ICASSP 2023 - 2023 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP) | 978-1-7281-6327-7/23/$31.00 ©2023 IEEE | DOI: 10.1109/ICASSP49357.2023.10096320

HIGH-DIMENSIONAL CONFIDENCE REGIONS IN SPARSE MRI

Frederik Hoppe[⋆] *Felix Krahmer*[†] *Claudio Mayrink Verdun*[†$] *Marion I. Menzel*[‡§¶] *Holger Rauhut*[⋆]

[⋆] Department of Mathematics, RWTH Aachen University, Aachen, Germany
[†] Department of Mathematics, Technical University of Munich, Munich, Germany
[‡] AImotion Bavaria, Technische Hochschule Ingolstadt, Ingolstadt, Germany
[§] Department of Physics, Technical University of Munich, Garching, Germany
[¶] GE Healthcare, Munich, Germany
[$] Chair of Theoretical Information Technology, Technical University of Munich, Munich, Germany

ABSTRACT

One of the most promising solutions for uncertainty quantification in high-dimensional statistics is the debiased LASSO that relies on unconstrained ℓ_1-minimization. The initial works focused on real Gaussian designs as a toy model for this problem. However, in medical imaging applications, such as compressive sensing for MRI, the measurement system is represented by a (subsampled) complex Fourier matrix. The purpose of this work is to extend the method to the MRI case in order to construct confidence intervals for each pixel of an MR image. We show that a sufficient amount of data is $n \gtrsim \max\{s_0 \log^2 s_0 \log p, s_0 \log^2 p\}$.

Index Terms— debiased LASSO, compressed sensing, confidence regions, MRI

1. INTRODUCTION

Several highly efficient methods for dealing with high-dimensional data have been proposed in the last decades. The idea of these methods is that the information contained in many natural datasets relies on statistics of much lower dimension than the original ambient one. This innovation in statistics, signal processing and machine learning became known as *sparse regression* (SR) [1, 2] in the statistical literature or *compressive sensing* (CS) [3, 4] in the signal processing literature. However, a framework that provides uncertainty quantification for guiding decision-making in certain critical applications, such as Magnetic Resonance Imaging (MRI), is still missing. Since reliable medical imaging procedures are pivotal for accurate interpretation and diagnostic tasks, a theory that quantifies the quality of such images based on sparse regression is of high relevance.

Thanks to Federal Ministry of Education and Research in the grant "SparseMRI3D+ (FZK 05M20WOA)". M.I.M. acknowledges partial funding by the project AI for Health Imaging Award "CHAIMELEON: Accelerating the Lab to Market Transition of AI Tools for Cancer Management [H2020-SC1-FA-DTS-2019-1 952172]".

Recently, a series of papers initiated a de-sparsified approach to sparse regression [5, 6, 7, 8]. This technique is able to characterize the distribution of a modified estimator based on the KKT conditions of the LASSO solution. For this modified estimator, sharp confidence intervals are derived for variable selection in the case of (sub-)Gaussian designs. Although these results provide fundamental theoretical insight for the theory of uncertainty quantification in the high-dimensional regime, such fully random matrices are of limited practical use. In MRI, for example, the measurement process is highly structured and can be described by a (subsampled) Fourier operator [9]. Moreover, structured matrices often allow for faster algorithmic processing by exploiting the fast Fourier transform (FFT) for matrix-multiplication as well as efficient storage.

The aim of this work is to close this gap by developing the theory for sharp confidence intervals for subsampled Fourier matrices since they are employed in the MR reconstruction pipeline.

2. BACKGROUND AND RELATED WORKS

Before we present our method for constructing confidence intervals in MRI, we introduce the underlying theory.

2.1. Sparse regression

For a design/measurement matrix $X \in \mathbb{C}^{n\times p}$ with rows $x_1^T, \dots, x_n^T$ and a data vector $y = (y_1, \dots y_n) \in \mathbb{C}^n$, we are interested in the high dimensional regression model

$$y = X\beta^0 + \varepsilon, \qquad p \gg n, \tag{1}$$

where $\beta^0 \in \mathbb{C}^p$ is s_0-sparse and the noise vector $\varepsilon \sim \mathcal{CN}(0, \sigma^2 I_{n\times n})$ is assumed to be a complex standard Gaussian vector whose components ε_i are independent. Note that we are considering complex-valued representations, since MRI measurement are typically modeled via complex numbers [10].

The main goal is to estimate $\beta^0 \in \mathbb{C}^p$ as well as to provide confidence regions for β^0 based on this estimator. A natural estimator is the LASSO [2, 11], denoted by $\hat{\beta}$, which is the minimizer of

$$\min_{\beta\in\mathbb{C}^p} \frac{1}{2n}\|X\beta - y\|_2^2 + \lambda\|\beta\|_1, \tag{2}$$

where $\lambda = \lambda(n, p, \sigma) \in \mathbb{R}$ is a tuning parameter to balance the data fidelity term and sparsity induced by the ℓ_1-norm. In order to stress the fact that all parameters are complex-valued, this problem is often referred to as the complex LASSO (c-LASSO) [11].

2.2. The desparsified LASSO

Following the works [6, 7, 8], we aim to derive confidence bounds for the c-LASSO estimator in the case that the design matrix is given by a random subsampled Fourier matrix, discussed in more details in Section 2.3. Note that this is a matrix with heavy-tailed rows where standard concentration techniques cannot be trivially applied [12].

Most previous contributions assumed the design to be a real (sub-)Gaussian matrix $X \in \mathbb{R}^{n\times p}$, i.e. a matrix with light tailed distribution [6, 7, 8]; only the work [8] provides also results for fixed (deterministic) designs and bounded random designs under strong assumptions.

The *debiased Lasso* estimator is constructed by "inverting" the KKT conditions [8] and is defined as

$$\hat{\beta}^u = \hat{\beta} + \frac{1}{n}MX^*(y - X\hat{\beta}), \tag{3}$$

where M may be chosen such that $M\hat{\Sigma} \approx I_{p\times p}$, where $\hat{\Sigma}$ is given by the sample covariance matrix, i.e., $\hat{\Sigma} = X^*X/n$. The difference of the debiased LASSO and the ground truth can then be decomposed into

$$\sqrt{n}(\hat{\beta}^u - \beta^0) = \frac{MX^*\varepsilon}{\sqrt{n}} - \sqrt{n}(M\hat{\Sigma} - I_{p\times p})(\hat{\beta} - \beta^0). \tag{4}$$

One of the important achievements of the desparsified LASSO theory is that it can be shown that the bias term $R := (M\hat{\Sigma} - I_{p\times p})(\hat{\beta} - \beta^0)$ asymptotically vanishes [8]. As $MX^*\varepsilon/\sqrt{n} \sim \mathcal{N}(0, \sigma^2\hat{\Sigma})$, this allows us to construct pointwise confidence intervals for β^0.

The previous approaches estimate the terms $(M\hat{\Sigma} - I_{p\times p})$ and $(\hat{\beta} - \beta^0)$ separately which leads to non-optimal bounds and therefore to a sample size $n \gtrsim s_0^2 \log^2 p$ [1]. The only exception requiring a sample size $n \gtrsim s_0 \log^2 p$ is the seminal paper [7] that, unlike the other works, uses a leave-one-out argument and strongly exploits the independence of $X\Sigma^{-1}e_i$ and X_{-i}. This independency holds, for example, for matrices with Gaussian rows but does not hold for heavy-tailed matrices such as a subsampled Fourier matrix.

[1] Here, the notation $a \gtrsim b$ means that there is a constant $C > 0$ such that $a \geq Cb$.

2.3. Subsampled Fourier matrices

As a result of the Bloch equation the magnetic resonance (MR) phenomenon can be modeled by Fourier measurements [10]. A Fourier matrix $F \in \mathbb{C}^{p\times p}$ is defined entrywise as $F_{l,k} = e^{2\pi i(l-1)(k-1)/p}$ with $l, k \in [p]$. The random subsampled Fourier matrix F_Ω, that plays a crucial role in fast MR images reconstruction, consists of the rows whose indices $j \in \Omega$ are obtained by n independently and uniformly selected points from $[p]$. We denote these rows by $f_1^T, \ldots, f_p^T$. Note that with probability larger than 0 some indices may be chosen more than once. In practice, however, the rows are sampled without replacement. The results differ only slightly, as discussed in [4, Chapter 12.6]. Due to the sampling pattern the rows of F_Ω are independent, but the entries within each row are not independent.

This type of measurement matrix falls into the class of bounded orthonormal systems [4]. In order to simplify the exposition, here we state the results for subsampled Fourier matrices, but note that they also hold for general matrices associated to bounded orthonormal systems as established in [13].

3. CONFIDENCE REGIONS IN THE SUBSAMPLED FOURIER CASE

As a consequence of the sampling pattern the second moment matrix $\mathbb{E}[f_j f_j^*]$ of any row f_j is the identity $I_{p\times p}$ [13]. In addition, a subsampled Fourier matrix satisfies $\mathbb{E}\big[\hat{\Sigma}\big] = I_{p\times p}$. Even though this is only in expectation, the estimation $|\hat{\Sigma}_{ij}| \leq 1$ for $i, j \in [p]$ shows, that the entries of the sample covariance are restricted to the range $[0, 1]$. Therefore, we choose $M = I_{p\times p}$. Then, the debiased LASSO from (3) takes the form

$$\hat{\beta}^u = \hat{\beta} + \frac{F_\Omega^*(y - F_\Omega\hat{\beta})}{n}. \tag{5}$$

Our main theoretical result, Theorem 1, states that conditioned on F_Ω, the debiased LASSO estimator is asymptotically normal, i.e.

$$\sqrt{n}(\beta^u - \beta^0) \mid F_\Omega \sim \mathcal{CN}(0, \sigma^2\hat{\Sigma}). \tag{6}$$

The crucial point for this asymptotic normality is that the bias term R vanishes. This is the case if

$$n \gtrsim \max\{s_0 \log^2 s_0 \log p, s_0 \log^2 p\}, \tag{7}$$

which is further discussed in Section 4. Then, for a consistent noise estimator $\hat{\sigma}$ (see Section 4.3), the confidence regions with significance level $\alpha \in (0, 1)$ for $\beta_i^0 \in \mathbb{C}$, estimated via the debiased LASSO,

$$J_i^\circ(\alpha) := \{z \in \mathbb{C} : |\hat{\beta}_i^u - z| \leq \delta_i^\circ(\alpha)\}, \tag{8}$$

with radius $\delta^\circ(\alpha) := \frac{\hat{\sigma}}{\sqrt{n}}\sqrt{\log(1/\alpha)}$ are asymptotically valid:

$$\lim_{n\to\infty} \mathbb{P}\left(\beta_i^0 \in J_i^\circ(\alpha)\right) = 1 - \alpha. \tag{9}$$

The results in [14] and [7] prove the optimality of the length of this type of confidence intervals construction. In particular they show, that the optimal radius should scale with $\frac{1}{\sqrt{n}}$. In this sense our confidence regions are optimal and their construction follows straightforwardly from the asymptotic normality. A more detailed discussion on this procedure in the complex case can be found in [13].

Algorithm 1 Confidence regions in Fourier case

Initialize α, F_Ω
Estimate noise $\hat{\sigma}$
$\lambda \leftarrow$ cross validation: test multiples of $\lambda_0 = \frac{\sigma\sqrt{K}}{\sqrt{n}}(2 + \sqrt{12\log p})$
Solve LASSO $\hat{\beta} \leftarrow \arg\min_\beta \frac{1}{2n}\|F_\Omega\beta - y\|_2^2 + \lambda\|\beta\|_1$
Compute $\hat{\beta}^u \leftarrow \hat{\beta} + F_\Omega^*(y - F_\Omega\hat{\beta})/n$
Compute $\delta^\circ(\alpha) \leftarrow \frac{\hat{\sigma}}{\sqrt{n}}\sqrt{\log(1/\alpha)}$

4. ASYMPTOTIC NORMALITY OF DEBIASED LASSO IN THE SUBSAMPLED FOURIER CASE

The asymptotic normality mentioned in (6) is the key property of the debiased LASSO in order to construct confidence intervals. It is stated rigorously in Theorem 1, which relies on the ℓ_2-consistency of the LASSO estimator.

4.1. ℓ_2-consistency of LASSO

One of the main sufficient conditions on the measurement matrix for establishing consistency of the LASSO estimator and optimal oracle inequalities [15] is the following: A matrix X satisfies the restricted isometry property (RIP) of order $1 \leq s \leq p$ with constant $\delta_s \in (0,1)$ if

$$(1-\delta_s)\|\beta\|_2^2 \leq \|X\beta\|_2^2 \leq (1+\delta_s)\|\beta\|_2^2 \tag{10}$$

for all s-sparse vectors $\beta \in \mathbb{C}^p$. Although not being described by a light tailed probabilistic model, (normalized) subsampled Fourier matrices still act as quasi-isometries on the subset of sparse vectors, i.e. they satisfy the RIP of order s_0 with high probability provided that $n \gtrsim s_0 \log^2 s_0 \log p$ [16].

This property can be used to show one of the key tools for asymptotic normality of desparsified estimators, namely, the existence of sharp ℓ_1 and ℓ_2 oracle estimates. In order to establish a bound for the bias term $R = (M\hat{\Sigma} - I_{p\times p})(\hat{\beta} - \beta^0)$, we start by stating oracle bounds for $\hat{\beta} - \beta^0$:

$$\|\hat{\beta} - \beta^0\|_2 \lesssim \frac{\sqrt{s_0 \log p}}{\sqrt{n}}, \qquad \|\hat{\beta} - \beta^0\|_1 \lesssim \frac{s_0\sqrt{\log p}}{\sqrt{n}}. \tag{11}$$

These results are widely available in the statistics literature, where it is usually assumed that the design matrix fulfills the restricted eigenvalue condition [1, Chapter 7] or the compatibility condition [17, Chapter 6]. As it is standard in the compressive sensing literature, the measurement matrix here is assumed to satisfy the (slightly stronger) RIP [4]. See [15] for a discussion about the different sufficient conditions for sparse regression and the relationship between them.

4.2. Main theoretical result

Theorem 1. *[13] Let $\frac{1}{\sqrt{n}}F_\Omega$ be a normalized subsampled Fourier matrix with $n \gtrsim s_0\log^2 s_0 \log p$ rows. Let further $\lambda \geq 2\lambda_0 := 2\frac{\sigma\sqrt{K}}{\sqrt{n}}(2 + \sqrt{12\log p})$. Then, the following decomposition holds*

$$\sqrt{n}(\hat{\beta}^u - \beta^0) = W + R, \tag{12}$$

where the debiased LASSO $\hat{\beta}^u$ is defined in (3) *and $W \mid F_\Omega \sim \mathcal{N}(0, \sigma^2\hat{\Sigma})$. Furthermore,*

$$\mathbb{P}\left(\|R\|_\infty \geq C(\sigma,\delta_t)\frac{\sqrt{s_0}\log p}{\sqrt{n}}\right) \leq 5p^{-2} \tag{13}$$

with $C(\sigma,\delta_t) \geq 0$ depending only on σ and $\delta_t < 1$.

In order to guarantee that the bias term R vanishes and hence, $\sqrt{n}(\hat{\beta}^u - \beta^0)$ is asymptotically Gaussian distributed, two sufficient conditions play a role - the fact that the measurement matrix satisfies the RIP with constant δ_t, which requires $n \gtrsim s_0\log^2 s_0 \log p$ samples [16] and the fact that the bias term R asymptotically vanishes if $n \gtrsim s_0 \log^2 p$, as stated in (13). Therefore, in very precise terms, our sample complexity reads as $n \gtrsim \max\{s_0\log^2 s_0\log p, s_0\log^2 p\}$.

4.3. Noise estimation

From the theoretical point of view, estimating the error variance for high-dimensional estimators is a non-trivial problem. The most common method used in the debiased LASSO literature, e.g. in [5, 18, 7, 19, 8] is the so-called scaled LASSO [20]. From the MRI practitioners' point of view, the noise can be measured directly during MR image acquisition (the so-called pre-scan procedure which is mandatory for every patient), yielding a direct, ground truth estimation of the noise [21]. Alternatively, it may be estimated retrospectively (indirectly) from the final image, in case the directly determined noise estimation is not accessible anymore. For a review of noise estimation methods see e.g. [21]. In any case, what is important for the construction of the confidence intervals is to have a noise estimator. Lemma 13 in [6] shows that this is exactly the case, i.e., they show that the asymptotic normality still holds when the true noise level is replaced by a consistent noise estimator.

5. NUMERICAL EXPERIMENTS

In this section we illustrate our theoretical results with numerical experiments using angiography brain image data from the Brain Vasculature (BraVa) database [22], which is

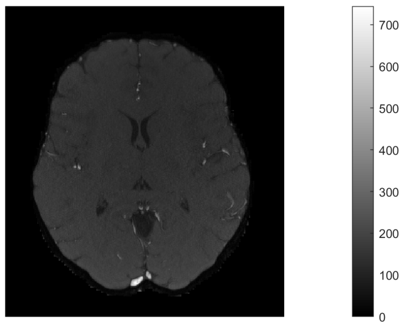

Fig. 1. Original non-sparse MR angiography; single brain slice with vessels lighting up; image intensities in arbitrary units [u.a.].

sparse in the canonical basis [23], as depicted in Figure 1. We use TFOCS [24] for the c-LASSO estimator in Algorithm 1. Throughout our experiments we assume for simplicity that σ is known and set $\alpha = 0.05$.

To artificially increase the sparsity of the input angiography image, highlighting only vessels, we set an image intensity threshold of 200 [a.u.], removing the brain background image and noise floor, such that each pixel with a magnitude lower than this threshold is set to 0. We obtain an $s_0 = 1282$-sparse image, which serves as the (unknown) ground truth $\beta^0 \in \mathbb{R}^{92160}$. Note that the theory as well as the algorithm allow for complex images, but due to better visualization, we stick to the real case where we simply set the imaginary part equal to 0. In the following, we simulate the image acquisition process in MRI by subsampling $n = 0.4p = 36864$ different rows of a full Fourier matrix. Then, we add complex noise with $\sigma = 1000$ to the measurement vector $F_\Omega \beta^0$ leading to a relative noise level $\frac{\|\varepsilon\|_2}{\|F_\Omega \beta^0\|_2} = 0.106$. From the model $y = F_\Omega \beta^0 + \varepsilon$, we know F_Ω, the subsampled data $y \in \mathbb{C}^{36864}$, and the noise level σ. The goal is to reconstruct the image β^0 and to provide lower and upper bounds for this estimate. Following Algorithm 1, with $\lambda = 25\lambda_0$ chosen via cross validation [25], we derive confidence regions. Figure 2 illustrates the confidence intervals based on the debiased LASSO estimator and the ground truth for the 68 pixels with largest magnitudes (vessels). In order to measure the performance of our method, we define the hitrate and the hitrate on the support S_0, respectively, as

$$h = \frac{1}{p}\sum_{i=1}^{p} \mathbb{1}_{\{\beta_i^0 \in J_i^\circ\}}, \quad h_{S_0} = \frac{1}{s_0}\sum_{i \in S} \mathbb{1}_{\{\beta_i^0 \in J_i^\circ\}}. \tag{14}$$

We calculate the average hitrates for 100 realizations of the subsampled Fourier matrix and the noise. The results are presented in Table 1. Furthermore, we change the threshold

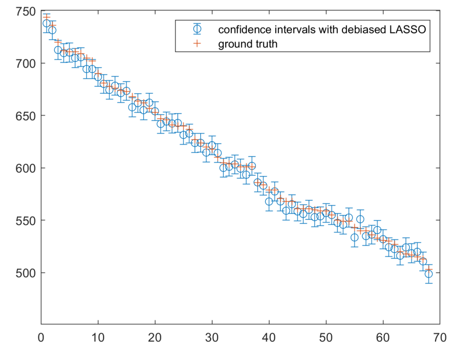

Fig. 2. Confidence intervals based on the debiased LASSO for the pixels with largest magnitude sorted in a descending order.

threshold	s_0	h_{S_0}	h	SSIM
210	648	0.942	0.955	0.967
200	1282	0.931	0.951	0.964
190	2789	0.901	0.941	0.954
180	5510	0.823	0.916	0.889

Table 1. Values of h_{S_0}, h and SSIM for different sparsity levels and constant $n = 0.4p$. The values are averaged over 100 realizations of F_Ω and ε.

leading to different sparsity levels in order to understand the role of the sparsity in the construction of the confidence intervals. Besides the hitrates, we calculate the similarity measure, SSIM, between the ground truth image and the estimated image. The smaller the sparsity, the better are the hitrates and the SSIM. We observe the same behavior if we fix the sparsity and increase the amount of data n. Even though the (sufficient) condition $n \gtrsim s_0 \log^2 p$ is not fulfilled for any threshold, the method still works well. Note that the hitrates do not depend on the noise level, since the radius of confidence regions in (8) scales with the noise level. For example, a threshold of 200 and a noise level of $\sigma = 2000$ lead to hitrates of, respectively, $h_{S_0} = 0.932$ and $h = 0.951$.

6. CONCLUSION AND FUTURE WORK

We derived confidence intervals for subsampled Fourier measurements that are used in compressive sensing for MRI reconstruction. The length of the confidence intervals decreases with the optimal rate $\frac{1}{\sqrt{n}}$. We showed that a sufficient amount of data for performing uncertainty quantification is given by $n \gtrsim \max\{s_0 \log^2 s_0 \log p, s_0 \log^2 p\}$. For this purpose we debiased the LASSO and showed its asymptotic normality. As an extension, we plan to derive confidence regions for quantitative multi-parametric MRI problems as well as to the case where the ground truth image is not trivially sparse but rather needs to be sparsified with a learned dictionary.

7. REFERENCES

[1] M. J. Wainwright, *High-dimensional statistics: A non-asymptotic viewpoint*, vol. 48, Cambridge University Press, 2019.

[2] R. Tibshirani, "Regression shrinkage and selection via the lasso," *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, vol. 58, no. 1, pp. 267–288, 1996.

[3] D. L. Donoho, "Compressed sensing," *IEEE Transactions on Information Theory*, vol. 52, no. 4, pp. 1289–1306, 2006.

[4] S. Foucart and H. Rauhut, *A Mathematical Introduction to Compressive Sensing*, Springer New York, New York, NY, 2013.

[5] C.-H. Zhang and S. S. Zhang, "Confidence intervals for low dimensional parameters in high dimensional linear models," *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, vol. 76, no. 1, pp. 217–242, 2014.

[6] A. Javanmard and A. Montanari, "Confidence intervals and hypothesis testing for high-dimensional regression," *Journal of Machine Learning Research*, vol. 15, pp. 2869–2909, 2014.

[7] A. Javanmard and A. Montanari, "Debiasing the lasso: Optimal sample size for Gaussian designs," *The Annals of Statistics*, vol. 46, no. 6A, 2018.

[8] S. van de Geer, P. Bühlmann, Y. Ritov, and R. Dezeure, "On asymptotically optimal confidence regions and tests for high-dimensional models," *The Annals of Statistics*, vol. 42, no. 3, 2014.

[9] M. Rudelson and R. Vershynin, "On sparse reconstruction from Fourier and Gaussian measurements," *Communications on Pure and Applied Mathematics*, vol. 61, no. 8, pp. 1025–1045, 2008.

[10] Z.-P. Liang and P. C. Lauterbur, *Principles of Magnetic Resonance Imaging: A Signal Processing Perspective*, SPIE Optical Engineering Press, 2000.

[11] A. Maleki, L. Anitori, Z. Yang, and R. G. Baraniuk, "Asymptotic analysis of complex lasso via complex approximate message passing (camp)," *IEEE Transactions on Information Theory*, vol. 59, no. 7, pp. 4290–4308, 2013.

[12] R. Vershynin, *High-Dimensional Probability: An Introduction with Applications in Data Science*, Cambridge University Press, 2018.

[13] F. Hoppe, F. Krahmer, C. Mayrink Verdun, M. I. Menzel, and H. Rauhut, "Uncertainty quantification for sparse Fourier recovery," *arXiv:2212.14864*, 2022.

[14] T. T. Cai and Z. Guo, "Confidence intervals for high-dimensional linear regression: Minimax rates and adaptivity," *The Annals of Statistics*, vol. 45, no. 2, 2017.

[15] S. A. van de Geer and P. Bühlmann, "On the conditions used to prove oracle results for the LASSO," *Electronic Journal of Statistics*, vol. 3, pp. 1360–1392, 2009.

[16] I. Haviv and O. Regev, "The restricted isometry property of subsampled Fourier matrices," in *Geometric Aspects of Functional Analysis: Israel Seminar (GAFA) 2014–2016*, Bo'az Klartag and Emanuel Milman, Eds., pp. 163–179. Springer International Publishing, Cham, 2017.

[17] P. Bühlmann and S. van de Geer, *Statistics for High-Dimensional Data*, Springer Berlin Heidelberg, Berlin, Heidelberg, 2011.

[18] P. C. Bellec and C.-H. Zhang, "De-biasing the lasso with degrees-of-freedom adjustment," *Bernoulli*, vol. 28, no. 2, pp. 713–743, 2022.

[19] S. Li, "Debiasing the debiased LASSO with bootstrap," *Electronic Journal of Statistics*, vol. 14, no. 1, 2020.

[20] T. Sun and C.-H. Zhang, "Scaled sparse linear regression," *Biometrika*, vol. 99, no. 4, pp. 879–898, 2012.

[21] S. Aja-Fernández and G. Vegas-Sánchez-Ferrero, *Statistical Analysis of Noise in MRI: Modeling, Filtering and Estimation*, Springer, Cham, 2016.

[22] S. N. Wright, P. Kochunov, F. Mut, M. Bergamino, K. M. Brown, J. C. Mazziotta, A. W. Toga, J. R. Cebral, and G. A. Ascoli, "Digital reconstruction and morphometric analysis of human brain arterial vasculature from magnetic resonance angiography," *NeuroImage*, vol. 82, pp. 170–181, 2013.

[23] M. Lustig, D. Donoho, and J. M. Pauly, "Sparse MRI: The application of compressed sensing for rapid MR imaging," *Magnetic Resonance in Medicine*, vol. 58, no. 6, pp. 1182–1195, 2007.

[24] S. R. Becker, E. J. Candès, and M. C. Grant, "Templates for convex cone problems with applications to sparse signal recovery," *Mathematical Programming Computation*, vol. 3, no. 3, pp. 165–218, 2011.

[25] D. Chetverikov, Z. Liao, and V. Chernozhukov, "On cross-validated LASSO in high dimensions," *The Annals of Statistics*, vol. 49, no. 3, pp. 1300 – 1317, 2021.