth.edu/ruey-s-tsay/research/analysis-of-financial-time-series-3rd-edition I will also upload the datasets to canvas course webpage.

I encourage you to include the R codes which might help you to get partial credits.

1. Let $a_t \sim iid\ N(0,1)$ (that is, a sequence of independent and identically distributed variables with marginal distribution being standard normal). Let $e_t = a_t a_{t-1}$. Show that e_t is a white noise sequence but not iid.

$\because E[e_t] = E[a_t a_{t-1}] = E[a_t]E[a_{t-1}]$ as a_t process is iid.

$= 0$

$Var[e_t] = E[a_t^2 a_{t-1}^2] - \cancel{(E[a_t a_{t-1}])^2}^{\,0}$

$= E[a_t^2] \times E[a_{t-1}^2]$

$= 1$

$E[e_t e_{t-k}] = E[a_t a_{t-1} a_{t-k} a_{t-k-1}] = 0$ $\forall k \in \mathbb{Z} - \{0\}$

$\therefore$ e_t is a white noise process as it has a finite variance with 0 mean and 0 covariance for all lags

$\because e_t = a_t a_{t-1} \rightarrow a_{t-1} = \dfrac{e_t}{a_t}$

$e_{t-1} = a_{t-1} a_{t-2} + a_{t-1} = \underrightarrow{e_{t-1}}$

a_{t-2}

e_t depends on e_{t-1} indicating non-iid.

2. Two random variables X and Y can be uncorrelated but dependent. (a) Let $X \sim N(0,1)$ and $Y = X^2 - 1$. Show that they are uncorrelated but dependent; (b) Give another example where X and Y are uncorrelated but dependent. Provide justification.

a. $E[x] = 0$, $E[Y] = 0$

$$E\left[(x - \cancel{E[x]}^{0})(y - \cancel{E[y]}^{0})\right] = E[x^3 - x]$$

$$= 0.$$

However, as Y can be written as $x^2 - 1$, Y and x are not independent to each other.

6. $Y = X^3 + k$; $X \overset{iid}{\sim} N(a, 1)$

$E[y] = 0$, $E[X] = 1$

$E[(X-1)(y-0)] = E[(X-1)(X^3+k)]$

$= E[X^3 - \underbrace{X^2}_{} + Xk - k]$

$\because \cancel{Var[X]}^{1} = E[X^2] - \cancel{E[X]}^{a^2}$

$\therefore E[X^2] = 1 + a^2$.

$= 0 - (1 + a^2) + ak - k$

$\therefore$ There exists a, k that give

$1 + a^2 = ak - k$ ($Cov(X, y) = 0$).

$a^2 - ak + (k+1) = 0 \rightarrow a = \dfrac{k \pm \sqrt{k^2 - 4(k+1)}}{2}$

However, ψ is clearly depends on x, which indicates dependence.

3. For a linear process $X_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$, where $\sum_{j=0}^{\infty} \psi_j^2 < \infty$ and ε_t are iid$(0, \sigma^2)$. Find the form of the autocorrelation function $\rho_l, l \geq 0$ and show that $\rho_l \to 0$ as $l \to \infty$.

$$\because \rho_L = \frac{Cov(X_t, X_{t-L})}{\sigma(X_t)\,\sigma(X_{t-L})} \;;\; E(X_t) = 0,$$

$$\because Var[X_t] = \sigma^2 \sum_{j=0}^{\infty} \psi_j = Var[X_{t-L}] = A;\ L \geq 0.$$

So, $$\rho_L = \frac{E[X_t X_{t-L}]}{A}$$

$$= \frac{E\left[\sum_{i=0}^{\infty} \psi_{i+L} \varepsilon_{t-i-L} \psi_i \varepsilon_{t-L-i}\right]}{A}$$

$$= E\left[\sum_{i=0}^{\infty} \psi_{i+L}\psi_i \times \varepsilon^2_{t-L-i}\right]$$

$$= \frac{\cancel{\sigma^2} \sum_{i=0}^{\infty} \psi_{i+L}\psi_i}{\cancel{\sigma^2} \sum_{i=0}^{\infty} \psi_i^2}$$

If we assume that ψ_j is decaying as j increases,

$\rho_L \to 0$ when $L \to \infty$

as $\psi_{i+L}\psi_i \ll \psi_i^2$ for $i = 0, 1, 2, 3, \ldots$

4. For the following AR(2) model: $X_t - (\sqrt{2}/2)X_{t-1} + (1/4)X_{t-2} = Z_t$, where $Z_t \sim \mathrm{WN}(0, \sigma^2)$, find a closed-form solution for its autocorrelation function $\rho(h)$, $h \in \mathbb{N}$. (Here $\mathrm{WN}(0, \sigma^2)$ stands for white noise, i.e., a sequence of uncorrelated random variables with mean zero and marginal variance σ^2).

$$X_t = \frac{\sqrt{2}}{2} X_{t-1} - \frac{1}{4} X_{t-2} + Z_t$$

$\because$ According to Yule-Walker equation

$$\gamma(0) = \phi_1 \gamma(1) + \phi_2 \gamma(2) + \sigma^2$$

$$\gamma(1) = \phi_1 \gamma(0) + \phi_2 \gamma(1)$$

$$\gamma(1) = \frac{\phi_1 \gamma(0)}{1 - \phi_2} \rightarrow \rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{\phi_1}{1 - \phi_2}$$

$$= \frac{\left(\frac{\sqrt{2}}{2}\right)}{1 + \frac{1}{4}} = \frac{2\sqrt{2}}{5}$$

$$\gamma(2) = \phi_1 \gamma(1) + \phi_2 \gamma(0).$$

$$= \left(\frac{\phi_1^2}{1-\phi_2} + \phi_2 \right) \gamma(0)$$

$$\therefore \rho(2) = \frac{\gamma(2)}{\gamma(0)} = \frac{\left(\frac{\sqrt{2}}{2}\right)^2}{1+\frac{1}{4}} - \frac{1}{4}$$

$$= \left(\frac{\cancel{4}}{5} \times \frac{2}{\cancel{4}} \right) - \frac{1}{4} = \frac{8-5}{20} = \frac{3}{20}$$

$$\therefore \rho(h) = \begin{cases} 1 \ ; & h=0 \\ \frac{2\sqrt{2}}{5} \ ; & h=1 \\ \frac{\sqrt{2}}{2}\rho(h-1) - \frac{1}{4}\rho(h-2) \ ; & h=2,3,\ldots \end{cases}$$

5. For the general causal AR(2) model: $X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = Z_t$, where $Z_t \sim \mathrm{WN}(0, \sigma^2)$, Denote by $\phi(z) = 1 - \phi_1 z - \phi_2 z^2$ the AR polynomial and its roots are $r_1 = r_2 = r$. In the case $r_1 = r_2$, $\rho(h) = r^{-h}(C_1 + C_2 h)$, where C_1 and C_2 are constants.

a. Verify that the solution provided above indeed satisfy the homogeneous difference equation of order 2, i.e., $\rho(h) - \phi_1\rho(h-1) - \phi_2\rho(h-2) = 0$ for $h \geq 1$.

b. Find the closed-from solution for its autocorrelation function $\rho(h)$ in term of ϕ_1 and ϕ_2. (Hint: So the key is to express C_1, C_2 and r in terms of ϕ_1, ϕ_2.).

c. Show the autocorrelation function of the following particular model: $X_t - 1.6X_{t-1} + 0.64X_{t-2} = Z_t$.

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + Z_t$$

Assume that X_t is at least weakly stationary. $\rightarrow E(X_t) = 0$

$$\therefore E\left[\frac{X_t X_{t-h}}{Var(X_t)}\right] = \phi_1 E\left[\frac{X_{t-1}X_{t-h}}{Var[X_t]}\right] + \phi_2\left[\frac{X_{t-2}X_{t-h}}{Var[X_t]}\right]$$

$$\rho(h) = \phi_1 \rho(h-1) + \phi_2 \rho(h-2); \ h \geq 1$$

$$\cancel{r^{-h}}(C_1 + C_2 h) = \phi_1 r^{(\cancel{-h}+1)}(C_1 + C_2(h-1)) + \phi_2 r^{(\cancel{-h}+2)}(C_1 + C_2(h-2))$$

Because we have repeated root r,

we have $r = \dfrac{\phi_1}{-2\phi_2}$ and $\boxed{\phi_1^2 + 4\phi_2 = 0}$

repeated root condition

From Q.4

$\therefore \rho(0) = c_1 = 1$

$$\rho(1) = \frac{(\phi_1^{-1} + c_2)}{r} = \frac{\phi_1}{1-\phi_2}$$

$$= \frac{1+c_2}{\left(\frac{\phi_1}{-2\phi_2}\right)} = \frac{\phi_1}{1-\phi_2}.$$

$$c_2 = \frac{\phi_1^2}{-2\phi_2(1-\phi_2)} - 1$$

So, $c_1 = 1$, $c_2 = \frac{\phi_1^2}{-2\phi_2(1-\phi_2)} - 1$

and $r = \frac{\phi_1}{-2\phi_2}$

$$\rho(h) = \phi_1 \rho(h-1) + \phi_2 \rho(h-2)$$

$$\cancel{r^{-h}}(c_1+c_2h) = \phi_1\left(\cancel{r^{-h+1}}(c_1+c_2(h-1))\right) + \phi_2\left(\cancel{r^{-h+2}}(c_1+c_2(h-2))\right)$$

$$c_1+c_2h = -\frac{\phi_1^2}{2\phi_2}(c_1+c_2h) + \cancel{\frac{\phi_1^2 c_2}{2\phi_2^2}}$$

$$+ \frac{\phi_1^2}{4\phi_2}(c_1+c_2h) - \cancel{\frac{\phi_1^2}{2\phi_2}c_2}.$$

$$\left(1+\frac{2\phi_1^2}{4\phi_2}-\cancel{\frac{\phi_1^2}{4\phi_2}}\right)(c_1+c_2h)=0$$

$$\left(\frac{\cancel{4\phi_2+\phi_1^2}}{4\phi_2}\right)(c_1+c_2h)=0$$

0 (by repeated root condition)

$\therefore\ \gamma(h)=\phi_1\gamma(h-1)+\phi_2\gamma(h-2)$

is satisfied under the given root equation.

c. $x_t = 1.6x_{t-1} - 0.64x_{t-2} + z_t$

$$1 - 1.6m + 0.64m^2 = 0$$

$$\therefore m = \frac{1.6 \pm \cancel{\sqrt{(1.6)^2 - 4\times 0.64}} \to 0}{2(0.64)}$$

$$= 1.25$$

As we have repeated root,

$$\rho(h) = 1.25^{-h}(c_1 + c_2 h)$$

$$\therefore \rho(0) = 1 = c_1$$

$$\rho(1) = \frac{\phi_1}{1-\phi_2} = \frac{1.6}{1.64} = 0.97561$$

$$= \frac{1}{1.25}\left[1 + c_2\right] \to c_2 = 0.21951$$

$$\therefore \rho(h) = 1.25^{-h}\left[1 + 0.21951h\right]$$

2.2. Suppose that the daily log return of a security follows the model

$$r_t = 0.01 + 0.2r_{t-2} + a_t,$$

where $\{a_t\}$ is a Gaussian white noise series with mean zero and variance 0.02. What are the mean and variance of the return series r_t? Compute the lag-1 and lag-2 autocorrelations of r_t. Assume that $r_{100} = -0.01$, and $r_{99} = 0.02$. Compute the 1- and 2-step-ahead forecasts of the return series at the forecast origin $t = 100$. What are the associated standard deviations of the forecast errors?

Assume stationarity:

$$E[r_t] = 0.01 + 0.2E[r_t] + 0$$

$$E[r_t] = 0.0125$$

$$Var[r_t] = 0 + 0.2^2 \cancel{E}\,Var[r_t] + 0.02$$

$$\therefore Var[r_t] = 0.020833$$

$$\gamma(0) = E[r_t^2] = Var[r_t] + E[r_t]^2$$

$$= 0.02098925$$

If $h \geq 1 \rightarrow \rho(h) = 0.01 E[r_t] + 0.2\, \rho(h-2)$

$$\rho(2) = 0.01 \times 0.0125 + 0.2 \times \cancel{\rho(0)} \rightarrow 1$$

$$= 0.200125.$$

$$\rho(1) = 0.01 \times 0.0125 + 0.2\, \rho(1)$$

$$\therefore\ \rho(1) = 0.00015625$$

$$\hat{r}_{101|F_{100}} = 0.01 + 0.2 \times 0.02$$

$$= 0.014.$$

$$\hat{r}_{102|F_{100}} = 0.01 + 0.2 \times \cancel{r_{100}} \rightarrow -0.01 = 0.008.$$