MCSC 6020G
Assignment 1

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Due 11:00 am, Thursday, October 5, 2017

Submit answers to the following questions clearly explaining your reasoning. Your write-up should be as professional as possible in the quality of mathematical argument and in the quality of written English. Typesetting in LaTeX is highly encouraged. Attach your code as a zip file as well if necessary. Submit electronically on Blackboard or as a private slack message by the due date.

1. B is called skew-symmetric when $B^T = -B$.
   (a) Provide an example of a 3×3 skew-symmetric matrix.
   (b) Let $A = (I + B)(I - B)^{-1}$ and show that $A^{-1} = A^T$ if B is skew-symmetric.

2. Assume that $\mathbf{x}$ is a vector in $\mathbb{C}^{n\times 1}$. Prove the following inequality and provide example of a vector for which equality is achieved. $\frac{1}{n}\|\mathbf{x}\|_1 \leq \|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2$

3. Let $A \in \mathbb{C}^{n\times n}$ be a nonsingular matrix.
   (a) Given $\mathbf{u}, \mathbf{v} \in \mathbb{C}^{n\times 1}$, prove that

$$\left[A - \mathbf{u}\mathbf{v}^H\right]^{-1} = A^{-1} + \frac{A^{-1}\mathbf{u}\mathbf{v}^H A^{-1}}{1 - \mathbf{v}^H A^{-1}\mathbf{u}}. \tag{1}$$

   This is *the Sherman-Morrison formula* that provides a means of computing the inverse of a rank-one correction of a matrix with known inverse. [Hint: multiply the right-hand side of (1) by $A - \mathbf{u}\mathbf{v}^H$ and expand].

   (b) Based on the Sherman-Morrison formula (1), write an algorithm (in pseudocode) to explain how you would compute the solution $\mathbf{x} \in \mathbb{C}^n$ of the system of equations

$$(A - \mathbf{u}\mathbf{v}^H)\,\mathbf{x} = \mathbf{b}.$$

   Assume that you have access to a function `Ainv` that accepts as input any vector $\mathbf{u} \in \mathbb{C}^n$ and returns the matrix-vector product $A^{-1}\mathbf{u}$, i.e., the line of pseudocode

$$\mathbf{v} \leftarrow \texttt{Ainv}(\mathbf{u})$$

   is equivalent to the assignment $\mathbf{v} := A^{-1}\mathbf{u}$. Assume that each call to `Ainv` costs n^2 flops. What asymptotic cost (in flops) do you expect for your proposed algorithm?

4. Which among the following approximations of π better limits the propagation of rounding errors? Compare using Matlab (or any tool) the obtained results as a function of the number of the terms in each of the following sums.

$$\pi = 4 \times \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots\right)$$

$$\pi = 6 \times \left(0.5 + \frac{(0.5)^3}{2 \times 3} + \frac{3 \times (0.5)^5}{2 \times 4 \times 5} + \frac{3 \times 5 \times (0.5)^7}{2 \times 4 \times 6 \times 7} + \dots\right)$$

5. Let $x, y, \widetilde{x}, \widetilde{y} \in \mathbb{C}$ be nonzero scalars. Show that

$$\left|\frac{xy - \widetilde{x}\widetilde{y}}{xy}\right| \leq (2 + \epsilon)\epsilon \quad \text{where} \quad \epsilon := \max\left\{\frac{|x - \widetilde{x}|}{|x|}, \frac{|y - \widetilde{y}|}{|y|}\right\}.$$

Moreover, if $\epsilon \leq 1$, show that

$$\left|\frac{xy - \widetilde{x}\widetilde{y}}{xy}\right| \leq 3\epsilon.$$

Therefore, the relative condition number of scalar multiplication is at most 3. We conclude that scalar multiplication is well-conditioned in the relative sense provided that the relative errors in the input data are sufficiently small.