THE DIVERGENCE THEOREM OF GAUSS states that if V is the volume bounded by a closed surface S and $\mathbf{A}$ is a vector function of position with continuous derivatives, then

$$\iiint_V \nabla\cdot\mathbf{A}\,dV = \iint_S \mathbf{A}\cdot\mathbf{n}\,dS = \oiint_S \mathbf{A}\cdot d\mathbf{S}$$

where $\mathbf{n}$ is the positive (outward drawn) normal to S.

18. Verify the divergence theorem for $\mathbf{A} = 4x\,\mathbf{i} - 2y^2\,\mathbf{j} + z^2\,\mathbf{k}$ taken over the region bounded by $x^2 + y^2 = 4$, $z = 0$ and $z = 3$.

$$\text{Volume integral} = \iiint_V \nabla\cdot\mathbf{A}\,dV = \iiint_V \left[\frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)\right] dV$$

$$= \iiint_V (4 - 4y + 2z)\,dV = \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{3} (4 - 4y + 2z)\,dz\,dy\,dx = 84\pi$$

The surface S of the cylinder consists of a base S_1 $(z = 0)$, the top S_2 $(z = 3)$ and the convex portion S_3 $(x^2 + y^2 = 4)$. Then

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$$\text{Surface integral} = \iint_S \mathbf{A}\cdot\mathbf{n}\,dS = \iint_{S_1} \mathbf{A}\cdot\mathbf{n}\,dS_1 + \iint_{S_2} \mathbf{A}\cdot\mathbf{n}\,dS_2 + \iint_{S_3} \mathbf{A}\cdot\mathbf{n}\,dS_3$$

On S_1 $(z=0)$, $\mathbf{n} = -\mathbf{k}$, $\mathbf{A} = 4x\,\mathbf{i} - 2y^2\,\mathbf{j}$ and $\mathbf{A}\cdot\mathbf{n} = 0$, so that $\iint_{S_1} \mathbf{A}\cdot\mathbf{n}\,dS_1 = 0$.

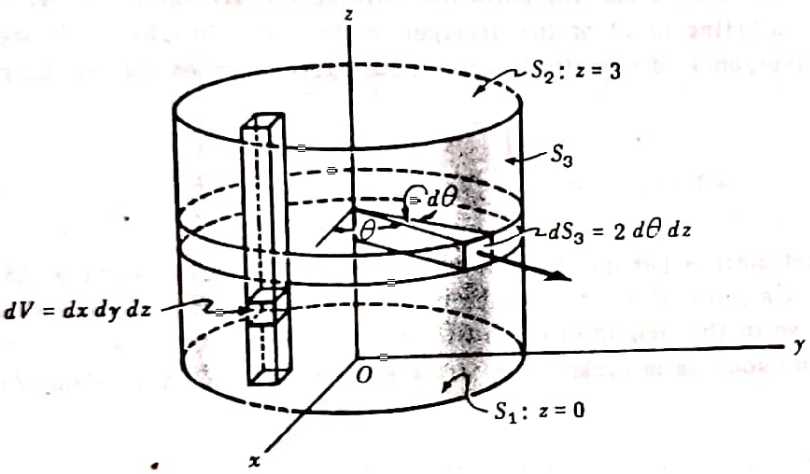
On S_2 $(z=3)$, $\mathbf{n} = \mathbf{k}$, $\mathbf{A} = 4x\,\mathbf{i} - 2y^2\,\mathbf{j} + 9\mathbf{k}$ and $\mathbf{A}\cdot\mathbf{n} = 9$, so that

$$\iint_{S_2} \mathbf{A}\cdot\mathbf{n}\,dS_2 = 9\iint_{S_2} dS_2 = 36\pi, \quad \text{since area of } S_2 = 4\pi$$

On S_3 $(x^2+y^2=4)$. A perpendicular to $x^2+y^2=4$ has the direction $\nabla(x^2+y^2) = 2x\,\mathbf{i} + 2y\,\mathbf{j}$.

Then a unit normal is $\mathbf{n} = \dfrac{2x\,\mathbf{i} + 2y\,\mathbf{j}}{\sqrt{4x^2+4y^2}} = \dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{2}$ since $x^2+y^2=4$.

$$\mathbf{A}\cdot\mathbf{n} = (4x\,\mathbf{i} - 2y^2\,\mathbf{j} + z^2\,\mathbf{k})\cdot\left(\frac{x\,\mathbf{i}+y\,\mathbf{j}}{2}\right) = 2x^2 - y^3$$



From the figure above, $x = 2\cos\theta$, $y = 2\sin\theta$, $dS_3 = 2\,d\theta\,dz$ and so

$$\iint_{S_3} \mathbf{A}\cdot\mathbf{n}\,dS_3 = \int_{\theta=0}^{2\pi}\int_{z=0}^{3} [2(2\cos\theta)^2 - (2\sin\theta)^3]\,2\,dz\,d\theta$$

$$= \int_{\theta=0}^{2\pi} (48\cos^2\theta - 48\sin^3\theta)\,d\theta = \int_{\theta=0}^{2\pi} 48\cos^2\theta\,d\theta = 48\pi$$

Then the surface integral $= 0 + 36\pi + 48\pi = 84\pi$, agreeing with the volume integral and verifying the divergence theorem.

Note that evaluation of the surface integral over S_3 could also have been done by projection of S_3 on the xz or yz coordinate planes.

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