Useful formulas

Basic definitions

$$0! = 1 \qquad \forall n \in \mathbb{N}^+, n! = n(n-1)! = n \cdot (n-1) \cdots 2 \cdot 1$$

$$(a)_n = \prod_{k=a}^{a+n-1} k = a \cdot (a+1) \cdots (a+n-1), \qquad (a)_0 = 1, \qquad (1)_n = n!$$

$$\forall n > k, \binom{n}{k} = \frac{n!}{n!(n-k)!} = \frac{(n-k+1)_k}{(1)_k} = \frac{(n-k+1)(n-k+2)\cdots n}{1 \cdot 2 \cdots k}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n, \qquad e \approx 2.718281828$$

$$i^2 = -1$$

$$e^{ix} = \cos x + i \sin x$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}, \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$\pi = 4\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\right) \approx 3.141592654$$

Binomial formulas

$$(x \pm y)^2 = x^2 \pm 2xy + y^2, \qquad (x \pm y)^3 = x^3 \pm 3x^2y + 3xy^2 \pm y^3$$

$$(x \pm y)^n = \sum_{k=0}^{n} \binom{n}{k} (\pm 1)^k x^{n-k} y^k$$

$$a_1 + a_2 + \cdots + a_m = n \implies \binom{n}{a_1, a_2, \ldots, a_m} = \frac{n!}{a_1! a_2! \cdots a_m!}$$

$$(x+y+z)^n = \sum_{i,j,k \in \mathbb{N}^3 | i+j+k=n} \binom{n}{i, j, k} x^i y^j z^k$$

$$(x_1+x_2+\cdots+x_m)^n = \sum_{a_1,a_2,\ldots,a_m \in \mathbb{N}^m | a_1+a_2+\cdots+a_m=n} \binom{n}{a_1, a_2, \ldots a_m} x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m}$$

$$\forall y \in \mathbb{R}, x \in (-1,1) \implies (1+x)^y = \sum_{k=0}^{\infty} \binom{y}{k} x^k = \sum_{k=0}^{\infty} \frac{(y-k+1)_k}{k!} x^k$$

$$\binom{n}{k} = \binom{n}{n-k}, \qquad \sum_{k=0}^{n} \binom{n}{k} = 2^n, \qquad \binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$

Trig identities

$$\sin^2 x + \cos^2 x = 1$$

$$\sin(-x) = -\sin x, \qquad \cos(-x) = \cos x, \qquad \tan(-x) = -\tan x$$

$$\sin\left(\frac{\pi}{2} \pm x\right) = \cos x, \qquad \sin(\pi \pm x) = \mp \sin x$$

$$\cos\left(\frac{\pi}{2} \pm x\right) = \mp \sin x, \qquad \cos(\pi \pm x) = -\cos x$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\forall x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \tan x = \sum_{n=1}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)x^{2n-1}}{(2n)!} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + O(x^9)$$

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

$$\sin(nx) = \sum_{k \text{ odd}} (-1)^{\frac{k-1}{2}} \binom{n}{k} \cos^{n-k} x \sin^k x$$

$$\cos(nx) = \sum_{k \text{ even}} (-1)^{\frac{k}{2}} \binom{n}{k} \cos^{n-k} x \sin^k x$$

$$\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

$$\tan\left(\frac{\pi}{4} \pm x\right) = \frac{\frac{1}{\tan x} \mp 1}{1 \pm \frac{1}{\tan x}}$$

$$\sin \frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}, \qquad \cos \frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$$

$$\tan \frac{x}{2} = \frac{\sin x}{1 + \cos x}$$

$$\tan\left(\frac{x \pm y}{2}\right) = \frac{\sin x \pm \sin y}{\cos x + \cos y}$$

$$\sin x \pm \sin y = 2 \sin\left(\frac{x \pm y}{2}\right) \cos\left(\frac{x \mp y}{2}\right)$$

Quadratic equations

For quadratic equations of the form $x^2+bx+c=0$, do a variable substitution $t=x+\frac{b}{2}$. Squaring both sides we get:

$$x^2=t^2-bx-\frac{b^2}{4} \implies t^2-bx-\frac{b^2}{4}+bx+c=0 \implies t^2=\frac{b^2}{4}-c$$

If $\frac{b^2}{4}>c$, there are 2 real solutions. If $\frac{b^2}{4}=c$, this is a degenerate case with 1 solution. If $\frac{b^2}{4}<c$, the equation has no real solutions. To find the solutions:

$$t=\pm\sqrt{\frac{b^2-4c}{4}}, \qquad x=t-\frac{b}{2}=\frac{-b\pm\sqrt{b^2-4c}}{2}$$

Cubic equations

For cubic equations of the form $x^3+bx^2+cx+d=0$, do a variable substitution to transform the equation into a depressed cubic w.r.t. t:

$$x=t-\frac{b}{3}, \quad x^2=t^2-\frac{2bt}{3}+\frac{b^2}{9}, \quad x^3=t^3-bt^2+\frac{b^2t}{3}-\frac{b^3}{27}$$

$$t^3-bt^2+\frac{b^2t}{3}-\frac{b^3}{27}+b\left(t^2-\frac{2bt}{3}+\frac{b^2}{9}\right)+c\left(t-\frac{b}{3}\right)+d=0$$

$$t^3-\frac{b^2t}{3}+\frac{2b^3}{27}-\frac{bc}{3}+d=0$$

And out equation now has the form $t^3+pt+q=0$. Let us perform the following variable substitution:

$$t=s-\frac{p}{3s}, \quad t^3=s^3-ps+\frac{p^2}{3s}-\frac{p^3}{27s^3}$$

$$s^3-ps+\frac{p^2}{3s}-\frac{p^3}{27s^3}+ps-\frac{p^2}{3s}+q=0$$

$$s^3+q-\frac{p^3}{27s^3}=0$$

$$(s^3)^2+q(s^3)-\frac{p^3}{27}=0$$

$$s^3=\frac{-q\pm\sqrt{q^2+\frac{4p^3}{27}}}{2}$$

Choosing any of the roots as a solution will work - there are 6 solutions for s, but when we recover t the non intjective functions of t as expressed in terms of s will reduce the amount of solutions to 3. Practically, choose the non-zero solution for s^3 in case one of them is zero, otherwise choose any of them.

Quartic equations

$$x^4+ax^3+bx^2+cx+d=0$$

$$x:=t-\frac{a}{4}, \quad x^2=t^2-\frac{at}{2}+\frac{a^2}{16}, \quad x^3=t^3-\frac{3at^2}{4}+\frac{3a^2t}{16}-\frac{a^3}{64}$$

$$x^4=t^4-at^3+\frac{3a^2t^2}{8}-\frac{a^3t}{16}+\frac{a^4}{256}$$

Substitution in the original equation:

$$t^4-at^3+\frac{3a^2t^2}{8}-\frac{a^3t}{16}+\frac{a^4}{256}+a\left(t^3-\frac{3at^2}{4}+\frac{3a^2t}{16}-\frac{a^3}{64}\right)+b\left(t^2-\frac{at}{2}+\frac{a^2}{16}\right)+c\left(t-\frac{a}{4}\right)+d=0$$

$$t^4+\left(b-\frac{3a^2}{8}\right)t^2+\left(\frac{a^3}{8}-\frac{ab}{2}+c\right)t+\left(d-\frac{3a^4}{256}+\frac{a^2b}{16}-\frac{ac}{4}\right)=0$$

Now we have an equation of the form

$$t^4+pt^2+qt+r=0$$

$$t^4+2pt^2+p^2+qt+r=pt^2+p^2$$

$$(t^2+p)^2+qt+r=pt^2+p^2$$

Inject a new free variable w into the first square and balance the equation:

$$(t^2+p+w)^2-2w(t^2+p)-w^2+qt+r=pt^2+p^2$$

$$(t^2+p+w)^2=(p+2w)t^2-qt+(w^2+2wp+p^2-r)$$

$$(t^2+p+w)^2=(p+2w)\left(t^2-\frac{qt}{p+2w}+\frac{w^2+2wp+p^2-r}{p+2w}\right)$$

$$\frac{(t^2+p+w)^2}{p+2w}=t^2-\frac{qt}{p+2w}+\frac{w^2+2wp+p^2-r}{p+2w}$$

Recall the descriminant condition for quadratic formula - in order for the right hand side to be a degenerate form and a perfect square, we must have - in there the condition was $\frac{b^2}{4}=c$, so the condition here is:

$$\frac{q^2}{4(p+2w)^2}=\frac{w^2+2wp+p^2-r}{p+2w}$$

$$q^2=4(p+2w)(w^2+2wp+p^2-r)$$

This is a cubic equation for w which is a free variable. Solve it and select a solution w_1. As we have solved for a perfect square,

$$\frac{(t^2+p+w_1)^2}{p+2w_1}=\left(t-\frac{q}{2(p+2w_1)}\right)^2$$

$$(t^2+p+w_1)^2=\left(\left(\sqrt{p+2w_1}\right)t-\frac{q}{2\sqrt{p+2w_1}}\right)^2$$

$$t^2+p+w_1=\pm\left(\left(\sqrt{p+2w_1}\right)t-\frac{q}{2\sqrt{p+2w_1}}\right)$$

From here, extract the 2 possible quadratic equations. Solve them to find the 4 solutions for the original quartic in terms of t, t_1,t_2,t_3,t_4.

$$x=t-\frac{a}{4}$$

$$x_1=t_1-\frac{a}{4}, x_2=t_2-\frac{a}{4}, x_3=t_3-\frac{a}{4}, x_4=t_4-\frac{a}{4}$$

If you have followed this derivation and solved an equation, you get +5 pts for determination.

In the early 19th century, Abel, Rufiini and Galois proved that a general quintic equation does not have an algebraic solution.

Matrices

If the eigenvalues of a a matrix M are $\lambda_1, \lambda_2, \ldots, \lambda_n$ than:

$$|M| = \prod_{m=1}^{n} \lambda_m, \qquad \operatorname{tr} M = \sum_{m=1}^{n} \lambda_m$$

Every square matrix over the complex plane can be expressed in jordan cannonical form:

$$J_{\lambda,1} = \begin{bmatrix}\lambda\end{bmatrix}, \qquad J_{\lambda,2} = \begin{bmatrix}\lambda & 1 \\ 0 & \lambda\end{bmatrix}, \qquad J_{\lambda,n} = \begin{bmatrix}\lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda\end{bmatrix}$$

$$\forall M \in \mathbb{M}_{n\times n}(\mathbb{C}), \exists P \in \mathbb{M}_{n\times n}(\mathbb{C}), M = P^{-1} \begin{bmatrix} J_{\lambda_1,n_1} & & & \\ & J_{\lambda_2,n_2} & & \\ & & \ddots & \\ & & & J_{\lambda_m,n_m}\end{bmatrix} P$$

$$e^M = \sum_{n=0}^{\infty} \frac{M^n}{n!}$$

$$\left|e^M\right| = e^{\operatorname{tr} M}$$

A random matrix over the complex field is diagonalizable with probability 1. If M is diagonalizable:

$$M^n = \left(P^{-1} \begin{bmatrix}\lambda_1 & 0 & \cdots & 0 & \\ 0 & \lambda_2 & \cdots & 0 & \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & & \lambda_m\end{bmatrix} P\right)^n = P^{-1} \begin{bmatrix}\lambda_1^n & 0 & \cdots & 0 \\ 0 & \lambda_2^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m^n\end{bmatrix} P$$

Formal power series

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty}\left(\left(\sum_{l=0}^{n} a_l b_{n-l}\right) x^n\right)$$

$$\left(\sum_{n\in\mathbb{Z}} a_n x^n\right)\left(\sum_{n\in\mathbb{Z}} b_n x^n\right) = \sum_{n,m\in\mathbb{Z}^2} a_n b_m x^{n+m}$$

Special functions

$$\text{Riemann's zeta:} \quad \forall z \in \mathbb{C}, \Re z > 1, \quad \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

$$\text{Dirichlet's eta:} \forall z \in \mathbb{C}, \Re z > 0, \quad \eta(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^z} = \left(1 - 2^{1-z}\right)\zeta(z)$$

Lambert's W - the inverse of $x \cdot e^x$, the bigger one if there are 2 inverses:

$$\forall x > -\frac{1}{e}, \quad W_0(xe^x) = x$$

The smaller branch of Lambert's W:

$$\forall x \in \left[-\frac{1}{e}, -1\right], \quad W_{-1}(xe^x) = x$$

$$\forall n \in \mathbb{N}, n! = \Gamma(n+1) = \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$