Exercises:

1-1

$$a_{ij} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}, \quad b_i = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

Calculate : a_{ii}, $a_{ij}a_{ij}$, $a_{ij}a_{jk}$, $a_{ij}b_j$, $a_{ij}b_i b_j$, $b_i b_j$, $b_i b_i$

i) $a_{ii} = a_{11} + a_{22} + a_{33} = 1 + 4 + 1 = 6 \Rightarrow$ Scalar

ii) $$a_{ij}a_{ij} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33}$$

$$= 1 + 1 + 1 + 0 + 16 + 4 + 0 + 1 + 1 = 25$$

$a_{ij}a_{ij} = 25 \Rightarrow$ scalar

iii) $$a_{ij}a_{jk} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= a_{i1}a_{1k} + a_{i2}a_{2k} + a_{i3}a_{3k}$$

$$= \begin{bmatrix} 1 & 16 & 4 \\ 0 & 18 & 10 \\ 0 & 5 & 3 \end{bmatrix} \text{ (matrix)}$$

iv $$a_{ij}b_j = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3$$

$$= \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} \text{ (vector)}$$

v) $$a_{ij}b_i b_j = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = a_{i1}b_i b_1 + a_{i2}b_i b_2 + a_{i3}b_i b_3$$

$$= a_{11}b_1b_1 + a_{12}b_1b_2 + a_{13}b_1b_3 + a_{21}b_2b_1 + a_{22}b_2b_2 + a_{23}b_2b_3 + a_{31}b_3b_1 + a_{32}b_3b_2 + a_{33}b_3b_3$$

$= 1+0+2+0+0+0+0+0+4 = 7$ (scalar)

vi) $b_i b_j = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = b_i b_1 + b_i b_2 + b_i b_3$

$= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix}$ (matric).

vii) $b_i b_i = b_1 b_1 + b_2 b_2 + b_3 b_3$

$= 1 + 0 + 4 = 5 =$ (scalar)

1-2 Express a_{ij} in exercise 1-1 in terms of the sum of symmetric and antisymmetric metrices.

$a_{(ij)}$ and $a_{[ij]}$

a) $a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$

$= \frac{1}{2}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

$a_{(ij)}$ and $a_{[ij]}$ satisfy the appropriate condition.

b)

$a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$

$= \frac{1}{2}\begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -3 \\ 0 & 3 & 6 \end{bmatrix}$

$a_{(ij)}$ and $a_{[ji]}$ satisfy the appropriate conditions.

1-3 If a_{ij} is Symmetric and b_{ij} is antisymmetric Prove in genral that $a_{ij}b_{ij}$ is Zero.

Solution

$$a_{ij}b_{ij} = -a_{ji}b_{ji} = -a_{ij}b_{ij}$$

$$= 2a_{ij}b_{ij} = 0 \Rightarrow a_{ij}b_{ij} = 0$$

a) $a_{(ij)}a_{[ij]} = \frac{1}{4}tr\left\{\begin{bmatrix} 2 & 1 & 1 \\ 1 & 8 & 3 \\ 1 & 3 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 4 \\ -1 & -1 & 0 \end{bmatrix}^T\right\} = 0$

b) $a_{(ij)}a_{[ij]} = \frac{1}{4}tr\left[\begin{bmatrix} 2 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 4 \end{bmatrix}\begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -3 \\ 0 & 3 & 0 \end{bmatrix}^T\right] = 0$

1-4 verify

$$\delta_{ij}a_j = a_i$$
$$\delta_{ij}a_{jk} = a_{ik}$$

Solution

$$\delta_{ij}a_j = \delta_{i1}a_1 + \delta_{i2}a_2 + \delta_{i3}a_3$$

$$= \begin{bmatrix} \delta_{11}a_1 + \delta_{12}a_2 + \delta_{13}a_3 \\ \delta_{21}a_1 + \delta_{22}a_2 + \delta_{23}a_3 \\ \delta_{31}a_1 + \delta_{32}a_2 + \delta_{33}a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = a_i$$

$$\delta_{ij}a_{jk} = \delta_{i1}a_{1k} + \delta_{i2}a_{2k} + \delta_{i3}a_{3k}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{ij}$$

1-5 Formally expand.

$$det(a_{ij}) = \varepsilon_{ijk}a_{1i}a_{2j}a_{3k}$$
$$= \varepsilon_{123}a_{11}a_{22}a_{33} + \varepsilon_{231}a_{12}a_{23}a_{31}$$
$$+ \varepsilon_{312}a_{13}a_{21}a_{32} + \varepsilon_{321}a_{13}a_{22}a_{31}$$

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$\varepsilon a_2 \qquad \varepsilon_{213}\, a_{12} a_{21} a_{33}$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$
$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31})$$
$$+ a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

1-6 Determine the components of b_i and a_{ij} in new coordinates system through a rotation of $45°$ about x_1- axies.

Solution:

$45°$ rotation about x_1-

$$Q_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$b_i' = Q_{ij} b_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$b_i' = \begin{bmatrix} 1 \\ \sqrt{2} \\ 2 \end{bmatrix}$$

$$a_{ij}' = Q_{ip} Q_{iq} a_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & \sqrt{2} & 0 \\ 0 & 4 & -1 \\ 0 & -2 & 1 \end{bmatrix}$$

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1-7 Consider two dimentional coordinates system.

$$Q_{ij} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$b_i = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$; $a_{ij} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ calculates their components in polar coordinates.

Sol:

$$Q_{ij} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$b_i' = Q_{ij} b_j = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$= \begin{bmatrix} b_1\cos\theta + b_2\sin\theta \\ -b_1\sin\theta + b_2\cos\theta \end{bmatrix}$$

$$a_{ij}' = Q_{ip} Q_{jq} a_{pq} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^T$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}\cos\theta + a_{21}\sin\theta & a_{12}\cos\theta + a_{22}\sin\theta \\ a_{11}\sin\theta + a_{21}\cos\theta & -a_{12}\sin\theta + a_{22}\cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}\cos^2\theta + a_{21}\sin^2\theta & -a_{12}\sin\theta\cos\theta + a_{22}\sin\theta\cos\theta \\ -a_{11}\sin\theta\cos\theta + a_{21}\sin\theta\cos\theta & a_{12}\sin^2\theta + a_{22}\cos^2\theta \end{bmatrix}$$

$$\begin{matrix} a_{11}' \\ a_{12}' \end{matrix} = \begin{bmatrix} a_{11}\cos^2\theta + (a_{12}+a_{21})\sin\theta\cos\theta + a_{22}\sin^2\theta \\ a_{12}\cos^2\theta - (a_{11}-a_{22})\sin\theta\cos\theta - a_{21}\sin^2\theta \end{bmatrix}$$

$$a_{21}' = a_{21}\cos^2\theta - (a_{11}-a_{22})\sin\theta\cos\theta - a_{12}\sin^2\theta$$

$$a_{22}' = a_{11}\sin^2\theta - (a_{12}+a_{21})\sin\theta\cos\theta + a_{22}\cos^2\theta.$$

1-8 2nd order tensor $a\delta_{ij}$, retains its form under any transformation. This form is then an isotropic tensor 2nd order.

Solution

tensor $a\delta_{ij}$

$$a'\delta'_{ij} = Q_{ip}Q_{jq}\, a\delta_{pq} = a\, Q_{ip}Q_{iq} = a\delta_{ij}.$$

1-9 4th order tensor

$$a\,\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$$

It remains same under any transformation.

Sol:

$$a\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$$

$$a'\delta'_{ij}\delta'_{kl} = Q_{im}Q_{jn}Q_{kp}Q_{lq}\, a\delta_{mn}\delta_{pq}$$

$$\beta'\delta'_{ik}\delta'_{jl} = Q_{im}Q_{kp}Q_{jn}Q_{lq}\,\beta\delta_{mp}\delta_{nq}$$

$$\gamma'\delta'_{il}\delta'_{jk} = Q_{im}Q_{lq}Q_{jn}Q_{kp}\,\gamma\delta_{mq}\delta_{np}$$

$$a'\delta'_{ij}\delta'_{kl} + \beta'\delta'_{ik}\delta'_{jl} + \gamma'\delta'_{il}\delta'_{jk}$$

$$\Rightarrow \cancel{Q_{im}Q_{jn}Q_{kp}Q_{lq}\, a\delta_{mn}\delta_{pq} + Q_{im}Q_{kp}Q_{jn}Q_{lq}}$$

$$Q_{im}Q_{jn}Q_{kp}Q_{lq}\left(a\delta_{mn}\delta_{pq} + \beta\delta_{mp}\delta_{nq} + \gamma\delta_{mq}\delta_{np}\right)$$

$$= a\,Q_{im}Q_{jm}Q_{kp}Q_{lp} + \beta Q_{im}Q_{jn}Q_{km}Q_{ln} + \gamma Q_{im}Q_{jn}Q_{kn}Q_{lm}$$

$$= a\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$$

1-10 If $\beta = \gamma$

$$C_{ijkl} = C_{klij}.$$

Solution

$$C_{ijkl} = a\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$$

$$\beta = \gamma$$

$$= a\delta_{ij}\delta_{kl} + \beta(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

$$= a\delta_{kl}\delta_{ij} + \beta(\delta_{ki}\delta_{lj} + \delta_{li}\delta_{kj}) = C_{klij}$$

1-11 Fundamental variants can be expressed in terms of the principle values.

Solution.

$$a = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$I_a = a_{ii} = \lambda_1 + \lambda_2 + \lambda_3$$

$$II_a = \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix} + \begin{vmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_3 \end{vmatrix} = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3$$

$$III_a = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{vmatrix} = \lambda_1\lambda_2\lambda_3$$

1-12 Determine the invariants and principle values and directions.

$$a_{ij} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I_a = -1 - \cancel{1} + \cancel{1} = -1$$

$$II_a = \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = -2$$

$$III_a = \begin{vmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

Charactrenstic eq/ rs

$$(a_{ij} - I\lambda) = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} -1-\lambda & 1 & 0 \\ 1 & -1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} \Rightarrow \det(a_{ij} - I\lambda) = \begin{vmatrix} -1-\lambda & 1 & 0 \\ 1 & -1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix}$$

$$= -\lambda^3 - \lambda^2 + 2\lambda = 0$$

$$\lambda(\lambda^2 + \lambda - 2) = 0 \Rightarrow \lambda(\lambda+2)(\lambda-1) = 0$$

Roots $\lambda = 0$; $\lambda = -2$; $\lambda = -1$.

$\lambda = -2$.

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \qquad \begin{matrix} n_1 + n_2 = 0 \\ n_3 = 0 \end{matrix}$$

$$\Rightarrow n_1 = -n_2 = \pm\sqrt{2}/2 \qquad n = \pm\sqrt{2}/2(-1, 1, 0)$$

$\lambda = 0$

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \qquad \begin{matrix} -n_1 + n_2 = 0 \\ n_3 = 0 \end{matrix}$$

$$n_1^2 + n_2^2 + n_3^2 = 1$$

$$n_1 = -n_2 = \pm\sqrt{2}/2$$

$$n = \pm\sqrt{2}/2(1, 1, 0)$$

$\lambda = -1$

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \qquad \begin{matrix} -2n_1 + n_2 = 0 \\ n_1 - 2n_2 = 0 \end{matrix}$$

$$n_1^2 + n_2^2 + n_3^2 = 1.$$

$$n_1 = n_2 = 0 \quad ; n_3^2 = 1 \quad = \pm(0, 0, 1)$$

$$Q_{ij} = \sqrt{2}/2 \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}$$

$$a'_{ij} = Q_{ip} Q_{jq} a_{pq}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2/\sqrt{2} \end{bmatrix}^T$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1-14 Calculate $\nabla \cdot u$, $\nabla \times u$, $\nabla^2 u$, ∇u, $tr(\nabla u)$

(a) $u = x_1 e_1 + x_2 e_2 x_1 + 2x_1 x_2 x_3 e_3$

$$\nabla : \begin{bmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \end{bmatrix}$$

$$\nabla \cdot u = U_{1,1} + U_{2,2} + U_{3,3}$$

$$= \left(e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + e_3 \frac{\partial}{\partial x_3}\right)(x_1 e_1 + x_2 x_1 e_2 + 2x_1 x_2 x_3 e_3)$$

$$= 1 + x_2 + 2x_2 x_3$$

$$\nabla \times U = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_1 & x_1 x_2 & 2x_1 x_2 x_3 \end{vmatrix} = 2x_1 x_3 e_1 + 2x_2 x_3 e_2 + x_2 e_3$$

$$\nabla^2 U = \left(e_1 \frac{\partial^2}{\partial x_1^2} + e_2 \frac{\partial^2}{\partial^2 x_2} + e_3 \frac{\partial^2}{\partial^2 x_3}\right)(x_1 e_1 + x_1 x_2 e_2 + 2x_1 x_2 x_3 e_3)$$

$$= 0 + 0 + 0 = 0$$

$$\nabla U = \begin{bmatrix} 1 & 0 & 0 \\ x_2 & x_1 & 0 \\ 2x_2 x_3 & 2x_1 x_3 & 2x_1 x_2 \end{bmatrix} - tr(\nabla U) = 1 + x_1 + 2x_1 x_2$$

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15 dual vector a_i of an antisymmetric 2nd order tensor a_{ij} is $a_i = -1/2\,\varepsilon_{ijk} a_{jk}$

$$a_{jk} = -\varepsilon_{ijk} a_i$$

Solution

$$a_i = -\frac{1}{2}\varepsilon_{ijk} a_{jk}$$

$$\varepsilon_{imn} a_i = -\frac{1}{2}\varepsilon_{ijk}\varepsilon_{imn} a_{jk}$$

$$= -\frac{1}{2}\begin{vmatrix} \delta_{ii} & \delta_{im} & \delta_{in} \\ \delta_{ji} & \delta_{jm} & \delta_{jn} \\ \delta_{ki} & \delta_{km} & \delta_{kn} \end{vmatrix} a_{jk}$$

$$= -\frac{1}{2}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) a_{jk}$$

$$= -\frac{1}{2}(a_{mn} - a_{nm}) = -\frac{1}{2}(a_{mn} + a_{mn})$$

$$= -\frac{2}{2} a_{mn} = -a_{mn}$$

$$\therefore a_{jk} = -\varepsilon_{ijk} a_i$$

1-16 Using index notation

a) I $\nabla(\phi\psi) = (\phi\psi)_{,k} = \phi_{,k}\psi = \nabla\phi\psi + \phi\nabla\psi$

$$\nabla^2(\phi\psi) = (\phi\psi)_{,kk} = (\phi\psi_{,k} + \phi_{,k}\psi)_{,k}$$

$$= \phi\psi_{,kk} + \phi_{,k}\psi_{,k} + \phi_{,k}\psi_{,k} + \phi_{,kk}\psi$$

$$= \phi_{,kk}\psi + \phi\psi_{,kk} + 2\phi_{,k}\psi_{,k}$$

$$= (\nabla^2\phi)\psi + \phi(\nabla^2\psi) + 2\nabla\phi\cdot\nabla\psi$$

$$\nabla\cdot(\phi u) = (\phi u_k)_{,k} = \phi u_{k,k} + \phi_{,k} u_k = \nabla\phi\cdot u + \phi(\nabla\cdot u)$$

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1-17 Determine the terms ∇f, $\nabla \cdot U$, $\nabla^2 f$ and $\nabla \times U$ for 3-D cilindrical coordinates system.

Solution.

Cylindrical coordinates.

$$\xi^1 = r \; ; \quad \xi^2 = \theta \; ; \quad \xi^3 = z$$

$$(ds)^2 = (dr)^2 + (r d\theta)^2 + (dz)^2$$

$$\Rightarrow h_1 = 1 \; ; \; h_2 = r \; ; \; h_3 = 1 .$$

$$\hat{e}_r = \cos\theta e_1 + \sin\theta e_2 \; ; \quad \hat{e}_\theta = -\sin\theta e_1 + \cos\theta e_2 \; , \quad \hat{e}_z = e_3$$

$$\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta \; ; \quad \frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r \; , \quad \frac{\partial \hat{e}_r}{\partial r} = \frac{\partial \hat{e}_\theta}{\partial r} = \frac{\partial \hat{e}_z}{\partial r} = \frac{\partial \hat{e}_z}{\partial \theta} = \frac{\partial \hat{e}_z}{\partial z} = 0$$

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z}$$

$$\nabla f = \hat{e}_r \frac{\partial f}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial f}{\partial \theta} + \hat{e}_z \frac{\partial f}{\partial z}$$

$$\nabla \cdot U = \frac{1}{r} \frac{\partial}{\partial r}(r U_r) + \frac{1}{r} \frac{\partial U_\theta}{\partial \theta} + \frac{\partial U_z}{\partial z}$$

$$\nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla \times U = \left[\frac{1}{r} \frac{\partial U_z}{\partial \theta} - \frac{\partial U_\theta}{\partial z} \right] \hat{e}_r + \left[\frac{\partial U_r}{\partial z} - \frac{\partial U_z}{\partial r} \right] \hat{e}_\theta$$

$$+ \frac{1}{r} \left[\frac{\partial}{\partial r}(r U_\theta) - \frac{\partial U_r}{\partial \theta} \right] \hat{e}_z$$

$$e_{xx} = \frac{\partial U_x}{\partial x} = \cos\theta \frac{\partial}{\partial r}\left(U_r \cos\theta - U_\theta \sin\theta\right) - \frac{\sin\theta}{r}\left(\frac{\partial}{\partial \theta}\left(U_r \cos\theta - U_\theta \sin\theta\right)\right)$$

$$= \frac{\partial U_r}{\partial r}\cos^2\theta - \frac{\partial U_\theta}{\partial r}\sin\theta\cos\theta - \frac{\partial U_r}{\partial \theta}\frac{\sin\theta\cos\theta}{r} + \frac{U_r}{r}\sin^2\theta + \frac{\partial U_\theta}{\partial \theta}\frac{\sin^2\theta}{r} + \frac{U_\theta}{r}\sin\theta\cos\theta$$

$$= \frac{\partial U_r}{\partial r}\cos^2\theta + \left(\frac{U_\theta}{r} - \frac{\partial U_\theta}{\partial r} - \frac{1}{r}\frac{\partial U_r}{\partial \theta}\right)\sin\theta\cos\theta + \left(\frac{U_r}{r} + \frac{1}{r}\frac{\partial U_\theta}{\partial \theta}\right)\sin^2\theta$$

$$e_{yy} = \frac{\partial U_y}{\partial y} = \sin\theta \frac{\partial}{\partial r}\left(U_r \sin\theta + U_\theta \cos\theta\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left(U_r \sin\theta + U_\theta \cos\theta\right)$$

$$e_{xy} = 2\left(\frac{\partial x}{\partial y} + \frac{\partial y}{\partial x}\right)$$

$$e_{rr} = \frac{\partial U_r}{\partial r} \; ; \; e_{\theta\theta} = \frac{1}{r}\left(U_r + \frac{\partial U_\theta}{\partial \theta}\right) \; ; \; \cancel{e_{r\theta} = \frac{1}{2}(}$$

$$e_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial U_r}{\partial \theta} + \frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r}\right) \; ; \; e_{zz} = \frac{\partial U_z}{\partial z}$$

Sphirical Coordinates.

$$R = \sqrt{x^2+y^2+z^2} \qquad x = R\cos\theta\cos\phi$$

$$\theta = \cos^{-1}\left(z/\sqrt{x^2+y^2+z^2}\right) \qquad y = R\sin\theta\sin\phi$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) \qquad z = R\cos\phi$$

$$\frac{\partial}{\partial x} = \frac{\partial R}{\partial x}\frac{\partial}{\partial R} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial y} = \frac{\partial R}{\partial y}\frac{\partial}{\partial R} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial z} = \frac{\partial R}{\partial y}\frac{\partial}{\partial R} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial z}\frac{\partial}{\partial \phi}$$

By using the process as we use for cylindrical...

$$e_R = \frac{\partial U_R}{\partial R} \; , \; e_\phi = \frac{1}{R}\left(U_R + \frac{\partial U_\phi}{\partial \phi}\right)$$

$$e_\theta = \frac{1}{R\sin\phi}\left(\frac{\partial U_\theta}{\partial \theta} + \sin\phi\, U_R + \cos\phi\, U_\phi\right)$$

Verify that the alternator e_{ijk} has the property that $e_{ijk} = Q_{ip} Q_{jq} Q_{kr} e_{pqr}$ for all proper orthogonal matrices $[Q]$

,

$$e_{ijk} \neq Q_{ip} Q_{jq} Q_{kr} e_{pqr}$$

for this the alternator is not an isotropic 3-tensor.

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Spherical coordinates: $\xi^1 = R,\ \xi^2 = \phi,\ \xi^3 = \theta$

$$x^1 = \xi^1 \sin\xi^2 \cos\xi^3,\quad x^2 = \xi^1 \sin\xi^2 \sin\xi^3,\quad x^3 = \cos\xi^2$$

Scale Factors

$$(h_1)^2 = \frac{\partial x^k}{\partial \xi^1}\frac{\partial x^k}{\partial \xi^1} = (\sin\phi\cos\theta)^2 + (\sin\phi\sin\theta)^2 + \cos^2\phi = 1 \Rightarrow h_1 = 1$$

$$(h_2)^2 = \frac{\partial x^k}{\partial \xi^2}\frac{\partial x^k}{\partial \xi^2} = R^2 \Rightarrow h_2 = R$$

$$(h_3)^2 = \frac{\partial x^k}{\partial \xi^3}\frac{\partial x^k}{\partial \xi^3} = R^2\sin^2\phi \Rightarrow h_3 = R\sin\phi$$

Unit Vectors:

$$\hat{e}_R = \cos\theta\sin\phi\, e_1 + \sin\theta\sin\phi\, e_2 + \cos\phi\, e_3$$

$$\hat{e}_\phi = \cos\theta\cos\phi\, e_1 + \sin\theta\cos\phi\, e_2 - \sin\phi\, e_3$$

$$\hat{e}_\theta = -\sin\theta\, e_1 + \cos\theta\, e_2$$

$$\frac{\partial \hat{e}_R}{\partial R} = 0,\quad \frac{\partial \hat{e}_R}{\partial \phi} = \hat{e}_\phi,\quad \frac{\partial \hat{e}_R}{\partial \theta} = \sin\phi\, \hat{e}_\theta$$

$$\frac{\partial \hat{e}_\phi}{\partial R} = 0,\quad \frac{\partial \hat{e}_\phi}{\partial \phi} = -\hat{e}_r,\quad \frac{\partial \hat{e}_\phi}{\partial \theta} = \cos\phi\, \hat{e}_\theta$$

$$\frac{\partial \hat{e}_\theta}{\partial R} = 0,\quad \frac{\partial \hat{e}_\theta}{\partial \phi} = 0,\quad \frac{\partial \hat{e}_\theta}{\partial \theta} = -\cos\phi\, \hat{e}_\phi$$

Using Equations

$$\nabla = \hat{e}_R \frac{\partial}{\partial R} + \hat{e}_\phi \frac{1}{R}\frac{\partial}{\partial \phi} + \hat{e}_\theta \frac{1}{R\sin\phi}\frac{\partial}{\partial \theta}$$

$$\nabla f = \hat{e}_R \frac{\partial f}{\partial R} + \hat{e}_\phi \frac{1}{R}\frac{\partial f}{\partial \phi} + \hat{e}_\theta \frac{1}{R}\frac{\partial f}{\partial \theta}$$

$$\nabla\cdot u = \frac{1}{R^2\sin\phi}\frac{\partial}{\partial R}(R^2\sin\phi\, u_R) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial \phi}(R\sin\phi\, u_\phi) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial \theta}(R\, u_\theta)$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 u_R) + \frac{1}{R\,\sin\phi}(\sin\phi\, u_\phi) + \frac{1}{R\sin\phi}\frac{\partial}{\partial\theta}(u_\theta)$$

$$\nabla^2 f = \frac{1}{R^2\sin\phi}\frac{\partial}{\partial R}\left(R^2\sin\phi\frac{\partial f}{\partial R}\right) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial f}{\partial\phi}\right)$$

$$+ \frac{1}{R^2\sin\phi}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\phi}\frac{\partial f}{\partial\theta}\right)$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial f}{\partial R}\right) + \frac{1}{R^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial f}{\partial\phi}\right) + \frac{1}{R^2\sin^2\phi}\frac{\partial^2 f}{\partial\theta^2}$$

$$\nabla\times u = \left(\frac{1}{R^2\sin\phi}\left[\frac{\partial}{\partial\phi}(R\sin\phi\, u_\theta) - \frac{\partial}{\partial\theta}(R u_\phi)\right]\right)\hat{e}_R + \left(\frac{1}{R\sin\phi}\left[\frac{\partial}{\partial\theta}(u_R) - \frac{\partial}{\partial R}(R\sin\phi\, u_\theta)\right]\right)\hat{e}_\phi$$

$$+ \left(\frac{1}{R}\frac{\partial}{\partial R}\left[(R u_\phi) - \frac{\partial}{\partial\phi}(u_R)\right]\right)\hat{e}_\theta$$

$$= \left[\frac{1}{R\sin\phi}\left(\frac{\partial}{\partial\phi}(\sin\phi\, u_\theta) - \frac{\partial u_\phi}{\partial\theta}\right)\right]\hat{e}_R + \left[\frac{1}{R\sin\phi}\frac{\partial u_R}{\partial\theta} - \frac{1}{R}\frac{\partial}{\partial R}(R u_\theta)\right]\hat{e}_\phi$$

$$+ \left[\frac{1}{R}\left(\frac{\partial}{\partial R}(R u_\phi) - \frac{\partial u_R}{\partial\phi}\right)\right]\hat{e}_\theta$$