Construction and Analysis of p^n Factorial Experiments

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Dedicated to

My parents

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Abstract

In this thesis paper we are mainly interested in symmetrical factorial experiments. First we propose the R code of the matrix method of designing layout in p^n factorial experiments. Though this method is the easiest method of designing layout in p^n factorial experiments, due to lack of computer program packages it had not been very popular. Then we propose two methods of analyzing symmetrical factorial experiments. One is a method of analyzing 2^n factorial experiment and another is a general method of analyzing p^n factorial experiments. The method of analyzing 2^n factorial experiment is performed by using sign rule or even versus odd rule and the general method of analyzing p^n factorial experiments is performed by using the solutions of the symbolic equations. A method of constructing a layout with single factorial effect confounded in a p^n (p is prime) factorial experiments was proposed by Jalil *et. al* (1990). This method has some notational mistakes. We make some corrections in these notational mistakes. This method is also restricted to p^n factorial experiments when p is prime. Here we propose a moderation to overcome this restriction. We also propose a general detection method of confounded effect in a given plan of a p^n (p is prime) factorial experiment with a single effect confounded. The method is described by introducing some new equations called detection equations.

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Chapter 1

Introduction

1.1 Concept of Factorial Experiment

A certain character under study may be influenced by a number of factors at different levels and hence it is necessary to test different combinations of the levels of the factors. Factorial experiments are experiments that study and investigate simultaneously the effects of two or more factors each at multiple levels on the response of a process. Factorial experiment provides study not only the individual effects of each factor but also their interactions. In these experiments, we require less resource to get same precision for each factorial effect. They give an exploratory work and hence they are widely used in research work.

1.2 Types of factorial experiment

We may define factorial experiments as experiments in which the effects (main effects and interactions) of more than one factor are studied together. In general if there are 'n' factors, say, $F_1, F_2, \dots, F_n$ and i^{th} factor has p_i levels, $i = 1, \dots, n$, then total number of treatment combinations is $\prod_{i=1}^{n} p_i$. Factorial experiments are of two types which are given below.

Experiments in which the number of levels of all the factors are same i.e all p_i's are equal are called symmetrical factorial experiments. If there are 'n' factors, say, $F_1, F_2, \dots, F_n$ and each has p levels, then the factorial experiment is denoted by p^n factorial experiment.

Experiments in which the number of levels of all factors are not same i.e at least two of the p_i's are different are called as asymmetrical factorial experiments. For example, $(p \times q)$ factorial experiment means first factor has p levels and second factor has q levels.

1.3 Motivation

Factorial experiment started it's works since 19th century. In 1937, F. Yates proposed a method of analyzing 2^n factorial experiments. Still now it is the easiest method and we use this method. We have methods of analyzing p^n factorial experiments only when p is prime. Moreover, the analysis method is not rigorous mathematically. We like to find whether a method can be found which is rigorous and general for p^n factorial experiments.

In 1990, Jalil *et. al* proposed two methods. One is matrix method of designing layout in p^n factorial experiments and another is a method of constructing layout with single factorial effect confounded in a p^n factorial experiment. There are three methods to represent the level combinations in a p^n factorial experiment. Among these three methods matrix method is the easiest method. Due to lack of computer program it had not been very popular. Additionally, when we worked with the method of constructing layout with single factorial effect confounded in a p^n factorial experiment, we saw some notational mistakes. The method is restricted to p^n factorial experiments when p is prime. These also motivated us.

A quick and easy method of detecting the confounded effect was proposed in 1994 (Jalil *et. al* 1994). It has been seen that the method would not work in p^n factorial experiments when $p > 3$. Also we have become interested whether the method can be moderated and can be applied for p^n factorial experiments, $p > 3$.

1.4 Objective of the study

Our study is conducted to serve the following purposes.

- To introduce matrix method of designing layout in p^n factorial experiments using computer program packages.
- To propose an alternative approach of analyzing 2^n factorial experiments.

- To propose a general method of analyzing p^n factorial experiments.
- To make corrections mistakes in the method of constructing layout with single factorial effect confounded in a p^n (p is prime) factorial experiment proposed by Jalil *et. al* (1990).
- To make moderation, in the method of constructing layout with single factorial effect confounded in a p^n (p is prime) factorial experiment proposed by Jalil *et. al* (1990), using which the method can be used in a p^n factorial experiment when p is a natural number.
- To propose a general method of detection of a confounded effect in a p^n (p is prime) factorial experiment.

1.5 Classification of the effects of a factorial experiment

The effects of a factorial experiment are broadly classified as main effects and interaction effects. The average change in the response produced by a change in the level of a factor is called the main effect of that factor. For example, consider a 2^2 factorial experiment given in Table 1.5.1. Here the factors are A, B and each has two levels 0, 1.

Table 1.5.1 Data from a 2^2 factorial experiment

		Factor B	
	Levels	b_0	b_1
Factor A	a_0	20	30
	a_1	40	52

The main effect of factor A can be thought of as the difference between the average response at level 0 of A and the average response at level 1 of A. Numerically, this is

$$A = \frac{40 + 52}{2} - \frac{30 + 20}{2} = 21$$

That is, increasing factor A from level 0 to level 1 causes an average increase in the response by 21 units. Similarly, the main effect of B is

$$B = \frac{30 + 52}{2} - \frac{20 + 40}{2} = 11$$

If the factors appear at more than two levels, the above procedure must be modified since there are many ways to express the differences between the average responses.

When the difference in response between the levels of one factor is not the same at all levels of the other factors, then there is an interaction between the factors. Interaction is the measure of dependence or correlation among a set of factors. In fact, two factors are said to be interact if the effects of a factor changes as the level of other factor changes. The interaction effects represent the manner in which each level of one main effect interaction with each level of other main effect. These effects are taken into account in the equation of the model by the addition of another term. For example, consider a 2^2 factorial experiment given in Table 1.5.2. Here the factors are A, B and each has two levels 0, 1.

Table 1.5.2 Data from a 2^2 factorial experiment

		Factor B	
	Levels	b_0	b_1
Factor A	a_0	20	40
	a_1	50	12

At level 0 of factor B (or b_0), the factor A effect is

$$A = 50 - 20 = 30$$

and At level 1 of factor B (or b_1), the factor A effect is

$$A = 12 - 40 = -28$$

Since the effect of A depends on the level chosen for factor B, we see that there is an interaction between A and B. The magnitude of the interaction effect is the average difference in these two A effects.

$$AB = \frac{(-28 - 30)}{2} = -29$$

The interaction is large in this experiment.

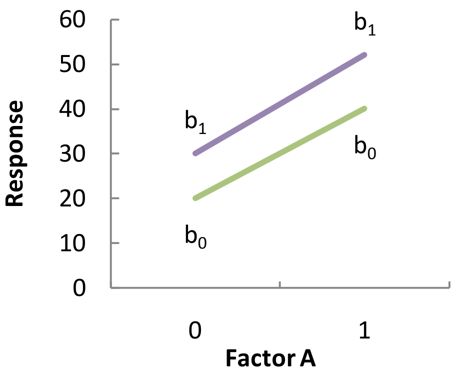

Fig. 1.5.1 A factorial experiment without interaction.

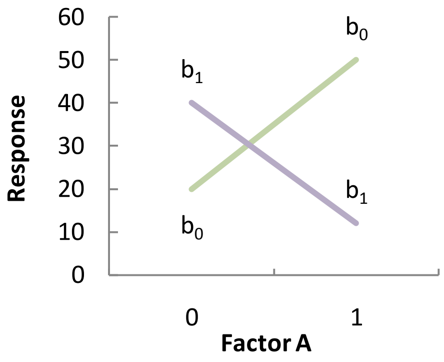

Fig. 1.5.2 A factorial experiment with interaction.

Generally, when an interaction is large, the corresponding main effects have little practical meaning. For the data of Table 1.5.2, we would estimate the main effect of A to be

$$A = \frac{50 + 12}{2} - \frac{20 + 40}{2} = 1$$

which is very small, and we are tempted to conclude that there is no effect due to A. However, when we examine the effects of A at different levels of factor B, we see that this is not the case. Factor A has an effect, but it depends on the level of factor B. That is, knowledge of AB interaction is more useful than knowledge of the main effect. A significant interaction will often mask the significance of main effects. In the presence of significant interaction, the experimenter must usually examine the levels of one factor, say A, with level of the other factors fixed to draw conclusions about the main effect of A.

1.6 Concept of confounding

When the number of factors and/or levels of the factors increase, the number of treatment combinations increase very rapidly and it is not possible to accommodate all these treatment combinations in a single homogeneous block. For example, a 2^5 factorial experiment would have 32 treatment combinations and blocks of 32 plots are quite big to ensure homogeneity within them. A new technique is therefore necessary for designing experiments with a large number of treatments. One such device is to take blocks of size less than the number of treatments and have more than one block per replication. The treatment combinations are then divided into as many groups as the number of blocks per replication. The different groups of treatments are allocated to the blocks.

In factorial experiments confounding is introduced in order to reduce the size of the blocks and obtain greater error control. The process of using incomplete blocks where important factorial effects are investigated and measured within the blocks while relatively unimportant factorial effects are deliberately mixed up or entangled with block effects is known as confounding. Preferably only higher order interactions are confounded, because their loss is immaterial. If a single factorial effect is confounded in a p^n factorial experiment, the treatment combinations are then divided into p blocks each containing p^{n-1} treatment combinations.

1.7 Types of confounding

In a replicated factorial experiment all replicates are confounded by same treatment effect or different replicates are confounded by different treatment effects. If all replicates are confounded by same treatment effect it is called total confounding or complete confounding and if different replicates are confounded by different treatment effects it is called partial confounding. In complete confounding all the information on confounded interactions are lost. But in partial confounding, the confounded interactions can be recovered from those replications in which they are not confounded.

For example, consider the following two replicates

Replicate 1	
Block 1	Block 2
(1)	abc
ac	a
ab	b
bc	c

Replicate 2	
Block 1	Block 2
(1)	abc
ac	a
ab	b
bc	c

Since in both replicate ABC is confounded, so abc is totally confounding or completely confounding. We cannot estimate the effect of ABC from any of the replicates.

If consider the following two replicates

Replicate 1	
Block 1	Block 2
(1)	abc
ac	a
ab	b
bc	c

Replicate 2	
Block 1	Block 2
(1)	a
ab	b
c	ac
abc	bc

Here, in Replicate 1 ABC is confounded but is unconfounded in Replicate 2. So the effect of ABC can be estimated from Replicate 2. And AB is confounded in Replicate 2 but is unconfounded in Replicate 1. So the effect of AB can be estimated from Replicate 1. Thus ABC and AB are partially confounded and partial information on AB and ABC are obtained from the replicates in which they are individually unconfounded.

1.8 Concept of detection of a confounded effect from a given plan

We can make a confounding plan taking any treatment effect as confounded. In this case first we take a treatment effect as confounded effect and then make a confounding plan. However, situations may arise when we have only a confounded plan or layout and no information about the confounded effect. The process of finding the confounded effect from this given plan or layout is called detection of confounded effect from a given plan.

1.9 Analysis of Variance (ANOVA)

Analysis of variance (ANOVA) is a well known statistical technique. The term "Analysis of Variance" was first introduced by Professor R. A. Fisher, one of the great founders of Statistics, to deal with the problem in the analysis of agricultural data. Analysis of Variance (ANOVA) is a well known and powerful statistical tool for tests of significance. The ANOVA technique has developed extensively and applied widely in diverse field of research in natural and social science such as agricultural, biological, psychological, medical, industrial and other allied fields.

According to R. A. Fisher, Analysis of Variance is the "Separation of variance ascribable to one group of causes from the variance ascribable to other group". It is a technique of separating the total variation in sample data into non-negative components where each of these components is a measure of the variation due to some specific independent cause for comparing the significance of the assignable causes.

In short, the ANOVA consists in the estimation of the amount of variation due to each of the independent factors (causes) separately and then comparing these estimates due to assignable factors (causes) with the estimate due to chance causes (commonly known as experimental error).

Suppose that we have one group of men drawn from a population in which the people are all of the same race and the variable studied is height in inches. A frequency distribution is drawn up and found to be approximately normal. Therefore, there is a good reason to assume that the variation in their group is approximately homogeneous. The same would apply to a group of women studied in the same manner. However, if data from the two groups are mixed up to form a new group, a second component of variation is brought in, namely the difference between the means of two groups. The difference would be large enough so that the frequency distribution for the combined group would probably show two peaks or modes. Carrying the analogy further, a third group might consist of boys from 13 to15 years of age, and a fourth group is the girls of similar age group. When all four groups are combined the frequency distribution might appear reasonably normal, but we know that two components of variations are actually present. One represents the variation within the groups and another represents the variation between the groups.

The arithmetic procedure of analysis of variance enables us to sort out and evaluate the components variation for such mixed populations. The variation in the data may arise due to random causes or due to some specific causes or both. The complete analysis of variance actually performs a double role. In first place we have to sorting out and estimation of the variance components and in the second place it provide for tests of significance.

The test procedures in the theory of classical Analysis of Variance (ANOVA) are quite complete under the following three fundamental assumptions of errors:

(1) normality
(2) homoscedasticity and
(3) independence

1.10 Some important concepts and definitions

We now proceed to define and explain some specific terms which will occur frequently in the context of this thesis.

- **Experiment :** An experiment is an action or investigation conducted to discover the underlying facts about a phenomenon. The usual purpose of an experiment is to establish the validity of some hypothesis about a population. An experiment is conducted either to estimate some parameters or to test some hypothesis. For example, one may be interested in estimating the true difference of the yielding capacities of two varieties of rice or one may wish to test the hypothesis that several teaching methods are equivalent in the long-run. In fact, an experiment is a device or means of data collection from some non-existent population to get answer to certain problem under consideration and also to discover some principles or effects or to test, establish or illustrate some suggested known truth.
- **Treatment :** The treatment denotes the particular set of experimental conditions which are imposed on the experimental units for purposes of comparison. It is a method or procedure whose effect is measured and compared with similar others. In medical experiment different doses of a medicine or diets are the treatments. Also, fertilizers, variety of a crop, method of cultivation are the examples of treatments.
- **Factor :** Factor is a criteria or attributes according to which observed data are classified. For example, rice is a factor.
- **Level :** Different types or different doses or different amounts of a factor are termed is levels. Different types of rice like Irri, Boro, Amon are levels

- **Treatment combinations :** Treatment combinations are the combinations of the levels of different factors. Let we have two factors such as fertilizer (A) and varity of rice (B). Fertilizer (A) has three levels as a_0, a_1, a_2 and varity of rice (B) has two levels as b_0, b_1. Then the treatment combinations are

A \ B	b_0	b_1
a_0	$a_0 b_0$	$a_0 b_1$
a_1	$a_1 b_0$	$a_1 b_1$
a_2	$a_2 b_0$	$a_2 b_1$

- **Layout :** The layout of a design indicates the placement of treatments to the experimental plots according to the condition of the design.

In factorial experiment total work can be partitioned into two parts. One is construction of a plan and another is the analysis procedure. In this thesis paper we will describe both construction plan and the analysis procedure that are applicable only for symmetrical factorial experiments.

1.11 Organization of Chapters

In this introductory chapter, we have discussed the concept of factorial experiments, types of factorial experiments, our motivation towards this study, purpose of this study, classification of the effects of a factorial experiment, concept of confounding, types of confounding, concept of detection of a confounded effect from a given plan, analysis of variance, and some important concepts and definitions relevant to this study. The remaining parts of this study are organized in next seven different chapters.

Review of some relevant literatures is presented in chapter two. Here, we get some literatures on confounding, but very few literatures on analysis procedure of p^n factorial experiments and on detection of the confounded effects.

There are three methods to represent the level combinations in a p^n factorial experiment. They are,

- Manipulating Method
- Matrix Product Method
- Matrix Method

The Matrix Method of designing layout in p^n factorial experiments was proposed by Jalil *et. al* (1990). It is the easiest method and also takes less time to present level combinations even when the number of factors as well as levels is large. The R-code of the Matrix Method is described in the third chapter.

An alternative method of analyzing 2^n factorial experiment is proposed in the fourth chapter and a general method of analyzing p^n factorial experiments will be proposed in fifth chapter. In the previous studies, there was only method of analyzing p^n (p is prime) factorial experiments.

A Method of constructing a layout with single factorial effect confounded was proposed by Jalil *et. al* (1990). This method has some notational mistakes. Corrections of these notational mistakes are described in the sixth chapter. This method has also limitation. The limitation is that the method can be used in p^n factorial experiments only when p is prime. For this reason, we will make moderation for which we can use this method in p^n factorial experiments, when p is any natural number. This moderation is also be described in this chapter.

An easy method, of detection of a confounded effect in p^n (p is prime) factorial experiments, was proposed by Jalil *et. al* (1994). However, if $p > 3$, the method does not work in detecting the confounded effect. For this purpose, we will make moderation. As such, we will be able to use this method in p^n factorial experiments, where p is prime. Detailed description of this moderation is described in the seventh chapter.

In chapter eight, we conclude our thesis by discussing all the finding of this study.

Chapter 2

Literature Review

A scantiness of literature on the analysis procedure of factorial experiment is observed so far. There are some literatures on confounding, but very few literatures on the analysis procedure of p^n factorial experiments and on detection of the confounded effect. A brief review of literature of these areas is given below.

Factorial designs were used in the 19th century by John Bennet Lawes and Joseph Henry Gilbert of the Rothamsted Experimental Station. Rothamsted Research, previously known as the Rothamsted Experimental Station and then the Institute of Arable Crops Research, is one of the oldest agricultural research institutions in the world, having been founded in 1843. It is located at Harpenden in the English county of Hertfordshire. Ronald Fisher argued in 1926 that "complex" designs (such as factorial designs) were more efficient than studying one factor at a time. Fisher wrote,

> "No aphorism is more frequently repeated in connection with field trials, than that we must ask Nature few questions, or, ideally, one question, at a time. The writer is convinced that this view is wholly mistaken."

Nature, he suggests, will best respond to "a logical and carefully thought out questionnaire". A factorial design allows the effect of several factors and even interactions between them to be determined with the same number of trials as are necessary to determine any one of the effects by itself with the same degree of accuracy. The term "factorial" may not have been used in print before 1935, when Fisher used it in his book “The Design of Experiments”.

Frank Yates was one of the pioneers of 20th century statistics. In 1931 Yates was appointed assistant statistician at Rothamsted Experimental Station by R.A. Fisher. In 1933 he became head of statistics when Fisher went to University College London. At Rothamsted he worked on the design of experiments, including contributions to the theory of analysis of variance and originating Yates's algorithm and the balanced incomplete block design.

The design and analysis of factorial experiments was described in 1937 by Yates' in considerable detail. In his treatment Yates described first the 2^n system and then went on to deal with 3^n experiments and experiments of the $2^m 3^n$ type. The 2^n system is capable of very easy explanation, but with experiments of higher order both design and analysis become of increasing complexity.

The most used analysis procedure of p^n factorial experiments, where p is prime, is done using I-total and J-total.

P. K. Batra and Seema Jaggi proposed two methods for analyzing 2^n and 3^n factorial experiments. For analyzing 3^n factorial experiment the effect of each factor is partitioned into linear and quadratic effects.

Bainbridge (1956) proposed a method for analyzing factorial experiments which is generalized to factors at any number of levels, not necessarily the same for each factor.

Bayhan (2004) proposed an alternative procedure for the estimation problem in $\mathbf{2^n}$ factorial experimental models.

Various system of confounding, using factor up to six in number, had been discussed by Barnard (1936) and Yates (1937). The classical method of arranging a p^n factorial experiment so that some effects are confounded with blocks is due to Yates (1937). This design uses projective geometry which restricts the value of p. Simple design and analysis results only when p is prime and is much more complicated for other values of p.

Nair (1938) developed a method of getting confounded arrangements in the p^n type of experiment, where n is a positive integer and p is prime or a power of a prime, based on his theory of interchanges derivable from the associated Hyper-Graeco-Latin

Squares. He has demonstrated the working of his method in obtaining confounded arrangements of the p^n type of experiment in sub-blocks of p^2 plots, for (1) $p = 3, n = 3$ and 4, (2) $p = 4, n = 3, 4$ and 5, and (3) $p = 5, n = 3$ and 4.

Bose and Kishan (1940), Bose (1947) described the construction of p^n factorial design using finite geometries. The treatments are represented by $n-$tuples $(a_1, a_2, \ldots, a_n)$ where a_1 are elements of $GF(p)$. The method is available only when p is prime or prime power.

Fisher (1942) discussed a method to develop the connection of the subject with that of Abelian groups, to prove a general proposition connecting the minimal size of block required with the number of factors involved, and to supply a catalogue of systems of confounding available up to fifteen factors.

A system of simultaneous confounding in 2^n factorial experiment has been described, where an intrablock subgroup is constructed with the common elements taken from the factorial effects of two incomplete blocks, each confounded with a single factorial effect (Kempthorne, 1947, 1952).

A method of confounding in asymmetrical factorial experiments was proposed by Nair and Rao (1948).

Zelen (1958) used group divisible incomplete block designs to construct confounding plans and to simplify the analysis for asymmetrical factorial experiments.

Bailey (1959) discussed the derivation of prime power factorial design, with confounding and fractional replication, using only comparatively elementary result in linear algebra. He also gave some further simplifications and systematization of the standard theory of factorial designs.

Oktaba (1963) proposed a method of constructing and arranging factorial field experiments and other experiments with 2, 3, 4 or 5 levels of each factor according to the so-called rule of complete confounding.

Das (1964) described an equivalent method of Bose in which some of the treatment factors are designated as basic factors and the others as added factors. Levels of added factors are derived by combination of levels of the basic factors over $GF(p)$. White and Hultquist (1965) extended the field method to design with number of levels of treatment factors.

David White and Robert A. Hultquist (1965) extended the use of finite fields for the construction of confounding plans to include asymmetrical or mixed factorials. He proposed a method of constructing confounding plans and a method of analyzing the data for asymmetrical or mixed factorials.

Sarah C. Cotter (1974) proposed a general method of confounding for symmetrical factorial experiments. This paper considered the problem of conducting a p^n factorial experiment in blocks size p^1. These designs can be constructed for all values of p, although for certain values better confounding patterns are available. Also analysis of these designs was given, showing which components of the sum of squares were confounded with blocks.

John and Dean (1975) described the construction of a particular class of single replicate block designs, which they call generalized cyclic designs. The essential feature of the method is that the $n-$tuples giving the treatments of a particular block constitute an Abelian group, the intrablock subgroup.

Patterson (1976) described a general computer algorithm, called DESIGN, in which levels of treatment factors are derived by linear combinations of levels of plot and block factors. The method provides finite-field, generalized cyclic and other designs.

Jalil, *et. al.*(1990) developed a matrix method of designing a single factorial effect confounded in a p^n - factorial experiment, where the level combinations are obtained by matrix operations of the levels. Construction method of simultaneous confounding has been developed independently for 3^n and 5^n factorial experiments (Jalil and Mallick, 2010, 2011). In 2012, Jalil proposed a general method of construction for simultaneous confounding in a p^n – (p is prime) factorial experiment. In 2012, Jalil also proposed a linear equation method of designing a p^n - factorial experiment of simultaneous confounding plan where we can confound k ($k \geq 1$) factorial effects.

A method of constructing the Principle Block (P.B.) and determination of the effects confounded from the elements of Principle Block (P.B.) for any number of factors in 2^n factorial experiments were proposed by Raja (1974).

An easy method of detection of the confounded effect in a given plan of a p^n (p is prime) factorial experiment with a single effect confounded was proposed by Jalil and Mallick (1994).

Chapter 3

R-code of Matrix Method of Designing Layout in p^n Factorial Experiments

3.1 Introduction

There are three methods to represent the level combinations in a p^n factorial experiment. These are:

(i) Manipulating Method, where all the level combinations are taken into account to layout the p^n factorial experiment.

(ii) Matrix Product Method, where the matrix of the product set is formed by the sets of level of the factors concerned. The product is to be taken as many times as the number of factors.

(iii) Matrix Method

When the number of factors is large, representation of level combinations will be time consuming as well as lengthy procedure by the first two methods. The Matrix Method was proposed by Jalil *et. al* (1990). It is the easiest method and also takes less time to present level combinations even when the number of factors as well as the number of levels is large. In this chapter, we will describe the R-code of the Matrix Method.

3.2 Matrix method of designing layout in p^n factorial experiments

The matrix method of designing layout in p^n factorial experiments was proposed by Jalil *et. al* in 1990. The method is described below.

3.2.1 The method

Consider the layout matrix, M_p of order $p^n \times n$ as,

$M_p = [V_1\{p^0\}, V_2\{p^1\}, \dots, V_n\{p^{n-1}\}]$

where,

$V_i\{p^{i-1}\} = p^{i-1}[0I_{p^{n-i}}, 1I_{p^{n-i}}, \dots, (p-1)I_{p^{n-i}}]'_{p^n \times 1}$

$i = 1, 2, \dots, n$

each is a column vector of dimension p^n.

$\{p^j\} = p^j$-times repeatation of the elements of V_i's in ascending order levels.

I_m = sum vector of dimension m.

3.2.2 Illustration with example

Suppose we want to write all the level combinations of a 3^3- factorial experiment.

Here $p = 3,\ n = 3$

The suggested matrix is,

$$M_3 = [V_1\{3^0\}, V_2\{3^1\}, V_3\{3^2\}]_{\ 3^3 \times 3}$$

$$= [V_1\{1\}, V_2\{3\}, V_3\{9\}]_{\ 27 \times 3}$$

where,

$$V_1\{1\} = 3^0[0I_9, 1I_9, 2I_9]'_{27 \times 1}$$

$$V_2\{3\} = 3^1[0I_3, 1I_3, 2I_3]'_{27 \times 1}$$

$$V_3\{9\} = 3^2[0I_0, 1I_0, 2I_0]'_{27 \times 1}$$

So that level combinations are, $M_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 2 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 0 & 2 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \\ 2 & 1 & 2 \\ 2 & 2 & 0 \\ 2 & 2 & 1 \\ 2 & 2 & 2 \end{bmatrix}$

3.3 R-code of matrix method of designing layout in p^n factorial experiments

R-code of matrix method of designing layout in p^n factorial experiments is given below:

```
mat <- function (p, n)
  {
   j <- c (0: (p-1))
   i <- 1: n
   m <- array (999, dim = c (p^(n), n))
   for (k in 1: length (i))
    {
```

```
    a <- array (rep (j, each = p^(n-i[k])), dim = c (p^(n),1))
    m [ , k] <- a
    k <- k + 1
   }
 print (m)
}
```

3.3.1 Illustration with example

Suppose we want to write the level combinations of a 3^3- factorial experiment.

Here $p = 3,\ n = 3$

So the R-code is,

```
>  mat <- function (p, n)
+    {
+     j <- c (0: (p-1))
+     i <- 1: n
+     m <- array (999, dim = c (p^(n), n))
+     for (k in 1: length(i))
+       {
+         a <- array (rep (j, each = p^(n-i[k])), dim = c (p^(n),1))
+         m [ , k] <- a
+         k <- k + 1
+       }
+     print (m)
+    }
```

```
> treat.combination <- mat (3, 3)

      [,1] [,2] [,3]
 [1,]    0    0    0
 [2,]    0    0    1
 [3,]    0    0    2
 [4,]    0    1    0
 [5,]    0    1    1
 [6,]    0    1    2
 [7,]    0    2    0
 [8,]    0    2    1
 [9,]    0    2    2
[10,]    1    0    0
[11,]    1    0    1
[12,]    1    0    2
[13,]    1    1    0
[14,]    1    1    1
[15,]    1    1    2
[16,]    1    2    0
[17,]    1    2    1
[18,]    1    2    2
[19,]    2    0    0
[20,]    2    0    1
[21,]    2    0    2
[22,]    2    1    0
[23,]    2    1    1
[24,]    2    1    2
[25,]    2    2    0
[26,]    2    2    1
[27,]    2    2    2
```

3.4 Summary

In this chapter, the R-code of the Matrix Method, proposed by Jalil *et. al* (1990), has been introduced. Matrix method, that has been the easiest method of designing layout in p^n factorial experiments, had not been very popular, mostly because of lack of computer program packages. Our main goal is simply to popularize this method by writing computer programs. We have written the function in R language. Anyone can use this function to create the designing layout of any p^n factorial experiments.

Chapter 4

An Alternative Approach of Analyzing 2^n Factorial Experiment

4.1 Introduction

F. Yates in 1937 developed a systematic tabular technique for obtaining factorial effects and their respective Sum of Squares (SS) in a 2^n factorial experiment. This is the most popular method of analyzing 2^n factorial experiment. After Yates, P. K. Batra and Seema Jaggi also propose a method of analyzing 2^n factorial experiment. However, in this chapter, we will describe an alternative approach of analyzing 2^n factorial experiment using sign rule or even versus odd rule. It is easy to calculate and less time consuming as well.

4.2 The Sign Rule & The Even Versus Odd Rule

The sign rule and the even versus odd rule are used primarily in constructing confounding plan in 2^n factorial experiment. The procedures of these two methods are described below.

4.2.1 The Sign Rule

In this method, positive signs are assigned to the treatment combinations, which are at the highest level of the corresponding factor, in the main effect. And negative signs are assigned to the treatment combinations, which are at the lowest level of the corresponding factor. In case of interaction effects, positive signs and negative signs are obtained by multiplying the corresponding signs of the contrasts of the interacting factors.

4.2.2 The Even versus odd rule

In the even versus odd rule every factorial effect is presented as simple comparison of two groups of treatment combinations, where one group receives positive sign (+) and the other group receives negative sign (-). If a factorial effect has an even number of letters, then the treatment combinations having an even number of letters common with the factorial effect occur with positive sign while the treatment combinations having an odd number of letters common enter with negative sign. Otherwise, if a factorial effect has an odd number of letters, entirely reverse rule of above happens.

4.3 Yates' method (Yates' Algorithm)

F. Yates in 1937 developed a systematic tabular technique for obtaining factorial effects and their respective Sum of Squares (SS) in a 2^n factorial design.

4.3.1 Algorithm

This routine procedure consists of the following steps:

Step 1: Treatment combinations in standard order are noted in the first column of table.

Step 2: Then corresponding total yields of all treatment combinations are put down in second column which is known as response column.

Step 3: Sum of total yields of each successive non-overlapping pairs of adjacent entries of response column is recorded in the 1^{st} half of column 1 and 2^{nd} half of column 1 is obtained by subtracting the first entry from the second of each non-overlapping pairs.

Step 4: Column 2 is obtained from column 1 in exactly same manner of sum and difference of pairs.

Step 5: For a 2^n factorial experiment, the process will be repeated n times until column n is obtained and this column contains top member as grand total and remaining entries as factorial effect totals in standard order.

Step 6: Then the estimates of factorial effects are obtained in the next column by dividing all the entries but first one of column n by $r.2^{n-1}$ where r is the number of replications.

Step 7: Finally SS of factorial effects are obtained in the last column by squaring the entries in column n, divided by $r.2^n$.

4.3.2 Illustration of Yates' method with example

The following data refer to the yields of a 2^3 factorial experiment with 3 factors A, B and C in a randomized block design.

Blocks	Treatment with yields							
1	(1) 257	a 232	ab 230	b 211	c 210	ac 176	abc 186	bc 175
2	ac 267	(1) 276	b 262	abc 220	a 256	c 269	ab 285	bc 272
3	b 188	bc 186	c 160	a 188	ab 164	(1) 214	abc 182	ac 166
4	ab 204	ac 206	(1) 239	bc 224	a 254	b 269	c 252	abc 301

Here,

Block totals: $B_1 = 1677, B_2 = 2117, B_3 = 1448, B_4 = 1949$

Treatment totals: $T_1 = 986, T_2 = 930, T_3 = 930, T_4 = 883, T_5 = 901, T_6 = 815,$

$T_7 = 857, T_8 = 889$

Grand total, G = 7191

Table 4.3.1 Estimation of main effects and interactions by Yates' method

Treatment combination	Total yield	Column 1	Column 2	Column 3	Main effects & interactions	SS
(1)	986	1916	3729	7191	- - - -	- - - -
a	930	1813	3452	-157	-9.8125 = A	770.28125
b	930	1716	-103	-73	-4.5625 = B	166.53125
ab	883	1786	-54	127	7.9375 = AB	504.03125
c	901	-56	-103	-267	-16.6875 = C	2227.78125
ac	815	-47	30	49	3.0625 = AC	75.03125
bc	857	-86	9	133	8.3125 = BC	552.78125
abc	889	32	118	109	6.8125 = ABC	371.28125

Total SS = $\sum\sum y_{ij}^2 - \frac{G^2}{32}$

$= 1667273 - \frac{7191^2}{32}$

$= 1667273 - 1615952.531$

$= 51320.469$

Block SS = $\frac{\sum B_i^2}{8} - \frac{G^2}{32}$

$= 1648665.54 - 1615952.531$

$= 32712.869$

Treatment SS = $\frac{\sum T_j^2}{4} - \frac{G^2}{32}$

$= 1620620.25 - 1615952.531$

$= 4667.719$

Error SS = Total SS − Block SS − Treatment SS

$= 51320.469 - 32712.869 - 4667.719$

$= 13939.881$

Table 4.3.2 ANOVA Table

Source of Variation	DF	SS	M.S.	F_c	p-value
Blocks Treatments	4-1=3 8-1=7	32712.869 4667.719			
A	1	770.2815	770.2815	1.160405	0.29360
B	1	166.5315	166.5315	0.250874	0.62167
AB	1	504.0315	504.0315	0.759307	0.39340
C	1	2227.7815	2227.7815	3.356083	0.08118
AC	1	75.0321	75.0321	0.113032	0.74005
BC	1	552.7815	552.7815	0.83275	0.37184
ABC	1	371.2815	371.2815	0.559324	0.46283
Error Total	21 31	19939.88	663.8039		

Test of hypothesis:

We want to test the following hypothesis

H_0 : All the main effects and interaction effects are insignificant.

H_1 : They are significant.

Level of significance: We want to test our hypothesis at 5% level of significance i.e. $\alpha = 0.05$.

Comment:

Since here p-values of all main effects and interactions are greater than the level of significance, α, so we may accept the null hypothesis. Hence, we can conclude that observed values of all main effects and interactions are insignificant.

4.4 Method of analyzing 2^n factorial experiment proposed by P. K. Batra and Seema Jaggi

A method of analyzing 2^n factorial experiment is proposed by P. K. Batra and Seema Jaggi.

4.4.1 Steps for Analysis

The steps of analyzing 2^n factorial experiment are given below:

Step 1: The Sum of Squares (S.S.) due to treatments, replications (in case randomized block design is used), due to rows and columns (in case a row-column design has been used), total S.S. and error S.S. is obtained as per established procedures. No replication S.S. is required in case of a completely randomized design.

Step 2: The treatment sum of squares is divided into different components viz. main effects and interactions each with single degrees of freedom (d.f). The S.S. due to these factorial effects is obtained by dividing the squares of the factorial effect total by $r.2^n$. For obtaining $2^n - 1$ factorial effects in a 2^n factorial experiment, the 'n' main effects is obtained by giving the positive signs to those treatment totals where the particular factor is at second level and minus to others and dividing the value so obtained by $r.2^n$ where r is the number of replications of the treatment combinations. All interactions can be obtained by multiplying the corresponding coefficients of main effects.

Step 3: Mean squares (M.S.) are obtained by dividing each S.S. by corresponding degrees of freedom.

Step 4: After obtaining the different S.S.'s, the usual Analysis of variance (ANOVA) table is prepared and the different effects are tested against error mean square and conclusions drawn.

Step 5: Standard error (S.E.'s) for main effects and two factor interactions:

S.E of difference between main effect means $= \sqrt{\frac{2MSE}{r.2^{n-1}}}$

S.E of difference between A means at same level of B = S.E of difference between B means at same level of A $= \sqrt{\frac{2MSE}{r.2^{n-2}}}$

In general,

S.E. for testing the difference between means in case of a r-factor interaction $= \sqrt{\frac{2MSE}{r.2^{n-r}}}$

The critical differences are obtained by multiplying the S.E. by the student's t value at α% level of significance at error degrees of freedom.

4.4.2 Illustration with example

The following data refer to the yields of a 2^3 factorial experiment with 3 factors A, B and C in a randomized block design.

Blocks	Treatment with yields							
1	(1) 257	a 232	ab 230	b 211	c 210	ac 176	abc 186	bc 175
2	ac 267	(1) 276	b 262	abc 220	a 256	c 269	ab 285	bc 272
3	b 188	bc 186	c 160	a 188	ab 164	(1) 214	abc 182	ac 166
4	ab 204	ac 206	(1) 239	bc 224	a 254	b 269	c 252	abc 301

Analysis

Step 1: The data is arranged in the following table:

Block	Treatment combinations								Total
	(1)	a	b	ab	c	ac	bc	abc	
B_1	257	232	211	230	210	176	175	186	1677 (B_1)
B_2	276	256	262	285	269	267	272	220	2117 (B_2)
B_3	214	188	188	164	160	166	186	182	1448 (B_3)
B_4	239	254	269	204	252	206	224	301	1949 (B_4)
Total	986 (T_1)	930 (T_2)	930 (T_3)	883 (T_4)	901 (T_5)	815 (T_6)	857 (T_7)	889 (T_8)	7191 (G)

Here,

Grand total, G = 7191

Number of observations $(n) = 32 = (r.2^n)$

Correction factor (C.F.) = $\frac{G^2}{n} = \frac{7191^2}{32} = 1615952.531$

Total SS = $\sum\sum y_{ij}^2$ $-$ C.F

= 1667273 – 1615952.531

= 51320.469

Block (Replication) SS = $\frac{\sum B_i^2}{8} -$ C.F

= 1648665.54 $-$ 1615952.531

= 32712.869

Treatment SS = $\frac{\sum T_j^2}{4} -$ C.F

= 1620620.25 $-$ 1615952.531

= 4667.719

Error SS = Total SS – Block SS – Treatment SS

= 51320.469 – 32712.869 – 4667.719

= 13939.881

Step 2: Main effects totals and interactions totals are obtained as follows:

$$A \quad = [abc] - [bc] + [ac] - [c] + [ab] - [b] + [a] - [1] = -157$$

$$B \quad = [abc] + [bc] - [ac] - [c] + [ab] + [b] - [a] - [1] = -73$$

$$C \quad = [abc] + [bc] + [ac] + [c] - [ab] - [b] - [a] - [1] = -267$$

$$AB \quad = [abc] - [bc] - [ac] + [c] + [ab] - [b] - [a] + [1] = 127$$

$$AC \quad = [abc] - [bc] + [ac] - [c] - [ab] + [b] - [a] + [1] = 49$$

$$BC \quad = [abc] + [bc] - [ac] - [c] - [ab] - [b] + [a] + [1] = 133$$

$$ABC \quad = [abc] - [bc] - [ac] + [c] - [ab] + [b] + [a] - [1] = 109$$

Factorial effects = $\frac{\text{Factorial effect total}}{r.2^{n-1}(=16)}$

Factorial effect SS $= \frac{(\text{Factorial effect total})^2}{r.2^n(=32)}$

Here Factorial Effects

$\mathrm{A} = -9.8125,\ \mathrm{B} = -4.5625, \mathrm{C} = -16.6875,\ \mathrm{AB} = 7.9375,\ \mathrm{AC} = 3.0625,$ $\mathrm{BC} = 8.3125,\ \mathrm{ABC} = 6.8125$

SS due to $A = 770.28125$

SS due to B =166.53125

SS due to C =2227.78125

SS due to AB =504.03125

SS due to $AC = 75.03125$

SS due to $BC = 552.78125$

SS due to $ABC = 371.28125$

Step 3: M.S. is obtained by dividing S.S.'s by respective degrees of freedom.

Table 4.4.1 ANOVA Table

Source of Variation	DF	SS	M.S.	F_c	p-value
Blocks	4-1=3	32712.869			
Treatments	8-1=7	4667.719			
A	1	770.2815	770.2815	1.160405	0.29360
B	1	166.5315	166.5315	0.250874	0.62167
AB	1	504.0315	504.0315	0.759307	0.39340
C	1	2227.7815	2227.7815	3.356083	0.08118
AC	1	75.0321	75.0321	0.113032	0.74005
BC	1	552.7815	552.7815	0.83275	0.37184
ABC	1	371.2815	371.2815	0.559324	0.46283
Error	21	19939.88	663.8039		
Total	31				

Test of hypothesis:

We want to test the following hypothesis

H_0 : All the main effects and interaction effects are insignificant.

H_1 : They are significant.

Level of significance: We want to test our hypothesis at 5% level of significance i.e. $\alpha = 0.05$.

Comment:

Since here p-values of all main effects and interactions are greater than the level of significance, α, so we may accept the null hypothesis. Hence, we can conclude that observed values of all main effects and interactions are insignificant.

4.5 Proposed alternative approach of analyzing 2^n factorial experiment

In this section, we are going to propose a new method of analyzing 2^n factorial experiments. This method is easy to apply and also less time consuming than the previously described methods.

4.5.1 Algorithm

The routine procedure for obtaining Sum of Squares (SS) of the factorial effects in a 2^n factorial experiment consists of the following steps:

Step 1. In the first column of the table, we will have to write the main effects and interaction effects (Table 4.5.1).

Step 2: All treatment combinations with their corresponding total yields will have to be listed in the first row of the second column. Thus, we will have 2^n sub-columns beneath this row (Table 4.5.1).

Step 3: These sub-columns will have to be filled up by using sign rule or even versus odd rule (Table 4.5.1).

Step 4: Sum of treatment yields with negative sign of each row and sum of treatment yields with positive sign of each row will have to be placed in the next two columns denoted by x_0 and x_1 respectively (Table 4.5.1). It should be noted that, we can calculate x_1 by subtracting x_0 from grant total, G, and vice versa.

Step 5: Finally, we will be able to obtain the sum of squares (SS) for each factorial effects by using the following formula

$$\frac{x_0^2 + x_1^2}{2^{n-1}.r} - C.T.$$

where,

r is the number of replicate,

C.T. = Correction term = $\frac{G^2}{2^n.r}$

G = Grand Total

Step 6: We can calculate the Sum of Squares (SS) due to treatments, replications, rows and columns, total SS, and error SS by using previously established procedures.

Using above algorithm we can construct the following table for calculating SS of the factorial effects in a 2^2 factorial experiment.

Table 4.5.1 Construction of table for calculating Sum of Square (SS) of factorial effects

Factorial effects	Treatment combination				Sum of treatment yields with (-) sign, x_0	Sum of treatment yields with (+) sign, x_1	Sum of square (SS) $\frac{x_0^2+x_1^2}{2^{n-1}\times r}$ - $C.T.$
	(1) (…)	a (…)	b (…)	ab (…)			
A	-	+	-	+	…	…	…
B	-	-	+	+	…	…	…
AB	+	-	-	+	…	…	…

4.5.2 Illustration of the alternative approach with example

The following data refer to the yields of a 2^3 factorial experiment with 3 factors A, B and C in a randomized block design.

Blocks	Treatment with yields							
1	(1) 257	a 232	ab 230	b 211	c 210	ac 176	abc 186	bc 175
2	ac 267	(1) 276	b 262	abc 220	a 256	c 269	ab 285	bc 272
3	b 188	bc 186	c 160	a 188	ab 164	(1) 214	abc 182	ac 166
4	ab 204	ac 206	(1) 239	bc 224	a 254	b 269	c 252	abc 301

Here,

Block totals: $B_1 = 1677, B_2 = 2117, B_3 = 1448, B_4 = 1949$

Treatment totals: $T_1 = 986,\ T_2 = 930,\ T_3 = 930,\ T_4 = 883,\ T_5 = 901,\ T_6 = 815,$
$T_7 = 857,\ T_8 = 889$

Grand total, G = 7191

C.T. = Correction term = $\frac{G^2}{2^n \times r} = \frac{7191^2}{32}$ = 1615952.531

Table 4.5.2 Construction of table for calculating Sum of Square (SS) of factorial effects

Factorial effects	Treatment combination								Sum of treatment yields with (-) sign, x_0	Sum of treatment yields with (+) sign, x_1	Sum of square (SS) $\frac{x_0^2+x_1^2}{2^{n-1}\times r}$ - $C.T.$
	(1) (986)	a (930)	b (930)	ab (883)	c (901)	ac (815)	bc (857)	abc (889)			
A	-	+	-	+	-	+	-	+	3674	3517	770.2815
B	-	-	+	+	-	-	+	+	3632	3559	166.5315
AB	+	-	-	+	+	-	-	+	3532	3659	504.0315
C	-	-	-	-	+	+	+	+	3729	3462	2227.7815
AC	+	-	+	-	-	+	-	+	3571	3620	75.0321
BC	+	+	-	-	-	-	+	+	3529	3662	552.7815
ABC	-	+	+	-	+	-	-	+	3541	3650	371.2815

Total SS = $\sum\sum y_{ij}^2 - $ C.T.

= 1667273 – 1615952.531

= 51320.469

Block SS = $\frac{\sum B_i^2}{8} -$ C.T.

= 1648665.54 $-$ 1615952.531

= 32712.869

Treatment SS = $\frac{\sum T_j^2}{4} -$ C.T.

= 1620620.25 $-$ 1615952.531

= 4667.719

Error SS = Total SS – Block SS – Treatment SS

= 51320.469 – 32712.869 – 4667.719

= 13939.881

Table 4.5.3 ANOVA Table

Source of Variation	DF	SS	M.S.	F_c	P-value
Blocks	4-1=3	32712.869			
Treatments	8-1=7	4667.719			
A	1	770.2815	770.2815	1.160405	0.29360
B	1	166.5315	166.5315	0.250874	0.62167
AB	1	504.0315	504.0315	0.759307	0.39340
C	1	2227.7815	2227.7815	3.356083	0.08118
AC	1	75.0321	75.0321	0.113032	0.74005
BC	1	552.7815	552.7815	0.83275	0.37184
ABC	1	371.2815	371.2815	0.559324	0.46283
Error	21	19939.88	663.8039		
Total	31				

Test of hypothesis:

We want to test the following hypothesis

H_0 : All the main effects and interaction effects are insignificant.

H_1 : They are significant.

Level of significance: We want to test our hypothesis at 5% level of significance i.e. $\alpha = 0.05$.

Comment:

Since here p-values of all main effects and interaction effects are greater than the level of significance, α, so we may accept the null hypothesis. Hence, we can conclude that observed values of all main effects and interaction effects are insignificant.

4.6 A comparative Study

In Yates method sum of total yields of each successive non-overlapping pairs of adjacent entities of response column is recorded in the 1st half of column 1 and 2nd half of column 1 is obtained by subtracting the first entry from the second of each non-overlapping pairs. Column 2 is obtained from column 1 in exactly the same manner of sum and difference of pairs. For a 2^n factorial experiment, the process will be repeated n times, until column n is obtained. So it is very difficult and time consuming technique. P. K. Batra and Seema Jaggi follow the same manner as Yates method. In Yates method, main effects total and interactions total are obtained in n^{th} column and P. K. Batra and Seema Jaggi calculate these total in a line. To calculate factorial effects and their SS they use the same rule as in Yates method. However, in this approach we have to sum treatment yields with negative sign for each row (denoted by x_0) and sum treatment yields with positive sign for each row (denoted by x_1), i.e. $x_1 = G - x_0$. So this approach is easier and less time consuming than the other methods.

4.7 Summary

In this chapter, a method of analyzing 2^n factorial experiment has been introduced. It is easier and less time consuming than any other method available for analyzing 2^n factorial experiment. This method is appropriate in general for any value of n, the number of factors. However, this method is restricted to p^n factorial experiments when $p = 2$.

Chapter 5

A General Method of Analyzing p^n Factorial Experiments

5.1 Introduction

In factorial experiments, we can analyze a p^n factorial experiment when p is prime. For 2^n factorial experiment we use Yates' method and the analysis of other factorial experiments is performed by using I-totals and J-totals. A method of analyzing 2^n and 3^n factorial experiment is also proposed by P.K. Batra and Seema Jaggi. There is no general method of analyzing p^n factorial experiments when p is a natural number. As such, in this chapter, we will propose a general method of analyzing p^n factorial experiments, when p is a natural number. It will be accomplished by using the solutions of symbolic equations. It is easier to calculate and less time consuming.

5.2 Illustration of the previous method with example

A 3^3 factorial experiment with three different exams of Lidocaine (A), three dosage levels (B) and three days (C) are used in this experiment. The experiment is r with two replications.

Analysis of data:

Lidocaine brand	Dosage strength	Replicate - 1			Replicate - 2		
		Day			Day		
		0	1	2	0	1	2
0	0	86	84	85	84	85	86
	1	94	99	98	95	97	90
	2	101	106	98	105	104	103
1	0	85	84	86	80	82	84
	1	95	98	97	93	99	95
	2	108	114	109	110	102	100
2	0	84	83	81	83	80	79
	1	95	97	93	92	96	93
	2	105	100	106	102	111	108

Solution: Here,

Level combination	Treatment total	Level combination	Treatment total	Level combination	Treatment total
000	170	001	169	002	171
010	189	011	196	012	188
020	206	021	210	022	201
100	165	101	166	102	170
110	188	111	197	112	192
120	218	121	216	122	209
200	167	201	163	202	160
210	187	211	193	212	186
220	207	221	211	222	214

Replicate total

Sum of Replicate-1, $R_1 = 2571$

Sum of Replicate-2, $R_2 = 2538$

Grand total, $G = \sum\sum\sum y_{ijk} = 5109$

Correction Term (C.T.) = $\frac{G^2}{3^n.r}$

$$= \frac{(5109)^2}{3^3.2}$$

$$= 483368.1667$$

Total SS = $\sum\sum\sum y_{ijk}^2 - C.T.$

$$= 488133 - 483368.1667$$

$$= 4768.8333$$

Replicate SS = $\frac{R_1^2+R_2^2}{3^n} - C.T.$

$$= \frac{(2571)^2+(2538)^2}{3^3} - 483368.1667$$

$$= 483388.3333 - 483368.1667$$

$$= 20.167$$

Treatment SS:

We construct the following table

Table 5.2.1 A × B Table

A \ B	0	1	2	Total
0	510	573	617	1700
1	501	577	643	1721
2	490	566	632	1688
Total	1501	1716	1892	5109

$$SS(A) = \frac{A_0^2 + A_1^2 + A_2^2}{18} - C.T.$$

$$= \frac{(1700)^2+(1721)^2+(1688)^2}{18} - 483368.1667$$

$$= 30.99$$

$$SS(B) = \frac{B_0^2 + B_1^2 + B_2^2}{18} - C.T.$$

$$= \frac{(1501)^2 + (1716)^2 + (1892)^2}{18} - 483368.1667$$

$$= 4260.778$$

Total SS for $AB = \frac{\sum\sum y_{ij}^2}{6} - C.T.$

$$= 4361.333$$

$SS(AB) =$ Total $SS(AB) - SS(A) - SS(B)$

$$= 4361.333 - 30.99 - 4260.778$$

$$= 69.5653$$

Now we consider B × C Table

Table 5.2.2 B × C Table

B \ C	0	1	2	Total
0	502	498	501	1501
1	564	586	566	1716
2	631	637	624	1892
Total	1697	1721	1691	5109

$$SS(C) = \frac{C_0^2 + C_1^2 + C_2^2}{18} - C.T.$$

$$= \frac{(1697)^2 + (1721)^2 + (1691)^2}{18} - 483368.1667$$

$$= 27.99$$

Total SS for $BC = \frac{\sum\sum y_{ij}^2}{6} - C.T.$

$$= 487693.8333 - 483368.1667$$

$$= 4325.667$$

$SS(BC) = \text{Total } SS(BC) - SS(B) - SS(C)$

$= 4325.667 - 4260.778 - 27.99$

$= 36.8986$

Now we consider B × C Table

Table 5.2.3 A × C Table

A \ C	0	1	2	Total
0	565	575	560	1700
1	571	579	571	1721
2	561	567	560	1688
Total	1697	1721	1691	5109

Total SS for $AC = \frac{\sum\sum y_{ij}^2}{6} - C.T.$

$= 483430.5 - 483368.1667$

$= 62.33$

$SS(AC) = \text{Total } SS(AC) - SS(A) - SS(C)$

$= 62.33 - 30.99 - 27.99$

$= 3.35$

Now we consider $A \times$ B Table with C_0 fixed

Table 5.2.4 $A \times$ B Table with C_0 fixed

A \ B	0	1	2
0	170	189	206
1	165	188	218
2	167	187	207

So, the I-totals and J-totals are

$$I_1 = 170 + 188 + 207 = 565$$
$$I_2 = 165 + 187 + 206 = 558$$
$$I_3 = 167 + 189 + 218 = 574$$

and

$$J_1 = 170 + 218 + 187 = 575$$
$$J_2 = 189 + 165 + 207 = 561$$
$$J_3 = 206 + 188 + 167 = 561$$

Now we consider $A \times$ B Table with C_1 fixed

Table 5.2.5 $A \times$ B Table with C_1 fixed

A \ B	0	1	2
0	169	196	210
1	166	197	216
2	163	193	211

So, the I-totals and J-totals are

$$I_1' = 169 + 197 + 211 = 577$$
$$I_2' = 166 + 193 + 210 = 569$$
$$I_3' = 163 + 196 + 216 = 575$$

and

$$J_1' = 169 + 216 + 193 = 578$$
$$J_2' = 196 + 166 + 211 = 573$$
$$J_3' = 210 + 197 + 163 = 570$$

Now we consider $A \times$ B Table with C_2 fixed

Table 5.2.6 $A \times$ B Table with C_2 fixed

A \ B	0	1	2
0	171	188	201
1	170	192	209
2	160	186	214

So, the I-totals and J-totals are

$$I_1'' = 171 + 192 + 214 = 577$$
$$I_2'' = 170 + 186 + 201 = 557$$
$$I_3'' = 160 + 188 + 209 = 557$$

and

$$J_1'' = 171 + 209 + 186 = 566$$
$$J_2'' = 188 + 170 + 214 = 572$$
$$J_3'' = 201 + 192 + 160 = 553$$

Now we can write I table as

$$\begin{bmatrix} 565 & 577 & 577 \\ 558 & 569 & 557 \\ 574 & 575 & 557 \end{bmatrix}$$

So, the I-totals and J-totals of I table are

$$(II)_0 = 565 + 569 + 557 = 1691$$
$$(II)_1 = 558 + 575 + 577 = 1710$$
$$(II)_2 = 574 + 577 + 557 = 1708$$

and

$$(IJ)_0 = 565 + 557 + 575 = 1697$$
$$(IJ)_1 = 577 + 558 + 557 = 1692$$
$$(IJ)_2 = 577 + 569 + 574 = 1720$$

Now we can write J table as

$$\begin{bmatrix} 575 & 578 & 566 \\ 561 & 573 & 572 \\ 561 & 570 & 553 \end{bmatrix}$$

So, the I-totals and J-totals of I table are

$$(JI)_0 = 575 + 573 + 553 = 1701$$
$$(JI)_1 = 561 + 570 + 566 = 1697$$
$$(JI)_2 = 561 + 578 + 572 = 1711$$

and

$$(JJ)_0 = 575 + 572 + 570 = 1717$$
$$(JJ)_1 = 578 + 561 + 553 = 1692$$
$$(JJ)_2 = 566 + 573 + 561 = 1700$$

$$SS(ABC) = \frac{(JJ)_0^2 + (JJ)_1^2 + (JJ)_2^2}{9.r} - C.T.$$

$$= \frac{(1717)^2+(1692)^2+(1700)^2}{9.2} - 483368.1667$$

$$= 18.111$$

$$SS(ABC^2) = \frac{(JI)_0^2 + (JI)_1^2 + (JI)_2^2}{9.r} - C.T.$$

$$= \frac{(1701)^2+(1697)^2+(1711)^2}{9.2} - 483368.1667$$

$$= 5.778$$

$$SS(AB^2C) = \frac{(IJ)_0^2 + (IJ)_1^2 + (IJ)_2^2}{9.r} - C.T.$$

$$= \frac{(1697)^2+(1692)^2+(1720)^2}{9.2} - 483368.1667$$

$$= 24.778$$

$$SS(AB^2C^2) = \frac{(II)_0^2 + (II)_1^2 + (II)_2^2}{9.r} - C.T.$$

$$= \frac{(1691)^2+(1710)^2+(1708)^2}{9.2} - 483368.1667$$

$$= 12.111$$

Treatment SS = $SS(A) + SS(B) + SS(C) + SS(AB) + SS(BC) + SS(AC) + SS(ABC) + SS(ABC^2) + SS(AB^2C) + SS(AB^2C^2)$

$$= 30.99 + 4260.778 + 27.99 + 69.5653 + 36.8986 + 3.35 + 18.111 + 5.778 + 24.778 + 12.111$$

$$= 4490.3499$$

Error SS = Total SS –Replicate SS – Treatment SS

$$= 4768.8333 - 20.167 - 4490.3499$$

$$= 258.32$$

Table 5.2.7 ANOVA Table

Source of Variation	DF	SS	MS	F_c	P-value
Replicate	$2-1=1$	20.167	20.167		
Treatment	$27-1=26$	4490.333	172.705		
A	2	30.99	15.495	1.5596	0.22926
B	2	4260.778	2130.389	214.4327	0.00000
C	2	27.99	13.995	1.4087	0.26251
AB	4	69.5653	17.391	1.7505	0.16928
AC	4	3.35	0.8375	0.0843	0.98653
BC	4	36.8986	9.22465	1.43982	0.24908
ABC	2	18.111	9.0556	0.9285	0.40786
ABC^2	2	5.778	2.889	0.2908	0.75007
AB^2C	2	24.778	12.389	1.247	0.30399
AB^2C^2	2	12.111	6.056	0.6096	0.55116
Error	26	258.32	9.935		
Total	$54-1=53$	4768.833			

Test of hypothesis:

We want to test the following hypothesis

H_0 : All the main effects and interactions are insignificant.

H_1 : They are significant.

Level of significance: We want to test our hypothesis at 5% level of significance i.e. $\alpha = 0.05$.

Comment:

Since here p-values of all main effects and interaction effects (except B) are greater than the level of significance, α, so we may accept the null hypothesis. Hence we conclude that observed values of all main effects and interaction effects (except B) are insignificant.

For B, p-value is less than the level of significance, α, so we may reject the null hypothesis. That is, we can say that this main effect is significant.

5.3 Proposed general method of analyzing p^n factorial experiments

In this section, we are going to propose a general method of analyzing p^n factorial experiments. This method is easy to apply and also less time consuming than the previously described methods.

5.3.1 Algorithm

The routine procedure for obtaining Sum of Squares (SS) of the factorial effects in a p^n factorial experiment consists of the following steps:

Step 1. In the first column of the table, we will have to write all the main effects and interaction effects (Table 5.3.1).

Step 2. In the second column, we will have to insert the left hand side (LHS) of the symbolic equations of the corresponding factorial effects (Table 5.3.1).

Step 3. All treatment combinations with their corresponding total yields will have to be listed in the first row of the third column. Thus, we will have p^n sub-columns beneath this row (Table 5.3.1).

Step 4. These p^n sub-columns will have to be filled with the solutions obtained by solving the LHS of the symbolic equations in mod p, with each treatment combination. The solutions are written in the corresponding intersection points (Table 5.3.1).

Step 5. Thus, these treatment combinations, in each row, are classified into p parts. We will have to fill the next p columns with,

Sum of treatment yields with value equal 0

Sum of treatment yields with value equal 1

………………………………………………..

Sum of treatment yields with value equal $(p-1)$.

We will denote these p columns by $x_0, x_1, \dots, x_{(p-1)}$respectively (Table 5.3.1).

Step 6. Finally, we will be able to obtain the sum of squares (SS) for each factorial effect by using the following formula,

$$\frac{x_0^2 + x_1^2 + \cdots\cdots + x_{p-1}^2}{p^{n-1}.r} - C.T.$$

where,

r is the number of replicates.

$C.T.$ is the correction term $= \frac{G^2}{p^n.r}$

G = Grand total

Step 7. We can calculate the Sum of Squares (SS) due to treatments, replications, rows and columns, total SS, and error SS by using previously established procedures.

Using above algorithm we can construct the following table for calculating SS of the factorial effects in a 3^2 factorial experiment.

Table 5.3.1 Construction of table for calculating Sum of Square (SS) of factorial effects

Factorial effects	LHS of the symbolic equation mod 3	Treatment combinations									Sum of treat-ment yields with value 0 (x_0)	Sum of treat-ment yields with value 1 (x_1)	Sum of treat-ment yields with value 2 (x_2)	SS= $\frac{x_0^2+x_1^2+x_2^2}{3^{n-1}\times r} - C.T.$
		00 (…)	01 (…)	02 (…)	10 (…)	11 (…)	12 (…)	20 (…)	21 (…)	22 (…)				
A	x_1	0	0	0	1	1	1	2	2	2	…	…	…	…
B	x_2	0	1	2	0	1	2	0	1	2	…	…	…	…
AB	x_1+x_2	0	1	2	1	2	0	2	0	1	…	…	…	…
AB^2	x_1+2x_2	0	2	1	1	0	2	2	1	0	…	…	…	…

5.3.2 Illustration of the general method of analyzing p^n factorial experiments with example

A 3^3 factorial experiment with three different exams of Lidocaine (A), three dosage levels (B) and three days (C) are used in this experiment. The experiment is r with two replications.

Analysis of data:

Lidocaine brand	Dosage strength	Replicate - 1			Replicate - 2		
		Day			Day		
		0	1	2	0	1	2
0	0	86	84	85	84	85	86
	1	94	99	98	95	97	90
	2	101	106	98	105	104	103
1	0	85	84	86	80	82	84
	1	95	98	97	93	99	95
	2	108	114	109	110	102	100
2	0	84	83	81	83	80	79
	1	95	97	93	92	96	93
	2	105	100	106	102	111	108

Solution:

Treatment totals

T_{000} = 170	T_{100} = 165	T_{200} = 167
T_{001} = 169	T_{101} = 166	T_{201} = 163
T_{002} = 171	T_{102} = 170	T_{202} = 160
T_{010} = 189	T_{110} = 188	T_{210} = 187
T_{011} = 196	T_{111} = 197	T_{211} = 193
T_{012} = 188	T_{112} = 192	T_{212} = 186
T_{020} = 206	T_{120} = 218	T_{220} = 207
T_{021} = 210	T_{121} = 216	T_{221} = 211
T_{022} = 201	T_{122} = 209	T_{222} = 214

Replicate total

Sum of Replicate-1, $R_1 = 2571$

Sum of Replicate-2, $R_2 = 2538$

Grand total, $G = \sum\sum\sum y_{ijk}$

$= 5109$

Correction Term (C.T.) $= \frac{G^2}{3^n.r}$

$= \frac{(5109)^2}{3^3.2}$

$= 483368.1667$

Here, $p = 3$ and $n = 3$. So, there will be $\frac{p^n-1}{p-1} = \frac{3^3-1}{3-1} = \frac{27-1}{2} = \frac{26}{2} = 13$ main effects and interactions and $p^n = 3^3 = 27$ treatment combinations. These 13 main effects and interaction effects are written in the first column of Table 5.3.2. The left hand sides (LHS) of the symbolic equations of the corresponding main effects and interaction effects are inserted in the second column. As for example, for interaction AB^2C^2, left hand side (LHS) of the symbolic equation is $x_1 + 2x_2 + 2x_3$, which is inserted in the second column.

27 treatment combinations in matrix method with their treatment totals are listed in the first row of the next $3^3 = 27$ sub-columns. These 27 columns are filled with the solutions that are obtained by solving the LHS of the symbolic equations with each treatment combination. The solutions are written in the corresponding intersection points. As for example, for interaction AB^2C^2 left hand side (LHS) of the symbolic equation is $x_1 + 2x_2 + 2x_3$.

For treatment combination 000 if we solve left hand side (LHS) of the symbolic equation $x_1 + 2x_2 + 2x_3$, we get $0 + 2 \times 0 + 2 \times 0 = 0$ mod 3. This solution is written in the intersection point of treatment combination 000 and left hand side (LHS) of the symbolic equation $x_1 + 2x_2 + 2x_3$.

For treatment combination 001 if we solve left hand side (LHS) of the symbolic equation $x_1 + 2x_2 + 2x_3$, we get $0 + 2 \times 0 + 2 \times 1 = 2$ mod 3. This solution is written in the intersection point of treatment combination 001 and left hand side (LHS) of the symbolic equation $x_1 + 2x_2 + 2x_3$.

These 27 columns are filled similarly.

Next $p = 3$ columns are filled with

Sum of treatment yields with value equal 0

Sum of treatment yields with value equal 1

Sum of treatment yields with value equal 2

These $p = 3$ columns are denoted by x_0, x_1, and x_2 respectively.

For interaction AB^2C^2,

$$x_0 = 170 + 188 + 210 + 166 + 188 + 209 + 160 + 193 + 207 = 1691$$

Similarly we get x_1 and x_2.

And in the last column we write Sum of Squares (SS) of factorial effects.

Table: 5.3.2

Construction of table for calculating Sum of Square (SS) of factorial effects:

Factor-ial effects	LHS of the symbo-lic Equati-on mod 3	Treatment combinations																											Sum of treat-ment yiel-ds with value 0 (x_0)	Sum of treat-ment yiel-ds with value 1 (x_1)	Sum of treat-ment yiel-ds with value 2 (x_2)	SS= $\frac{x_0^2+x_1^2+x_2^2}{3^{n-1}\times r} - C.T.$
		000 = 170	001 = 169	002 = 171	010 = 189	011 = 196	012 = 188	020 = 206	021 = 210	022 = 201	100 = 165	101 = 166	102 = 170	110 = 188	111 = 197	112 = 192	120 = 218	121 = 216	122 = 209	200 = 167	201 = 163	202 = 160	210 = 187	211 = 193	212 = 186	220 = 207	221 = 211	222 = 214				
A	x_1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	1700	1721	1688	30.99
B	x_2	0	0	0	1	1	1	2	2	2	0	0	0	1	1	1	2	2	2	0	0	0	1	1	1	2	2	2	1501	1716	1892	4260.778
C	x_3	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	1697	1721	1691	27.99
AB	$x_1 + x_2$	0	0	0	1	1	1	2	2	2	1	1	1	2	2	2	0	0	0	2	2	2	0	0	0	1	1	1	1719	1706	1684	34.778
AB^2	$x_1 + 2x_2$	0	0	0	2	2	2	1	1	1	1	1	1	0	0	0	2	2	2	2	2	2	1	1	1	0	0	0	1719	1684	1706	34.778
AC	$x_1 + x_3$	0	1	2	0	1	2	0	1	2	1	2	0	1	2	0	1	2	0	2	0	1	2	0	1	2	0	1	1703	1706	1700	0.99997
AC^2	$x_1 + 2x_3$	0	2	1	0	2	1	0	2	1	1	0	2	1	0	2	1	0	2	2	1	0	2	1	0	2	1	0	1704	1698	1707	2.333
BC	$x_2 + x_3$	0	1	2	1	2	0	2	0	1	0	1	2	1	2	0	2	0	1	0	1	2	1	2	0	2	0	1	1705	1686	1718	28.778
BC^2	$x_2 + 2x_3$	0	2	1	1	0	2	2	1	0	0	2	1	1	0	2	2	1	0	0	2	1	1	0	2	2	1	0	1712	1702	1695	8.111
ABC	$x_1 + x_2 + x_3$	0	1	2	1	2	0	2	0	1	1	2	0	2	0	1	0	1	2	2	0	1	0	1	2	1	2	0	1717	1692	1700	18.111
ABC^2	$x_1 + x_2 + 2x_3$	0	2	1	1	0	2	2	1	0	1	0	2	2	1	0	0	2	1	2	1	0	0	2	1	1	0	2	1701	1697	1711	5.778
AB^2C	$x_1 + 2x_2 + x_3$	0	1	2	2	0	1	1	2	0	1	2	0	0	1	2	2	0	1	2	0	1	1	2	0	0	1	2	1697	1692	1720	24.778
AB^2C^2	$x_1 + 2x_2 + 2x_3$	0	2	1	2	1	0	1	0	2	1	0	2	0	2	1	2	1	0	2	1	0	1	0	2	0	2	1	1691	1710	1708	12.111

Total SS = $\sum\sum\sum y_{ijk}^2 \ - C.T.$

$= 488133 - \ 483368.1667 = 4768.8333$

Replicate SS = $\frac{R_1^2+R_2^2}{3^n} \ - C.T.$

$= \frac{(2571)^2+(2538)^2}{3^3} - \ 483368.1667$

$= 483388.3333 - 483368.1667$

$= 20.167$

Treatment SS = $\frac{\sum_{i=1}^{27} T_i^2}{r} - \ C.T.$

$= \frac{975717}{2} - \ 483368.1667$

$= 4490.333$

Error SS = Total SS –Replicate SS – Treatment SS

$= 4768.8333 - 20.167 - 4490.33$

$= 258.333$

Table 5.3.3 ANOVA Table

Source of Variation	DF	SS	MS	F_c	P-value
Replicate	$2-1=1$	20.167	20.167		
Treatment	$27-1=26$	4490.333	172.705		
A	2	30.99	15.495	1.55049	0.23113
B	2	4260.778	2130.389	213.17533	0.00000
C	2	27.99	13.995	1.4004	0.26448
AB	2	34.778	17.389	1.740014	0.19534
AB^2	2	34.778	17.389	1.740014	0.19534
AC	2	0.99997	0.4999	0.05002	0.95130
AC^2	2	2.333	1.1667	0.11674	0.89028
BC	2	28.778	14.389	1.43982	0.25525
BC^2	2	8.111	4.0556	0.40582	0.67058
ABC	2	18.111	9.0556	0.90614	0.41646
ABC^2	2	5.778	2.889	0.28909	0.75132
AB^2C	2	24.778	12.389	1.23969	0.30602
AB^2C^2	2	12.111	6.056	0.60599	0.55306
Error	26	258.333	9.9936		
Total	$54-1=53$	4768.833			

Test of hypothesis:

We want to test the following hypothesis

H_0 : All the main effects and interactions are insignificant.

H_1 : They are significant.

Level of significance: We want to test our hypothesis at 5% level of significance i.e. $\alpha = 0.05$.

Comment:

Since here p-values of all main effects and interaction effects (except B) are greater than the level of significance, α, so we may accept the null hypothesis. Hence we conclude that observed values of all main effects and interaction effects (except B) are insignificant.

For B, p-value is less than the level of significance, α, so we may reject the null hypothesis. That is, we can say that this main effect is significant.

5.4 A comparative study

Previously, there was no general method of analyzing p^n factorial experiments, when p is a natural number. Only for p being prime, methods of analyzing p^n factorial experiments existed. 2^n factorial experiment is still analyzed using Yates' method and other p^n (p is prime) factorial experiments are still analyzed using I-totals and J-totals. These methods are difficult and time consuming. In this study we propose a general method of analyzing p^n factorial experiments, when p is a natural number. Also it is easy to calculate and less time consuming.

5.5 Summary

In this chapter, we have introduced a general method of analyzing p^n factorial experiments. It is easier and rewarding than any method available in analyzing p^n (p is prime) factorial experiments. The method is appropriate in general for any value of n and for any possible value of p as well, where, n is the number of factors and p is the levels of the factors. However, this method is restricted to p^n symmetrical factorial experiments only.

Chapter 6

A General Method of Constructing Layout with Single Factorial Effect Confounded in p^n Factorial Experiments

6.1 Introduction

A method to construct a plan for a factorial effect to be confounded in a p^n factorial experiment (p is prime) was proposed in 1990 (Jalil *et. al* 1990). This method had some notational mistakes and also it was possible only to construct the plan when p is prime. In this chapter, we will make corrections of these notational mistakes and also make moderation using which we can construct a plan for a factorial effect to be confounded in a p^n factorial experiment, when p is a natural number.

6.2 Method of constructing confounding plan with single factorial effect in p^n (p is prime) factorial experiments

For constructing confounding plan with single factorial effect in p^n (p is prime) factorial experiment, an easy method was developed in 1990 (Jalil *et. al*, 1990). This method for constructing confounding plan with single factorial effect in a p^n (p is prime) factorial experiment is described below.

6.2.1 The method

Consider a matrix M of order $p^{n-1} \times np$, which represents the construction method of a p^n f.e. confounded with a factorial effects as,

$$M = \left[M_0\ M_1 \ldots \ldots M_{p-1}\right]_{p^{n-1} \times np} \tag{6.2.1}$$

Where, M_u; $u = 0, 1, \ldots \ldots \ldots, (p-1)$ resepresent incomplete block defined as,

$M_u \ = [V_1\{p^0\},\ V_2\{p^1\}, \ldots \ldots \ldots,\ V_{n-1}\{p^{p-1}\}, a_u]_{p^{n-1} \times n}$ With

$V_i\{p^j\} = p^j \left[0I_{p^{(n-1)-i}}\ , 1I_{p^{(n-1)-i}}\ , \ldots \ldots \ldots, (p-1)I_{p^{(n-1)-i}}\ \right]$,

each is a column vector of dimension p^{n-1} .

$\{p^j\} = p^j$ times repetitions of the elements of V_i's in ascending ordered levels.

$u = 0, 1, 2, \ldots, (p-1)$; $\quad i = 1, 2, 3, \ldots, (n-1)$; $\quad j = 0, 1, 2, \ldots, (p-1)$; with the restriction that $i = j + 1$

I_m: sum vector of dimension m; and

$a_u = \left[a_{u1}, a_{u2}, \ldots, a_{up}\right]'$ is called the adjustment vector.

At $u = 0$, the adjustment vector a_0 is called the key vector and resulting incomplete block represented by this key vector is called the key incomplete block. The key vector a_0 determines other adjustment vectors a_u and hence the incomplete blocks M_u , for all $u > 0$. The elements of the key vector can be obtained by solving the symbolic equation such that $\sum_i a_i F_i + a_k F_k = 0$ mod p where F_k represents the adjustment vector.

6.3 Corrections of the notational mistakes

1. Here the adjustment vector (a_u) is defined as

   $$a_u = \left[a_{u1}, a_{u2}, \ldots, a_{up}\right]'$$

   But the adjustment vector (a_u) is a $p^{n-1} \times 1$ order matrix. So the adjustment vector (a_u) will be defined as

   $$a_u = \left[a_{u1}, a_{u2}, \ldots, a_{up^{n-1}}\right]'$$

2. Here F_k is defined as the adjustment vector. Since F_i indicates the i^th factor, so F_k will be the adjustment factor.

   Here the adjustment vector is denoted by a_u.

3. Here M_u is defined as

$M_u \ = [V_1\{p^0\}, V_2\{p^1\}, \dots \dots \dots, V_{n-1}\{p^{p-1}\}, a_u]_{p^{n-1}\times n}$

But it will be

$M_u \ = [V_1\{p^0\}, V_2\{p^1\}, \dots \dots \dots, V_{n-1}\{p^{n-2}\}, a_u]_{p^{n-1}\times n}$

4. Here we get the following notation

$$V_i\{p^j\} = p^j \left[0I_{p^{(n-1)-i}}\ , 1I_{p^{(n-1)-i}}\ , \dots \dots \dots, (p-1)I_{p^{(n-1)-i}}\ \right]$$

$i = 1, 2, 3, \dots, (n-1)$; $j = 0, 1, 2, \dots, (p-1)$; with the restriction that $i = j + 1$.

Since $i = j + 1$ that is $j = i - 1$ and $i = 1, 2, 3, \dots, (n-1)$
so $j = 0, 1, 2, \dots, (n-2)$.

It will be better to write as follows

$$V_i\{p^{i-1}\} = p^{i-1} \left[0I_{p^{(n-1)-i}}\ , 1I_{p^{(n-1)-i}}\ , \dots \dots \dots, (p-1)I_{p^{(n-1)-i}}\ \right],$$

each is a column vector of dimension p^{n-1} .

$\{p^j\} = p^j$ times repetitions of the elements of V_i's in ascending ordered levels.

where $i = 1, 2, 3, \dots, (n-1)$ $and\, j = 0, 1, 2, \dots, (n-2)$; with the restriction that $i = j + 1$.

6.4 Method of constructing layout with single factorial effect confounded after correction

Consider a matrix M of order $p^{n-1} \times np$, which represents the construction method of a p^n f.e. confounded with a factorial effects as,

$$M = \left[M_0\ M_1 \dots \dots M_{p-1}\right]_{p^{n-1}\times np} \tag{6.4.1}$$

Where, M_u; $u = 0, 1, \dots \dots \dots, (p-1)$ resepresent incomplete block defined as,

$M_u \ = [V_1\{p^0\}, V_2\{p^1\}, \dots \dots \dots, V_{n-1}\{p^{n-2}\}, a_u]_{p^{n-1}\times n}$ With

$$V_i\{p^{i-1}\} = p^{i-1} \left[0I_{p^{(n-1)-i}}\ , 1I_{p^{(n-1)-i}}\ , \dots \dots \dots, (p-1)I_{p^{(n-1)-i}}\ \right],$$

each is a column vector of dimension p^{n-1} .

$\{p^{i-1}\} = p^{i-1}$ times repetitions of the elements of V_i's in ascending ordered levels.

$u = 0, 1, 2, \dots, (p-1)$; $i = 1, 2, 3, \dots, (n-1)$;

I_m: sum vector of dimension m; and

$a_u = \left[a_{u1}, a_{u2}, \dots, a_{up^{n-1}}\right]'$ is called the adjustment vector.

At $u = 0$, the adjustment vector a_0 is called the key vector and resulting incomplete block represented by this key vector is called the key incomplete block. The key vector a_0 determines other adjustment vectors a_u and hence the incomplete blocks M_u , for all $u > 0$. The elements of the key vector can be obtained by solving the symbolic equation such that $\sum_i a_i F_i + a_k F_k = 0 \bmod p$ where F_k represents the adjustment factor.

6.5 Limitation of this method

This method is applicable in a p^n factorial experiment only if p is prime. When p is not a prime number, we cannot apply this method. As for example, suppose the factorial effect $F_1F_2F_3F_4$ is confounded in a 3^4 factorial experiment. In this case we can apply this method, because here $p = 3$ which is a prime number, but if we want to use this method to confound $F_1F_2F_3^2$ in 4^3 factorial experiment considering F_3 as adjustment factor, we cannot solve the following equation

$$\sum_i a_i F_i + a_k F_k = 0 \bmod p,$$

where, F_k represents the adjustment factor.

6.6 Illustration with example

Suppose the factorial effect $F_1F_2F_3F_4$ is to be confounded in a 3^4 factorial experiment. Here, $n = 4$; $F_1,\ F_2,\ F_3\ and\ F_4$; $p = 3$; $0, 1\ and\ 2$.

From the equation, $M = \left[M_0\ M_1 \dots \dots M_{p-1}\right]_{p^{n-1} \times np}$, we have in this case, The suggested matrix is,

$$M = [M_0\ M_1\ M_2]_{3^{4-1} \times 4*3} = [M_0\ M_1\ M_2]_{27\times 12}$$

where, $M_u \ = [V_1\{p^0\}, V_2\{p^1\}, \dots \dots \dots, V_{n-1}\{p^{n-2}\}, a_u]_{p^{n-1}\times n}$ with

$V_i\{p^{i-1}\} = p^{i-1}\left[0I_{p^{(n-1)-i}}\ , 1I_{p^{(n-1)-i}}\ , \dots \dots \dots, (p-1)I_{p^{(n-1)-i}}\ \right]$,

each is a column vector of dimension p^{n-1} . Therefore, we have,

$M_0 = [V_1\{3^0\}, V_2\{3^1\}, V_2\{3^2\}, a_0]_{\ 3^3\times 4}$;

$M_1 = [V_1\{3^0\}, V_2\{3^1\}, V_2\{3^2\}, a_1]_{\ 3^3\times 4}$;

$M_2 = [V_1\{3^0\}, V_2\{3^1\}, V_2\{3^2\}, a_2]_{\ 3^3\times 4}$

with

$v_1\{1\} = 1[0I_9, 1I_9, 2I_9]\,'_{27\times 1}$

$v_2\{3\} = 3[0I_3, 1I_3, 2I_3]\,'_{27\times 1}$

$v_3\{9\} = 9[0I_1, 1I_1, 2I_1]\,'_{27\times 1}$

Now the elements of the adjustment vector are obtained from the equation $\sum_i a_i F_i + a_k F_k = 0 \text{ mod } p$, considering $F_k = F_3$ we have,

$$v_1\{1\} = \begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\1\\1\\1\\1\\1\\1\\1\\1\\2\\2\\2\\2\\2\\2\\2\\2\\2\end{bmatrix} \quad v_2\{3\} = \begin{bmatrix}0\\0\\0\\1\\1\\1\\2\\2\\2\\0\\0\\0\\1\\1\\1\\2\\2\\2\\0\\0\\0\\1\\1\\1\\2\\2\\2\end{bmatrix} \quad v_3\{9\} = \begin{bmatrix}0\\1\\2\\0\\1\\2\\0\\1\\2\\0\\1\\2\\0\\1\\2\\0\\1\\2\\0\\1\\2\\0\\1\\2\\0\\1\\2\end{bmatrix} \quad \text{and hence, } a_0 = \begin{bmatrix}0\\2\\1\\2\\1\\0\\1\\0\\2\\2\\1\\0\\1\\0\\2\\0\\2\\1\\1\\0\\2\\0\\2\\1\\2\\1\\0\end{bmatrix}$$

[Solving $\sum_i a_i F_i + a_k F_k = 0 \bmod p$]

Therefore

$$M_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 1 \\ 2 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 2 & 0 & 2 & 2 \\ 2 & 1 & 0 & 0 \\ 2 & 1 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 2 & 2 & 0 & 2 \\ 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 0 \end{bmatrix}, M_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 1 & 2 & 2 & 2 \\ 2 & 0 & 0 & 2 \\ 2 & 0 & 1 & 1 \\ 2 & 0 & 2 & 0 \\ 2 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 2 & 1 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 2 & 1 & 2 \\ 2 & 2 & 2 & 1 \end{bmatrix} \text{ and } M_2 = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 0 & 0 \\ 0 & 2 & 1 & 2 \\ 0 & 2 & 2 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 2 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 0 & 2 \\ 1 & 2 & 1 & 1 \\ 1 & 2 & 2 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & 1 & 2 \\ 2 & 0 & 2 & 1 \\ 2 & 1 & 0 & 2 \\ 2 & 1 & 1 & 1 \\ 2 & 1 & 2 & 0 \\ 2 & 2 & 0 & 1 \\ 2 & 2 & 1 & 0 \\ 2 & 2 & 2 & 2 \end{bmatrix}$$

So the desired plan is

$$M = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 \\
0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 2 & 1 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 0 \\
0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 2 & 0 & 1 & 1 & 0 \\
0 & 1 & 2 & 0 & 0 & 1 & 2 & 1 & 0 & 1 & 2 & 2 \\
0 & 2 & 0 & 1 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 \\
0 & 2 & 1 & 0 & 0 & 2 & 1 & 1 & 0 & 2 & 1 & 2 \\
0 & 2 & 2 & 2 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 1 \\
1 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & 1 & 2 & 1 & 0 & 1 & 0 \\
1 & 0 & 2 & 0 & 1 & 0 & 2 & 1 & 1 & 0 & 2 & 2 \\
1 & 1 & 0 & 1 & 1 & 1 & 0 & 2 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\
1 & 1 & 2 & 2 & 1 & 1 & 2 & 0 & 1 & 1 & 2 & 1 \\
1 & 2 & 0 & 0 & 1 & 2 & 0 & 1 & 1 & 2 & 0 & 2 \\
1 & 2 & 1 & 2 & 1 & 2 & 1 & 0 & 1 & 2 & 1 & 1 \\
1 & 2 & 2 & 1 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 0 \\
2 & 0 & 0 & 1 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 2 & 0 & 1 & 1 & 2 & 0 & 1 & 2 \\
2 & 0 & 2 & 2 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 1 \\
2 & 1 & 0 & 0 & 2 & 1 & 0 & 1 & 2 & 1 & 0 & 2 \\
2 & 1 & 1 & 2 & 2 & 1 & 1 & 0 & 2 & 1 & 1 & 1 \\
2 & 1 & 2 & 1 & 2 & 1 & 2 & 2 & 2 & 1 & 2 & 0 \\
2 & 2 & 0 & 2 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 1 \\
2 & 2 & 1 & 1 & 2 & 2 & 1 & 2 & 2 & 2 & 1 & 0 \\
2 & 2 & 2 & 0 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2
\end{bmatrix}$$

Let us consider the factorial effect $F_1F_2F_3^2$ is to be confounded in a 4^3 factorial experiment.

Here, $n = 3$; F_1, F_2 *and* F_3; $p = 4$; $0, 1, 2$ *and* 3

From the equation, $M = \left[M_0\ M_1\ \dots\dots M_{p-1}\right]_{p^{n-1}\times np}$, we have in this case,

$M = [M_0\ \ M_1\ \ M_2\ \ M_3\ ]_{4^{3-1}\times 3*4} = [M_0\ \ M_1\ \ M_2\ \ M_3\ ]_{16\times 12}$;

where, $M_u\ = [V_1\{p^0\}, V_2\{p^1\}, \dots\dots\dots, V_{n-1}\{p^{n-2}\}, a_u]_{p^{n-1}\times n}$ with

$V_i\{p^{i-1}\} = p^{i-1}\left[0I_{p^{(n-1)-i}}\ , 1I_{p^{(n-1)-i}}\ , \dots\dots\dots, (p-1)I_{p^{(n-1)-i}}\ \right]$,

each is a column vector of dimension p^{n-1} . Therefore, we have,

$M_0\ = [V_1\{4^0\}, V_2\{4^1\}, a_0]\ _{4^2\times 3}$; $M_1\ = [V_1\{4^0\}, V_2\{4^1\}, a_1]\ _{4^2\times 3}$;

$M_2\ = [V_1\{4^0\}, V_2\{4^1\}, a_2]\ _{4^2\times 3}$; $M_3\ = [V_1\{4^0\}, V_2\{4^1\}, a_3]\ _{4^2\times 3}$;

with

$V_1\{4^0\} = 1\left[0I_4\,, 1I_4\ \,, 2I_4\,, 3I_4\,,\right]_{16\times1}, V_2\{4^1\} = 4[0I_1\,, 1I_1\,, 2I_1\,, 3I_1\,]_{16\times1}$

Thus,

$$V_1\{4^0\} = \begin{bmatrix}0\\0\\0\\0\\1\\1\\1\\1\\2\\2\\2\\2\\3\\3\\3\\3\end{bmatrix} \quad V_2\{4^1\} = \begin{bmatrix}0\\1\\2\\3\\0\\1\\2\\3\\0\\1\\2\\3\\0\\1\\2\\3\end{bmatrix}$$

Now a_u is column vector of dimension 16, i.e.

$$a_u = [a_{u1}, a_{u2}, \dots, a_{u16}]'$$

So to get the elements of the key adjustment vector, a_0, we have to solve the following equation

$$x_1 + x_2 + 2x_3 = 0\ (mod\ 4)$$

Here we consider F_3 as adjustment factor. Thus we get,

$$0 + 0 + 2x_3 = 0\ (mod\ 4) ==> a_{01} = 0$$

$$0 + 1 + 2x_3 = 0\ (mod\ 4)$$

We cannot calculate a_{02}. Thus we cannot calculate the key adjustment vector, a_0.

The reason is discussed in the next section.

6.7 The reason

We know that if we multiply an even number by any natural number the result will be an even number. For this reason if we take adjustment factor, F_k, whose exponent is an even number, we cannot calculate the key adjustment vector a_0. So we have to take adjustment factor F_k whose exponent is odd number.

6.8 Moderation

In this method, proposed by Jalil *et. al* (1990), we have to take that factor as adjustment factor whose exponent is odd. If there is more than one factor whose exponents are odd, then any one can be considered as adjustment factor. After moderating, this method can be applied in p^n factorial experiments, when p is any natural number.

6.9 Illustrate with Example

Construct a confounding plan where the confounded effect is $F_1F_2F_3^2$ in a 4^3 f.e..

Solution: Here, $n = 3;\ F_{1,}\ F_2\ and\ F_3; p = 4;\ 0, 1, 2\ and\ 3$

From the equation, $M = \left[M_0\ M_1 \ldots \ldots M_{p-1}\right]_{p^{n-1} \times np}$, we have in this case,

$$M = [M_0\ \ M_1\ \ M_2\ \ M_3\ ]_{4^{3-1} \times 3*4} = [M_0\ \ M_1\ \ M_2\ \ M_3\ ]_{16 \times 12};$$

where, $M_u\ = [V_1\{p^0\}, V_2\{p^1\}, \ldots \ldots \ldots, V_{n-1}\{p^{n-2}\}, a_u]_{p^{n-1} \times n}$ with

$$V_i\{p^{i-1}\} = p^{i-1}\left[0I_{p^{(n-1)-i}}\ , 1I_{p^{(n-1)-i}}\ , \ldots \ldots \ldots, (p-1)I_{p^{(n-1)-i}}\ \right],$$

each is a column vector of dimension p^{n-1} . Therefore, we have,

$$M_0 = [V_1\{4^0\}, V_2\{4^1\}, a_0]\ _{4^2 \times 3};\ \ M_1 = [V_1\{4^0\}, V_2\{4^1\}, a_1]\ _{4^2 \times 3};$$

$$M_2 = [V_1\{4^0\}, V_2\{4^1\}, a_2]\ _{4^2 \times 3}\ ;\ \ M_3 = [V_1\{4^0\}, V_2\{4^1\}, a_3]\ _{4^2 \times 3};$$

with

$$V_1\{4^0\} = 1[0I_4\ , 1I_4\ , 2I_4\ , 3I_4\ ]_{16 \times 1}, V_2\{4^1\} = 4[0I_1\ , 1I_1\ , 2I_1\ , 3I_1\ ]_{16 \times 1}$$

Now the elements of the adjustment vector are obtained from the equation $\sum_i a_iF_i + a_kF_k = 0$ mod p where F_k represents the adjustment factor. Since the exponents of both $F_1\ and\ F_2$ are odd, anyone can be considered as adjustment factor. Here F_1 is considered as adjustment factor. Now we have

$$V_1\{4^0\} = \begin{bmatrix}0\\0\\0\\0\\1\\1\\1\\1\\2\\2\\2\\2\\3\\3\\3\\3\end{bmatrix} \quad V_2\{4^1\} = \begin{bmatrix}0\\1\\2\\3\\0\\1\\2\\3\\0\\1\\2\\3\\0\\1\\2\\3\end{bmatrix} \quad \text{and hence, } a_0 = \begin{bmatrix}0\\2\\0\\2\\3\\1\\3\\1\\2\\0\\2\\0\\1\\3\\1\\3\end{bmatrix}$$

[Solving $\sum_i a_i F_i + a_k F_k = 0 \bmod p$]

$a_1, a_2\ and\ a_3$ can be obtained by solving $a_{uk} = a_{0k} + u$ where $k = 0, 1, 2, \ldots, p^{n-1}$;

Therefore

$$M_0 = \begin{bmatrix}0&0&0\\2&0&1\\0&0&2\\2&0&3\\3&1&0\\1&1&1\\3&1&2\\1&1&3\\2&2&0\\0&2&1\\2&2&2\\0&2&3\\1&3&0\\3&3&1\\1&3&2\\3&3&3\end{bmatrix}, \quad M_1 = \begin{bmatrix}1&0&0\\3&0&1\\1&0&2\\3&0&3\\0&1&0\\2&1&1\\0&1&2\\2&1&3\\3&2&0\\1&2&1\\3&2&2\\1&2&3\\2&3&0\\0&3&1\\2&3&2\\0&3&3\end{bmatrix}, \quad M_2 = \begin{bmatrix}2&0&0\\0&0&1\\2&0&2\\0&0&3\\1&1&0\\3&1&1\\1&1&2\\3&1&3\\0&2&0\\2&2&1\\0&2&2\\2&2&3\\3&3&0\\1&3&1\\3&3&2\\1&3&3\end{bmatrix}, \quad M_3 = \begin{bmatrix}3&0&0\\1&0&1\\3&0&2\\1&0&3\\2&1&0\\0&1&1\\2&1&2\\0&1&3\\1&2&0\\3&2&1\\1&2&2\\3&2&3\\0&3&0\\2&3&1\\0&3&2\\2&3&3\end{bmatrix}$$

So the desired plan is

$$M = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 3 & 0 & 0 \\ 2 & 0 & 1 & 3 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 0 & 2 & 2 & 0 & 2 & 3 & 0 & 2 \\ 2 & 0 & 3 & 3 & 0 & 3 & 0 & 0 & 3 & 1 & 0 & 3 \\ 3 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 2 & 1 & 0 \\ 1 & 1 & 1 & 2 & 1 & 1 & 3 & 1 & 1 & 0 & 1 & 1 \\ 3 & 1 & 2 & 0 & 1 & 2 & 1 & 1 & 2 & 2 & 1 & 2 \\ 1 & 1 & 3 & 2 & 1 & 3 & 3 & 1 & 3 & 0 & 1 & 3 \\ 2 & 2 & 0 & 3 & 2 & 0 & 0 & 2 & 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 1 & 2 & 1 & 2 & 2 & 1 & 3 & 2 & 1 \\ 2 & 2 & 2 & 3 & 2 & 2 & 0 & 2 & 2 & 1 & 2 & 2 \\ 0 & 2 & 3 & 1 & 2 & 3 & 2 & 2 & 3 & 3 & 2 & 3 \\ 1 & 3 & 0 & 2 & 3 & 0 & 3 & 3 & 0 & 0 & 3 & 0 \\ 3 & 3 & 1 & 0 & 3 & 1 & 1 & 3 & 1 & 2 & 3 & 1 \\ 1 & 3 & 2 & 2 & 3 & 2 & 3 & 3 & 2 & 0 & 3 & 2 \\ 3 & 3 & 3 & 0 & 3 & 3 & 1 & 3 & 3 & 2 & 3 & 3 \end{bmatrix}$$

6.10 Summary

In this chapter, we have made some corrections of the notational mistakes in the method of constructing confounding plan with single factorial effect in p^n (p is prime) factorial experiments proposed in 1990 (Jalil *et. al* 1990). This method worked only when p is prime in p^n factorial experiments. We have made a modification in taking adjustment factor, using which we can construct confounding plan with single factorial effect in p^n factorial experiment. The method is appropriate in general for any value of n, the number of factors and for any possible value of p, the levels of the factors. However, this method is restricted to p^n symmetrical factorial experiments.

Chapter 7

A General Method of Detection of a Confounded Effect in p^n (p is prime) Factorial Experiments

7.1 Introduction

If we are given a plan of a p^n factorial experiment where a single factorial effect is confounded and then if we are to detect the confounded effect, we can do it by the manipulating method of symbolic equations considering its mod value at the level (p) of the experiment (Kemthrone, 1952). A quick and easier method of detecting the confounded effect was proposed in 1994 (Jalil *et. al* 1994). The method works nicely in detection the confounded effect for a p^n factorial experiment when p equals 2 or 3. If $p > 3$, the method does not work in detecting the confounded effect. In this chapter, moderation is made in detection of the confounded effect by introducing some new equations called detection equations. The moderated method can be used in detecting the confounded effect for a p^n (p is prime) factorial experiment.

7.2 An easy method of detection of a confounded effect in p^n (p is prime) factorial experiments

Suppose we have a confounding plan with level combinations, confounded with a factorial effect in a p^n factorial experiment into p incomplete blocks and no information is given about which factorial effect is confounded. Now, to detect the confounded effect we have the manipulating method where the symbolic equations are solved equating the levels under mod p in a trial and error manner. If we have a plan of p^n factorial experiment where a single factorial effect is confounded will have p incomplete blocks. For detection which factorial effect is confounded by manipulating method we have to solve the following p equations in a trial and error fashion for each of the incomplete blocks.

$$\left.\begin{array}{l} a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0 \\ a_1x_1 + a_2x_2 + \cdots + a_nx_n = 1 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_1x_1 + a_2x_2 + \cdots + a_nx_n = (p-1) \end{array}\right\} \bmod p$$

where, x_i is the level of the i-th factor (F_i) and will take on values from 0 to $(p-1)$. a_i will also take on values from 0 to $(p-1)$ but not all equal to zero. There is a restriction that the coefficient of the first a_i that is not zero to be equal to unity (Kemthrone, 1952).

For a quick and easier detection, an algebraic method was developed in 1994 (Jalil et. al, 1994). This method of detection of the confounded effect in a given plan of a p^n (p is prime) factorial experiment with a single factorial effect is described below.

Among the p- incomplete blocks, the incomplete block with the level combinations of n- factors each at their lowest level is to be ignored. Any of the rest $(p-1)$ incomplete blocks is to be considered for the detection purpose. From the considered block, select those treatment combinations where all but one at their lowest level.Let the n factors in a p^n f.e. be denoted by $F_1, F_2, \dots, F_n$ and in general, F_t be the t-th factor. The level combinations of a treatment combination with the i-th level for first factor, j-th level for 2nd factor, k-th level for 3rd factor and finally the r-th level for n-th factor can symbolically be written as (Das and Giri, (1984)):

$((F_1)_i (F_2)_j (F_3)_k \dots (F_n)_r)$

Now, to find the confounded effect the selected treatments are multiplied in such a way that the various level combinations of each factor is raised to its power. The arithmetic of this multiplication can be expressed as,

$((F_1)_i (F_2)_j \dots (F_n)_r)\ ((F_1)_i (F_2)_j \dots (F_n)_r)\ \dots\ ((F_1)_i (F_2)_j \dots (F_n)_r)$

$$=F_1^{\sum i}\ \ F_2^{\sum j}\ \ \dots F_n^{\sum r} \qquad (7.2.1)$$

7.3 Problem of this method

As stated the method works in detecting the confounded effect in a p^n (p is prime) factorial experiment. Practically, the method works when $p = 2 \; or \; 3$ and it does not work in detection the confounded effect when $p > 3$ as shown in the examples described below.

It is considered a plan of 3^4 factorial experiment where a single factorial effect is confounded. Here, $p = 3$; so the levels are $0, 1, 2$ and $n = 4$; the factors are $F_1, F_2, F_3 \; and \; F_4$.

A plan of 3^4 factorial experiment where a single factorial effect is confounded is given below:

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\
0 & 0 & 1 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & 0 & 0 & 1 & 2 \\
0 & 0 & 2 & 2 & 0 & 0 & 2 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{2} & \mathbf{0} \\
0 & 1 & 0 & 1 & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & 0 & 1 & 0 & 2 \\
0 & 1 & 1 & 2 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 1 & 2 & 0 & 0 & 1 & 2 & 2 & 0 & 1 & 2 & 1 \\
0 & 2 & 0 & 2 & 0 & 2 & 0 & 1 & \mathbf{0} & \mathbf{2} & \mathbf{0} & \mathbf{0} \\
0 & 2 & 1 & 0 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 1 \\
0 & 2 & 2 & 1 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 2 \\
1 & 0 & 0 & 1 & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 & 0 & 0 & 2 \\
1 & 0 & 1 & 2 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 2 & 0 & 1 & 0 & 2 & 2 & 1 & 0 & 2 & 1 \\
1 & 1 & 0 & 2 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 \\
1 & 1 & 2 & 1 & 1 & 1 & 2 & 0 & 1 & 1 & 2 & 2 \\
1 & 2 & 0 & 0 & 1 & 2 & 0 & 2 & 1 & 2 & 0 & 1 \\
1 & 2 & 1 & 1 & 1 & 2 & 1 & 0 & 1 & 2 & 1 & 2 \\
1 & 2 & 2 & 2 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 0 \\
2 & 0 & 0 & 2 & 2 & 0 & 0 & 1 & \mathbf{2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
2 & 0 & 1 & 0 & 2 & 0 & 1 & 2 & 2 & 0 & 1 & 1 \\
2 & 0 & 2 & 1 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 2 \\
2 & 1 & 0 & 0 & 2 & 1 & 0 & 2 & 2 & 1 & 0 & 1 \\
2 & 1 & 1 & 1 & 2 & 1 & 1 & 0 & 2 & 1 & 1 & 2 \\
2 & 1 & 2 & 2 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 0 \\
2 & 2 & 0 & 1 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 2 \\
2 & 2 & 1 & 2 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 0 \\
2 & 2 & 2 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1
\end{bmatrix}
$$

Since in block-1, the level combinations of all the factors are at their lowest level, block-1 is to be ignored. Any one of the rest two blocks can be considered. If we consider block-2 the required treatments as described in the method will be: (0002), (0010), (0100) and (1000)

In the given plan the resulting confounded effect could be found with the equation (7.2.1) as,

$((F_1)_0(F_2)_0(F_3)_0(F_4)_2)\ ((F_1)_0(F_2)_0(F_3)_1(F_4)_0)\ ((F_1)_0(F_2)_1(F_3)_0(F_4)_0)$

$((F_1)_1(F_2)_0(F_3)_0(F_4)_0)$

$= F_1^{(0+0+0+1)}\ F_2^{(0+0+1+0)}\ F_3^{(0+1+0+0)}\ F_4^{(2+0+0+0)}$

$= F_1^1 F_2^1 F_3^1 F_4^2 = F_1 F_2 F_3 F_4^2$

Again if block-3 were considered, then the selected treatments would be (0001), (0020), (0200) and (2000) and therefore, the confounded effect can be found as

$((F_1)_0(F_2)_0(F_3)_0(F_4)_1)\ ((F_1)_0(F_2)_0(F_3)_2(F_4)_0)\ ((F_1)_0(F_2)_2(F_3)_0(F_4)_0)$

$((F_1)_2(F_2)_0(F_3)_0(F_4)_0)$

$= F_1^{(0+0+0+2)}\ F_2^{(0+0+2+0)}\ F_3^{(0+2+0+0)}\ F_4^{(1+0+0+0)}$

$= F_1^2 F_2^2 F_3^2 F_4^1 = F_1 F_2 F_3 F_4^2$ (mod 3)

Now let us consider a plan of 5^3 factorial experiment where the effect $F_1 F_2 F_3^2$ is confounded. Here $p = 5$, which is a prime number; so the levels are $0, 1, 2, 3, 4$ and $n = 3$; the factors are $F_1, F_2\ and\ F_3$. The construction layout of the confounding plan becomes:

$$
\begin{bmatrix}
0 & 0 & 0 & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{2} & \mathbf{0} & \mathbf{0} & \mathbf{3} & \mathbf{0} & \mathbf{0} & \mathbf{4} & \mathbf{0} & \mathbf{0} \\
3 & 0 & 1 & 4 & 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{1} & 1 & 0 & 1 & 2 & 0 & 1 \\
1 & 0 & 2 & 2 & 0 & 2 & 3 & 0 & 2 & 4 & 0 & 2 & \mathbf{0} & \mathbf{0} & \mathbf{2} \\
4 & 0 & 3 & \mathbf{0} & \mathbf{0} & \mathbf{3} & 1 & 0 & 3 & 2 & 0 & 3 & 3 & 0 & 3 \\
2 & 0 & 4 & 3 & 0 & 4 & 4 & 0 & 4 & \mathbf{0} & \mathbf{0} & \mathbf{4} & 1 & 0 & 4 \\
4 & 1 & 0 & \mathbf{0} & \mathbf{1} & \mathbf{0} & 1 & 1 & 0 & 2 & 1 & 0 & 3 & 1 & 0 \\
2 & 1 & 1 & 3 & 1 & 1 & 4 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 2 & 1 & 1 & 2 & 2 & 1 & 2 & 3 & 1 & 2 & 4 & 1 & 2 \\
3 & 1 & 3 & 4 & 1 & 3 & 0 & 1 & 3 & 1 & 1 & 3 & 2 & 1 & 3 \\
1 & 1 & 4 & 2 & 1 & 4 & 3 & 1 & 4 & 4 & 1 & 4 & 0 & 1 & 4 \\
3 & 2 & 0 & 4 & 2 & 0 & \mathbf{0} & \mathbf{2} & \mathbf{0} & 1 & 2 & 0 & 2 & 2 & 0 \\
1 & 2 & 1 & 2 & 2 & 1 & 3 & 2 & 1 & 4 & 2 & 1 & 0 & 2 & 1 \\
4 & 2 & 2 & 0 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 3 & 2 & 2 \\
2 & 2 & 3 & 3 & 2 & 3 & 4 & 2 & 3 & 0 & 2 & 3 & 1 & 2 & 3 \\
0 & 2 & 4 & 1 & 2 & 4 & 2 & 2 & 4 & 3 & 2 & 4 & 4 & 2 & 4 \\
2 & 3 & 0 & 3 & 3 & 0 & 4 & 3 & 0 & \mathbf{0} & \mathbf{3} & \mathbf{0} & 1 & 3 & 0 \\
0 & 3 & 1 & 1 & 3 & 1 & 2 & 3 & 1 & 3 & 3 & 1 & 4 & 3 & 1 \\
3 & 3 & 2 & 4 & 3 & 2 & 0 & 3 & 2 & 1 & 3 & 2 & 2 & 3 & 2 \\
1 & 3 & 3 & 2 & 3 & 3 & 3 & 3 & 3 & 4 & 3 & 3 & 0 & 3 & 3 \\
4 & 3 & 4 & 0 & 3 & 4 & 1 & 3 & 4 & 2 & 3 & 4 & 3 & 3 & 4 \\
1 & 4 & 0 & 2 & 4 & 0 & 3 & 4 & 0 & 4 & 4 & 0 & \mathbf{0} & \mathbf{4} & \mathbf{0} \\
4 & 4 & 1 & 0 & 4 & 1 & 1 & 4 & 1 & 2 & 4 & 1 & 3 & 4 & 1 \\
2 & 4 & 2 & 3 & 4 & 2 & 4 & 4 & 2 & 0 & 4 & 2 & 1 & 4 & 2 \\
0 & 4 & 3 & 1 & 4 & 3 & 2 & 4 & 3 & 3 & 4 & 3 & 4 & 4 & 3 \\
3 & 4 & 4 & 4 & 4 & 4 & 0 & 4 & 4 & 1 & 4 & 4 & 2 & 4 & 4
\end{bmatrix}
$$

Since in block-1, the level combinations of all the factors are at their lowest level, block-1 is to be ignored. Any one of the rest four blocks can be considered. If we consider block-2, the required treatments as described in the method will be: (100), (003) and (010)

In the given plan the resulting confounded effect could be founded with the equation (7.2.1) as,

$$((F_1)_1(F_2)_0(F_3)_0)\,((F_1)_0(F_2)_0(F_3)_3)\,((F_1)_0(F_2)_1(F_3)_0)$$

$$= F_1^{1+0+0} F_2^{0+0+1} F_3^{0+3+0}$$

$$= F_1^1 F_2^1 F_3^3 = F_1 F_2 F_3^3$$

Here, in the given layout the factorial effect $F_1F_2F_3^2$ confounded. But when we are detecting the confounded effect using the quick method (Equation 7.2.1), we are getting $F_1F_2F_3^3$ as the confounded effect, which is not true. Using the other blocks we also get $F_1F_2F_3^3$ as the confounded effect. Clearly, the method fails to detect the confounded effect.

7.4 The reason

In the construction of a confounded plan symbolic equations are used to construct the level combinations of the incomplete blocks. But in the detection, we need the coefficients of symbolic equations, because the exponent of factors in the factorial effect to be confounded becomes the coefficient in the symbolic equations. For example, if in 3^3 factorial experiment $F_1F_2F_3^2$ is the factorial effect to be confounded, the symbolic equation becomes as

$$x_1 + x_2 + 2x_3 = 0, = 1, = \ 2 \ (mod \ 3)$$

Let from the considered block we select a treatment combination where all but x_i at their lowest level. So using the detection method we will get the i-th factor in the confounded effect as $F_i^{x_i}$ but actually the i-th factor in the confounded effect becomes as $F_i^{a_i}$. For p^n factorial experiment we get the value of the coefficient, a_i, by solving any one of the following equations

$$\left.\begin{array}{l} a_i \times x_i = 1 \\ a_i \times x_i = 2 \\ \dots\dots\dots\dots\dots\dots \\ a_i \times x_i = (p-1) \end{array}\right\} \qquad (7.4.1)$$

Here $a_i \ and \ x_i$ take values from $1, 2, \dots, (p-1)$.

So first we take any one equation from (7.4.1). Then we have to solve this equation for all values of $a_i \ and \ x_i$. Thus we will get $(p-1)$ equations.

We name the equations in (7.4.1) as the detection equations.

7.5 The moderated detecting method

To detect confounded effect in a p^n factorial experiment, where n denotes number of factors and p (prime) denotes the number of levels; first we take any one equation from (7.4.1). Then we have to solve this equation for all values of $a_i \ and \ x_i$. Thus we will get $(p-1)$ equations.

Then using the method proposed by Jalil et. al (1994) we get a treatment effect.

If x_i is the power of a factor in the treatment effect then we have to replace x_i by a_i to get the confounded effect. Similarly we have to replace the power of all factors obtained in the treatment effect. After interchanging the power of all factors we get the confounded effect.

7.6 Illustration with example

To illustrate the method, let us consider a plan of 3^4 factorial experiment where a single factorial effect is confounded.

A plan of 3^4 factorial experiment where a single factorial effect is confounded is given below:

$$\begin{bmatrix}
0 & 0 & 0 & 0 & & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{2} & & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\
0 & 0 & 1 & 1 & & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & & 0 & 0 & 1 & 2 \\
0 & 0 & 2 & 2 & & 0 & 0 & 2 & 1 & & \mathbf{0} & \mathbf{0} & \mathbf{2} & \mathbf{0} \\
0 & 1 & 0 & 1 & & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & & 0 & 1 & 0 & 2 \\
0 & 1 & 1 & 2 & & 0 & 1 & 1 & 1 & & 0 & 1 & 1 & 0 \\
0 & 1 & 2 & 0 & & 0 & 1 & 2 & 2 & & 0 & 1 & 2 & 1 \\
0 & 2 & 0 & 2 & & 0 & 2 & 0 & 1 & & \mathbf{0} & \mathbf{2} & \mathbf{0} & \mathbf{0} \\
0 & 2 & 1 & 0 & & 0 & 2 & 1 & 2 & & 0 & 2 & 1 & 1 \\
0 & 2 & 2 & 1 & & 0 & 2 & 2 & 0 & & 0 & 2 & 2 & 2 \\
1 & 0 & 0 & 1 & & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & & 1 & 0 & 0 & 2 \\
1 & 0 & 1 & 2 & & 1 & 0 & 1 & 1 & & 1 & 0 & 1 & 0 \\
1 & 0 & 2 & 0 & & 1 & 0 & 2 & 2 & & 1 & 0 & 2 & 1 \\
1 & 1 & 0 & 2 & & 1 & 1 & 0 & 1 & & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & & 1 & 1 & 1 & 2 & & 1 & 1 & 1 & 1 \\
1 & 1 & 2 & 1 & & 1 & 1 & 2 & 0 & & 1 & 1 & 2 & 2 \\
1 & 2 & 0 & 0 & & 1 & 2 & 0 & 2 & & 1 & 2 & 0 & 1 \\
1 & 2 & 1 & 1 & & 1 & 2 & 1 & 0 & & 1 & 2 & 1 & 2 \\
1 & 2 & 2 & 2 & & 1 & 2 & 2 & 1 & & 1 & 2 & 2 & 0 \\
2 & 0 & 0 & 2 & & 2 & 0 & 0 & 1 & & \mathbf{2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
2 & 0 & 1 & 0 & & 2 & 0 & 1 & 2 & & 2 & 0 & 1 & 1 \\
2 & 0 & 2 & 1 & & 2 & 0 & 2 & 0 & & 2 & 0 & 2 & 2 \\
2 & 1 & 0 & 0 & & 2 & 1 & 0 & 2 & & 2 & 1 & 0 & 1 \\
2 & 1 & 1 & 1 & & 2 & 1 & 1 & 0 & & 2 & 1 & 1 & 2 \\
2 & 1 & 2 & 2 & & 2 & 1 & 2 & 1 & & 2 & 1 & 2 & 0 \\
2 & 2 & 0 & 1 & & 2 & 2 & 0 & 0 & & 2 & 2 & 0 & 2 \\
2 & 2 & 1 & 2 & & 2 & 2 & 1 & 1 & & 2 & 2 & 1 & 0 \\
2 & 2 & 2 & 0 & & 2 & 2 & 2 & 2 & & 2 & 2 & 2 & 1
\end{bmatrix}$$

Here $p = 3$, which is a prime number; so the levels are $0, 1, 2$ and $n = 4$; the factors are F_1, F_2, F_3 and F_4. For $p = 3$ the values of a_i and x_i are $1, 2$. From equation (7.4.1) we can write

$$a_i \times x_i = 1$$

$$a_i \times x_i = 2$$

Now we have to take any one equation from these two equations. Here we take

$$a_i \times x_i = 1 \quad (7.6.1)$$

Solving this equation for all values of a_i and x_i, we get

$$1 \times 1 = 1$$

$$2 \times 2 = 4 = 1(mod\ 3)$$

Since in block-1, the level combinations of all the factors are at their lowest level, block-1 is to be ignored. Any one of the rest two blocks can be considered.

For block-2

If we consider block-2 the required treatments as described in the method will be: (0002), (0010), (0100) and (1000)

In the given plan the resulting confounded effect could be found with the equation (7.2.1) as,

$((F_1)_0(F_2)_0(F_3)_0(F_4)_2)$ $((F_1)_0(F_2)_0(F_3)_1(F_4)_0)$ $((F_1)_0(F_2)_1(F_3)_0(F_4)_0)$

$((F_1)_1(F_2)_0(F_3)_0(F_4)_0)$

$= F_1^{(0+0+0+1)}\ F_2^{(0+0+1+0)}\ F_3^{(0+1+0+0)}\ F_4^{(2+0+0+0)}$

$= F_1^1 F_2^1 F_3^1 F_4^2 = F_1 F_2 F_3 F_4^2$

Since the power of F_1 in the treatment effect is 1 so using the solutions of the equation (7.6.1) we have to replace 1 by 1. In F_2 the power is 1 so we have to replace 1 by 1, in F_3 the power is 1 so we have to replace 1 by 1 and in F_4 the power is 2 so we have to replace 2 by 2.

So the confounded effect is

$$F_1^1 F_2^1 F_3^1 F_4^2 = F_1 F_2 F_3 F_4^2$$

For block-3

Again if block-3 were considered, then the selected treatments would be (0001), (0020), (0200) and (2000) and therefore, the confounded effect can be found as

$((F_1)_0(F_2)_0(F_3)_0(F_4)_1)\ ((F_1)_0(F_2)_0(F_3)_2(F_4)_0)\ ((F_1)_0(F_2)_2(F_3)_0(F_4)_0)$

$((F_1)_2(F_2)_0(F_3)_0(F_4)_0)$

$= F_1^{(0+0+0+2)}\ F_2^{(0+0+2+0)}\ F_3^{(0+2+0+0)}\ F_4^{(1+0+0+0)}$

$= F_1^2F_2^2F_3^2F_4^1 = F_1F_2F_3F_4^2\ (mod\ 3)$

Since the power of F_1 in the treatment effect is 1 so using the solutions of the equation (7.6.1) we have to replace 1 by 1. In F_2 the power is 1 so we have to replace 1 by 1, in F_3 the power is 1 so we have to replace 1 by 1 and in F_4 the power is 2 so we have to replace 2 by 2.

So the confounded effect is

$$F_1^1F_2^1F_3^1F_4^2 = F_1F_2F_3F_4^2$$

If we take

$$a_i \times x_i = 2 \tag{7.6.2}$$

Solving this equation for all values of $a_i\ and\ x_i$, we get

$$1 \times 2 = 2$$

$$2 \times 1 = 2$$

Since in block-1, the level combinations of all the factors are at their lowest level, block-1 is to be ignored. Any one of the rest two blocks can be considered.

For block-2

If we consider block-2 the required treatments as described in the method will be: (0002), (0010), (0100) and (1000)

In the given plan the resulting confounded effect could be found with the equation (7.2.1) as,

$((F_1)_0(F_2)_0(F_3)_0(F_4)_2)\ ((F_1)_0(F_2)_0(F_3)_1(F_4)_0)\ ((F_1)_0(F_2)_1(F_3)_0(F_4)_0)$

$((F_1)_1(F_2)_0(F_3)_0(F_4)_0)$

$= F_1^{(0+0+0+1)}\ F_2^{(0+0+1+0)}\ F_3^{(0+1+0+0)}\ F_4^{(2+0+0+0)}$

$= F_1^1 F_2^1 F_3^1 F_4^2 = F_1 F_2 F_3 F_4^2$

Since the power of F_1 in the treatment effect is 1 so using the solutions of the equation (7.6.2) we have to replace 1 by 2. In F_2 the power is 1 so we have to replace 1 by 2, in F_3 the power is 1 so we have to replace 1 by 2 and in F_4 the power is 2 so we have to replace 2 by 1.

So the confounded effect is

$$F_1^2 F_2^2 F_3^2 F_4^1 = F_1 F_2 F_3 F_4^2\ (mod\ 3)$$

For block-3

Again if block-3 were considered, then the selected treatments would be (0001), (0020), (0200) and (2000) and therefore, the confounded effect can be found as

$((F_1)_0(F_2)_0(F_3)_0(F_4)_1)\ ((F_1)_0(F_2)_0(F_3)_2(F_4)_0)\ ((F_1)_0(F_2)_2(F_3)_0(F_4)_0)$

$((F_1)_2(F_2)_0(F_3)_0(F_4)_0)$

$= F_1^{(0+0+0+2)}\ F_2^{(0+0+2+0)}\ F_3^{(0+2+0+0)}\ F_4^{(1+0+0+0)}$

$= F_1^2 F_2^2 F_3^2 F_4^1 = F_1 F_2 F_3 F_4^2\ (mod\ 3)$

Since the power of F_1 in the treatment effect is 1 so using the solutions of the equation (7.6.2) we have to replace 1 by 2. In F_2 the power is 1 so we have to replace 1 by 2, in F_3 the power is 1 so we have to replace 1 by 2 and in F_4 the power is 2 so we have to replace 2 by 1.

So the confounded effect is

$$F_1^2 F_2^2 F_3^2 F_4^1 = F_1 F_2 F_3 F_4^2\ (mod\ 3)$$

To illustrate the method, now we consider a plan of 5^3 factorial experiment where a single factorial effect is confounded.

A plan of 5^3 factorial experiment where a single factorial effect is confounded is given below:

$$\begin{bmatrix}
0 & 0 & 0 & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{2} & \mathbf{0} & \mathbf{0} & \mathbf{3} & \mathbf{0} & \mathbf{0} & \mathbf{4} & \mathbf{0} & \mathbf{0} \\
3 & 0 & 1 & 4 & 0 & 1 & \mathbf{0} & \mathbf{0} & \mathbf{1} & 1 & 0 & 1 & 2 & 0 & 1 \\
1 & 0 & 2 & 2 & 0 & 2 & 3 & 0 & 2 & 4 & 0 & 2 & \mathbf{0} & \mathbf{0} & \mathbf{2} \\
4 & 0 & 3 & \mathbf{0} & \mathbf{0} & \mathbf{3} & 1 & 0 & 3 & 2 & 0 & 3 & 3 & 0 & 3 \\
2 & 0 & 4 & 3 & 0 & 4 & 4 & 0 & 4 & \mathbf{0} & \mathbf{0} & \mathbf{4} & 1 & 0 & 4 \\
4 & 1 & 0 & \mathbf{0} & \mathbf{1} & \mathbf{0} & 1 & 1 & 0 & 2 & 1 & 0 & 3 & 1 & 0 \\
2 & 1 & 1 & 3 & 1 & 1 & 4 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 2 & 1 & 1 & 2 & 2 & 1 & 2 & 3 & 1 & 2 & 4 & 1 & 2 \\
3 & 1 & 3 & 4 & 1 & 3 & 0 & 1 & 3 & 1 & 1 & 3 & 2 & 1 & 3 \\
1 & 1 & 4 & 2 & 1 & 4 & 3 & 1 & 4 & 4 & 1 & 4 & 0 & 1 & 4 \\
3 & 2 & 0 & 4 & 2 & 0 & \mathbf{0} & \mathbf{2} & \mathbf{0} & 1 & 2 & 0 & 2 & 2 & 0 \\
1 & 2 & 1 & 2 & 2 & 1 & 3 & 2 & 1 & 4 & 2 & 1 & 0 & 2 & 1 \\
4 & 2 & 2 & 0 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 3 & 2 & 2 \\
2 & 2 & 3 & 3 & 2 & 3 & 4 & 2 & 3 & 0 & 2 & 3 & 1 & 2 & 3 \\
0 & 2 & 4 & 1 & 2 & 4 & 2 & 2 & 4 & 3 & 2 & 4 & 4 & 2 & 4 \\
2 & 3 & 0 & 3 & 3 & 0 & 4 & 3 & 0 & \mathbf{0} & \mathbf{3} & \mathbf{0} & 1 & 3 & 0 \\
0 & 3 & 1 & 1 & 3 & 1 & 2 & 3 & 1 & 3 & 3 & 1 & 4 & 3 & 1 \\
3 & 3 & 2 & 4 & 3 & 2 & 0 & 3 & 2 & 1 & 3 & 2 & 2 & 3 & 2 \\
1 & 3 & 3 & 2 & 3 & 3 & 3 & 3 & 3 & 4 & 3 & 3 & 0 & 3 & 3 \\
4 & 3 & 4 & 0 & 3 & 4 & 1 & 3 & 4 & 2 & 3 & 4 & 3 & 3 & 4 \\
1 & 4 & 0 & 2 & 4 & 0 & 3 & 4 & 0 & 4 & 4 & 0 & \mathbf{0} & \mathbf{4} & \mathbf{0} \\
4 & 4 & 1 & 0 & 4 & 1 & 1 & 4 & 1 & 2 & 4 & 1 & 3 & 4 & 1 \\
2 & 4 & 2 & 3 & 4 & 2 & 4 & 4 & 2 & 0 & 4 & 2 & 1 & 4 & 2 \\
0 & 4 & 3 & 1 & 4 & 3 & 2 & 4 & 3 & 3 & 4 & 3 & 4 & 4 & 3 \\
3 & 4 & 4 & 4 & 4 & 4 & 0 & 4 & 4 & 1 & 4 & 4 & 2 & 4 & 4
\end{bmatrix}$$

Here $p = 5$, which is a prime number; so the levels are $0, 1, 2, 3, 4$ and $n = 3$; the factors are $F_1, F_2 \; and \; F_3$. For $p = 5$ the values of $a_i \; and \; x_i$ are $1, 2, 3, 4$. From equation (7.4.1) we can write

$$a_i \times x_i = 1$$

$$a_i \times x_i = 2$$

$$a_i \times x_i = 3$$

$$a_i \times x_i = 4$$

Now we have to take any one equation from these four equations. Here we take

$$a_i \times x_i = 1 \tag{7.6.3}$$

Solving this equation for all values of a_i and x_i, we get

$$1 \times 1 = 1$$

$$2 \times 3 = 6 = 1 \ (mod\ 5)$$

$$3 \times 2 = 6 = 1 \ (mod\ 5)$$

$$4 \times 4 = 16 = 1 \ (mod\ 5)$$

Since block-1 contains the level combination of all factors each at their lowest level, so block-1 is ignored. Any one of the remaining four blocks can be considered.

For block-2

If we consider block-2 the required treatments as described in the method will be: (100), (003) and (010).

In the given plan the resulting confounded effect could be founded with the equation (7.2.1) as,

$((F_1)_1(F_2)_0(F_3)_0)\ ((F_1)_0(F_2)_0(F_3)_3)\ ((F_1)_0(F_2)_1(F_3)_0)$

$= F_1^{1+0+0} F_2^{0+0+1} F_3^{0+3+0}$

$= F_1^1 F_2^1 F_3^3 = F_1 F_2 F_3^3$

Since the power of F_1 in the treatment effect is 1 so using the solutions of the equation (7.6.3) we have to replace 1 by 1. In F_2 the power is 1 so we have to replace 1 by 1 and in F_3 the power is 3 so we have to replace 3 by 2.

So the confounded effect is

$$F_1^1 F_2^1 F_3^2 = F_1 F_2 F_3^2$$

For block-3

If block-3 was considered, then the selected treatments would be (200), (001) and (020) and therefore, the confounded effect can be found as

$((F_1)_2(F_2)_0(F_3)_0)\ ((F_1)_0(F_2)_0(F_3)_1)\ ((F_1)_0(F_2)_2(F_3)_0)$

$= F_1^{2+0+0} F_2^{0+0+2} F_3^{0+1+0}$

$= F_1^2 F_2^2 F_3^1 = F_1 F_2 F_3^3 \ (mod\ 5)$.

Since the power of F_1 in the treatment effect is 1 so using the solutions of the equation (7.6.3) we have to replace 1 by 1. In F_2 the power is 1 so we have to replace 1 by 1 and in F_3 the power is 3 so we have to replace 3 by 2.

So the confounded effect is

$$F_1^1 F_2^1 F_3^2 = F_1 F_2 F_3^2$$

Similarly from block-4 and block-5 we get $F_1 F_2 F_3^2$ as the confounded effect.

Now if we take

$$a_i \times x_i = 2 \qquad (7.6.4)$$

Solving this equation for all values of $a_i\ and\ x_i$, we get

$$1 \times 2 = 2$$

$$2 \times 1 = 2\ (mod\ 5)$$

$$3 \times 4 = 12 = 2\ (mod\ 5)$$

$$4 \times 3 = 12 = 2\ (mod\ 5)$$

Since block-1 contains the level combination of all factors each at their lowest level, so block-1 is ignored. Any one of the remaining four blocks can be considered.

For block-2

If we consider block-2 the required treatments as described in the method will be: (100), (003) and (010).

In the given plan the resulting confounded effect could be founded with the equation (7.2.1) as,

$((F_1)_1 (F_2)_0 (F_3)_0)\ ((F_1)_0 (F_2)_0 (F_3)_3)\ ((F_1)_0 (F_2)_1 (F_3)_0)$

$= F_1^{1+0+0} F_2^{0+0+1} F_3^{0+3+0}$

$= F_1^1 F_2^1 F_3^3 = F_1 F_2 F_3^3$

Since the power of F_1 in the treatment effect is 1 so using the solutions of the equation (7.6.4) we have to replace 1 by 2. In F_2 the power is 1 so we have to replace 1 by 2 and in F_3 the power is 3 so we have to replace 3 by 4.

So the confounded effect is

$$F_1^2F_2^2F_3^4 = F_1F_2F_3^2\ (mod\ 5)$$

For block-3

If block-3 was considered, then the selected treatments would be (200), (001) and (020) and therefore, the confounded effect can be found as

$((F_1)_2(F_2)_0(F_3)_0)\ ((F_1)_0(F_2)_0(F_3)_1)\ ((F_1)_0(F_2)_2(F_3)_0)$

= $F_1^{2+0+0}F_2^{0+0+2}F_3^{0+1+0}$

= $F_1^2F_2^2F_3^1$ = $F_1F_2F_3^3\ (mod\ 5)$.

Since the power of F_1 in the treatment effect is 1 so using the solutions of the equation (7.6.4) we have to replace 1 by 2. In F_2 the power is 1 so we have to replace 1 by 2 and in F_3 the power is 3 so we have to replace 3 by 4.

So the confounded effect is

$$F_1^2F_2^2F_3^4 = F_1F_2F_3^2\ (mod\ 5)$$

Similarly from block-4 and block-5 we get $F_1F_2F_3^2$ as the confounded effect.

Now if we take

$$a_i \times x_i = 3 \qquad (7.6.5)$$

Solving this equation for all values of $a_i\ and\ x_i$, we get

$$1 \times 3 = 3$$

$$2 \times 4 = 8 = 3\ (mod\ 5)$$

$$3 \times 1 = 3\ (mod\ 5)$$

$$4 \times 2 = 8 = 3\ (mod\ 5)$$

Since block-1 contains the level combination of all factors each at their lowest level, so block-1 is ignored. Any one of the remaining four blocks can be considered.

For block-2

If we consider block-2 the required treatments as described in the method will be: (100), (003) and (010)

In the given plan the resulting confounded effect could be founded with the equation (7.2.1) as,

$((F_1)_1(F_2)_0(F_3)_0)\,((F_1)_0(F_2)_0(F_3)_3)\,((F_1)_0(F_2)_1(F_3)_0)$

$= F_1^{1+0+0}F_2^{0+0+1}F_3^{0+3+0}$

$= F_1^1F_2^1F_3^3 = F_1F_2F_3^3$

Since the power of F_1 in the treatment effect is 1 so using the solutions of the equation (7.6.5) we have to replace 1 by 3. In F_2 the power is 1 so we have to replace 1 by 3 and in F_3 the power is 3 so we have to replace 3 by 1.

So the confounded effect is

$$F_1^3F_2^3F_3^1 = F_1F_2F_3^2\ (mod\ 5)$$

For block-3

If block-3 was considered, then the selected treatments would be (200), (001), (020) and therefore, the confounded effect can be found as

$((F_1)_2(F_2)_0(F_3)_0)\,((F_1)_0(F_2)_0(F_3)_1)\,((F_1)_0(F_2)_2(F_3)_0)$

$= F_1^{2+0+0}F_2^{0+0+2}F_3^{0+1+0}$

$= F_1^2F_2^2F_3^1 = F_1F_2F_3^3\ (mod\ 5).$

Since the power of F_1 in the treatment effect is 1 so using the solutions of the equation (7.6.5) we have to replace 1 by 3. In F_2 the power is 1 so we have to replace 1 by 3 and in F_3 the power is 3 so we have to replace 3 by 1.

So the confounded effect is

$$F_1^3 F_2^3 F_3^1 = F_1 F_2 F_3^2 \ (mod\ 5)$$

Similarly from block-4 and block-5 we get $F_1 F_2 F_3^2$ as the confounded effect.

Now if we take

$$a_i \times x_i = 4 \qquad (7.6.6)$$

Solving this equation for all values of $a_i\ and\ x_i$, we get

$$1 \times 4 = 4$$

$$2 \times 2 = 4$$

$$3 \times 3 = 9 = 4\ (mod\ 5)$$

$$4 \times 1 = 4 = 4\ (mod\ 5)$$

Since block-1 contains the level combination of all factors each at their lowest level, so block-1 is ignored. Any one of the remaining four blocks can be considered.

For block-2

If we consider block-2 the required treatments as described in the method will be: (100), (003) and (010)

In the given plan the resulting confounded effect could be founded with the equation (7.2.1) as,

$((F_1)_1(F_2)_0(F_3)_0)\ ((F_1)_0(F_2)_0(F_3)_3)\ ((F_1)_0(F_2)_1(F_3)_0)$

$= F_1^{1+0+0} F_2^{0+0+1} F_3^{0+3+0}$

$= F_1^1 F_2^1 F_3^3 = F_1 F_2 F_3^3$

Since the power of F_1 in the treatment effect is 1 so using the solutions of the equation (7.6.6) we have to replace 1 by 4. In F_2 the power is 1 so we have to replace 1 by 4 and in F_3 the power is 3 so we have to replace 3 by 3.

So the confounded effect is

$$F_1^4 F_2^4 F_3^3 = F_1 F_2 F_3^2 \ (mod\ 5)$$

For block-3

If block-3 was considered, then the selected treatments would be (200), (001), (020) and therefore, the confounded effect can be found as

$((F_1)_2(F_2)_0(F_3)_0)\ ((F_1)_0(F_2)_0(F_3)_1)\ ((F_1)_0(F_2)_2(F_3)_0)$

$= F_1^{2+0+0} F_2^{0+0+2} F_3^{0+1+0}$

$= F_1^2 F_2^2 F_3^1 = F_1 F_2 F_3^3\ (mod\ 5)$.

Since the power of F_1 in the treatment effect is 1 so using the solutions of the equation (7.6.6) we have to replace 1 by 4. In F_2 the power is 1 so we have to replace 1 by 4 and in F_3 the power is 3 so we have to replace 3 by 3.

So the confounded effect is

$$F_1^4 F_2^4 F_3^3 = F_1 F_2 F_3^2 \ (mod\ 5)$$

Similarly from block-4 and block-5 we get $F_1 F_2 F_3^2$ as the confounded effect.

7.7 Summary

In this chapter, a detection method has been developed for p^n (p is prime) factorial experiment where a single factorial effect is confounded. The method is appropriate in general for any value of n, the number of factors and for any possible value of p, the levels of the factors. The method is restricted to p^n symmetrical factorial experiment when p is prime.

Chapter 8

Discussion & Conclusion

In this thesis paper we are mainly interested in symmetrical factorial experiments. Here we describe both construction plan and the analysis procedure that are applicable only for symmetrical factorial experiments.

There are three methods to represents the level combinations in a p^n factorial experiment. These are Manipulating Method, Matrix Product Method And Matrix Method. Among these, Matrix Method is the easiest method. This method was proposed Jalil *et. al* (1990). But due to lack of computer program packages, this method has not been very popular. Even it has not been well introduced. Our main goal is simply to introduce and popularize this method by writing computer programs. In chapter three, we have written the R-code of the Matrix Method. We have written the function in R language. Anyone can use this function to create the designing layout of any p^n factorial experiments.

In 1937, F. Yates developed a method for obtaining factorial effects and their respective Sum of Squares (SS) in a 2^n factorial experiment. After Yates, P. K. Batra and Seema Jaggi also propose a method of analyzing 2^n factorial experiment. However, in chapter four, a method of analyzing 2^n factorial experiment has been introduced. It is easier and less time consuming than any other methods available for analyzing 2^n factorial experiment. This method is appropriate in general for any value of n, the number of factors. However, this method is restricted to p^n factorial experiment when $p = 2$.

Though factorial experiment started its work since 19^{th} century, there is no procedure of analyzing p^n factorial experiments when p is a natural number. We have methods of analyzing p^n factorial experiments only when p is prime. In chapter five, we have introduced a general method of analyzing p^n factorial experiments. It is easier and rewarding than any method available in analyzing p^n (p is prime) factorial experiment. The method is appropriate in general for any value of n and for any possible value of p as well, where, n is the number of factors and p is the levels of the factors. However, this method is restricted to p^n symmetrical factorial experiments only.

In 1990, Jalil *et. al* proposed a method of constructing confounding plan with single factorial effect in p^n (p is prime) factorial experiments. This method has some notational mistakes. In chapter six, we have made corrections of these mistakes. Also the method is restricted to p^n factorial experiments when p is prime. We have made moderation in taking adjustment factor, using which we can construct confounding plan with single factorial effect in p^n factorial experiment when p is a natural number. The method is appropriate in general for any value of n, the number of factors and for any possible value of p, the levels of the factors. However, this method is restricted to p^n symmetrical factorial experiment.

A quick and easy method of detection of a confounded effect in p^n (p is prime) factorial experiments was proposed in 1994 (Jalil *et. al* 1994). It has been seen that the method would not work in p^n factorial experiments when $p > 3$. In chapter seven, we have made moderation, using which we can detect a confounded effect in p^n (p is prime) factorial experiments. The method is appropriate in general for any value of n, the number of factors and for any possible value of p, the levels of the factors. The method is restricted to p^n symmetrical factorial experiment when p is prime.

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