# Useful formulas

## Basic definitions

$$0! = 1 \qquad \forall n \in \mathbb{N}^+, n! = n(n-1)! = n \cdot (n-1) \cdots 2 \cdot 1$$

$$(a)_n = \prod_{k=a}^{a+n-1} k = a \cdot (a+1) \cdots (a+n-1), \qquad (a)_0 = 1, \qquad (1)_n = n!$$

$$\forall n > k, \binom{n}{k} = \frac{n!}{n!(n-k)!} = \frac{(n-k+1)_k}{(1)_k} = \frac{(n-k+1)(n-k+2)\cdots n}{1 \cdot 2 \cdots k}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n, \qquad e \approx 2.718281828$$

$$i^2 = -1$$

$$e^{ix} = \cos x + i \sin x$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}, \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$\pi = 4\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\right) \approx 3.141592654$$

## Binomial formulas

$$(x \pm y)^2 = x^2 \pm 2xy + y^2, \qquad (x \pm y)^3 = x^3 \pm 3x^2 y + 3xy^2 \pm y^3$$

$$(x \pm y)^n = \sum_{k=0}^{n} \binom{n}{k}(\pm 1)^k x^{n-k} y^k$$

$$a_1 + a_2 + \cdots + a_m = n \implies \binom{n}{a_1, a_2, \ldots, a_m} = \frac{n!}{a_1! a_2! \cdots a_m!}$$

$$(x + y + z)^n = \sum_{i,j,k \in \mathbb{N}^3 | i+j+k=n} \binom{n}{i,j,k} x^i y^j z^k$$

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{a_1, a_2, \ldots, a_m \in \mathbb{N}^m | a_1 + a_2 + \cdots + a_m = n} \binom{n}{a_1, a_2, \ldots a_m} x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m}$$

$$\forall y \in \mathbb{R}, x \in (-1, 1) \implies (1+x)^y = \sum_{k=0}^{\infty} \binom{y}{k} x^k = \sum_{k=0}^{\infty} \frac{(y-k+1)_k}{k!} x^k$$

$$\binom{n}{k} = \binom{n}{n-k}, \qquad \sum_{k=0}^{n} \binom{n}{k} = 2^n, \qquad \binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$

## Trig identities

$$\sin^2 x + \cos^2 x = 1$$

$$\sin(-x) = -\sin x, \qquad \cos(-x) = \cos x, \qquad \tan(-x) = -\tan x$$

$$\sin\left(\frac{\pi}{2} \pm x\right) = \cos x, \qquad \sin(\pi \pm x) = \mp \sin x$$

$$\cos\left(\frac{\pi}{2} \pm x\right) = \mp \sin x, \qquad \cos(\pi \pm x) = -\cos x$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\forall x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \tan x = \sum_{n=1}^{\infty} \frac{B_{2n}(-4)^n(1-4^n) x^{2n-1}}{(2n)!} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + O(x^9)$$

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

$$\sin(nx) = \sum_{k \text{ odd}} (-1)^{\frac{k-1}{2}} \binom{n}{k} \cos^{n-k} x \sin^k x$$

$$\cos(nx) = \sum_{k \text{ even}} (-1)^{\frac{k}{2}} \binom{n}{k} \cos^{n-k} x \sin^k x$$

$$\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

$$\tan\left(\frac{\pi}{4} \pm x\right) = \frac{\frac{1}{\tan x} \mp 1}{1 \pm \frac{1}{\tan x}}$$

$$\sin \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{2}}, \qquad \cos \frac{x}{2} = \pm \sqrt{\frac{1 + \cos x}{2}}$$

$$\tan \frac{x}{2} = \frac{\sin x}{1 + \cos x}$$

$$\tan\left(\frac{x \pm y}{2}\right) = \frac{\sin x \pm \sin y}{\cos x + \cos y}$$

$$\sin x \pm \sin y = 2 \sin\left(\frac{x \pm y}{2}\right) \cos\left(\frac{x \mp y}{2}\right)$$

## Quadratic equations

For quadratic equations of the form $x^2 + bx + c = 0$, do a variable substitution $t = x + \frac{b}{2}$. Squaring both sides we get:

$$x^2 = t^2 - bx - \frac{b^2}{4} \implies t^2 - bx - \frac{b^2}{4} + bx + c = 0 \implies t^2 = \frac{b^2}{4} - c$$

If $\frac{b^2}{4} > c$, there are 2 real solutions. If $\frac{b^2}{4} = c$, this is a degenerate case with 1 solution. If $\frac{b^2}{4} < c$, the equation has no real solutions. To find the solutions:

$$t = \pm\sqrt{\frac{b^2 - 4c}{4}}, \qquad x = t - \frac{b}{2} = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

## Cubic equations

Full derivation of the cubic formula is provided as using it in tests without derivation been known to lead to score detraction. For cubic equations of the form $x^3 + bx^2 + cx + d = 0$, do a variable substitution to transform the equation into a depressed cubic w.r.t. $t$:

$$x = t - \frac{b}{3}, \quad x^2 = t^2 - \frac{2bt}{3} + \frac{b^2}{9}, \quad x^3 = t^3 - bt^2 + \frac{b^2 t}{3} - \frac{b^3}{27}$$

$$t^3 - bt^2 + \frac{b^2 t}{3} - \frac{b^3}{27} + b\left(t^2 - \frac{2bt}{3} + \frac{b^2}{9}\right) + c\left(t - \frac{b}{3}\right) + d = 0$$

$$t^3 - \frac{b^2 t}{3} + \frac{2b^3}{27} - \frac{bc}{3} + d = 0$$

And out equation now has the form $t^3 + pt + q = 0$. Let us perform the following variable substitution:

$$t = s - \frac{p}{3s}, \quad t^3 = s^3 - ps + \frac{p^2}{3s} - \frac{p^3}{27s^3}$$

$$s^3 - ps + \frac{p^2}{3s} - \frac{p^3}{27s^3} + ps - \frac{p^2}{3s} + q = 0$$

$$s^3 + q - \frac{p^3}{27s^3} = 0$$

$$(s^3)^2 + q(s^3) - \frac{p^3}{27} = 0$$

$$s^3 = \frac{-q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2}$$

Choosing any of the roots as a solution will work - there are 6 solutions for $s$, but when we recover $t$ the non intjective functions of $t$ as expressed in terms of $s$ will reduce the amount of solutions to 3. Practically, choose the non-zero solution for $s^3$ in case one of them is zero, otherwise choose any of them.

## Quartic equations

$$x^4 + ax^3 + bx^2 + cx + d = 0$$

$$x := t - \frac{a}{4}, \quad x^2 = t^2 - \frac{at}{2} + \frac{a^2}{16}, \quad x^3 = t^3 - \frac{3at^2}{4} + \frac{3a^2 t}{16} - \frac{a^3}{64}$$

$$x^4 = t^4 - at^3 + \frac{3a^2 t^2}{8} - \frac{a^3 t}{16} + \frac{a^4}{256}$$

Substitution in the original equation:

$$t^4 - at^3 + \frac{3a^2 t^2}{8} - \frac{a^3 t}{16} + \frac{a^4}{256} + a\left(t^3 - \frac{3at^2}{4} + \frac{3a^2 t}{16} - \frac{a^3}{64}\right) + b\left(t^2 - \frac{at}{2} + \frac{a^2}{16}\right) + c\left(t - \frac{a}{4}\right) + d = 0$$

$$t^4 + \left(b - \frac{3a^2}{8}\right)t^2 + \left(\frac{a^3}{8} - \frac{ab}{2} + c\right)t + \left(d - \frac{3a^4}{256} + \frac{a^2 b}{16} - \frac{ac}{4}\right) = 0$$

Now we have an equation of the form

$$t^4 + pt^2 + qt + r = 0$$

$$t^4 + 2pt^2 + p^2 + qt + r = pt^2 + p^2$$

$$(t^2 + p)^2 + qt + r = pt^2 + p^2$$

Inject a new free variable $w$ into the first square and balance the equation:

$$(t^2 + p + w)^2 - 2w(t^2 + p) - w^2 + qt + r = pt^2 + p^2$$

$$(t^2 + p + w)^2 = (p + 2w)t^2 - qt + (w^2 + 2wp + p^2 - r)$$

$$(t^2 + p + w)^2 = (p + 2w)\left(t^2 - \frac{qt}{p + 2w} + \frac{w^2 + 2wp + p^2 - r}{p + 2w}\right)$$

$$\frac{(t^2 + p + w)^2}{p + 2w} = t^2 - \frac{qt}{p + 2w} + \frac{w^2 + 2wp + p^2 - r}{p + 2w}$$

Recall the descriminant condition for quadratic formula - in order for the right hand side to be a degenerate form and a perfect square, we must have - in there the condition was $\frac{b^2}{4} = c$, so the condition here is:

$$\frac{q^2}{4(p + 2w)^2} = \frac{w^2 + 2wp + p^2 - r}{p + 2w}$$

$$q^2 = 4(p + 2w)(w^2 + 2wp + p^2 - r)$$

This is a cubic equation for $w$ which is a free variable. Solve it and select a solution $w_1$. As we have solved for a perfect square,

$$\frac{(t^2 + p + w_1)^2}{p + 2w_1} = \left(t - \frac{q}{2(p + 2w_1)}\right)^2$$

$$(t^2 + p + w_1)^2 = \left(\left(\sqrt{p + 2w_1}\right)t - \frac{q}{2\sqrt{p + 2w_1}}\right)^2$$

$$t^2 + p + w_1 = \pm\left(\left(\sqrt{p + 2w_1}\right)t - \frac{q}{2\sqrt{p + 2w_1}}\right)$$

There are 2 resulting quadratic equations requlting for this. Solve them to find the 4 solutions for the original quartic in terms of $t$, $t_1, t_2, t_3, t_4$.

$$x = t - \frac{a}{4}$$

$$x_1 = t_1 - \frac{a}{4}, x_2 = t_2 - \frac{a}{4}, x_3 = t_3 - \frac{a}{4}, x_4 = t_4 - \frac{a}{4}$$

If you have followed this formula and solved an equation, give yourself +5 pts for determination.

## Matrices

If the eigenvalues of a a matrix $M$ are $\lambda_1, \lambda_2, \ldots, \lambda_n$ than:

$$|M| = \prod_{m=1}^{n} \lambda_m, \qquad \operatorname{tr} M = \sum_{m=1}^{n} \lambda_m$$

Every square matrix over the complex plane can be expressed in jordan cannonical form:

$$J_{\lambda,1} = \begin{bmatrix} \lambda \end{bmatrix}, \qquad J_{\lambda,2} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \qquad J_{\lambda,n} = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}$$

$$\forall M \in \mathbb{M}_{n \times n}(\mathbb{C}), \exists P \in \mathbb{M}_{n \times n}(\mathbb{C}), M = P^{-1} \begin{bmatrix} J_{\lambda_1, n_1} & & & \\ & J_{\lambda_2, n_2} & & \\ & & \ddots & \\ & & & J_{\lambda_m, n_m} \end{bmatrix} P$$

$$e^M = \sum_{n=0}^{\infty} \frac{M^n}{n!}$$

$$\left| e^M \right| = e^{\operatorname{tr} M}$$

A random matrix over the complex field is diagonalizable with probability 1. If $M$ is diagonalizable:

$$M^n = \left( P^{-1} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m \end{bmatrix} P \right)^n = P^{-1} \begin{bmatrix} \lambda_1^n & 0 & \cdots & 0 \\ 0 & \lambda_2^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m^n \end{bmatrix} P$$

## Formal power series

$$\left( \sum_{n=0}^{\infty} a_n x^n \right) \left( \sum_{n=0}^{\infty} b_n x^n \right) = \sum_{n=0}^{\infty} \left( \left( \sum_{l=0}^{n} a_l b_{n-l} \right) x^n \right)$$

$$\left( \sum_{n \in \mathbb{Z}} a_n x^n \right) \left( \sum_{n \in \mathbb{Z}} b_n x^n \right) = \sum_{n,m \in \mathbb{Z}^2} a_n b_m x^{n+m}$$

## Special functions

$$\text{Riemann's zeta:} \quad \forall z \in \mathbb{C}, \Re z > 1, \quad \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

$$\text{Dirichlet's eta:} \forall z \in \mathbb{C}, \Re z > 0, \quad \eta(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^z} = \left( 1 - 2^{1-z} \right) \zeta(z)$$

Lambert's W - the inverse of $x \cdot e^x$, the bigger one if there are 2 inverses:

$$\forall x > -\frac{1}{e}, \quad W_0(xe^x) = x$$

The smaller branch of Lambert's W:

$$\forall x \in \left[ -\frac{1}{e}, -1 \right], \quad W_{-1}(xe^x) = x$$

$$\forall n \in \mathbb{N}, n! = \Gamma(n+1) = \Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt$$