Useful formulas

Basic definitions

$$0! = 1 \qquad \forall n \in \mathbb{N}^+, n! = n(n-1)! = n\cdot(n-1)\cdots 2\cdot 1$$

$$(a)_n = \prod_{k=a}^{a+n-1} k = a\cdot(a+1)\cdots(a+n-1), \qquad (a)_0 = 1, \qquad (1)_n = n!$$

$$\forall n > k, \binom{n}{k} = \frac{n!}{n!(n-k)!} = \frac{(n-k+1)_k}{(1)_k} = \frac{(n-k+1)(n-k+2)\cdots n}{1\cdot 2\cdots k}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n, \qquad e \approx 2.718281828$$

$$i^2 = -1$$

$$e^{ix} = \cos x + i \sin x$$

$$\sin x = \frac{e^{ix}-e^{-ix}}{2i}, \qquad \cos x = \frac{e^{ix}+e^{-ix}}{2}$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$\pi = 4\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\right) \approx 3.141592654$$

Binomial formulas

$$(x\pm y)^2 = x^2 \pm 2xy + y^2, \qquad (x\pm y)^3 = x^3 \pm 3x^2y + 3xy^2 \pm y^3$$

$$(x\pm y)^n = \sum_{k=0}^{n}\binom{n}{k}(\pm 1)^k x^{n-k}y^k$$

$$a_1 + a_2 + \cdots + a_m = n \implies \binom{n}{a_1, a_2, \ldots, a_m} = \frac{n!}{a_1!a_2!\cdots a_m!}$$

$$(x+y+z)^n = \sum_{i,j,k\in\mathbb{N}^3 | i+j+k=n}\binom{n}{i,j,k}x^i y^j z^k$$

$$(x_1+x_2+\cdots+x_m)^n = \sum_{a_1,a_2,\ldots,a_m\in\mathbb{N}^m | a_1+a_2+\cdots+a_m=n}\binom{n}{a_1,a_2,\ldots a_m}x_1^{a_1}x_2^{a_2}\cdots x_m^{a_m}$$

$$\forall y\in\mathbb{R}, x\in(-1,1) \implies (1+x)^y = \sum_{k=0}^{\infty}\binom{y}{k}x^k = \sum_{k=0}^{\infty}\frac{(y-k+1)_k}{k!}x^k$$

$$\binom{n}{k} = \binom{n}{n-k}, \qquad \sum_{k=0}^{n}\binom{n}{k} = 2^n, \qquad \binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$

Trig identities

$$\sin^2 x + \cos^2 x = 1$$

$$\sin(-x) = -\sin x, \qquad \cos(-x) = \cos x, \qquad \tan(-x) = -\tan x$$

$$\sin\left(\frac{\pi}{2}\pm x\right) = \cos x, \qquad \sin(\pi \pm x) = \mp \sin x$$

$$\cos\left(\frac{\pi}{2}\pm x\right) = \mp\sin x, \qquad \cos(\pi\pm x) = -\cos x$$

$$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad \cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

$$\forall x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right), \tan x = \sum_{n=1}^{\infty}\frac{B_{2n}(-4)^n(1-4^n)x^{2n-1}}{(2n)!} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + O(x^9)$$

$$\sin 2x = 2\sin x\cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

$$\sin(nx) = \sum_{k \text{ odd}}(-1)^{\frac{k-1}{2}}\binom{n}{k}\cos^{n-k}x\sin^k x$$

$$\cos(nx) = \sum_{k \text{ even}}(-1)^{\frac{k}{2}}\binom{n}{k}\cos^{n-k}x\sin^k x$$

$$\sin(x\pm y) = \sin x\cos y \pm \sin y\cos x$$

$$\cos(x\pm y) = \cos x\cos y \mp \sin x\sin y$$

$$\tan(x\pm y) = \frac{\tan x\pm\tan y}{1\mp\tan x\tan y}$$

$$\tan\left(\frac{\pi}{4}\pm x\right) = \frac{\frac{1}{\tan x}\mp 1}{1\pm\frac{1}{\tan x}}$$

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}, \qquad \cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$$

$$\tan\frac{x}{2} = \frac{\sin x}{1+\cos x}$$

$$\tan\left(\frac{x\pm y}{2}\right) = \frac{\sin x\pm\sin y}{\cos x+\cos y}$$

$$\sin x\pm\sin y = 2\sin\left(\frac{x\pm y}{2}\right)\cos\left(\frac{x\mp y}{2}\right)$$

Matrices

If the eigenvalues of a a matrix M are $\lambda_1, \lambda_2, \ldots, \lambda_n$ than:

$$|M| = \prod_{m=1}^{n} \lambda_m, \qquad \operatorname{tr} M = \sum_{m=1}^{n} \lambda_m$$

Every square matrix over the complex plane can be expressed in jordan cannonical form:

$$J_{\lambda,1} = \left[\lambda\right], \qquad J_{\lambda,2} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \qquad J_{\lambda,n} = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}$$

$$\forall M \in \mathbb{M}_{n\times n}(\mathbb{C}), \exists P \in \mathbb{M}_{n\times n}(\mathbb{C}), M = P^{-1} \begin{bmatrix} J_{\lambda_1,n_1} & & & \\ & J_{\lambda_2,n_2} & & \\ & & \ddots & \\ & & & J_{\lambda_m,n_m} \end{bmatrix} P$$

$$e^M = \sum_{n=0}^{\infty} \frac{M^n}{n!}$$

$$\left|e^M\right| = e^{\operatorname{tr} M}$$

A random matrix over the complex field is diagonalizable with probability 1. If M is diagonalizable:

$$M^n = \left(P^{-1} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 & \\ 0 & \lambda_2 & \cdots & 0 & \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & & \lambda_m \end{bmatrix} P \right)^n = P^{-1} \begin{bmatrix} \lambda_1^n & 0 & \cdots & 0 \\ 0 & \lambda_2^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m^n \end{bmatrix} P$$

Formal power series

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty}\left(\left(\sum_{l=0}^{n} a_l b_{n-l}\right) x^n\right)$$

$$\left(\sum_{n\in\mathbb{Z}} a_n x^n\right)\left(\sum_{n\in\mathbb{Z}} b_n x^n\right) = \sum_{n,m\in\mathbb{Z}^2} a_n b_m x^{n+m}$$

Special functions

$$\text{Riemann's zeta:} \quad \forall z \in \mathbb{C}, \Re z > 1, \quad \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

$$\text{Dirichlet's eta:} \forall z \in \mathbb{C}, \Re z > 0, \quad \eta(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^z} = \left(1 - 2^{1-z}\right)\zeta(z)$$

Lambert's W - the inverse of $x \cdot e^x$, the bigger one if there are 2 inverses:

$$\forall x > -\frac{1}{e}, \quad W_0(xe^x) = x$$

The smaller branch of Lambert's W:

$$\forall x \in \left[-\frac{1}{e}, -1\right], \quad W_{-1}(xe^x) = x$$

$$\forall n \in \mathbb{N}, n! = \Gamma(n+1) = \Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt$$