MCSC 6020G
Assignment 2

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Due Tuesday, October 24, 2017, at 11 am

Submit answers to the following questions clearly explaining your reasoning. Your write-up should be as professional as possible in the quality of mathematical argument and in the quality of written English. Typesetting in LaTeX is highly encouraged. Submit electronically as a Blackboard message to the instructor along with a zip file containing all files (pdf, codes, figures, latex source) by the due date.

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1. A matrix $A \in \mathbb{C}^{n\times n}$ is said to be *strictly column diagonally dominant* if

$$|a_{j,j}| > \sum_{\substack{i=1\\ i\neq j}} |a_{i,j}| \quad (j = 1\colon\ n).$$

Prove that a strictly column diagonally dominant matrix is always nonsingular. [Hint: let $D = \mathrm{diag}(a_{1,1}, a_{2,2}, \ldots, a_{n,n})$ be the diagonal matrix of diagonal entries of A; show that A can be factored as $A = (I+E)D$ for a suitable matrix E and argue that the factors are nonsingular by construction.]

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2. A polynomial $p(x) = \sum_{j=1}^{n} c_j x^{j-1}$ of degree at most $n-1$ that fits a prescribed function $f(x)$ can be determined by sampling the function at m points (assume $m \geq n$). This leads to a system of linear equations

$$p(x_k) = \sum_{j=1}^{n} c_j x_k^{j-1} = f(x_k) \qquad (k = 1, 2, \ldots, m)$$

for the unknown coefficients c_j ($j = 1, 2, \ldots, n$). When $m = n$, the polynomial interpolates the samples; when $m > n$, the system of equations is overdetemined and the resulting polynomial fits the data in a least-squares sense.

In particular, when $m > n$, this overdetermined linear system in matrix form is

$$A\mathbf{c} = \mathbf{f} \text{ with } A = \begin{pmatrix} 1 & x_1^1 & x_1^2, & \ldots & x_1^{n-1} \\ 1 & x_2^1 & x_2^2, & \ldots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_m^1 & x_m^2, & \ldots & x_m^{n-1} \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \quad \text{and} \quad \mathbf{f} = \begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_m) \end{pmatrix}. \tag{1}$$

For this problem, you will write a program to construct a least-squares polynomial fitting certain data by distinct computational methods. In particular, you will find the least-squares polynomial of degree 11 (i.e., $n = 12$) fitting $m = 50$ points obtained by sampling the function $f(x) = \cos(4x)$ at m points uniformly spaced across the interval $[0, 1]$. That is, the programs should produce the 12 coefficients of the least-squares polynomial obtained by "solving" the over-determined linear system of equations (1).

(a) Write a program to compute the coefficient matrix A and the right-hand side vector $\mathbf{f}$ from (1). Assume $n = 12$, $m = 50$, and the function f and the sample points $\{x_k\}_{k=1}^{m}$ are as specified above. (In MATLAB, the utilities `linspace`, `vander`, and `fliplr` may be of some use).

(b) Modify your program from part (2a) to compute the coefficients $\mathbf{c}^{(\text{Chol})}$ obtained when solving the normal equations $A^T A\mathbf{c} = A^T\mathbf{f}$ by computing the Cholesky decomposition of $A^T A$ (i.e., $A^T A = L^T L$). Use whatever software you like to compute the Cholesky decomposition (just be explicit about what it is).

(c) Modify your program from part (2b) to compute the coefficients $\mathbf{c}^{(\text{QR})}$ obtained using the QR decompostion of A (i.e., $A = QR$). Use whatever software you like to compute the QR decomposition (just be explicit about what it is).

(d) Modify your program from part (2c) to compute the coefficients $\mathbf{c}^{(\text{SVD})}$ obtained using the singular value decompostion of A (i.e., $A = U\Sigma V^T$). Use whatever software you like to compute the SVD (just be explicit about what it is).

(e) Plot the absolute residuals $|\mathbf{r} - A\mathbf{c}|$ for each of the three solutions $\mathbf{c}$ computed against x. That is, plot the residuals computed for each of the coefficient vectors $\mathbf{c}^{(\text{Chol})}$ from part (2b), $\mathbf{c}^{(\text{QR})}$ from part (2c), and $\mathbf{c}^{(\text{SVD})}$ from part (2d) on the same axes (you will probably need to use some kind of logarithmic scaling to see the trends). These matrix-vector products actually show the absolute misfit $|f(x_k) - p(x_k)|$ ($k = 1, \ldots, m$) at each point, i.e., the amount by which the least-squares polynomial fails to interpolate the actual data. Comment on your results.

(f) Finally, print a table to 16 digits of precision showing the coefficients of the least-squares polynomials in parts (2b) through (2d). Your table should have $n = 12$ rows and 3 columns. Shade, underline, or highlight in some manner the digits that appear to be incorrect. You do not have to explain the results, but do comment on any differences you observe.

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3. (a) Let $A \in C^{n\times n}$ be an outer product of two vectors, i.e., $A = \mathbf{y}\mathbf{x}^H$ for some vectors $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$. Show that $\|A\|_2 = \|\mathbf{x}\|_2 \|\mathbf{y}\|_2$, i.e., the (matrix) ℓ_2-norm of A is the product of the (vector) ℓ_2-norms of $\mathbf{x}$ and $\mathbf{y}$.

   (b) Let $K_p(A) = \|A\|_p \left\|A^{-1}\right\|_p$ be the condition number of matrix inversion in the p-norm. If $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times n}$ are both invertible, prove that $K_p(AB) \leq K_p(A)K_p(B)$.

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4. The determinant of a triangular matrix is the product of its diagonal entries. Use this fact to develop a MATLAB function `ludet` for computing the determinant of any arbitrary $A \in \mathbb{C}^{n\times n}$ using its LU decomposition. You can use MATLAB's built-in `lu` function or any other library code, or you may program your own routine. How can you determine the proper sign for the determinant? To avoid risk of overflow or underflow, you may wish to consider computing the logarithm of the determinant rather than the determinant itself. Test your function on a variety of random square matrices (include some that are singular to make sure your program can detect singular matrices). Submit the code for `ludet.m` and a MATLAB diary file documenting your experiments.