Ellipse $r = \dfrac{\omega}{1+\epsilon\cos\theta}$

LINE INTEGRALS

6. If $\mathbf{A} = (3x^2+6y)\,\mathbf{i} - 14yz\,\mathbf{j} + 20xz^2\,\mathbf{k}$, evaluate $\int_C \mathbf{A}\cdot d\mathbf{r}$ from (0,0,0) to (1,1,1) along the following paths C:
(a) $x = t,\ y = t^2,\ z = t^3$.
(b) the straight lines from (0,0,0) to (1,0,0), then to (1,1,0), and then to (1,1,1).
(c) the straight line joining (0,0,0) and (1,1,1).

$$\int_C \mathbf{A}\cdot d\mathbf{r} = \int_C [(3x^2+6y)\,\mathbf{i} - 14yz\,\mathbf{j} + 20xz^2\mathbf{k}]\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$
$$= \int_C (3x^2+6y)\,dx - 14yz\,dy + 20xz^2\,dz$$

(a) If $x=t,\ y=t^2,\ z=t^3$, points (0,0,0) and (1,1,1) correspond to $t=0$ and $t=1$, respectively. Then

$$\int_C \mathbf{A}\cdot d\mathbf{r} = \int_{t=0}^{1} (3t^2+6t^2)\,dt - 14(t^2)(t^3)\,d(t^2) + 20(t)(t^3)^2\,d(t^3)$$
$$= \int_{t=0}^{1} 9t^2\,dt - 28t^6\,dt + 60t^9\,dt$$
$$= \int_{t=0}^{1} (9t^2-28t^6+60t^9)\,dt = 3t^3 - 4t^7 + 6t^{10}\Big|_0^1 = 5$$

Another Method.

Along C, $\mathbf{A} = 9t^2\mathbf{i} - 14t^5\mathbf{j} + 20t^7\mathbf{k}$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ and $d\mathbf{r} = (\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k})\,dt$.

Then $$\int_C \mathbf{A}\cdot d\mathbf{r} = \int_{t=0}^{1} (9t^2\,\mathbf{i} - 14t^5\,\mathbf{j} + 20t^7\,\mathbf{k})\cdot(\mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k})\,dt$$
$$= \int_0^1 (9t^2 - 28t^6 + 60t^9)\,dt = 5$$

(b) Along the straight line from (0,0,0) to (1,0,0) $y=0,\ z=0,\ dy=0,\ dz=0$ while x varies from 0 to 1. Then the integral over this part of the path is

$$\int_{x=0}^{1} (3x^2+6(0))\,dx - 14(0)(0)(0) + 20x(0)^2(0) = \int_{x=0}^{1} 3x^2\,dx = x^3\Big|_0^1 = 1$$

Along the straight line from (1,0,0) to (1,1,0) $x=1,\ z=0,\ dx=0,\ dz=0$ while y varies from 0 to 1. Then the integral over this part of the path is

$$\int_{y=0}^{1} (3(1)^2+6y)\,0 - 14y(0)\,dy + 20(1)(0)^2\,0 = 0$$

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Along the straight line from (1,1,0) to (1,1,1) $x=1$, $y=1$, $dx=0$, $dy=0$ while z varies from 0 to 1. Then the integral over this part of the path is

$$\int_{z=0}^{1} (3(1)^2+6(1))\,0 \;-\; 14(1)\,z(0) \;+\; 20(1)\,z^2\,dz \;=\; \int_{z=0}^{1} 20z^2\,dz \;=\; \frac{20z^3}{3}\Big|_0^1 \;=\; \frac{20}{3}$$

Adding,
$$\int_C \mathbf{A}\cdot d\mathbf{r} \;=\; 1 + 0 + \frac{20}{3} \;=\; \frac{23}{3}$$

(c) The straight line joining (0,0,0) and (1,1,1) is given in parametric form by $x=t$, $y=t$, $z=t$. Then

$$\int_C \mathbf{A}\cdot d\mathbf{r} \;=\; \int_{t=0}^{1} (3t^2+6t)\,dt \;-\; 14(t)(t)\,dt \;+\; 20(t)(t)^2\,dt$$

$$= \int_{t=0}^{1} (3t^2+6t-14t^2+20t^3)\,dt \;=\; \int_{t=0}^{1} (6t-11t^2+20t^3)\,dt \;=\; \frac{13}{3}$$

7. Find the total work done in moving a particle in a force field given by $\mathbf{F} = 3xy\,\mathbf{i} - 5z\,\mathbf{j} + 10x\,\mathbf{k}$ along the curve $x=t^2+1$, $y=2t^2$, $z=t^3$ from $t=1$ to $t=2$.

$$\text{Total work} \;=\; \int_C \mathbf{F}\cdot d\mathbf{r} \;=\; \int_C (3xy\,\mathbf{i} - 5z\,\mathbf{j} + 10x\,\mathbf{k})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \int_C 3xy\,dx \;-\; 5z\,dy \;+\; 10x\,dz$$

$$= \int_{t=1}^{2} 3(t^2+1)(2t^2)\,d(t^2+1) \;-\; 5(t^3)\,d(2t^2) \;+\; 10(t^2+1)\,d(t^3)$$

$$= \int_1^2 (12t^5 + 10t^4 + 12t^3 + 30t^2)\,dt \;=\; 303$$

8. If $\mathbf{F} = 3xy\,\mathbf{i} - y^2\,\mathbf{j}$, evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the curve in the xy plane, $y=2x^2$, from (0,0) to (1,2).

Since the integration is performed in the xy plane ($z=0$), we can take $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} \;=\; \int_C (3xy\,\mathbf{i} - y^2\,\mathbf{j})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j})$$

$$= \int_C 3xy\,dx \;-\; y^2\,dy$$

First Method. Let $x=t$ in $y=2x^2$. Then the parametric equations of C are $x=t$, $y=2t^2$. Points (0,0) and (1,2) correspond to $t=0$ and $t=1$ respectively. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} \;=\; \int_{t=0}^{1} 3(t)(2t^2)\,dt \;-\; (2t^2)^2\,d(2t^2) \;=\; \int_{t=0}^{1} (6t^3-16t^5)\,dt \;=\; -\frac{7}{6}$$

Second Method. Substitute $y=2x^2$ directly, where x goes from 0 to 1. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} \;=\; \int_{x=0}^{1} 3x(2x^2)\,dx \;-\; (2x^2)^2\,d(2x^2) \;=\; \int_{x=0}^{1} (6x^3-16x^5)\,dx \;=\; -\frac{7}{6}$$

Note that if the curve were traversed in the opposite sense, i.e. from (1,2) to (0,0), the value of the integral would have been $7/6$ instead of $-7/6$.

10. (a) If $\mathbf{F} = \nabla\phi$, where ϕ is single-valued and has continuous partial derivatives, show that the work done in moving a particle from one point $P_1 \equiv (x_1, y_1, z_1)$ in this field to another point $P_2 \equiv (x_2, y_2, z_2)$ is independent of the path joining the two points.

(b) Conversely, if $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of the path C joining any two points, show that there exists a function ϕ such that $\mathbf{F} = \nabla\phi$.

(a) Work done $= \int_{P_1}^{P_2} \mathbf{F} \cdot d\mathbf{r} = \int_{P_1}^{P_2} \nabla\phi \cdot d\mathbf{r}$

$$= \int_{P_1}^{P_2} \left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right) \cdot (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \int_{P_1}^{P_2} \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= \int_{P_1}^{P_2} d\phi = \phi(P_2) - \phi(P_1) = \phi(x_2, y_2, z_2) - \phi(x_1, y_1, z_1)$$

Then the integral depends only on points P_1 and P_2 and not on the path joining them. This is true of course only if $\phi(x, y, z)$ is single-valued at all points P_1 and P_2.

18. Suppose that the surface S has projection R on the xy plane (see figure of Prob. 17). Show that

$$\iint_S \mathbf{A}\cdot\mathbf{n}\,dS = \iint_R \mathbf{A}\cdot\mathbf{n}\,\frac{dx\,dy}{|\mathbf{n}\cdot\mathbf{k}|}$$

19. Evaluate $\iint_S \mathbf{A} \cdot \mathbf{n} \, dS$, where $\mathbf{A} = 18z\,\mathbf{i} - 12\mathbf{j} + 3y\,\mathbf{k}$ and S is that part of the plane $2x + 3y + 6z = 12$ which is located in the first octant.

The surface S and its projection R on the xy plane are shown in the figure below.

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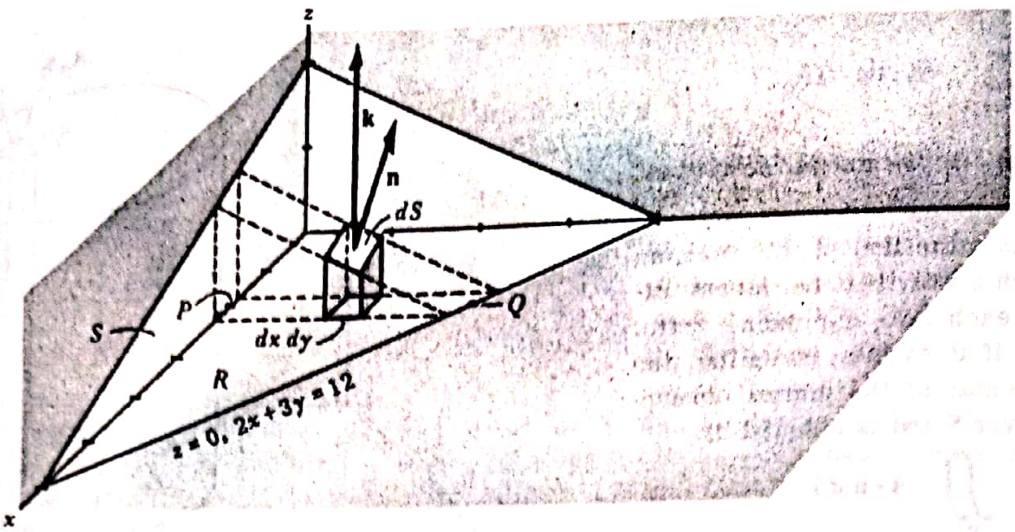

From Problem 17,

$$\iint_S \mathbf{A}\cdot\mathbf{n}\,dS = \iint_R \mathbf{A}\cdot\mathbf{n}\,\frac{dx\,dy}{|\mathbf{n}\cdot\mathbf{k}|}$$

To obtain $\mathbf{n}$ note that a vector perpendicular to the surface $2x+3y+6z = 12$ is given by $\nabla(2x+3y+6z) = 2\mathbf{i}+3\mathbf{j}+6\mathbf{k}$ (see Problem 5 of Chapter 4). Then a unit normal to any point of S (see figure above) is

$$\mathbf{n} = \frac{2\mathbf{i}+3\mathbf{j}+6\mathbf{k}}{\sqrt{2^2+3^2+6^2}} = \frac{2}{7}\mathbf{i}+\frac{3}{7}\mathbf{j}+\frac{6}{7}\mathbf{k}$$

Thus $\mathbf{n}\cdot\mathbf{k} = (\frac{2}{7}\mathbf{i}+\frac{3}{7}\mathbf{j}+\frac{6}{7}\mathbf{k})\cdot\mathbf{k} = \frac{6}{7}$ and so $\frac{dx\,dy}{|\mathbf{n}\cdot\mathbf{k}|} = \frac{7}{6}\,dx\,dy$.

Also $\mathbf{A}\cdot\mathbf{n} = (18z\,\mathbf{i}-12\mathbf{j}+3y\,\mathbf{k})\cdot(\frac{2}{7}\mathbf{i}+\frac{3}{7}\mathbf{j}+\frac{6}{7}\mathbf{k}) = \frac{36z-36+18y}{7} = \frac{36-12x}{7}$, using the fact that $z = \frac{12-2x-3y}{6}$ from the equation of S. Then

$$\iint_S \mathbf{A}\cdot\mathbf{n}\,dS = \iint_R \mathbf{A}\cdot\mathbf{n}\,\frac{dx\,dy}{|\mathbf{n}\cdot\mathbf{k}|} = \iint_R \left(\frac{36-12x}{7}\right)\frac{7}{6}\,dx\,dy = \iint_R (6-2x)\,dx\,dy$$

To evaluate this double integral over R, keep x fixed and integrate with respect to y from $y=0$ (P in the figure above) to $y = \frac{12-2x}{3}$ (Q in the figure above); then integrate with respect to x from $x=0$ to $x=6$. In this manner R is completely covered. The integral becomes

$$\int_{x=0}^{6}\int_{y=0}^{(12-2x)/3} (6-2x)\,dy\,dx = \int_{x=0}^{6}\left(24-12x+\frac{4x^2}{3}\right)dx = 24$$

If we had chosen the positive unit normal $\mathbf{n}$ opposite to that in the figure above, we would have obtained the result -24.

20. Evaluate $\iint_S \mathbf{A}\cdot\mathbf{n}\, dS$, where $\mathbf{A} = z\,\mathbf{i} + x\,\mathbf{j} - 3y^2 z\,\mathbf{k}$ and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$.

Project S on the xz plane as in the figure below and call the projection R. Note that the projection of S on the xy plane cannot be used here. Then

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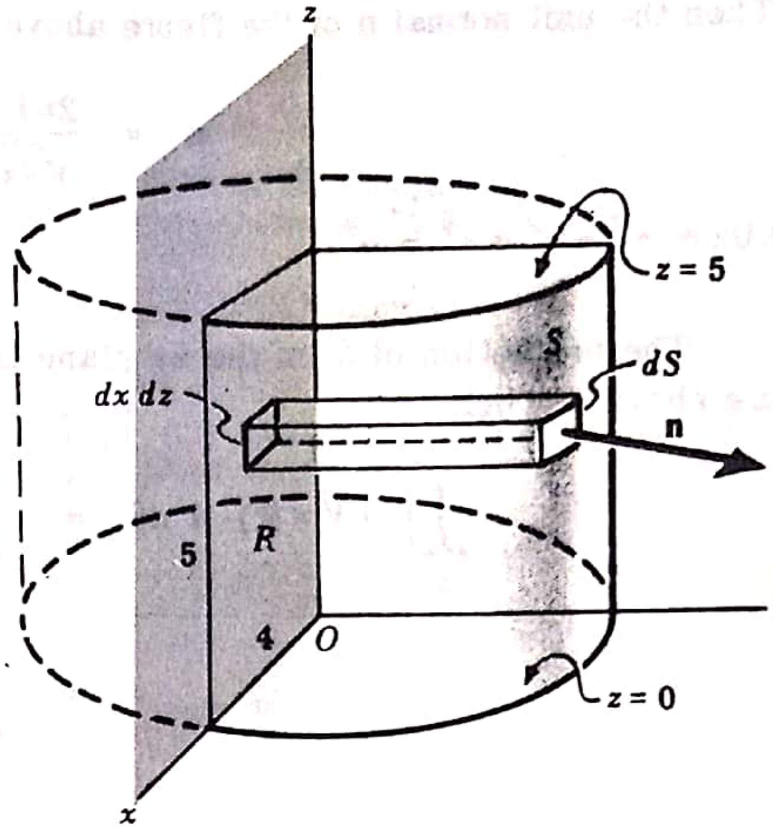
$$\iint_S \mathbf{A}\cdot\mathbf{n}\, dS = \iint_R \mathbf{A}\cdot\mathbf{n}\, \frac{dx\, dz}{|\mathbf{n}\cdot\mathbf{j}|}$$

A normal to $x^2+y^2 = 16$ is $\nabla(x^2+y^2) = 2x\mathbf{i}+2y\mathbf{j}$. Thus the unit normal to S as shown in the adjoining figure, is



$$\mathbf{n} = \frac{2x\,\mathbf{i} + 2y\,\mathbf{j}}{\sqrt{(2x)^2 + (2y)^2}} = \frac{x\,\mathbf{i} + y\,\mathbf{j}}{4}$$

since $x^2+y^2 = 16$ on S.

$$\mathbf{A}\cdot\mathbf{n} = (z\,\mathbf{i} + x\,\mathbf{j} - 3y^2z\,\mathbf{k})\cdot(\frac{x\,\mathbf{i}+y\,\mathbf{j}}{4}) = \frac{1}{4}(xz + xy)$$

$$\mathbf{n}\cdot\mathbf{j} = \frac{x\,\mathbf{i}+y\,\mathbf{j}}{4}\cdot\mathbf{j} = \frac{y}{4}.$$

Then the surface integral equals

$$\iint_R \frac{xz + xy}{y}\, dx\, dz = \int_{z=0}^{5}\int_{x=0}^{4} (\frac{xz}{\sqrt{16-x^2}} + x)\, dx\, dz = \int_{z=0}^{5} (4z+8)\, dz = 90$$