DEFINITION Let $f(x)$ be defined on an open interval about c, except possibly at c itself. We say that the **limit of $f(x)$ as x approaches c is the number L**, and write

$$\lim_{x \to c} f(x) = L,$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

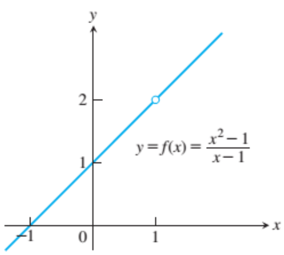
2.2 Limit of a Function and Limit Laws

HISTORICAL ESSAY
Limits
www.goo.gl/5V45iK

In Section 2.1 we saw how limits arise when finding the instantaneous rate of change of a function or the tangent line to a curve. We begin this section by presenting an informal definition of the limit of a function. We then describe laws that capture the behavior of limits. These laws enable us to quickly compute limits for a variety of functions, including polynomials and rational functions. We present the precise definition of a limit in the next section.

Limits of Function Values

Frequently when studying a function $y = f(x)$, we find ourselves interested in the function's behavior *near* a particular point c, but not *at* c itself. An important example occurs when the process of trying to evaluate a function at c leads to division by zero, which is undefined. We encountered this when seeking the instantaneous rate of change in y by considering the quotient function $\Delta y/h$ for h closer and closer to zero. In the next example we explore numerically how a function behaves near a particular point at which we cannot directly evaluate the function.



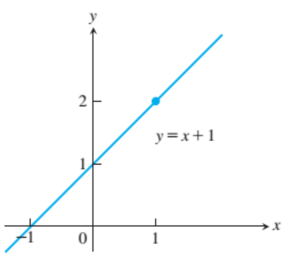


FIGURE 2.7 The graph of f is identical with the line $y = x + 1$ except at $x = 1$, where f is not defined (Example 1).

EXAMPLE 1 How does the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

behave near $x = 1$?

Solution The given formula defines f for all real numbers x except $x = 1$ (since we cannot divide by zero). For any $x \neq 1$, we can simplify the formula by factoring the numerator and canceling common factors:

$$f(x) = \frac{(x - 1)(x + 1)}{x - 1} = x + 1 \qquad \text{for} \qquad x \neq 1.$$

The graph of f is the line $y = x + 1$ with the point $(1, 2)$ *removed*. This removed point is shown as a "hole" in Figure 2.7. Even though $f(1)$ is not defined, it is clear that we can make the value of $f(x)$ as close as we want to 2 by choosing x close enough to 1 (Table 2.2). ■

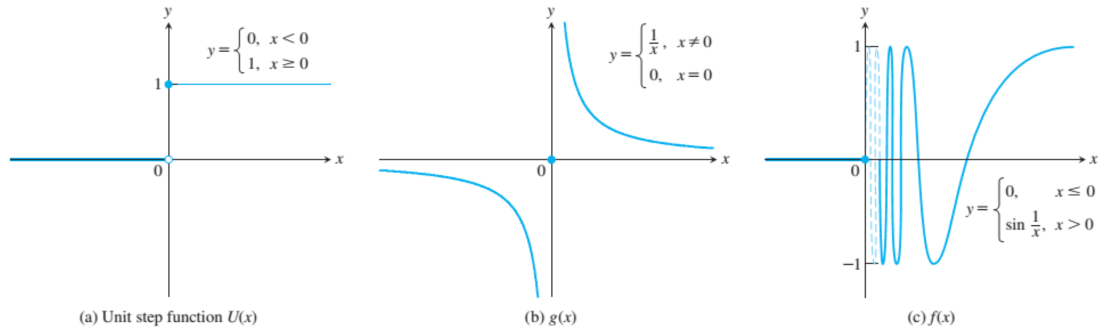


FIGURE 2.10 None of these functions has a limit as x approaches 0 (Example 4).

EXAMPLE 4 Discuss the behavior of the following functions, explaining why they have no limit as $x \to 0$.

(a) $U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$

(b) $g(x) = \begin{cases} \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(c) $f(x) = \begin{cases} 0, & x \leq 0 \\ \sin\dfrac{1}{x}, & x > 0 \end{cases}$

Solution

(a) The function *jumps*: The **unit step function** $U(x)$ has no limit as $x \to 0$ because its values jump at $x = 0$. For negative values of x arbitrarily close to zero, $U(x) = 0$. For positive values of x arbitrarily close to zero, $U(x) = 1$. There is no *single* value L approached by $U(x)$ as $x \to 0$ (Figure 2.10a).

(b) The function *grows too "large" to have a limit*: $g(x)$ has no limit as $x \to 0$ because the values of g grow arbitrarily large in absolute value as $x \to 0$ and therefore do not stay close to *any* fixed real number (Figure 2.10b). We say the function is *not bounded*.

(c) The function *oscillates too much to have a limit*: $f(x)$ has no limit as $x \to 0$ because the function's values oscillate between $+1$ and -1 in every open interval containing 0. The values do not stay close to any single number as $x \to 0$ (Figure 2.10c). ■

THEOREM 1—Limit Laws
If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x \to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x \to c} [f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $f(x) \geq 0$ for x in an interval containing c.)

EXAMPLE 5 Use the observations $\lim_{x \to c} k = k$ and $\lim_{x \to c} x = c$ (Example 3) and the limit laws in Theorem 1 to find the following limits.

(a) $\lim_{x \to c}(x^3 + 4x^2 - 3)$

(b) $\lim_{x \to c} \dfrac{x^4 + x^2 - 1}{x^2 + 5}$

(c) $\lim_{x \to -2} \sqrt{4x^2 - 3}$

Solution

(a)
$$\begin{aligned}
\lim_{x \to c}(x^3 + 4x^2 - 3) &= \lim_{x \to c} x^3 + \lim_{x \to c} 4x^2 - \lim_{x \to c} 3 && \text{Sum and Difference Rules} \\
&= c^3 + 4c^2 - 3 && \text{Power and Multiple Rules}
\end{aligned}$$

(b)
$$\begin{aligned}
\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5} &= \frac{\lim_{x \to c}(x^4 + x^2 - 1)}{\lim_{x \to c}(x^2 + 5)} && \text{Quotient Rule} \\
&= \frac{\lim_{x \to c} x^4 + \lim_{x \to c} x^2 - \lim_{x \to c} 1}{\lim_{x \to c} x^2 + \lim_{x \to c} 5} && \text{Sum and Difference Rules} \\
&= \frac{c^4 + c^2 - 1}{c^2 + 5} && \text{Power or Product Rule}
\end{aligned}$$

(c)
$$\begin{aligned}
\lim_{x \to -2} \sqrt{4x^2 - 3} &= \sqrt{\lim_{x \to -2}(4x^2 - 3)} && \text{Root Rule with } n = 2 \\
&= \sqrt{\lim_{x \to -2} 4x^2 - \lim_{x \to -2} 3} && \text{Difference Rule} \\
&= \sqrt{4(-2)^2 - 3} && \text{Product and Multiple Rules and limit of a constant function} \\
&= \sqrt{16 - 3} \\
&= \sqrt{13}
\end{aligned}$$

Evaluating Limits of Polynomials and Rational Functions

Theorem 1 simplifies the task of calculating limits of polynomials and rational functions. To evaluate the limit of a polynomial function as x approaches c, just substitute c for x in the formula for the function. To evaluate the limit of a rational function as x approaches a point c *at which the denominator is not zero*, substitute c for x in the formula for the function. (See Examples 5a and 5b.) We state these results formally as theorems.

THEOREM 2—Limits of Polynomials
If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0.$$

THEOREM 3—Limits of Rational Functions
If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

EXAMPLE 6 The following calculation illustrates Theorems 2 and 3:

$$\lim_{x \to -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{0}{6} = 0$$

Since the denominator of this rational expression does not equal 0 when we substitute -1 for x, we can just compute the value of the expression at $x = -1$ to evaluate the limit. ■

Eliminating Common Factors from Zero Denominators

Theorem 3 applies only if the denominator of the rational function is not zero at the limit point c. If the denominator is zero, canceling common factors in the numerator and denominator may reduce the fraction to one whose denominator is no longer zero at c. If this happens, we can find the limit by substitution in the simplified fraction.

Identifying Common Factors
If $Q(x)$ is a polynomial and $Q(c) = 0$, then $(x - c)$ is a factor of $Q(x)$. Thus, if the numerator and denominator of a rational function of x are both zero at $x = c$, they have $(x - c)$ as a common factor.

EXAMPLE 7 Evaluate

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}.$$

Solution We cannot substitute $x = 1$ because it makes the denominator zero. We test the numerator to see if it, too, is zero at $x = 1$. It is, so it has a factor of $(x - 1)$ in common with the denominator. Canceling this common factor gives a simpler fraction with the same values as the original for $x \neq 1$:

$$\frac{x^2 + x - 2}{x^2 - x} = \frac{(x - 1)(x + 2)}{x(x - 1)} = \frac{x + 2}{x}, \qquad \text{if } x \neq 1.$$

Using the simpler fraction, we find the limit of these values as $x \to 1$ by evaluating the function at $x = 1$, as in Theorem 3:

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \to 1} \frac{x + 2}{x} = \frac{1 + 2}{1} = 3.$$

THEOREM 4—The Sandwich Theorem
Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

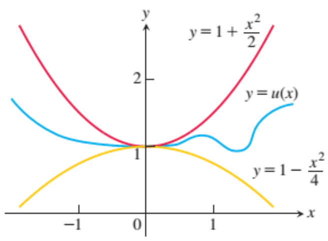


FIGURE 2.13 Any function $u(x)$ whose graph lies in the region between $y = 1 + (x^2/2)$ and $y = 1 - (x^2/4)$ has limit 1 as $x \to 0$ (Example 10).

The Sandwich Theorem is also called the Squeeze Theorem or the Pinching Theorem.

EXAMPLE 10 Given a function u that satisfies

$$1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{2} \qquad \text{for all } x \neq 0,$$

find $\lim_{x \to 0} u(x)$, no matter how complicated u is.

Solution Since

$$\lim_{x \to 0} (1 - (x^2/4)) = 1 \qquad \text{and} \qquad \lim_{x \to 0} (1 + (x^2/2)) = 1,$$

the Sandwich Theorem implies that $\lim_{x \to 0} u(x) = 1$ (Figure 2.13). ■

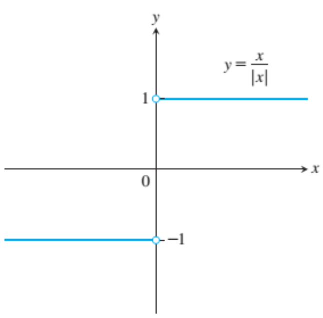


FIGURE 2.24 Different right-hand and left-hand limits at the origin.

Approaching a Limit from One Side

Suppose a function f is defined on an interval that extends to both sides of a number c. In order for f to have a limit L as x approaches c, the values of $f(x)$ must approach the value L as x approaches c from either side. Because of this, we sometimes say that the limit is **two-sided**.

If f fails to have a two-sided limit at c, it may still have a one-sided limit, that is, a limit if the approach is only from one side. If the approach is from the right, the limit is a **right-hand limit** or **limit from the right**. From the left, it is a **left-hand limit** or **limit from the left**.

The function $f(x) = x/|x|$ (Figure 2.24) has limit 1 as x approaches 0 from the right, and limit -1 as x approaches 0 from the left. Since these one-sided limit values are not the same, there is no single number that $f(x)$ approaches as x approaches 0. So $f(x)$ does not have a (two-sided) limit at 0.

Precise Definitions of One-Sided Limits

The formal definition of the limit in Section 2.3 is readily modified for one-sided limits.

DEFINITIONS (a) Assume the domain of f contains an interval (c, d) to the right of c. We say that $f(x)$ has **right-hand limit L at c**, and write

$$\lim_{x \to c^+} f(x) = L$$

if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad c < x < c + \delta.$$

(b) Assume the domain of f contains an interval (b, c) to the left of c. We say that f has **left-hand limit L at c**, and write

$$\lim_{x \to c^-} f(x) = L$$

if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad c - \delta < x < c.$$

THEOREM 7—Limit of the Ratio $\sin \theta/\theta$ as $\theta \to 0$

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \qquad (\theta \text{ in radians}) \tag{1}$$

EXAMPLE 5 Show that **(a)** $\lim_{y\to 0} \frac{\cos y - 1}{y} = 0$ and **(b)** $\lim_{x\to 0} \frac{\sin 2x}{5x} = \frac{2}{5}$.

Solution

(a) Using the half-angle formula $\cos y = 1 - 2\sin^2(y/2)$, we calculate

$$\begin{aligned}
\lim_{h\to 0} \frac{\cos y - 1}{y} &= \lim_{h\to 0} -\frac{2\sin^2(y/2)}{y} \\
&= -\lim_{\theta\to 0} \frac{\sin\theta}{\theta}\sin\theta && \text{Let } \theta = y/2. \\
&= -(1)(0) = 0. && \text{Eq. (1) and Example 11a in Section 2.2}
\end{aligned}$$

(b) Equation (1) does not apply to the original fraction. We need a $2x$ in the denominator, not a $5x$. We produce it by multiplying numerator and denominator by $2/5$:

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin 2x}{5x} &= \lim_{x\to 0} \frac{(2/5)\cdot \sin 2x}{(2/5)\cdot 5x} \\
&= \frac{2}{5}\lim_{x\to 0} \frac{\sin 2x}{2x} && \text{Eq. (1) applies with } \theta = 2x. \\
&= \frac{2}{5}(1) = \frac{2}{5}.
\end{aligned}$$

EXAMPLE 6 Find $\lim_{t\to 0} \frac{\tan t \sec 2t}{3t}$.

Solution From the definition of $\tan t$ and $\sec 2t$, we have

$$\begin{aligned}
\lim_{t\to 0} \frac{\tan t \sec 2t}{3t} &= \lim_{t\to 0} \frac{1}{3}\cdot\frac{1}{t}\cdot\frac{\sin t}{\cos t}\cdot\frac{1}{\cos 2t} \\
&= \frac{1}{3}\lim_{t\to 0} \frac{\sin t}{t}\cdot\frac{1}{\cos t}\cdot\frac{1}{\cos 2t} \\
&= \frac{1}{3}(1)(1)(1) = \frac{1}{3}. && \text{Eq. (1) and Example 11b in Section 2.2}
\end{aligned}$$

EXAMPLE 7 Show that for nonzero constants A and B.

$$\lim_{\theta\to 0} \frac{\sin A\theta}{\sin B\theta} = \frac{A}{B}.$$

Solution

$$\begin{aligned}
\lim_{\theta\to 0} \frac{\sin A\theta}{\sin B\theta} &= \lim_{\theta\to 0} \frac{\sin A\theta}{A\theta} A\theta \frac{B\theta}{\sin B\theta}\frac{1}{B\theta} && \text{Multiply and divide by } A\theta \text{ and } B\theta. \\
&= \lim_{\theta\to 0} \frac{\sin A\theta}{A\theta}\frac{B\theta}{\sin B\theta}\frac{A}{B} && \lim_{u\to 0}\frac{\sin u}{u} = 1, \text{ with } u = A\theta \\
&= \lim_{\theta\to 0} (1)(1)\frac{A}{B} && \lim_{v\to 0}\frac{v}{\sin v} = 1, \text{ with } v = B\theta \\
&= \frac{A}{B}.
\end{aligned}$$