5. Mensuration (Perimeters, Areas & Volumes)

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5.1 2D PERIMETERS & AREAS

5.1.1 AREA - FORMULAE

What is area and why do we need to calculate it?

- **Area** is the amount of space taken up by a two-dimensional shape
- **Volume** deals with three-dimensional shapes and space
- Some of the uses of area are a little more obvious than some areas of maths
- Examples include working out the area of a floor if laying or purchasing a new carpet or the amount of land needed if designing a sports field

Area – using formulae

- There are some basic formulae you should know and be comfortable using
- Be aware that some area formulae use distances that aren't necessarily one of the sides of the shape
- Make sure you know what the different letters in each formula are referring to

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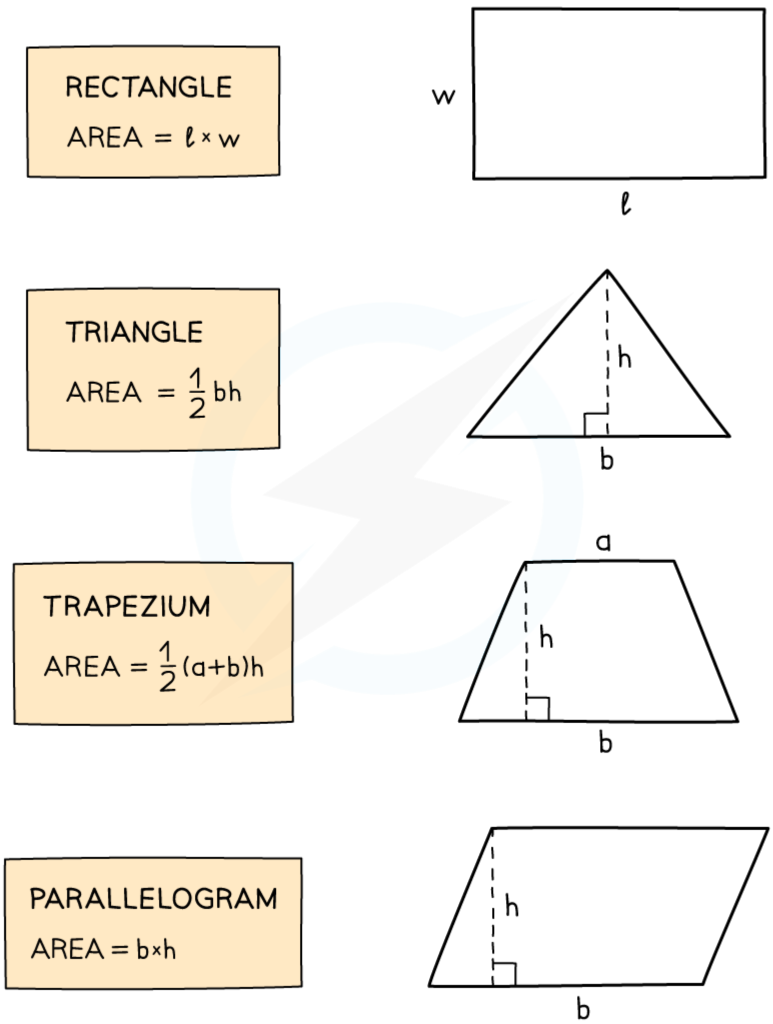
- These formulae are essential – anything more complicated will be given in the exam:



Exam Tip

You may have to do some work to find the lengths first – using Pythagoras Theorem, Trigonometry (SOHCAHTOA) etc. so make sure you look out for that!

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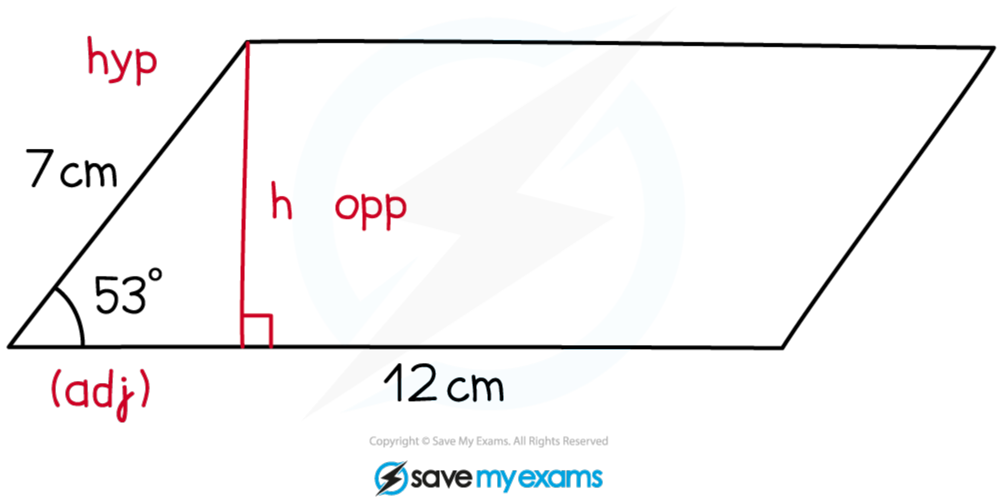
Worked Example

1. *Find the area of this parallelogram.*

   Give your answer to 1 decimal place.



The area of a parallelogram is $A = b \times h$ where h is the perpendicular height (Marked in red on the diagram)

Before we can find the Area we must find the height using SOHCAHTOA on the right angled triangle that has been created

$\sin\theta = \frac{opp}{hyp}$	*Labelling sides show that the sine is needed*
$opp = hyp \times \sin\theta$	*You can substitute in first then rearrange*
$h = 7 \times \sin 53°$	*so $\sin 53° = \frac{h}{7}$ and then $h = 7 \times \sin 53°$*
$\text{Area of parallelogram} = 12 \times 7 \times \sin 53°$	*No need to work h out on its own, put it*
$= 67.08538 \ldots$	*directly into the calculator*
$\text{Area of parallelogram} = 67.1 \text{ cm}^2$	*Final answer should be rounded to 1 decimal place − and include units if not given*

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5.1.2 AREA - ADDING & SUBTRACTING

What do we mean by an awkward shape (a compund shape)?

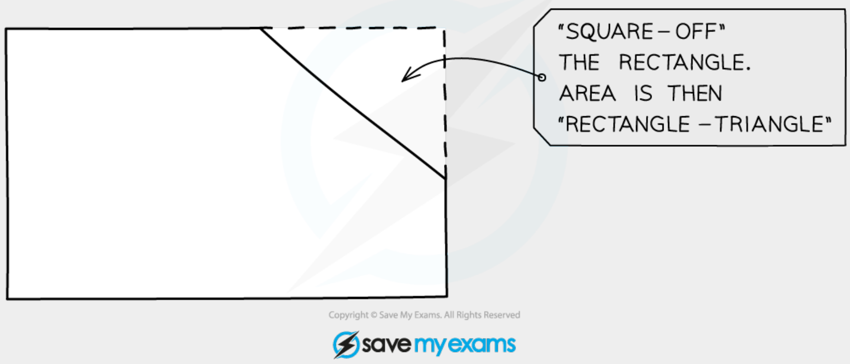
- Sometimes the shape we want to find the area of isn't one of the standard shapes in Area – Formulae
- However, the area may be found by using a combination of standard shapes
- These are often called Compound Shapes and you may see Compound Area mentioned too

Finding the area of an awkward shape (compound area)

- When you are asked to find the area of an "awkward" shape, split the shape into standard shapes first, and then add them together

Exam Tip

Take a moment to think about how to split up the shape into the easiest shapes possible – there will probably be more than one way to do it!
Occasionally it may be easier to add an extra shape to the diagram and subtract the area of the extra shape from the new bigger shape.
For example, for this shape you might complete the rectangle by putting a triangle in the top left corner. Then the area of the whole shape is the rectangle minus the triangle:



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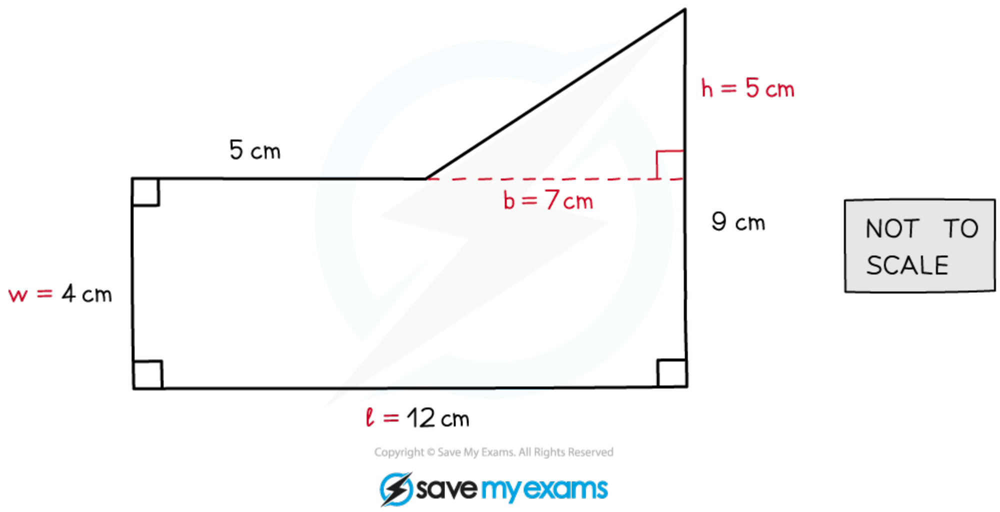
Worked Example

1. *Find the area of this pentagon.*



Firstly do not be put off by the word 'pentagon'
Just because the question is using the correct name for the shape (it's not a regular pentagon by the way) it doesn't necessarily mean there's a formula that will find the area directly.

The easiest way to deal with this shape is to split into a rectangle and a triangle
Do show this on the diagram (in red above) and fill in any sides/distances you will need as you work them out

$b = 12 - 5 = 7\ cm$ — *The base of the triangle*

$h = 9 - 4 = 5\ cm$ — *The height of the triangle*

$Total\ Area\ =\ Area\ of\ Rectangle\ +\ Area\ of\ Triangle$

$Total\ Area\ =\ l \times w\ +\ \frac{1}{2} \times b \times h$

$Total\ Area = 12 \times 4 + \frac{1}{2} \times 7 \times 5$ — *Substitute values for l, w, b and h from your diagram*

$Total\ Area = 65.5\ cm^2$ — *Remember units with your final answer if not given*

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5.1.3 PROBLEM SOLVING WITH AREAS

What is problem solving?

- **Problem solving**, as far as GCSE Mathematics is concerned, usually has two key features:
  - A question is given as a **real-life scenario** (eg. Mary is painting a bedroom in her house ...)
  - There is normally **more than one topic of maths** you will need in order to answer the question (eg. Area and Percentages)

Problem solving with areas

- Area is a commonly used topic of maths in the real world
- Laying a carpet, painting a house, designing a sports field, building a patio or decking all involve area
- Also, doing each of these things has a cost – so a lot of area problems also involve calculations with money

How to solve problems

- The key to getting started on problem-solving questions is to not focus only on what the question asks you to find out but thinking about what you can do with the information given
- Often this will lead you to think of something else you can do and then eventually you may be able to see your way to answering the original question
- These questions could appear on either a non-calculator paper or calculator paper, depending on how awkward they decide to make the numbers involved!

Exam Tip

Even if you never get to a final answer always try to do some maths with the information from the question – you are likely to score some extra marks!

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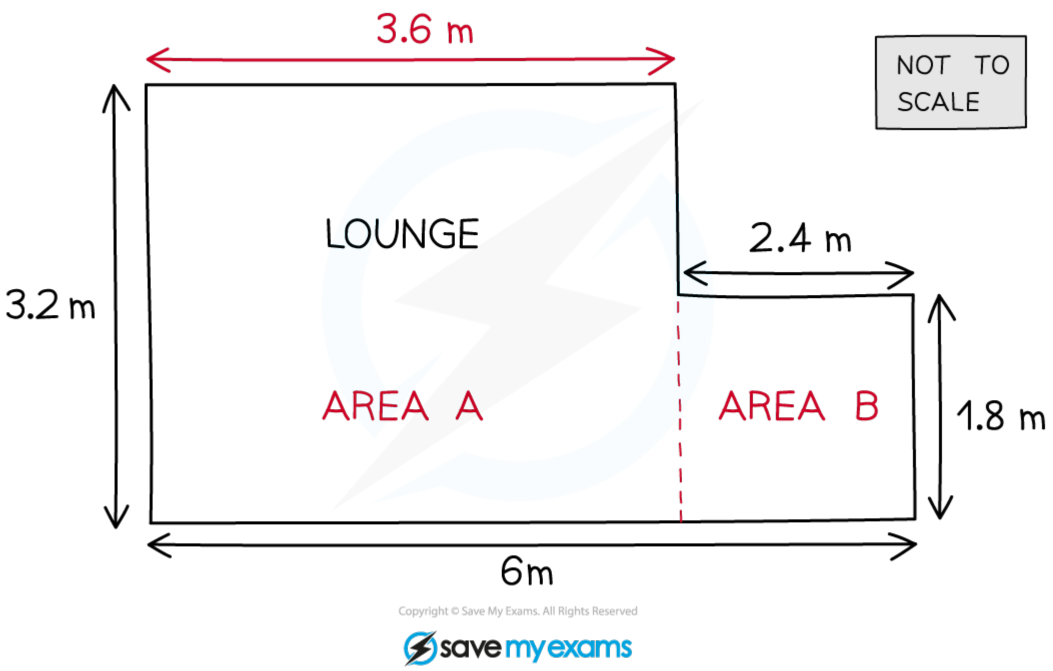
Worked Example

1. John wants a new carpet for the lounge in his house – a sketch of his lounge is given below.



He gets quotes from two local companies, Company A and Company B.

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The amount they charge for laying a carpet is given below:

- Company A: Fixed price of £5.50 per square metre
- Company B: £6 per square metre for the first ten square metres, then £4 per square metre for anything over that.

Which company should John choose in order to keep the cost of laying the carpet to a minimum?

Notice the question doesn't mention the word area but it is implied by the "square metres" mentioned in the costs

$6 - 2.4 = 3.6$

The first thing we can do is work out the "awkward" area of the lounge – add lengths, etc to the diagram (in red above)

$Area\ A = 3.2 \times 3.6 = 11.52$

$Area\ B = 1.8 \times 2.4 = 4.32$

$Total\ area\ of\ Lounge = 11.52 + 4.32 = 15.84$

Company A:

$Total\ Cost = 5.50 \times 15.84 = £87.12$

Now we can work out the cost of each company charges to lay the carpet

Company B:

$First\ ten\ square\ metres:\ 10 \times 6 = £60$

Company B's Total Cost is split into two parts - the first 10 m^2 at £6 …

$Remaining\ area = 15.84 - 10 = 5.84$

$Cost\ of\ remaining\ area = 5.84 \times 4 = £23.36$

$Total\ Cost = 60 + 23.36 = £83.36$

… and the remaining area at £4

Remember to find the total cost too!

John should choose Company B to keep the cost of laying the carpet to a minimum

Make sure you answer the question with a concluding sentence!

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5.2 CIRCLE PROBLEMS

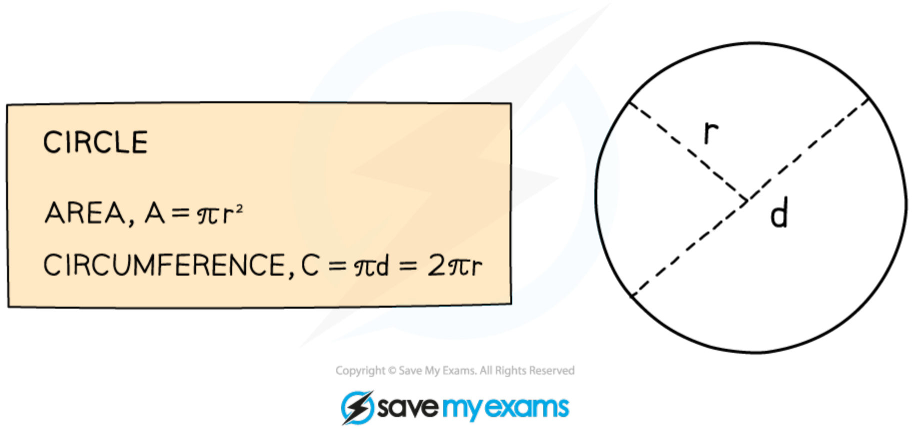
5.2.1 CIRCLES - AREA & CIRCUMFERENCE

Why are circles different to other 2D shapes?

- Circles just have their own language!
- The **circumference** of a circle is its perimeter
- **π** is the number (3.14159 …) that links a circle's **diameter** to its circumference
- Diameter (**d**) is twice the radius (**r**)
  - You may be asked to give an area answer to a certain number of decimal places or significant figures
  - Alternatively you may be asked to give the exact value – or "give your answer in terms of π" – so this topic could crop up on the non-calculator paper!

Working with circles

- You must know the **formulas** for the **area and circumference of a circle**
- There are two versions for the circumference and it is important not to get the radius and diameter confused
- Remember that **d = 2r**
  But you may prefer to remember the formulas by having different letters involved



- Other than that, working with circle formulas is just like working with any other formula:

1. **WRITE DOWN** – what you know (what you want to know)
2. Pick correct **FORMULA**
3. **SUBSTITUTE** and SOLVE

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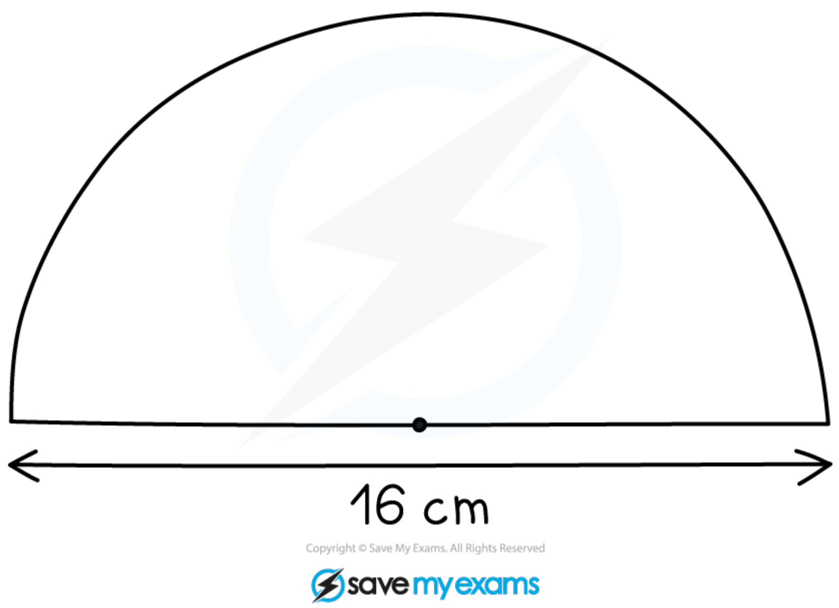
Worked Example

1. *The circumference of a circle is 201 cm.*
Find the radius and the area of the circle.
Give your answers to the nearest whole number.

$C = 201$	1 − *We know the circumference, C*
r	1 − *We are trying to find the radius, r (and area, but can't do that without the radius first)*
$C = 2\pi r$	2 − *Pick the formula for circumference*
$201 = 2\pi r$	3 − *Substitute and solve*
$r = \frac{201}{2\pi}$	
$r = 31.99014\ ...$	*This will be stored under the ANS memory key on your calculator*
$r = 32\ cm$	
$r = 31.99014\ ...$	1 − *Starting the process again for the area, we know the radius*
A	1 − *We are trying to find the area, A*
$A = \pi r^2$	2 − *Pick the correct formula*
$A = \pi \times (31.99\ ...)^2$	3 − *Substitute and solve but use ANS key on your calculator, so you would type* $\pi \times ANS^2$
$A = 3215.009428\ ...$	
$A = 3215\ cm^2$	

2. *Find the perimeter and area of the semicircle shown in the diagram.*
Give your answers in terms of π.



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Okay, so it's a semicircle, don't let that put you off
Work with a whole circle and adapt afterwards

$d = 16$	1 – *We know the diameter, d*
C	1 – *We are trying to find (half the) circumference*
$C = \pi d$	2 – *Pick the formula for circumference*
$C = 16\pi$	3 – *Substitute but do NOT use calculator to go to decimals as "answers in terms of* π*"*
$P = \frac{1}{2}C + d$	2 – *This is not a formula you should try to remember but the question should suggest it*
$P = \frac{1}{2} \times 16\pi + 16$	3 – *Substitute and simplify (rather than solve)*
$P = 8\pi + 16\ cm$	

$d = 16 \rightarrow r = 8$	1 – *We know the diameter but need radius for area*
A	1 – *We are trying to find the area (of whole circle)*
$A = \pi r^2$	2 – *Pick the formula for area*
$A = \pi \times 8^2$	3 – *Substitute but do NOT use calculator to go to decimals as "answers in terms of* π*"*
$A = 64\pi$	*Remember this is the area of a whole circle*
Area of semicircle $= 32\pi\ cm^2$	*so halve it to get the area of the semicircle*

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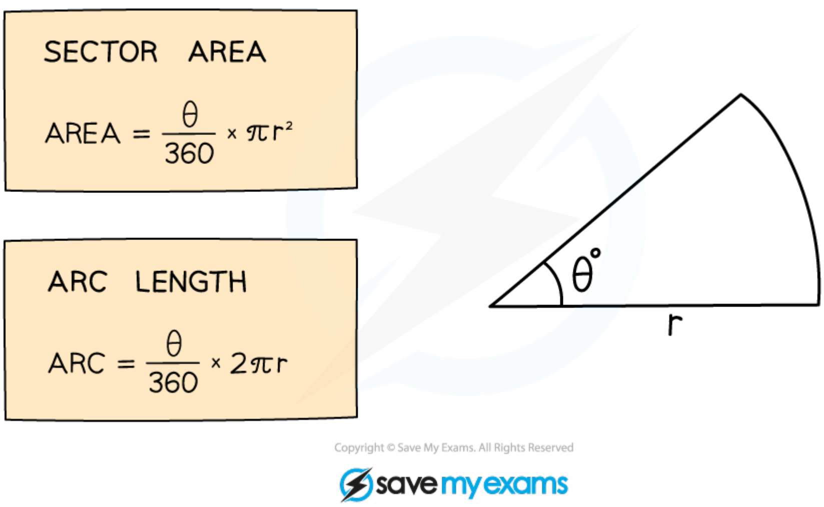
5.2.2 CIRCLES - SECTOR AREAS & ARC LENGTHS

What is a sector? What is an arc?

- A maths book will (correctly) tell you that an **arc** is part of the **circumference** of a circle and a **sector** is part of a circle enclosed by two **radii** (radiuses) and an arc
- It's much easier to think of a sector as the shape of a slice of a circular pizza (or cake, or pie, or …) and an arc as the curvy bit at the end of it (where the crust is)

- If the angle of the slice is **θ** (that's the Greek letter "theta") then the formulas for the **area of a sector** and the **length of an arc** are just fractions of the **area and circumference of a circle**:



- If you are not too good at remembering formulae there is a logic to these two
- You'll need to remember the circumference and area formulas
- After that we are just finding a fraction of the whole circle – "θ out of 360"
- Other than that, working with s**ector and arc formulas** is just like working with any other formula:

1. **WRITE DOWN** – what you know (what you want to know)
2. Pick correct **FORMULA**
3. **SUBSTITUTE** and SOLVE

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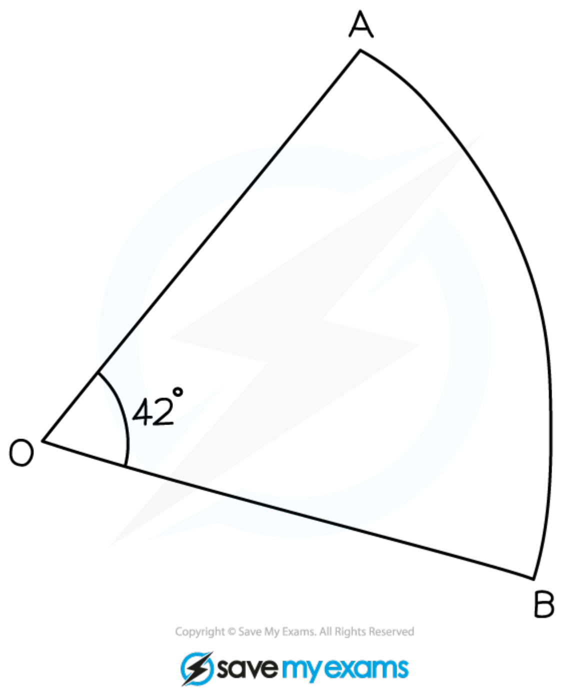
Exam Tip

If you’re under pressure and can’t remember which formula is which, remember that **area is always measured in square units** (cm^2, m^2 etc.) so the formula with r^2 in it is the one for area.

Worked Example

1. *OAB is a sector of a circle with angle* 42°, *as shown.*
The area of the sector AOB is 28 *cm*^{2}.
Find the radius OA and the arc length AB.
Give your answers to the nearest whole number.



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$A = 28,\ \theta = 42$ — 1 – *We know the area (A) and angle (θ) and want to find the radius (r)*

r

$Area = \frac{\theta}{360} \times \pi r^2$ — 2 – *Pick the correct formula*

$28 = \frac{42}{360} \times \pi r^2$ — 3 – *Substitute and solve*

$r^2 = (28 \times \frac{360}{42}) \div \pi$

$r = \sqrt{(28 \times \frac{360}{42}) \div \pi}$

$r = 8.740387\ ...$ — *This will be stored under the ANS memory key on your calculator*

$r = 9\ cm$

$r = 8.74\ ...,\ \theta = 42$ — 1 – *We know the radius (r) and angle (θ) but want to find arc length (which we've decided to call l)*

l

$l = \frac{\theta}{360} \times 2\pi r$ — 2 – *Pick the correct formula*

$l = \frac{42}{360} \times 2 \times \pi \times 8.74\ ...$ — 3 – *Substitute and solve, using ANS on your calculator rather than 8.74 ...*

$l = 6.40703...$

$l = 6\ cm$

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5.3 3D AREAS & VOLUMES

5.3.1 3D SHAPES - SURFACE AREA

What is surface area?

- A face is one of the flat or curved surfaces that make up a 3D shape
- The surface area of a 3D shape is the sum of the areas of all the faces that make up the shape
- Note how we are carrying a 2D idea (area) into 3 dimensions here

Surface area – cuboids, pyramids, and prisms

- In cuboids, polygonal-based pyramids, and polygonal-based prisms (ie. pyramids and prisms whose bases have straight sides), all the faces are flat
- The surface area is found simply by adding up the areas of these flat faces

- When calculating surface area, it can be very helpful to **draw a 2D net for the 3D shape** in question
- For example:
    - The base of a square-based pyramid is 15 cm on a side
    - The triangular faces are identical isosceles triangles, each with a height (from the base to the top of the pyramid) of 23 cm
    - Find the total surface area of the pyramid
    - Draw a net for the shape:

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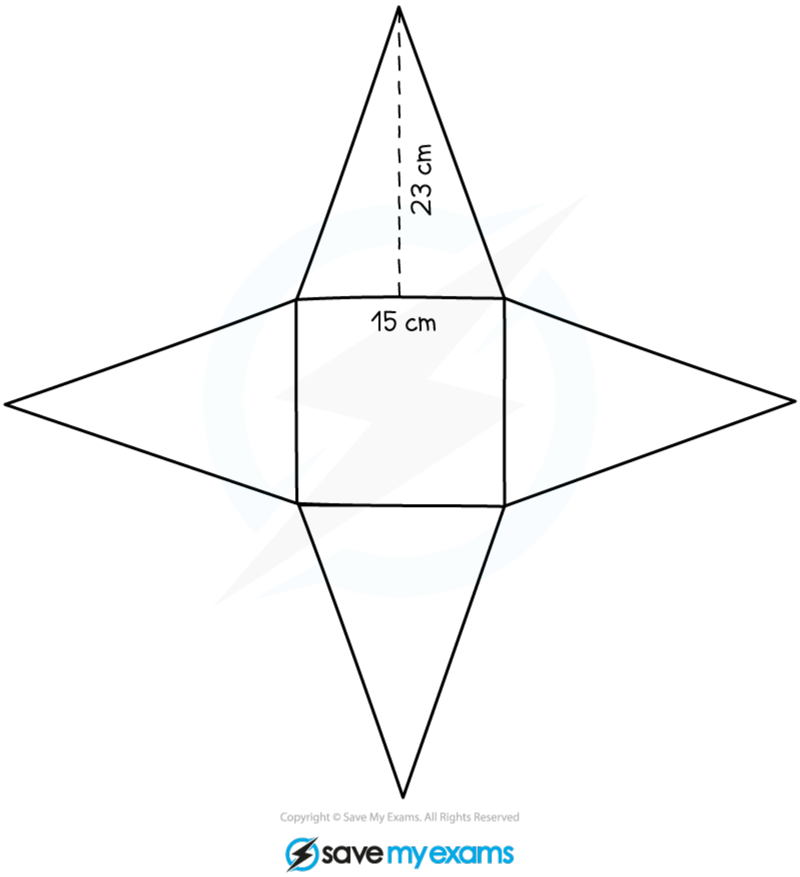

Area of square base $= 15^2 = 225\text{ cm}^2$

Area of one triangular face = ½ base × height $= \frac{1}{2} \times 15 \times 23 = 172.5\text{ cm}^2$

Total surface area $= 225 + 4 \times 172.5 = 915\text{ cm}^2$

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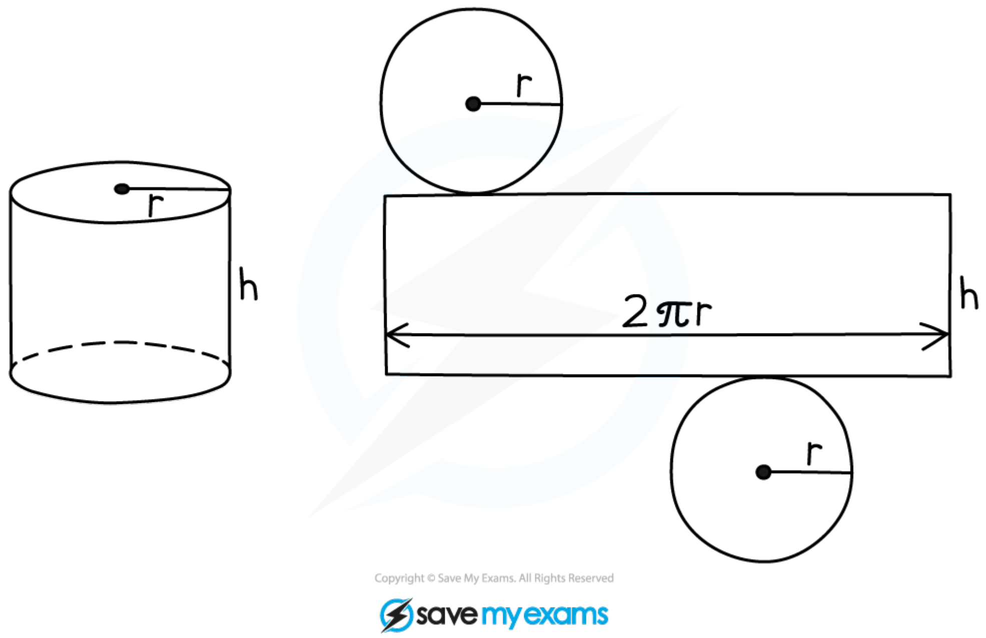
Surface area – cylinders, cones, and spheres

- All three of these shapes have curved faces, so we have to be a little more careful when calculating their surface areas

1. The net of a cylinder consists of two circles and a rectangle:



- The total surface area of a cylinder with base radius r and height h is therefore given by:

  Total surface area of a cylinder= $2\pi r^2 + 2\pi rh$

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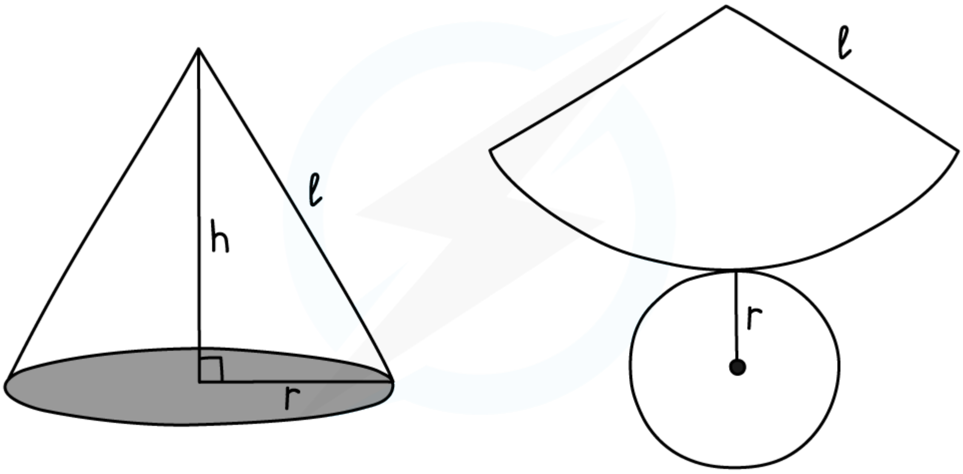
2. The net of a cone consists of the circular base along with the curved surface area:



- The length l in that diagram is known as the **slant height** (while h is the **vertical height** of the cone)
- To find the surface area of a cone with base radius r and slant height l, we use the formulas: Curved surface area of a cone = πrl

  Total surface area of a cone = $\pi r^2 + \pi rl$

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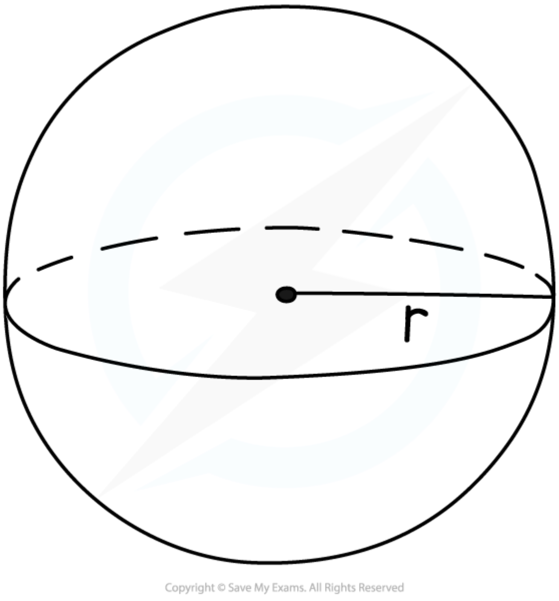
3. To find the surface area of a sphere with radius r, use the formula:

Surface area of a sphere$=4\pi r^2$



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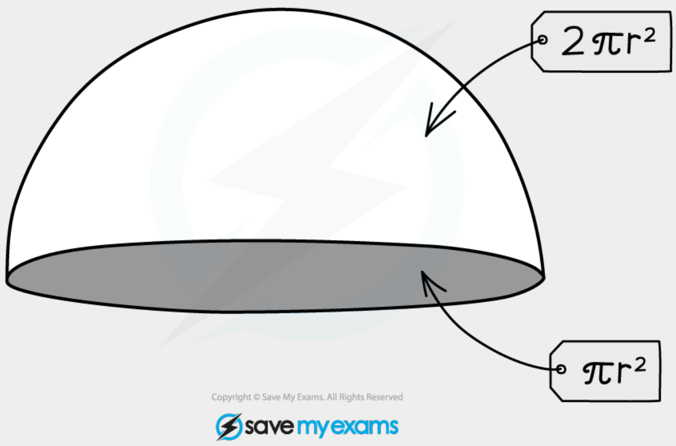
Exam Tip

The formula for the surface area of a sphere or the curved surface area of a cone will be given to you in an exam question if you need it. The rest of the formulas here come from what you should already know about areas of rectangles, triangles, and circles.

Be careful when calculating the surface area of a hemisphere:



The total surface area consists of the curved part (half of a sphere) PLUS the flat circular face – so the total surface area is $3\pi r^2$

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Worked Example

The base radius, r of a cone is the same as the radius of a hemisphere. The total surface area of the cone is equal to the total surface area of the hemisphere.

a) Find the slant height, l, of the cone in terms of r.

b) Given that $r = 19\ cm$, find the curved surface area of the cone. Give your answer accurate to 1 d.p.

a) Hemisphere has curved surface (half a sphere) plus a flat circular face, so:

3. $\quad Surface\ area\ of\ hemisphere = \frac{1}{2}\times 4\pi r^2 + \pi r^2 = 3\pi r^2$

Cone consists of curved surface area plus circular base, so:

2. $\quad Surface\ area\ of\ cone = \pi rl + \pi r^2 = \pi r(l + r)$

But surface areas are equal, so:

$$\pi r\,(l + r) = 3\pi r^2$$
$$l + r = 3r$$
$$l = 2r$$

b) If $r = 19\ cm$, then $l = 2\times 19 = 38\ cm$.

2. $\quad Curved\ surface\ area\ of\ cone = \pi rl$

$= \pi\times 19\times 38 = 722\pi = 2268.2298\ldots = 2268.2\ cm^2\ \ (to\ 1\ d.p.)$

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5.3.2 3D SHAPES - VOLUME

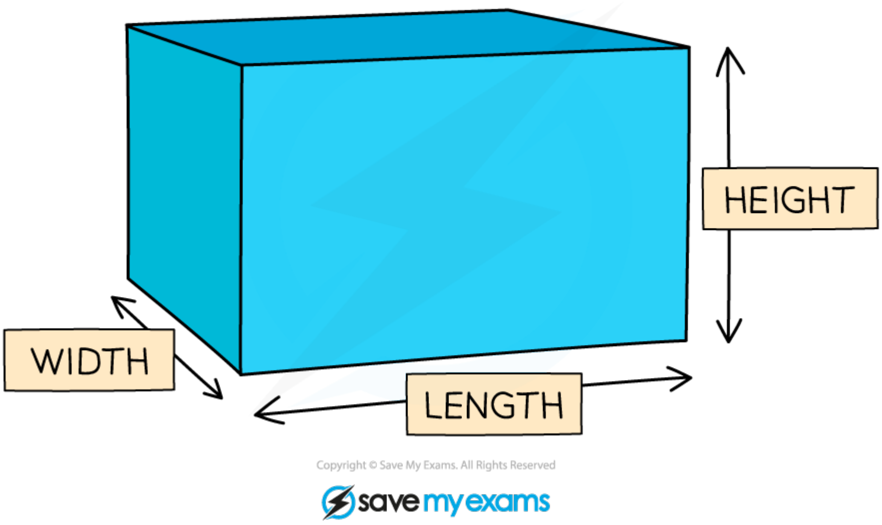
What is volume?

- The **volume** of a 3D shape is a measure of how much 3-D space it takes up
- You need to be able to calculate the volumes of a number of common shapes

Volume – cuboids, prisms, and cylinders

1. To find the volume of a **cuboid** use the formula:

Volume of a cuboid = length × width × height



- You will sometimes see the terms ‘depth’ or ‘breadth’ instead of ‘width’
- Note that a cuboid is in fact a rectangular-based **prism**

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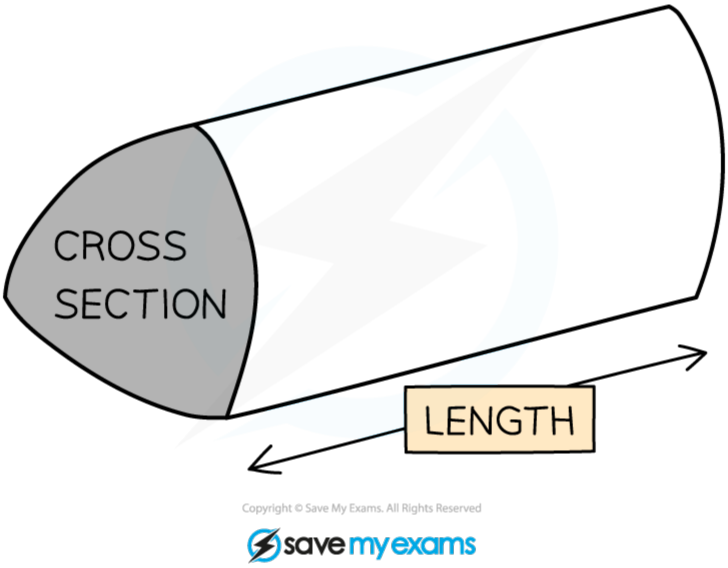
2. To find the volume of a **prism** use the formula:

Volume of a prism = area of cross-section × length



- Note that the cross-section can be any shape, so:
  As long as you know its **area and length**, you can calculate the **volume** of the prism
  Or if you know the **volume and length** of the prism, you can calculate the **cross-section area**

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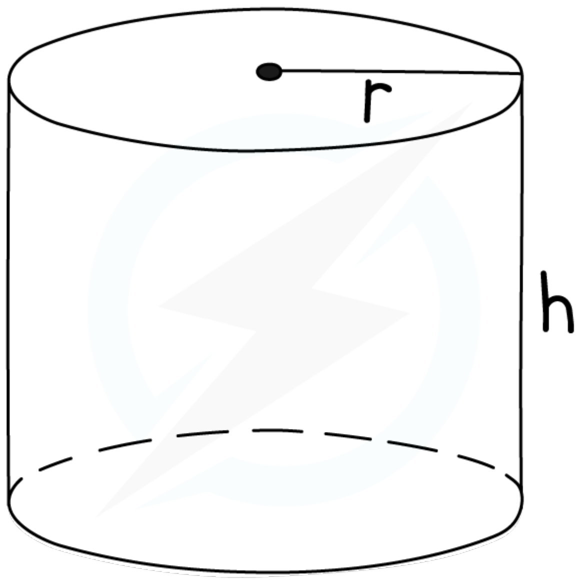
3. To calculate the volume of a **cylinder** with radius and height, use the formula:

Volume of a cylinder = $\pi r^2 h$



- Note that a cylinder is in fact a circular-based **prism**: its cross-section is a circle with area πr^2, and its length is h

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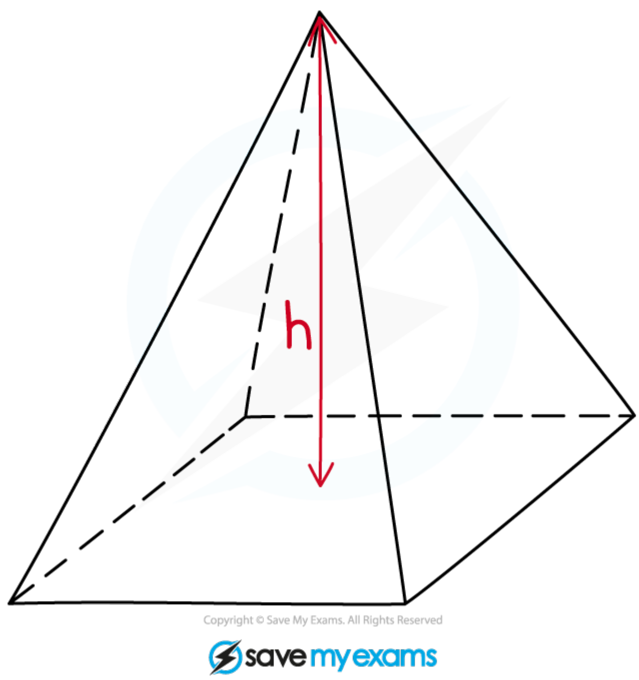
Volume – pyramids, cones, & spheres

4. To calculate the volume of a **pyramid** with height h, use the formula:

Volume of a pyramid = 1/3 × area of base × h



- Note that to use this formula the height must be a line from the top of the pyramid that is **perpendicular** to the base

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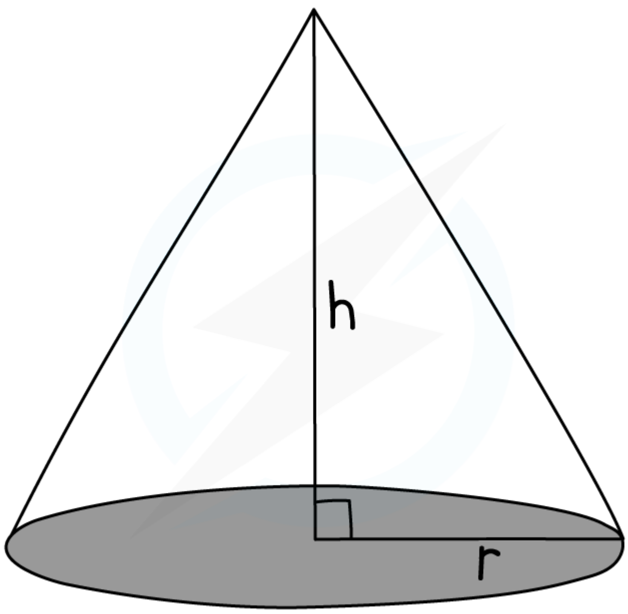
5. To calculate the volume of a **cone** with base radius and height h, use the formula:

Volume of a cone = 1/3 $\pi r^2 h$



- Note that a **cone** is in fact a circular-based **pyramid**: as with a pyramid, to use the cone volume formula the height must be a line from the top of the cone that is **perpendicular** to the base

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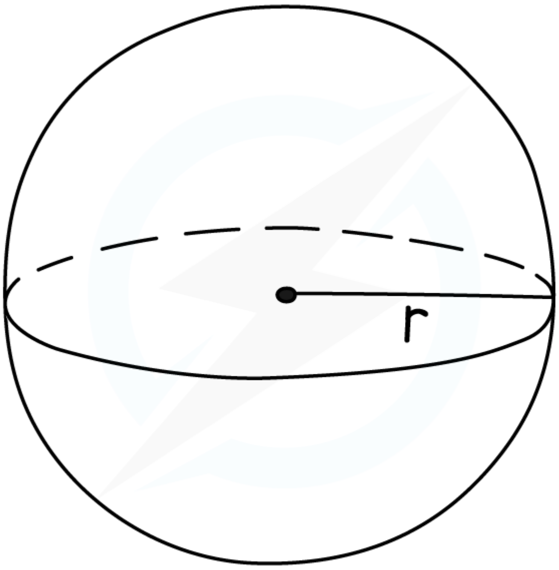
6. To calculate the volume of a **sphere** with radius r, use the formula:

Volume of a sphere = 4/3 πr^3



Exam Tip

The formula for volume of a sphere or volume of a cone will be given to you in an exam question if you need it. You need to memorise the other volume formulas!

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Worked Example

A sculptor has a block of marble in the shape of a cuboid, with a square base 35 cm on a side, and a height of 2 m.

He carves the block into a cone, with the same height as the original block, and with a base diameter equal to the side length of the original square base.

What is the volume of the marble he *removes* from the block while carving the cone. Give your answer in m^3 , rounded to 3 significant figures.

First find the volume of the original cuboid, remembering to convert cm into m:

1. $Volume\ of\ cuboid = length \times width \times height$

$$= 2 \times 0.35 \times 0.35$$
$$= 0.245\ m^3$$

Now find the volume of the cone – but remember that we need the *radius* of the base:

$$radius\ of\ cone = 0.35 \div 2 = 0.175\ m$$

5. $Volume\ of\ cone = \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3} \times \pi \times 0.175^2 \times 2 = 0.0641409\ cm^3$$

Now we can find the volume of marble *removed* by the sculptor:

$$Volume\ removed = volume\ of\ cuboid - volume\ of\ cone = 0.245 - 0.0641409 = 0.1808591 = 0.181\ m^3$$
$$(to\ 3\ sig\ figs.)$$