Lecture 8: Variational Methods

The Variational Approach

Create energy $E[u]$

Calculate first variation ∇E

Minimize by steepest descent $\frac{\partial u}{\partial t} = -\nabla E$

Discretize and code

$$u^{n+1} = u^n + \Delta t(-\nabla E)$$

- The challenge is to design "better" energies and faster minimization schemes.

Total Variation

- *(Rudin-Osher-Fatemi, 1992)* proposed the Total Variation (TV) denoising model.

$$E_{TV}[u|f] = \int_{\Omega} |\nabla u| d\vec{x} + \lambda \int_{\Omega} (u - f)^2 d\vec{x}$$

- TV is the most famous and widely used image restoration model. Its simplicity and adaptability makes it very useful.

Black & White TV

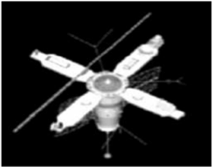
- Text and barcode images are black & white. We want a denoising method that produces black & white images.
- Suppose we want our final result to be either 0 (black) or 255 (white).
- We could add a "double-well" term that penalizes values of u that are not 0 or 255 *(Esedoglu, 2005)*.

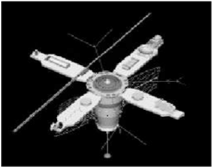
$$E_{TV}[u|f] = \int_{\Omega} |\nabla u| + \lambda(u-f)^2 + \beta u^2 (255-u)^2 \, d\vec{x}$$

TV Deblurring

- The best part of the TV model is that it can be adapted to other problems.
- Image blurred by kernel K *(Chang-Chien-Wang-Xu, 2008)*.

$$\min E_{TV}[u|f] = \int_{\Omega} |\nabla u| d\vec{x} + \boxed{\lambda \int_{\Omega} (K * u - f)^2 d\vec{x}}$$

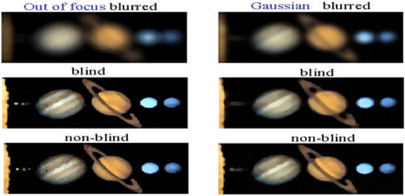




TV Deblurring

- If we don't know the blur kernel K, then this becomes a much harder *blind deconvolution* problem.
- *(Chan-Wong, 1998)* suggested that K should also be smooth.

$$\min E_{TV}[u, K|f] = \int_{\Omega} |\nabla u| d\vec{x} + \boxed{\lambda \int_{\Omega} (K * u - f)^2 d\vec{x} + \beta \int_{\Omega} |\nabla K| d\vec{x}}$$



Bayesian Interpretation

- From a statistical perspective, we are trying to find the most likely ideal image *u* given the noisy image *f*.

$$\max P(u|f)$$

- Applying Baye's Theorem, we get

$$\boxed{\max P(u|f) = \max \frac{P(u)P(f|u)}{P(f)}}$$

- Since f is given, *P(f)* is constant. So we can ignore it in our optimization.

- Taking the negative log likelihood, our maximization problem becomes a minimization problem.

$$\max P(u)P(f|u) \;\rightarrow\; \min\{-\log P(u) - \log P(f|u)\}$$

Bayesian Interpretation

$$\min\{-\log P(u) - \log P(f|u)\}$$

- The first term is a *prior* describing how likely the image u is to occur in the “real world”.
- So this can correspond to our TV regularization.

$$\int_{\Omega} |\nabla u| d\vec{x}$$

Bayesian Interpretation

$$\min\{-\log P(u) - \log P(f|u)\}$$

- The second term describes the probability that an ideal image u gave rise to the noisy image f.
- So this term describes the noise process.
- Assuming Gaussian noise

$$P(f|u) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{(u-f)^2}{2\sigma^2}\right)$$

$$-\log P(f|u) = \frac{1}{2\sigma^2}(u-f)^2 + const$$

Bayesian Interpretation

$$\min\{-\log P(u) - \log P(f|u)\}$$

$$\min \int_{\Omega} |\nabla u| d\vec{x} + \frac{1}{2\sigma^2} \int_{\Omega} (u-f)^2 d\vec{x}$$

- So the standard TV model is really designed for Gaussian noise.

$$\min E_{TV}[u|f] = \int_{\Omega} |\nabla u| d\vec{x} + \lambda \int_{\Omega} (u-f)^2 d\vec{x}$$

- The parameter λ should be inversely proportional to the amount of noise.

Poisson Noise

- We can repeat this derivation for other types of noise.
- If we had Poisson noise

$$\boxed{P(f|u) = \frac{e^{-u}u^{f}}{u!}}$$

$$-\log P(f|u) = u - f\log u + const$$

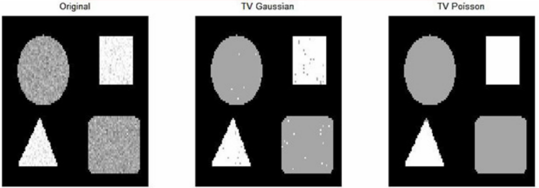
- So the TV model for Poisson noise should be

$$\min E_{TV}[u|f] = \int_{\Omega} |\nabla u| d\vec{x} + \lambda \int_{\Omega} (u - f\log u) d\vec{x}$$

Poisson Noise

- The TV-Poisson model is better suited for denoising images corrupted by Poisson noise, such as PET images *(Jonsson-Huang-Chan, 1998)*.

$$\min E_{TV}[u|f] = \int_{\Omega} |\nabla u| d\vec{x} + \lambda \int_{\Omega} (u - f \log u) d\vec{x}$$



Noise Models

- Gaussian Noise *(Rudin-Osher-Fatemi, 1992)*

$$\min E_{TV}[u|f] = \int_{\Omega} |\nabla u| d\vec{x} + \lambda \int_{\Omega} (u - f)^2 d\vec{x}$$

- Poisson Noise *(Jonsson-Huang-Chan, 1998)*

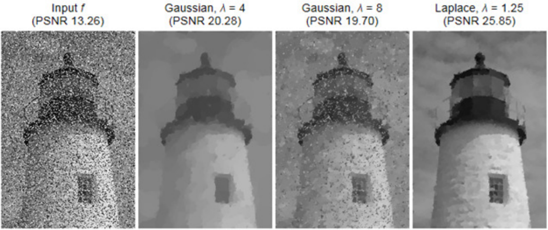
$$\min E_{TV}[u|f] = \int_{\Omega} |\nabla u| d\vec{x} + \lambda \int_{\Omega} (u - f \log u) d\vec{x}$$

- Laplace Noise *(Chan-Esedoglu)*

$$\min E_{TV}[u|f] = \int_{\Omega} |\nabla u| d\vec{x} + \lambda \int_{\Omega} |u - f| d\vec{x}$$

Noise Models

- Guessing the closest noise model can improve results.
- The lighthouse below was corrupted by Salt & Pepper noise, which resembles Laplace noise *(Getreuer, 2012)*.



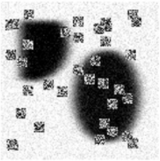
TV Inpainting

- Inpainting: Fill in a damaged region D.
- We don't have data on D, so we turn off the fidelity for the pixels in D *(Chan-Shen, 1998)*.

$$\min E_{TV}[u|f] = \int_{\Omega} |\nabla u| d\vec{x} + \lambda \int_{\Omega \setminus D} (u-f)^2 d\vec{x}$$

- Note that the expresssion is not convex anymore. It helps to initialize the damaged region D by filling it with noise or a checkerboard texture.

Noisy motion-blurred image with missing data



TV restoration and inpainting

TV Inpainting

- *(Chan-Kang, 2005)* established error bounds for TV inpainting.

- Assuming the functions below are in $C^2(\Omega)$
  - Original image *f*
  - TV solution *u*
  - "True" image u_{TRUE}

- If *D* has a smooth boundary and can be covered by an ellipse with minor diameter *d*, then

$$\boxed{|u - u_{TRUE}| \le 2Ld^2}$$

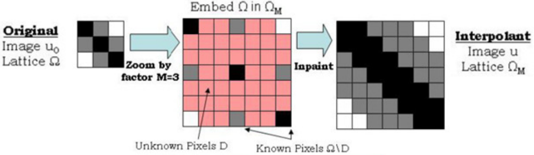
where L is a constant satisfying $\boxed{L \le |\Delta u_{TRUE}|.}$

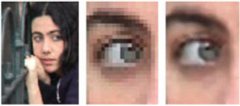
TV Inpainting

- TV is best at repairing long thin regions.
- In the picture below, the 3 ellipses all have the same area.

TV Zooming

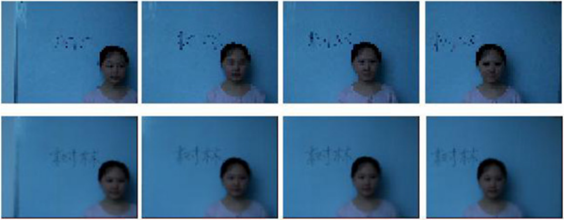
- If we think of image zooming as filling in pixels "in between" pixels, then an inpainting method defines a zooming method *(Bertalmio-Bertozzi-Sapiro, 2001)*.





TV Super-resolution

Original sequence



TV SR sequence

TV Super-resolution