4. Geometry

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CONTENTS

4.1 SIMILARITY

4.1.1 SIMILARITY - LENGTHS

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What are similar shapes?

- Two shapes are **mathematically similar** if one is an enlargement of the other
- Equivalent **angles** in the two shapes will be equal
- Equivalent **lengths** in the two shapes will be in the same ratio and are linked by a **scale factor** (which you will normally have to find)

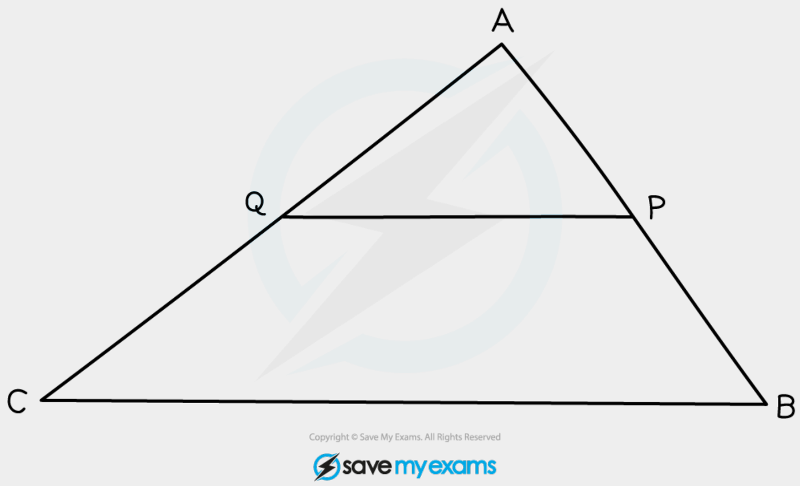
Working with similar shapes

1. Identify **equivalent** known lengths
2. Establish **direction** (getting bigger or smaller?)
3. Find **scale factor** = Second Length ÷ First Length
   (Check that SF > 1 if getting bigger and SF < 1 if getting smaller)
4. Use **scale factor** to find length

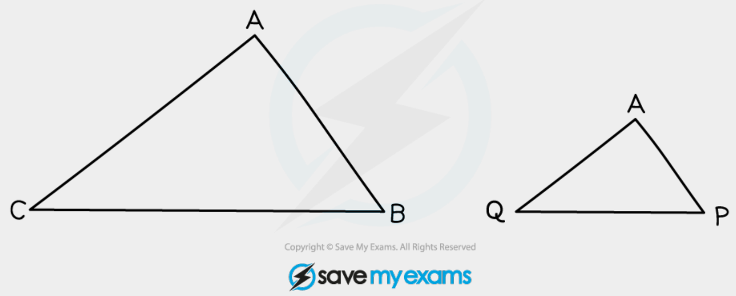
Exam Tip

If similar shapes overlap on the diagram (or are not clear) draw them separately!

For example, in this diagram the triangles ABC and APQ are similar:



So we would redraw them separately before we start:



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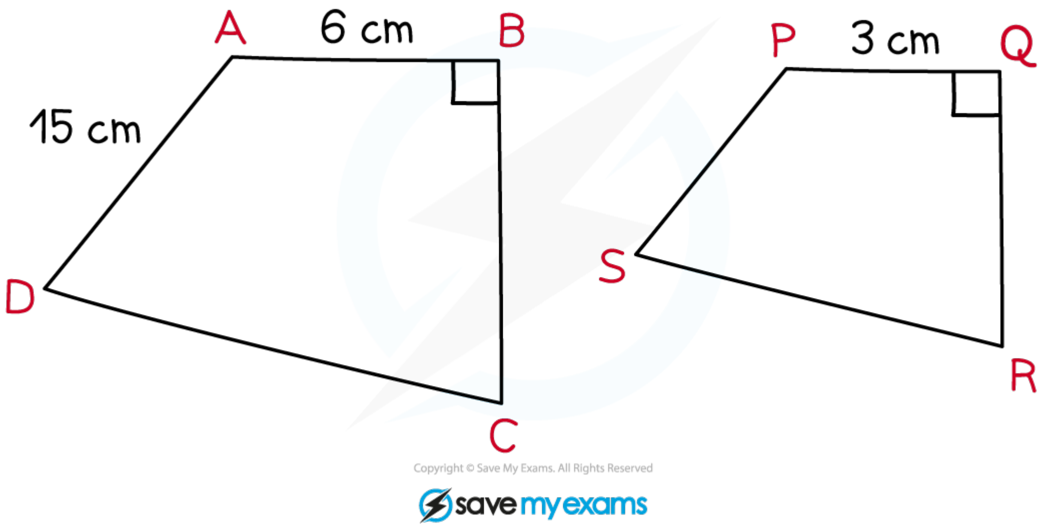
Worked Example

$ABCD$ and $PQRS$ are similar shapes.



Find the length of PS.

1. AB and PQ are the equivalent known lengths
2. To find PS we are working from $ABCD$ to $PQRS$ so that is "getting smaller"
3. $Scale\ Factor = \frac{PQ}{AB} = \frac{3}{6} = 0.5$ ($SF < 1$ as expected)
4. $$PS = 0.5 \times AD$$ $$= 0.5 \times 15$$ $$= 7.5\ cm$$

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4.1.2 SIMILARITY - AREAS & VOLUMES

What are similar shapes?

- See Similarity – Lengths and know that:
- Equivalent **areas** are linked by an **area factor** (see below)
- Equivalent **volumes** are linked by a **volume factor** (see below)

Working with similar shapes

1. Identify **equivalent** known quantities (Lengths, Areas or Volumes)
2. Establish **direction** (getting bigger or smaller?)
3. Find **Scale/Area/Volume Factor** = Second Quantity ÷ First Quantity
   (Check Factor > 1 if getting bigger and Factor < 1 if getting smaller)
4. **Convert** between Scale/Area/Volume Factors using

   Area Factor = $(\text{Scale Factor})^2$ (Scale Factor = $\sqrt{(\text{Area Factor})}$)

   Volume Factor = $(\text{Scale Factor})^3$ (Scale Factor = $\sqrt[3]{(\text{Volume Factor})}$)
5. Use **Scale/Area/Volume Factor** to find new quantity

Worked Example

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Solid A and solid B are mathematically similar.

The volume of solid A is $32\ cm^3$.

The volume of solid B is $108\ cm^3$.

The height of solid A is $10\ cm$.

Find the height of solid B.

1. *The equivalent known quantities are the volumes*
2. *To find the height of B we are working from A to B so that is "getting bigger"*
3. $Volume\ Factor = \frac{Volume\ B}{Volume\ A} = \frac{108}{32} = \frac{27}{8}$

We want the Scale Factor (as we are dealing with height, which is a length)

4. $Scale\ Factor = \sqrt[3]{Volume\ Factor} = \sqrt[3]{\frac{27}{8}} = \frac{3}{2}$
5. $Height\ of\ B = \frac{3}{2} \times Height\ of\ A$

$$= \frac{3}{2} \times 10$$

$$= 15\ cm$$

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4.2 CONGRUENT TRIANGLES

4.2.1 CONGRUENT TRIANGLES

What are congruent triangles?

- Two triangles are **CONGRUENT** if they are the same **SIZE** and **SHAPE** (although they may be reflections, rotations of each other)

How do we prove that two triangles are congruent?

- Happily we don't have to show that all 3 sides and all 3 angles are the same in each triangle
- We only need to show that 3 of the 6 things are the same – as long as they are the right three!
- To do this we MUST use one of the **5 standard tests**:

1. **SAS** - Side Angle Side

Note: The angle MUST be between the two sides to use this test. There is no 'ASS' test!

2. **ASA** - Angle Side Angle

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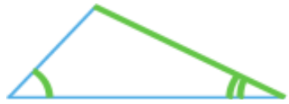
3. **AAS** - Angle Angle Side



4. **SSS** - Side Side Side

5. **RHS** - Right-angle Hypotenuse Side

Exam Tip

To find equivalent angles/sides you may need to use a variety of skills:
eg Parallel Lines, Symmetry, Isosceles Triangles, Circle Theorems, etc.
You may also have to be inventive and keep an open mind as to which of the five standard tests you are going to end up using!

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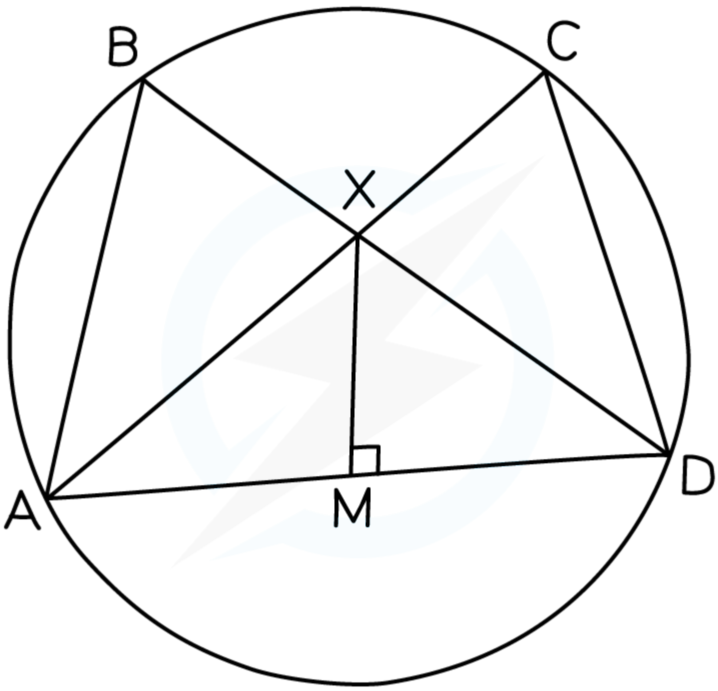
Worked Example

In the diagram A, B, C and D are four points on a circle.

X is the point of intersection of the lines AC and BD.

M is the midpoint of AD.

XM is perpendicular to AD.



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Prove that triangles AXB and DXC are congruent.

Because $AM = MD$, *and* XM *is perpendicular to* AD, *the triangles* AXM *and* MXD *are similar ('Side Angle Side' test). Therefore :*

$$AX = XD$$

Using the Circle Theorem that says "angles in the same segment are equal",

we see that : angle ABX = angle XCD

and also that : angle BAX = angle XDC

We now have 'Angle Angle Side' being equal in each triangle and so triangles AXB *and* DXC *are congruent*

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4.3 SYMMETRY

4.3.1 SYMMETRY

What is symmetry?

- **Symmetry** in mathematics can refer to one of two types:
  - **Line symmetry** which deals with **reflections** and **mirror images** of shapes or parts of shapes
  - Rotational symmetry which deals with how often a shape looks identical (**congruent**) when it has been rotated

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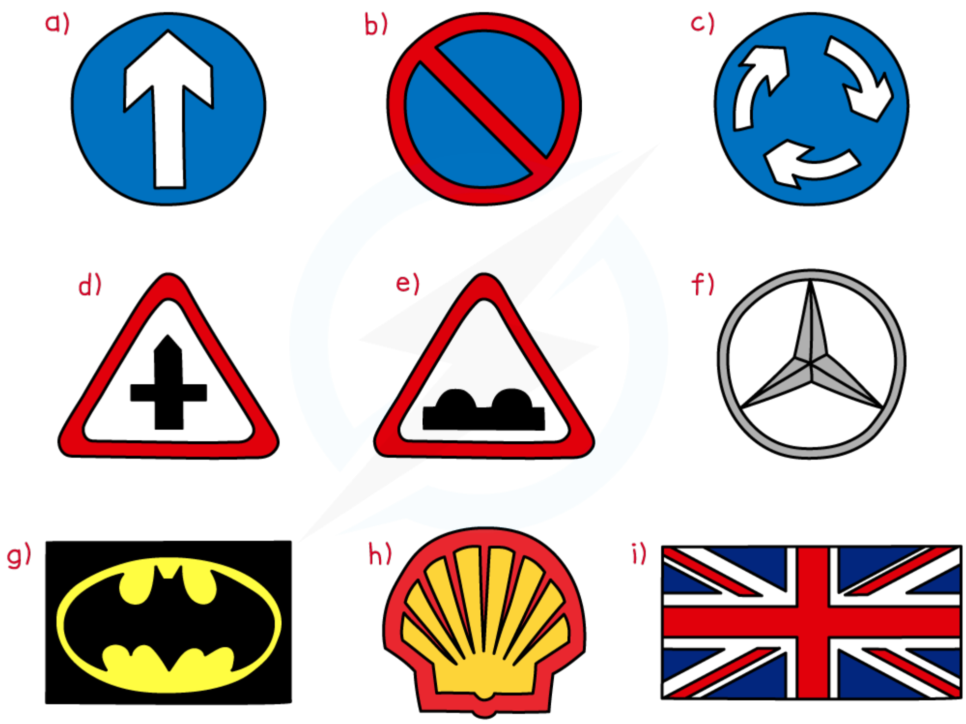
LINE AND ROTATIONAL SYMMETRY ARE OFTEN INCORPORATED INTO ROAD SIGNS, FLAGS AND LOGOS

SEE IF YOU CAN SPOT LINE AND ROTATIONAL SYMMETRY IN THE SIGNS & LOGOS BELOW.



LINE SYMMETRY:
a)1 b)2 c)0 d)1 e)1 f)3 g)1 h)1 i)2

ORDER OF ROTATIONAL SYMMETRY:
a)1 b)2 c)3 d)1 e)1 f)3 g)1 h)1 i)2

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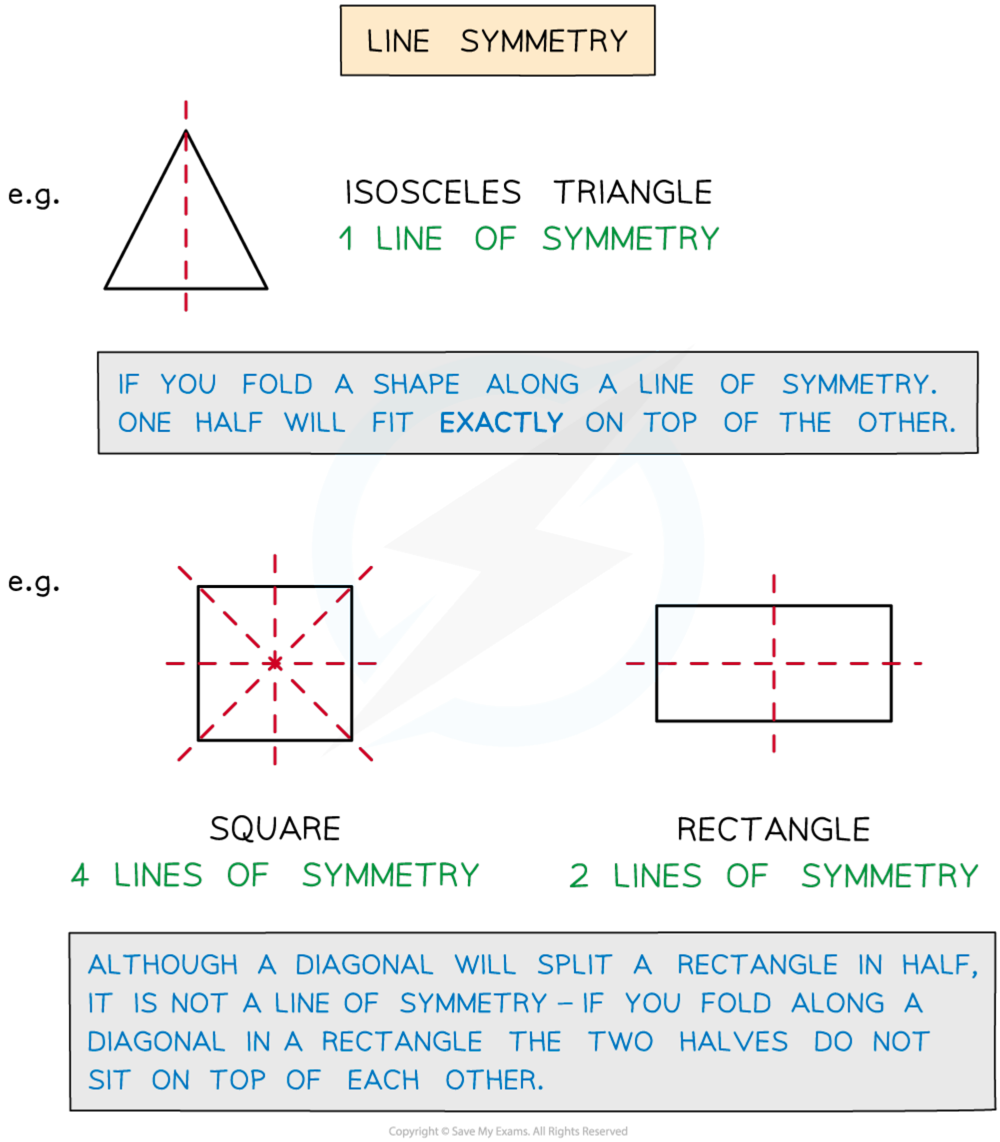
YOUR NOTES

What is line symmetry?

- **Line symmetry** refers to shapes that can have **mirror** lines added to them
  - Each side of the line of symmetry is a **reflection** of the other side
- Lines of symmetry can be thought of as a **folding** line too
  - **Folding** a shape along a line of symmetry results in the two parts sitting **exactly** on top of each other



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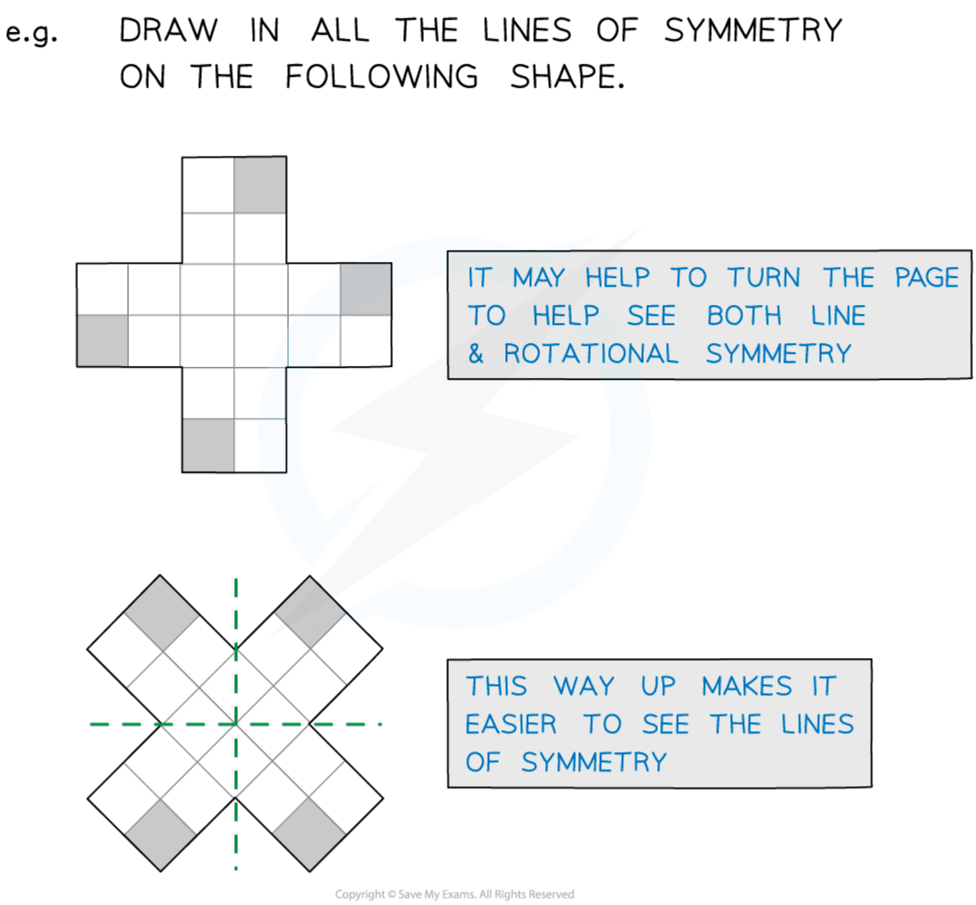
- It can help to look at shapes from different angles – turn the page to do this



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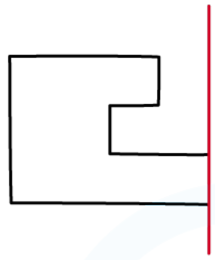
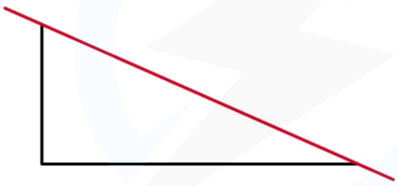
- Some questions will provide a shape and a line of symmetry
    - In these cases you need to complete the shape
- Be careful with **diagonal** lines of symmetry!
- "**Two-way**" **reflections** occur if the line of symmetry passes **through** the shape

e.g. COMPLETE THE SHAPES BELOW USING THE LINES OF SYMMETRY

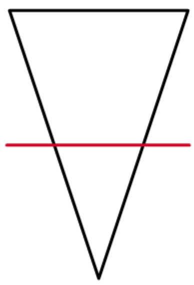
a)



b)



c)



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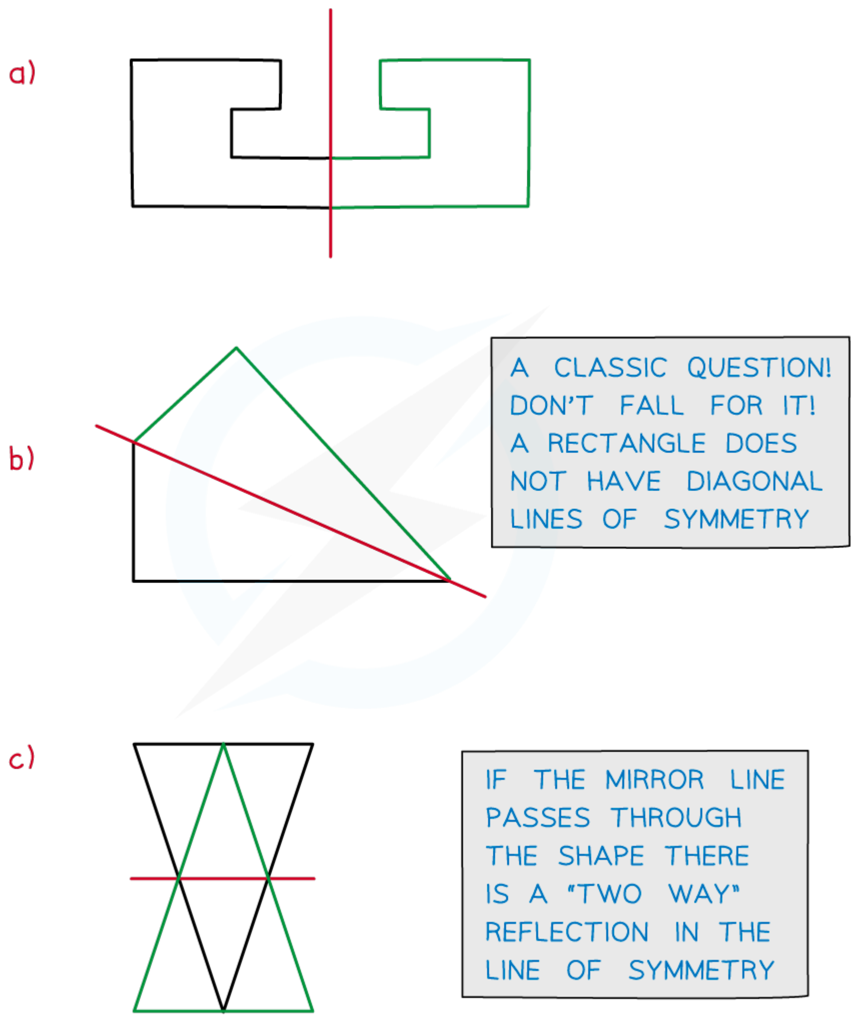

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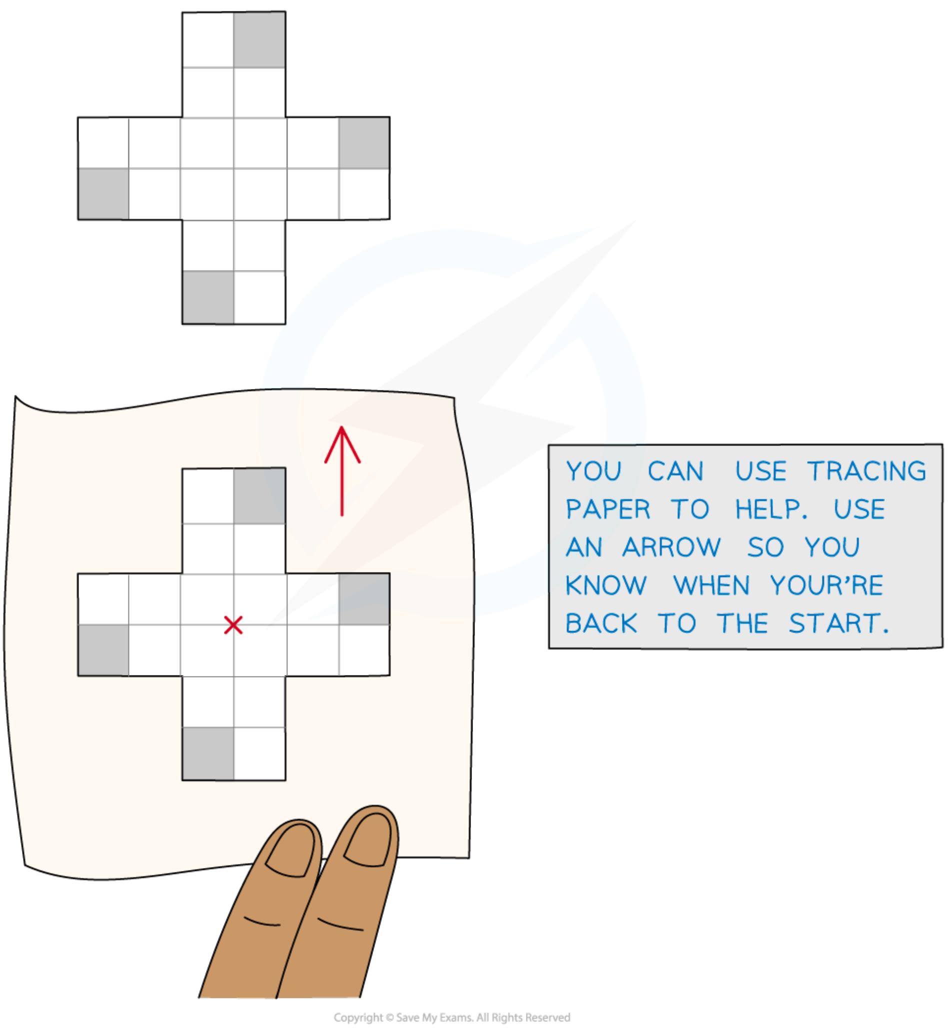
What is (the order of) rotational symmetry?

- **Rotational symmetry** refers to the number of times a shape looks the same as it is **rotated 360°** about its **centre**
- This number is called the **order** of **rotational symmetry**
- Tracing paper can help work out the order of rotational symmetry
  - Draw an arrow on the tracing paper so you can easily tell when you have turned it through 360°

e.g. WRITE DOWN THE ORDER OF ROTATIONAL SYMMETRY OF THE FOLLOWING SHAPE.



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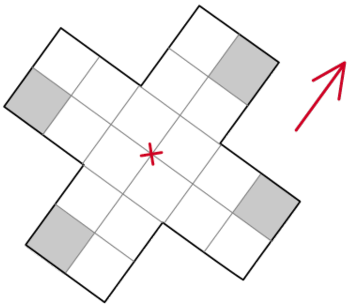
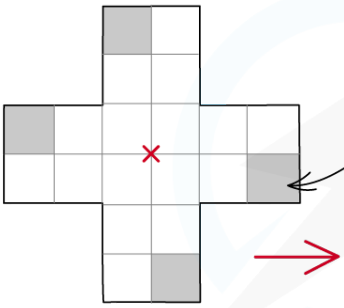
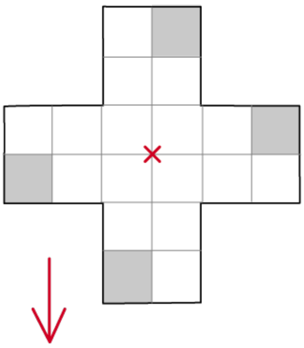


...AS YOU TURN THE TRACING PAPER COUNT HOW MANY TIMES THE SHAPE LOOKS THE SAME AS THE ORIGINAL.



BE CAREFUL HERE. THIS SHADED SQUARE IS NOT QUITE THE SAME AS THE ORIGINAL.



THIS IS THE SAME AS THE ORIGINAL (SO "1" SO FAR")

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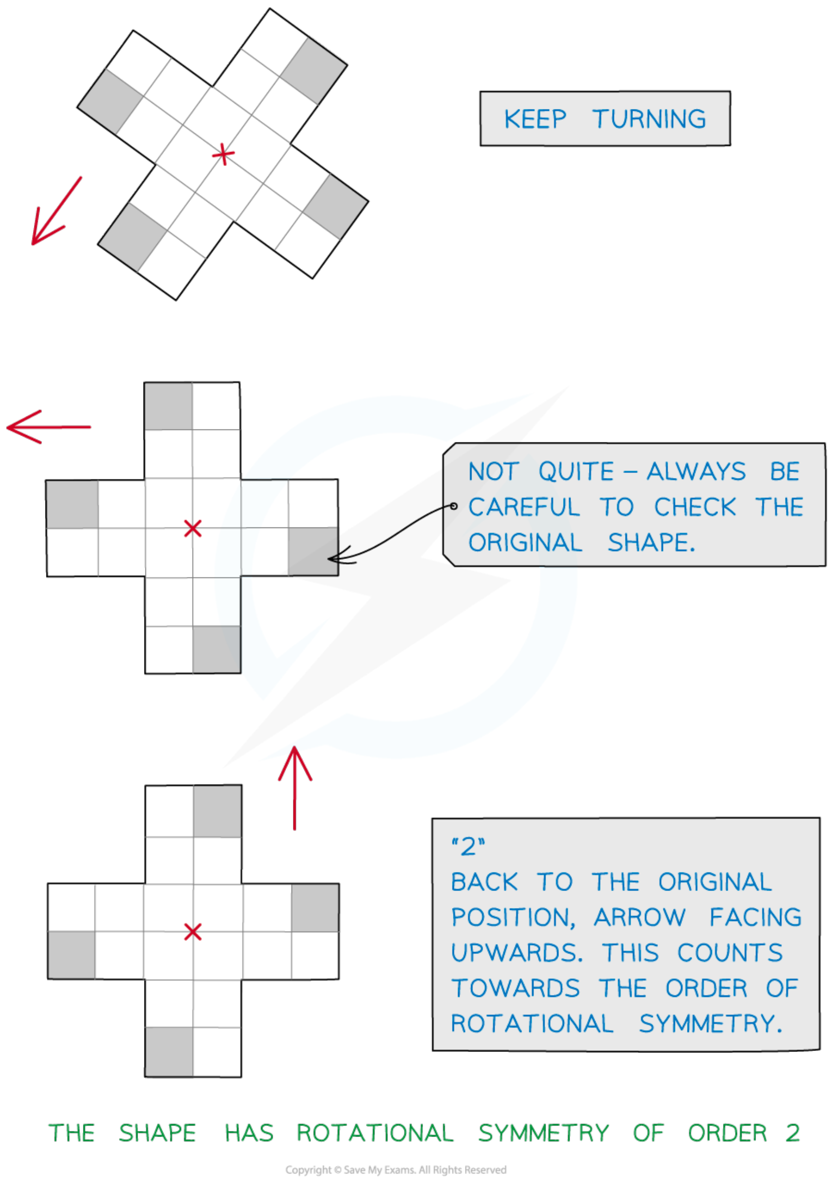

- Notice that returning to the original shape contributes 1 to the order
  - This means a shape can never have order 0
  - A shape with **rotational symmetry order 1** may be described as **not** having any rotational symmetry
    (The only time it looks the same is when you get back to the start)

How do I solve problems involving symmetry?

- Symmetry can be used to help solve missing length and angle problems

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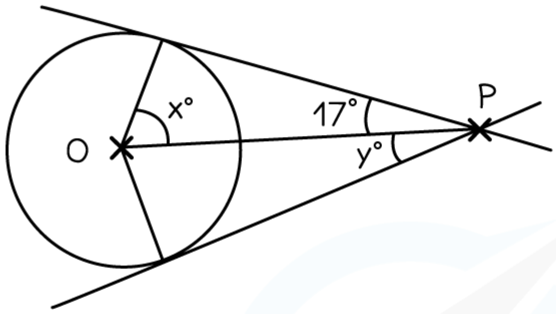
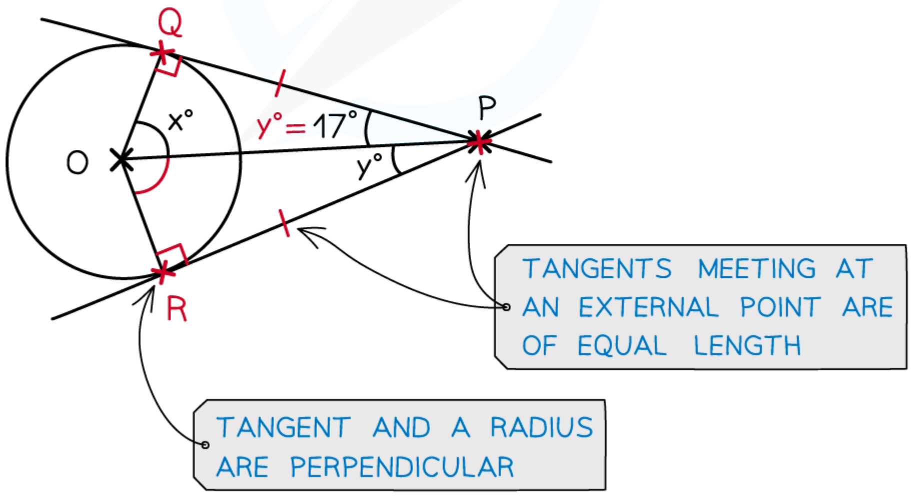
e.g. FIND THE SIZE OF THE ANGLES LABELLED x AND y IN THE DIAGRAM BELOW



ADD ANY INFORMATION TO THE DIAGRAM



OQPR IS A KITE SO THE LINE OP IS A LINE OF SYMMETRY

$y = 17$

$x + 90 + 17 = 180$

$x = 73$

ANGLES IN A TRIANGLE SUM 180°

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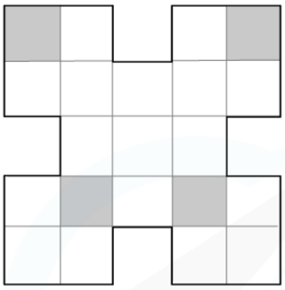
YOUR NOTES

Exam Tip

It may help to draw a diagram and add lines of symmetry to it – or add to a diagram if one is given in a question.
Tracing paper may help for rotational symmetry. One trick is to draw an arrow facing upwards so that when you rotate the tracing paper you know when it is back to its original position.

Worked Example

For the shape below



(a) write down the number of lines of symmetry the shape has,

(b) write down the order of rotational symmetry of the shape,

(c) shade exactly 4 more squares so the shape has 4 lines of symmetry and rotational symmetry order 4.

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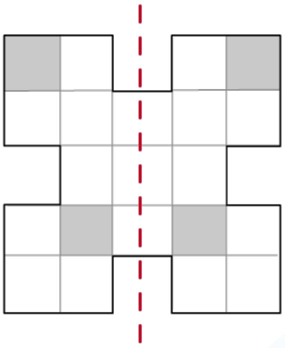
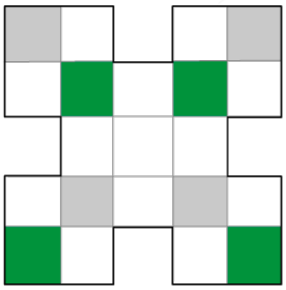


a) 1 THERE IS ONLY ONE LINE OF SYMMETRY THAT CAN BE DRAWN ON THE DIAGRAM

b) 1 THE SHAPE HAS NO ROTATIONAL SYMMETRY DUE TO THE LOCATION OF THE SHADED SQUARES

c)



THIS SHAPE IS BASED ON A SQUARE THAT HAS 4 LINES OF SYMMETRY AND ROTATIONAL SYMMETRY ORDER 4.

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4.4 ANGLES IN POLYGONS

4.4.1 ANGLES IN POLYGONS

What is a polygon?

- A **polygon** is a flat (plane) shape with n straight sides
  For example:
  A triangle is a polygon with 3 sides
  A quadrilateral polygon with 4 sides
  A pentagon is a polygon with 5 sides
- In a **regular** polygon all the sides are the same length and all the angles are the same:
  A regular polygon with 3 sides is an equilateral triangle
  A regular polygon with 4 sides is a square

Working with angles in polygons

1. TOTAL OF INTERIOR ANGLES = 180° × (n – 2)
(because the polygon can be split into n -2 triangles)
2. TOTAL OF EXTERIOR ANGLES = 360°
(this exterior angles rule is the same for ANY number of sides!)
eg For a hexagon (n=6):

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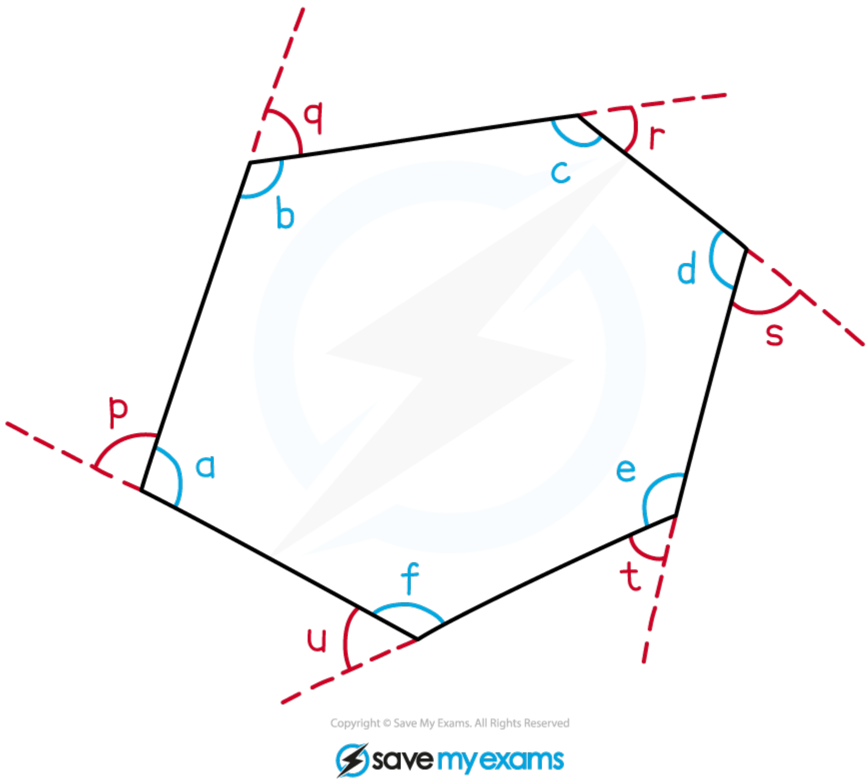
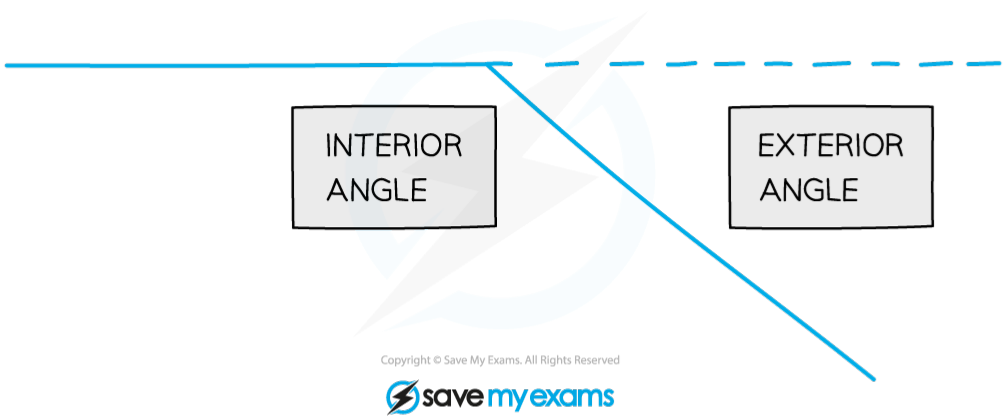


a + b + c + d + e + f = 180° × (6 – 2) = 720°

p + q + r + s + t + u = 360°



For **REGULAR** Polygons:

3. EXTERIOR ANGLE = 360 ÷ n

4. INTERIOR ANGLE = 180 – Exterior Angle

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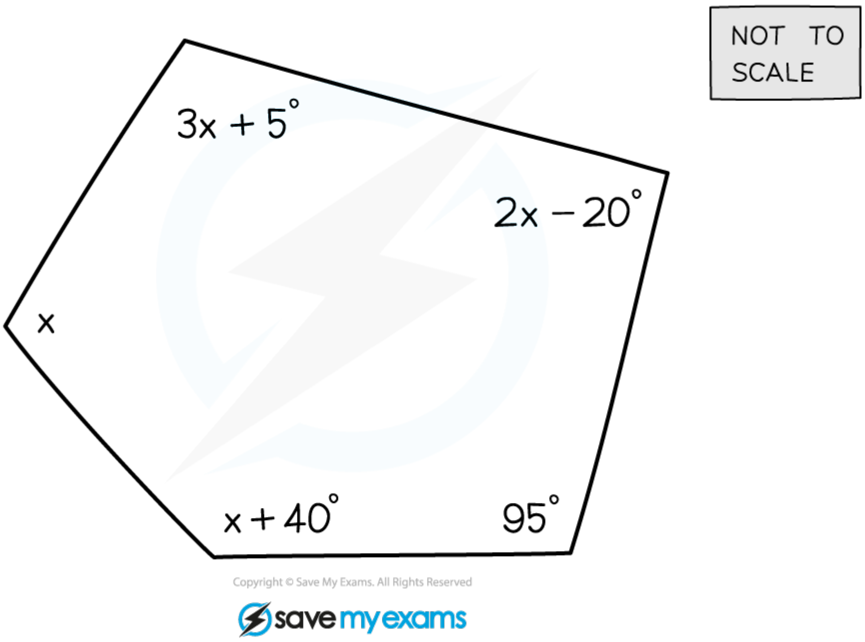
Exam Tip

Make sure you know whether you are dealing with a regular or irregular polygon before you start a question.

Worked Example

The diagram shows an irregular pentagon.
Work out the value of x.



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$This\ is\ an\ irregular\ pentagon\ (5\ sides)\ so:$

1. $$Total\ Interior\ Angles = 180 \times (5-2) = 540°$$

$Putting\ in\ the\ information\ from\ the\ diagram\ gives:$

$$(3x+5)+x+(x+40)+95+(2x-20)=540$$

$Simplifying:$

$$7x+120=540$$

$And\ solving:$

$$7x=420$$

$$x=60$$

4. Geometry

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4.5 CIRCLE THEOREMS

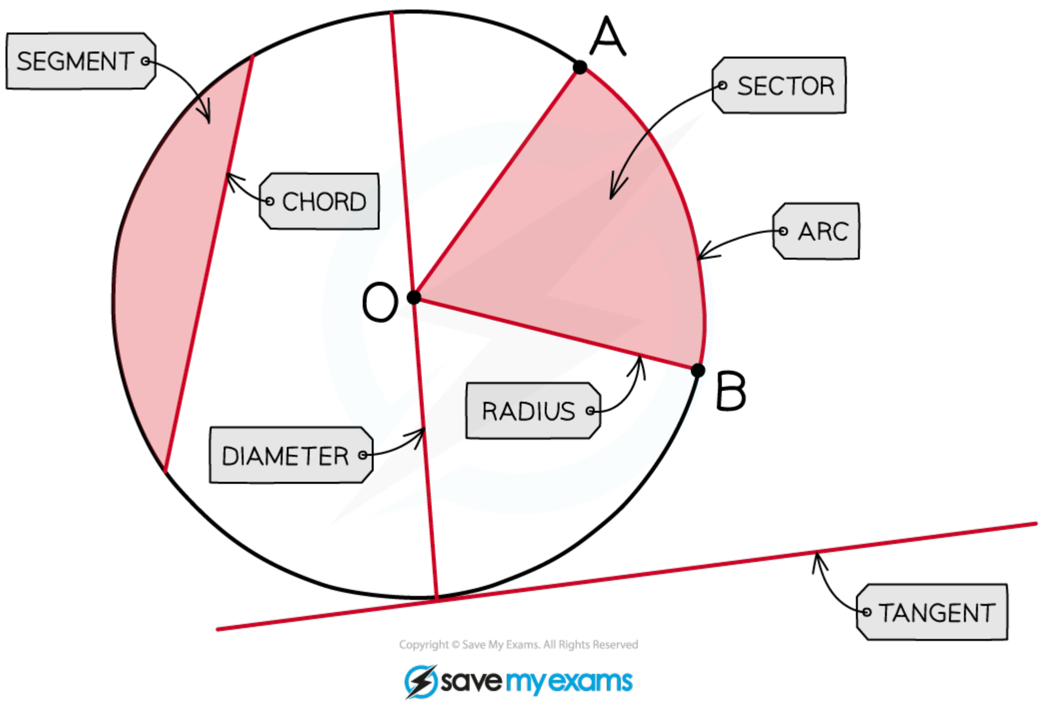
4.5.1 CIRCLE THEOREMS - ANGLES AT CENTRE & CIRCUMFERENCE

What are circle theorems?

- You will have learned a lot of angle facts for your GCSE – angles in polygons, angles with parallel lines
- **Circle Theorems** deal with angle facts that occur with shapes and lines drawn within and connected to a circle

What do I need to know?

- Be familiar with the names of parts of a circle and lines within and outside of a circle:



- To solve hard problems you may need to use the angle facts you are already familiar with from triangles, polygons, and parallel lines
- You may also have to use **circumference** and **area** formulas, so ensure you're familiar with:
  C = πd
  and
  A = πr²

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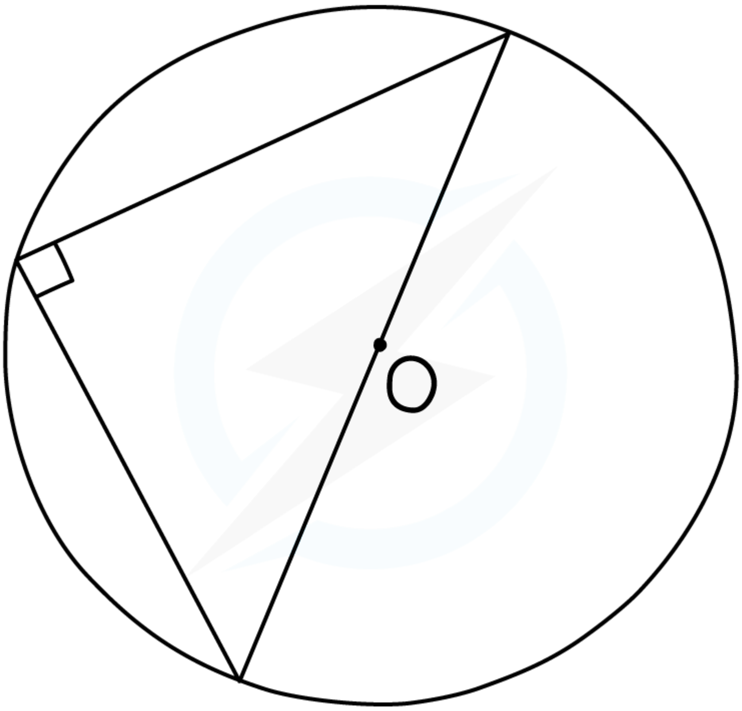
The first 3 circle theorems

1. The angle at the circumference in a semicircle is a right angle



- This theorem should be self-explanatory form its name/title
- The semicircle arises if you ignore the right-hand side of the diameter in the diagram above
- Look out for triangles hidden among other lines/shapes within the circle

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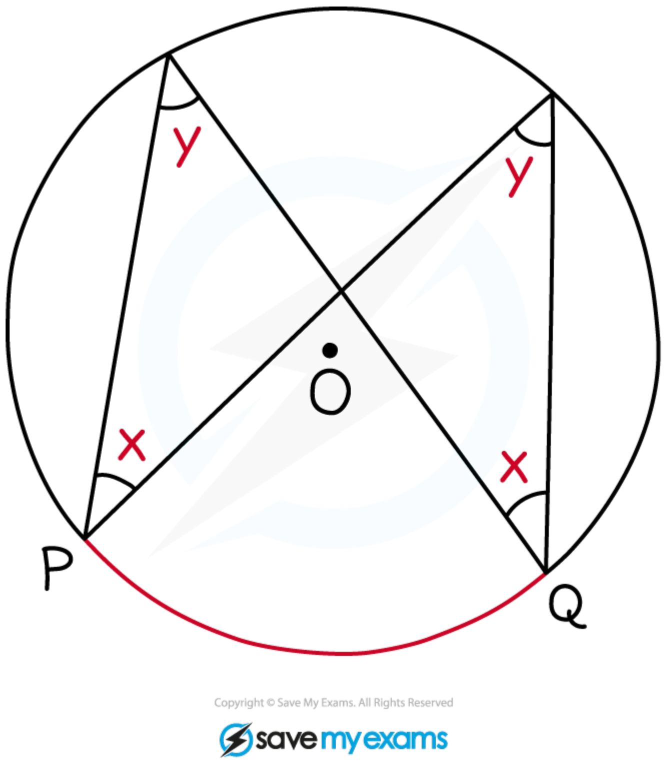
2. Angles at the circumference subtended by the same arc are equal



- **Subtended** means the equal angles are created by drawing chords from the ends of the arc PQ
- Theses chords may or may not pass through the centre
- Both pairs of angles are equal

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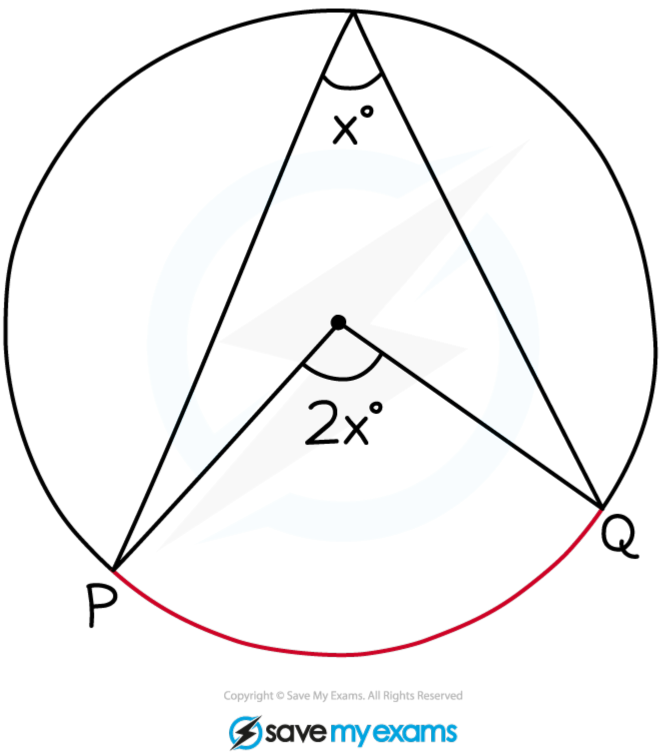
3. Angle subtended by an arc at the centre is twice the angle at the circumference



- Similar to above, the chords (radii) to the centre and the chords to the circumference are both drawn from (subtended by) the ends of the arc PQ
- This theorem can also happen when the 'triangle parts' overlap:

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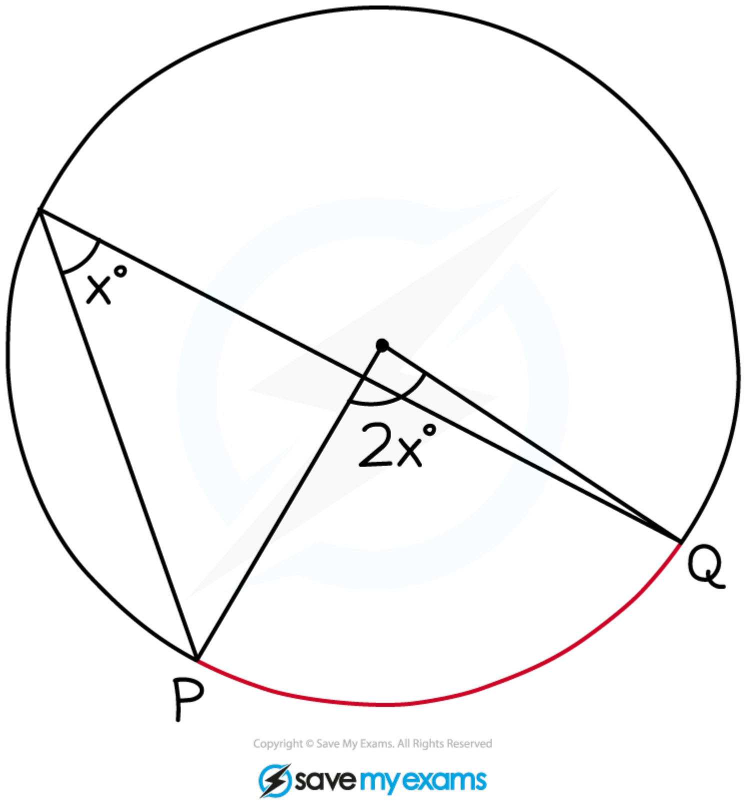

Exam Tip

Add anything you can to a diagram you have been given – write in any angles and lengths you can work out, even if they don't seem relevant to the actual question.

For each angle you work out, try to assign an angle fact or circle theorem to it – questions often ask for "reasons" and the names/titles/phrases for each of these is exactly what they are after.

When asked to "give reasons" aim to quote an angle fact or circle theorem for **every** angle you find, not just one for the final answer.

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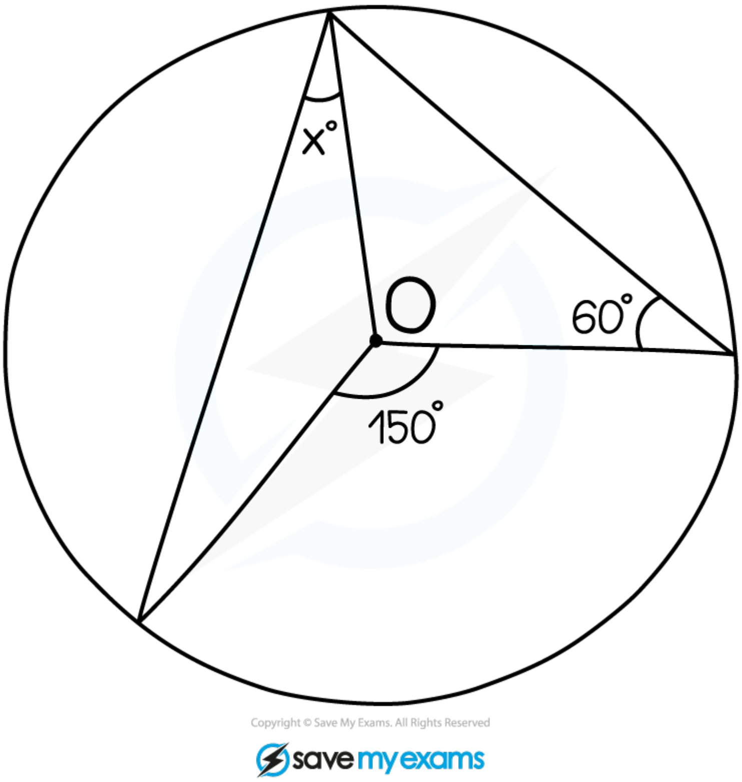
Worked Example

1. Find the value of x in the diagram below.



You should spot that we have three radii which create two isosceles triangles. This means that the angle next to x must be 60°.

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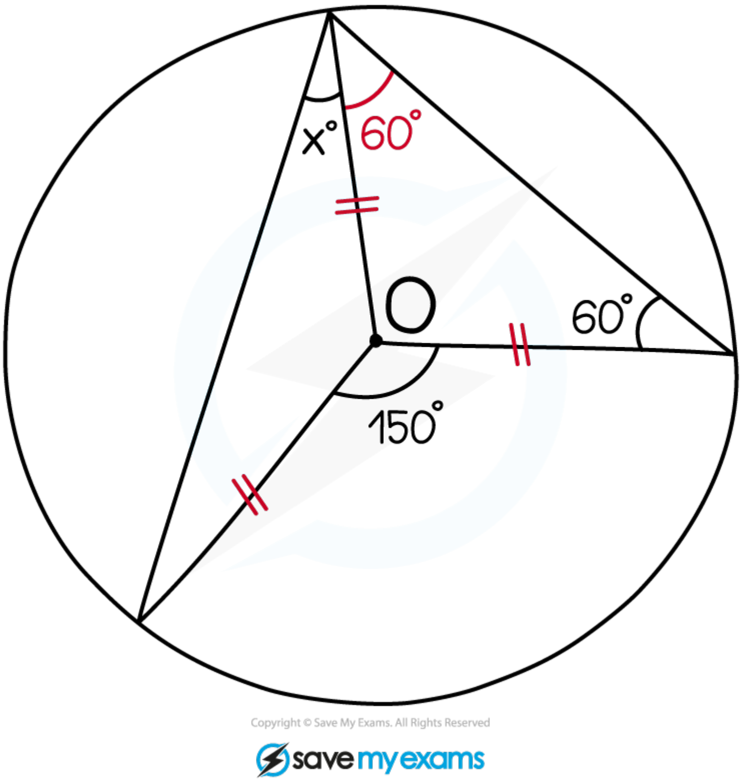

3 - Now using the circle theorem "Angle subtended by an arc at the centre is twice the angle at the circumference" we can create an equation for x:

$2(x + 60) = 150$

$2x + 120 = 150$ ... which we can then rearrange and solve

$2x = 30$

$\boldsymbol{x = 15°}$

2. Find the value of θ in the diagram below giving reasons for your answers.

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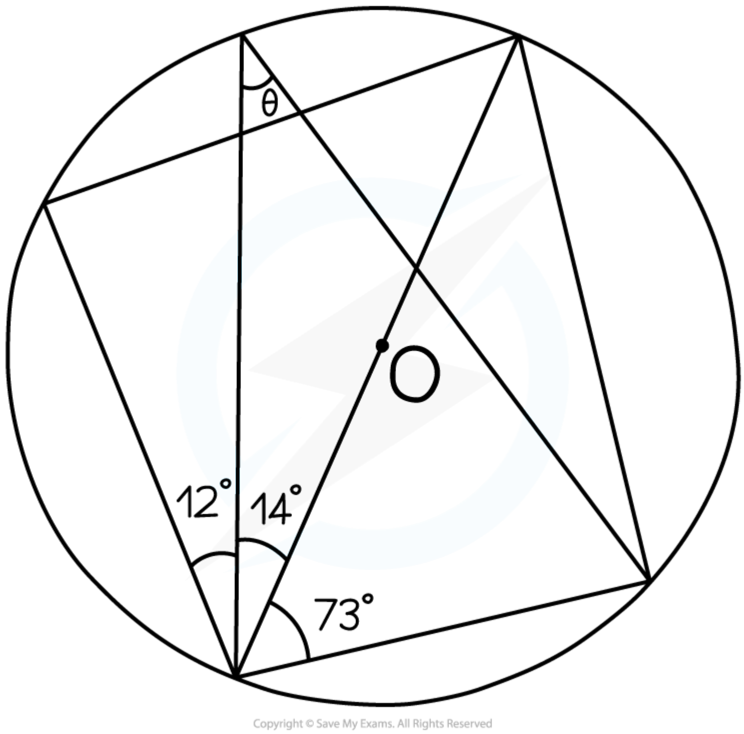

2 - There is a lot going on in this diagram, but you should spot there is a "triangle in a semi-circle" on both sides of the diameter.

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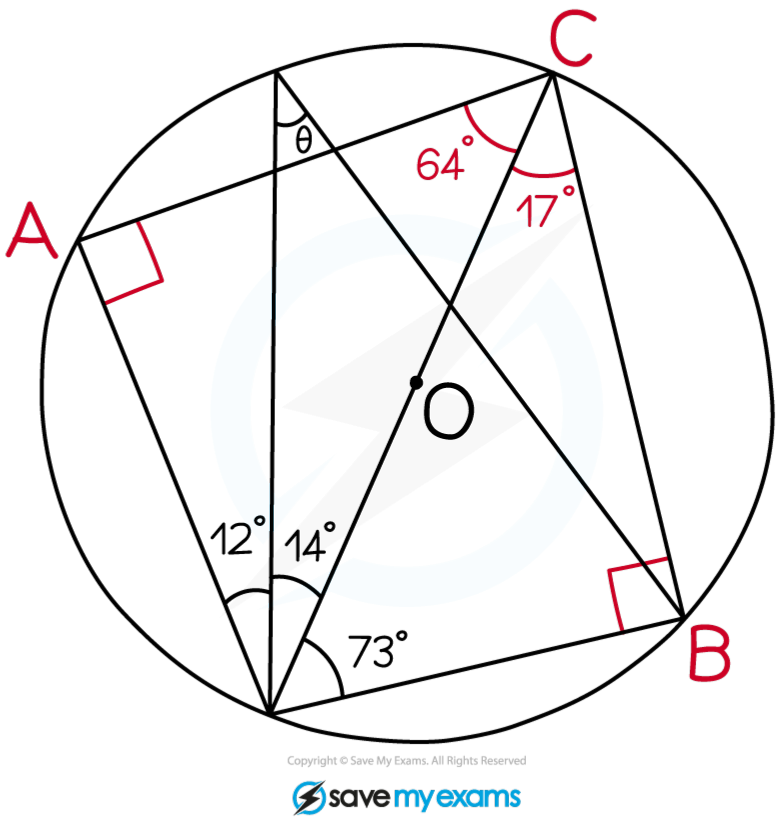

Angles at A and B are 90° because the angle at the circumference in a semicircle is a right angle

The angles at C are 64°and 17° because the angles in a triangle add up to 180°

1 - We can now find θ using one of the circle theorems …

$\theta = 17°$

Angles at the circumference subtended by the same arc are equal

Notice in this question that all the "working" (red) is on the diagram and final answers (green) include the phrases/titles of the circle theorems and angle facts used – this is what the question means when it says "give reasons"

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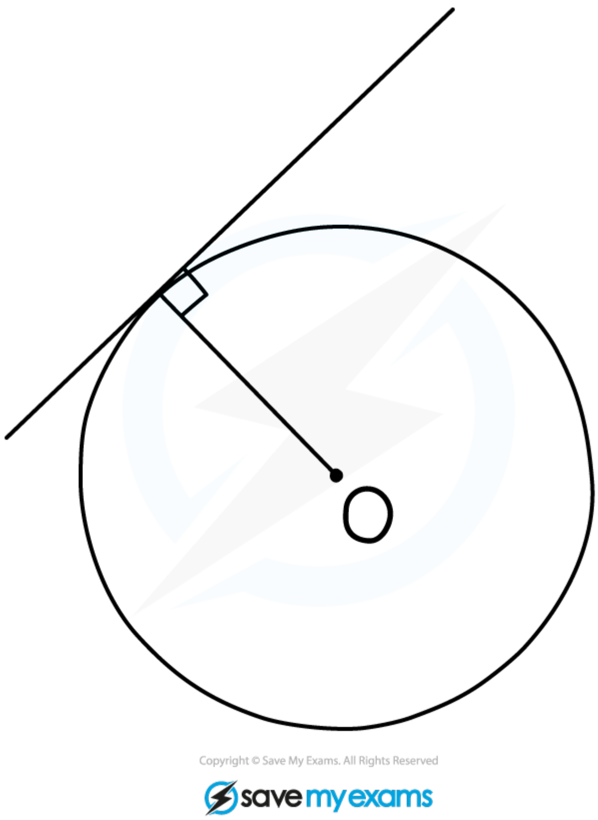
4.5.2 CIRCLE THEOREMS - TANGENTS

What do I need to know about tangents

1. A radius and a tangent are perpendicular



- Note that **perpendicular** lines meet at right angles

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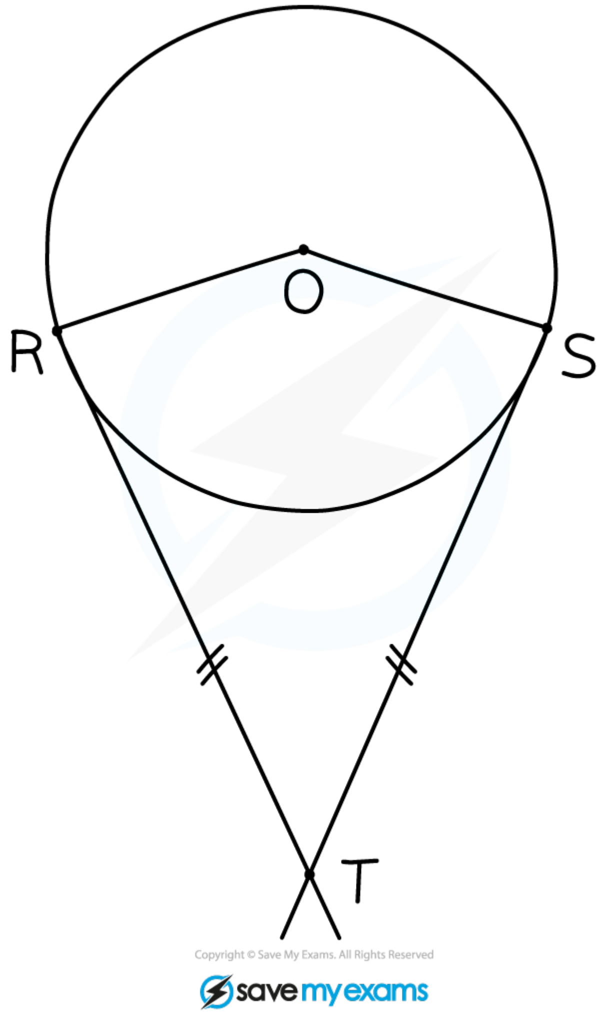
YOUR NOTES

2. Not strictly a circle theorem but a very important fact for solving some problems



- In this sense the tangents end at two points – the first point is where the two tangents meet and the other end is where each one touches the circle
- Notice because of the circle theorem above that the quadrilateral ROST is a kite with two right angles

4. Geometry

YOUR NOTES
⬇

Exam Tip

Add anything you can to a diagram you have been given – write in any angles and lengths you can work out, even if they don't seem relevant to the actual question.

Also, when asked to "give reasons" (this is very common), aim to quote an angle fact or circle theorem for **every** angle you find, not just one for the final answer.

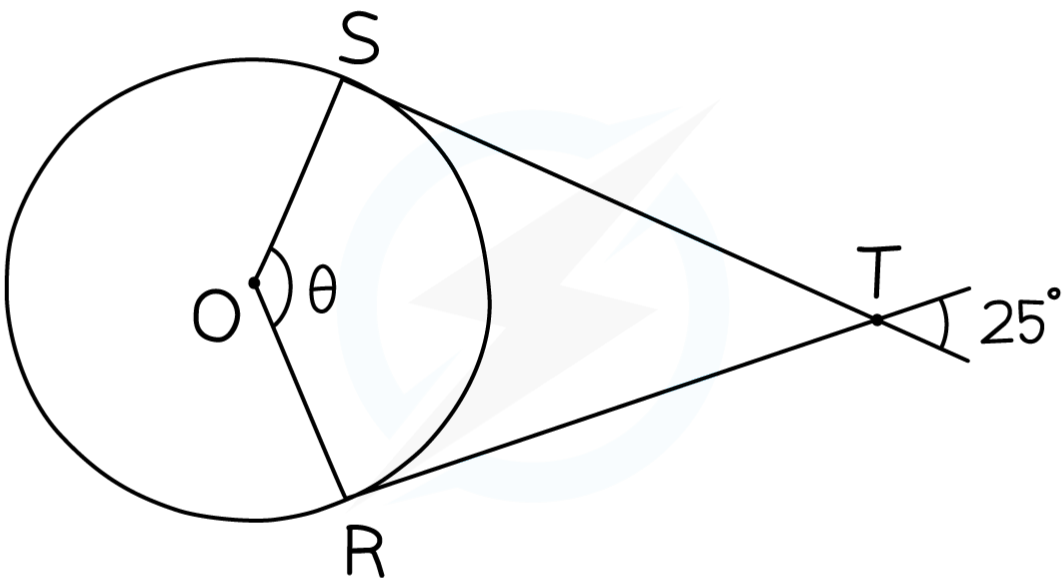
Likewise, for each angle you work out, try to assign an angle fact or circle theorem to it – giving the names/titles/phrases for each of these is exactly what the examiners want to see.

4. Geometry

YOUR NOTES

Worked Example

(a) Find the value of θ in the diagram below giving reasons for your answer.

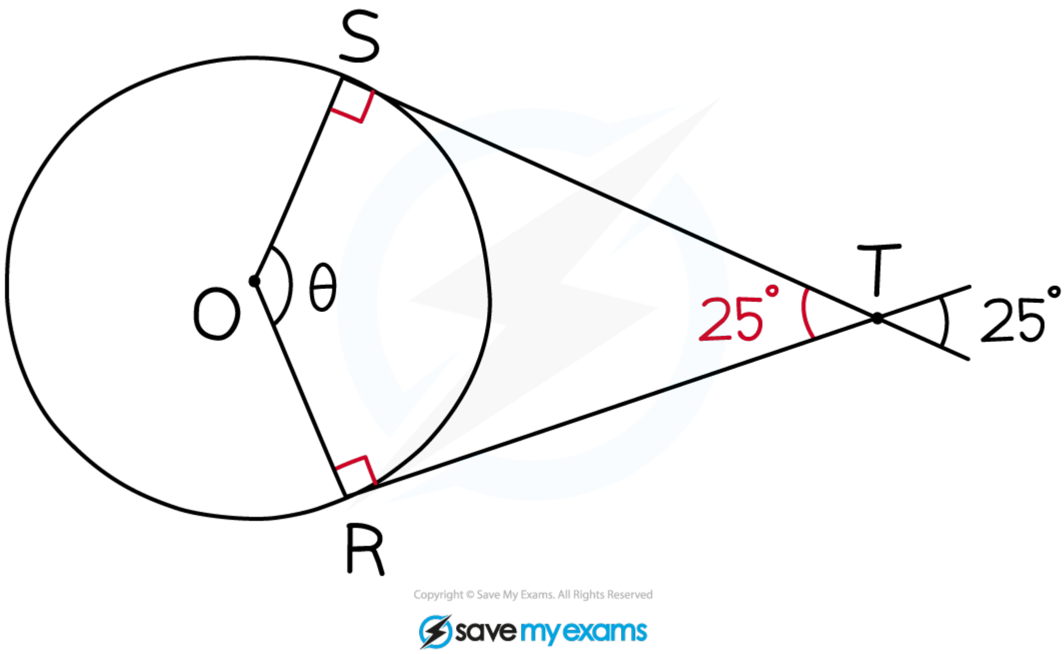


Missing angle at T is 25° because vertically opposite angles are equal

(See revision notes on Vertically Opposite Angles)

4. Geometry

YOUR NOTES
⬇



$$\theta = 360 - 2 \times 90 - 25$$

Angles at S and R are 90° since a radius and a tangent are perpendicular

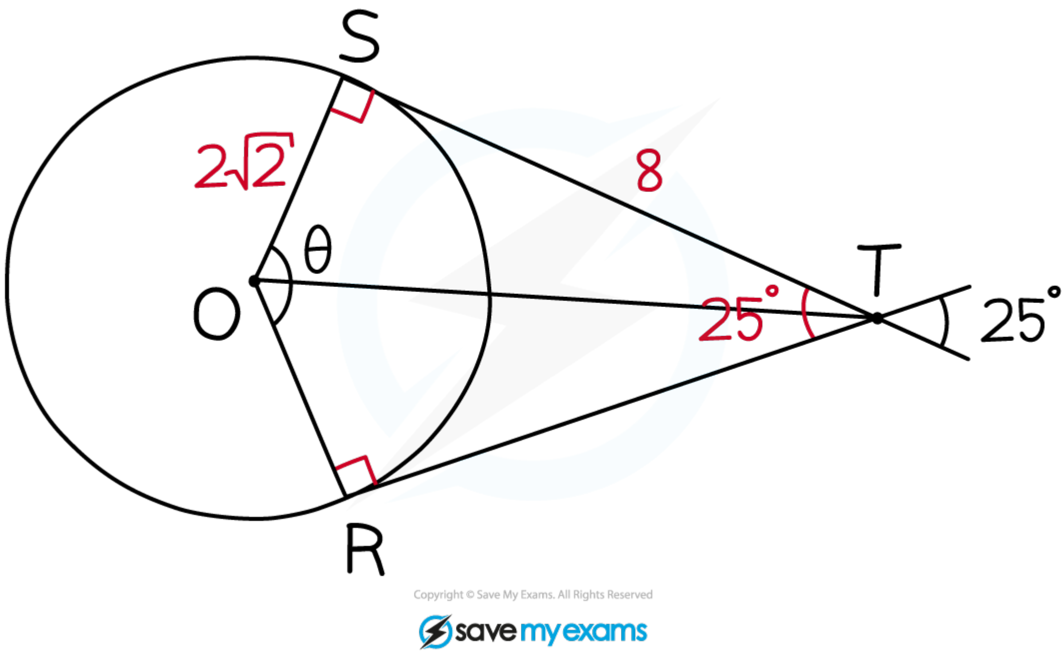
Angles in a quadrilateral add up to 360°

$$\theta = \mathbf{155°}$$

(b) Given that the radius of the circle is $2\sqrt{2}\ cm$ and ST is 8 cm find the length of OT, giving your answer in the form $a\sqrt{2}$, where a is an integer.

4. Geometry

YOUR NOTES
⬇



OS is a radius (as is OR) and OST is a right-angled triangle (as is ORT) and we know the lengths of two sides so using Pythagoras' Theorem we can find the third. (See revision notes on Pythagoras' Theorem)

$(OT)^2 = (2\sqrt{2})^2 + 8^2$	Apply Pythagoras' …
$(OT)^2 = 8 + 64$	… number crunch …
$(OT)^2 = 72$	
$OT = \sqrt{72}$	… and put final answer in correct form.
$\boldsymbol{OT = 6\sqrt{2}\ cm}$	Your calculator may do this for you, if allowed!
	(See revision notes on Surds)

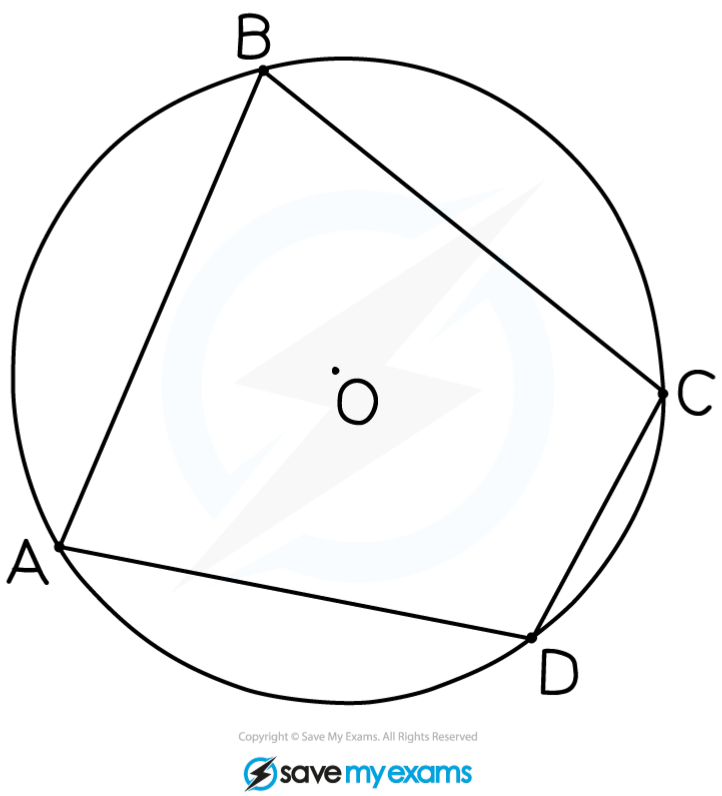
4. Geometry

YOUR NOTES

4.5.3 CIRCLE THEOREMS - CYCLIC QUADRILATERALS

More advanced circle theorems

1. Opposite angles in a cyclic quadrilateral add up to 180°



- Double-check is that all 4 vertices of the quadrilateral are on the circumference
- The diagram below shows a common scenario that is **NOT** a cyclic quadrilateral:

4. Geometry

YOUR NOTES
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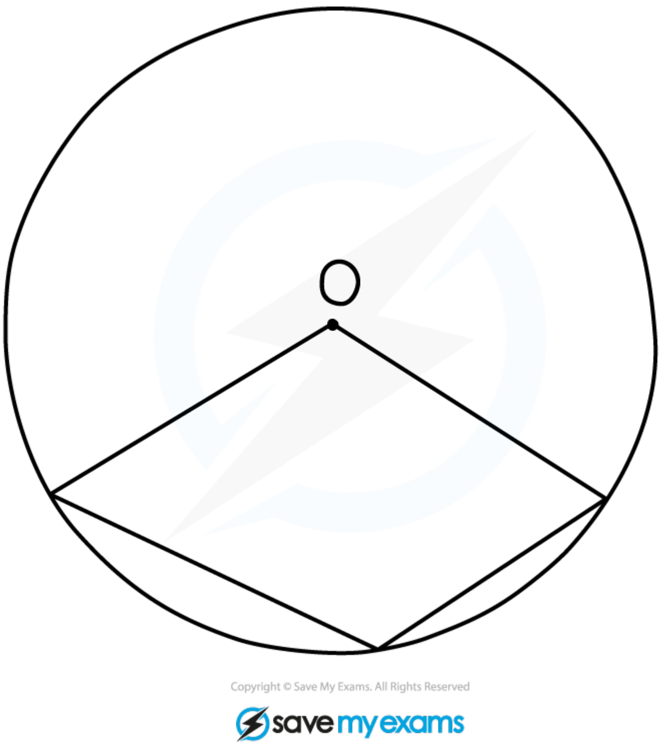
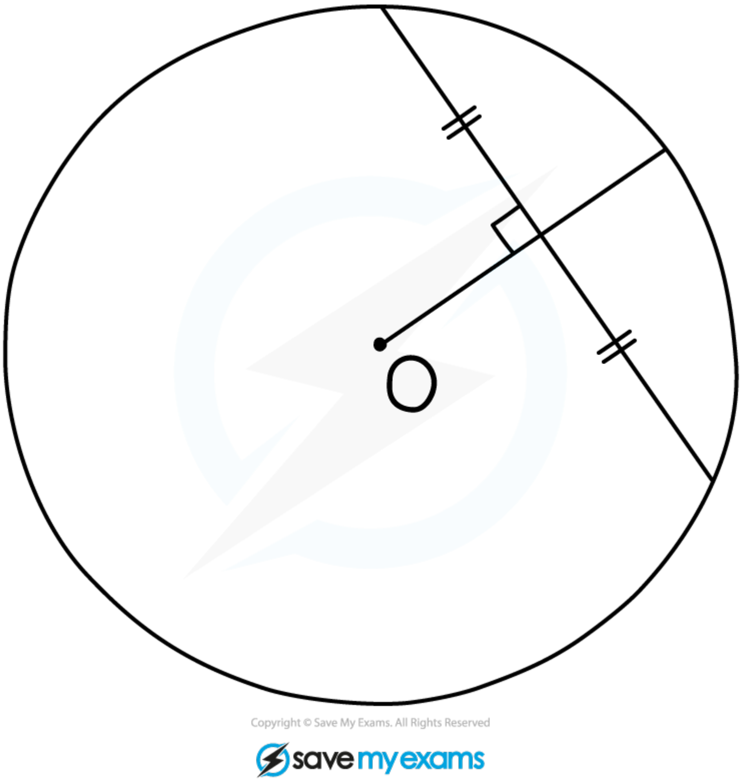


2. The perpendicular bisector of a chord is a radius

4. Geometry

YOUR NOTES



- This is also easier to see than remember from its description
- Problems here involve the radii being joined to the end of the chords and so creating two congruent triangles

Exam Tip

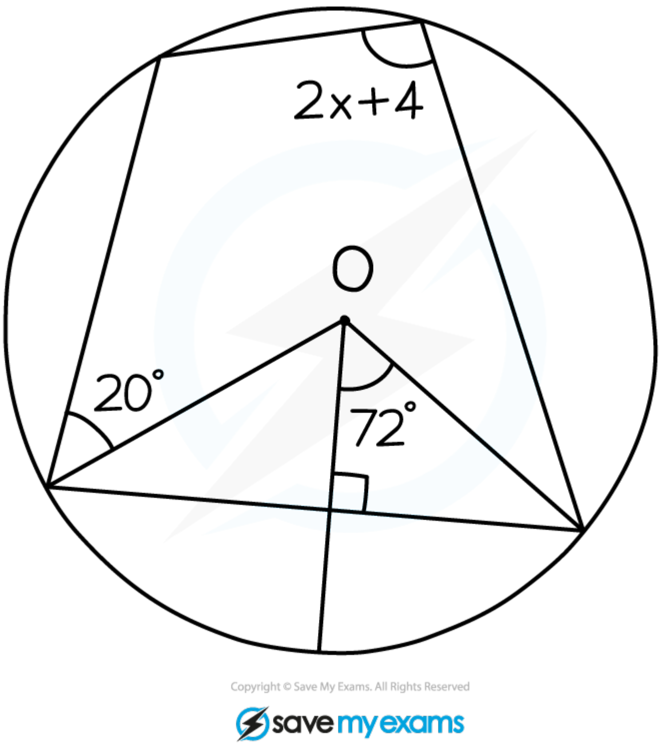
Remember to add anything you can to a diagram you have been given – write in any angles and lengths you can work out, even if they don't seem relevant to the actual question.

4. Geometry

YOUR NOTES

Worked Example

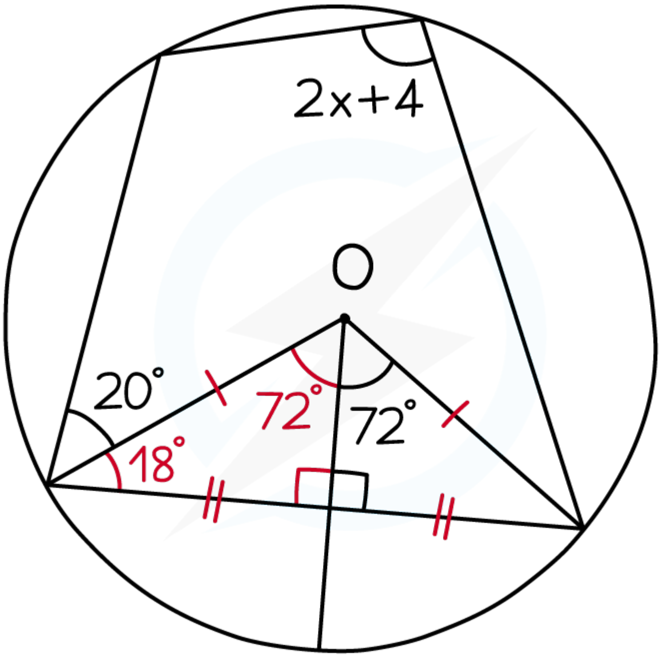
1. Find the value of x.



1, 2 - A busy diagram but we have both a cyclic quadrilateral and a radius that is perpendicular to a chord. Add to the diagram as you work through the problem.

4. Geometry

YOUR NOTES
⬇



The radius bisects the chord and so creates two congruent triangles

We can work out the 72° (equal to equivalent angle in other triangle) and 18° (angles in a triangle add up to 180°)

$2x + 4 + 20 + 18 = 180$ Now use the cyclic quadrilateral circle theorem

$2x = 138$

$x = 69°$

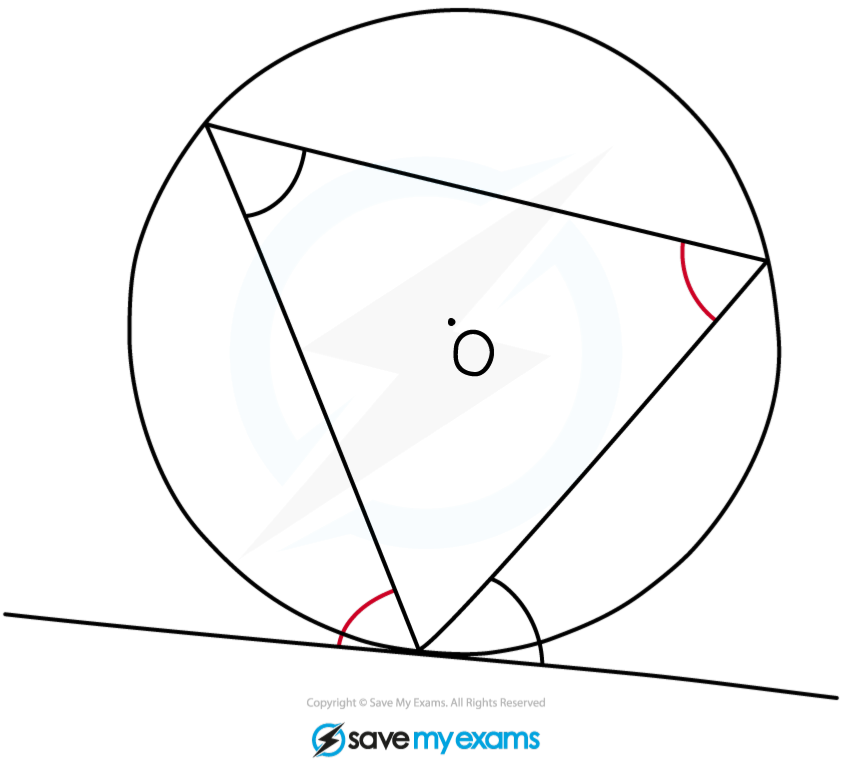
4. Geometry

YOUR NOTES
⬇

4.5.4 CIRCLE THEOREMS - ALTERNATE SEGMENT

What you need to know

1. **Alternate Segment Theorem**



- The Alternate Segment Theorem states that the angle between a chord and a tangent is equal to the angle in the alternate segment
- You can spot this circle theorem by looking for a "cyclic triangle"
ie. all 3 vertices lie on the circumference) but one vertex lies on a tangent – look for where 2 chords meet a tangent

4. Geometry

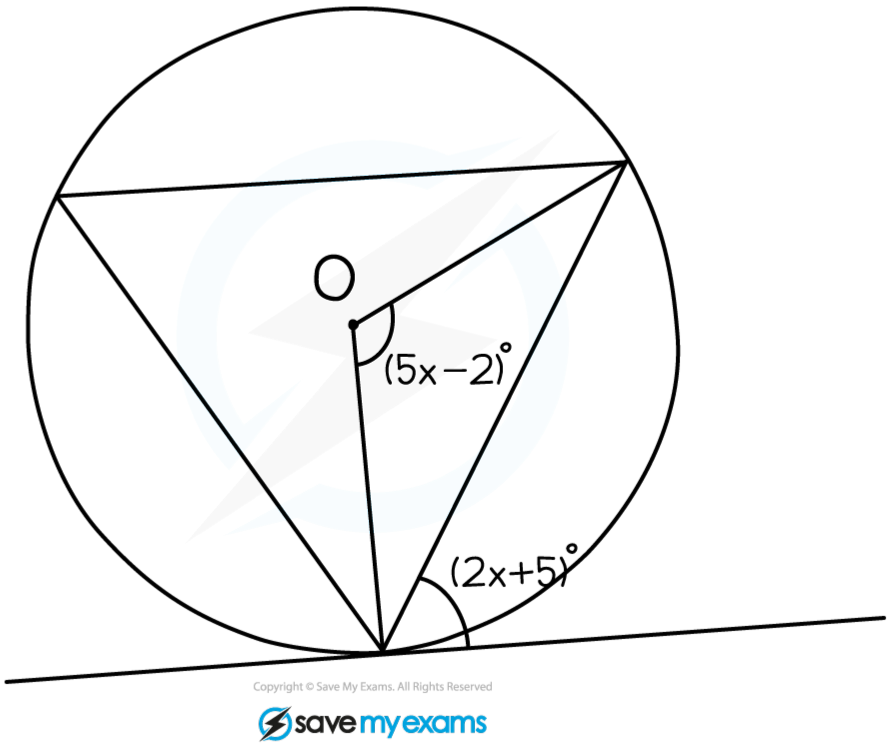
YOUR NOTES
⬇

Exam Tip

Questions often ask for “reasons” – aim to quote an angle fact or circle theorem for **every** angle you find, not just one for the final answer.

Worked Example

1. Find the value of x, stating any angle facts and circle theorems you use.



4. Geometry

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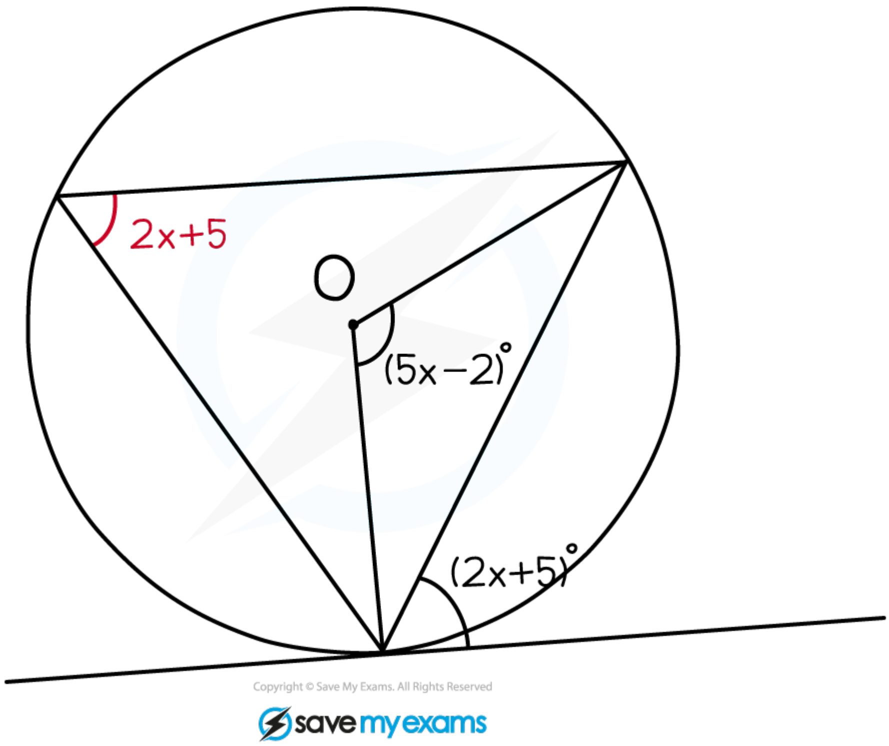


$2x + 5$ The angle "top left" is $2x + 5$ due to the …

Alternate Segment Theorem

$5x - 2 = 2(2x + 5)$ Another circle theorem sets up an equation in x

Angle subtended by an arc at the centre is twice the angle at the circumference

$5x - 2 = 4x + 10$

$\boldsymbol{x = 12}$

4. Geometry

YOUR NOTES

4.6 PARALLEL LINES

4.6.1 ANGLES IN PARALLEL LINES

What are parallel lines?

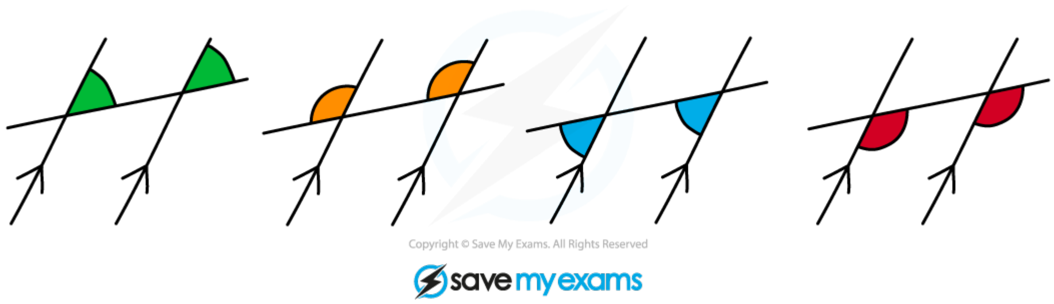
- Parallel lines are lines that are always equidistant (ie the same distance apart) – no matter how far the lines are extended in either direction, they will never meet.

Working with angles in parallel lines

- There are 3 main rules:

1. Corresponding angles are equal

- A line cutting across two parallel lines creates four pairs of equal corresponding angles, as in the diagram below:



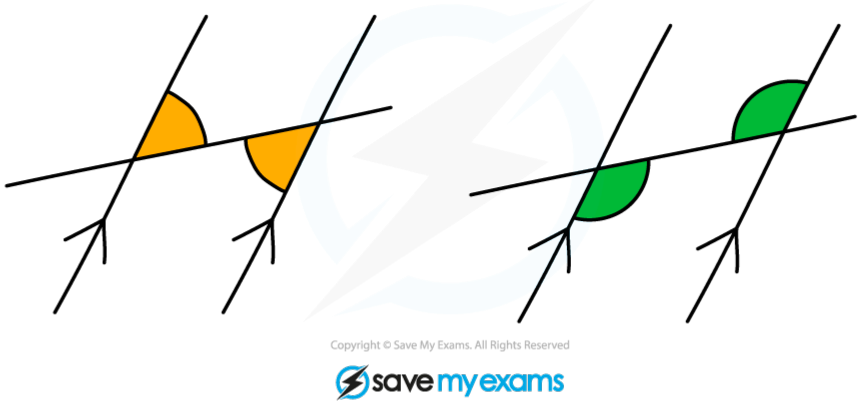
- Note: You may also have heard these referred to as 'F angles' – do not use that term in an exam or you will lose marks!

4. Geometry

YOUR NOTES
⬇

2. Alternate angles are equal

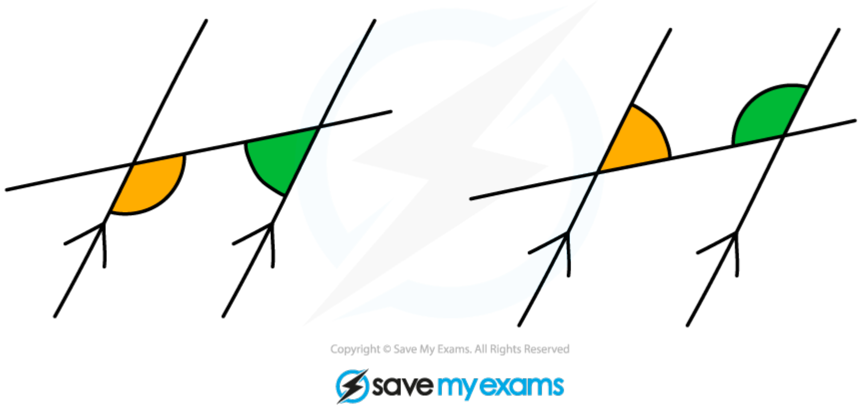
- A line cutting across two parallel lines creates two pairs of equal alternate angles, as in the diagram below:



- Note: You may also have heard these referred to as 'Z angles' – do not use that term on an exam or you will lose marks!

3. Co-interior angles add to 180°

- A line cutting across two parallel lines creates two pairs of co-interior angles
- In the diagram below, the two coloured angles on the left add up to 180°, as do the two coloured angles on the right:



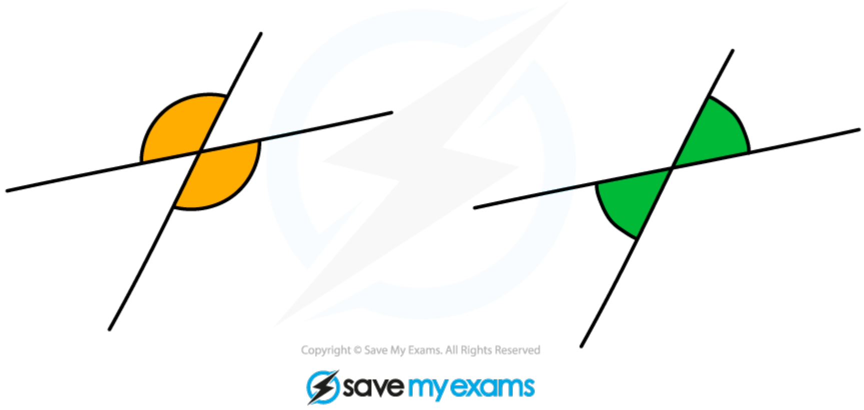
- Note: These are sometimes referred to as allied angles, which is fine. You may also have heard these referred to as 'C angles' – do not use that term on an exam or you will lose marks!

4. Geometry

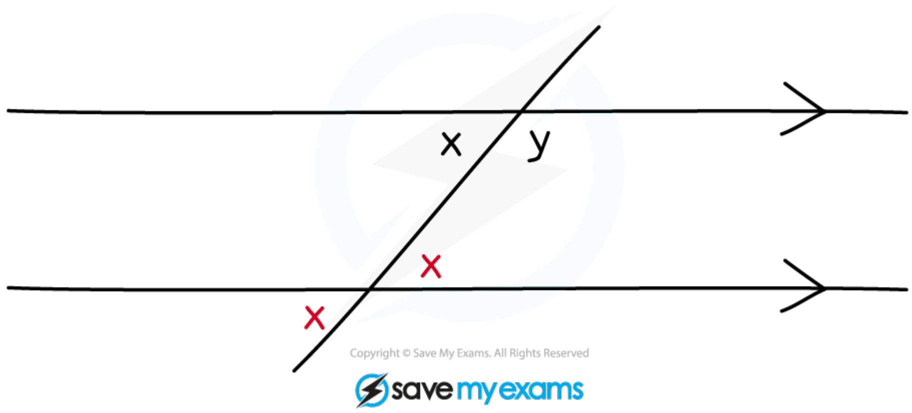
YOUR NOTES ⬇

4. Vertically opposite angles are equal

- Whenever two straight lines cross, they create two pairs of equal vertically opposite angles, as in the diagram below:



- Don't forget this rule when answering parallel line questions! For example, in the following diagram the highlighted angles are equal:



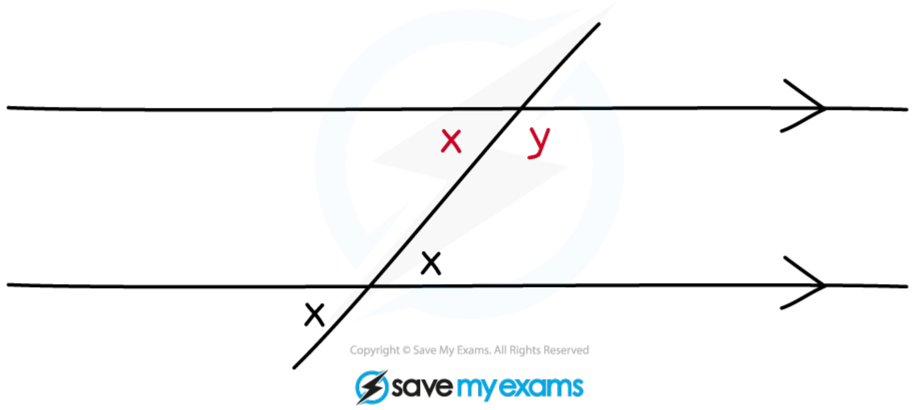
- Note: vertically opposite angles are sometimes simply called opposite angles. Either term will get you the marks

4. Geometry

YOUR NOTES

5. Angles on a line add to 180°

- This rule is also still true with parallel line questions! In the following diagram, for example, the highlighted angles add up to 180°:



Then:

6. Just **angle chase**!

Exam Tip

Do not forget to give reasons for each step of your working in an angles question.
These are often needed to get full marks!

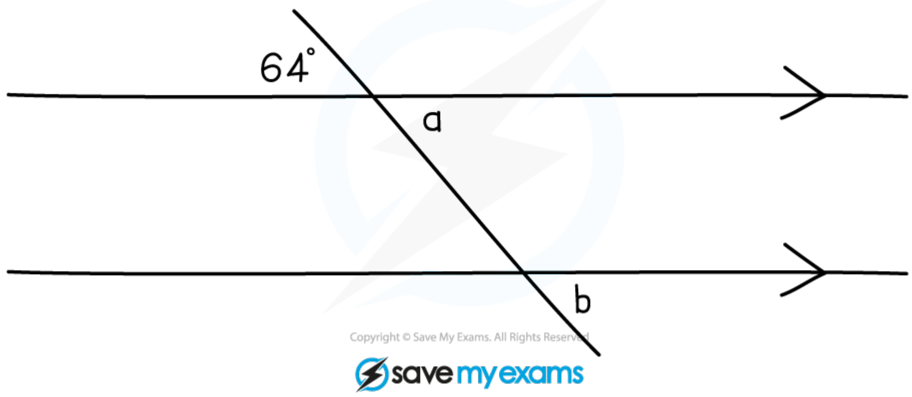
4. Geometry

YOUR NOTES

Worked Example

Find the size of the angles marked with letters.

Give a reason for each step in your working.



We can pretty much write down the answers

– but we MUST give reasons!

$a = 64°$ (*because vertically opposite angles are equal*)

$b = 64°$ (*because corresponding angles are equal*)