Strain - Displacement Relation from Cylindrical to Spherical Coordinate System

The relation between cylindrical and spherical coordinate is:

$\gamma = \rho \sin\phi$, $z = \rho\cos\phi$, $\theta = \theta$

Where $\rho = \sqrt{\gamma^2+z^2}$, $\theta = \tan^{-1}(y/x)$, $\phi = \arccos\left(\frac{z}{\rho}\right)$

The partial derivatives for the above equations are

$$\frac{\partial}{\partial r} = \frac{\partial\rho}{\partial r}\cdot\frac{\partial}{\partial\rho} + \frac{\partial\phi}{\partial r}\cdot\frac{\partial}{\partial\phi}$$

$$= \sin\phi\frac{\partial}{\partial\rho} + \frac{\gamma^2}{\sqrt{\gamma^2-z^2}\cdot\rho^{3/2}}\cdot\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial z} = \frac{\partial\rho}{\partial z}\cdot\frac{\partial}{\partial\rho} + \frac{\partial\phi}{\partial z}\cdot\frac{\partial}{\partial\phi}$$

$$= \cos\phi\frac{\partial}{\partial\rho} + \frac{\gamma z}{\sqrt{\gamma^2-z^2}\cdot\rho^{3/2}}\cdot\frac{\partial}{\partial\phi}$$

Now

$$u_r = u_\rho\sin\phi + u_\phi\frac{\gamma^2}{\sqrt{\gamma^2-z^2}\cdot\rho^{3/2}}, \quad u_z = u_\rho\cos\phi + u_\phi\frac{\gamma z}{\sqrt{\gamma^2-z^2}\cdot\rho^{3/2}}; \quad u_\theta = u_\theta$$

Calculating $e_\rho = \frac{\partial u_r}{\partial r}$

$$\hat{e}_\rho = \sin\phi\left[\frac{\partial}{\partial\rho}\left(u_\rho\sin\phi + u_\phi\frac{\gamma^2}{\sqrt{r^2-z^2}\cdot\rho^{3/2}}\right)\right] + \frac{\gamma^2}{\sqrt{r^2-z^2}\cdot\rho^{3/2}}\frac{\partial}{\partial\phi}\left[u_\rho\sin\phi + u_\phi\frac{\gamma^2}{\sqrt{r^2-z^2}\cdot\rho^{3/2}}\right]$$

$$= \left[\frac{\partial u_\rho}{\partial\rho}\sin\phi + \frac{\partial u_\phi}{\partial\rho}\cdot\frac{\gamma^2\sin\phi}{\rho^{3/2}\sqrt{r^2-z^2}} + \frac{u_\phi\gamma^2}{\sqrt{r^2-z^2}}\cdot\frac{\sin\phi}{\rho^{-5/2}} + \frac{\partial u_\rho}{\partial\phi}\cdot\frac{\sin\phi\,\gamma^2}{\sqrt{r^2-z^2}\cdot\rho^{3/2}} + \frac{\gamma^2 u_\rho\cos\phi}{\sqrt{r^2-z^2}}\right.$$

$$\left. + \frac{\partial u_\phi}{\partial\phi}\cdot\frac{\gamma^4}{\rho^3\cdot(r^3-z^3)}\right]$$

$$\hat{e}_\rho = \frac{\partial u_\rho}{\partial\rho}\sin\phi + \left(\frac{\partial u_\phi}{\partial\rho}\cdot\frac{1}{\rho^{3/2}} + \frac{u_\phi}{\rho^{-5/2}} + \frac{\partial u_\rho}{\partial\phi}\cdot\frac{1}{\rho^{3/2}}\right)\frac{\gamma^3\cdot\sin\phi}{\sqrt{\gamma^2-z^2}}$$

$$+ \left(\left(u_\rho\cos\phi + \frac{\partial u_\phi}{\partial\phi}\cdot\frac{1}{\rho^3}\right)\cdot\frac{\gamma^2}{\sqrt{\gamma^2-z^2}}\right)\frac{\gamma^2}{\sqrt{r^2-z^2}}$$

$$e_\rho = \frac{\partial u_z}{\partial z}$$

$$\hat{e}_\rho = \cos\phi \frac{\partial}{\partial \rho}\left[u_\rho \cos\phi + u_\phi \cdot \frac{rz}{\rho^{3/2}\cdot\sqrt{r^2-z^2}}\right] + \frac{rz}{\sqrt{r^2-z^2}\cdot\rho^{3/2}}\frac{\partial}{\partial\phi}\left[u_\rho\cos\phi + u_\phi \frac{rz}{\sqrt{r^2-z^2}\cdot\rho^{3/2}}\right]$$

$$= \frac{\partial u_\rho}{\partial \rho}\cos^2\phi + \frac{\partial u_\rho}{\partial \rho}\cdot\frac{rz\cos\phi}{\rho^{3/2}\cdot\sqrt{r^2-z^2}} + \frac{u_\phi\, rz}{\sqrt{r^2-z^2}}\cdot\frac{\cos\phi}{\rho^{-5/2}} + \frac{\partial u_\rho}{\partial\phi}\frac{\cos\phi\, rz}{\rho^{3/2}\cdot\sqrt{r^2-z^2}} - \frac{\sin\phi \cdot rz\, u_\rho}{\sqrt{r^2-z^2}\cdot\rho^{5/2}}$$

$$+ \frac{\partial u_\phi}{\partial\phi}\cdot\frac{r^2z^2}{(r^2-z^2)\rho^3}$$

$$\partial\phi = \frac{\partial u_\rho}{\partial\rho}\cos^2\phi + \left(\frac{\partial u_\rho}{\partial\rho}\cdot\frac{1}{\rho^{3/2}} + \frac{u_\phi}{\rho^{5/2}} + \frac{\partial u_\rho}{\partial\phi}\cdot\frac{1}{\rho^{3/2}}\right)\frac{\cos\phi\cdot rz}{r^2-z^2}$$

$$+\left(\frac{\partial u_\phi}{\partial\phi}\cdot\frac{rz}{\sqrt{r^2-z^2}}\rho^{3/2} - \frac{u_\rho\sin\phi}{\rho^{3/2}}\right)\frac{rz}{\sqrt{r^2-z^2}}$$

Therefore the strain-displacement relation becomes

$$e_\rho = \frac{\partial u_r}{\partial r}, \quad e_\phi = \frac{1}{\rho}\left(u_r + \frac{\partial u_z}{\partial\phi}\right)$$

$$e_\theta = \frac{1}{\rho\sin\phi}\left(\frac{\partial u_\theta}{\partial\theta} + \sin\phi\, u_r + \cos\phi\, u_z\right)$$

$$e_{\rho\phi} = \frac{1}{2}\left(\frac{1}{\rho}\frac{\partial u_r}{\partial\phi} + \frac{\partial u_z}{\partial\rho} - \frac{u_z}{\rho}\right)$$

$$e_{\phi\theta} = \frac{1}{2\rho}\left(\frac{1}{\sin\theta}\cdot\frac{\partial u_z}{\partial\theta} + \frac{\partial u_\theta}{\partial\theta} - \cot\phi\, u_\theta\right)$$

$$e_{\theta\rho} = \frac{1}{2}\left(\frac{1}{\rho\sin\phi}\cdot\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial\rho} - \frac{u_\theta}{\rho}\right)$$