# 2. Algebra & Graphs

YOUR NOTES ⬇

CONTENTS

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## 2.1 USING ALGEBRA

### 2.1.1 REARRANGING FORMULAE - BASICS

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## Choosing a method

- There are lots of different ways of remembering what to do when rearranging formulae – every teacher (and student) has their favourite, this is my recommendation….

## GROF GROBLET FIND ANSWER

- This is a mnemonic (way of remembering something) deals with everything except roots and powers (see the following set of notes for that) and can also be used for solving equations…

1. **GROF** – Get Rid Of Fractions
2. **GROB** – Get Rid Of Brackets
3. **LET** – Lump Everything Together (or Let's Examine Terms)
4. **FIN** – Factorise If Necessary
5. **D** – Divide

**ANSWER**!

### Worked Example

*Make x the subject of* $p = \frac{2-ax}{x-b}$

$p(x-b) = 2 - ax$ — *1 – GROF, multiply both sides by* $(x-b)$

$px - pb = 2 - ax$ — *2 – GROB, expand brackets*

$px + ax = 2 + pb$ — *3 – LET, we want x to be the subject so lump all x terms together on one side and lump everything else on the other side*

$x(p+a) = 2 + pb$ — *4 – FIN, we need to factorise to isolate x*

$x = \frac{2+pb}{p+a}$ — *5 – D, divide both sides by* $(p+a)$

*ANSWER!*

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## 2.1.2 REARRANGING FORMULAE - HARDER

### SIR! GROF GROBLET FIND ANSWER!

- This is a mnemonic (way of remembering something) developed to deal with everything including roots and powers (for a simpler version see the previous set of notes) and can also be used for solving equations…

  **SIR**! – Squares, Indices, Roots

1. **SIR**! – Squares, Indices, Roots (can be used at any point)
2. **GROF** – Get Rid Of Fractions
3. **GROB** – Get Rid Of Brackets
4. **LET** – Lump Everything Together (or Let's Examine Terms)
5. **FIN** – Factorise if Necessary
6. **D** – Divide
   **ANSWER**!

### Worked Example

$Make\ x\ the\ subject\ of\quad p = \sqrt{\frac{2-ax^2}{x^2-b}}$

$p^2 = \frac{2-ax^2}{x^2-b}$ $\quad$ $0 - SIR!\ Square\ both\ sides$

$p^2(x^2-b) = 2 - ax^2$ $\quad$ $1 - GROF,\ multiply\ both\ sides\ by\ (x^2-b)$

$p^2x^2 - p^2b = 2 - ax^2$ $\quad$ $2 - GROB,\ expand\ brackets$

$p^2x^2 + ax^2 = 2 + p^2b$ $\quad$ $3 - LET,\ x\ terms\ together,\ everything\ else\ on\ other\ side$

$x^2(p^2+a) = 2 + p^2b$ $\quad$ $4 - FIN,\ factorise\ to\ isolate\ x\ term$

$x^2 = \frac{2+p^2b}{p^2+a}$ $\quad$ $5 - GROF,\ multiply\ both\ sides\ by\ (x^2-b)$

$x = \sqrt{\frac{2+p^2b}{p^2+a}}$ $\quad$ $0 - SIR!\ Square\ root\ both\ sides$

$ANSWER!$

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## 2.1.3 SUBSTITUTION

### What is substitution?

- Substitution is where we replace letters in a formula with their values
- This allows you to find one other value that is in the formula

### How do we substitute?

- Write down the FORMULA if not clearly stated in question
- SUBSTITUTE the numbers given – use ( ) around negative numbers
- SIMPLIFY if you can
- REARRANGE if necessary – it is usually easier to substitute first
- Do the CALCULATION – use a calculator if allowed

### Worked Example

1. *Find the value of the expression* $2x(x+3y)$ *when* $x=2$ *and* $y=-4$

$2 \times 2 \times (2 + 3 \times (-4))$ — *1 – Substitute the numbers given, use () around negatives*
$= 2 \times 2 \times (2 - 12)$ — *You don't need all these lines of working but we've included*
$= 2 \times 2 \times (-10)$ — *them here to remind you of order of operations*
$= -40$ — *Use a calculator if allowed*

2. *The formula* $P = 2l + 2w$ *is used to find the perimeter,* $P$ *of a rectangle of length* $l$ *and width* $w$
*Given that a rectangle has a perimeter of 20 cm and a width of 4 cm, find its length.*

$20 = 2 \times l + 2 \times 4$ — *1 – Substitute the numbers given, no negatives*
$20 = 2l + 8$ — *2 – Simplify*
$2l = 12$ — *3, 4 – Rearrange and do the calculation*
$l = 6$
*Length is 6 cm*

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## 2.1.4 EQUATIONS & PROBLEM SOLVING

### What is problem solving?

- Problem solving in mathematics involves using several stages, across a variety of topics, to answer a question
- In this set of notes all the problems will involve **equations**
    - These could be **linear** equations, **quadratic** equations or **simultaneous** equations and **other**, relatively straightforward equations

e.g. $3x + 24 = 48$

$x = 8$

LINEAR EQUATION

e.g. 2 $x^2 + 8x - 9 = 0$

$x = -9$

$x = 1$

QUADRATIC EQUATION

e.g. 3 $3x + 2y = 12$

$x + 5y = 17$

$x = 2$

$y = 3$

SIMULTANEOUS EQUATIONS

OTHER EQUATIONS

e.g. 4 ... COULD INVOLVE FRACTIONS

$\frac{2}{x} + 3 = 8$

$x = 0.4$

e.g. 5 ... COULD INVOLVE CUBES OR HIGHER POWERS OF x

$x^5 = 32$

$x = 2$

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- You may notice there are not many subheadings in these notes
- That is deliberate so the examples are not labelled or dealt with in an order
  - This is the nature of problem solving questions!
  - You never know exactly what's coming ... !

e.g. AT A FIREWORKS STALL A CUSTOMER PAYS £9 FOR SIX BANGERS AND TWELVE SPARKLERS. ANOTHER CUSTOMER BUYS NINE BANGERS AND TEN SPARKLERS FOR £12.30. FIND THE COST OF 5 BANGERS AND 15 SPARKLERS.

TWO "THINGS" – B's AND S's
TWO UNKNOWNS – COST OF B's AND S's
TWO PIECES OF INFO – TWO EQUATIONS
SIMULTANEOUS EQUATIONS

NO "NICE" LINK FROM 6 OR 9 TO 5, NOR FROM 12 OR 10 TO 15, SO FIND COST OF 1 B AND 1 S

NOTICE HOW MUCH WORK AND THINKING SHOULD HAPPEN IN YOUR HEAD BEFORE ATTEMPTING A SOLUTION!

LET B BE THE PRICE OF BANGERS, IN POUNDS.
LET S BE THE PRICE OF SPARKLERS, IN POUNDS.

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$$6B + 12S = 9 \quad (1)$$
$$9B + 10S = 12.3 \quad (2)$$

$(1) \times 3$: $18B + 36S = 27 \quad (3)$

$(2) \times 2$: $18B + 20S = 24.6 \quad (4)$

$(3) - (4)$: $16S = 2.4$

$$S = \frac{2.4}{16}$$
$$S = 0.15$$

SUB. IN (3): $18B + 36 \times 0.15 = 27$

$$18B = 21.6$$
$$B = \frac{21.6}{18}$$
$$B = 1.2$$

SO, $5B + 15S = 5 \times 1.2 + 15 \times 0.15$
$= 8.25$

FIVE BANGERS AND FIFTEEN SPARKLERS WILL COST £8.25

A GOOD ALTERNATIVE QUESTION HERE TO TEST INTERPRETATION OF YOUR ANSWER WOULD BE TO FIND THE CHANGE FROM £10. (£1.75 !)



- In an ordinary mathematics question you would be given an equation to solve
- In a problem solving question you would have to **generate** the equation …
  - … using information from the question
  - … using your knowledge of standard mathematical results

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e.g. AN IRONMAN COMPETITION CONSISTS OF A SWIM, A BICYCLE RIDE AND A MARATHON WITH THE DISTANCES OF EACH SPLIT IN THE RATIO $\frac{x}{10} : 2(2x+5) : x$

COMPETITOR'S COVER A TOTAL DISTANCE OF 137.5 MILES.
FIND THE DISTANCE OF EACH COMPONENT OF THE COMPETITION.

RATIO – TOTAL GIVEN

$$\frac{x}{10} + 2(2x+5) + x = 137.5$$

$$x + 20(2x+5) + 10x = 1375$$

x10 TO "GET RID OF FRACTIONS" (GROF)

$$x + 40x + 100 + 10x = 1375$$

EXPAND TO "GET RID OF BRACKETS" (GROB)

$$51x + 100 = 1375$$
$$51x = 1275$$
$$x = 25$$

SO $\frac{x}{10} : 2(2x+5) : x = \frac{25}{10} : 2(2 \times 25 + 5) : 25$

$= 2.5 : 110 : 25$

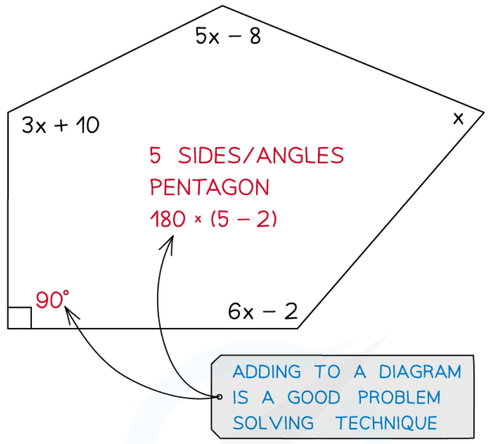
SWIM IS 2.5 MILES
BICYCLE RIDE IS 110 MILES
MARATHON IS 25 MILES

AN ALTERNATIVE HERE WOULD BE TO ASK HOW MANY TIMES FURTHER IS THE BICYCLE RIDE THAN THE SWIM (44)



- A key feature of problem solving questions is to **interpret** the answer in **context**
- An answer on a calculator may be **2**
  - If the question was about money then your final answer should be **£1.20**
- A **quadratic** equation can have **two** solutions
  - Only **one** may be valid if **only** positive values are relevant (eg distance)

e.g. THE DIAGRAM BELOW SHOWS AN IRREGULAR POLYGON.
ALL ANGLES ARE IN DEGREES.
FIND THE VALUE OF x.




NOT TO SCALE – CAN'T MEASURE IT!
**IRREGULAR** POLYGON, BUT, 5 SIDES PENTAGON
**INTERIOR** ANGLES GIVEN
∟ IS A RIGHT-ANGLE, 90°
SUM OF INTERIOR ANGLES IN **ANY** POLYGON

$$\text{SUM OF ANGLES} = 180(n-2) = 180(5-2) = 180 \times 3 = 540$$

$3x + 10 + 5x - 8 + x + 6x - 2 + 90 = 540$

$15x + 90 = 540$

$15x = 450$

$x = 30°$

A GOOD ALTERNATIVE QUESTION HERE WOULD BE TO FIND THE SMALLEST OR LARGEST ANGLE (30°, 178°!)



- In problem solving questions you are typically given less information about the type of maths involved
- It is impossible to list every type of problem solving question you could see
  - There are endless contexts questions can be set in
  - There is no one-fits-all step-by-step method to solving problems
- **Practice**, **experience** and **familiarity** are the keys to solving problems successfully

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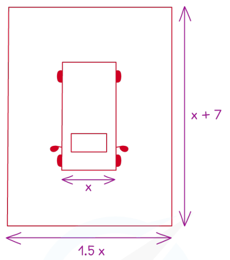
e.g. A TYPICAL DISABLED CAR PARKING SPACE HAS TO BE ONE ANS A HALF TIMES THE WIDTH OF AN AVERAGE CAR WHEN MEASURED IN METRES.
THE LENGTH OF THE PARKING SPACE HAS TO BE 7 m LONGER THAN THE WIDTH OF AVERAGE CAR.
GIVEN THAT THE AREA OF A TYPICAL DISABLED CAR PARKING SPACE IS 25 m² FIND THE WIDTH OF AN AVERAGE CAR.

WIDTHS, LENGTHS, AREAS...
DRAW A DIAGRAM !!

LET x m BE THE WIDTH OF AN AVERAGE CAR.




AREA = LENGTH × WIDTH

$1.5x(x + 7) = 25$

$1.5x^2 + 10.5x = 25$ QUADRATIC

$1.5x^2 + 10.5x - 25 = 0$ MAKE = 0

$"b^2 - 4ac" = (10.5)^2 - 4 \times 1.5 \times -25$
$= 260.25$

$x = \dfrac{-10.5 \pm \sqrt{260.25}}{2 \times 1.5}$

NUMBERS AWKWARD SO USE FORMULA

$x = 1.877421...$ $x = -8.877421...$

WRITE DOWN MORE DIGITS THAN NEEDED AND INTERPRET AND ROUND AFTERWARDS

x IS A DISTANCE SO CAN'T BE NEGATIVE.
REJECT $x = -8.877421...$ AS A SOLUTION

$x = 1.88$ m (2 dp)

METRES SO 2 dp IS CENTIMETRES AND SENSIBLE

THE WIDTH OF AN AVERAGE CAR IS 1.88m

A GOOD ALTERNATIE HERE WOULD BE TO ASK ABOUT THE LENGTH OF THE DISABLED PARKING SPACE (8.88m !)



- Do **not** necessarily expect whole number (**integer**) or “nice” solutions
  - Especially where a calculator is allowed
- **Rounding appropriately** may be one of the skills being tested
  - eg Rounding a value in **cm** only needs to be to one decimal place; so it indicates **mm**

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## Exam Tip

Do **not** start by focusing on what the question has asked you to find, but on what maths you **can** do.

If your attempt turns out to be unhelpful, that's fine, you may still pick up some marks.

If your attempt is relevant it could nudge you towards the full solution – and full marks!

Add information to a **diagram** as you work through a problem. If there is no diagram, try **sketching** one.

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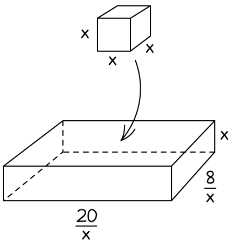
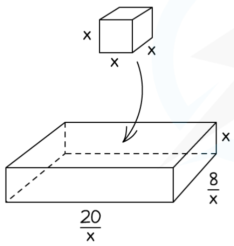
## Worked Example

Cubes of side length $x$ cm are placed into a cuboid measuring $\frac{20}{x}$ cm by $\frac{8}{x}$ cm by $x$ cm.

The cuboid holds exactly 10 of the cubes.

Find the volume of one of the cubes.




$V_1 = x^3$ ← ADD TO THE DIAGRAM

$V_2 = \frac{20}{x} \times \frac{8}{x} \times x$

$V_2 = \frac{160x}{x^2}$

$V_2 = \frac{160}{x}$

$V_2 = 10V_1$ ← CUBOID HOLDS 10 CUBES

$\frac{160}{x} = 10x^3$

$16 = x^4$ ← MULTIPLY (BOTH SIDES) BY x, DIVIDE (BOTH SIDES) BY 10

$x = 2$ — $x = -2$ TOO BUT $x$ IS A DISTANCE SO $x > 0$

VOLUME OF ONE CUBE $= 2 \times 2 \times 2$

$= 8\text{ cm}^3$

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## 2.2 FACTORISING & EXPANDING

### 2.2.1 EXPANDING ONE BRACKET

#### What is expanding one bracket?

- When we expand brackets in algebra we 'get rid of' the brackets by multiplying out
- Expanding one bracket is done by multiplying everything inside the bracket by the factor that is outside the bracket

#### Beware of minus signs

Remember the basic rules:
" $- \times - = +$ "
" $- \times + = -$ "

#### How to multiply out

1. Identify the factor outside the bracket
2. Identify the terms inside the bracket
3. Link the outside factor to the inside terms
4. Follow the links to multiply the outside factor by each term inside the bracket

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Worked Example

1. *Expand* $4x(2x - 3)$

1. *Factor outside is* "$4x$"
2. *Terms inside are* "$2x$" *and* "$-3$"
3. $4x(2x - 3)$
4. $= 4x \times 2x + 4x \times (-3)$

$= 8x^2 - 12x$

1. *Expand* $-7x(4 - 5x)$

1. *Factor outside is* "$-7x$"
2. *Terms inside are* "$4$" *and* "$-5x$"
3. $-7x(4 - 5x)$
4. $= (-7x) \times 4 + (-7x) \times (-5x)$

$= -28x + 35x^2$

# 2. Algebra & Graphs

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## 2.2.2 SIMPLE FACTORISATION

### What is simple factorisation?

Simple Factorisation is the opposite of Expanding One Bracket

### Beware of minus signs

Remember the basic rules:

" $-\times-=+$ "

" $-\times+=-$ "

### How to factorise

- Write each term in factorised form
- Identify the common factors
- Write those factors outside a bracket
- Write everything else inside the bracket
  You can check your answer by multiplying out (in your head!)

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## Worked Example

1. $Factorise\ 6x^2 - 15x$

1. $6x^2 - 15x = 2 \times 3 \times x \times x - 3 \times 5 \times x$

2. $Common\ Factors:\ 3 \times x$

3&4. $6x^2 - 15x = 3 \times x \times (2 \times x - 5)$

$= 3x(2x - 5)$

2. $Factorise\ 42x^2 - 18x$

1. $42x^2 - 18x = 7 \times 6 \times x \times x - 3 \times 6 \times x$

2. $Common\ Factors:\ 6 \times x$

3&4. $42x^2 - 18x = 6 \times x \times (7 \times x - 3)$

$= 6x(7x - 3)$

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## 2.2.3 EXPANDING TWO BRACKETS - BASICS

### What is expanding two brackets?

- When we expand brackets in algebra we 'get rid of' the brackets by multiplying out
- When we expand two brackets, each term in one bracket gets multiplied by each term in the other bracket

### Beware of minus signs

" $- \times - = +$ "

" $- \times + = -$ "

### How to multiply out two brackets

- Write FOIL above your working space
  F – First term in each bracket
  O – Outer pair of terms
  I – Inner pair of terms
  L – Last term in each bracket
- Write down the four multiplications
- Careful here – you have to be sure to include any minus signs!
- Simplify by collecting like terms (if there are any)

### Worked Example

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1. *Expand* $(2x - 3)(x + 4)$

1.

2. $(2x - 3)(x + 4) = \overbrace{2x \times x}^{F} + \overbrace{2x \times 4}^{O} - \overbrace{3 \times x}^{I} - \overbrace{3 \times 4}^{L}$

$= 2x^2 + 8x - 3x - 12$

3. $= 2x^2 + 5x - 12$

2. *Expand* $(x - 3)(3x - 5)$

1.

2. $(x - 3)(3x - 5) = \overbrace{x \times 3x}^{F} + \overbrace{x \times (-5)}^{O} - \overbrace{3 \times 3x}^{I} - \overbrace{3 \times (-5)}^{L}$

$= 3x^2 - 5x - 9x + 15$

3. $= 3x^2 - 14x + 15$

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## 2.2.4 EXPANDING TWO BRACKETS - HARDER

### Hidden FOIL (aka spot the hidden brackets)

- Always write $(a + b)^2$ as $(a + b)(a + b)$

### Beware of minus signs

- Remember the basic rules:
  " − × − = + "
  " − × + = − "

### How to multiply out two brackets

- Write FOIL above your working space
  F – First term in each bracket
  O – Outer pair of terms
  I – Inner pair of terms
  L – Last term in each bracket
- Write down the four MULTIPLICATIONS
  Careful here – you have to be sure to include any minus signs!
- Simplify by collecting LIKE TERMS (if there are any)

### Worked Example

$Expand\ (2x+3)^2$

$First\ write:\quad (2x+3)^2 = (2x+3)\,(2x+3)$

1. $\qquad F \qquad O \qquad I \qquad L$

2. $(2x+3)\,(2x+3) = \underbrace{2x \times 2x}_{F} + \underbrace{2x \times 3}_{O} + \underbrace{3 \times 2x}_{I} + \underbrace{3 \times 3}_{L}$

$$= 4x^2 + 6x + 6x + 9$$

3. $= 4x^2 + 12x + 9$

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## 2.2.5 EXPANDING THREE BRACKETS

### Expanding three brackets

- We're still following the basic principle of expanding here – get rid of the brackets by multiplying out
- We just need a good method to make sure everything gets multiplied by everything else in the correct way

### Beware of minus signs

- Remember the basic rules:
  " $- \times - = +$ "
  " $- \times + = -$ "

### How to multiply out three brackets

- Multiply out one pair of brackets using FOIL (as for two brackets)
- If the new bracket has two terms use FOIL (again!)
- Otherwise link each term in the smaller bracket to each term in the larger bracket (that's six links in total)
- Write down the six multiplications
- Careful here – you have to be sure to include any minus signs!
- Simplify by collecting like terms (if there are any)

# 2. Algebra & Graphs

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## Worked Example

1. *Expand* $(2x - 3)(x + 4)(3x - 1)$

1. $$(2x - 3)(x + 4) = \underbrace{2x \times x}_{F} + \underbrace{2x \times 4}_{O} - \underbrace{3 \times x}_{I} - \underbrace{3 \times 4}_{L}$$
$$= 2x^2 + 5x - 12$$

2. $(2x - 3)(x + 4)(3x - 1) = (2x^2 + 5x - 12)(3x - 1)$

3. $(2x - 3)(x + 4)(3x - 1) = 2x^2 \times 3x + 5x \times 3x - 12 \times 3x + 2x^2 \times (-1) + 5x \times (-1) - 12 \times (-1)$
$$= 6x^3 + 15x^2 - 36x - 2x^2 - 5x + 12$$

4. $$= 6x^3 + 13x^2 - 41x + 12$$

2. *Expand* $(x - 3)(x + 2)(2x - 1)$

1. $$(x - 3)(x + 2) = \underbrace{x \times x}_{F} + \underbrace{x \times 2}_{O} - \underbrace{3 \times x}_{I} - \underbrace{3 \times 2}_{L}$$
$$= x^2 - x - 6$$

2. $(x - 3)(x + 2)(2x - 1) = (x^2 - x - 6)(2x - 1)$

3. $(x - 3)(x + 2)(2x - 1) = x^2 \times 2x - x \times 2x - 6 \times 2x + x^2 \times (-1) - x \times (-1) - 6 \times (-1)$
$$= 2x^3 - 2x^2 - 12x - x^2 + x + 6$$

4. $$= 2x^3 - 3x^2 - 11x + 6$$

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## 2.2.6 FACTORISING QUADRATICS - BASICS

### What is a quadratic expression?

- A quadratic expression looks like this:

  $ax^2 + bx + c$ (as long as $a \neq 0$)

- Note: If there are any higher powers of x (like $x^3$ say) then it is not a quadratic!

### Factorising a 3-term quadratic expression (a = 1)

- Signs in quadratic determine signs in brackets:
  - if c is **positive** then both signs are the same as the sign of b
  - if c is **negative** then the signs are different and bigger number has the sign of b
- Using those signs find numbers p and q which
  - **multiply** to give c
  - **add** to give b
- Write down the brackets $(x + p)(x + q)$ – that's your answer!
  Don't forget – both p and q here can be negative!

#### Exam Tip

Make sure you know if you are being asked to:

• Solve an equation (look for the "=") or

• Factorise an expression (no "=")

Do not confuse the two things.

When the quadratic expression only has two terms check for:

• Simple factorisation (no number term, ie. when $c = 0$ )

• Difference Of Two Squares (no x term, ie. when $b = 0$)

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## Worked Example

1. $Factorise\ x^2 - 2x - 8$

1. $"-8"$ *means the signs will be different*
$"-2"$ *means the bigger number will be negative*

2. *The only numbers which multiply to give* $-8$ *and follow the sign rules in* 1. *are*

$$-8 \times 1 \quad and \quad -4 \times 2$$

*But only the second pair add to give* $-2$

$$-4 \times 2 = -8 \quad and \quad -4 + 2 = -2$$

*So we have found* $p = -4$ *and* $q = 2$

3. $$x^2 - 2x - 8 = (x-4)(x+2)$$

2. $Factorise\ x^2 - 5x + 6$

1. $"+6"$ *means the signs will be the same*
$"-5"$ *means that both signs will be negative*

2. *The only numbers which multiply to give* $+6$ *and follow the sign rules in* 1. *are*

$$-2 \times -3 \quad and \quad -1 \times -6$$

*But only the first pair add to give* $-5$

$$-2 \times (-3) = 6 \quad and \quad -2 + (-3) = -5$$

*So we have found* $p = -2$ *and* $q = -3$

3. $$x^2 - 5x + 6 = (x-2)(x-3)$$

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## 2.2.7 FACTORISING QUADRATICS - HARDER

### What is a quadratic expression?

- A quadratic expression looks like this:

  $ax^2 + bx + c$ (as long as $a \neq 0$)

- Note: If there are any higher powers of x (like $x^3$ say) then it is not a quadratic!

### Factorising a 3 term quadratic expression ($a \neq 1$)

- Signs in quadratic determine signs in brackets:
  - If c is **positive** then both signs are the same as the sign of b
  - If c is **negative** then the signs are different and bigger number has the sign of b
- Using those signs find numbers p and q which
  - **Multiply** to give $a \times c$
  - **Add** to give b
  - Cheat by writing brackets as $(ax + p)(ax + q)$ – Yes, I know! This is **not** correct!
  - Uncheat by cancelling common factors in each bracket

### Exam Tip

Check for a common factor – life will be a lot easier if you take it out before you try to factorise.
If a is negative take out out a factor of -1 – that will help, too.

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## Worked Example

1. $Factorise\ 6x^2 - 7x - 3$

1. $"-3"$ *means the signs will be different*
   $"-7"$ *means the bigger number will be negative*

2. $a \times c = 6 \times (-3) = -18$

   *The only numbers which multiply to give* $-18$ *and follow the sign rules in 1. are*

   $-18 \times 1$ *and* $-9 \times 2$ *and* $-6 \times 3$

   *But only the second pair add to give* $-7$

   $-9 \times 2 = -18$ *and* $-9 + 2 = -7$

   *So we have found* $p = -9$ *and* $q = 2$

3. *Cheat :* $(6x - 9)(6x + 2)$ *This is not correct!*
   $= 3(2x - 3) \times 2(3x + 1)$

4. *Uncheat :* $6x^2 - 7x - 3 = (2x - 3)(3x + 1)$

2. $Factorise\ 10x^2 + 9x - 7$

1. $"-7"$ *means the signs will be different*
   $"+9"$ *means the bigger number will be positive*

2. $a \times c = 10 \times (-7) = -70$

   *The only numbers which multiply to give* $-70$ *and follow the sign rules in 1. are*

   $-1 \times 70$ *and* $-2 \times 35$ *and* $-5 \times 14$ *and* $-7 \times 10$

   *But only the third pair add to give* $+9$

   $-5 \times 14 = -70$ *and* $-5 + 14 = 9$

   *So we have found* $p = -5$ *and* $q = 14$

3. *Cheat :* $(10x - 5)(10x + 14)$ *This is not correct!*
   $= 5(2x - 1) \times 2(5x + 7)$

4. *Uncheat :* $10x^2 + 9x - 7 = (2x - 1)(5x + 7)$

# 2. Algebra & Graphs

YOUR NOTES

## 2.2.8 DIFFERENCE OF TWO SQUARES

### What is the difference of two squares?

- A "Difference Of Two Squares" is any expression in the form $p^2 - q^2$

### Why is it so important?

- It comes up a lot – recognising it can save a lot of time

### How does it factorise?

- $p^2 - q^2$ can always be factorised in the same way:

  $p^2 - q^2 = (p + q)(p - q)$
  (Try multiplying it out to see why…)

### Worked Example

1. *Factorise* $9x^2 - 16$

$$9x^2 - 16 = (3x)^2 - 4^2 = (3x+4)(3x-4)$$

2. *Factorise* $4x^2 - 25$

$$4x^2 - 25 = (2x)^2 - 5^2 = (2x+5)(2x-5)$$

# 2. Algebra & Graphs

YOUR NOTES
⬇

## 2.2.9 FACTORISING QUADRATICS - GENERAL

### What is a quadratic expression?

- A quadratic expression looks like this:

  $ax^2 + bx + c$ (as long as $a \neq 0$)

- Note: If there are any higher powers of x (like $x^3$ say) then it is not a quadratic!

### Does it factorise?

- It is sensible to check before trying to factorise a quadratic that it actually does factorise!
- A quadratic will factorise if (and only if) the discriminant ($b^2 - 4ac$) is a perfect square
- If it does factorise then you have to decide what to do:

### Decisions, decisions…

- Always check for a Common (numerical) Factor before you factorise – spotting it early will make life a lot easier
- Take it out and just leave it in front of a big bracket
- When the quadratic expression only has two terms check for:
  - Simple factorisation (no number term, ie. when $c = 0$ )
  - Difference Of Two Squares (no x term, ie. when $b = 0$)
- Once you have decided that it is a factorisable three term quadratic with no common numerical factors then you can follow the normal rules (see other Notes for those!)

# 2. Algebra & Graphs

YOUR NOTES

## Worked Example

*Factorise* $-8x^2 + 100x - 48$

*Spot the common factor of* $-4$ : $-8x^2 + 100x - 48 = -4(2x^2 - 25x + 12)$

*Check the discriminant for* $2x^2 - 25x + 12$ : $b^2 - 4ac = (-25)^2 - 4 \times 2 \times 12 = 529$

$529 = 23^2$ *is a perfect square so it factorises*

*Proceed with* $2x^2 - 25x + 12$ *as you would for factorising a harder quadratic (ie where* $a \neq 1$*).*

1. "$+12$" *means the signs will be the same*

   "$-25$" *means that both signs will be negative*

2. $a \times c = 2 \times 12 = 24$

   *The only numbers which multiply to give 24 and follow the sign rules in 1. are*

   $-1 \times (-24)$ *and* $-2 \times (-12)$ *and* $-3 \times (-8)$ *and* $-4 \times (-6)$

   *But only the first pair add to give* $-25$

   $-1 \times (-24) = 24$ *and* $-1 + (-24) = -25$

   *So we have found* $p = -1$ *and* $q = -24$

3. *Cheat* : $(2x - 24)(2x - 1)$ *This is not correct!*

   $= 2(x - 12) \times (2x - 1)$

4. *Uncheat* : $2x^2 - 25x + 12 = (x - 12)(2x - 1)$

*Now just put the whole thing back together* :

$$-8x^2 + 100x - 48 = -4(x - 12)(2x - 1)$$

# 2. Algebra & Graphs

YOUR NOTES
⬇

## 2.3 ALGEBRAIC FRACTIONS

### 2.3.1 ALGEBRAIC FRACTIONS - ADDING & SUBTRACTING

#### What is an algebraic fraction?

- An algebraic fraction is simply a fraction with an algebraic expression on the top (numerator) and/or the bottom (denominator)

#### Adding & subtracting

- Rules same as numeric fractions:

1. Find the lowest common bottom (denominator)
2. Write fractions with new bottoms
3. Multiply tops by the same as bottoms
4. Write as a single fraction (take care if subtracting)
5. Simplify the top

- Always leave your answer in as simple a form as possible (see Algebraic Fractions – Simplifying)

#### Exam Tip

Leaving the top and bottom of the fraction in factorised form will help you see if anything cancels.

# 2. Algebra & Graphs

YOUR NOTES ⬇

## Worked Example

1. *Express* $\frac{x}{x+4} - \frac{3}{x-1}$ *as a single fraction*

1. $LCD = (x+4)(x-1)$

2&3. $\frac{x}{x+4} - \frac{3}{x-1} = \frac{x(x-1)}{(x+4)(x-1)} - \frac{3(x+4)}{(x+4)(x-1)}$

4. $= \frac{x(x-1)-3(x+4)}{(x+4)(x-1)}$

5. $= \frac{x^2-x-3x-12}{(x+4)(x-1)}$

$= \frac{x^2-4x-12}{(x+4)(x-1)}$

*Now factorise the top to see if anything cancels :*

$= \frac{(x+2)(x-6)}{(x+4)(x-1)}$

*Nothing cancels so that's the final answer.*

2. *Express* $\frac{x-4}{2(x-3)} - \frac{x-1}{2x}$ *as a single fraction*

1. $LCD = 2x(x-3)$

2&3. $\frac{x-4}{2(x-3)} - \frac{x-1}{2x} = \frac{x(x-4)}{2x(x-3)} - \frac{(x-1)(x-3)}{2x(x-3)}$

4. $= \frac{x(x-4)-(x-1)(x-3)}{2x(x-3)}$

5. $= \frac{x^2-4x-(x^2-4x+3)}{2x(x-3)}$

$= \frac{x^2-4x-x^2+4x-3}{2x(x-3)}$

$= \frac{-3}{2x(x-3)}$

*There's nothing to factorise on the top, and nothing cancels, so that's the final answer.*

# 2. Algebra & Graphs

YOUR NOTES
⬇

## 2.3.2 ALGEBRAIC FRACTIONS - SIMPLIFYING FRACTIONS

### What is an algebraic fraction?

- An algebraic fraction is simply a fraction with an algebraic expression on the top (numerator) and/or the bottom (denominator)

### How do you simplify an algebraic fraction?

- When you have a single Algebraic Fraction (or two multiplied together) you may be able to simplify things by cancelling common factors
- Factorise top and bottom
- Cancel common factors
  That's it!

### Exam Tip

If you are asked to simplify an algebraic fraction and have to factorise the top or bottom it is very likely that one of the factors will be the same on the top and the bottom – you can use this to help you factorise difficult quadratics!

### Worked Example

$$Simplify \quad \frac{4x+6}{2x^2-7x-15}$$

1. $Top : 4x + 6 = 2\,(2x + 3)$

*Use the fact that there is a factor of* $(2x + 3)$ *to help factorise the bottom :*

$$Bottom : 2x^2 - 7x - 15 = (2x + 3)\,(x - 5)$$

$$so \qquad \frac{4x+6}{2x^2-7x-15} = \frac{2(2x+3)}{(2x+3)(x-5)}$$

2. $$= \frac{2}{(x-5)}$$

# 2. Algebra & Graphs

YOUR NOTES

## 2.3.3 ALGEBRAIC FRACTIONS - MULTIPLYING & DIVIDING

### What is an algebraic fraction?

- An algebraic fraction is simply a fraction with an algebraic expression on the top (numerator) and/or the bottom (denominator)

### Dividing algebraic fractions

- **Never** try to divide fractions
- **Instead** "flip'n'times"
- So "÷a/b" becomes "×b/a"
  and then follow the rules for multiplying…

### Multiplying algebraic fractions

1. **Simplify** by factorising and cancelling (ignore the × between the fractions)
2. Multiply the **tops** (numerators)
3. Multiply the **bottoms** (denominators)
4. **Simplify** by factorising and cancelling (if you missed something earlier).

# 2. Algebra & Graphs

YOUR NOTES ⬇

### Worked Example

*Divide* $\frac{x+3}{x-4}$ *by* $\frac{2x+6}{x^2-16}$, *giving your answer as a simplified fraction.*

*First "flip'n'times" :*

$$\frac{x+3}{x-4} \div \frac{2x+6}{x^2-16} = \frac{x+3}{x-4} \times \frac{x^2-16}{2x+6}$$

*Now follow rules for multiplying :*

1. $$\frac{\cancel{x+3}}{x-4} \times \frac{\cancel{x^2-16}}{2x+6} = \frac{\cancel{x+3}}{x-4} \times \frac{(x-4)(x+4)}{2(x+3)}$$

*Cancel the* $(x+3)$*s and the* $(x-4)$*s :*

2&3. $$= \frac{1}{1} \times \frac{1\times(x+4)}{2\times 1}$$

$$= \frac{x+4}{2}$$

4. *No need to factorise and cancel again as that was done in 1.*

# 2. Algebra & Graphs

YOUR NOTES
⬇

## 2.4 INDICES

### 2.4.1 ROOTS & INDICES - BASICS

#### What are indices?

- An Index (plural = indices) is just a power that a number (called the base) is raised to:


#### Laws of indices – what you need to know

- There are lots of very important laws (or rules)
- It is important that you know and can apply these
- Understanding the explanations will help you remember them:

# 2. Algebra & Graphs

YOUR NOTES
⬇

| Laws | Explanations |
|---|---|
| $a^1 = a$ | |
| $a^p \times a^q = a^{p+q}$ | $a^3 \times a^2$ $= (a \times a \times a) \times (a \times a)$ $= a^5$ |
| $a^p \div a^q = a^{p-q}$ | $a^5 \div a^3$ $= \frac{a \times a \times a \times a \times a}{a \times a \times a}$ $= a^2$ |
| $(a^p)^q = a^{p \times q}$ | $(a^3)^2$ $= (a \times a \times a) \times (a \times a \times a)$ $= a^6$ |
| $a^0 = 1$ | $a^0 = a^{2-2} = a^2 \div a^2 = \frac{a^2}{a^2} = 1$ |
| $a^{-p} = \frac{1}{a^p}$ | $a^{-3} = a^{0-3} = a^0 \div a^3 = \frac{a^0}{a^3} = \frac{1}{a^3}$ |
| $a^{\frac{1}{n}} = \sqrt[n]{a}$ | $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^1 = a = \sqrt{a} \times \sqrt{a}$ |
| $a^{\frac{p}{q}} = (\sqrt[q]{a})^p = \sqrt[q]{(a)^p}$ | $a^{\frac{3}{2}} = a^{\frac{1}{2} \times 3} = (a^{\frac{1}{2}})^3 = (\sqrt{a})^3$ $a^{\frac{3}{2}} = a^{3 \times \frac{1}{2}} = (a^3)^{\frac{1}{2}} = \sqrt{(a)^3}$ |

## Exam Tip

Take it slowly and apply the laws one at a time.

# 2. Algebra & Graphs

YOUR NOTES

Worked Example

*Simplify* $\sqrt{\dfrac{p^3 \times p^7}{p^6}}$

*Use the second law from above on the top of the fraction :*

$$\sqrt{\frac{p^3 \times p^7}{p^6}} = \sqrt{\frac{p^{3+7}}{p^6}} = \sqrt{\frac{p^{10}}{p^6}}$$

*Use the third law from above on the whole fraction :*

$$\sqrt{\frac{p^{10}}{p^6}} = \sqrt{p^{10-6}} = \sqrt{p^4}$$

*Use the seventh law from above to change the square root into a power :*

$$\sqrt{p^4} = (p^4)^{\frac{1}{2}}$$

*Use the fourth law from above to finish :*

$$(p^4)^{\frac{1}{2}} = p^{4 \times \frac{1}{2}} = p^2$$

# 2. Algebra & Graphs

YOUR NOTES

## 2.4.2 ROOTS & INDICES - HARDER

### What are indices?

- An Index (plural = indices) is just a power that a number (called the base) is raised to:


### Laws of indices – what you need to know

- There are lots of very important laws (or rules)
- It is important that you know and can apply these
- Understanding the explanations will help you remember them:

# 2. Algebra & Graphs

YOUR NOTES
⬇

| Laws | Explanations |
|---|---|
| $a^1 = a$ | |
| $a^p \times a^q = a^{p+q}$ | $a^3 \times a^2$<br>$= (a \times a \times a) \times (a \times a)$<br>$= a^5$ |
| $a^p \div a^q = a^{p-q}$ | $a^5 \div a^3$<br>$= \frac{a \times a \times a \times a \times a}{a \times a \times a}$<br>$= a^2$ |
| $(a^p)^q = a^{p \times q}$ | $(a^3)^2$<br>$= (a \times a \times a) \times (a \times a \times a)$<br>$= a^6$ |
| $a^0 = 1$ | $a^0 = a^{2-2} = a^2 \div a^2 = \frac{a^2}{a^2} = 1$ |
| $a^{-p} = \frac{1}{a^p}$ | $a^{-3} = a^{0-3} = a^0 \div a^3 = \frac{a^0}{a^3} = \frac{1}{a^3}$ |
| $a^{\frac{1}{n}} = \sqrt[n]{a}$ | $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^1 = a = \sqrt{a} \times \sqrt{a}$ |
| $a^{\frac{p}{q}} = (\sqrt[q]{a})^p = \sqrt[q]{(a)^p}$ | $a^{\frac{3}{2}} = a^{\frac{1}{2} \times 3} = (a^{\frac{1}{2}})^3 = (\sqrt{a})^3$<br>$a^{\frac{3}{2}} = a^{3 \times \frac{1}{2}} = (a^3)^{\frac{1}{2}} = \sqrt{(a)^3}$ |

## Exam Tip

Write numbers in the question with the same BASE if possible.

# 2. Algebra & Graphs

YOUR NOTES

## Worked Example

*Express* $\sqrt{\dfrac{8^2\times 2^7}{4^3}}$ *as a single power of* 2.

*Express* 8 *as* $2^3$ *and* 4 *as* $2^2$ :

$$\sqrt{\frac{8^2\times 2^7}{4^3}} = \sqrt{\frac{(2^3)^2\times 2^7}{(2^2)^3}}$$

*Use the fourth law from above to simplify* :

$$\sqrt{\frac{(2^3)^2\times 2^7}{(2^2)^3}} = \sqrt{\frac{2^{3\times 2}\times 2^7}{2^{2\times 3}}} = \sqrt{\frac{2^6\times 2^7}{2^6}}$$

*Use the second law from above on the top of the fraction* :

$$\sqrt{\frac{2^6\times 2^7}{2^6}} = \sqrt{\frac{2^{6+7}}{2^6}} = \sqrt{\frac{2^{13}}{2^6}}$$

*Use the third law from above on the whole fraction* :

$$\sqrt{\frac{2^{13}}{2^6}} = \sqrt{2^{13-6}} = \sqrt{2^7}$$

*Use the seventh law from above to change the square root into a power* :

$$\sqrt{2^7} = \left(2^7\right)^{\frac{1}{2}}$$

*Use the fourth law (again) from above to finish* :

$$\left(2^7\right)^{\frac{1}{2}} = 2^{7\times\frac{1}{2}} = 2^{\frac{7}{2}}$$

# 2. Algebra & Graphs

YOUR NOTES
⬇

## 2.5 LINEAR EQUATIONS

### 2.5.1 COLLECTING LIKE TERMS

#### Collecting like terms

- Expand brackets first!

1. TERMS are separated by + or –
2. “LIKE” terms must have exactly the same LETTER bit (the NUMBER bit can be different)
3. Add the COEFFICIENTS of like terms

#### Exam Tip

A “Coefficient” answers the question “how many?”

For example:

the coefficient of x in $2x^2 - 5x + 2$ is -5

and:

the coefficient of x in $ax^2 + bx + c$ is b

#### Worked Example

*Simplify* $x^2 - 3xy + 2x^2 + 4x - 2xy - x + 7$

$x^2 + 2x^2 - 3xy - 2xy + 4x - x + 7$

$= 3x^2 - 5xy + 3x + 7$

*1 – There are 7 terms to deal with*

*2 – Grouping like terms – "$x^2$"s, "$xy$"s, "$x$"s and constants*

*3 – Add the coefficients*

# 2. Algebra & Graphs

YOUR NOTES

## 2.5.2 SOLVING LINEAR EQUATIONS

### Solving equations…

… is just like rearranging formula so we'll use exactly the same method …

### GROF GROBLET FIND ANSWER!

- This is a mnemonic (way of remembering something) developed to deal with everything except roots and powers (see Rearranging Formulae – Extra Bits notes for that) and can also be used for rearranging formulae…

1. **GROF** – Get Rid Of Fractions
2. **GROB** – Get Rid Of Brackets
3. **LET** – Lump Everything Together (or Let's Examine Terms)
4. **FIN** – Factorise If Necessary
5. **D** – Divide
   ANSWER!

### Worked Example

*Solve the equation* $\frac{3x-2}{4-x} = -1$

| | |
|---|---|
| $3x-2=-1(4-x)$ | $1 - GROF$, *multiply both sides by* $(4-x)$, *careful with that* $-1$ ! |
| $3x-2=-4+x$ | $2 - GROB$, *expand brackets* |
| $2x=-2$ | $3 - LET$, *x terms on one side, numbers on the other*, "$-x$" *and* "$+2$" |
| | $4 - FIN$ − *you shouldn't need this step when solving equations* |
| $x=-1$ | $5 - D$, *divide both sides by* 2 |
| | *ANSWER*! |

# 2. Algebra & Graphs

## 2.6 LINEAR SIMULTANEOUS EQUATIONS

### 2.6.1 LINEAR SIMULTANEOUS EQUATIONS

#### What are linear simultaneous equations?

- When there are two unknowns (say x **and** y) in a problem, we need two equations to be able to find them both: these are called **Simultaneous Equations**
- If they just have x and y in them (no $x^2$ or $y^2$) then they are **Linear** Simultaneous Equations (They can be represented by two straight lines on a graph – the two answers are the x and y coordinates of the point of intersection of the lines)
- You may have to use the information in the question to write down the equations
- Look for an "is" (or equivalent) in the question and make sure you know what the letters you are using stand for
    - For example, if the question says:
      "The cost of 5 apples and 3 bananas is $1.35"
      then you can write down the equation: $5a + 3b = 135$
      where a is the cost of an apple in cents (and b is the cost of a banana in cents)

#### How do you solve linear simultaneous equations?

- The method described here is called the Elimination (or Balance) Method:**Label** the equations **A** and **B**

1. **Multiply A** and/or **B** by numbers to make the coefficients of x or y the same size
2. **Add** or **subtract** appropriate equations to eliminate that variable
   (Note: if the coefficients you made the same size in 1. have different signs you will be adding. If they have the same signs you will be subtracting.)
3. **Solve** the resulting equation
4. **Substitute** back into **A** or **B** and solve to find the other variable
5. **Check** your answer by substituting into the equation you didn't use in 4

#### Exam Tip

If one of the equations is written in the form **y = …** or **x = …** then you can use the method of **Substitution** (see Simultaneous Equations – Quadratic).

# 2. Algebra & Graphs

YOUR NOTES

## Worked Example

*Solve the simultaneous equations :*

$$3x - y = 13 \qquad A$$
$$2x + 3y = 5 \qquad B$$

1. $A \times 3:$ $\quad 9x - 3y = 39$

*The coefficients of y in the last two equations are the same size but have different signs so we can add these equations together to eliminate y :*

2. $B + A \times 3:$ $\quad 11x = 44$
3. *Solving gives :* $\quad x = 4$
4. *Sub x into A :* $\quad 3 \times 4 - y = 13$

   *and solve :* $\quad -y = 1$

   $y = -1$
5. *Check in B :* $\quad 2 \times 4 + 3 \times (-1) = 5$ ✓ *It works!*

*So the answer is*

$$x = 4, \; y = -1$$

# 2. Algebra & Graphs

YOUR NOTES

## 2.7 LINEAR INEQUALITIES

### 2.7.1 SOLVING INEQUALITIES - LINEAR

#### What is a linear inequality?

- An **Inequality** tells you that one expression is greater than (“>”) or less than (“<”) another
  - “⩾” means “greater than or equal to”
  - “⩽” means “less than or equal to”
- A **Linear Inequality** just has an **x** (and/or a **y**) etc in it and no $\mathbf{x^2}$ or similar
- For example, $3x + 4 \geqslant 7$ would be read “3x + 4 is greater than or equal to 7”

#### Solving linear inequalities

- Solving linear inequalities is just like **Solving Linear Equations** (so review these notes first)
- You also need to know how to use **Number Lines** and deal with **“Double” Inequalities**

1. Same rules as solving equations: **GROF GROBLET FIND ANSWER!**
But do **NOT** multiply (or divide) by negative numbers
2. When drawing **NUMBER LINES:**

< or > use an open circle ○ (end points are excluded)
⩽ or ⩾ use a closed circle ● (end points are included)

3. For “double” inequalities just do the same thing to all three parts

# 2. Algebra & Graphs

YOUR NOTES

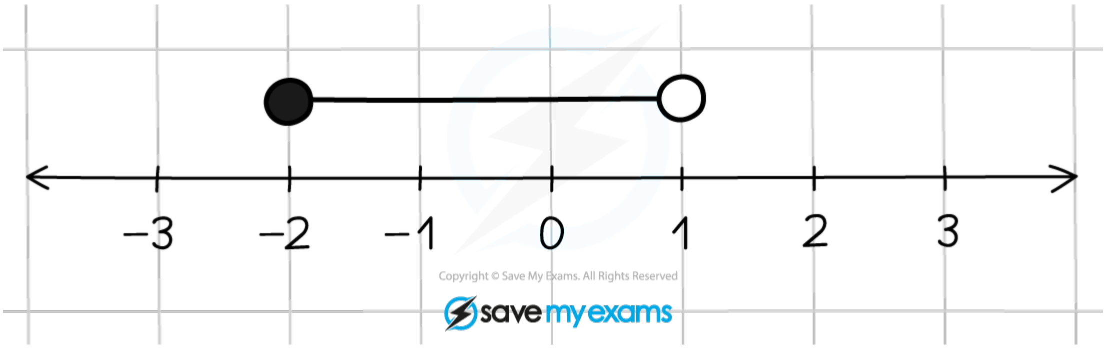
## Worked Example

1. *Solve the inequality* $-7 \leq 3x - 1 < 2$ *illustrating your answer on a number line.*

$-7 \leq 3x - 1 < 2$ — 3 – *First thing to notice is that this is a double inequality*

$-6 \leq 3x < 3$ — 1 – *We are trying to isolate x in the middle so first step is to add 1 to all three parts*

$-2 \leq x < 1$ — 1 – *Divide all three parts by 3*




2 – *Illustrate your final answer on a number line, using open and closed circles as appropriate*

# 2. Algebra & Graphs

YOUR NOTES

## 2.8 QUADRATIC EQUATIONS

### 2.8.1 QUADRATIC FORMULA

#### What is a quadratic equation?

- A quadratic equation looks like (or can/should be made to look like) this:

  $ax^2 + bx + c = 0$ (as long as $a \neq 0$)

#### What is the quadratic formula?

- The Quadratic Formula is a way of finding the roots (also called solutions) of a quadratic equation:

$$If\ ax^2 + bx + c = 0\ then\ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- The part of the formula under the square root (b2 – 4ac) is called the discriminant and it tells you a lot about the roots:
  - If $b^2 - 4ac > 0$ then there are two distinct (different) real roots (*)
  - If $b^2 - 4ac = 0$ then there is one real root (two repeated roots)
  - If $b^2 - 4ac < 0$ then there are no real roots (and the equation cannot be solved)
- Also (and not a lot of people know this!):
- If $b^2 - 4ac$ is a perfect square ( 1, 4, 9, 16, ...) then the quadratic can be factorised
  (*) We have to call them "real" roots for a very mathsy reason – don't worry about it too much – all the numbers you come across at GCSE are real!

# 2. Algebra & Graphs

YOUR NOTES
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## A quick note about calculators

If you have an "advanced" scientific calculator (like ones from the Casio Classwiz range) you may find it will solve quadratic equations for you. However, at GCSE, you should avoid using this feature as marks are awarded for method. At best you can use them to check your final answers but be aware that such calculators often show solutions even when the roots are not real.

## When do you use the quadratic formula?

You can use the Quadratic Formula whenever you need to solve a quadratic equation UNLESS you are specifically asked to do it by Factorising or Completing the Square. (You could still use the formula to check your answer!)

If the question asks you to give your answers to a certain degree of accuracy (eg 3 significant figures or 2 decimal places etc) then you will not be able to factorise and you MUST use the formula.

## How do you use the quadratic formula?

- The safest way to avoid errors is to break the formula down into stages:

1. WRITE DOWN the values of a, b and c – ie a = … , b = … and c = …
2. Work out the DISCRIMINANT, $d = b^2 - 4ac$
3. Substitute values into the QUADRATIC FORMULA including d
4. Work out the TWO ANSWERS on your calculator
   (Type in the formula twice, once with the "+" and once with the "-")

### Exam Tip

Write down more digits than you need from your calculator display.

Then do the rounding as a separate stage.

This will avoid errors and could save you a mark.

# 2. Algebra & Graphs

YOUR NOTES

## Worked Example

*Find the roots of the equation* $3x^2 - 2x - 4 = 0$, *giving your answers to 2 decimal places.*

| | |
|---|---|
| | *Exam Tip 2 – we have to use the formula* |
| $a = 3,\ b = -2,\ c = -4$ | 1 – *Write down the values of a, b and c* |
| $d = b^2 - 4ac$ | 2 – *Work out the value of d, the discriminant* |
| $d = (-2)^2 - 4 \times 3 \times (-4)$ | *Always be careful with negatives, use () around them* |
| $d = 52$ | |
| $x = \frac{2 \pm \sqrt{52}}{2 \times 3}$ | 3 – *Substitute values into the quadratic formula* |
| $x = 1.53518375\ldots$ *or* $-0.86851709\ldots$ | 4 – *Using the "+" and using the "−"* |
| | *Exam Tip 3 – write down more digits than required* |
| $x = 1.54$ *or* $-0.87$ *to 2 dp* | *Round to 2 decimal places for final answer* |

# 2. Algebra & Graphs

YOUR NOTES
⬇

## 2.8.2 COMPLETING THE SQUARE

### What is completing the square?

- "Completing the square" is something that can be done to a quadratic expression (to make it easier to work with or more useful in some way)
- It involves writing the quadratic expression $x^2 + px + q$ in the form $(x + a)^2 + b$

#### Exam Tip

A question on this topic (see the question below) may use the **Identity Symbol** "≡" instead of an **Equals Sign** "=".

This tells you that what is on the left is exactly the same as what is on the right (no matter what value any letters take).

You should never try to solve an identity like you might try to solve an equation.

#### Worked Example

# 2. Algebra & Graphs

YOUR NOTES

1. *Find integers a and b such that* $x^2 - 2x - 4 \equiv (x+a)^2 + b$.

| | |
|---|---|
| $a = \frac{1}{2} \times (-2) = -1$ | $1 - $ *Halve the coefficient of the x term :* $a = \frac{1}{2}p$ |
| $b = -4 - (-1)^2 = -5$ | $2 - $ *Subtract the square of a from q :* $b = q - a^2$ |
| $x^2 - 2x - 4 \equiv (x-1)^2 - 5$ | $3 - $ *Write down your answer using a and b* |

2. *Solve the equation* $4x^2 - 16x + 15 = 0$ *by completing the square.*

| | |
|---|---|
| $4x^2 - 16x + 15 = 4(x^2 - 4x + \frac{15}{4})$ | *Special case :* $m \neq 1$, *divide by* 4 |
| $= 4((x-2)^2 + \frac{15}{4} - (-2)^2)$ | $1, 2 - $ *Halve* $-4$ *and subtract* $(-2)^2$ *from q* |
| $= 4((x-2)^2 - \frac{1}{4})$ | $3 - $ *Finish completing the square* |
| $= 4(x-2)^2 - 1$ | *Special case : Multiply by m* |

| | |
|---|---|
| $4(x-2)^2 - 1 = 0$ | *Now we can solve by rearranging* |
| $4(x-2)^2 = 1$ | |
| $(x-2)^2 = \frac{1}{4}$ | |
| $x - 2 = \pm \frac{1}{2}$ | $\sqrt{\frac{1}{4}} = \pm \frac{1}{2}$ |
| $x = 2 \pm \frac{1}{2}$ | |
| $x = \frac{5}{2}$ *or* $x = \frac{3}{2}$ | *Decimals or fractions fine for final answers unless a specific format is asked for* |

# 2. Algebra & Graphs

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## 2.8.3 QUADRATIC SIMULTANEOUS EQUATIONS

### What are quadratic simultaneous equations?

- When there are two unknowns (say x and y) in a problem, we need two equations to be able to find them both: these are called **Simultaneous Equations**
- If there is an x<sup>2</sup> or y<sup>2</sup> in one of the equations then they are **Quadratic** (or **Non-Linear**) Simultaneous Equations
  (They can be represented by a straight line and a curve on a graph – the two pairs of answers are the points of intersection of the line and the curve)

### How do you solve quadratic simultaneous equations?

- This is called the **Substitution Method**:**Label** the equations **A** and **B**

1. Rearrange the **linear** equation to **y = …** (or **x = …**)
2. **Substitute** for **y** (or **x**) in the **quadratic** equation
3. **Multiply out** brackets
4. **Rearrange** to “quadratic = 0”
5. **Solve** using appropriate method (Factorisation or Formula)
6. **Substitute** back into the linear equation to find the other variable
7. **Check** your answer by substituting into the equation you didn’t use in 6

### Exam Tip

If the resulting quadratic has a **repeated root** then the line is a **tangent** to the curve.
If the resulting quadratic has **no roots** then the line does not intersect with the curve – or you have made a mistake!
When giving your final answer, make sure you indicate which x and y values go together. If you don’t make this clear you can lose marks for an otherwise correct answer.

# 2. Algebra & Graphs

YOUR NOTES

Worked Example

*Solve the simultaneous equations :*

$$3x - y = 13 \quad A$$
$$x^2 + y^2 = 17 \quad B$$

1. *Rearrange A :* $y = 3x - 13$
2. *Substitute into B :* $x^2 + (3x - 13)^2 = 17$
3. *Multiply out :* $x^2 + 9x^2 - 78x + 169 = 17$
4. *Rearrange :* $10x^2 - 78x + 152 = 0$
5. *Solve (÷ 2 first) :* $5x^2 - 39x + 76 = 0$

   *Using the Formula or Factorising gives :*

   $$x = 4 \quad or \quad x = \frac{19}{5}$$

6. *Substitute into A :* $y = -1 \quad or \quad y = -\frac{8}{5}$
7. *Now check in B :* $4^2 + (-1)^2 = 17$ ✓ *That one works!*

   $\left(\frac{19}{5}\right)^2 + \left(-\frac{8}{5}\right)^2 = 17$ ✓ *So does that one!*

*So the answers are*

$$x = 4,\ y = -1$$
$$or \quad x = \frac{19}{5},\ y = -\frac{8}{5}$$

# 2. Algebra & Graphs

YOUR NOTES

## 2.9 GRAPHICAL INEQUALITIES

### 2.9.1 INEQUALITIES ON GRAPHS - DRAWING

#### How do we draw inequalities on a graph?

- First, see Straight Line Graphs and Linear Inequalities

Once you do…

1. **DRAW** the line (as if using "=") for each inequality
   Use a **solid** line for ≤ or ≥ (to indicate the line is included)
   Use **dotted** line for < or > (to indicate the line is not included)
2. **DECIDE** which side of line is wanted:
   **Below** line if ≤ or <
   **Above** line if ≥ or >
   (Use the point (0, 0) as a test if unsure)
3. Shade **UNWANTED** side of each line (unless the question says otherwise)
   This is because it is easier, with pen/pencil/paper at least, to see which region has not been shaded at all than it is to look for a region that has been shaded 2-3 times or more
   (Graphing software often shades the area that is required but this is easily overcome by reversing the inequality sign)

#### Worked Example

1. *On the axes given show the region that satisfies the three inequalities*

$$3x + 2y \geq 12 \qquad y < 2x \qquad x < 3$$

*Label the region R.*

$1, 2, 3$ − *Draw* $3x + 2y = 12$ *with a solid line and shade below it*

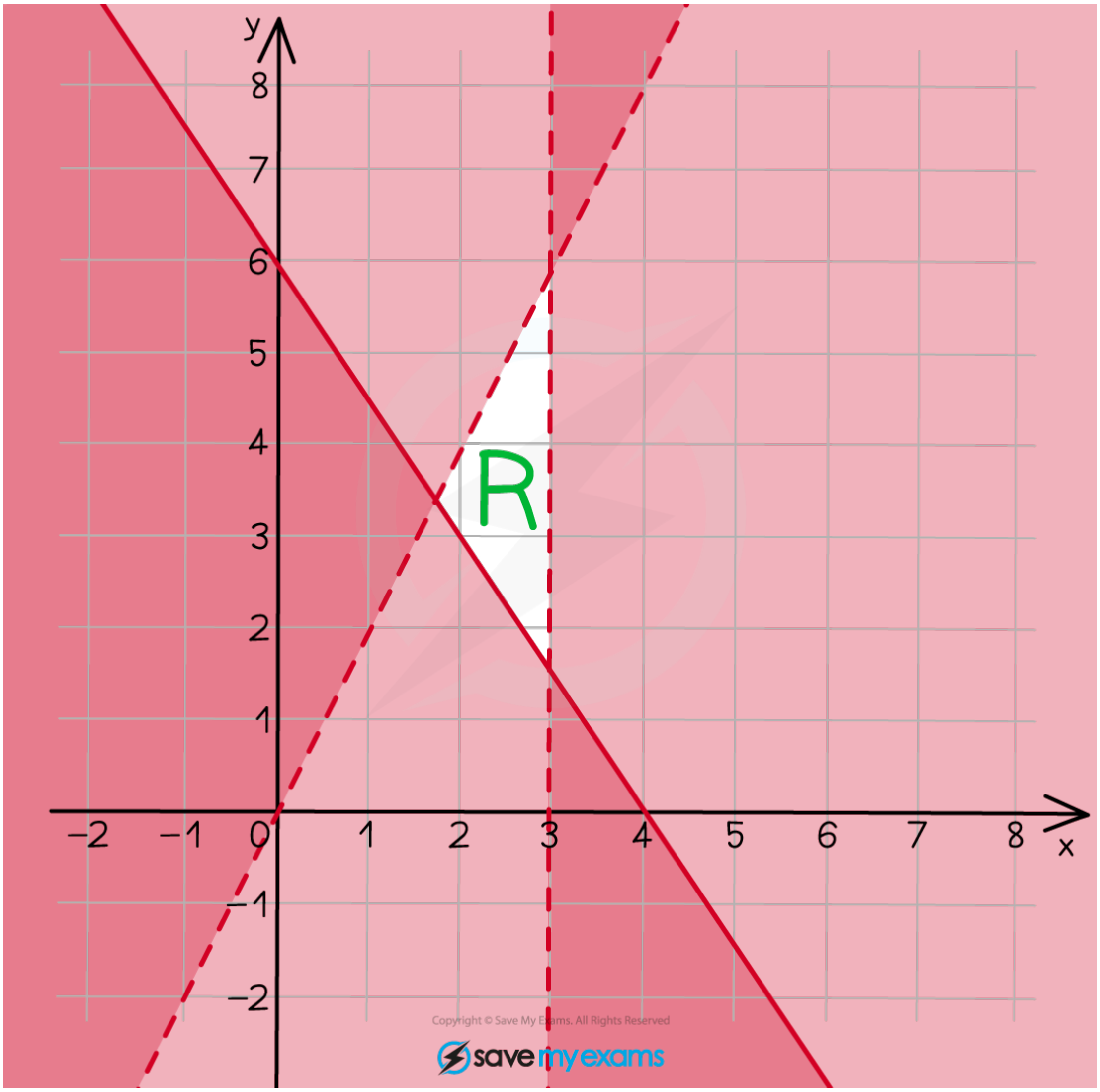
*(The point* $(0, 0)$ *is below this line)*

$1, 2, 3$ − *Draw* $y = 2x$ *with a dotted line and shade above it*

$1, 2, 3$ − *Draw* $x = 3$ *with a dotted line and shade to the right of it*

# 2. Algebra & Graphs

YOUR NOTES
⬇



# 2. Algebra & Graphs

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## 2.9.2 INEQUALITIES ON GRAPHS - INTERPRETING

### How do we interpret inequalities on a graph?

- First, see Inequalities and Straight Line Graphs

Once you do...

1. Write down the **EQUATION** of each line on the graph
2. **REMEMBER** that lines are drawn with:
   A **solid** line for ≤ or ≥ (to indicate line included in region)
   A **dotted** line for < or > (to indicate line not included)
3. **REPLACE** = sign with:
   ≤ or < if shading **below** line ≥ or
   > if shading **above** line
   (Use a point to test if not sure)

### Worked Example

1\. *Write down the three inequalities which define the shaded region on the axes below.*

# 2. Algebra & Graphs

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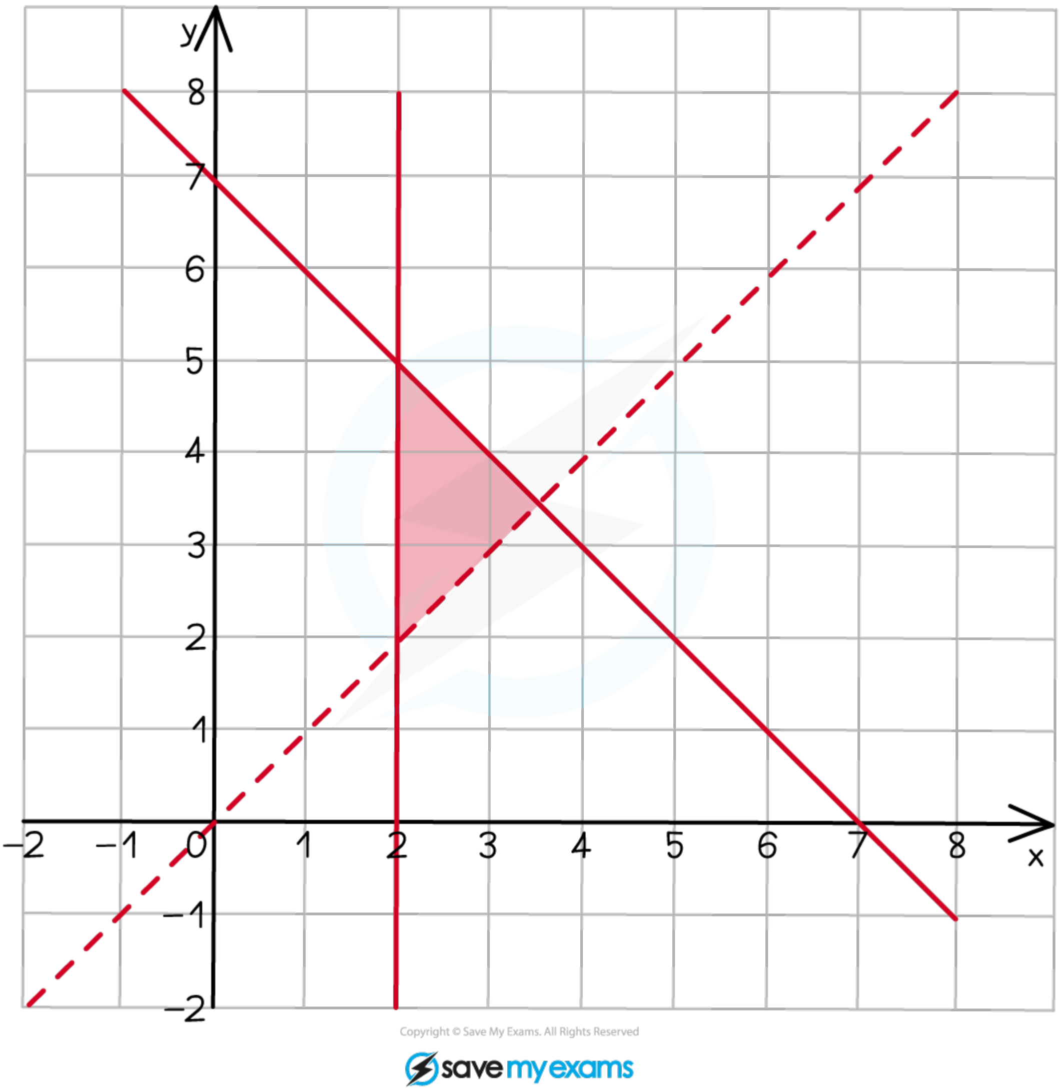


| | |
|---|---|
| $y > x$ | *1 − The dotted line has equation $y = x$* |
| | *2, 3 − Dotted line and region is above it so $>$* |
| | *(If we tested the point $(3, 4)$ from the shaded region* |
| | *then $4 > 3$ and so $y > x$)* |
| $x \geq 2$ | *1 − The solid vertical line is $x = 2$* |
| | *2, 3 − Solid line and region is to the right of it so $\geq$* |
| $x + y \leq 7$ | *1 − The other solid line has equation $x + y = 7$* |
| | *2, 3 − Solid line and region is below it so $\leq$* |

# 2. Algebra & Graphs

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## 2.10 SEQUENCES

### 2.10.1 SEQUENCES - LINEAR

#### What is a sequence?

- A sequence is simply a set of numbers (or objects) in an order

#### What is a linear sequence?

- A linear sequence is one where the numbers go up (or down) by the same amount each time
  - eg 1, 4, 7, 10, 13, ... (add 3 to get the next term)
  - 15, 10, 5, 0, -5, ... (subtract 5 to get the next term)
- If we look at the differences between the terms, we see that they are **constant**

#### What can we do with linear sequences?

- You should be able to recognise and continue a linear sequence
- You should also be able to find a formula for the **$n^{th}$ term** of a linear sequence in terms of n
- This formula will be in the form:

  $n^{th}$ term = dn + b
  where
  d is the common difference, b is a constant that makes the first term "work"

#### How to find the $n^{th}$ term formula for a linear sequence

- Find the common **difference** between the terms – this is d
- Put the first term and n=1 into the formula, then solve to find b

# 2. Algebra & Graphs

YOUR NOTES

## Worked Example

*For the sequence* 5, 7, 9, 11, 13, ..., *find the next three terms and a formula for the* $n^{th}$ *term.*

*Looking at the differences between the terms, we see that they are all* 2,
*so this is a Linear Sequence with common difference* 2.

*The next three terms are* :

$13 + 2 = 15, \quad 15 + 2 = 17, \quad 17 + 2 = 19$

*To find the* $n^{th}$ *term formula* :

1. *The common difference between the terms is* 2, *so* $d = 2$. *This means*

$n^{th}\ term \ = \ 2n + b$

2. *The first term is* 5, *so we put this and* $n = 1$ *into the formula* :

$5 \ = \ 2 \times 1 + b$

$b = 5 - 2 = 3$

$n^{th}\ term \ = \ 2n + 3$

# 2. Algebra & Graphs

YOUR NOTES

## 2.10.2 SEQUENCES - QUADRATIC

### What is a sequence?

- A sequence is simply a set of numbers (or objects) in an **order**

### What is a quadratic sequence?

- Unlike in a linear sequence, in a quadratic sequence the differences between the terms (the **first differences**) are not constant
- However, the differences between the differences (the **second differences**) are constant
- Another way to think about this is that in a quadratic sequence, the sequence of differences is a linear sequence
  eg Sequence 2, 3, 6, 11, 18, …
  1st Differences 1 3 5 7 (a Linear Sequence)
  2nd Differences 2 2 2 (Constant)
- Because the second differences there are constant, we know that the example is a 1quadratic sequence

### What can we do with quadratic sequences?

- You should be able to recognise and continue a quadratic sequence
- You should also be able to find a formula for the $n^{th}$ term of a quadratic sequence in terms of n
- This formula will be in the form:
  $n^{th}$ term = an2 + bn + c
  (The process for finding a, b, and c is given below)

### How to find the nth term formula for a quadratic sequence

1. Work out the sequences of first and second differences
   Note: check that the first differences are not constant and the second differences are constant, to make sure you have a quadratic sequence!
2. Use the first and second differences to find a, b, and c in the $n^{th}$ term formula
   Follow these steps in order:
   **2a** is the 1st second difference (or any second difference, as in a quadratic sequence they are all the same!)
   **3a + b** is the 1st first difference
   **a + b + c** is the 1st term

# 2. Algebra & Graphs

YOUR NOTES

## Exam Tip

Before doing the very formal process to find the $n^{th}$ term, try comparing the sequence to the square numbers 1, 4, 9, 16, 25, … and see if you can spot the formula.

For example:

Sequence 4, 7, 12, 19, 28, …

Square Numbers 1 4 9 16 25

We can see that each term of the sequence is 3 more than the equivalent square number so the formula is:

$n^{th}$ term = n2 + 3

This could save you a lot of time!

# 2. Algebra & Graphs

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## Worked Example

*For the sequence* $5,\ 7,\ 11,\ 17,\ 25,\ \ldots,$ *find the next three terms and a formula for the* $n^{th}$ *term.*

1. *Sequence* $5,\ 7,\ 11,\ 17,\ 25,\ \ldots$

   $1^{st}$ *Differences* $2 \quad 4 \quad 6 \quad 8$ *(a Linear Sequence)*

   $2^{nd}$ *Differences* $2 \quad 2 \quad 2$ *(Constant)*

   *So we know this is a Quadratic Sequence.*

   *The first differences increase by 2 each time, so the next three terms are :*

   $25 + 10 = 35,\ \ 35 + 12 = 47,\ \ 47 + 14 = 61$

2. $n^{th}\ term\ =\ an^2 + bn + c$

   *Looking at the* $1^{st}$ *second difference :*

   $2a = 2\ \ so\ \ a = 1$

   *Looking at the* $1^{st}$ *first difference :*

   $3a + b = 2\ \ so\ \ 3 \times 1 + b = 2\ \ so\ \ b = -1$

   *Looking at the* $1^{st}$ *term :*

   $a + b + c = 5\ \ so\ \ 1 - 1 + c = 5\ \ so\ \ c = 5$

   *Which gives :*

   $n^{th}\ term\ =\ n^2 - n + 5$

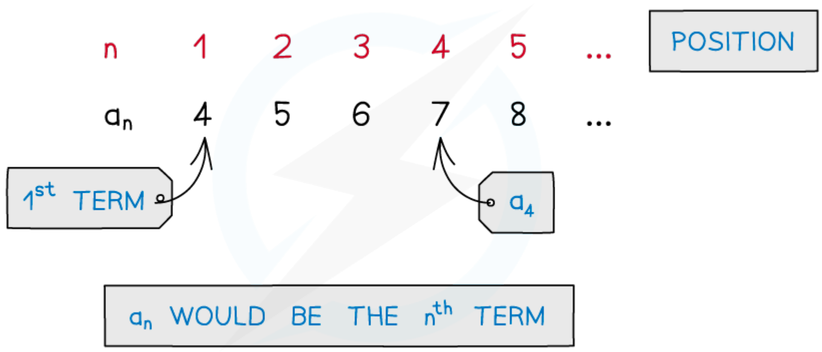
# 2. Algebra & Graphs

YOUR NOTES

## 2.10.3 SEQUENCES - BASICS

### What are sequences?

- A sequence is an order set of (usually) numbers
- Each number in a sequence is called a **term**
- The **location** of a **term** within a **sequence** is called its **position**
  - The letter **n** is often used for (an unknown) **position**
- Subscript notation is used to talk about a particular term
  - $\mathbf{a_1}$ would be the **first** term in a sequence
  - $\mathbf{a_7}$ would be the **seventh** term
  - $\mathbf{a_n}$ would be the $\mathbf{n^{th}}$ term






### What is a position-to-term rule?

- A **position-to-term** rule gives the $\mathbf{n^{th}}$ term of a sequence in terms of **n**
  - This is a very powerful piece of mathematics
  - With a position-to-term rule the $\mathbf{100^{th}}$ term of a sequence can be found without having to know or work out the first 99 terms!

# 2. Algebra & Graphs

YOUR NOTES
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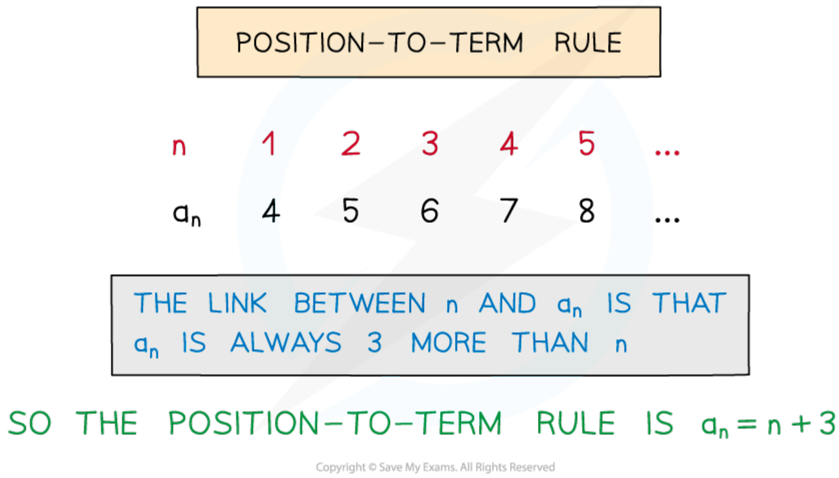
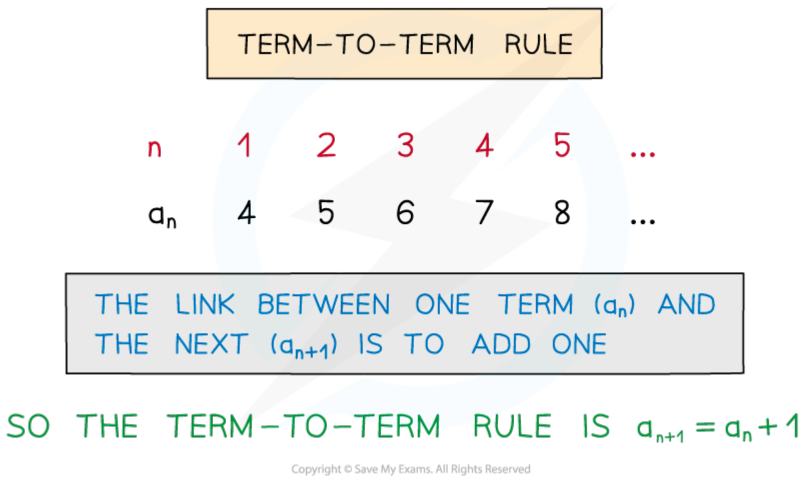



## What is a term-to-term rule?

- A **term-to-term** rule gives the **(n+1)**$^{\text{th}}$ term in terms of the **n**$^{\text{th}}$ term
    - ie $\mathbf{a_{n+1}}$ is given in terms of $\mathbf{a_n}$
    - If a term is known, the next one can be worked out



# 2. Algebra & Graphs

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## How do I use the position-to-term and term-to-term rules?

- These can be used to generate a sequence
- From a given sequence the rules can be deduced
- Recognising and being aware of the types of sequences helps
  - Linear and quadratic sequences
  - Geometric sequences
  - Fibonacci sequences
  - Other sequences

e.g. THE POSITION-TO-TERM RULE FOR A SEQUENCE IS $a_n = 2n - 3$

a) FIND THE FIRST FIVE TERMS OF THE SEQUENCE

$n = 1, \quad a_1 = 2 \times 1 - 3 = -1$

$n = 2, \quad a_2 = 2 \times 2 - 3 = 1$

$n = 3, \quad a_3 = 2 \times 3 - 3 = 3$

$n = 4, \quad a_4 = 2 \times 4 - 3 = 5$

$n = 5, \quad a_5 = 2 \times 5 - 3 = 7$

FIRST FIVE TERMS ARE:
$-1, 1, 3, 5, 7$

b) WRITE DOWN THE TERM-TO-TERM RULE FOR THE SEQUENCE

$-1 \xrightarrow{+2} 1 \xrightarrow{+2} 3 \xrightarrow{+2} 5, 7$

$a_{n+1} = a_n + 2$



### Exam Tip

Write the position numbers above (or below) each term in a sequence. This will make it much easier to recognise and spot common types of sequence.

# 2. Algebra & Graphs

YOUR NOTES

(a) Find the first 5 terms of the following sequences

(i) $n^{th}$ term $= 20 - 2n$ (ii) $n^{th}$ term $= 2n^2 - 3$

(iii) $n^{th}$ term $= \dfrac{1}{2n}$

(b) Find the 2nd, 3rd and 4th terms in each of these sequences

(i) $a_{n+1} = a_n + 6$ with $a_1 = 3$

(ii) $a_{n+1} = 3a_n{}^2 - 1$ with $a_1 = 2$

(iii) $a_{n+2} = a_{n+1} + a_n$ with $a_1 = 4$ and $a_2 = 6$

a) "$n^{th}$ TERM = ..." MEANS WE HAVE THE POSITION-TO-TERM RULE

i)

| $n$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $a_n$ | 18 | 16 | 14 | 12 | 10 |

$20 - 2 \times 1$

THE CALCULATIONS ARE USUALLY EASY SO YOU CAN DO THEM MENTALLY

YOU MAY SPOT THIS IS A LINEAR SEQUENCE

ii)

| $n$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $a_n$ | $-1$ | 5 | 15 | 29 | 47 |

$2 \times 3^2 - 3$

YOU MAY SPOT THIS IS A QUADRATIC SEQUENCE

iii)

| $n$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $a_n$ | 1/2 | 1/4 | 1/6 | 1/8 | 1/10 |

$\dfrac{1}{2 \times 4}$

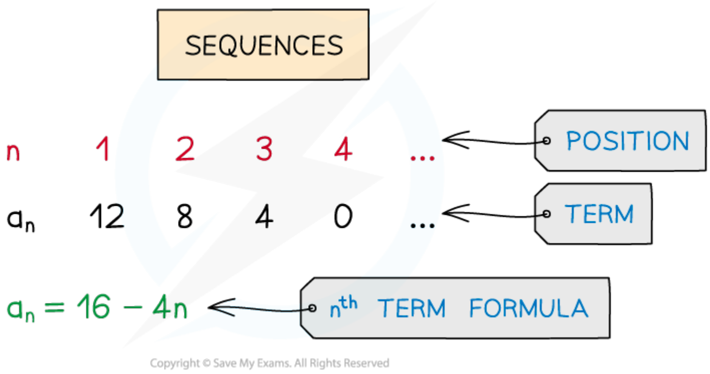
# 2. Algebra & Graphs

YOUR NOTES

## 2.10.4 SEQUENCES - IDENTIFYING

### What are sequences?

- A sequence is an order set of (usually) numbers
- Make sure you are familiar with the **basics** and **notation** used with sequences




### How do I identify a sequence?

- Is it obvious?
- Does it tell you in the question?
- Is there a number that you multiply to get from one term to the next?
  - If so then it is a **geometric** sequence
- Next, look at the **differences** between the terms

  If **1st differences** are constant – it is a **linear** sequence

  If **2nd differences** are constant – it is a **quadratic** sequence

# 2. Algebra & Graphs

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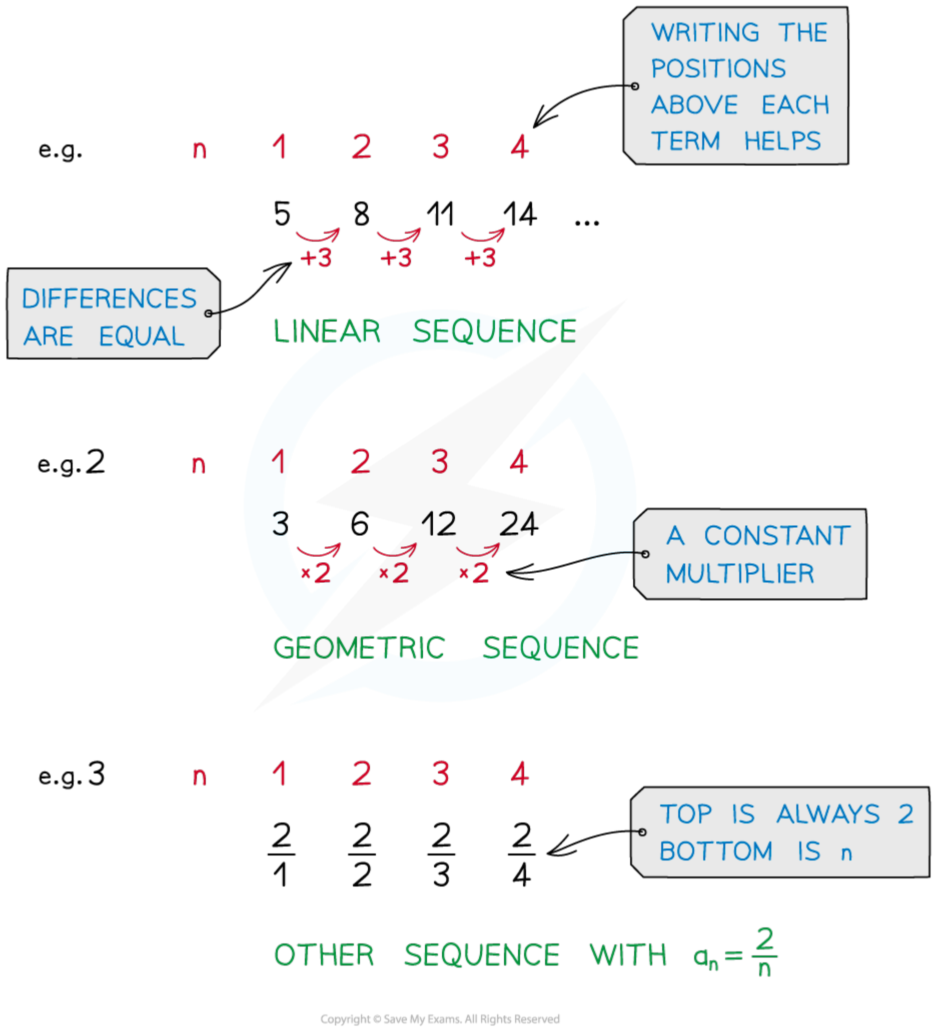


- Special cases to be aware of:
- If the differences **repeat** the original sequence
    - It is a **geometric** sequence with **common ratio** 2
- **Fibonacci** sequences also have **differences** that **repeat** the original sequence
    - However questions usually indicate if a Fibonacci sequence is involved

# 2. Algebra & Graphs

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e.g.

| n | 1 | 2 | 3 | 4 | 5 | ... |
|---|---|---|---|---|---|---|
| | 4 | 8 | 16 | 32 | 64 | ... |

Differences: +4, +8, +16, +32
×2, ×2, ×2, ×2

DIFFERENCES ARE REPEATING ORIGINAL SEQUENCE.
THERE IS A COMMON RATIO OF 2 SO MUST BE GEOMETRIC

GEOMETRIC SEQUENCE

e.g.2

3 7 10 17 27 ...

Differences: +3, +7, +10

DIFFERENCES ARE REPEATING THE ORIGINAL SEQUENCE
NO COMMON RATIO SO MUST BE FIBONACCI

FIBONACCI SEQUENCE

## Worked Example

(a) Identify the types of sequence below

(i) 4, 5, 9, 14, 23, 37, 60, ...

(ii) 6, 10, 16, 24, 34

(iii) 12, 7, 2, -3, ...

(b) Write down a formula for the $n^{th}$ term (in terms of $n$) for each of the following sequences

(i) $\frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \frac{4}{6},$

(ii) 2, 6, 12, 20, 30, 42, ...

a) i) 4 5 9 14 23 37 60 ...

+4 +5 +9 +14 +23

DIFFERENCES ARE REPEATING THE ORIGINAL SEQUENCE
GEOMETRIC – NO, AS NO COMMON RATIO.
FIBONACCI ✓

FIBONACCI SEQUENCE

ii) 6 10 16 24 34

+4 +6 +8 +10

+2 +2 +2

1st DIFFERENCES ARE **NOT** EQUAL

2nd DIFFERENCES ARE EQUAL

QUADRATIC SEQUENCE

iii) 12 7 2 −3

−5 −5 −5

1st DIFFERENCES ARE EQUAL

SEQUENCE IS GOING DOWN SO DIFFERENCES ARE NEGATIVE

LINEAR SEQUENCE

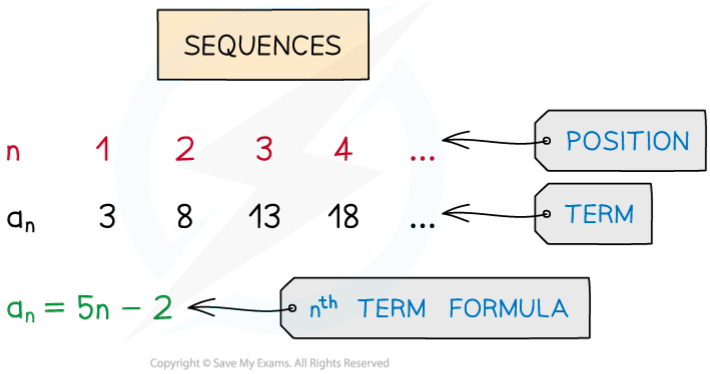
# 2. Algebra & Graphs

YOUR NOTES ⬇

## 2.10.5 SEQUENCES - OTHERS

### What are sequences?

- A sequence is an order set of (usually) numbers
- Make sure you are familiar with the **basics** and **notation** used with sequences




### What are “other” sequences?

- **Linear** and **quadratic** sequences are particular types of sequence covered in previous notes
- “Other” sequences include geometric and Fibonacci sequences, both briefly mentioned here Sequences – Identifying
- Geometric and Fibonacci are looked at in more detail below
- But other sequences can include fractions, decimals
  - Anything that makes the position-to-term and/or the term-to-term rule easy to spot

# 2. Algebra & Graphs

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## TYPES OF SEQUENCES

**LINEAR**

e.g. 2, 6, 10, 14, 18 ...

+4 +4 +4 +4

FIRST DIFFERENCES CONSTANT

**QUADRATIC**

e.g. 5, 9, 15, 23, 33 ...

+4 +6 +8 +10

+2 +2 +2

SECOND DIFFERENCES CONSTANT

**GEOMETRIC**

e.g. 4, 12, 36, 108, 324 ...

×3 ×3 ×3 ×3

CONSTANT MULTIPLIER (COMMON RATIO)

**FIBONACCI**

e.g. 2, 4, 6, 10, 16, 26, 42 ...

ADD THE PREVIOUS TWO TERMS

**OTHER**

e.g.

| n | 1 | 2 | 3 | 4 | |
|---|---|---|---|---|---|
| | 1 | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | ... |

$n^{th}$ TERM, $a_n = \frac{1}{n}$

SUCH SEQUENCES DON'T FALL INTO ANY CATEGORY BUT THE LINK BETWEEN n AND $a_n$ IS FAIRLY EASY TO SPOT

# 2. Algebra & Graphs

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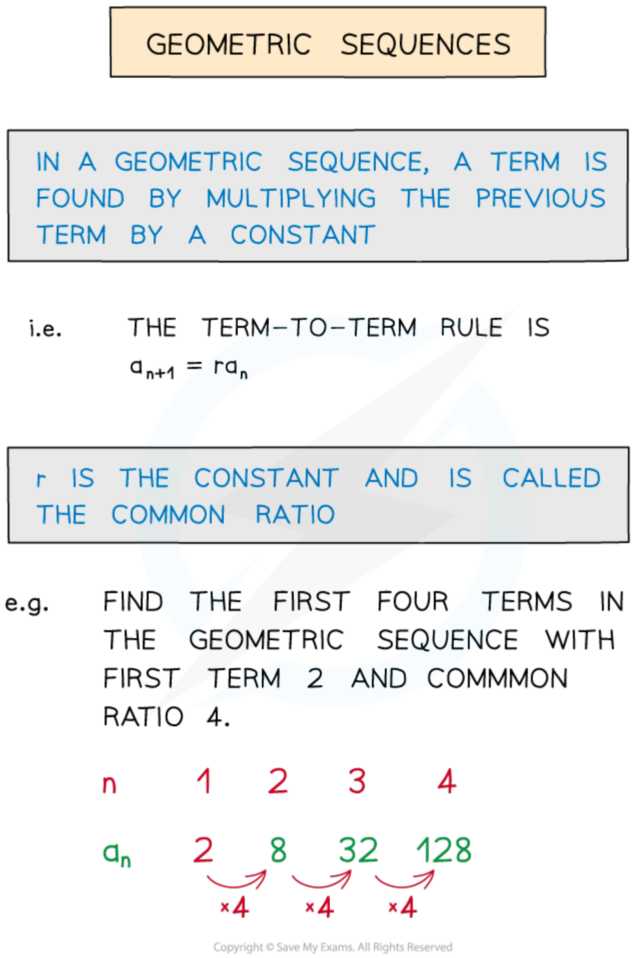
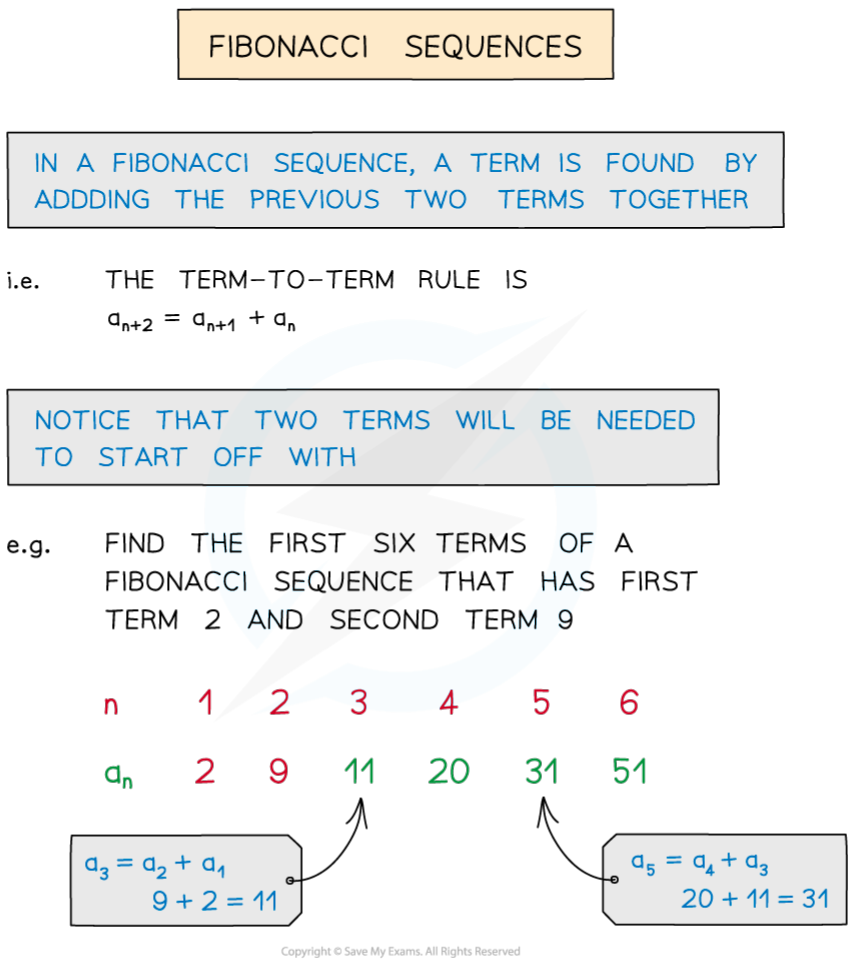
## What is a geometric sequence?

- In a geometric sequence, the term-to-term rule would be to multiply by a constant
  - $\mathbf{a_{n+1} = ra_n}$
- **r** is called the **common ratio**
- In the sequence 4, 8, 16, 32, 64, ... the common ratio would be 2



## What is a Fibonacci sequence?

- **THE** Fibonacci sequence is **1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...**
- The sequence starts with the **first two terms** as **1**
- Each subsequent term is the **sum** of the **previous two**
  - ie The term-to-term rule is $\mathbf{a_{n+2} = a_{n+1} + a_n}$
  - Notice that two terms are needed to start a Fibonacci sequence
- Any sequence that has the term-to-term rule of adding the previous two terms is called a **Fibonacci** sequence but the first two terms will not both be 1
- Fibonacci sequences occur a lot in nature such as the number of petals of flowers



# 2. Algebra & Graphs

YOUR NOTES

## Problem solving and sequences

- When the type of sequence is known it is possible to find unknown terms within the sequence
- This can lead to problems involving setting up and solving equations
  - Possibly simultaneous equations
- Other problems may involve sequences that are related to common number sequences such as square numbers, cube numbers and triangular numbers

e.g. IN A FIBONACCI SEQUENCE THE $4^{th}$ TERM IS $2a$, AND THE $5^{th}$ TERM IS $b + 1$

a) WRITE DOWN EXPRESSIONS FOR THE $6^{th}$ AND $7^{th}$ TERMS

$6^{th}$ TERM $= (b + 1) + 2a$

$a_6 = a_5 + a_4$

$6^{th}$ TERM $= 2a + b + 1$

$7^{th}$ TERM $= (2a + b + 1) + (b + 1)$

$a_7 = a_6 + a_5$

$7^{th}$ TERM $= 2a + 2b + 2$

b) GIVEN $a_6 = 20$ AND $a_7 = 32$ FIND THE VALUES OF $a$ AND $b$

$2a + b + 1 = 20$

$2a + 2b + 2 = 32$

SOLVE AS SIMULTANEOUS EQUATIONS

$2a + b = 19$

$2a + 2b = 30$

$b = 11$

SUBTRACT

$2a + 11 = 19$

$2a = 8$

$a = 4$

SUBSTITUTE VALUE OF $b$ INTO ANY EQUATION USED

$a = 4$ AND $b = 11$

# 2. Algebra & Graphs

YOUR NOTES

## Worked Example

(a) Three consecutive terms in a geometric sequence are

$5x+9$, $2x$ and $x-3$

Find the two possible values of $x$.

(b) The 3rd and 6th terms in a Fibonacci sequence are 7 and 31 respectively.
Find the 1st and 2nd terms of the sequence.

(c) Write down a formula for $a_n$ for the following sequences

(i) 6, 10, 15, 21, 28, ...
(ii) 0, 7, 26, 63, 124, ...
(iii) 5, 15, 45, 135, 405, ...

a) $r(5x+9) = 2x$

$r(2x) = x-3$

$r = \dfrac{2x}{5x+9}$ $\qquad r = \dfrac{x-3}{2x}$

SO, $\dfrac{2x}{5x+9} = \dfrac{x-3}{2x}$

ONCE YOU ARE FAMILIAR WITH GEOMETRIC SEQUENCES YOU CAN START WITH THE LINE ABOVE. DO TRY TO UNDERSTAND WHERE IT HAS COME FROM

$(2x)(2x) = (x-3)(5x+9)$

MULTIPLY TO GET RID OF FRACTIONS (GROF)

$4x^2 = 5x^2 - 6x - 27$

EXPAND (USING FOIL MAYBE?) TO GET RID OF BRACKETS (GROB)

$x^2 - 6x - 27 = 0$ QUADRATIC EQUATION

$(x-9)(x+3) = 0$

$x = 9, \qquad x = -3$

IN THIS QUESTION, BOTH ANSWERS ARE VALID. ON ANOTHER QUESTION, YOU MAY BE TOLD THAT $x > 0$, IN WHICH CASE **ONLY** $x = 9$ WOULD BE VALID

# 2. Algebra & Graphs

YOUR NOTES ⬇

b) WRITE AT WHAT YOU DO KNOW ABOUT THE SEQUENCE

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $a_n$ | | $x$ | 7 | $x+7$ | $x+14$ | 31 | |

$a_{n+2} = a_{n+1} + a_n$ FIBONACCI – TOLD IN QUESTION

$a_3 = a_2 + a_1$ $\quad a_2 + a_1 = 7$

$a_6 = a_5 + a_4$ $\quad a_5 + a_4 = 31$

LET $a_2 = x$, THEN, $a_4 = x + 7$

AND $a_5 = (x+7)+7$

$= x + 14$

SO $(x+14)+(x+7) = 31$ $\quad \boxed{a_6 = a_5 + a_4 = 31}$

$2x + 21 = 31$

$2x = 10$

$x = 5$ $\quad a_2 = 5$

$a_3 = a_2 + a_1$

$7 = 5 + a_1$ $\quad a_1 = 2$

# 2. Algebra & Graphs

YOUR NOTES

c) i)

| $n$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $a_n$ | 6 | 10 | 15 | 21 | 28 |

**CAREFUL!** IF YOU IMMEDIATELY CONSIDER DIFFERENCES THIS WILL LEAD TO A QUADRATIC SEQUENCE. WHICH IS CORRECT BUT ITS STLL TRICKY TO GET THE $n^{th}$ TERM FORMULA AS IT IS NOT "OBVIOUS"

YOU SHOULD SPOT THAT 6, 10, 15, ... ARE TRIANGULAR NUMBERS, SO COMPARE THE SEQUENCE TO THOSE

| $n$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $\frac{1}{2}n(n+1)$ | 1 | 3 | 6 | 10 | 15 |
| $a_n$ | 6 | 10 | 15 | 21 | 28 |

$$a_n = \frac{1}{2}(n+2)(n+2+1)$$

THE TRIANGULAR NUMBERS ARE "2 AHEAD" SO REPLACE "$n$" WITH "$n+2$" IN THE $n^{th}$ TERM FORMULA

$$a_n = \frac{1}{2}(n+2)(n+3)$$

ii)

| $n$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $a_n$ | 0 | 7 | 26 | 63 | 124 |

NUMBERS GROW QUICKLY SO NOT LIKELY LINEAR OR QUADRATIC.
NOT FIBONACCI, NOTHING LIKE TRIANGULAR.
TRY CUBE NUMBERS

| $n$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $n^3$ | 1 | 8 | 27 | 64 | 125 |
| $a_n$ | 0 | 7 | 26 | 63 | 124 |

a – ha!

$$a_n = n^3 - 1$$

iii)

| $n$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $a_n$ | 5 | 15 | 45 | 135 | 405 |

×3 ×3 ×3 ×3

GEOMETRIC SEQUENCE

TERM–TO–TERM IS $a_{n+1} = 3a_n$

POSITION–TO–TERM IS NOT THAT OBVIOUS – UNTIL YOU'VE SEEN IT ONCE! HERE GOES...

$a_2 = 3a_1$
$a_3 = 3a_2 = 3(3a_1)$
$a_4 = 3a_3 = 3(3(3a_1))$
$a_5 = 3a_4 = 3(3(3(a_1)))$ etc

i.e. $a_2 = 3a_1$
$a_3 = 3^2a_1$
$a_4 = 3^3a_1$
$a_5 = 3^4a_1$

WE KNOW $a_1 = 5$, SO...

$$a_n = 5 \times 3^{n-1}$$

# 2. Algebra & Graphs

YOUR NOTES

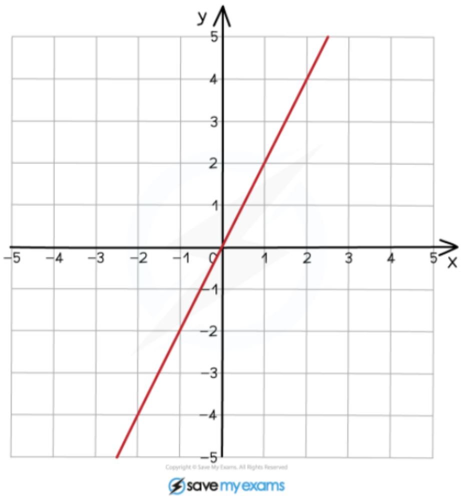
## 2.11 DIRECT & INVERSE PROPORTION

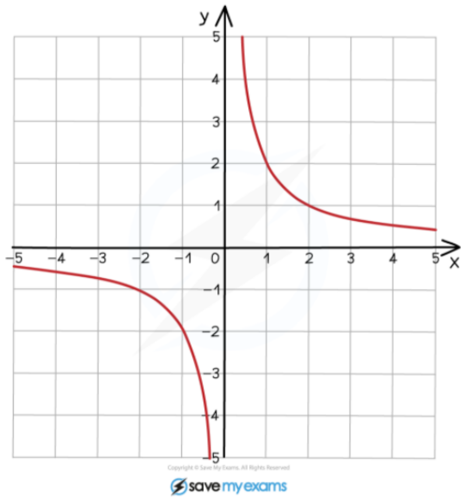
### 2.11.1 DIRECT & INVERSE PROPORTION

#### What is proportion?

- **Proportion** is a way of talking about how two **variables** are related to each other
- **Direct** proportion: as one variable goes **up** the other goes up by the same **factor**
  eg. If one variable is multiplied by 3, so is the other
- **Inverse** proportion: as one variable goes up the other goes **down** by the same **factor**
  eg. If one variable is multiplied by 3, the other is **divided** by 3

You should know what the graphs look like:





#### How do we deal with proportion questions?

1. Identify the two **VARIABLES** (called **A** and **B** below)
2. Choose **TYPE** of proportion:
   **A** is **DIRECTLY** proportional to **B** use formula **A = kB**
   **A** is **INVERSELY** proportional to **B** use formula **A = k÷B**
3. **FIND k** using the values given in the question
4. Write **FORMULA** for **A** in terms of **B** (using your value of **k**)
5. **USE** formula to find the required quantity

# 2. Algebra & Graphs

YOUR NOTES

### Exam Tip

Even if the question doesn't ask for a formula (Step 4. above) it is always worth working one out and using it in all but the simplest cases.

### Worked Example

1. *$y$ is directly proportional to the square of $x$*

*When $x = 3$, $y = 18$*

*Find the value of $y$ when $x = 4$.*

| | |
|---|---|
| $y,\ x^2$ | 1 − *Identify the two variables* |
| $y = kx^2$ | 2 − *We are told this is DIRECT proportion* |
| $18 = k \times 3^2$ | 3 − *We can now find $k$ using $y = 18$ when $x = 3$* |
| $k = \frac{18}{3^2} = 2$ | |
| $y = 2x^2$ | 4 − *We can now write the full equation in $x$ and $y$* |
| $y = 2 \times 4^2$ | 5 − *And we can use this formula to find the value of $y$ when $x = 4$* |
| $y = 32$ | |

# 2. Algebra & Graphs

YOUR NOTES

## 2.12 FUNCTIONS

### 2.12.1 FUNCTIONS - BASICS

#### What is a function?

- A function is simply a mathematical "machine" that takes one set of numbers and changes them into another set of numbers according to a set rule
  eg. If the function (rule) is "double the number and add 1"
  - Putting 3 in to the function would give $2 \times 3 + 1 = 7$ out
  - Putting -4 in would give $2 \times (-4) + 1 = -7$ out
  - Putting a in would give $2a + 1$ out
- The number being put into the function is often called the input
- The number coming out of the function is often called the output

#### What does a function look like?

- A function f can be written as:
  $f(x) = \ldots$ or $f : x \mapsto \ldots$
  which mean exactly the same thing
- eg. The function with the rule "triple the number and subtract 4" would be written:
  $f(x) = 3x - 4$ or $f{:}x \mapsto 3x - 4$
- In such cases, x would be the input and f would be the output
- Sometimes functions don't have names like f and can be written as $y = \ldots$
  eg. $y = 3x - 4$

#### How does a function work?

1. A function has an INPUT (x) and OUTPUT (f or y)
2. Whatever goes in the bracket (instead of x)with f, replaces the x on the other side
   eg. For the function $f(x) = 2x + 1$
   - $f(3) = 2 \times 3 + 1=7$
   - $f(-4) = 2 \times (-4) + 1 = -7$
   - $f(a) = 2a + 1$

# 2. Algebra & Graphs

YOUR NOTES

## Worked Example

$A\ function\ is\ defined\ as\ f(x) = 3x^2 - 2x + 1$

$(a)\ Find\ f(7)$

$(b)\ \ Find\ f(x+3)$

$(a)\ \ f(7) = 3 \times 7^2 - 2 \times 7 + 1$ $\quad 2 -$ *Substitute x's for 7's on the right hand side*

$= 147 - 14 + 1$ $\quad$ *Use a calculator if allowed, if not take your*

$= 134$ $\quad$ *time and be accurate.*

$(b)\ \ f(x+3) = 3(x+3)^2 - 2(x+3) + 1$ $\quad 2 -$ *Substitute x's for (x + 3)'s*

$= 3(x^2 + 6x + 9) - 2x - 6 + 1$ $\quad$ *Expand and simplify*

$= 3x^2 + 18x - 2x + 27 - 5$

$= 3x^2 + 16x + 22$

# 2. Algebra & Graphs

YOUR NOTES

## 2.12.2 COMPOUND FUNCTIONS

### What is a compound function?

- A compound function is one function applied to the output of another function

### What do compound functions look like?

- The notation you will see is:
  $fg(x)$
- it can be written as:
  $f(g(x))$
  and means "f applied to the output of g(x)" – ie. g(x) happens FIRST !

### How does a compound function work?

- If you are putting a number into fg(x):

1. Put the number into g(x)
2. Put the output into f(x)
   eg. if $f(x) = 2x + 1$ and $g(x) = 1/x$
   then $fg(2) = f(½) = 2 \times ½ + 1 = 2$
   and $gf(2) = g(2 \times 2 + 1) = g(5) = 1/5$

- If you are using algebra:

1. For fg(x) put g(x) wherever you see x in f(x)
2. Substitute g(x) with the right hand side of g(x)=…
3. SIMPLIFY if necessary
   eg. if $f(x) = 2x + 1$ and $g(x) = 1 / x$
   then $fg(x) = f(1/x) = 2 \times 1 / x + 1 = 2 / x +1$
   and $gf(x) = g(2 \times x + 1) = g(2x + 1) = 1 / 2x+1$

### Exam Tip

Make sure you are applying the functions in the correct order.

The letter nearest the bracket is the function applied first.

# 2. Algebra & Graphs

YOUR NOTES

## Worked Example

*In this question* $f(x) = 2x - 1$ *and* $g(x) = (x + 2)^2$

(a) Find $fg(4)$

(b) Find $gf(x)$

$$\begin{aligned}(a)\ fg(4) &= f(g(4)) && 1 - g(x) \text{ happens first, put the number, 4, into that}\\ &= f((4+2)^2) && 2 - \text{The output is } 6^2 \text{ (or 36) so this is the input for } f(x)\\ &= 2 \times 36 - 1 &&\\ &= 71 &&\end{aligned}$$

$$\begin{aligned}(b)\ gf(x) &= g(f(x)) && 3 - f(x) \text{ happens first and we are working with algebra}\\ &= g(2x-1) && 4 - \text{Substitute } f(x) \text{ with the right hand side } (2x-1)\\ &= ((2x-1)+2)^2 &&\\ &= (2x+1)^2 && 5 - \text{And simplify}\end{aligned}$$

# 2. Algebra & Graphs

YOUR NOTES

## 2.12.3 INVERSE FUNCTIONS

### What is an inverse function?

- An inverse function does the exact opposite of the function it came from
- Eg. if the function "doubles the number and adds 1"
  then its inverse will "subtract 1 and halve the result"
- It is the **INVERSE** operations in the **reverse** order

### What do inverse functions look like?

- An inverse function f-1 can be written as:

  $f^{-1}(x) = \dots$ or $f^{-1} : x \mapsto \dots$
- Eg. if f(x) = 2x + 1 its inverse can be written as:

  $f^{-1}(x) = x - 1 / 2$ or $f^{-1}: x \mapsto x - 1 / 2$

### What do you find an inverse function?

1. Write the function in the form y=…
2. SWAP the xs and ys to get x=…
3. REARRANGE to give y=… again
4. Write as $f^{-1}(x) = \dots$ (or $f^{-1} : x \mapsto \dots$)

- Eg. if f(x)=2x+1 its inverse can be found as follows …
  y= 2x + 1 1 – 1. Write the function as y=…
  x = 2y + 1 2 – 2. Swap the xs and ys
  x – 1= 2y 3 – 3. Rearrange into the form y=… (or …=y)
  x – 1 / 2 = y

  $f^{-1}(x) = x - 1 / 2$ – 4. Write as $f^{-1}(x)=\dots$

# 2. Algebra & Graphs

YOUR NOTES

## Worked Example

1. *Find the inverse of the function* $f(x) = 5 - 3x$

*Give your answer in the form* $f^{-1}(x) = \ldots$

| | |
|---|---|
| $y = 5 - 3x$ | 1 – *Write the function as* $y = \ldots$ |
| $x = 5 - 3y$ | 2 – *Swap the xs and ys* |
| $3y = 5 - x$ | 3 – *Rearrange into the form* $y = \ldots$ |
| $y = \frac{5-x}{3}$ | |
| $f^{-1}(x) = \frac{5-x}{3}$ | 4 – *Write final answer in the required format* |

# 2. Algebra & Graphs

YOUR NOTES
⬇

## 2.13 SPEED, DISTANCE & TIME

### 2.13.1 DISTANCE-TIME GRAPHS

#### What is a distance-time graph?

- **Distance-time** graphs show distance from a fixed point at different times
  - Distance is on the vertical axis, and time is on the horizontal axis.
- The gradient of the graph is the speed:
  Speed =RISE / RUN=DISTANCE / TIME
- **Straight** line = Constant speed
- **Horizontal** line = Stationary (not moving!)

#### Exam Tip

It is easy to get confused between different types of graph.

Look at the label on the vertical axis to make sure you are looking at a DISTANCE-time graph (not speed-time).

# 2. Algebra & Graphs

YOUR NOTES

## Worked Example

*One afternoon Mary cycled to her grandparents's house, 8 km from her own home.*

*Part of the travel graph for her journey is shown below.*

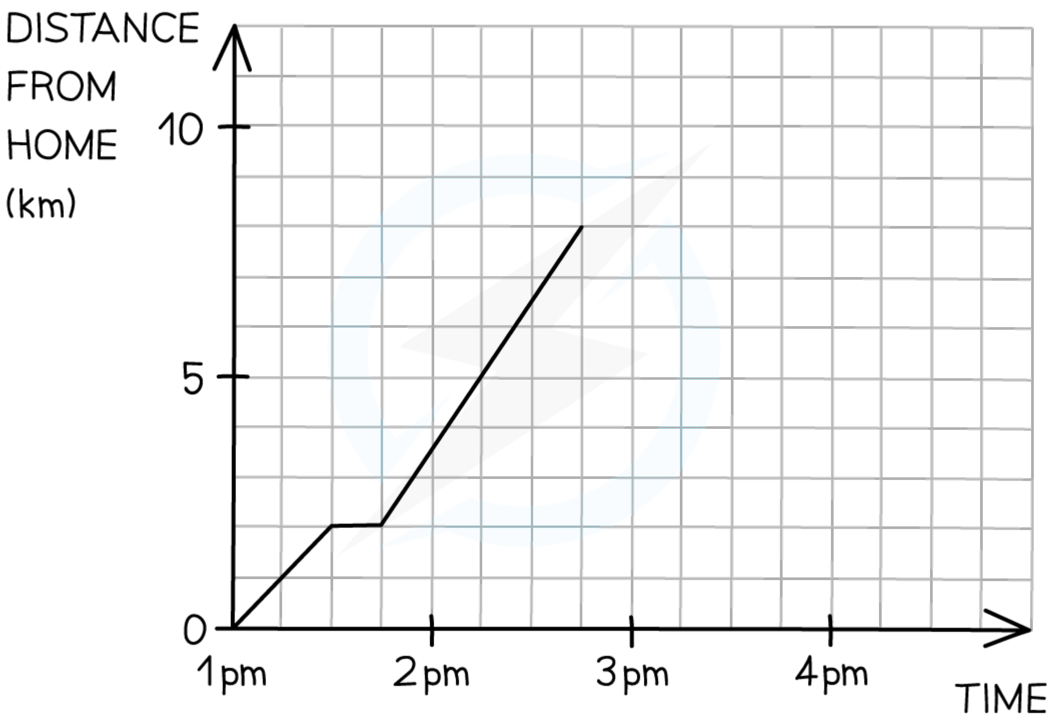
*Mary stayed at her grandparent's house for half an hour.*

*She then cycled home at a steady speed, without stopping, arriving home at 4pm.*

(*a*) *Complete the travel graph for Mary's journey.*

(*b*) *For how long did Mary stop on the way to her grandparent's house?*

(*c*) *What is Mary's speed between 1.45pm and 2.45pm?*





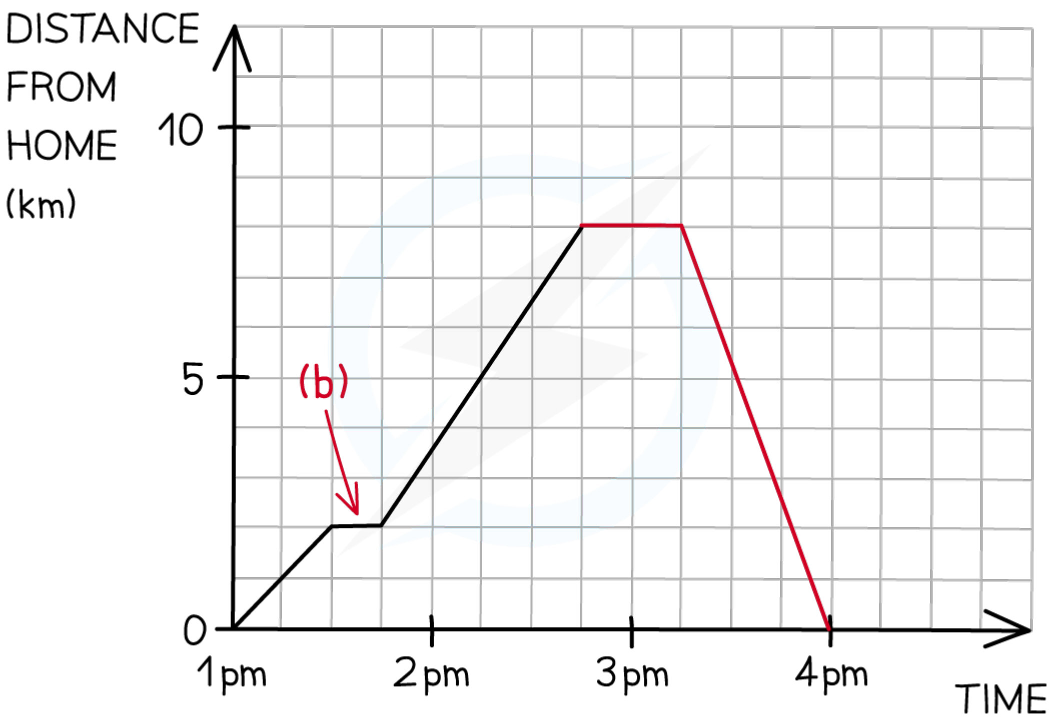

*Note the scale on the time axis – one square is 15 minutes.*

(*a*) *She is not moving for half an hour so we draw a horizontal line for 4 squares.*

*Her cycle home is represented by a straight line (steady speed) drawn from the end of her stay to 4pm on the time axis (where the distance from home is zero).*

## 2. Algebra & Graphs

YOUR NOTES




(b) *Mary's stop on the way is the short horizontal line indicated.*

*It is 1 square long so Time Stopped = 15 minutes*

(c) $Speed = \frac{RISE}{RUN} = \frac{DISTANCE}{TIME}$

$Speed = \frac{6\ km}{1\ hour} = 6\ km/h$

# 2. Algebra & Graphs

YOUR NOTES
⬇

## 2.13.2 SPEED-TIME GRAPHS

### What is a speed-time graph?

- **Speed-time** graphs show speed at different times
  - Speed is on the vertical axis, and time is on the horizontal axis.
- The **gradient** of the graph is the **acceleration**:
  **Acceleration** = RISE / RUN = SPEED / TIME
  **Area** under graph = Distance covered
  **Horizontal** line = Constant speed (so zero acceleration)

### Exam Tip

It is easy to get confused between different types of graph.

Look at the label on the vertical axis to make sure you are looking at a SPEED-time graph (not distance-time).

# 2. Algebra & Graphs

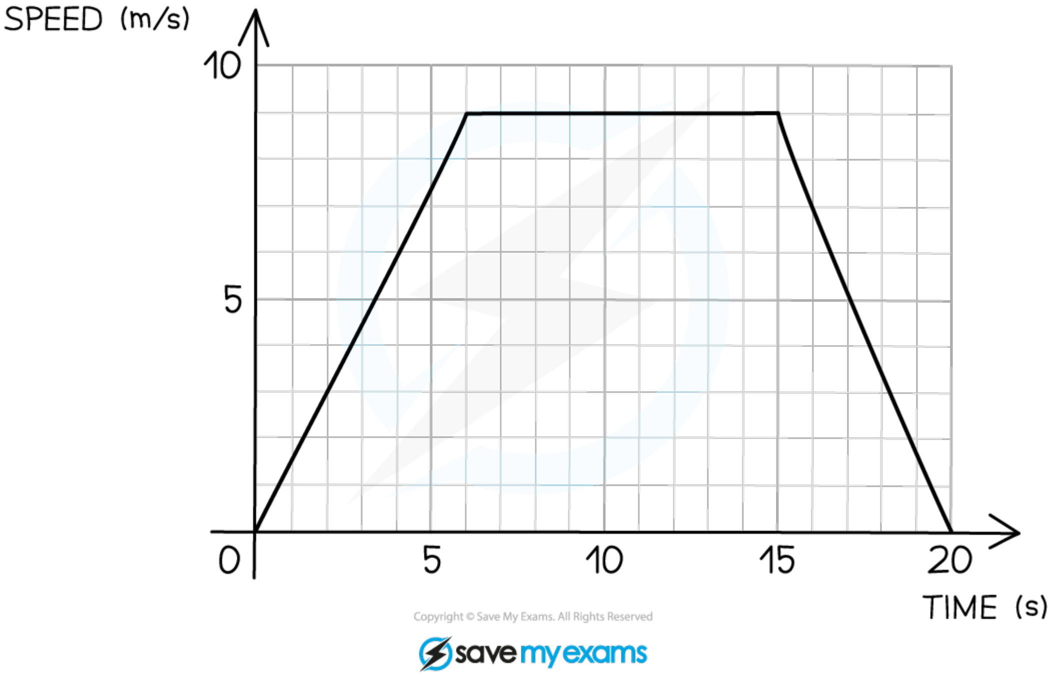
YOUR NOTES

## Worked Example

*The speed – time graph for a car travelling between two sets of traffic lights is shown below.*

*(a) Calculate the acceleration in the first 6 seconds.*

*(b) Work out the distance covered by the car.*




(a) $Acceleration = \frac{RISE}{RUN} = \frac{SPEED}{TIME}$

$Acceleration = \frac{9\ m/s}{6\ s} = 1.5\ ms^{-2}$

(b) $Distance = Area\ under\ the\ graph$

$In\ this\ case\ it's\ a\ trapezium\ so\ A = \frac{1}{2}(a+b)h$

$Distance = \frac{1}{2} \times (9 + 20) \times 9$

$Distance = 130.5\ m$

# 2. Algebra & Graphs

YOUR NOTES

## 2.14 WORKING WITH GRAPHS

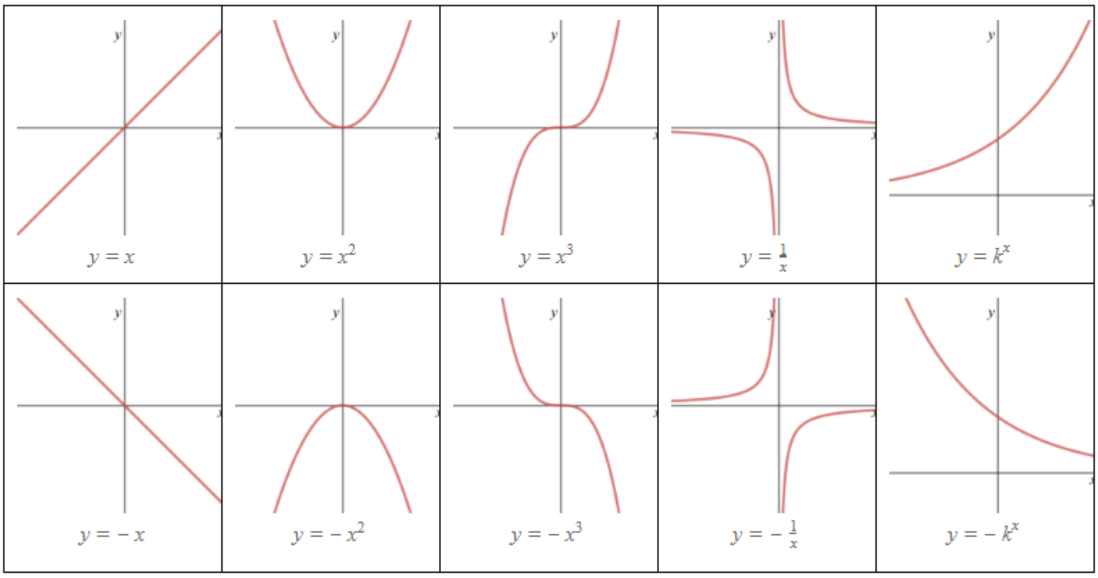
### 2.14.1 DRAWING GRAPHS - SHAPES

#### Why do we need to know what graphs look like?

- Graphs are used in various aspects of mathematics – but in the real world they can take on specific meanings
- For example a **linear (straight line)** graph could be the path a ship needs to sail along to get from one port to another
- An **exponential** graph ($y=k^x$) can be used to model population growth – for instance to monitor wildlife conservation projects

#### Drawing graphs – shapes

- Recalling facts alone won't do much for boosting your GCSE Mathematics grade!
- But being familiar with the general shapes of graphs will help you quickly recognise the sort of maths you are dealing with and features of the graph a question may refer to



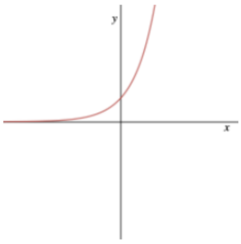
# 2. Algebra & Graphs

YOUR NOTES

## Worked Example

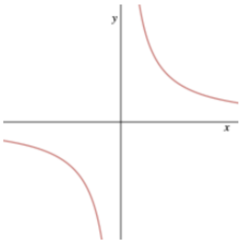
1. *Match the graphs to the equations.*

(A)



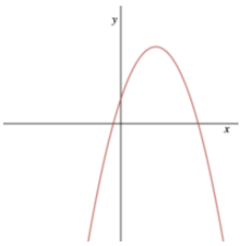

(1) $y = 0.6x + 2$

(B)




(2) $y = 3^x$

(C)

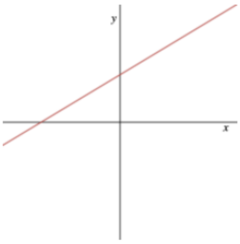



(3) $y =- 0.7x^3$

# 2. Algebra & Graphs

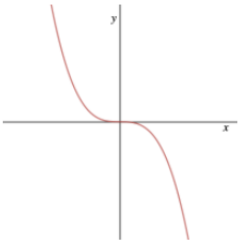
YOUR NOTES
⬇

(D)




(4) $y = \frac{4}{x}$

(E)




(5) $y = -x^2 + 3x + 2$

$Graph\ A \rightarrow Equation\ 2$
$Graph\ B \rightarrow Equation\ 4$
$Graph\ C \rightarrow Equation\ 5$
$Graph\ D \rightarrow Equation\ 1$
$Graph\ E \rightarrow Equation\ 3$

# 2. Algebra & Graphs

YOUR NOTES

## 2.14.2 DRAWING GRAPHS - USING A TABLE

### How do we draw a graph using a table of values?

- Use your calculator
- THINK what the graph might look like – see the previous notes on being familiar with shapes of graphs
- Find the TABLE function on your CALCULATOR
- Enter the FUNCTION – f(x)
  (use ALPHA button and x or X, depending on make/model)
  (Press = when finished)
  (If you are asked for another function, g(x), just press enter again)
- Enter **Start**, **End** and **Step** (gap between **x** values)
- Press = and scroll up and down to see **y** values
- PLOT POINTS and join with a SMOOTH CURVE

- If your calculator does not have a TABLE function then you will have to work out each y value separately using the normal mode on your calculator
- To avoid errors always put negative numbers in brackets and use the (-) key rather than the subtraction key

### Worked Example

1. (*a*) *Complete the table of values for the function* $f(x) = x^3 - 5x + 2$

(*b*) *On the graph paper provided draw the graph of* $y = x^3 - 5x + 2$

(*a*)

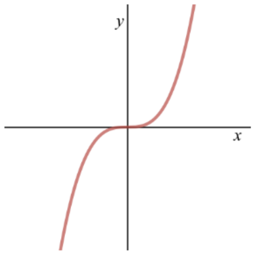
| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | **-10** | 4 | **6** | **2** | **-2** | **0** | 14 |

*2, 3, 4, 5 − Complete the table using you calculator*

(*b*)

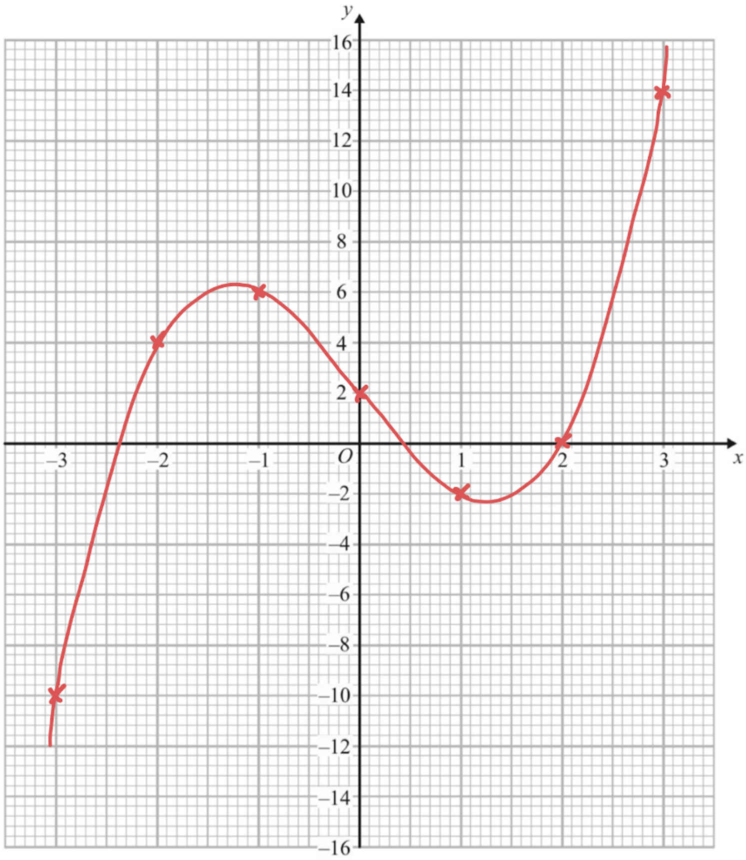
# 2. Algebra & Graphs

YOUR NOTES
⬇




*1 – It's a positive cubic graph so you know it should look a bit like this*

*It is not essential to sketch the graph but you should've at least thought about it*




*6 – Use the table from part (a) to plot the points carefully and join them up wit a smooth curve*
*If you don't/can't get a smooth curve flowing nicely through the points you may have made a mistake so go back and double check your values and plotting*

# 2. Algebra & Graphs

YOUR NOTES

## 2.14.3 SOLVING EQUATIONS USING GRAPHS

### How do we use graphs to solve equations?

- Solutions are always read off the **x-axis**
- Solutions of **f(x) = 0** are where the graph of **y = f(x)** cuts the **x**-axis
- If given g(x)=0 instead (Q: "by drawing a suitable straight line") then:
  - Rearrange into **f(x) = mx + c** and draw the line **y = mx + c**
  - Solutions are the **x**-coordinates of where the **line** crosses the **curve**
    Note that **solutions** may also be called **roots**

### Worked Example

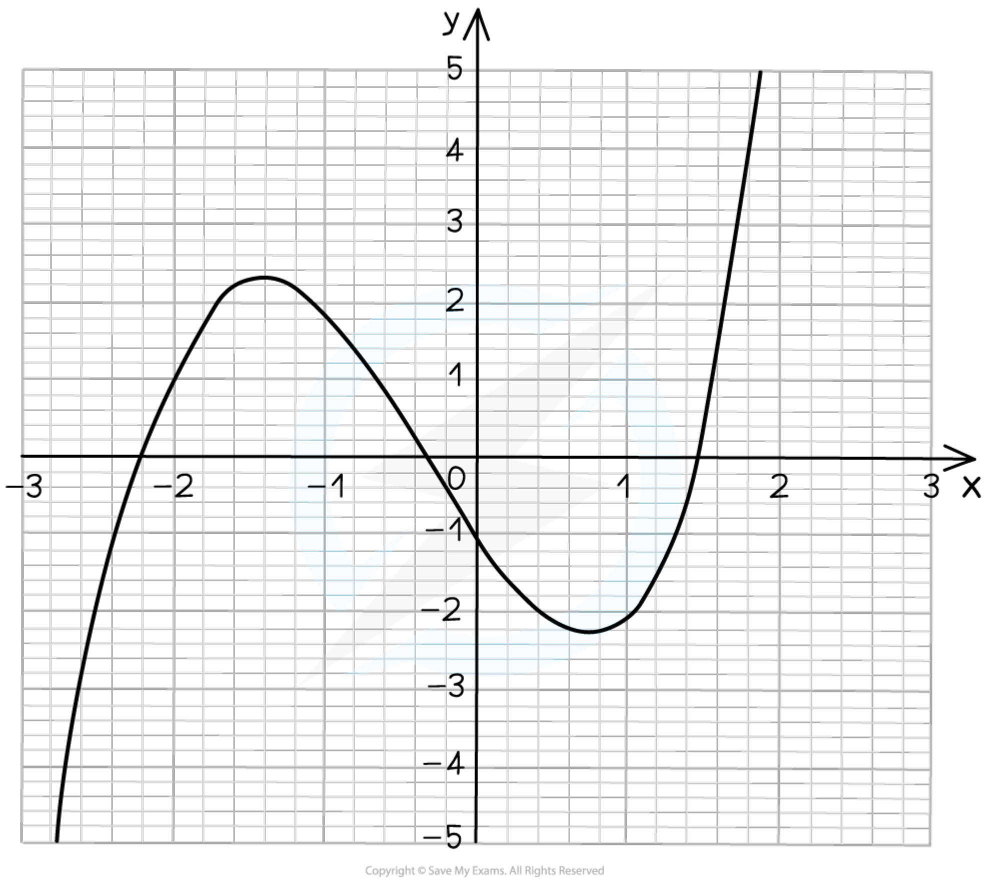
*The graph of* $y = x^3 + x^2 - 3x - 1$ *is shown below.*

*Use the graph to estimate the solutions (to 1 d.p.) of the equation*

$$x^3 + x^2 - 4x = 0.$$

# 2. Algebra & Graphs

YOUR NOTES




3. *We are given a different equation to the one plotted so we must rearrange it to* $f(x) = mx + c$.

$$x^3 + x^2 - 4x = 0$$

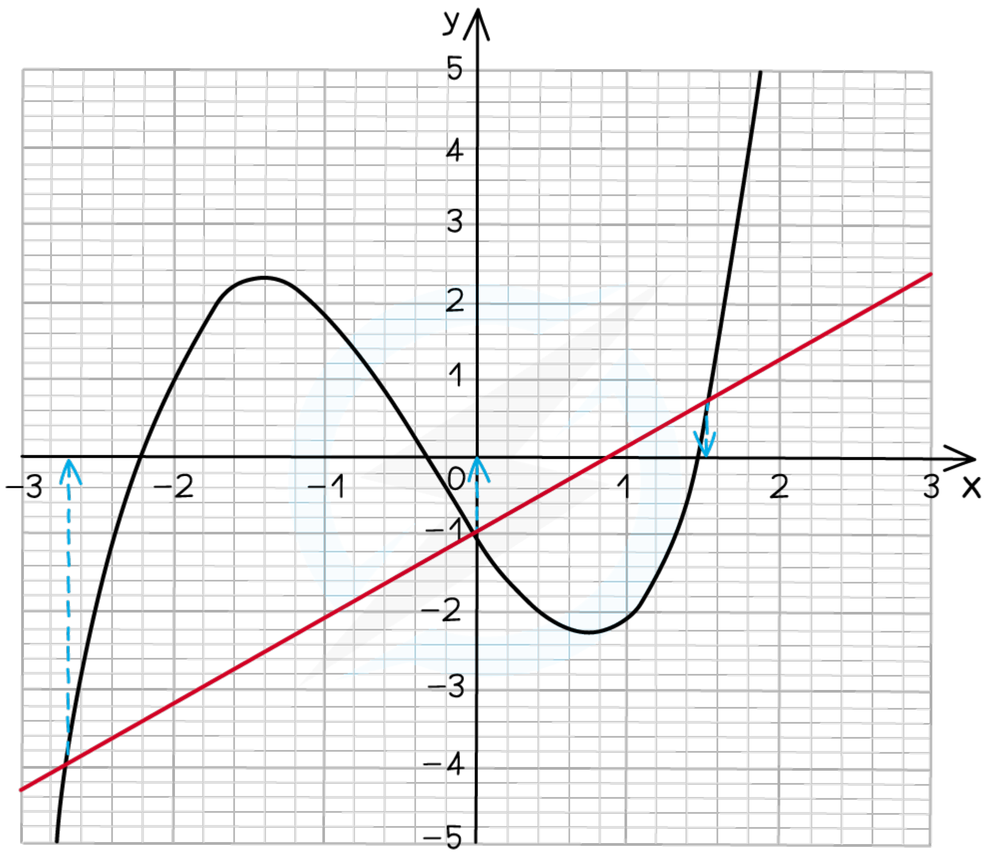
*Add* $x - 1$ *to both sides :*

$$x^3 + x^2 - 3x - 1 = x - 1$$

*Now plot* $y = mx + c$ *(which is* $y = x - 1$ *here) on the graph* $-$ *this is the solid red line on the graph below.*

# 2. Algebra & Graphs

YOUR NOTES




*The solutions are the x coordinates of the crossing points so :*

$x = -2.6,\ x = 0,\ x = 1.6$

# 2. Algebra & Graphs

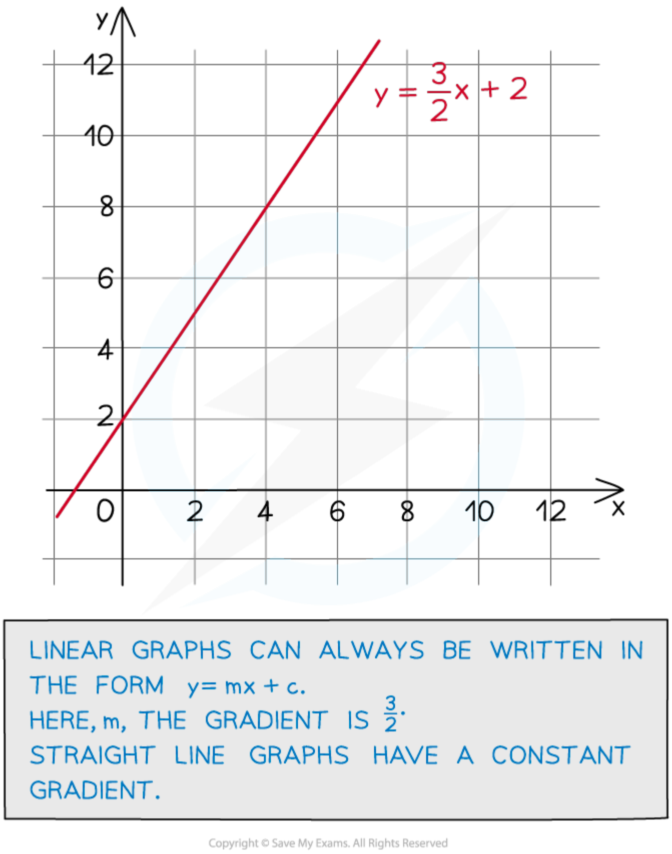
YOUR NOTES

## 2.15 TANGENTS & GRADIENTS

### 2.15.1 FINDING GRADIENTS OF NON-LINEAR GRAPHS

#### What is a non-linear graph?

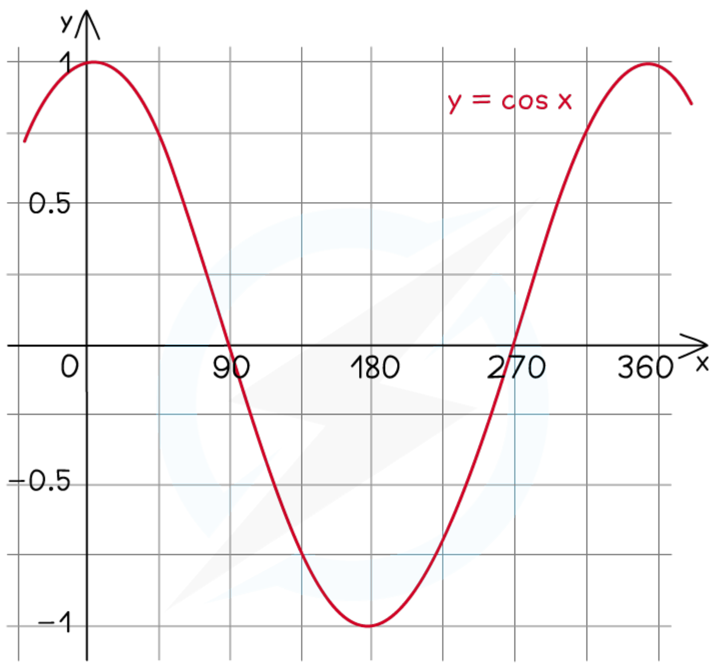
- A **linear** graph is a **straight-line graph**
- These are easily identified as their equations can always be written in the form **y = mx + c**, where **m** is the **gradient** and **c** is the **y**-axis intercept



# 2. Algebra & Graphs

YOUR NOTES
⬇

- All other graphs are **non-linear** – ie. **curves**
- The equations of non-linear graphs take various forms
- Here are a few you could plot quickly using graphing software
    - **$y = x^2$ - 4x + 3** (a **quadratic** graph – called a **parabola**)
    - **y = sin x** (a **trigonometric** graph)
    - **$y = x^3 + 2x^2$ - 4** (a **cubic** graph)
    - **y = 1 / x** (a **reciprocal** graph)




NON-LINEAR GRAPHS CAN TAKE MANY FORMS. THIS IS THE TRIGONOMETRIC GRAPH $y = \cos x$.



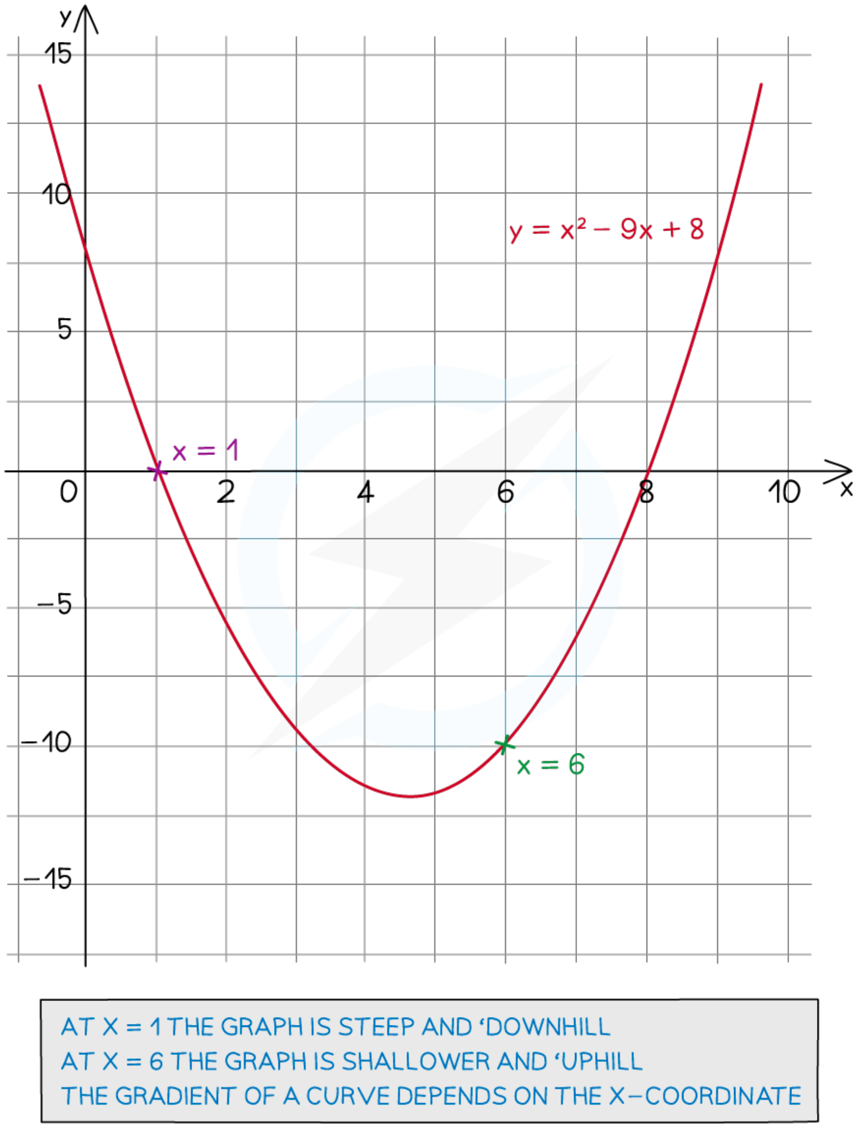
## What is a gradient?

- **Gradient** means **steepness**
- Another way of thinking about gradient is how **y** changes as **x** changes
- On a graph this means how steep the graph is at a certain point on it
    - ie. how is y changing at a particular value of **x**
- For a **linear** graph the **gradient** is **constant** – the value of **x** is irrelevant

# 2. Algebra & Graphs

YOUR NOTES
⬇

- For a **non-linear** graph, the **gradient** is **dependent** on the **x**-coordinate





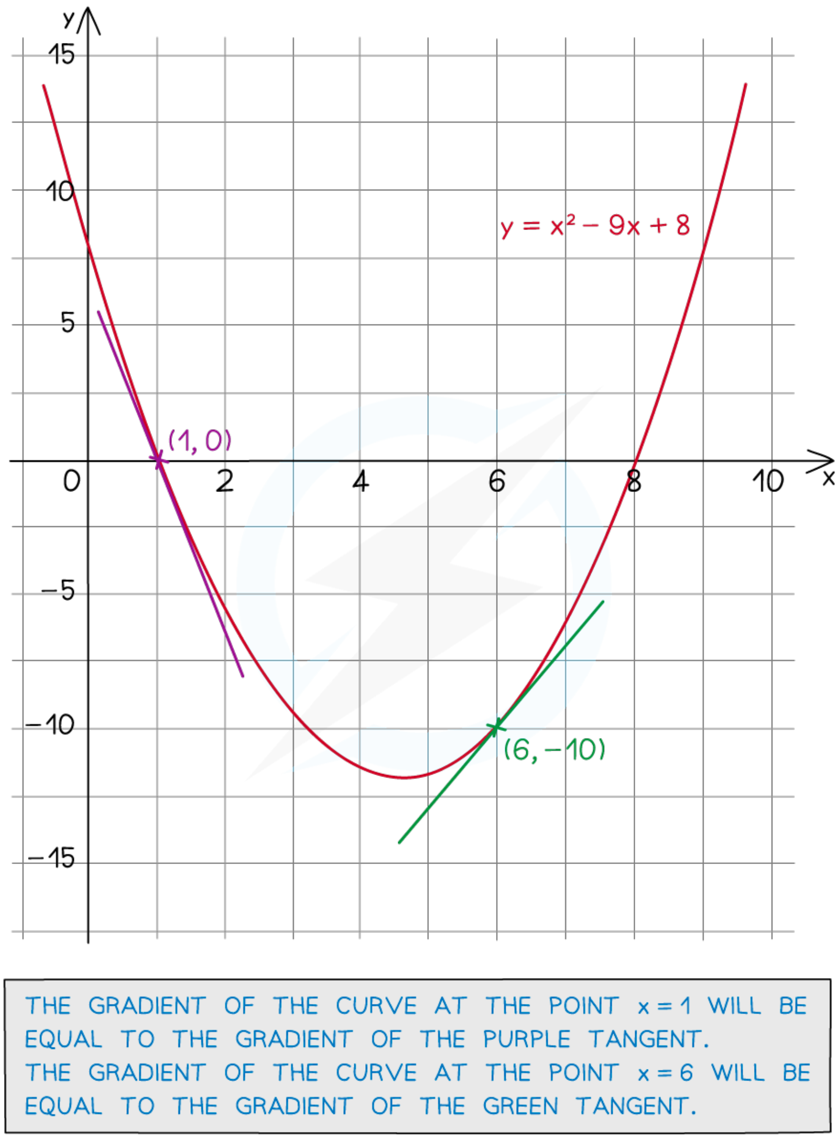

## How do I find the gradient of a non-linear graph?

- Using a copy of the graph it will only be possible to find an **estimate** of a **gradient**
- **Differentiation** allows gradients to be found exactly for certain graphs
- First, a **tangent** to the curve must be drawn
  - A **tangent** to a curve is a **straight line** that touches it at **one point only**
- The **gradient** of a **curve**, at point **(x , y)** is **equal** to the **gradient** of the **tangent** at point

# 2. Algebra & Graphs

YOUR NOTES

**(x , y)**





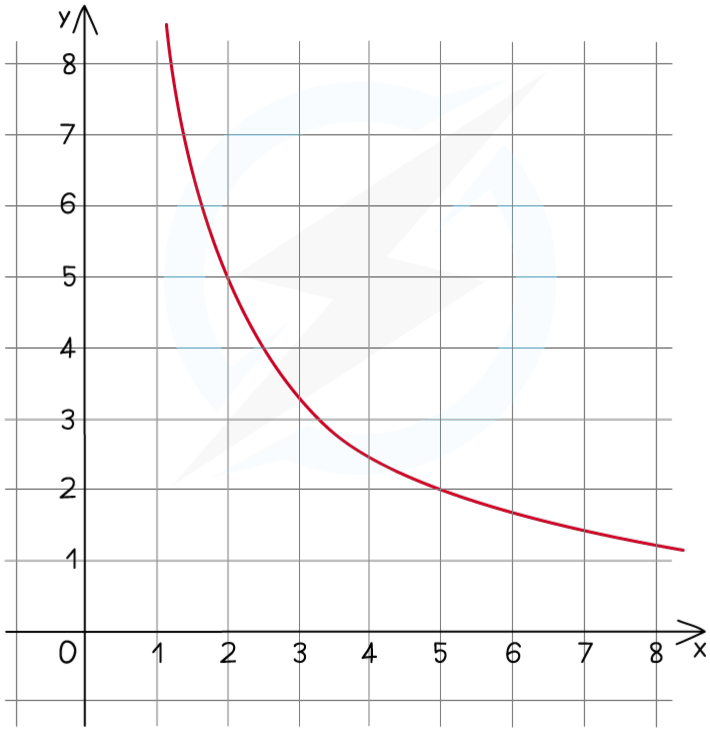

- STEP 1 Draw a **tangent** to the curve at the required **x**-coordinate
- STEP 2 Turn the tangent into a **right**-**angled** triangle
- STEP 3 **Measure**/Read off (some **estimating** usually involved here) the **rise** and the **run**
- STEP 4 The **gradient** is given by **rise** ÷ **run**
  (Alternatively this is "**Change in y**" ÷ "**Change in x**")

# 2. Algebra & Graphs

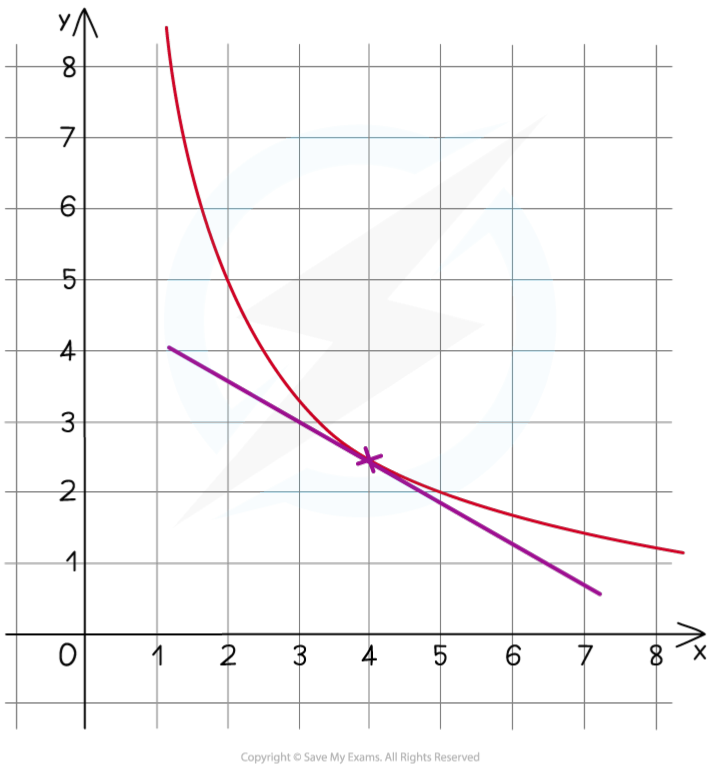
YOUR NOTES ⬇

e.g. THE DIAGRAM BELOW SHOWS A GRAPH OF $y = \frac{10}{x}$ WHERE $x > 0$

FIND AN ESTIMATE OF THE GRADIENT AT THE POINT WHERE $x = 4$



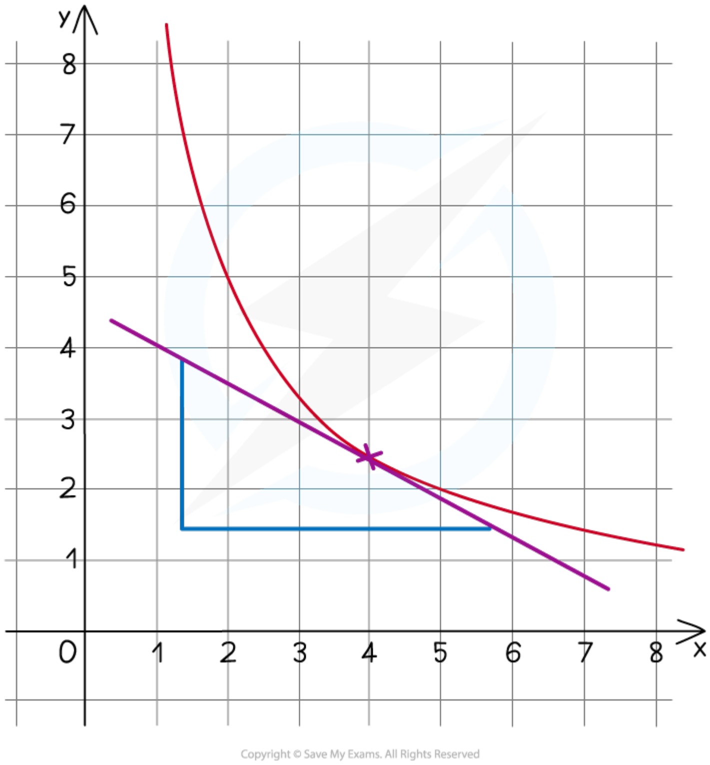
STEP 1: DRAW A TANGENT TO THE CURVE AT THE REQUIRED x COORDINATE



# 2. Algebra & Graphs

YOUR NOTES

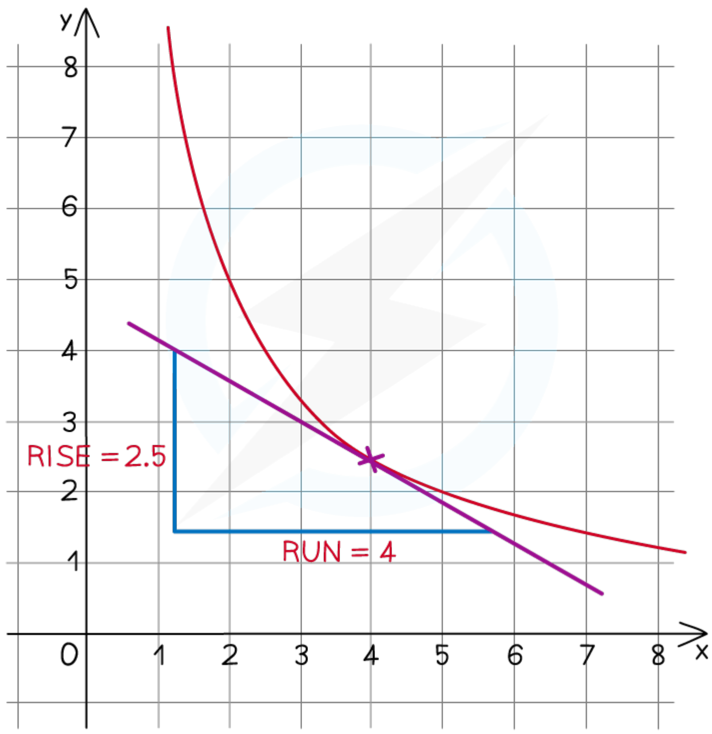
STEP 2: TURN THE TANGENT INTO A RIGHT ANGLED TRIANGLE



# 2. Algebra & Graphs

YOUR NOTES

STEP 3: MEASURE/READ-OFF (ESTIMATE) THE RISE AND RUN






STEP 4: THE GRADIENT IS GIVEN BY $\frac{\text{RISE}}{\text{RUN}}$

GRADIENT AT $x = 4$ IS $\frac{2.5}{4} = 0.625$

GRADIENT $= 0.63$ (2 dp)

IT'S AN ESTIMATE ANYWAY SO NO NEED TO BE FUSSY WHEN ROUNDING

# 2. Algebra & Graphs

YOUR NOTES

### Exam Tip

A sharp pencil helps – but not too sharp – pencil markings made with very sharp pencils are difficult for examiners to see once papers have been scanned into a computer.

Remember your answer is an estimate so can vary a fair amount from someone else's attempt.

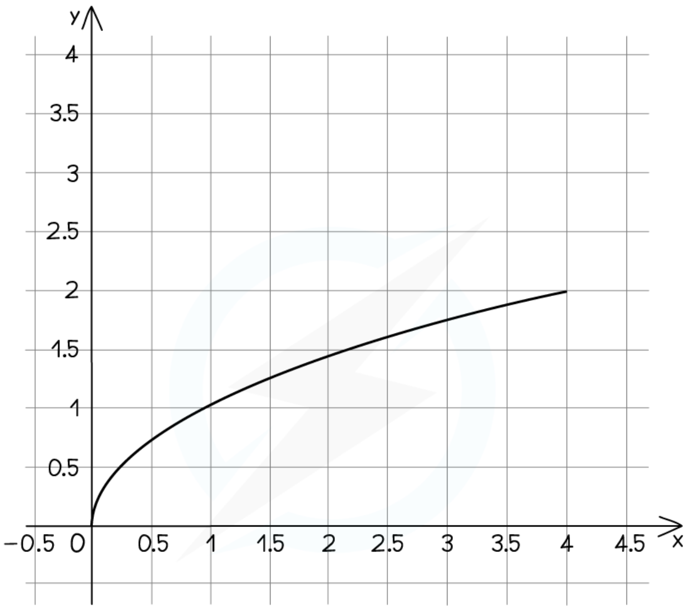
Make your working clear – your tangent, right-angled triangle and your rise/run values should all be clear in your working.

### Worked Example

# 2. Algebra & Graphs

YOUR NOTES

The diagram below shows the graph of $y = \sqrt{x}$ for $0 \leq x \leq 4$.

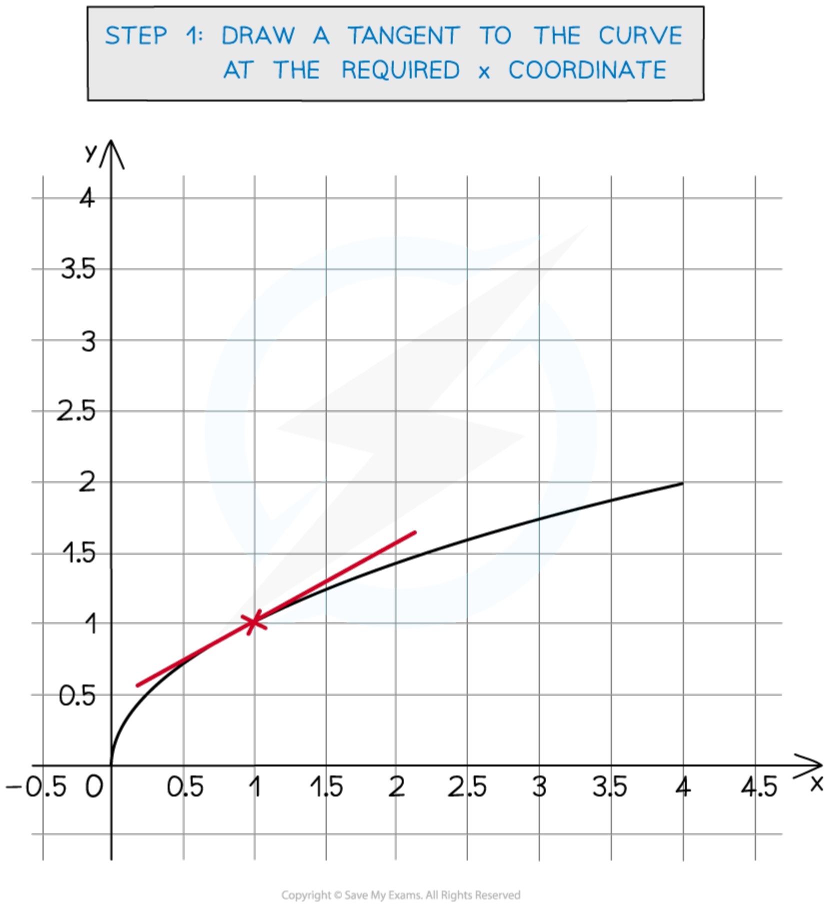



(a) Find an estimate of the gradient of the curve at the point where $x = 1$.

(b) A friend suggests the gradient at the point where $x = 2$ is half the gradient at the point where $x = 1$. By estimating the gradient at $x = 2$ comment on the friend's suggestion.

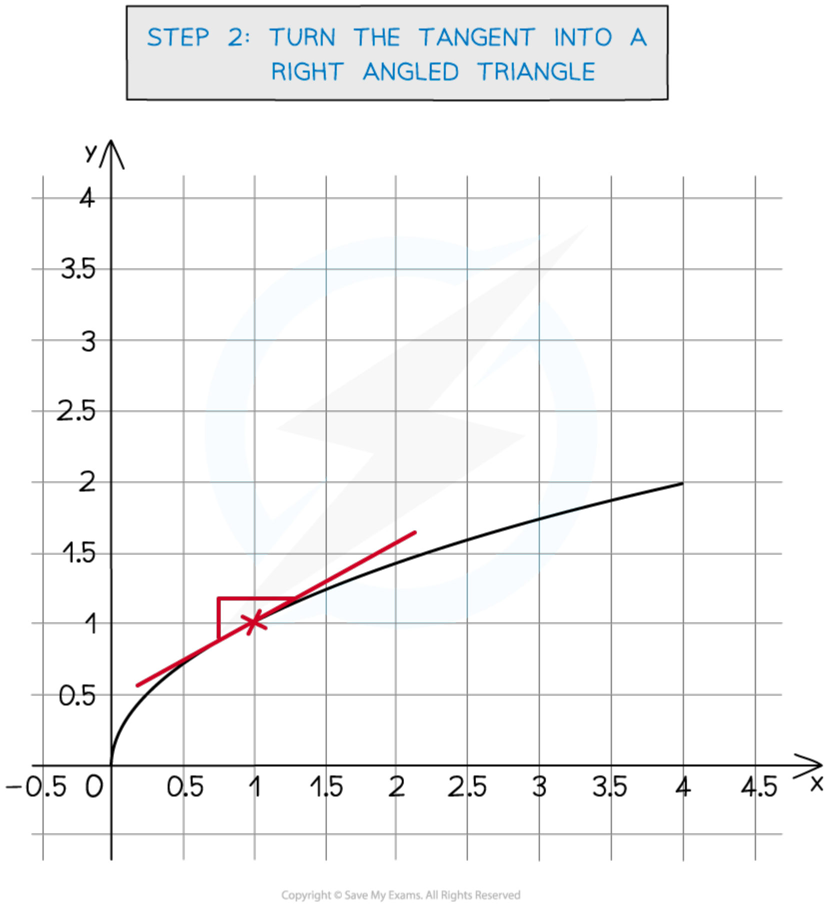
# 2. Algebra & Graphs

YOUR NOTES
⬇



# 2. Algebra & Graphs

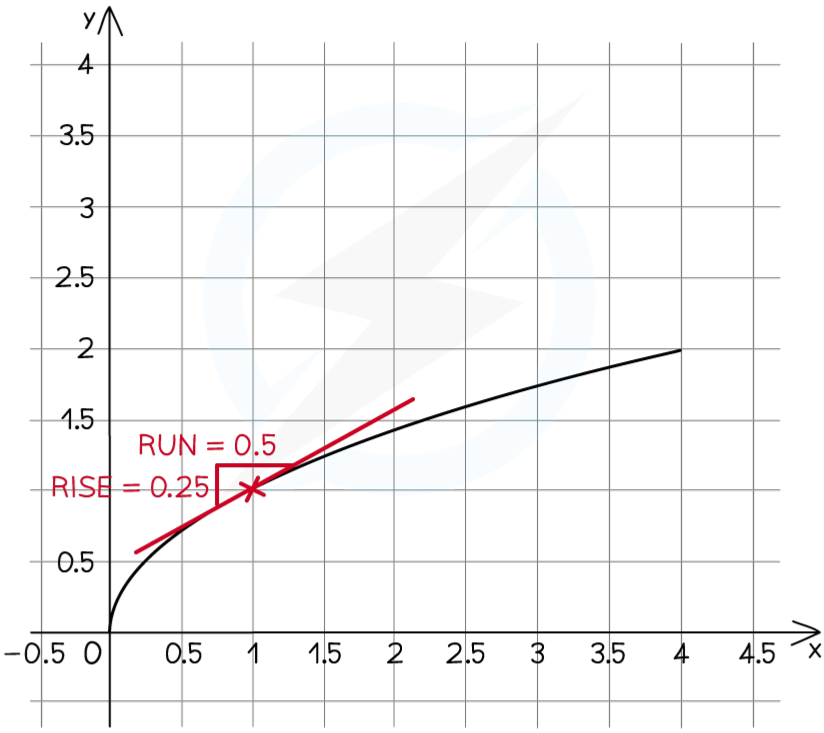
YOUR NOTES



# 2. Algebra & Graphs

YOUR NOTES

STEP 3: MEASURE/READ-OFF (ESTIMATE) THE RISE AND RUN





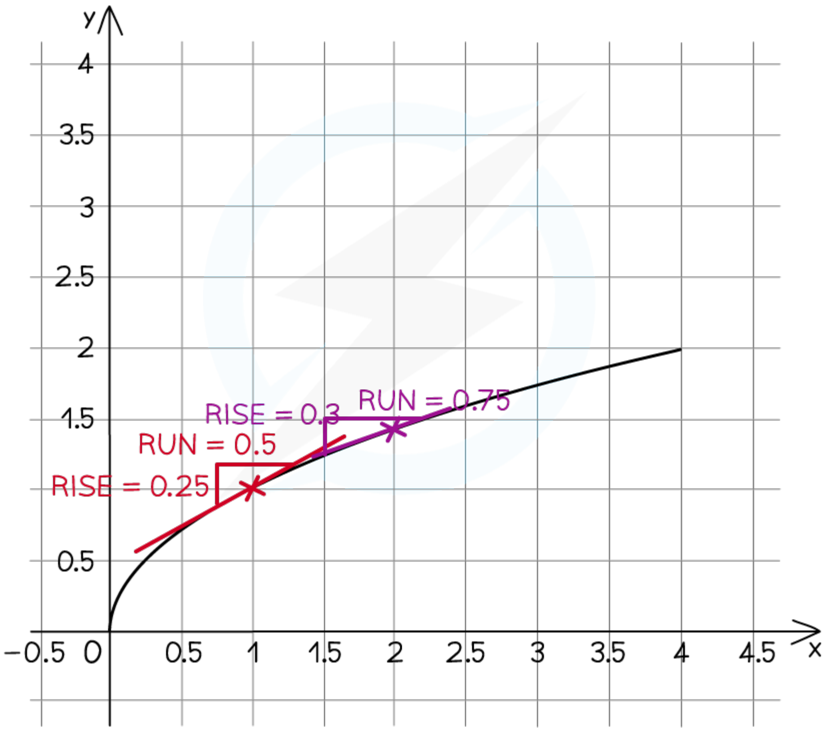

STEP 4: THE GRADIENT IS GIVEN BY $\frac{\text{RISE}}{\text{RUN}}$

GRADIENT AT $x = 1$ IS $\frac{0.25}{0.5}$

GRADIENT $= 0.5$

## 2. Algebra & Graphs

YOUR NOTES ⬇






GRADIENT AT $x = 2$ IS $\frac{0.3}{0.75} = 0.4$

GRADIENT AT $x = 2$ IS 0.4

GRADIENT AT $x = 1$ IS 0.5

FRIEND'S SUGGESTION IS VERY LIKELY TO BE INCORRECT BASED ON MY ESTIMATES ABOVE.
$\frac{1}{2}$ OF $0.5 = 0.25$ BUT GRADIENT AT $x = 2$ WAS ESTIMATED AS 0.4

# 2. Algebra & Graphs

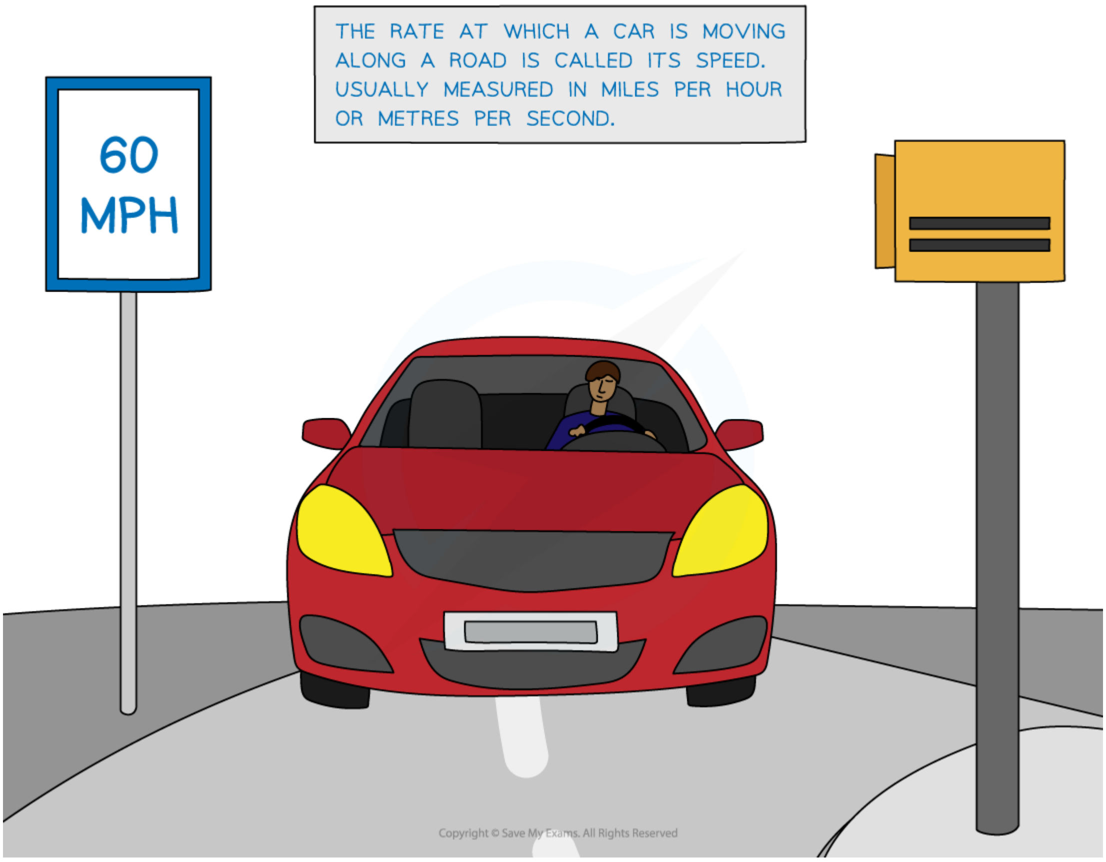
YOUR NOTES

## 2.16 DIFFERENTIATION

### 2.16.1 DIFFERENTIATION - BASICS

#### What is differentiation?

- **Differentiatio**n is part of the branch of mathematics called Calculus
- It is concerned with the **rate** at which changes takes place – so has lots of real-world uses:
  - The **rate** at which a car is moving – ie. its speed
  - The **rate** at which a virus spreads amongst a population



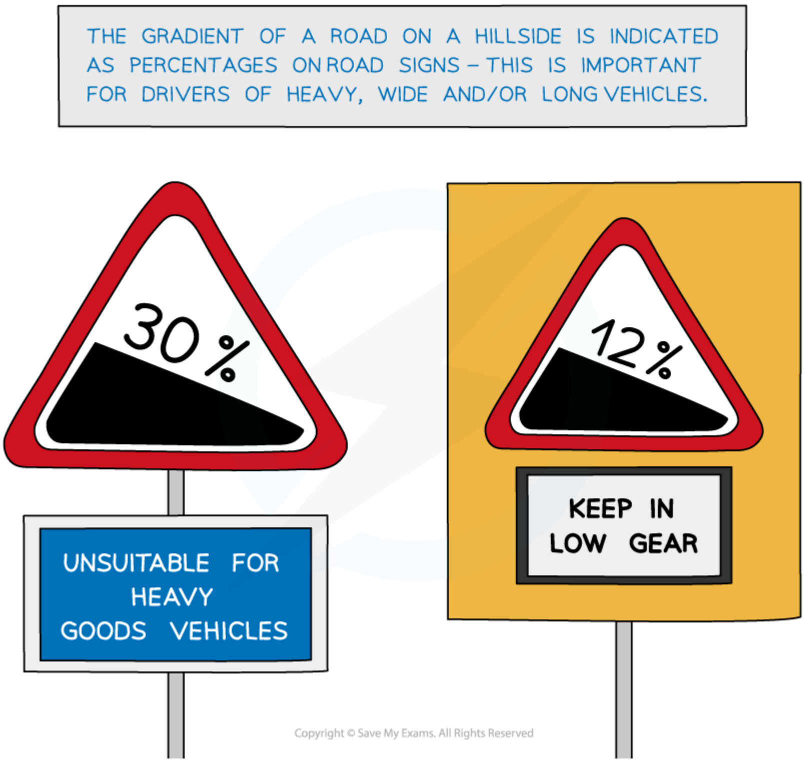

- To begin to understand differentiation you'll need to understand **gradient** (see Finding Gradients of Non-Linear Graphs)

# 2. Algebra & Graphs

YOUR NOTES

## Gradient

- **Gradient** generally means **steepness**.
    - For example, the gradient of a road up the side of a hill is important to lorry drivers



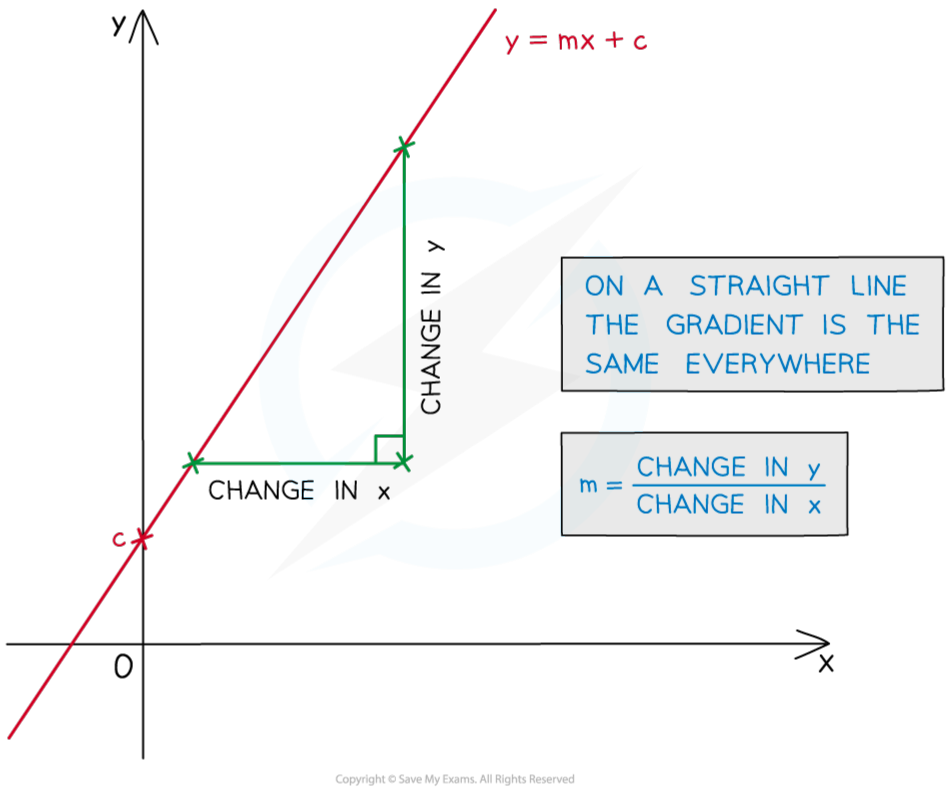

- On a graph the gradient refers to how steep a line or a curve is
    - It is really a way of measuring how fast **y** changes as **x** changes
    - This may be referred to as the **rate** at which **y**
- So **gradient** is a way of describing the **rate** at which **change** happens

# 2. Algebra & Graphs

YOUR NOTES

## Straight lines and curves

- For a straight line the gradient is always the same (constant)
  - Recall **y= mx + c**, where **m** is the gradient (see Straight Lines – Finding Equations)

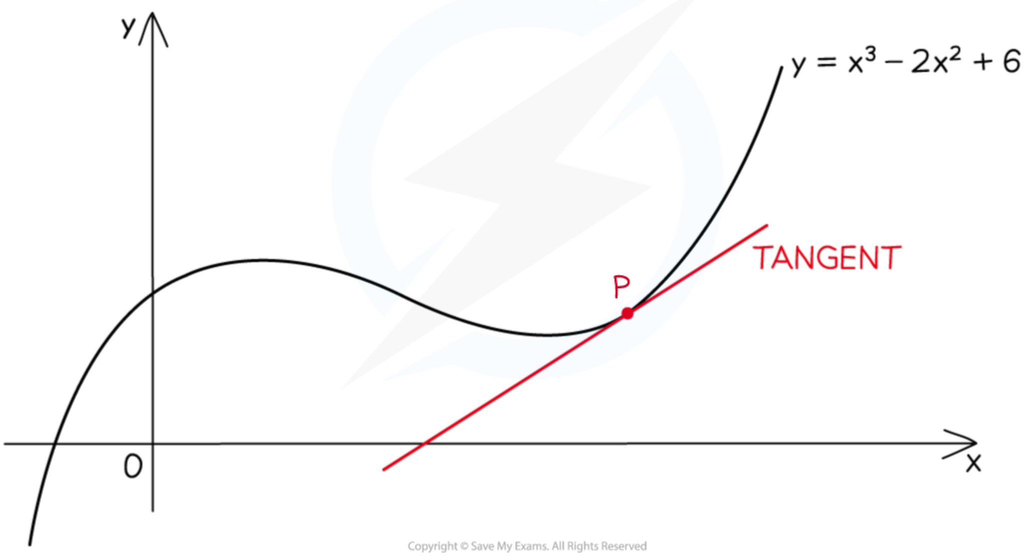



- For a **curve** the **gradient** changes as the value of **x** changes
- At any point on the **curve**, the **gradient** of the curve is **equal** to the **gradient** of the **tangent** at that point
  - A **tangent** is a straight line that touches the curve at one point

# 2. Algebra & Graphs

YOUR NOTES
⬇

THE GRADIENT OF THE CURVE AT POINT P IS EQUAL TO THE GRADIENT OF THE TANGENT TO THE CURVE AT POINT P



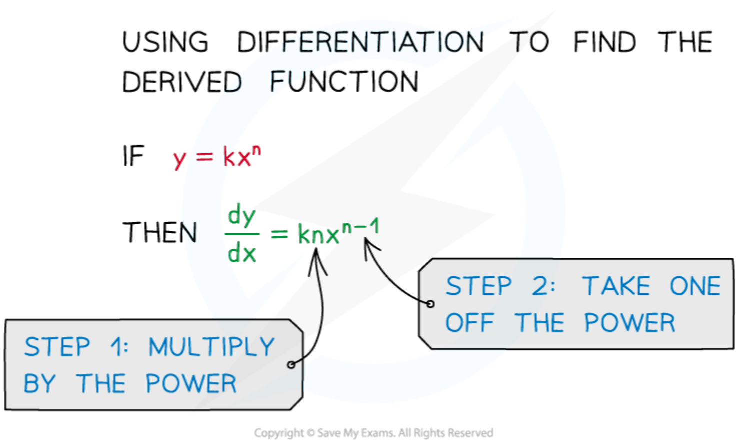

- The **derived function** (aka **gradient function**) is an expression that allows the gradient to be calculated anywhere along a curve
- The **derived function** is also called the derivative

# 2. Algebra & Graphs



## How do I find the derived function or derivative?

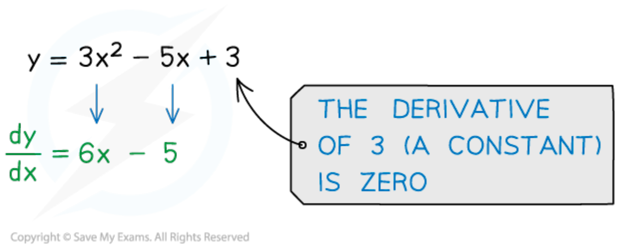
- This is really where the fun with differentiation begins!
- The **derived function** (**dy/dx**) is found by **differentiating y**




- This looks worse than it is!
- For powers of x …

STEP 1 Multiply by the power

STEP 2 Take one off the power



# 2. Algebra & Graphs

YOUR NOTES

## How do I find the value of a gradient?

- Substitute the **x** value into the **expression** for the **derived function** and **evaluate** it

e.g. FIND THE GRADIENT OF THE CURVE $y = 3x^2 - 2x$ AT THE POINT (2, 8)

$\frac{dy}{dx} = 6x - 2$ ← FIND THE DERIVED FUNCTION

WHEN $x = 2$, $\frac{dy}{dx} = 6 \times 2 - 2$ ← SUBSTITUTE IN THE x−COORDINATE

$\frac{dy}{dx} = 10$

THE GRADIENT AT THE POINT (2, 8) IS 10



### Exam Tip

When differentiating long, awkward expressions, write each step out fully and simplify afterwards.

### Worked Example

# 2. Algebra & Graphs

YOUR NOTES

A curve has equation $y = x^3 - 3x^2 - 9x + 1$.

(a) Use differentiation to find the derived function $\frac{dy}{dx}$.

(b) Find the gradient of the curve when $x = -2$.

(c) For which values of $x$ is the gradient equal to 0.

a) $y = x^3 - 3x^2 - 9x^1 + 1$

$x = x^1$

$\frac{dy}{dx} = 3x^2 - 6x^1 - 9$

$-9x^0 = -9$

$x = x^1$

STEP 1: MULTIPLY BY THE POWER

STEP 2: TAKE ONE OFF THE POWER

CONSTANT TERMS HAVE ZERO GRADIENT

$\frac{dy}{dx} = 3x^2 - 6x - 9$

b) WHEN $x = -2$, $\frac{dy}{dx} = 3(-2)^2 - 6(-2) - 9$

SUBSTITUTE $x = -2$ INTO THE DERIVATIVE

$= 12 + 12 - 9$

$\frac{dy}{dx} = 15$

c) $\frac{dy}{dx} = 0$ WHEN $3x^2 - 6x - 9 = 0$

THIS IS A QUADRATIC EQUATION

$3(x^2 - 2x - 3) = 0$

$(x - 3)(x + 1) = 0$

$x = 3, \quad x = -1$

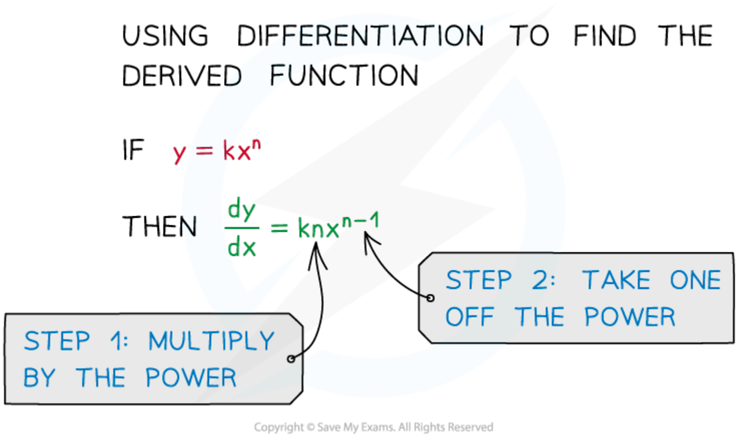
# 2. Algebra & Graphs

YOUR NOTES
⬇

## 2.16.2 DIFFERENTIATION - TURNING POINTS

### Remind me of the rule for differentiating powers …

- If $y = kx^n$ then $dy/dx = knx^{n-1}$

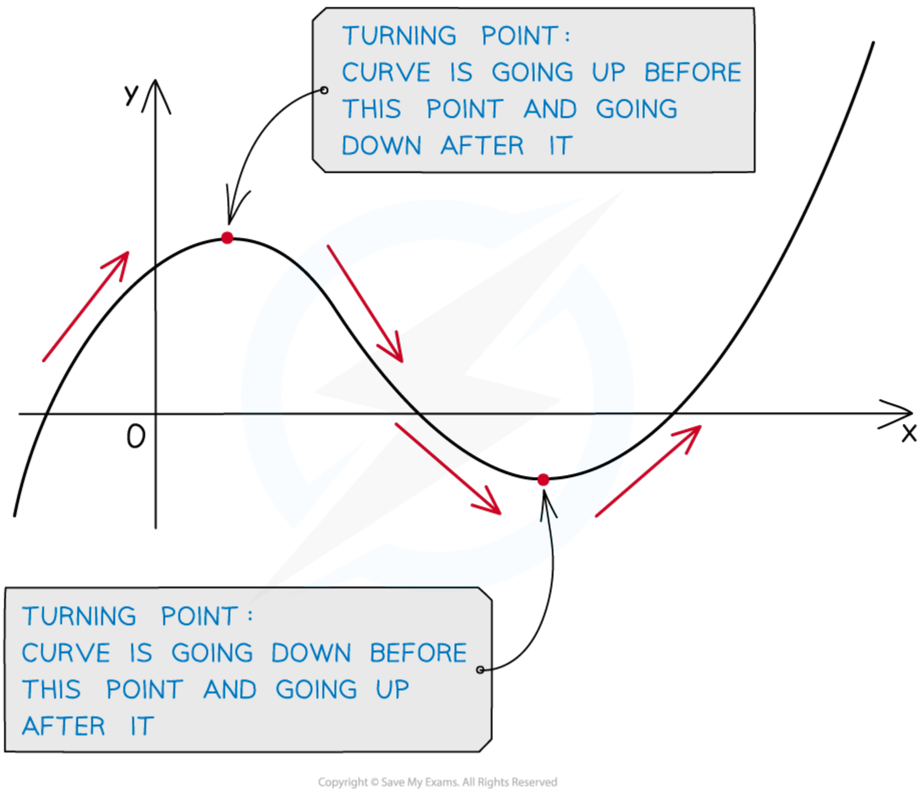



### What is a turning point?

- The easiest way to think of a turning point is that it is a point at which a curve **changes** from **moving upwards** to **moving downwards**, or vice versa
- Turning points are also called **stationary points**
- Ensure you are familiar with **Differentiation – Basics** before moving on

# 2. Algebra & Graphs

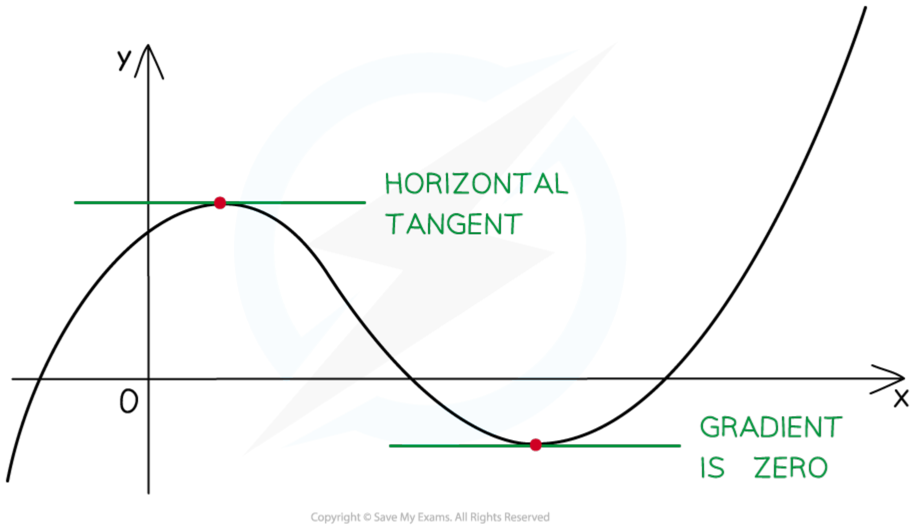
YOUR NOTES



# 2. Algebra & Graphs

YOUR NOTES

- At a **turning point** the **gradient** of the curve is **zero**.
  - If a **tangent** is drawn at a turning point it will be a **horizontal line**
  - Horizontal lines have a gradient of zero
- This means at a turning point the **derived function** (aka **gradient function** or **derivative**) equals zero

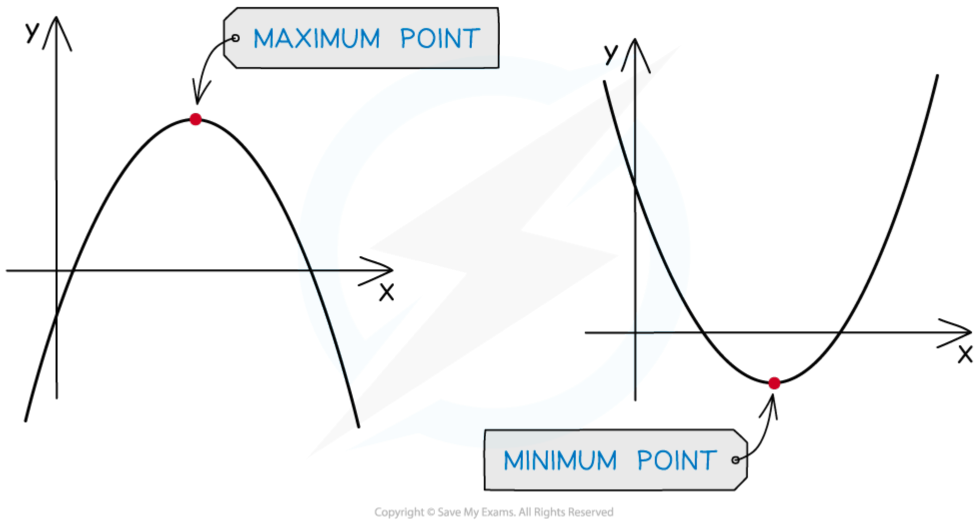


# 2. Algebra & Graphs

YOUR NOTES

## How do I know if a curve has turning points?

- You can see from the shape of a curve whether it has turning points or not
- At IGCSE, two types of turning point are considered:
  - **Maximum points** – this is where the graph reaches a "**peak**"
  - **Minimum points** – this is where the graph reaches a "**trough**"




- These are sometimes called **local** maximum/minimum points as other parts of the graph may still reach higher/lower values

# 2. Algebra & Graphs

YOUR NOTES

## How do I find the coordinates of a turning point?

- STEP 1 Solve the equation of the **derived function** (derivative) equal to zero
  ie. solve **dy/dx = 0**
  This will find the **x**-coordinate of the turning point
- STEP 2 To find the **y**-coordinate substitute the **x**-coordinate into the equation of the graph
  ie. substitute **x** into "**y = ...**"

e.g. FIND THE COORDINATES OF THE TURNING POINT ON THE CURVE WITH EQUATION $y = 2x^2 + 8x - 9$

$$\frac{dy}{dx} = 4x + 8$$

USE DIFFERENTIATION TO FIND THE DERIVATIVE

STEP 1 SOLVE $\frac{dy}{dx} = 0$

$4x + 8 = 0$

$4x = -8$

$x = -2$

STEP 2 SUBSTITUTE x–COORDINATE INTO "y = ..."

$y = 2(-2)^2 + 8(-2) - 9$

$y = -17$

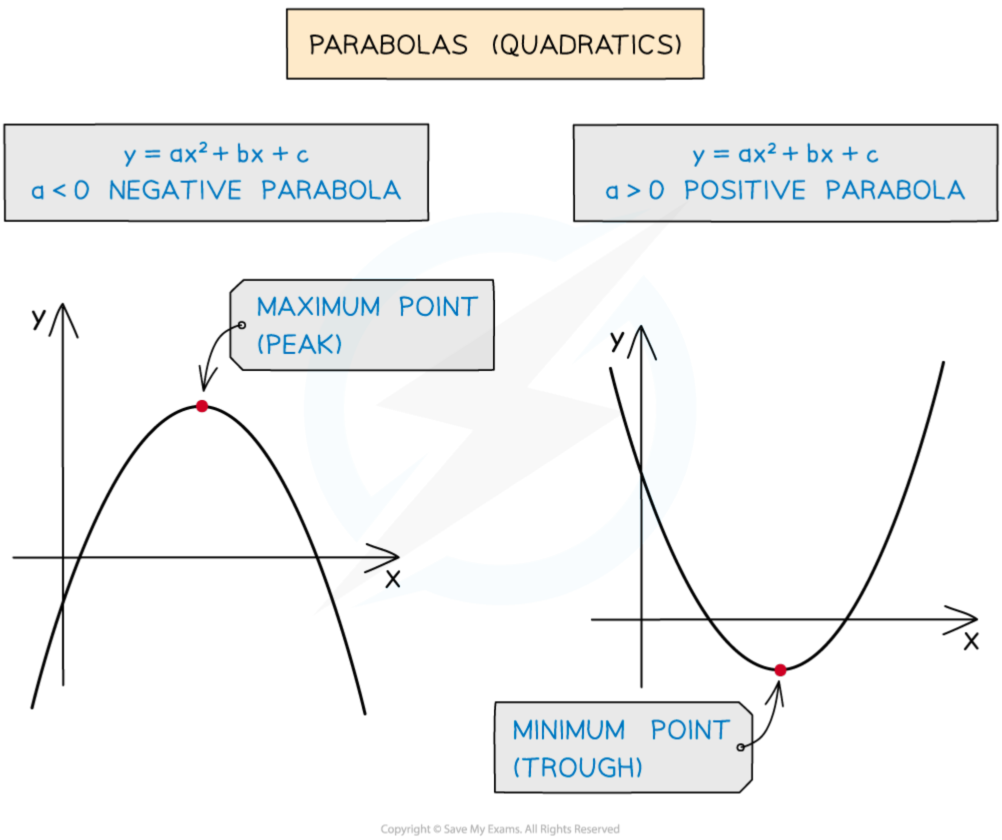
THE TURNING POINT HAS COORDINATES $(-2, -17)$

# 2. Algebra & Graphs

YOUR NOTES

## How do I know which point is a maximum and which is a minimum?

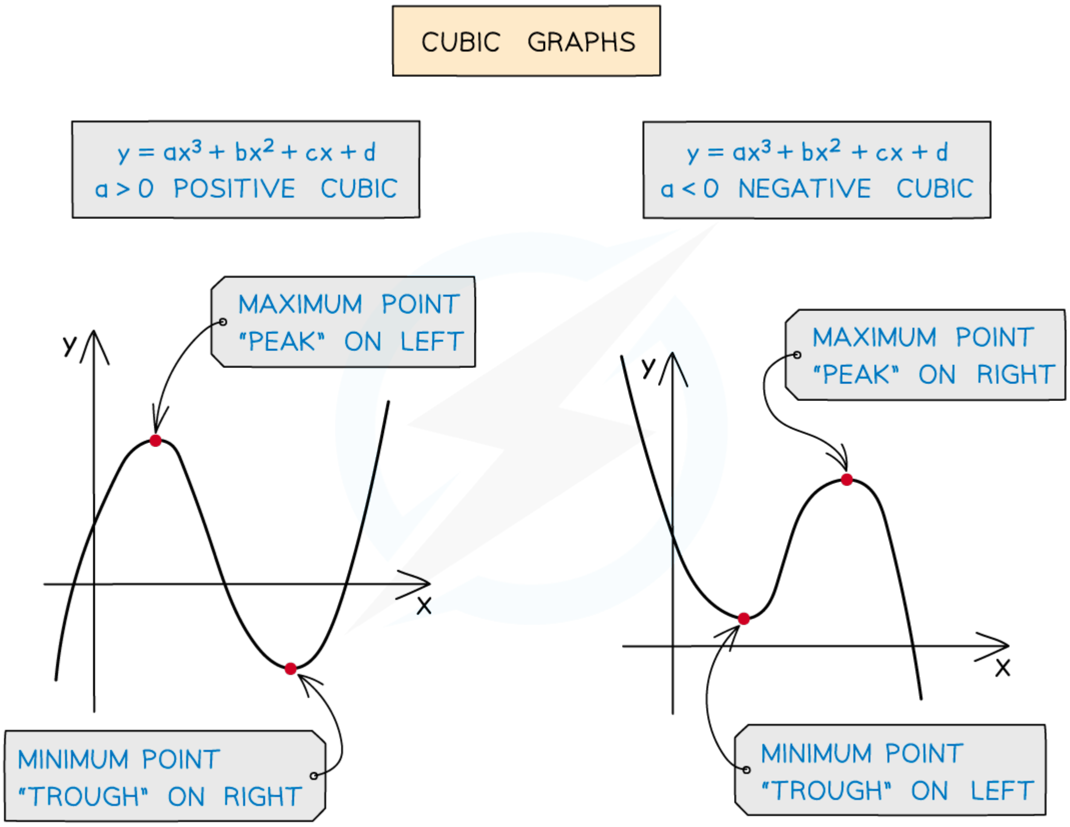
- There are several ways to do this
    - Exam questions may ask you to "give reasons" or "justify" why you think a particular turning point is, say, a maximum
- The **easiest** way to do this is to **recognise** the **shape** of the **curve**
    - ... either from a given **sketch** of the **curve**
    - ... a **sketch** of the **curve** you can quickly draw yourself (You may even be asked to do this as part of a question)
    - ... the **equation** of the **curve**
- For **parabolas** (quadratics) it should be obvious ...
    - ... a **positive** parabola (**positive $x^2$** term) has a **minimum** point
    - ... a **negative** parabola (**negative $x^2$** term) has a **maximum** point



# 2. Algebra & Graphs

YOUR NOTES

- **Cubic** graphs are also easily recognisable ...
  - ... a **positive** cubic has a **maximum** point on the **left**, **minimum** on the **right**
  - ... a **negative** cubic has a **minimum** on the **left**, **maximum** on the **right**






## Exam Tip

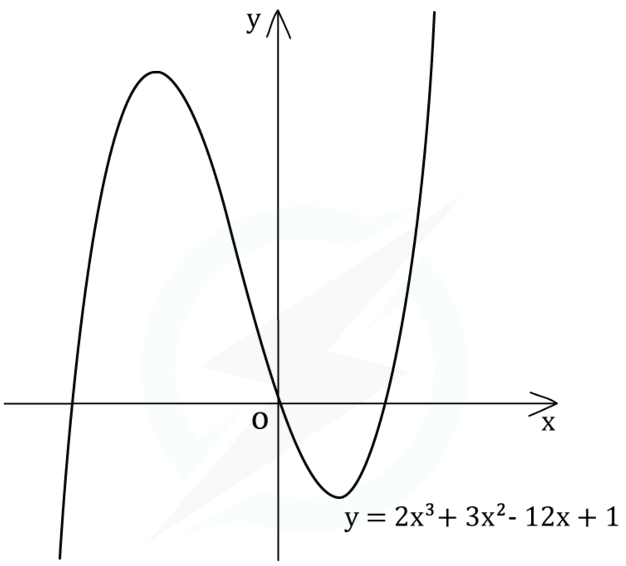
Remember to read the questions carefully – sometimes only the **x**-coordinate of a turning point is required.
Differentiating accurately is crucial in leading to equations you can work with and solve.

## Worked Example

# 2. Algebra & Graphs



A sketch of the graph of the curve with equation $y = 2x^3 + 3x^2 - 12x + 1$ is shown below.




(a) How many turning points does the graph have?

(b) Find an expression for the derived function, $\frac{dy}{dx}$.

(c) Find the coordinates of any turning points on the curve.

(d) Determine the nature of any turning points you found in part (c), justifying your answer.

# 2. Algebra & Graphs

YOUR NOTES

a) 2

THERE ARE TWO TURNING POINTS
1. MAXIMUM IN "TOP LEFT"
2. MINIMUM IN "BOTTOM RIGHT"

b) $\frac{dy}{dx} = 6x^2 + 6x - 12$

DIFFERENTIATE

MULTIPLY BY POWER

TAKE ONE OFF THE POWER

c) $\frac{dy}{dx} = 0$

STEP 1: SOLVE THE DERIVATIVE EQUAL TO ZERO

$6x^2 + 6x - 12 = 0$ ← FACTOR OF 6

$6(x^2 + x - 2) = 0$

$x^2 + x - 2 = 0$ ← QUADRATIC EQUATION

$(x + 2)(x - 1) = 0$

$x = -2, \quad x = 1$

STEP 2: QUESTION REQUIRES "COORDINATES" SO y−VALUES ALSO NEEDED

AT $x = -2$

$y = 2(-2)^3 + 3(-2)^2 - 12(-2) + 1$

$y = 21$

AT $x = 1$

$y = 2(1)^3 + 3(1)^2 - 12(1) + 1$

$y = -6$

∴ TURNING POINTS ARE $(-2, 21)$ AND $(1, -6)$

d) FROM THE GRAPH (WHICH IS A **POSITIVE CUBIC**)

IF THE GRAPH HAD NOT BEEN GIVEN YOU SHOULD RECOGNISE IT IS A POSITIVE CUBIC FROM ITS EQUATION $y = 2x^3 \ldots$

$(-2, 21)$ MUST BE ON THE LEFT OF $(1, -6)$

∴ $(-2, 21)$ IS A MAXIMUM POINT
$(1, -6)$ IS A MINIMUM POINT

# 2. Algebra & Graphs

YOUR NOTES
⬇

## 2.16.3 DIFFERENTIATION - PROBLEM SOLVING

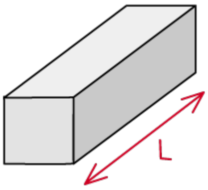
### What problems could involve differentiation?

- **Differentiation** allows analysis of how one quantity **changes** as another does
- The **derived function** (**derivative**) gives a measure of the **rate of change**
- Problems involving a **variable quantity** can involve differentiation
  - How the **area** of a rectangle changes as its **length** varies
  - How the **volume** of a cylinder changes as its **radius** varies
  - How the **position** of a car changes over **time** (ie its **speed**)
- Problems based on the graph of a curve may also arise
  - The distance between two turning points
  - The area of a shape formed by points on the curve such as turning points and axes intercepts
- Ensure you are familiar with Differentiation – Basics and Differentiation – Turning Points

# 2. Algebra & Graphs

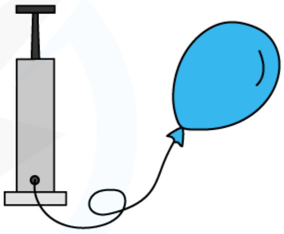
YOUR NOTES
⬇

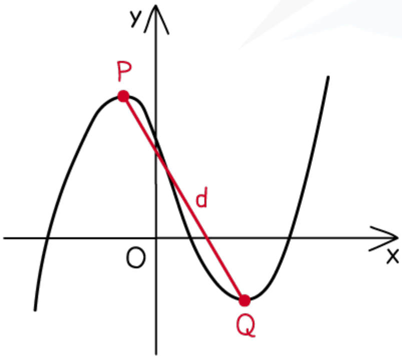
## PROBLEMS INVOLVING DIFFERENTIATION




THE VOLUME OF A CUBOID WILL CHANGE AS ITS LENGTH VARIES

THE SURFACE AREA OF A BALLOON BEING INFLATED WILL CHANGE OVER TIME





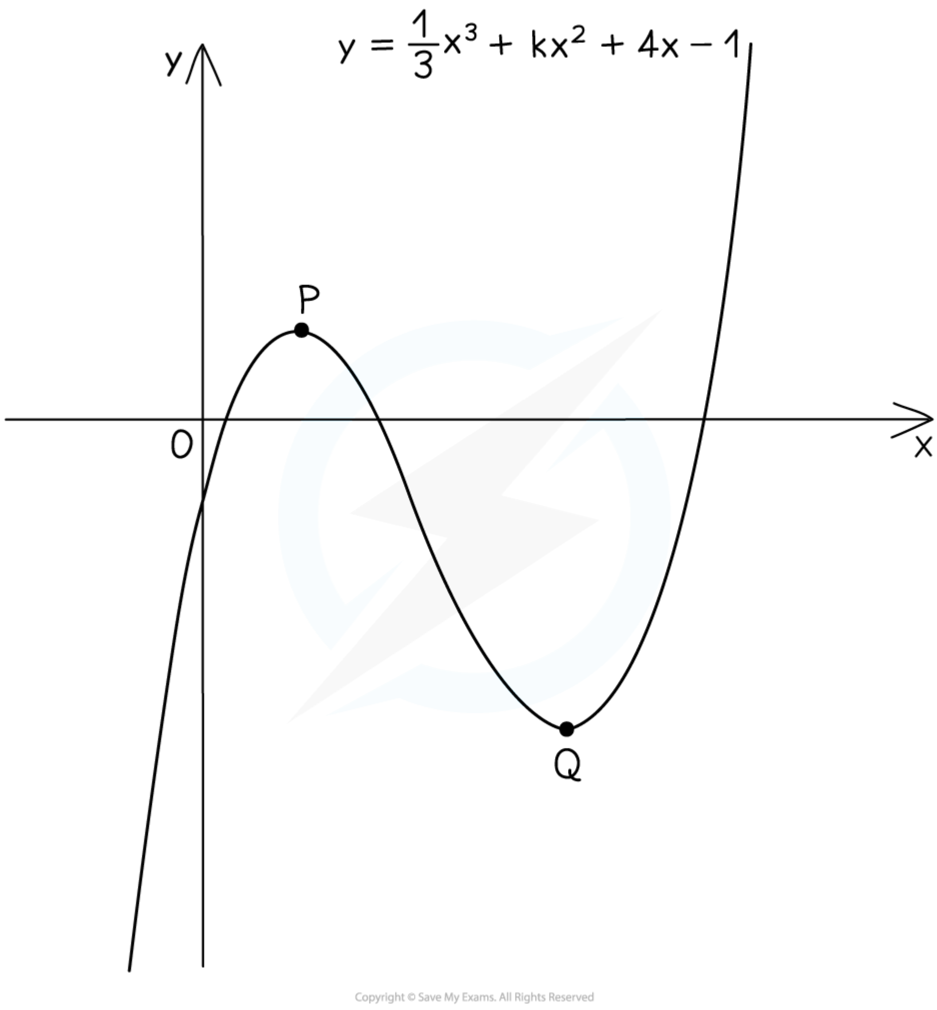

THE DISTANCE, d, BETWEEN TWO TURNING POINTS, P AND Q

# 2. Algebra & Graphs

YOUR NOTES

## How do I solve problems involving differentiation?

- Problems generally fall into two categories:**1. Graph based problems**
    - These problems are based around the graph of a curve and its turning points



THE GRAPH ABOVE SHOWS THE CURVE WITH EQUATION $y = \frac{1}{3}x^3 + kx^2 + 4x - 1$, WHERE $k$ IS A CONSTANT.
P AND Q ARE STATIONARY POINTS.

STATIONARY POINTS ARE ANOTHER NAME FOR TURNING POINTS

a) GIVEN THE x-COORDINATE OF P IS 1, FIND THE VALUE OF k.

b) FIND THE x-COORDINATE OF Q.

c) R IS THE POINT SUCH THAT PQR FORMS A RIGHT ANGLED TRIANGLE WITH HYPOTENUSE PQ.
FIND THE AREA OF TRIANGLE PQR.

a) AT P, $\frac{dy}{dx} = 0$ $\quad \frac{dy}{dx} = x^2 + 2kx + 4$

$3 - 1 = 2$

$\frac{1}{3} \times 3 = 1$

AT $x = 1$, $\quad (1)^2 + 2k(1) + 4 = 0$

$2k + 5 = 0$

$k = -\frac{5}{2}$

b) $\frac{dy}{dx} = x^2 - 5x + 4 = 0$

$(x-1)(x-4) = 0$

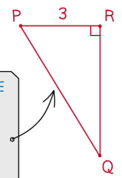
SINCE $x = 1$ IS A SOLUTION ALREADY KNOWN

$x = 1, \quad x = 4$

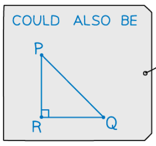
POINT P

Q HAS x-COORDINATE 4

c)




COULD ALSO BE




TO FIND THE "HEIGHT" OF THE TRIANGLE FIND THE y-COORDINATES OF P AND Q...

AT $x = 1$,

$y = \frac{1}{3} - \frac{5}{2} + 4 - 1 = \frac{5}{6}$

AT $x = 4$,

$y = \frac{64}{3} - 40 + 16 - 1 = \frac{-11}{3}$

SO, THE DISTANCE QR IS $\frac{5}{6} + \frac{11}{3} = \frac{9}{2}$

AREA OF $\triangle PQR = \frac{1}{2} \times 3 \times \frac{9}{2}$

$= \frac{27}{4}$ SQUARE UNITS.



### 2. Maximum/Minimum problems

- The maximum or minimum values have a meaning in the question
  eg. the maximum volume of a box made from a flat sheet of material
  eg. the minimum height of water in a reservoir
- These are sometimes called **optimisation** problems
  The maximum or minimum value gives the **optimal** (**ideal/best**) solution to the problem

# 2. Algebra & Graphs

YOUR NOTES

e.g. PART OF THE PROFILE OF A ROLLERCOASTER IS MODELLED BY THE QUADRATIC EQUATION

$$h = x^2 - 12x + 40$$

WHERE h IS THE HEIGHT IN METRES ABOVE THE GROUND AND x IS THE HORIZONTAL DISTANCE ALONG THE GROUND FROM A FIXED POINT 0.

a) FIND AN EXPRESSION FOR THE DERIVED FUNCTION, $\frac{dh}{dx}$,

b) FIND THE MINIMUM HEIGHT THIS PART OF THE ROLLERCOASTER REACHES ABOVE THE GROUND

c) HOW DO YOU KNOW YOUR ANSWER TO PART b) IS A MINIMUM VALUE?

a) $\frac{dh}{dx} = 2x - 12$

DIFFERENTIATE
TRY NOT TO BE PUT OFF BY UNUSUAL LETTERS

b) FOR A MINIMUM, $\frac{dh}{dx} = 0$

$2x - 12 = 0$

$x = 6$

THIS IS THE VALUE OF x AT THE MINIMUM, BUT QUESTION ASKS FOR MINIMUM HEIGHT

$h = 6^2 - 12 \times 6 + 40$

SUBSTITUTE $x = 6$ INTO h

MINIMUM HEIGHT IS 4m

c) THE CURVE OF THE GRAPH OF $h = x^2 - 12x + 40$ IS A **POSITIVE QUADRATIC** (PARABOLA) SO IT WILL HAVE A MINIMUM POINT.

# 2. Algebra & Graphs

YOUR NOTES

### Exam Tip

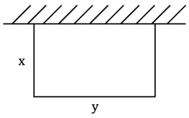
Diagrams can help – if you are not given one, draw one (a sketch is usually good enough – add to the diagram as you go through the question).
Make sure you know how to find the areas and volumes of basic shapes, eg. area of squares, rectangles, triangles, circles, volume of cubes, cuboids, cylinders.
Early parts of questions often ask you to “show that” a result is true – even if you can’t do this part of the question, the result may still be used in other parts allowing you to score some marks.

### Worked Example

# 2. Algebra & Graphs

YOUR NOTES

A farmer wishes to fence off an animal pen next to an existing wall. The pen is to be in the shape of a rectangle measuring $x$ m by $y$ m, as shown in the diagram below




The farmer has 50 m of fence and wishes to use all of it in for this pen.

(a) Show that $2x + y = 50$.

(b) Show that the area, $A$ m², of the pen is given by

$A = 50x - 2x^2$

(c) Find the maximum area of the pen

(d) Justify that your answer to part (c) is a maximum

a) PERIMETER $= x + y + x = 50$

ONLY 3 SIDES AS THE OTHER IS A WALL

FARMER WANTS TO USE ALL THE FENCE

$2x + y = 50$

b) $A = xy = x(50 - 2x)$

REARRANGING ANSWER TO a)

$A = 50x - 2x^2$

EXPAND BRACKETS

c) $\frac{dA}{dx} = 50 - 4x$

DIFFERENTIATE

FOR A MAXIMUM, $\frac{dA}{dx} = 0$

$50 - 4x = 0$

$x = 12.5$

THIS IS THE VALUE OF x AT THE MAXIMUM BUT QUESTION ASKS FOR THE MAXIMUM AREA

$A = 12.5(50 - 2 \times 12.5)$

$A = 12.5 \times 25$

MAXIMUM AREA $= 312.5$ m²

d) THIS IS A MAXIMUM POINT AS THE CURVE FOR A IS A NEGATIVE QUADRATIC (PARABOLA).

$A = 50x - 2x^2$