Definitions:-

- **Function of two variables:**
If a quantity z has a unique, finite value for every pair of values of x and y, then z is called a function of two variables x and y. A function of two variables x and y is symbolically written as

$$f(x,y) \text{ or } F(x,y) \text{ or } \phi(x,y$$

- **Domain:** Domain of a function of two variables is a subset of $\mathbb{R}^2 = R \times R = \{(x,y); x,y \in R\}$

Range:
Range is subset of R.
Thus a function f of two variable is denoted by
$$f: S \rightarrow R \text{ where } S \subset R^2$$

Similarly, a function f of three variables is denoted as $F: S \rightarrow R$ where $S \subset R^3$.

- **Partial derivatives of first order:**
Let $z = f(x,y)$ be a function of two independent variables x and y. If y is kept constant and x alone is allowed to vary, then z becomes a function of x only. The derivative of z with respect to x, treating y as constant, is called partial derivative of z w.r.t x is denoted by $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or f_x.

$$\frac{dz}{dx} = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

Similary

$$\frac{dz}{dy} = \lim_{k \to 0} \frac{f(x, y+k) - f(x,y)}{k}$$

$\frac{dz}{dx}$ and $\frac{dz}{dy}$ are called first order partial derivatives of z

★ Composite functions:-

If $z = f(x,y)$ where $x = \phi(u,v)$, $y = \psi(u,v)$ then z is called a composite function of (two variables) u and v so that we can find $\frac{dz}{du}$ and $\frac{dz}{dv}$.

Diff of composite function:-

If u is composite function of t, defined by the relations $u = f(x,y)$; $x = \phi(t)$, $y = \psi(t)$ then

$$\frac{du}{dt} = \frac{du}{dx} \cdot \frac{dx}{dt} + \frac{du}{dy} \cdot \frac{dy}{dt}$$

$\frac{du}{dt}$ is called the total derivative of u to distinguish it from partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$

limit:

$$\lim_{(x,y)\to(a,b)} \underbrace{f(x,y)}_{z} = L \text{ (number)}$$

$z = f(x,y)$

SVC

$$\lim_{x\to a} f(x) = L$$

$$\iff \lim_{x\to a^+} f(x) = L$$

$$= \lim_{x\to a^-} f(x)$$

Continuity:

A function of two variable is said to be continous at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = \underbrace{f(a,b)}_{\text{must be defined}}$$

⇓

Limit = L

$\neq$ f is discontinou at (a,b)

yey

4y2

Partial derivatives

const

$$f(x,y)\ , \quad \left.\frac{df}{dx}\right|_{x=x_0} = \lim_{\substack{\cancel{x\to x_0} \\ h\to 0}} \frac{f(x_0+h,\ y_0) - f(x_0, y_0)}{h}$$

here y is const

const

$$\left.\frac{df}{dy}\right|_{y=y_0} = \lim_{\substack{\cancel{y\to y_0} \\ h\to 0}} \frac{f(x_0,\ y_0+h) - f(x_0, y_0)}{h}$$

here x is const

exist

Taylor's theorem

* Taylor's theorem about the (a,b) for $f(x,y)$

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right]$$

$$+ \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b)\right.$$

$$\left. + (y-b)^2 f_{yy}(a,b)\right]$$

$$+ \frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b)f_{xxy}(a,b)\right.$$

$$\left. + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\right]$$

$$+ \cdots$$

If $(a,b) = (0,0) \Rightarrow$ ~~This whole is know~~n as [Maclaurin's theorem for two variable]

* Jacobians.

If u and v are functions of two independent variables x and y, then the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$ is called Jacobian of u, v with

respect to x, y and is denoted by the

symbol $J\left[\frac{u,v}{x,y}\right]$ or $\frac{\partial(u,v)}{\partial(x,y)}$

* Properties

1. If u, v are functions of r, s where r, s are functions of x, y then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}$$

2. If J_1 is the Jacobian of u, v w.r.to x, y and J_2 is the Jacobian of x, y with respect to u, v then $J_1 J_2 = 1$ i.e $\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1$

3. Jacobian of Implicit function

$$\frac{\partial(u_1, u_2, u_3 \dots u_n)}{\partial(x_1, x_2, \dots x_n)} = (-1)^n \frac{\frac{\partial(f_1, f_2 \dots f_n)}{\partial(x_1, x_2 \dots x_n)}}{\frac{\partial(f_1, f_2 \dots f_n)}{\partial(u_1, u_2 \dots u_n)}}$$

* Vector functions

If to each value of a scalar variable t, there corresponds a value of a vector $\vec{r}$, then $\vec{r}$ is called a vector function of the scalar variable t and we write $\vec{r} = \vec{r}(t)$

or $\vec{r} = \vec{f}(t)$

Since every vector can be uniquely expressed as a linear combination of three fixed non-coplanar vectors, we may write $\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$

$\hat{i}, \hat{j}, \hat{k}$ — unit vectors along the axis of x, y, z

$f_1(t)$, $f_2(t)$ and $f_3(t)$ — compon— of vector $\vec{f}(t)$

* Derivative of a constant vector:

A vector is said to be constant if both its magnitude and direction are fixed. If either of these changes, the vector is not constant.

1.

2.

3.