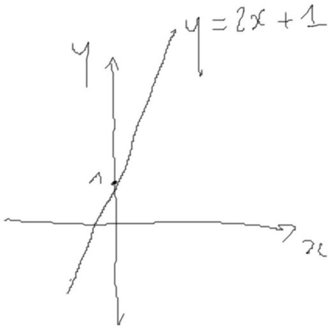
Espaces et sev

Exercice 1

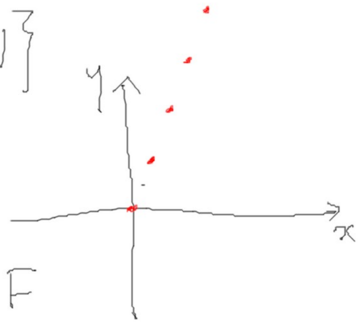
1) $F = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 ,\ y = 2x + 1 \right\}$



. $0_{\mathbb{R}^2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\qquad 2 \times 0 + 1 = 1 \neq 0$

$0_{\mathbb{R}^2} \notin F$

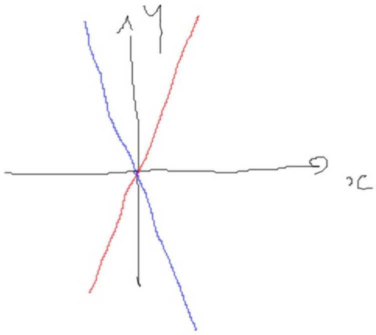
2) $F = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \quad y = 2x ,\ x \in \mathbb{N} ,\ y \in \mathbb{N} \right\}$



. $0_{\mathbb{R}^2} \in F$

. $u = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \in F$ et $\frac{1}{2} u = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \notin F$

3) $F = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 ,\ y = 2x \ \text{ ou } \ y = -3x \right\}$



. $0_{\mathbb{R}^2} \in F$

. On prend $u = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ et $v = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

$u + v = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \notin F$

Exercice 2

1) $F = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 ,\ x + y + z = 0 \right\}$

$u, v \in F, \lambda \in \mathbb{R} \quad \lambda u + v \in F$

- $\lambda \overset{\in F}{u} + \overset{\in F}{v} \in F$?

$$\lambda u + v = \lambda \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \lambda x + x' \\ \lambda y + y' \\ \lambda z + z' \end{pmatrix}$$

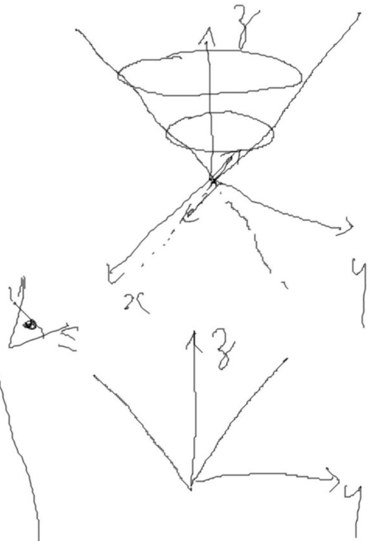
$$\lambda x + x' + \lambda y + y' + \lambda z + z' = \lambda x + \lambda y + \lambda z + \underbrace{x' + y' + z'}_{=0}$$
$$= \lambda(\underbrace{x + y + z}_{=0})$$
$$= 0$$

Donc $\lambda u + v \in F$

Finalement F est un sev de $\mathbb{R}^3$

2) $F = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \quad x^2 + y^2 - z^2 = 0 \right\}$

$x^2 + y^2 = z^2$



- $0_{\mathbb{R}^3} \in F$
- Soit $\lambda = -1$ et $u = \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix}$

$\lambda u = \begin{pmatrix} -1 \\ -1 \\ -\sqrt{2} \end{pmatrix}$

$(-1)^2 + (-1)^2 = 2$ et $(-\sqrt{2})^2 = 2$

$\lambda u \in F$

$u = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad 1^2 + 0^2 = 1^2 \quad \in F$

$v = \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix} \in F$

$u + v = \begin{pmatrix} 2 \\ 1 \\ 1+\sqrt{2} \end{pmatrix} \quad 2^2 + 1^2 = 5 \quad u + v \notin F$

$(1+\sqrt{2})^2 \neq 3 + 2\sqrt{2}$

3) $F = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \ , \ x = y = z \right\} = \left\{ \begin{pmatrix} x \\ x \\ x \end{pmatrix} \in \mathbb{R}^3, x \in \mathbb{R} \right\}$

$= \left\{ x \times \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, x \in \mathbb{R} \right\} = \mathrm{Vect} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

. $0_{\mathbb{R}^3} \in F$.

. Soient $u = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in F$ et $v \in \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \in F$ et $\lambda \in \mathbb{R}$

. $\lambda u + v = \begin{pmatrix} \lambda x + x' \\ \lambda y + y' \\ \lambda z + z' \end{pmatrix}$

. $\lambda x + x' - (\lambda y + y') = \lambda x + x' - \lambda y - y'$

$= \lambda(\underbrace{x - y}_{=0}) + \underbrace{x' - y'}_{=0} = 0$

. $\lambda y + y' - (\lambda z + z') = \lambda(y - z) + y' - z' = 0$

Donc $\lambda u + v \in F$

Donc F est un sev de $\mathbb{R}^3$

4) $F = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \ , \ x + y + z = 0 \ \text{et} \ x - 2y + 3z = 0 \right\}$

. $0_{\mathbb{R}^3} \in F$

. Soient $u = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in F$, $v = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \in F$ et $\lambda \in \mathbb{R}$

$\lambda u + v = \begin{pmatrix} \lambda x + x' \\ \lambda y + y' \\ \lambda z + z' \end{pmatrix}$. D'après 1), il suffit de vérifier la 2e équation

$\lambda x + x' - 2 \times (\lambda y + y') + 3 \times (\lambda z + z')$

$= \lambda x - \lambda \times 2y + \lambda \times 3z + \underbrace{x' - 2y' + 3z'}_{=0}$

$= \lambda(x - 2y + 3z) = 0$

Exercice 3.

$n=3$.

1) $F = \left\{ \begin{bmatrix} \lambda_1 & & (0) \\ & \lambda_2 & \\ (0) & & \lambda_3 \end{bmatrix} \in M_3(\mathbb{R}),\ \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \right\}$.

. $0_{M_3(\mathbb{R})} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in F$

. Soit $u = \begin{pmatrix} \lambda_1 & & (0) \\ & \lambda_2 & \\ (0) & & \lambda_3 \end{pmatrix} \in F$, $v = \begin{pmatrix} \lambda_1' & & (0) \\ & \lambda_2' & \\ (0) & & \lambda_3' \end{pmatrix} \in F$ $\lambda \in \mathbb{R}$

$$\lambda u + v = \begin{pmatrix} \lambda.\lambda_1 + \lambda_1' & & (0) \\ & \lambda.\lambda_2 + \lambda_2' & \\ (0) & & \lambda.\lambda_3 + \lambda_3' \end{pmatrix} \in F$$

F est un sev de $M_3(\mathbb{R})$

2) $F = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \in M_3(\mathbb{R}),\ a,b,c,d,e,f \in \mathbb{R} \right\}$

. $0_{M_3(\mathbb{R})} \in F$.

. Avec $u, v \in F$ et $\lambda \in \mathbb{R}$, on a bien $\lambda u + v \in F$

F est un sev de $M_3(\mathbb{R})$

3) . M symétrique $\iff M^T = M$

$0_{M_3(\mathbb{R})} \in F$.

. Soient $u, v \in F$, $\lambda \in \mathbb{R}$

$$\begin{aligned} (\lambda u + v)^T &= (\lambda u)^T + v^T \\ &= \lambda u^T + v^T \\ &= \lambda u + v \end{aligned}$$

4) F = ensemble des matrices inversibles.

. $0_{M_n(\mathbb{R})} \notin F \rightarrow F$ n'est pas un sev.

5) $0_{M_2(\mathbb{R})} \notin F$.

Familles de vecteurs, bases et dimension

Exercice 1

$u = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix}$ $v = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ $w = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 4 \end{pmatrix}$

1) Soient $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ tels que :

$$\lambda_1 u + \lambda_2 v + \lambda_3 w = 0_{\mathbb{R}^4} \iff \begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = 0 \\ 2\lambda_1 + \lambda_2 + 2\lambda_3 = 0 \\ -\lambda_1 + \lambda_2 = 0 \\ 3\lambda_1 + \lambda_2 + 4\lambda_3 = 0 \end{cases} \iff \begin{cases} 2\lambda_1 + \lambda_3 = 0 \\ 3\lambda_1 + 2\lambda_3 = 0 \\ \lambda_2 = \lambda_1 \\ 4\lambda_1 + 4\lambda_3 = 0 \end{cases}$$

$$\iff \begin{cases} -\lambda_3 = 0 \\ 3\lambda_1 + 2\lambda_3 = 0 \\ \lambda_2 = \lambda_1 \\ \lambda_1 = -\lambda_3 \end{cases} \iff \lambda_1 = \lambda_2 = \lambda_3 = 0$$

Donc $\{u, v, w\}$ est libre dans $\mathbb{R}^4$

2) $Rg(u, v, w) = \dim(\mathrm{Vect}(u, v, w))$

Comme $\{u, v, w\}$ est libre, $Rg(u, v, w) = 3$

3) $Rg(u, v, w) < \dim \mathbb{R}^4$, $\{u, v, w\}$ n'est pas une base de $\mathbb{R}^4$.

Exercice 2

Base canonique de $\mathbb{R}^n$: $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $\cdots$ $e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$ $\quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \cdot e_1 + y \cdot e_2 + z \cdot e_3$

1) $M_{2,3}(\mathbb{R}) \ni M = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$ $\quad e_{1,1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $e_{1,2} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $e_{1,3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$e_{2,1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $e_{2,2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $e_{2,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

. libre ? Soient $\lambda_{1,1}, \ldots, \lambda_{2,3} \in \mathbb{R}$ tels que :

$$\lambda_{1,1} e_{1,1} + \cdots + \lambda_{2,3} e_{2,3} = 0_{M_{2,3}(\mathbb{R})} \iff \begin{pmatrix} \lambda_{11} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda_{2,3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} \lambda_{1,1} & \lambda_{1,2} & \lambda_{1,3} \\ \lambda_{2,1} & \lambda_{2,2} & \lambda_{2,3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \iff \lambda_{1,1} = \cdots = \lambda_{2,3} = 0$$

Donc la famille de vecteurs est libre dans $M_{2,3}(\mathbb{R})$

. Génératrice ?

Soit $M = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \in M_{2,3}(\mathbb{R})$

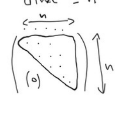
$$M = a \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b \times \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
$$+ d \times \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + e \times \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + f \times \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Finalement $\{\underbrace{e_{1,1}, \ldots, e_{2,3}}_{2 \times 3 = 6 \text{ matrices}}\}$ forme une base de $M_{2,3}(\mathbb{R})$

2) $\dim M_{n,p}(\mathbb{R}) = n \times p$

3) . $\begin{pmatrix} \lambda_1 & & (0) \\ & \lambda_2 & \\ (0) & & \ddots & \\ & & & \lambda_n \end{pmatrix}$ $\quad \underbrace{\begin{pmatrix} 1 & & (0) \\ & 0 & \\ (0) & & \ddots & 0 \end{pmatrix} \begin{pmatrix} 0 & & (0) \\ & 1 & \\ (0) & & \ddots & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & & (0) \\ & \ddots & \\ (0) & & 1 \end{pmatrix}}_{n \text{ éléments.}}$

dim $= n$

. 

$n = 3$: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$n = 3 \to \dim 6$

$$\underset{\times}{n} + n - 1 + n - 2 + \cdots + 2 + \underset{\times}{1} = n + 1 + n + 1 + \cdots n + 1 = \frac{n(n+1)}{2}$$

Exercice 3

. libre ? Soient $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ tels que $\lambda_1 u + \lambda_2 v + \lambda_3 w = 0_{\mathbb{R}^3}$

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = 0 \\ \lambda_2 + \lambda_3 = 0 \\ \lambda_3 = 0 \end{cases} \iff \lambda_1 = \lambda_2 = \lambda_3 = 0$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Génératrice ?

Soit $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$ On cherche $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ tels que

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda_1 u + \lambda_2 v + \lambda_3 w \iff \begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = x \\ \lambda_2 + \lambda_3 = y \\ \lambda_3 = z \end{cases}$$

$$\iff \begin{cases} \lambda_1 + \lambda_2 + z = x \\ \lambda_2 + z = y \\ \lambda_3 = z \end{cases} \iff \begin{cases} \lambda_1 + y - z = x \\ \lambda_2 = y - z \\ \lambda_3 = z \end{cases} \iff \begin{cases} \lambda_1 = x + z - y \\ \lambda_2 = y - z \\ \lambda_3 = z \end{cases}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (x + z - y) u + (y - z) v + z w$$

Donc $\{u, v, w\}$ génératrice.

Exercice 4

$$F = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \,,\ \begin{matrix} x - w + z = 0 \\ x + z = w \end{matrix} \right\} = \left\{ \begin{pmatrix} x \\ y \\ z \\ x+z \end{pmatrix} \in \mathbb{R}^4 \,,\ x, y, z \in \mathbb{R} \right\}$$

On cherche une base de F :

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \in F \ ; \ v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \in F \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \in F$$

- Soient $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ tels que $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0_{\mathbb{R}^4}$

$$\Leftrightarrow \begin{cases} \lambda_1 & = 0 \\ \quad \lambda_2 & = 0 \\ \qquad \lambda_3 & = 0 \\ \lambda_1 + \lambda_3 & = 0 \end{cases} \Leftrightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$$

Donc $\{v_1, v_2, v_3\}$ est libre.

- Soit $v = \begin{pmatrix} x \\ y \\ z \\ x+z \end{pmatrix} \in F$. On cherche $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ tels que :

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = v \Leftrightarrow \begin{cases} \lambda_1 & = x \\ \quad \lambda_2 & = y \\ \qquad \lambda_3 & = z \\ \lambda_1 + \lambda_3 & = x + z \end{cases}$$

$$v = x \cdot v_1 + y \cdot v_2 + z v_3$$

Donc $F = \mathrm{Vect}(v_1, v_2, v_3)$

Finalement $\{v_1, v_2, v_3\}$ est une base de F : $\dim F = 3$

Exercice 4

$$F = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \ , \ \begin{matrix} x - w + z = 0 \\ x + z = w \end{matrix} \right\} = \left\{ \begin{pmatrix} x \\ y \\ z \\ x+z \end{pmatrix} \in \mathbb{R}^4, \ x, y, z \in \mathbb{R} \right\}$$

On cherche une base de F :

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \in F \ ; \ v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \in F \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \in F$$

- Soient $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ tels que $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0_{\mathbb{R}^4}$

$$\Leftrightarrow \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0 \\ \lambda_3 = 0 \\ \lambda_1 + \lambda_3 = 0 \end{cases} \Leftrightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$$

Donc $\{v_1, v_2, v_3\}$ est libre.

- Soit $v = \begin{pmatrix} x \\ y \\ z \\ x+z \end{pmatrix} \in F$. On cherche $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ tels que :

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = v \Leftrightarrow \begin{cases} \lambda_1 = x \\ \lambda_2 = y \\ \lambda_3 = z \\ \lambda_1 + \lambda_3 = x + z \end{cases}$$

$$v = x \cdot v_1 + y \cdot v_2 + z v_3$$

Donc $F = \mathrm{Vect}(v_1, v_2, v_3)$

Finalement $\{v_1, v_2, v_3\}$ est une base de F : $\dim F = 3$

Exemple du cours :

$$f\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} \right) = \begin{pmatrix} y - x \\ 2z + y \end{pmatrix} \qquad M(f) = \begin{array}{c} \begin{matrix} f(e_1) & f(e_2) & f(e_3) \end{matrix} \\ \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \end{array} \in M_{2,3}(\mathbb{R})$$

$$f : \underset{E}{\mathbb{R}^3} \longrightarrow \underset{F}{\mathbb{R}^2} \qquad e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \ e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \ e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$u = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad f(u) = \begin{pmatrix} 2 - 1 \\ 2 \times 3 + 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \times 1 + 1 \times 2 + 0 \times 3 \\ 0 \times 1 + 1 \times 2 + 2 \times 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \end{pmatrix}$$

Applications linéaires, matrices.

Exercice 1

Soient $u = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$, $v = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \in \mathbb{R}^3$, $\lambda \in \mathbb{R}$

1) $f : \mathbb{R}^3 \to \mathbb{R}^2$
$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} y \\ x \end{pmatrix}$

$\checkmark$. $f(u+v) = f\left(\begin{pmatrix} x+x' \\ y+y' \\ z+z' \end{pmatrix}\right) = \begin{pmatrix} y+y' \\ x+x' \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix} + \begin{pmatrix} y' \\ x' \end{pmatrix} = f(u) + f(v)$

$\checkmark$. $f(\lambda u) = f\left(\begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \end{pmatrix}\right) = \begin{pmatrix} \lambda y \\ \lambda x \end{pmatrix} = \lambda . \begin{pmatrix} y \\ x \end{pmatrix} = \lambda f(u)$

f est une application linéaire. $f \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$

2) $f : \mathbb{R}^3 \to \mathbb{R}^2$
$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x \\ 0 \end{pmatrix}$

$\checkmark$. $f(u+v) = \begin{pmatrix} x+x' \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} x' \\ 0 \end{pmatrix} = f(u) + f(v)$

$\checkmark$. $f(\lambda u) = \begin{pmatrix} \lambda x \\ 0 \end{pmatrix} = \lambda . \begin{pmatrix} x \\ 0 \end{pmatrix} = \lambda f(u)$

$f \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$

3) $f : \mathbb{R}^3 \to \mathbb{R}^2$
$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x \\ 1 \end{pmatrix}$

Avec : $u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ et $v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$:

$f(u+v) = f\left(\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ et $f(u) + f(v) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$f \notin \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$

4) $f : \mathbb{R}^3 \to \mathbb{R}^2$
$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x^2+y^2 \\ z \end{pmatrix}$

$u = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$ $v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$f(u+v) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ or $f(u) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ et $f(v) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$f(u) + f(v) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$f \notin \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$

5) $f : \mathbb{R}^3 \to \mathbb{R}^2$
$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

. $f(u+v) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = f(u) + f(v)$

. $f(\lambda u) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \lambda f(u)$

$f \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$

6) $f(u+v) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ et $f(u) + f(v) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \neq f(u+v)$

$f \notin \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$

7) $f\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x \\ yz \end{pmatrix}$ $\quad u = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\quad v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$f(u+v) = f\left(\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ or $f(u) + f(v) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$\neq f(u+v)$

$f \notin \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$

8) $f\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x \\ x+y+z \end{pmatrix}$

· $f(u+v) = \begin{pmatrix} x+x' \\ x+x'+y+y'+z+z' \end{pmatrix} = \begin{pmatrix} x \\ x+y+z \end{pmatrix} + \begin{pmatrix} x' \\ x'+y'+z' \end{pmatrix}$

$= f(u) + f(v)$

· $f(\lambda u) = \begin{pmatrix} \lambda x \\ \lambda x + \lambda y + \lambda z \end{pmatrix} = \lambda f(u)$

Exercice 2

1) $f : \mathbb{R}^3 \to \mathbb{R}^2$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} y \\ x \end{pmatrix}$ $\qquad e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$f(e_1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\qquad f(e_3) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$M_{B,B'}(f) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $\qquad f(e_2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

2) $f : \mathbb{R}^3 \to \mathbb{R}^2$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x \\ 0 \end{pmatrix}$ $\qquad f(e_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $f(e_2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $f(e_3) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$M_{B,B'}(f) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

5) $M(f) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

8) $f : \mathbb{R}^3 \to \mathbb{R}^2$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ x+y+z \end{pmatrix}$ $\qquad f(e_1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $f(e_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $f(e_3) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$M(f) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

Exercice 3

$f : \mathbb{R}^3 \to \mathbb{R}^3$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$

1) Endomorphisme = linéaire et $E \to E$

Soient $u, v \in \mathbb{R}^3$ et $\lambda \in \mathbb{R}$; $u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $v = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$

· $f(\lambda u + v) = \begin{pmatrix} \lambda x + x' \\ \lambda y + y' \\ 0 \end{pmatrix}$

$= \lambda \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} x' \\ y' \\ 0 \end{pmatrix} = \lambda f(u) + f(v)$

Donc $f : \mathbb{R}^3 \to \mathbb{R}^3$ est linéaire : $f \in \mathcal{L}(\mathbb{R}^3)$

2) $f(e_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $f(e_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $f(e_3) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$M_B(f) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\qquad$ Remarque : $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

3) . Injective ?

Contre exemple : $f\left(\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $\qquad f\left(\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

f non injective.

. Surjective ?

$f : \mathbb{R}^3 \to \mathbb{R}^3$

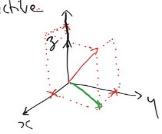
On cherche $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$ n'ayant pas d'antécédants par f :

$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$ avec $x, y \in \mathbb{R}$ n'admet aucun antécédant par f.

f non surjective.

. f non bijective

4)



Exercice 4

$f : M_2(\mathbb{R}) \to M_2(\mathbb{R})$, $M \to AM$ $\qquad A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

1) Soient $M_1, M_2 \in M_2(\mathbb{R})$ et $\lambda \in \mathbb{R}$

$f(\lambda M_1 + M_2) = A(\lambda M_1 + M_2) = \lambda . AM_1 + AM_2$

$= \lambda f(M_1) + f(M_2)$

$f \in \mathcal{L}(M_2(\mathbb{R}))$ $\qquad \dim(M_{n,p}) = n \times p$

2) $B = \left\{ \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{e_1}, \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{e_2}, \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{e_3}, \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{e_4} \right\}$ base canonique de $M_2(\mathbb{R})$

· $f(e_1) = Ae_1 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix} = 1 \times e_1 + 0 \times e_2 + 3 \times e_3 + 0 \times e_4$

La représentation de $f(e_1)$ dans B est $\begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}$: $f(e_1)|_B = \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}$

· $f(e_2) = Ae_2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix} = 0 \times e_1 + 1 \times e_2 + 0 \times e_3 + 3 \times e_4$ $\qquad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 3 \end{pmatrix}$

· $f(e_3) = Ae_3 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix}$ $\qquad f(e_3)|_B = \begin{pmatrix} 2 \\ 0 \\ 4 \\ 0 \end{pmatrix}$

· $f(e_4) = Ae_4 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 4 \end{pmatrix}$ $\qquad f(e_4)|_B = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 4 \end{pmatrix}$

$M_B(f) = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{pmatrix}$