Multi Variable calculus

Vector function :-

If to each value of a scalar variable t, there corresponds a value of a vector $\vec{r}$, then $\vec{r}$ is called a vector function of scalar variable t.

$$\vec{r} = \vec{r}(t) \quad \text{(or)} \quad \vec{r} = \vec{f}(t)$$

Every vector can be uniquely expressed as a linear combination of three fixed non-coplanar vectors,

$$\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$$

$f_1(t), f_2(t), f_3(t) \longrightarrow$ Components of vector $\vec{f}(t)$ along co-ordinate axis

Derivative of vector function with respect to scalar :-

let $\vec{r} = \vec{f}(t)$ to be a vector function of scalar variable t.

let δt be small increment in t and δr be increment in $\vec{r}$.

Then $\vec{r} + \delta\vec{r} = \vec{f}(t+\delta t)$ ~~$\vec{f}(t+\delta t)$~~

so that $\delta\vec{r} = \vec{f}(t+\delta t) - \vec{f}(t)$

$$\frac{\delta\vec{r}}{\delta t} = \frac{\vec{f}(t+\delta t) - \vec{f}(t)}{\delta t}$$

* If $\lim\limits_{\delta t \to 0} \frac{\delta\vec{r}}{\delta t} = \lim\limits_{\delta t \to 0} \frac{\vec{f}(t+\delta t) - \vec{f}(t)}{\delta t}$ exists, then the value of limit is denoted by $\frac{d\vec{r}}{dt}$ and is called derivative of $\vec{r}$ with respect to t.

General rules for differentiation :-

If $\vec{a}, \vec{b}$ and $\vec{c}$ are vector functions of a scalar t and ϕ is a scalar function of t, then

(i) $$\frac{d}{dt}(\vec{a} \pm \vec{b}) = \frac{d\vec{a}}{dt} \pm \frac{d\vec{b}}{dt}$$

(ii) $$\frac{d}{dt}(\vec{a}\cdot\vec{b}) = \vec{a}\cdot\frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt}\cdot\vec{b}$$

(iii) $$\frac{d}{dt}(\vec{a}\times\vec{b}) = \vec{a}\times\frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt}\times\vec{b}$$

(iv) $$\frac{d}{dt}(\phi\vec{a}) = \phi\frac{d\vec{a}}{dt} + \frac{d\phi}{dt}\vec{a}$$

(v) $$\frac{d}{dt}[\vec{a}\ \vec{b}\ \vec{c}] = \left[\frac{d\vec{a}}{dt}\ \vec{b}\ \vec{c}\right] + \left[\vec{a}\ \frac{d\vec{b}}{dt}\ \vec{c}\right] + \left[\vec{a}\ \vec{b}\ \frac{d\vec{c}}{dt}\right]$$

(vi) $$\frac{d}{dt}\left[\vec{a}\times(\vec{b}\times\vec{c})\right] = \frac{d\vec{a}}{dt}\times(\vec{b}\times\vec{c}) + \vec{a}\times\left(\frac{d\vec{b}}{dt}\times\vec{c}\right) + \vec{a}\times\left(\vec{b}\times\frac{d\vec{c}}{dt}\right)$$

Derivative of a constant vector :-

A vector is said to be constant if both its magnitude and direction are fixed. If either of these changes vector is not constant.

Let $\vec{r}$ be a constant vector function of scalar variable t $\vec{r} = \vec{f}(t)$, then $\vec{r} = \vec{f}(t+\delta t)$ so that $\vec{f}(t+\delta t) - \vec{f}(t) = \vec{0}$

$$\frac{d\vec{r}}{dt} = \lim_{\delta t\to 0}\frac{\vec{f}(t+\delta t) - \vec{f}(t)}{\delta t} = \lim_{\delta t\to 0}\vec{0} = \vec{0}$$

derivative of a constant is equal to null vector.

$$\frac{d\hat{i}}{dt} = \frac{d\hat{j}}{dt} = \frac{d\hat{k}}{dt} = \vec{0}$$

* If $\vec{F}(t)$ has a constant magnitude, then $\vec{F} \cdot \frac{d\vec{F}}{dt} = 0$

* If $\vec{F}(t)$ has a constant direction, then $\vec{F} \times \frac{d\vec{F}}{dt} = \vec{0}$

Velocity :-

If the scalar variable t donotes the time and $\vec{r}$ is the position vector of a moving particle P, then $\delta\vec{r}$ is the displacement of particle in time δt. The vector $\frac{\delta\vec{r}}{\delta t}$ is the average velocity of particle during the interval δt. If $\vec{v}$ represents the velocity vector of particle at P, then $\vec{v} = \lim_{\delta t \to 0} \frac{\delta\vec{r}}{\delta t} = \frac{d\vec{r}}{dt}$ and its direction its direction is along the tangent at P.

Acceleration :-

If $\delta\vec{v}$ be change in velocity $\vec{v}$ during the time δt, then $\frac{\delta\vec{v}}{\delta t}$ is the average acceleration of particle during the interval δt. If $\vec{a}$ represents the acceleration of particle at P, then

$$\vec{a} = \lim_{\delta t \to 0} \frac{\delta\vec{v}}{\delta t} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d^2\vec{r}}{dt^2}$$

* **Scalar point function** :- Let R be a region of space at each point of which a scalar $\phi = \phi(x,y,z)$ is given, then ϕ is called a scalar function. R is called scalar field.

Example :- The temperature distribution in a medium, the distribution of atmospheric in space.

* **Vector point function** :- Let R be a region of space at each point of which a vector $\vec{v} = \vec{v}(x,y,z)$ is given, then $\vec{v}$ is called a vector point function and R is called a vector field.

Ex :- The velocity of moving fluid at any instant, the gravitational force.

* **Gradient of scalar field** :- Let $\phi(x,y,z)$ be a function defining a scalar field, then vector $\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$ is called gradient of scalar field ϕ and denoted by grad ϕ.

$$\text{grad}\,\phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

gradient is denoted by symbol ∇, read as del (nabla).

$$\text{grad}\,\phi = \nabla \phi$$

* Directional derivative :-

Let $PQ = \delta r$, then $\lim_{\delta r \to 0} \frac{\delta \phi}{\delta r} = \frac{\partial \phi}{\partial r}$ is called directional derivative of ϕ at P in direction PQ.

Let $\hat{N}'$ be a unit vector in direction PQ then $\delta r = \frac{\delta n}{\cos\theta}$

$$\delta r = \frac{\delta n}{\hat{N} \cdot \hat{N}'}$$

$$\frac{\partial \phi}{\partial r} = \hat{N}' \cdot \nabla \phi$$

* Properties of gradient :-

(a) If ϕ is a constant scalar point function, then $\nabla \phi = \vec{0}$

(b) If ϕ_1 and ϕ_2 are two scalar point functions, then

(i) $\nabla(\phi_1 \pm \phi_2) = \nabla \phi_1 \pm \nabla \phi_2$

(ii) $\nabla(c_1\phi_1 + c_2\phi_2) = c_1 \nabla \phi_1 + c_2 \nabla \phi_2$, where c_1, c_2 are constant

(iii) $\nabla(\phi_1 \phi_2) = \phi_1 \nabla \phi_2 + \phi_2 \nabla \phi_1$

(iv) $\nabla\left(\frac{\phi_1}{\phi_2}\right) = \frac{\phi_2 \nabla \phi_1 - \phi_1 \nabla \phi_2}{\phi_2^2}$, $\phi_2 \neq 0$

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* Divergence of vector point function :-

The divergence of a differentiable vector point function $\vec{V}$ is denoted by div $\vec{V}$ and is defined as

$$\text{div } \vec{V} = \nabla \cdot \vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \vec{V} = \hat{i}\cdot\frac{\partial \vec{V}}{\partial x} + \hat{j}\cdot\frac{\partial \vec{V}}{\partial y} + \hat{k}\cdot\frac{\partial \vec{V}}{\partial z}$$

If $\vec{V} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ then

$$\text{div}\,\vec{V} = \nabla\cdot\vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}\right)$$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

since $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

** The divergence of a vector point function is a "scalar point function"

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* CURL of a vector point function :-

The curl (or rotation) of a differentiable vector point function $\vec{V}$ is denoted by curl $\vec{V}$ and defined as

$$\text{curl}\,\vec{V} = \nabla\times\vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\vec{V}$$

$$= \hat{i}\frac{\partial \vec{V}}{\partial x} + \hat{j}\frac{\partial \vec{V}}{\partial y} + \hat{k}\frac{\partial \vec{V}}{\partial z}$$

If $\vec{V} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

then $\text{curl}\,\vec{V} = \nabla\times\vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\left[v_1\hat{i} + v_2\hat{j} + v_3\hat{k}\right]$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

** The curl of a vector point function is vector point function.

* Note :-

If Div of vector $v = 0$, then vector v is called solenoidal vector point function

* Note :-

If curl $\vec{v} = 0$, then $\vec{v}$ is said to be an irrotational vector otherwise rotational.

→ Properties of divergence and curl :-

① For a constant vector $\vec{a}$, $\text{div}\,\vec{a} = 0$, $\text{curl}\,\vec{a} = 0$

② $\nabla\cdot(\vec{A}+\vec{B}) = \nabla\cdot\vec{A} + \nabla\cdot\vec{B}$

③ $\nabla\times(\vec{A}+\vec{B}) = \nabla\times\vec{A} + \nabla\times\vec{B}$

④ If $\vec{A}$ is vector function and ϕ is a scalar function, then

$\text{div}(\phi\vec{A}) = \phi\,\text{div}\,\vec{A} + (\text{grad}\,\phi)\cdot\vec{A}$

$\text{curl}(\phi\vec{A}) = \text{grad}\,\phi\times\vec{A} + \phi\,\text{curl}\,\vec{A}$

⑤ $\nabla(\vec{A}\cdot\vec{B}) = (\vec{A}\cdot\nabla)\vec{B} + (\vec{B}\cdot\nabla)\vec{A} + \vec{A}\times(\nabla\times\vec{B}) + \vec{B}\times(\nabla\times\vec{A})$

⑥ $\text{div}(\vec{A}\times\vec{B}) = \vec{B}\cdot\text{curl}\,\vec{A} - \vec{A}\cdot\text{curl}\,\vec{B}$

⑦ $\nabla\times(\vec{A}\times\vec{B}) = (\nabla\cdot\vec{B})\vec{A} - (\nabla\cdot\vec{A})\vec{B} + (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{B}$

Repeated operations by ∇ :-

① $\text{Div}(\text{grad}\,\phi) = \nabla^2\phi$ $\qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

② $\text{curl}(\text{grad}\,\phi) = \nabla\times\nabla\phi = \vec{0}$

③ $\text{Div}(\text{curl}\,\vec{v}) = \nabla\cdot(\nabla\times\vec{v}) = 0$

④ $\text{Curl}(\text{curl}\,\vec{v}) = \text{grad}\,\text{div}\,\vec{v} - \nabla^2\vec{v}$

Integration of Vector functions :-

Let $\vec{f}(t)$ and $\vec{F}(t)$ be two vector functions of a scalar variable such that $\frac{d}{dt}\vec{F}(t) = \vec{f}(t)$, then $\vec{F}(t)$ is called an integral of $\vec{f}(t)$ with respect to t

$\vec{F}(t)$ is called indefinite integral of $\vec{f}(t)$

The definite integral of $\vec{f}(t)$ between the limits $t = a$ and $t = b$ is written as

$$\int_a^b \vec{f}(t)\,dt = \left[\vec{F}(t)\right]_a^b = \vec{F}(b) - \vec{F}(a)$$

Line integral :-

Any integral which is to be evaluated along a curve is called a line integral.

$$A = P_0, P_1, P_2, \text{———} P_n = B$$

Circulation :-

In fluid dynamics, if $\vec{V}$ represents the velocity of a fluid particle and c is closed curve then integral $\oint_C \vec{V}\cdot d\vec{R}$ called the circulation of $\vec{V}$ around curve C.

If the circulation of $\vec{V}$ around every closed curve in region D vanishes, then $\vec{V}$ is said to be irrotational in D.

Work done by force:-

Let $\vec{F}$ represent the force acting on a particle moving along an arc AB. The work done during a small displacement $\vec{\delta R}$ is $\vec{F} \cdot \vec{\delta R}$.

The total workdone by $\vec{F}$ during displacement from A to B is given by $\int_A^B \vec{F} \cdot \vec{dR}$

If the force $\vec{F}$ is conservative, then there exists a scalar function ϕ such that

$$\vec{F} = \nabla\phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

$\therefore$ The workdone by $\vec{F}$ during displacement from A to B

$$= \int_A^B \vec{F} \cdot \vec{dR}$$

$$= \int_A^B \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right) \times (\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= \int_A^B \left(\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz\right) = \int_A^B d\phi = [\phi]_A^B$$

$$\phi_B - \phi_A$$

Surface integral :-

Any integral which is to be evaluated over a surface is called a surface integral.

let $\vec{F}(P)$ be a continuous vector point function and S a two sided surface. Divide S into a finite number of sub surfaces $\delta S_1, \delta S_2, \ldots \delta S_k$. let P_i be any point in δS_i and $\hat{n}_i$ be the unit vector at P_i in the direction of outward drawn normal to surface at P_i. Then the limit of sum

$\sum_{i=1}^{k} \vec{F}(P_i) \cdot \hat{n}_i \, \delta S_i$, as $k \to \infty$ and each $\delta S_i \to 0$ is called the normal surface integral of $\vec{F}(P)$ over S and is denoted by $\iint_S \vec{F} \cdot \hat{n} \, dS$.

Surface integral can be written as

$$\iint_S \vec{F} \cdot d\vec{S}$$

If $\vec{F}$ represent the velocity of a fluid at any point P on a closed surface S, then $\vec{F} \cdot \hat{n}$ is the normal component of $\vec{F}$ at P

$\iint_S \vec{F} \cdot \hat{n} \, dS = \iint_S \vec{F} \cdot d\vec{S}$ is a measure of volume emerging from S per unit time it measures flux of $\vec{F}$ over S.

Let R be orthogonal projection of s on xy-plane

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_R \vec{F}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{k}\cdot\hat{n}|}$$

Volume integrals :-

Any integral which is to be evaluated over a volume is called a volume integral

$$\iiint_V \phi\,dv \quad \text{and} \quad \iiint_V \vec{F}\,dV$$

(scalar) (vector)

$$\iiint_V \vec{F}\,dv = \hat{i}\iiint_V f_1(x,y,z)\,dx\,dy\,dz + \hat{j}\iiint f_2(x,y,z)\,dx\,dy\,dz + \hat{k}\iiint_V f_3(x,y,z)\,dx\,dy\,dz$$

$$\iiint_V \phi\,dv = \iiint_V \phi(x,y,z)\,dx\,dy\,dz$$

Green's theorem :-

If $M(x,y)$ and $N(x,y)$ be continuous functions of x and y having continuous partial derivatives $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ in a region R of xy-plane bounded by a closed curve 'c', then

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

where c is traversed in counterclockwise direction

Applications:-

This theorem useful for changing a line integral around a closed curve C into a double integral over region R enclosed by C.

Divergence theorem of Gauss :-

(Relation between surface and volume integrals)

If $\vec{F}$ is a vector point function having continuous first order partial derivatives in the region V bounded by a closed surface S, then $\iiint_V \nabla \cdot \vec{F} dv = \iint_S \vec{F} \cdot \hat{n} ds$,

where $\hat{n}$ is outwards drawn unit normal vector to surface S.

[The volume integral of divergence of a vector point function $\vec{F}$ taken over the volume V enclosed by a surface S, is equal to surface integral of normal component of $\vec{F}$ taken over the closed surface S.]

$$\iiint_V \nabla \cdot \vec{F} dv = \iint_R \vec{F} \cdot \hat{n} ds$$

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx\, dy\, dz$$

Applications:- Electrostatic fields.

Stoke's theorem :- (Reduction of surface integral to line integral)

The line integral of tangential component of a vector f taken around a simple closed curve C is equal to the surface integral of normal component of curl of taken over S having C as its boundary

Stoke's theorem in plane :-

$$\oint_C (f_1 dx + f_2 dy) = \iint_S \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\,dy$$

This form of stoke's theorem is also known as Green's theorem in plane.

Stoke's theorem in space :-

$$\oint_C \left((f_1 dx + f_2 dy + f_3 dz) \right) = \iint_S \left[\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \cos\alpha + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) \cos\beta + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \cos\gamma \right] ds.$$

Applications :-

Used in evaluating curl of a vector field.