THE UNIVERSITY OF ILLINOIS
Department of Statistics
STATISTICS 556
Advanced Time Series Analysis
Fall 2024

Homework 2 (Due: September 25, 12pm)

1. Show that (a) for two random variables X and Y, $E(X|Y) = 0$ and the existence of second moment of X and Y implies that $cov(X, Y) = 0$. (b) Suppose X_t is a sequence of martingale difference, i.e., $E(X_t|\mathrm{F}_{t-1}) = 0$ where F_t is the information up to time t. Then X_t is white noise provided that the second moment of X_t are finite.

2. Suppose we want to forecast X_t based on the information F_{t-1}. Show that the best point forecast is $E(X_t|\mathrm{F}_{t-1})$ in the mean squared error sense. That is, assuming the finiteness of second moment for X_t, we want to show that

$$\mathrm{argmin}_g E[(X_t - g(\mathrm{F}_{t-1}))^2|\mathrm{F}_{t-1}] = E(X_t|\mathrm{F}_{t-1})$$

   Or equivalently for any g,

$$E[(X_t - g(\mathrm{F}_{t-1}))^2|\mathrm{F}_{t-1}] \geq E[(X_t - E(X_t|\mathrm{F}_{t-1}))^2|\mathrm{F}_{t-1}]$$

3. In this problem, we shall study the ARCH(2) model with t innovation, i.e., $r_t = \sigma_t\epsilon_t$, where $\epsilon_t \sim iid\ t(v)/\sqrt{v/(v-2)}$, $v > 2$ and $\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2 + \alpha_2 r_{t-2}^2$ with $\alpha_0 > 0$, $\alpha_1 \geq 0$ and $\alpha_2 > 0$. We shall assume that the model adimits a strictly stationary solution and has finite fourth moments, which implies that $E(r_t^2), E(r_t^4)$ and $E(r_t^2 r_{t-1}^2)$ are all finite and do not depends on t.

   (a) Derive $var(r_t)$ and discuss what conditions we need so it is finite.

   (b) Derive $E(r_t^4)$. (assuming $v > 4$)

   (c) What is the kurtosis of r_t and discuss the impact of v and other model coefficients on the kurtosis.

4. AoFTS, Ex 3.3

5. AoFTS, Ex 3.5 (Hint: try t-innovation and $N(0, 1)$ innovation in GARCH(1,1) model, and see which one fits better. Pick the model with smaller AIC and then use that for volatility forecasting.)

6. AoFTS, Ex 3.6

7. AoFTS, Ex 3.8 (a) (c) (d) (Hint: you may fix the order of GARCH and EGARCH model to be (1,1))

1a $Cov(x,y) = 0$

$$\because E[x] = E[\cancelto{0}{E[x|y]}]$$

$$= 0$$

$$\therefore E[xy] = E[E[xy|y]]$$

$$= E[y\cancelto{0}{E[x|y]}]$$

$$= 0$$

$$\therefore Cov(x,y) = E[(x - E[x])(y - E[y])]$$

$$= \cancelto{0}{E[xy]} - \cancelto{0}{E[x]E[y]}$$

$$= 0$$

1.6.

To be a white noise

① $E[x] = 0$.

② $Var[x] = \sigma^2$ (constant) → given.

③ $\gamma(k) = c_k$, c_k is a constant

$\forall k = 1, 2, \ldots$

$\therefore E[x_t] = E[\cancel{E[x_t | \mathcal{F}_{t-1}]}] = 0$

∴ ① is satisfied!

As x_t is an MDS, it is an iid, meaning

$$\gamma(k) = 0 \quad \forall k = 1, 2, 3, \ldots$$

∴ ③ is also satisfied.

∴ x_t is a WN process.

2. Suppose we want to forecast X_t based on the information F_{t-1}. Show that the best point forecast is $E(X_t|F_{t-1})$ in the mean squared error sense. That is, assuming the finiteness of second moment for X_t, we want to show that

$$\text{argmin}_g E[(X_t - g(F_{t-1}))^2|F_{t-1}] = E(X_t|F_{t-1})$$

Or equivalently for any g,

$$E[(X_t - g(F_{t-1}))^2|F_{t-1}] \geq E[(X_t - E(X_t|F_{t-1}))^2|F_{t-1}]$$

As we do conditioning on F_{t-1}, define $g(F_{t-1})|F_{t-1} = g_t$

$$E[\cancel{x_t^2} - 2g_t x_t + g_t^2 | F_{t-1}] \geq E[\cancel{x_t^2} - 2x_t E(x_t|F_{t-1}) + E^2[x_t|F_{t-1}]|F_{t-1}]$$

$$\therefore -2g_t E[x_t|F_{t-1}] + g_t^2 \geq -\cancel{2}E^2[x_t|F_{t-1}] \cancel{+ E^2[x_t|F_{t-1}]}$$

So, we have

$$E^2[x_t|F_{t-1}] - 2g_t E(x_t|F_{t-1}) + g_t^2 \geq 0$$

$$(g_t - E[x_t|F_{t-1}])^2 \geq 0$$

This function is always true,

So we already prove that

$$E[(x_t - g(\mathcal{F}_{t-1}))^2 \mid \mathcal{F}_{t-1}] \geq E[(x_t - E(x_t \mid \mathcal{F}_{t-1}))^2 \mid \mathcal{F}_{t-1}].$$

for any g.

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3. In this problem, we shall study the ARCH(2) model with t innovation, i.e., $r_t = \sigma_t \epsilon_t$, where $\epsilon_t \sim iid\ t(v)/\sqrt{v/(v-2)}$, $v > 2$ and $\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2 + \alpha_2 r_{t-2}^2$ with $\alpha_0 > 0$, $\alpha_1 \geq 0$ and $\alpha_2 > 0$. We shall assume that the model adimits a strictly stationary solution and has finite fourth moments, which implies that $E(r_t^2), E(r_t^4)$ and $E(r_t^2 r_{t-1}^2)$ are all finite and do not depends on t.

(a) Derive $var(r_t)$ and discuss what conditions we need so it is finite.

(b) Derive $E(r_t^4)$. (assuming $v > 4$)

(c) What is the kurtosis of r_t and discuss the impact of v and other model coefficients on the kurtosis.

3.a $\therefore \sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2 + \alpha_2 r_{t-2}^2$

$$E[\sigma_t^2] = \alpha_0 + \alpha_1 E(r_{t-1}^2) + \alpha_2 E[r_{t-2}^2]$$

with stationarity,

$$Var[r_t] = \alpha_0 + \alpha_1 Var[r_t] + \alpha_2 Var[r_t]$$

$$Var[r_t] = \frac{\alpha_0}{1-\alpha_1-\alpha_2} = E(r_t^2) = \sigma^2$$

3.b $\therefore E(r_t^4) = E[\sigma_t^4] \times E[\varepsilon_t^4]$

$$\therefore E[\varepsilon_t^4] = E\left[\frac{t(v)^4}{\left(v^2/(v-2)^2\right)}\right]$$

$$= \frac{\left(\dfrac{3\cancel{\nu^2}}{\cancel{(\nu-2)}(\nu-4)}\right)}{\left(\dfrac{\cancel{\nu^2}}{(\nu-2)^{\cancel{2}}}\right)}$$

$$= \left(\frac{\nu-2}{\nu-4}\right); \ \nu > 4.$$

$$\therefore \ E\left[\sigma_t^4\right] = E\left(\alpha_0 + \alpha_1 r_{t-1}^2 + \alpha_2 r_{t-2}^2\right)^2$$

$$= E\big[\alpha_0^2 + \alpha_0\alpha_1 r_{t-1}^2 + \alpha_0\alpha_2 r_{t-2}^2$$

$$+ \alpha_1\alpha_0 r_{t-1}^2 + \alpha_1^2 r_{t-1}^4 + \alpha_1\alpha_2 r_{t-1}^2 r_{t-2}^2$$

$$+ \alpha_0\alpha_2 r_{t-2}^2 + \alpha_1\alpha_2 r_{t-1}^2 r_{t-2}^2$$

$$+ \alpha_2^2 r_{t-2}^4]$$

$$= \alpha_0^2 + 2\alpha_0 E[r_t^2](\alpha_1 + \alpha_2)$$

$$+ 2\alpha_1\alpha_2 E[r_t^2 r_{t-1}^2]$$

$$+ (\alpha_1^2 + \alpha_2^2) E[r_t^4]$$

$$\therefore E[r_t^2 r_{t-1}^2] = \sigma^4 E[\varepsilon_t^2 \varepsilon_{t-1}^2]$$

(because of iid condition) $= \sigma^4 E(\varepsilon_t^2) E[\varepsilon_{t-1}^2]$

$$= \sigma^4 \left(E\left[\frac{t^2(v)}{\left(v/(v-2)\right)} \right]^2 \right)$$

$$= \sigma^4$$

$$\therefore E[\sigma_t^4] = \alpha_0^2 + 2\alpha_0 \left[\frac{\alpha_0}{1-\alpha_1-\alpha_2} \right] \times (\alpha_1+\alpha_2)$$

$$+ 2\alpha_1\alpha_2[\sigma^4] + (\alpha_1^2+\alpha_2^2) E[r_t^4]$$

$$\therefore E(r_t^4) = E[\sigma_t^4]\, E[\varepsilon_t^4]$$

$$= \left(\frac{v-2}{v-4}\right)\left(\alpha_0^2 + \frac{2\alpha_0^2(\alpha_1+\alpha_2)}{1-\alpha_1-\alpha_2} + 2\alpha_1\alpha_2\sigma^4 + (\alpha_1^2+\alpha_2^2) E(r_t^4)\right)$$

$$E\left(r_t^4\right) \times \left[1 - \frac{(V-2)}{(V-4)}\left(\alpha_1^2+\alpha_2^2\right)\right]$$

$$= \left(\frac{V-2}{V-4}\right)\left(\alpha_0^2 + \frac{2\alpha_0^2(\alpha_1+\alpha_2)}{1-\alpha_1-\alpha_2} + 2\alpha_1\alpha_2\sigma^4\right)$$

$$\therefore E\left(r_t^4\right) = \frac{\left(\frac{V-2}{\cancel{V-4}}\right)\left(\alpha_0^2 + \frac{2\alpha_0^2(\alpha_1+\alpha_2)}{1-\alpha_1-\alpha_2} + 2\alpha_1\alpha_2\sigma^4\right)}{\left[\frac{V-4}{\cancel{V-4}} - \left(\frac{V-2}{\cancel{V-4}}\right)\left(\alpha_1^2+\alpha_2^2\right)\right]}$$

$$= \frac{(V-2)\left(\alpha_0^2 + \frac{2\alpha_0^2(\alpha_1+\alpha_2)}{1-\alpha_1-\alpha_2} + 2\alpha_1\alpha_2\sigma^4\right)}{(V-2)(1-\alpha_1^2-\alpha_2^2) - 2}$$

$$= \frac{(V-2)\left(\frac{\alpha_0^2(1+\alpha_1+\alpha_2)}{(1-\alpha_1-\alpha_2)} + 2\alpha_1\alpha_2\sigma^4\right)}{(V-2)(1-\alpha_1^2-\alpha_2^2) - 2}$$

As $\sigma^4 = \left(\frac{\alpha_0}{1-\alpha_1-\alpha_2}\right)^2$.

$$\therefore E\left[r_t^4\right] = \frac{(v-2)\left(\frac{\alpha_0^2(1+\alpha_1+\alpha_2)}{1-\alpha_1-\alpha_2} + \frac{2\alpha_0^2\alpha_1\alpha_2}{(1-\alpha_1-\alpha_2)^2}\right)}{(v-2)(1-\alpha_1^2-\alpha_2^2)-2}$$

$$= v-2\,\frac{\left(\alpha_0^2\left(1-\overbrace{(\alpha_1+\alpha_2)^2}^{\alpha_1^2+\cancel{2\alpha_1\alpha_2}+\alpha_2^2}\right) + \cancel{2\alpha_0^2\alpha_1\alpha_2}\right)}{\left((v-2)(1-\alpha_1^2-\alpha_2^2)-2\right)(1-\alpha_1-\alpha_2)^2}$$

$$= \frac{(v-2)\alpha_0^2\left(1-\alpha_1^2-\alpha_2^2\right)}{\left((v-2)(1-\alpha_1^2-\alpha_2^2)-2\right)(1-\alpha_1-\alpha_2)^2}$$

$$\therefore \text{kurtosis} = \frac{E[r_t^4]}{\left(E[r_t^2]\right)^2}$$

$$= \frac{\left(\dfrac{(v-2)\cancel{\alpha_0^2}(1-\alpha_1^2-\alpha_2^2)}{\left[(v-2)(1-\alpha_1^2-\alpha_2^2)-2\right]\cancel{(1-\alpha_1-\alpha_2)}}\right)}{\cancel{\dfrac{\alpha_0^2}{(1-\alpha_1-\alpha_2)^2}}}$$

$$= \frac{(v-2)(1-\alpha_1^2-\alpha_2^2)-2+2}{(v-2)(1-\alpha_1^2-\alpha_2^2)-2}$$

$$= 1 + \frac{2}{(v-2)(1-\alpha_1^2-\alpha_2^2)-2}$$

So, we have

Parameter	Param size	kurtosis change
V	Large	Small
	Small	Large
$\alpha_1^2 + \alpha_2^2$	Large	Small
	Small	Large

(close to 1)

close to 0.

4. 3.3. Suppose that $r_1, \ldots, r_n$ are observations of a return series that follows the AR(1)-GARCH(1,1) model

$$r_t = \mu + \phi_1 r_{t-1} + a_t, \qquad a_t = \sigma_t \epsilon_t, \qquad \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2,$$

where ϵ_t is a standard Gaussian white noise series. Derive the conditional log-likelihood function of the data.

Likelihood function $= P(r_1) P(r_2 | r_1) \times P(r_3 | r_1, r_2) \times \sim \times P(r_n | r_1, \sim, r_n)$.

$\because r_t = \mu + \phi_1 r_{t-1} + \sigma_t \varepsilon_t$; $\varepsilon_t \sim N(0, 1)$

$\therefore r_t | F_{t-1} \sim N(\mu + \phi_1 r_{t-1}, \sigma_t^2)$.

$\therefore$ The conditional log-likelihood function =

$$\sum_{t=2}^{n} \log\left[\frac{1}{\sqrt{2\pi\sigma_t^2}} \times \exp\left[-\frac{(r_t - \mu - \phi r_{t-1})^2}{2(\sigma_t^2)}\right]\right]$$

$$= -\frac{(n-1)}{2}\log(2\pi) - \sum_{t=2}^{n}\left[\left(\frac{1}{2}\right)(\sigma_t^2) + \frac{[r_t - \mu - \phi r_{t-1}]^2}{2\sigma_t^2}\right]$$

5.

3.5. Consider the monthly simple returns of Intel stock from January 1973 to December 2008 in `m-intc7308.txt`. Transform the returns into log returns. Build a GARCH model for the transformed series and compute 1-step- to 5-step-ahead volatility forecasts at the forecast origin December 2008.

$R = \log\left(\frac{P_{t+1}}{P_t}\right)$, use AIC as the main information criteria.

See the attached file. (Q5_GARCH.html)

6.

3.6. The file `m-mrk4608.txt` contains monthly simple returns of Merck stock from June 1946 to December 2008. The file has two columns denoting date and simple return. Transform the simple returns to log returns.

(a) Is there any evidence of serial correlations in the log returns? Use autocorrelations and 5% significance level to answer the question. If yes, remove the serial correlations.

(b) Is there any evidence of ARCH effects in the log returns? Use the residual series if there are serial correlations in part (a). Use Ljung–Box statistics for the squared returns (or residuals) with 6 and 12 lags of autocorrelations and 5% significance level to answer the question.

(c) Identify an ARCH model for the data and fit the identified model. Write down the fitted model.

See the attached file. (Q6_ARCH_EFFECT.html)

7.

3.8. The file `m-gmsp5008.txt` contains the dates and monthly simple returns of General Motors stock and the S&P 500 index from 1950 to 2008.

(a) Build a GARCH model with Gaussian innovations for the log returns of GM stock. Check the model and write down the fitted model.

(c) Build a GARCH model with Student-t distribution for the log returns of GM stock, including estimation of the degrees of freedom. Write down the fitted model. Let v be the degrees of freedom of the Student-t distribution. Test the hypothesis $H_0 : v = 6$ versus $H_a : v \neq 6$, using the 5% significance level.

(d) Build an EGARCH model for the log returns of GM stock. What is the fitted model?

See the attached file, (Q7_EGARCH.html)