Differential entropy

- A continuous random variable X characterized by a probability density function $f_X(x)$ has differential entropy
$$h(X) = -\int f_X(x)\log_2 f_X(x)\,dx$$
- The concept of differential entropy derives from quantization
- X is quantized over intervals of amplitude Δ: $[k\Delta, (k+1)\Delta)$ for $k \in \mathbb{Z}$
- Then, we can map X to $\mathbb{Z}$ by the rule $Q: \mathbb{R} \mapsto \mathbb{Z}$ defined by
$$Q(x) = \mathrm{mod}\left(\frac{x}{\Delta}, 1\right)$$
- For example, if $\Delta = 0.2$ and $x = 1.1$,
$$Q(x) = \mathrm{mod}\left(\frac{1.1}{0.2}, 1\right) = \mathrm{mod}(5.5,1) = 5$$

Differential entropy

- Then, we can argue that the entropy of the continuous random variable X is represented by $H\big(Q(x)\big)$, i.e.,

$$H\big(Q(X)\big) = -\sum_k p_k \log_2 p_k$$

- The probabilities p_k are the probabilities that $X \in [k\Delta, (k+1)\Delta)$:

$$p_k = \int_{k\Delta}^{(k+1)\Delta} f_X(x)dx$$

- When Δ is very small, using the mean-value theorem for integrals, they can be approximated by

$$p_k \approx f_X\left(\left(k + \frac{1}{2}\right)\Delta\right) \cdot \Delta$$

Differential entropy

- In this way,

$$
\begin{aligned}
H\big(Q(X)\big) &= -\sum_k p_k \log_2 p_k \\
&\approx -\sum_k f_X\left(\left(k+\frac{1}{2}\right)\Delta\right)\cdot\Delta \log_2\left\{f_X\left(\left(k+\frac{1}{2}\right)\Delta\right)\cdot\Delta\right\} \\
&= -\sum_k f_X\left(\left(k+\frac{1}{2}\right)\Delta\right)\cdot\Delta \log_2\left\{f_X\left(\left(k+\frac{1}{2}\right)\Delta\right)\right\} \\
&\quad -\sum_k f_X\left(\left(k+\frac{1}{2}\right)\Delta\right)\cdot\Delta \log_2\{\Delta\} \\
&\approx -\int f_X(x)\log_2 f_X(x)\,dx - \int f_X(x)\log_2\{\Delta\}\,dx \\
&= -\int f_X(x)\log_2 f_X(x)\,dx - \log_2\{\Delta\}
\end{aligned}
$$

Differential entropy

- Summarizing, the entropy of $Q(X)$ is approximated by
$$H\big(Q(X)\big) \approx -\log_2\{\Delta\} - \int f_X(x)\log_2 f_X(x)\,dx$$
- The first term corresponds to the precision of the approximation in bits. For example, if $\Delta = \frac{1}{8}$, then $-\log_2\{\Delta\} = 3$ and $\Delta = (0.001)_2$
- The second term corresponds to the difference
$$H\big(Q(X)\big) - (-\log_2\{\Delta\})$$
- When $\Delta \downarrow 0$, the difference converges to the differential entropy defined as
$$h(X) \stackrel{\text{def}}{=} -\int f_X(x)\log_2 f_X(x)\,dx$$

Properties of the differential entropy

- The differential entropy can be positive or negative
- Consider the uniform distribution over (a, b), such that

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

- In this case,

$$h(X) = -\int_a^b \frac{1}{b-a} \log_2\left(\frac{1}{b-a}\right) dx = \log_2(b-a)$$

- Thus, for example, if $b - a = 1$, $h(X) = 0$, if $b - a = \frac{1}{2}$, $h(X) = 1$, and , if $b - a = 2$, $h(X) = 1$

Properties of the differential entropy

- The differential entropy is invariant to translations of the pdf:
$$h(X) = h(a + X)$$
- Applying this result to $Y = X + Z$, one obtains
$$h(Y|X) = h(X + Z|X) = h(Z)$$
- Multiplication by a constant changes the differential entropy as follows:
$$h(aX) = \log_2 |a| + h(X)$$
- The differential entropy of a random variable with given variance σ^2 is maximum when the pdf is Gaussian:
$$h(X) \leq h\big(\mathcal{N}(\mu, \sigma^2)\big)$$

Differential entropy of a Gaussian random variable

- The differential entropy of a Gaussian random variable is calculated as follows:

$$h\big(\mathcal{N}(\mu,\sigma^2)\big) = -\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \log_2\left\{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right\} dx$$

- Changing the integration variable by $u = x - \mu$,

$$h\big(\mathcal{N}(\mu,\sigma^2)\big) = -\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{u^2}{2\sigma^2}} \log_2\left\{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{u^2}{2\sigma^2}}\right\} du$$

$$= -\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{u^2}{2\sigma^2}} \log_2\left\{\frac{1}{\sqrt{2\pi\sigma^2}}\right\} du$$

$$= -\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{u^2}{2\sigma^2}} \log_2\left\{e^{-\frac{u^2}{2\sigma^2}}\right\} du$$

Differential entropy of a Gaussian random variable

- The first integral is
$$-\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{u^2}{2\sigma^2}} \log_2\left\{\frac{1}{\sqrt{2\pi\sigma^2}}\right\} du = \frac{1}{2}\log_2(2\pi\sigma^2)$$
- The second is
$$-\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{u^2}{2\sigma^2}} \log_2\left\{e^{-\frac{u^2}{2\sigma^2}}\right\} du = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{u^2}{2\sigma^2}} \frac{u^2}{2\sigma^2} du \log_2 e = \frac{1}{2}\log_2 e$$
- Summarizing,
$$h\big(\mathcal{N}(\mu,\sigma^2)\big) = \frac{1}{2}\log_2(2\pi e\sigma^2)$$
- As expected, the result is independent of μ