Information Theory and Applications
Lossy compression

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Lossy compression

- The complete description of a real number requires an infinite number of bits and is impossible to achieve
- An approximate description can be obtained by using a finite number of bits but implies an error that is called distortion
- The relationship between the rate (# bits/symbol) and the average distortion is studied by Rate Distortion Theory (RDT)

- We begin by considering the problem of representing a single continuous random variable through a finite number of bits

Quantization

- Consider a Gaussian random variable $X \sim \mathcal{N}(0,1)$ and its quantized representation $\hat{X}$ over $R = 1$ bit:
$$\hat{X} = \begin{cases} +a & X \geq 0 \\ -a & X < 0 \end{cases}$$
- We use the mean-square error as a distortion measure:
$$D = E\left[\left(X - \hat{X}\right)^2\right]$$
- The minimum distortion is obtained by setting $a = \sqrt{\frac{2}{\pi}} \approx 0.79$
- The minimum distortion is
$$D_{\min} = 1 - \frac{2}{\pi} = 0.3634$$

Calculation of $D_{\min}$

- Recall that $X \sim \mathcal{N}(0,1)$, $D = E\left[\left(X - \hat{X}\right)^2\right]$, $\hat{X} = \begin{cases} +a & X \geq 0 \\ -a & X < 0 \end{cases}$
- Then, the distortion is

$g(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$$D = \int_{-\infty}^{0} (x+a)^2 g(x)dx + \int_{0}^{\infty} (x-a)^2 g(x)dx$$

- To minimize D we take the derivative with respect to a:

$$\frac{\partial D}{\partial a} = 2\int_{-\infty}^{0} (x+a) g(x)dx - 2\int_{0}^{\infty} (x-a) g(x)dx = 2a - 2\int_{-\infty}^{+\infty} |x| g(x)dx$$

- Setting the derivative equal to 0 we obtain:

$u = \frac{x^2}{2}$

$$a = \int_{-\infty}^{+\infty} |x| g(x)dx = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} x e^{-x^2/2} dx = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-u} du = \sqrt{\frac{2}{\pi}} \approx 0.79$$

$du = x\,dx$

- Since $D = 1 + a^2 - 2a\int_{-\infty}^{+\infty} |x| g(x)dx = 1 + \frac{2}{\pi} - 2 \cdot \frac{2}{\pi}$, the minimum distortion is

$$D_{\min} = 1 - \frac{2}{\pi} = 0.3634$$

Quantization

- Increasing the rate to $R = 2$ bits makes the problem more difficult
- A general algorithm exists which minimizes the average distortion for an arbitrary number of representative points (Lloyd algorithm)
- The Lloyd algorithm is iterative: starting from an initial quantization, this is progressively improved to minimize the average distortion

Lloyd Algorithm

- Consider a random variable X and its quantization characterized by
  - $(N-1)$ thresholds $\boldsymbol{t} = (t_1, \dots, t_{N-1})$ dividing the real line into N intervals, $(-\infty, t_1), (t_1, t_2), \dots, (t_{N-1}, \infty)$ increasing
  - N quantized values $\boldsymbol{x} = (x_1, \dots, x_N)$
- The distortion is the average mean-square error:

$$D(\boldsymbol{t}, \boldsymbol{x}) = \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} (x - x_i)^2 f_X(x) dx$$

- In this expression we define $t_0 \stackrel{\text{def}}{=} -\infty, t_N \stackrel{\text{def}}{=} \infty$
- The distortion is nonnegative and unbounded so that it can have e minimum with respect to the vectors $\boldsymbol{t}, \boldsymbol{x}$

Lloyd Algorithm

$$\frac{\partial}{\partial x}\int_a^x f(u)du = f(x)$$
$$\frac{\partial}{\partial x}\int_x^b f(u)du = -f(x)$$

- A necessary condition for the minimum is that the partial derivatives are all equal to 0
- The partial derivatives with respect to the thresholds are

$$\frac{\partial D}{\partial t_i} = \frac{\partial}{\partial t_i}\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}(x-x_j)^2 f_X(x)dx = [(t_i-x_i)^2-(t_i-x_{i+1})^2]f_X(t_i) = 0$$

- Since $(t_i-x_i)^2-(t_i-x_{i+1})^2 = (2t_i-x_i-x_{i+1})(x_i-x_{i+1})$, we get the following equations:

$$t_i = \frac{x_i+x_{i+1}}{2}, \qquad i = 1,\dots,N-1$$

Lloyd Algorithm

- Next, we consider the partial derivatives with respect to the quantized values:

$$\frac{\partial D}{\partial x_i} = \frac{\partial}{\partial x_i} \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} (x - x_j)^2 f_X(x) dx = \int_{t_{i-1}}^{t_i} 2(x_i - x) f_X(x) dx = 0$$

- Then, we get the following equations:

$$x_i = \frac{\int_{t_{i-1}}^{t_i} x f_X(x) dx}{\underbrace{\int_{t_{i-1}}^{t_i} f_X(x) dx}_{P(X \in (t_{i-1}, t_i))}} = E[X \mid X \in (t_{i-1}, t_i)], \qquad i = 1, \dots, N$$

Lloyd Algorithm

- The resulting iterative algorithm is described by the following steps:
- Initialize the quantized values: $\boldsymbol{x}^{(0)} = \boldsymbol{x}_0$
- Repeat the following steps for $n = 0,1,2 \ldots$

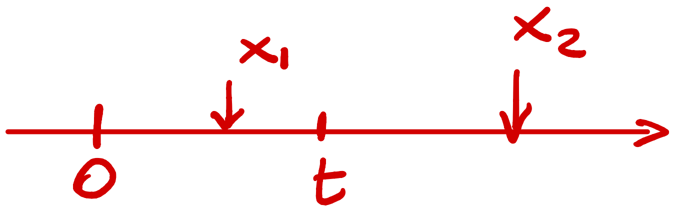
$$t_i^{(n+1)} = \frac{x_i^{(n)} + x_{i+1}^{(n)}}{2}, \qquad i = 1, \ldots, N-1$$

$$x_i^{(n+1)} = E[X \mid X \in (t_{i-1}^{(n+1)}, t_i^{(n+1)})], \qquad i = 1, \ldots, N$$

- Until $\left\|\boldsymbol{x}^{(n+1)} - \boldsymbol{x}^{(n)}\right\|^2$ falls below a convergence threshold

Lloyd Algorithm example

- We apply the algorithm to the optimum quantization of $X \sim \mathcal{N}(0,1)$ with $R = 2$ bits (4 levels)
- By symmetry, the threshold are $(-t, 0, t)$ and the quantized values are $(-x_2, -x_1, x_1, x_2)$
- The iterative equations are



$$t^{(n+1)} = \frac{x_1^{(n)} + x_2^{(n)}}{2}$$

$$x_1^{(n+1)} = \frac{1 - \exp\left(-\frac{\left(t^{(n+1)}\right)^2}{2}\right)}{\sqrt{2\pi}\left(\frac{1}{2} - Q\left(t^{(n+1)}\right)\right)}$$

$$x_2^{(n+1)} = \frac{\exp\left(-\frac{\left(t^{(n+1)}\right)^2}{2}\right)}{\sqrt{2\pi}Q\left(t^{(n+1)}\right)}$$

$$\frac{1}{\sqrt{2\pi}}\int_0^t x e^{-x^2/2}\,dx = \frac{1}{\sqrt{2\pi}}\int_0^{t^2/2} e^{-u}\,du \qquad u = \frac{x^2}{2}$$

$$du = x\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left(1 - e^{-t^2/2}\right)$$

$$\int_0^t \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx = P(0 < N(0,1) < t)$$

$$= P(0 < N(0,1) < \infty) - P(N(0,1) > t)$$

$$= \frac{1}{2} - Q(t)$$

$$E[X \mid 0 < X < t] = \frac{1 - e^{-t^2/2}}{\sqrt{2\pi}\left(\frac{1}{2} - Q(t)\right)}$$

$$E[X \mid X > t]$$

$$= \frac{\int_t^\infty \frac{1}{\sqrt{2\pi}} x e^{-x^2/2}\, dx}{\int_t^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx} = \frac{\frac{1}{\sqrt{2\pi}} \int_{t^2/2}^\infty e^{-u}\, du}{Q(t)}$$

$$= \frac{e^{-t^2/2}}{\sqrt{2\pi}\, Q(t)}$$

Lloyd Algorithm example

- We can also calculate the distortion:

$$D = 2\int_0^t (x - x_1)^2 \frac{\exp(-x^2/2)}{\sqrt{2\pi}}\,dx + 2\int_t^\infty (x - x_2)^2 \frac{\exp(-x^2/2)}{\sqrt{2\pi}}\,dx$$
$$= 1 - 2\sqrt{\frac{2}{\pi}}\,x_1 + x_1^2 - 2\sqrt{\frac{2}{\pi}}\,e^{-t^2/2}\,(x_2 - x_1) + 2\,(x_2^2 - x_1^2)Q(t)$$

Lloyd Algorithm example (evolution of x_1, x_2)

Iter.	x_1	t	x_2	D
0	1.0000	1.5000	2.0000	
1	0.6220	1.5000	1.9387	0.1627
2	0.5582	1.2803	1.7540	0.1342
3	0.5169	1.1561	1.6515	0.1235
4	0.4913	1.0842	1.5930	0.1196
5	0.4758	1.0422	1.5590	0.1182
6	0.4665	1.0174	1.5391	0.1178
7	0.4609	1.0028	1.5274	0.1176
8	0.4576	0.9942	1.5205	0.1175
9	0.4557	0.9890	1.5164	0.1175
10	0.4545	0.9860	1.5139	0.1175

Iter.	x_1	t	x_2	D
11	0.4538	0.9842	1.5125	0.1175
12	0.4534	0.9831	1.5117	0.1175
13	0.4531	0.9825	1.5112	0.1175
14	0.4530	0.9821	1.5109	0.1175
15	0.4529	0.9819	1.5107	0.1175
16	0.4529	0.9818	1.5106	0.1175
17	0.4528	0.9817	1.5105	0.1175
18	0.4528	0.9817	1.5105	0.1175
19	0.4528	0.9816	1.5104	0.1175
20	0.4528	0.9816	1.5104	0.1175

Optimum multidimensional quantization

- The random vector $\boldsymbol{x}$ has pdf $f_{\boldsymbol{x}}(\boldsymbol{x})$ over the multidimensional region $\mathcal{R}$
- Partition $\mathcal{R}$ into sub-regions $\mathcal{R}_i$, $i = 1, \dots, N$
- Write the distortion as a sum of integrals:

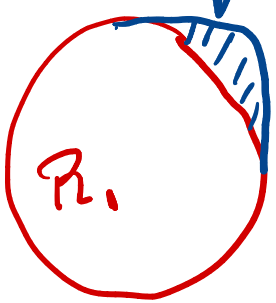
$$\delta(\|\underline{x}\|^2) = \delta(\underline{x}\cdot\underline{x}) = \underline{x}\cdot\delta\underline{x} + \delta\underline{x}\cdot\underline{x} = 2\underline{x}\cdot\delta\underline{x}$$

$$D = \sum_{i=1}^{N} \int_{\mathcal{R}_i} \|\boldsymbol{x} - \widehat{\boldsymbol{x}}_i\|^2 f_{\boldsymbol{x}}(\boldsymbol{x}) d\boldsymbol{x} = E\left[\|\underline{x}-\hat{\underline{x}}\|^2\right] = D(\hat{\underline{x}}_1, \dots \hat{\underline{x}}_N, R_1, \dots R_N)$$

- Calculate the variation of D:

$$\delta\{D\} = -2 \sum_{i=1}^{N} \int_{\mathcal{R}_i} (\boldsymbol{x} - \widehat{\boldsymbol{x}}_i) \cdot \delta\{\widehat{\boldsymbol{x}}_i\} f_{\boldsymbol{x}}(\boldsymbol{x}) d\boldsymbol{x}$$
$$+ \sum_{i=1}^{N} \int_{\delta\{\mathcal{R}_i\}} \|\boldsymbol{x} - \widehat{\boldsymbol{x}}_i\|^2 f_{\boldsymbol{x}}(\boldsymbol{x}) d\boldsymbol{x} = 0$$

$R_1 \cup \delta\{R_1\}$

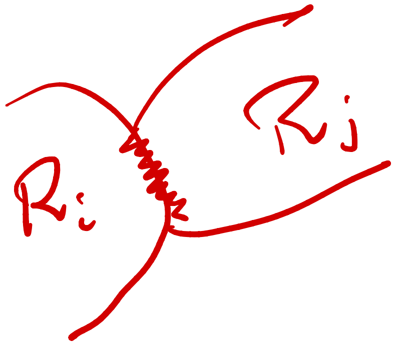


Optimum multidimensional quantization

- The variation of D is equal to 0 if the following conditions hold:
  - $\int_{\mathcal{R}_i} (\boldsymbol{x} - \widehat{\boldsymbol{x}}_i) f_{\boldsymbol{x}}(\boldsymbol{x}) d\boldsymbol{x} = \boldsymbol{0}$
  - $\sum_{i=1}^{N} \int_{\delta\{\mathcal{R}_i\}} \|\boldsymbol{x} - \widehat{\boldsymbol{x}}_i\|^2 f_{\boldsymbol{x}}(\boldsymbol{x}) d\boldsymbol{x} = 0$
- The former condition implies that

$$\widehat{\boldsymbol{x}}_i = \frac{\int_{\mathcal{R}_i} \boldsymbol{x}\, f_{\boldsymbol{x}}(\boldsymbol{x}) d\boldsymbol{x}}{\int_{\mathcal{R}_i} f_{\boldsymbol{x}}(\boldsymbol{x}) d\boldsymbol{x}} = E[\boldsymbol{x} | \boldsymbol{x} \in \mathcal{R}_i]$$

- The latter condition is more subtle

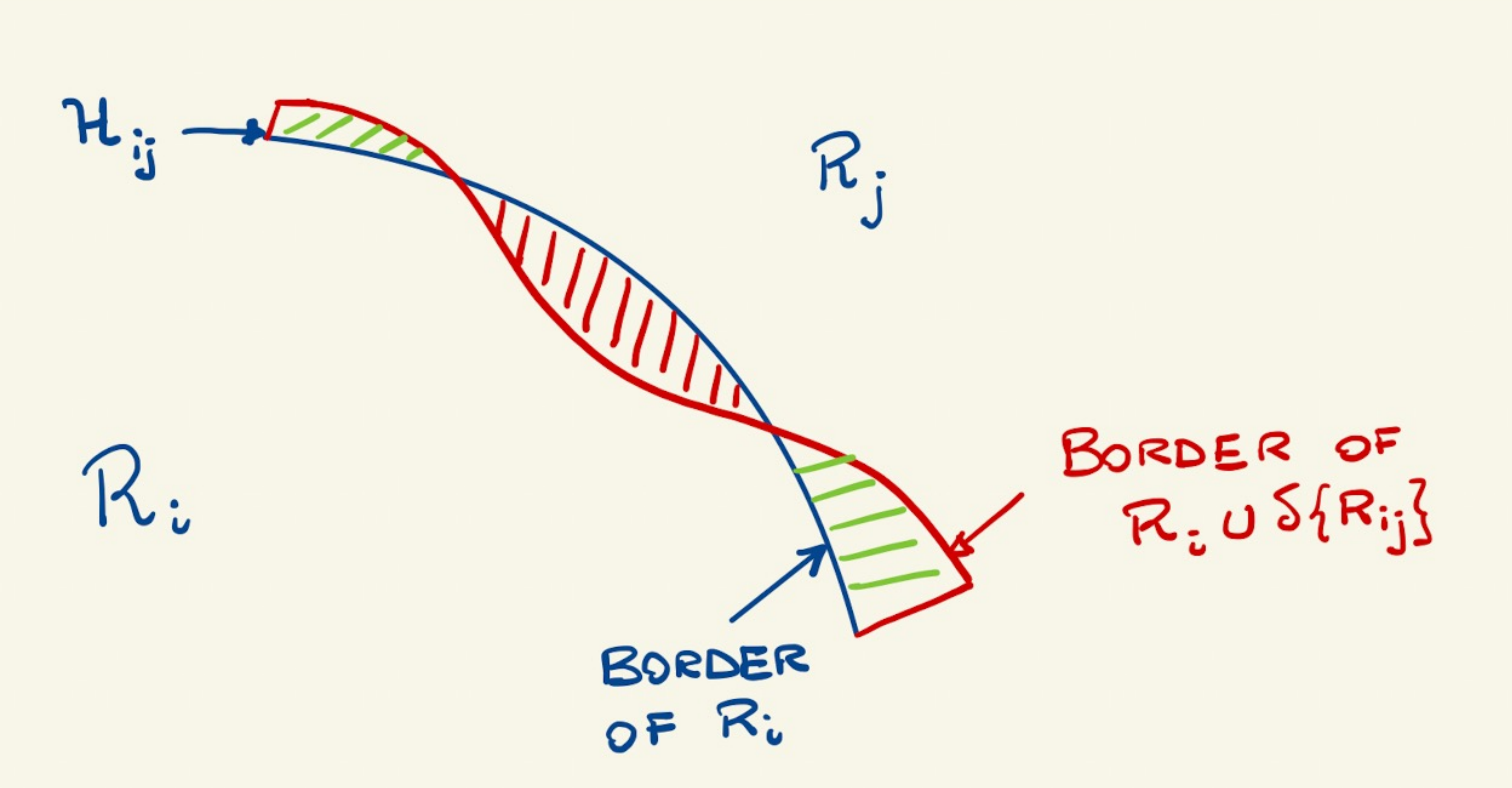


$$\int_{R_i \cup \delta\{R_i\}} \|\underline{x} - \hat{\underline{x}}_i\|^2 f_{\underline{x}}(\underline{x}) d\underline{x} - \int_{R_i} \|\underline{x} - \hat{\underline{x}}_i\|^2 f_{\underline{x}}(\underline{x}) d\underline{x}$$

Optimum multidimensional quantization

- Expanding the region $\mathcal{R}_i$ to $\mathcal{R}_i \cup \delta\{\mathcal{R}_i\}$ implies a contraction of the neighboring regions and *vice versa*
- Let the regions $\mathcal{R}_i$ and $\mathcal{R}_j$ have neighboring hypersurface $\mathcal{H}_{ij}$
- Let $\delta\{\mathcal{R}_{ij}\}$ represent the expansion of $\mathcal{R}_i$ stemming from the hypersurface $\mathcal{H}_{ij}$ so that $\delta\{\mathcal{R}_i\}$ is the union of the $\delta\{\mathcal{R}_{ij}\}$ over all the indexes j corresponding to a neighboring region (with a nonempty hypersurface $\mathcal{H}_{ij}$)
- The condition $\sum_{i=1}^{N} \int_{\delta\{\mathcal{R}_i\}} \|\boldsymbol{x} - \hat{\boldsymbol{x}}_i\|^2 f_{\boldsymbol{x}}(\boldsymbol{x}) d\boldsymbol{x} = 0$ is satisfied if and only if, for every i, j corresponding to neighboring regions,

$$\underbrace{\int_{\delta\{\mathcal{R}_{ij}\}} \|\boldsymbol{x} - \hat{\boldsymbol{x}}_i\|^2 f_{\boldsymbol{x}}(\boldsymbol{x}) d\boldsymbol{x}}_{\text{exp. of } \mathcal{R}_i} + \underbrace{\int_{\delta\{\mathcal{R}_{ji}\}} \left\|\boldsymbol{x} - \hat{\boldsymbol{x}}_j\right\|^2 f_{\boldsymbol{x}}(\boldsymbol{x}) d\boldsymbol{x}}_{\text{exp. of } \mathcal{R}_j} = 0$$


Hij
Rj
Ri
BORDER OF
Ri ∪ δ{Rij}
BORDER
OF Ri

Optimum multidimensional quantization

- Since the expansion of $\delta\{\mathcal{R}_{ij}\}$ corresponds to a contraction of $\delta\{\mathcal{R}_{ji}\}$, the two integrals have opposite signs if considers as measures, so that:
$$\int_{\delta\{\mathcal{R}_{ij}\}} \|\boldsymbol{x} - \widehat{\boldsymbol{x}}_i\|^2 f_{\boldsymbol{x}}(\boldsymbol{x}) d\boldsymbol{x} = \int_{\delta\{\mathcal{R}_{ij}\}} \left\|\boldsymbol{x} - \widehat{\boldsymbol{x}}_j\right\|^2 f_{\boldsymbol{x}}(\boldsymbol{x}) d\boldsymbol{x}$$
- This condition requires that, over the neighboring hypersurface $\mathcal{H}_{ij}$,
$$\|\boldsymbol{x} - \widehat{\boldsymbol{x}}_i\| = \left\|\boldsymbol{x} - \widehat{\boldsymbol{x}}_j\right\|$$
- This condition means that all the points of the neighboring hypersurface $\mathcal{H}_{ij}$ are equidistant from the quantization points $\widehat{\boldsymbol{x}}_i$ and $\widehat{\boldsymbol{x}}_j$
- Finally, this implies that the quantization regions are the minimum distance regions corresponding to the quantization points:
$$\mathcal{R}_i \stackrel{\text{def}}{=} \left\{\boldsymbol{x}: \|\boldsymbol{x} - \widehat{\boldsymbol{x}}_i\| < \left\|\boldsymbol{x} - \widehat{\boldsymbol{x}}_j\right\| \ \forall\, j \neq i\right\}$$

LBG (Linde-Buzo-Gray) algorithm

- The algorithm derived is a multidimensional generalization of the Lloyd algorithm, called LBG algorithm
- The optimum quantization is obtained by an iterative algorithm

1. Choose randomly N initial quantization points $\hat{\boldsymbol{x}}_i$, $i = 1, \ldots, N$
2. Let the quantization regions be
$$\mathcal{R}_i = \{\boldsymbol{x}: \|\boldsymbol{x} - \hat{\boldsymbol{x}}_i\| < \|\boldsymbol{x} - \hat{\boldsymbol{x}}_j\| \; \forall\, j \neq i\}$$
3. Recalculate the quantization points as the centroids of the quantization regions:
$$\hat{\boldsymbol{x}}_i = E[\boldsymbol{x} | \boldsymbol{x} \in \mathcal{R}_i]$$
4. Go to step 2 unless the new quantization points are sufficiently close to the old ones

Information Theory and Applications
Rate distortion function

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Joint entropy, conditional entropy, mutual information

- Given two random variables, X, Y:
  - The joint entropy is $H(X,Y) = -\sum_{x,y} p(x,y) \log_2 p(x,y) = H(Y,X)$
  - The conditional entropy is
  $$H(X|Y) = -\sum_{x,y} p(x,y) \log_2 p(x|y) = H(X,Y) - H(Y)$$
  - The mutual information is
  $$I(X;Y) = H(X) - H(X|Y) = H(Y) - H(Y|X) = I(Y;X)$$
- If $X \perp Y$, $H(X,Y) = H(X) + H(Y)$
- Mutual information inequality: $I(X;Y) \geq 0$
- Conditional entropy:
$$H(X|Y) = H(X,Y) - H(Y) = H(X) - I(X;Y)$$
- Conditioning inequality:
$$H(X|Y) \leq H(X)$$

Rate Distortion Function

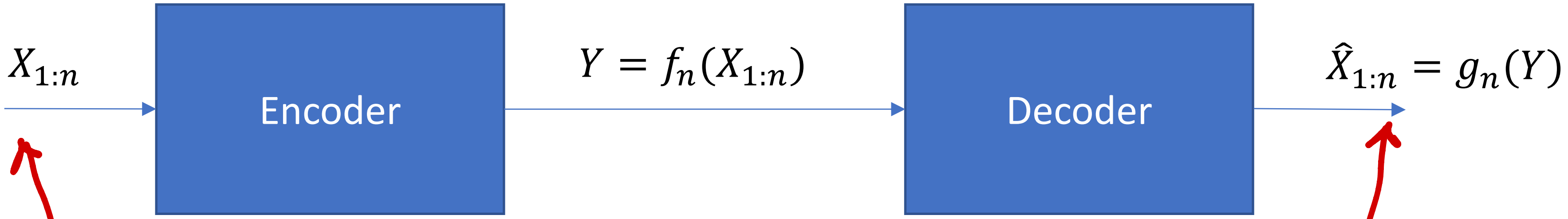
- Consider the source encoding/decoding problem with the sequence of iid variables $X_{1:n}$
- The variables are represented by nR bits, so that R is the number of bits/symbol for the encoding
- Though the random variables are independent, joint encoding achieves a lower distortion
- A sequence $X_{1:n}$ is encoded by an index in $[1,2^{nR}]$ $\lceil 2^{nR} \rceil$
- We assume that $p(x)$ is the probability distribution and each

$$X_i \in \mathcal{X} = \{\xi_1, \dots, \xi_M\}$$

Rate Distortion Function

With lossless source encoding $\hat{X}_{1:n} \equiv X_{1:n}$

- Source encoding/decoding:



- Distortion function:

$$0 \leq d\left(X_{1:n}, \hat{X}_{1:n}\right) < \infty$$

- Hamming distortion function:

$$d_H(x, \hat{x}) = \begin{cases} 1 & x \neq \hat{x} \\ 0 & x = \hat{x} \end{cases}$$

Rate Distortion Function

- Squared error distortion function:

$$d_S(x,\hat{x}) = (x-\hat{x})^2$$

- Average Hamming distortion:

$$= P(x=\hat{x})\cdot 0 + P(x\neq\hat{x})\cdot 1$$

$$E\left[d_H(X,\hat{X})\right] = P(X\neq\hat{X})$$

Error probability

- Sequence distortion:

$$d(x_{1:n},\hat{x}_{1:n}) = \frac{1}{n}\sum_{i=1}^{n} d(x_i,\hat{x}_i)$$

- Encoding function:

$$\lceil 2^{nR}\rceil$$

$$f_n\colon \mathcal{X}^n \mapsto \{1\colon 2^{nR}\}$$

- Decoding function:

$$g_n\colon \{1\colon 2^{nR}\} \mapsto \mathcal{X}^n$$

Rate Distortion Function

- Source code average distortion:

$$D_n = E[d(X_{1:n}, g_n(f_n(X_{1:n})))]$$

- The rate distortion pair (R, D) is *achievable* if there is a sequence of encoding/decoding functions f_n, g_n such that

$$\lim_{n\to\infty} D_n < D$$

- The relationship between R and D is called
  - Rate-distortion function $R(D)$ or
  - Distortion-rate function $D(R)$

- <u>$R(D)$ gives the minimum rate R necessary to achieve distortion D</u>

Rate Distortion Function

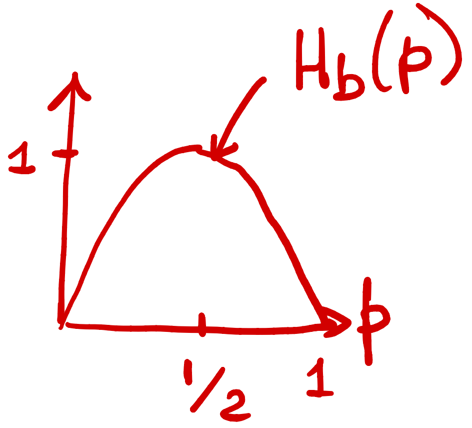
- Theorem: *The rate distortion function can be calculated as*

$$R(D) = \min_{E[d(X,\hat{X})] \le D} I(X;\hat{X})$$

- The rate distortion function of a binary source with Hamming distortion and probability of generating one of the two symbols p (with $p < \frac{1}{2}$) is

$$R(D) = \max\{0, H_b(p) - H_b(D)\}$$

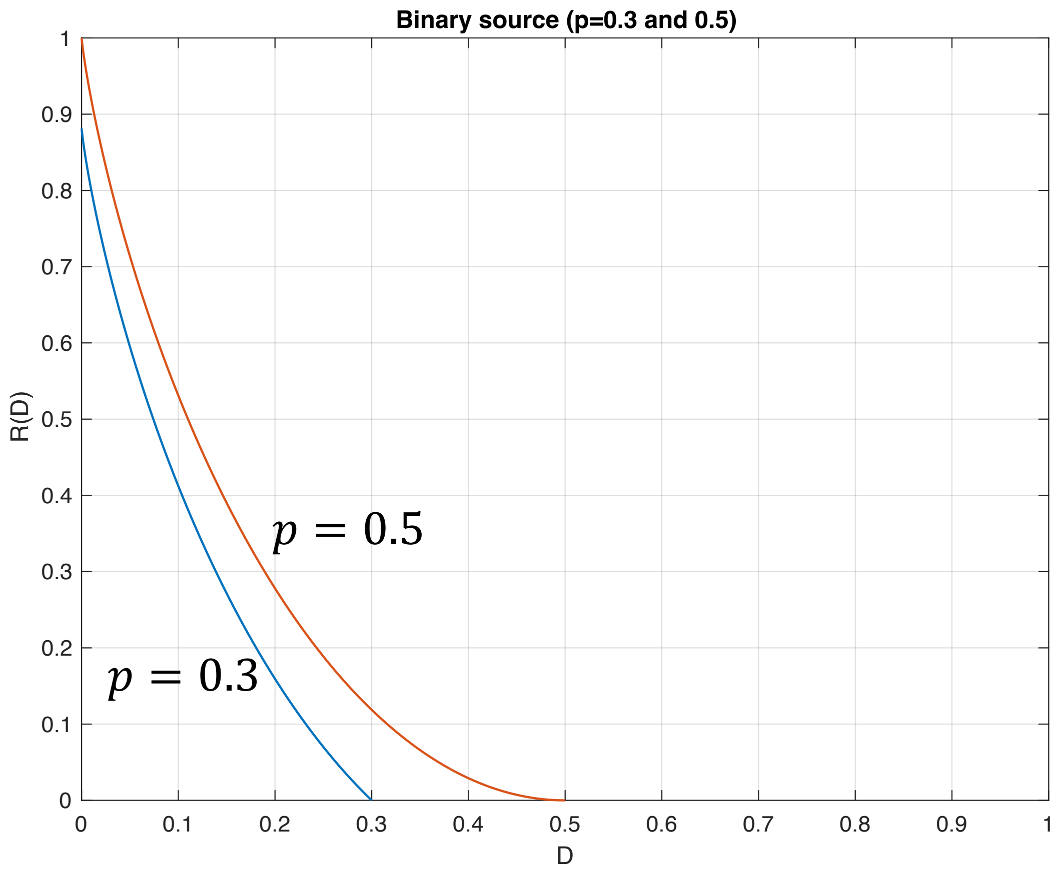
$$H_b(p) = H(p, 1-p) = -p\log_2 p - (1-p)\log_2(1-p)$$

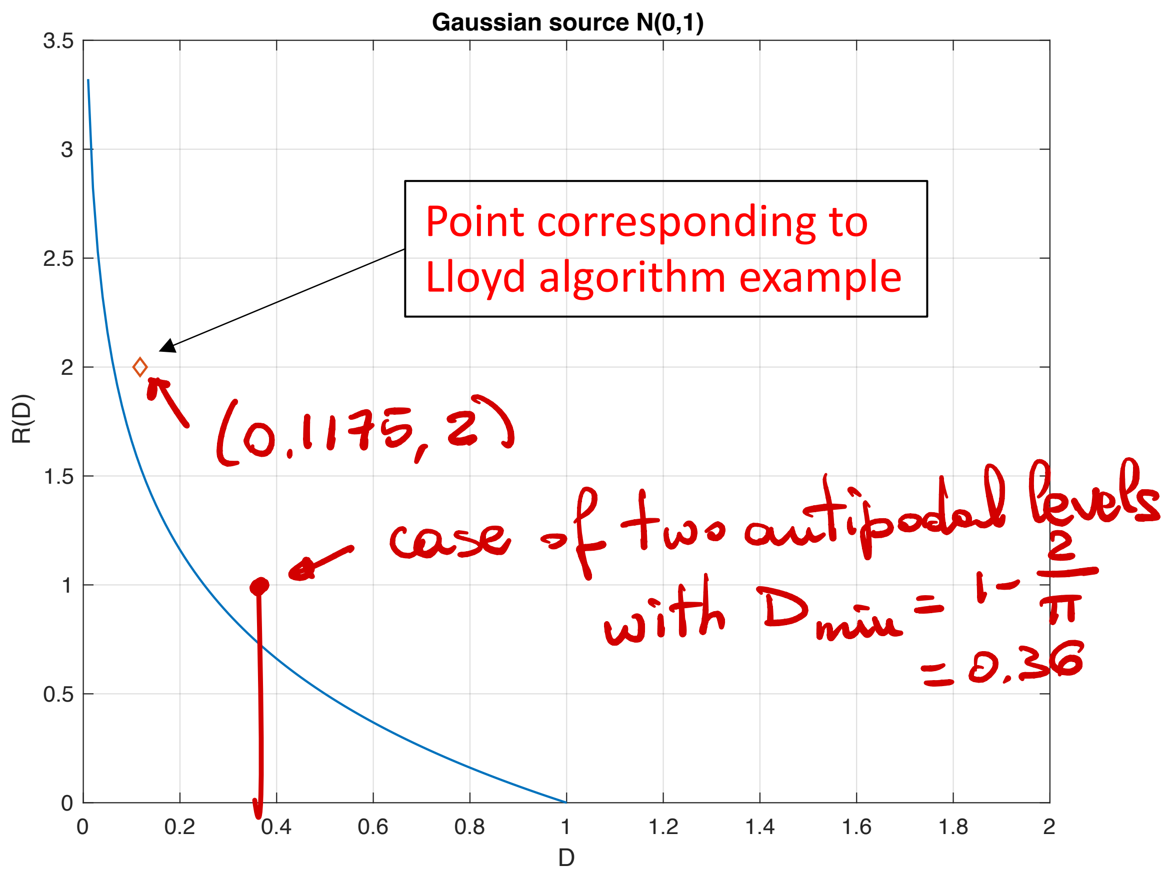


Rate Distortion Function

- The rate distortion $R(D)$ decreases with D:

distortion fct
$E[(x-\hat{x})^2]$





Interpretation of the results

- Consider a stationary binary source with $p = 0.3$
- If we need lossless compression ($D = 0$), the required coding rate is
$$R(0) = H_b(0.3) = 0.8813$$
- This is the achievable compression factor of a lossless compression algorithm
- If instead we can allow an error probability (distortion) $D = 0.1$, the required coding rate is
$$R(0.1) = H_b(0.3) - H_b(0.1) = 0.8813 - 0.4690 = 0.4123$$
- This is the achievable compression factor of a lossy compression algorithm with error probability 0.1, corresponding to an asymptotic reduction in the number of compressed bits up to
$$\frac{0.4690}{0.8813} = 0.5322 = 53.22\%$$

Calculation of $R(D)$ for the binary source

XOR	0	1
0	0	1
1	1	0

- Our goal is to calculate

$$R(D) = \min_{P(X \neq \hat{X}) \leq D} I(X; \hat{X})$$

XOR

- If the symbols are 0,1, then $P(X \neq \hat{X}) = \mathrm{E}[\mathrm{X} \oplus \hat{X}]$
- We have:

$$\begin{aligned} I(X; \hat{X}) &= H(X) - H(X|\hat{X}) && \text{By definition} \\ &= H_b(p) - H(X \oplus \hat{X}|\hat{X}) && \text{Because of conditioning clause} \\ &\geq H_b(p) - H_b(X \oplus \hat{X}) && \text{Because of conditioning inequality} \\ &\geq H_b(p) - H_b(D) && \text{Because } P(X \oplus \hat{X} = 1) \leq D \end{aligned}$$

Calculation of $R(D)$ for the binary source

- Finally, we must show there is a joint distribution of X and $\hat{X}$ such that $I(X;\hat{X}) = H_b(p) - H_b(D)$ if $D \le p \le \frac{1}{2}$

$X \to \boxed{BSC(D)} \to \hat{X}$

- We assume
  - $P(X=0) = \frac{1-p-D}{1-2D}$ and $P(X=1) = \frac{p-D}{1-2D}$
  - $\hat{X}$ is the output of a binary symmetric channel with error probability D

$\frac{1}{2} \ge p \ge D$!

$p + D \le 1$

- By the previous assumptions,

$$P(\hat{X}=0) = \frac{1-p-D}{1-2D} \times (1-D) + \frac{p-D}{1-2D} \times D = 1-p$$

- Therefore, with this distribution,

$$I(X;\hat{X}) = H(\hat{X}) - H(\hat{X}|X) = H_b(p) - H_b(D)$$

Gaussian source

- Now we extend the previous results to the case of a Gaussian source
- The source symbols are Gaussian random variables
- Recall the differential entropy of $X \sim \mathcal{N}(\mu, \sigma^2)$:

$$h(X) = \frac{1}{2}\log_2(2\pi e\sigma^2)$$

Rate Distortion Function

- Theorem: *The rate distortion function can be calculated as*

$$R(D) = \min_{E[d(X,\hat{X})]\leq D} I(X;\hat{X})$$

- The rate distortion function of a Gaussian source $\mathcal{N}(0,\sigma^2)$ with ~~squared-error~~ distortion $D = E\left[\left(X - \hat{X}\right)^2\right]$ is

MSE

$$R(D) = \max\left\{0, \frac{1}{2}\log_2 \frac{\sigma^2}{D}\right\}$$

Gaussian source $X \sim \mathcal{N}(0, \sigma^2)$

- In this case,

$$R(D) = \min_{E[(X-\hat{X})^2] \le D} I(X;\hat{X})$$

- We have

$$\begin{aligned}
I(X;\hat{X}) &= h(X) - h(X|\hat{X}) \\
&= \frac{1}{2}\log_2(2\pi e\sigma^2) - h(X - \hat{X}|\hat{X}) \\
&\geq \frac{1}{2}\log_2(2\pi e\sigma^2) - h(X - \hat{X}) \\
&\geq \frac{1}{2}\log_2(2\pi e\sigma^2) - \frac{1}{2}\log_2\left(2\pi e\, \overbrace{E\left[(X - \hat{X})^2\right]}^{\le D}\right) \\
&\geq \frac{1}{2}\log_2(2\pi e\sigma^2) - \frac{1}{2}\log_2(2\pi e D) = \frac{1}{2}\log_2\frac{\sigma^2}{D}
\end{aligned}$$

Gaussian source $X \sim \mathcal{N}(0, \sigma^2)$

- To check the tightness of the lower bound, assume $D < \sigma^2$, $X = \hat{X} + Z$, and $Z \sim \mathcal{N}(0, D)$, independent of $\hat{X} \sim \mathcal{N}(0, \sigma^2 - D)$
- By this assumption,
  - $h(X) = \frac{1}{2}\log_2(2\pi e \sigma^2)$
  - $h(X|\hat{X}) = h(Z) = \frac{1}{2}\log_2(2\pi e D)$
- In this way, the rate is

$$I(X;\hat{X}) = h(X) - h(X|\hat{X}) = \frac{1}{2}\log_2(2\pi e \sigma^2) - \frac{1}{2}\log_2(2\pi e D) = \frac{1}{2}\log_2\left(\frac{\sigma^2}{D}\right)$$

- The joint distribution of X and $\hat{X}$ is

$$f_{X\hat{X}}(x, \hat{x}) = f_{\hat{X}}(\hat{x}) f_Z(x - \hat{x}) = \frac{1}{\sqrt{2\pi(\sigma^2 - D)}} \frac{1}{\sqrt{2\pi D}} e^{-\left(\frac{\hat{x}^2}{2(\sigma^2 - D)} + \frac{(x - \hat{x})^2}{2D}\right)}$$

Parallel Gaussian sources

- Consider a source represented by set of independent (parallel) Gaussian random variables $X_k \sim \mathcal{N}(0, \sigma_k^2)$ for $k = 1, \dots, K$
- Each source is encoded by $\hat{X}_k$
- The quadratic distortion of the source is defined by

$$D_k \stackrel{\text{def}}{=} E\left[\left(X_k - \hat{X}_k\right)^2\right], \qquad \sum_{k=1}^{K} D_k \leq D$$

overall quadratic distortion

$(x)_+ \triangleq \max\{0, x\}$

Minimum distortion

- The following inequalities hold:

$$I(X_1, \dots, X_K; \hat{X}_1, \dots, \hat{X}_K) = h(X_1, \dots, X_K) - h(X_1, \dots, X_K | \hat{X}_1, \dots, \hat{X}_K)$$

$$= \sum_{k=1}^{K} \{h(X_k) - h(X_k | X_1, \dots, X_{k-1}, \hat{X}_1, \dots, \hat{X}_K)\}$$

$$\left(\log_2\left(\frac{\sigma_k^2}{D}\right)\right)_+$$

$$\geq \sum_{k=1}^{K} \{h(X_k) - h(X_k | \hat{X}_k)\} = \sum_{k=1}^{K} I(X_k; \hat{X}_k) = \frac{1}{2} \sum_{k=1}^{K} \cancel{\left(\frac{\sigma_k^2}{D_k}\right)_+}$$

- The minimum is attained when $X_k = \hat{X}_k + Q_k, D_k \leq \sigma_k^2$,

$$\hat{X}_k \perp Q_k, \qquad \hat{X}_k \sim \mathcal{N}(0, \sigma_k^2 - D_k), \qquad Q_k \sim \mathcal{N}(0, D_k)$$

$$h(x_1, x_2) = h(x_1) + h(x_2 | x_1)$$

$$h(x_1, x_2, x_3) = h(x_1) + \overbrace{h(x_2, x_3 | x_1)}^{=}$$

$$= h(x_1) + \underbrace{h(x_2 | x_1) + h(x_3 | x_1, x_2)}_{=}$$

$$h(x_1 \ldots x_k) = \sum_{k=1}^{k} h(x_k | x_{1:k-1})$$

telescopic entropy sum

Minimum distortion

- The resulting minimum distortion depends on the D_k:

$$\phi(D_1, \dots, D_K) = \frac{1}{2}\sum_{k=1}^{K} \cancel{\left(\frac{\sigma_k^2}{D_k}\right)}_+ \qquad \left(\log_2\left(\frac{\sigma_k^2}{D_k}\right)\right)_+$$

- Given the σ_k^2 and the constraint $D = \sum_{k=1}^{K} D_k$, the minimum set of distortions is found to be $D_k = \min\{\lambda, \sigma_k^2\}$
- The parameter λ is found by solving

$$D = \sum_{k=1}^{K} \min\{\lambda, \sigma_k^2\} \le \sum_{k=1}^{K} \sigma_k^2$$

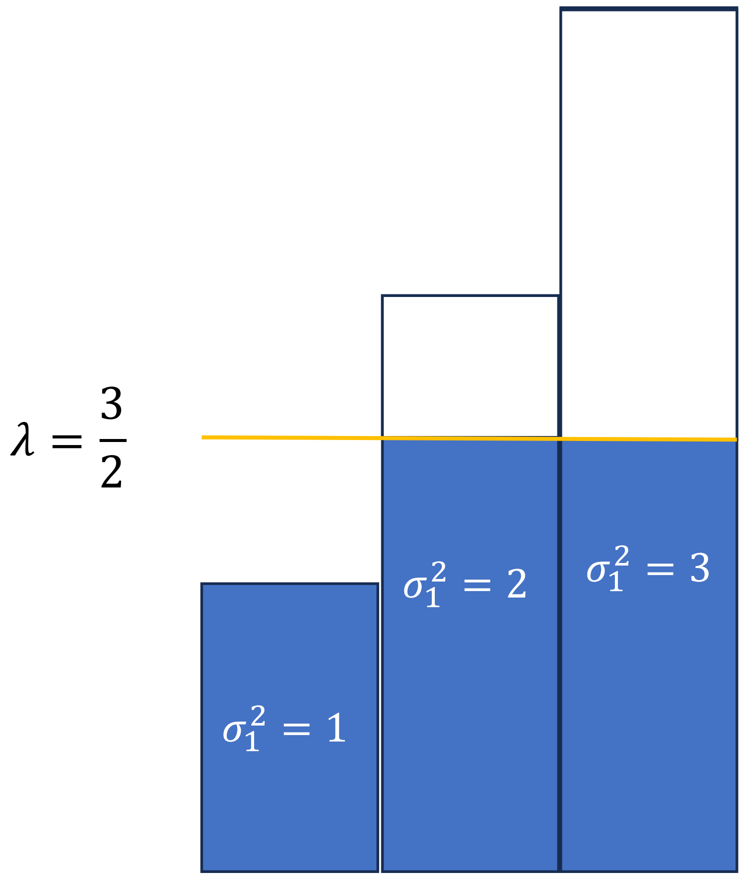
Example

- Let $\sigma_1^2 = 1, \sigma_2^2 = 2, \sigma_3^2 = 3, D = 4$
- The parameter λ is found by solving

$$D = \sum_{k=1}^{K} \min\{\lambda, \sigma_k^2\}$$

  - Assume $\lambda < 1 \Rightarrow 4 = 3\lambda \Rightarrow \lambda = \frac{4}{3}$, invalid solution
  - Assume $1 < \lambda < 2 \Rightarrow 4 = 1 + 2\lambda \Rightarrow \lambda = \frac{3}{2}$, valid solution
  - Therefore, $D_1 = 1, D_2 = D_3 = \frac{3}{2}$

Example (cont.)



- The optimum distortion allocation algorithm is called “reverse water-filling”
- It finds application in audio and video coding in conjunction with transform techniques such as
  - DCT
  - Wavelet transforms
  - Sub-band decomposition
  - Prediction
  - Motion compensation, *etc*