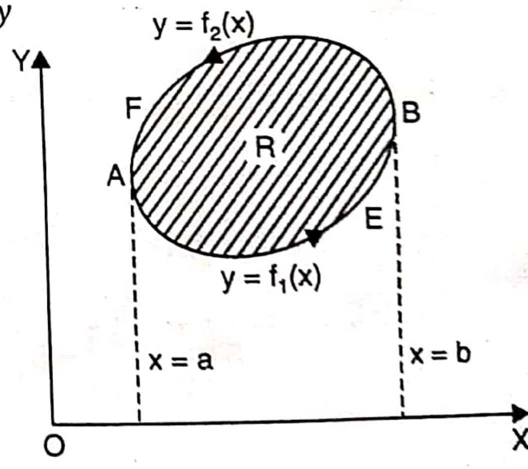
13.28. GREEN'S THEOREM IN THE PLANE

If $M(x, y)$ and $N(x, y)$ be continuous functions of x and y having continuous partial derivatives $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ in a region R of the xy-plane bounded by a closed curve C, then

$$\oint_C (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

where C is traversed in the counterclockwise direction.



Let us assume that the region R is such that any line parallel to either axes meets the boundary curve C in at most two points.

[The proof can be easily extended to other cases.]

Suppose the region R is bounded between the lines $x = a$, $x = b$ and two arcs AEB and BFA whose equations are $y = f_1(x)$ and $y = f_2(x)$ respectively such that $f_2(x) > f_1(x)$.

Now $\iint_R \frac{\partial M}{\partial y}\, dx\, dy = \int_a^b \left[\int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y}\, dy\right] dx$

$$= \int_a^b \left[M(x, y)\right]_{f_1(x)}^{f_2(x)} dx = \int_a^b [M(x, f_2) - M(x, f_1)]\, dx$$

$$= \int_a^b M(x, f_2)\, dx - \int_a^b M(x, f_1)\, dx = -\int_b^a M(x, f_2)\, dx - \int_a^b M(x, f_1)\, dx$$

$$= -\left[\int_a^b M(x, f_1)\, dx + \int_b^a M(x, f_2)\, dx\right] = -\oint_C M dx$$

or $$\oint_C M dx = -\iint_R \frac{\partial M}{\partial y}\, dx\, dy \quad \ldots(1)$$

Similarly, we can show that $\oint_C N dy = \iint_R \frac{\partial N}{\partial x}\, dx\, dy \quad \ldots(2)$

Adding (1) and (2), we have $\oint_C (M dx + N dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$

This theorem is useful for changing a line integral around a closed curve C into a double integral over the region R enclosed by C.

ILLUSTRATIVE EXAMPLES

Example 1. *Verify Green's theorem in the plane for* $\oint_C (3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy$ *where C is the boundary of the region defined by*

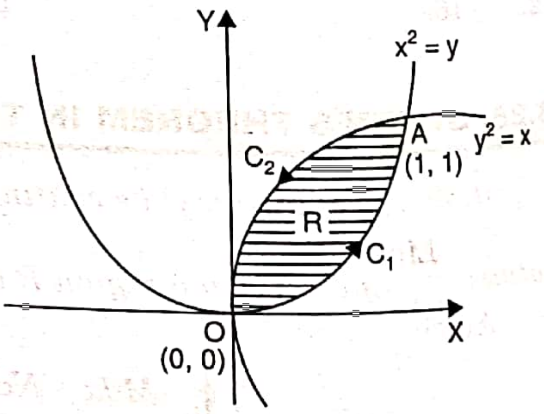
(a) $y = \sqrt{x}$, $y = x^2$ *(b)* $x = 0, y = 0, x + y = 1$.

Sol. *(a)* $y = \sqrt{x}$ *i.e.,* $y^2 = x$ and $y = x^2$ are two parabolas intersecting at O(0, 0) and A(1, 1).

Here $M = 3x^2 - 8y^2,\ N = 4y - 6xy$

$$\frac{\partial M}{\partial y} = -16y, \quad \frac{\partial N}{\partial x} = -6y$$

$$\therefore \quad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 10y$$



If R is the region bounded by C, then

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} 10y\, dy\, dx = \int_0^1 5\left[y^2\right]_{x^2}^{\sqrt{x}} dx$$

$$= 5\int_0^1 (x - x^4)\, dx = 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$$

$$= 5\left(\frac{1}{2} - \frac{1}{5}\right) = 5\left(\frac{3}{10}\right) = \frac{3}{2} \quad \ldots(1)$$

Also, $\oint_C (Mdx + Ndy) = \int_{C_1} (Mdx + Ndy) + \int_{C_2} (Mdx + Ndy)$

Along C_1, $x^2 = y$, $\therefore$ $2x\,dx = dy$ and the limits of x are from 0 to 1.

$\therefore$ Line integral along $C_1 = \int_{C_1} (Mdx + Ndy)$

$$= \int_0^1 (3x^2 - 8x^4)\,dx + (4x^2 - 6x\,.\,x^2)\,2x\,dx = \int_0^1 (3x^2 + 8x^3 - 20x^4)\,dx$$

$$= \left[x^3 + 2x^4 - 4x^5\right]_0^1 = -1$$

Along C_2, $y^2 = x$, $\therefore$ $2y\,dy = dx$ and the limits of y are from 1 to 0.

$\therefore$ Line integral along $C_2 = \int_{C_2} (Mdx + Ndy)$

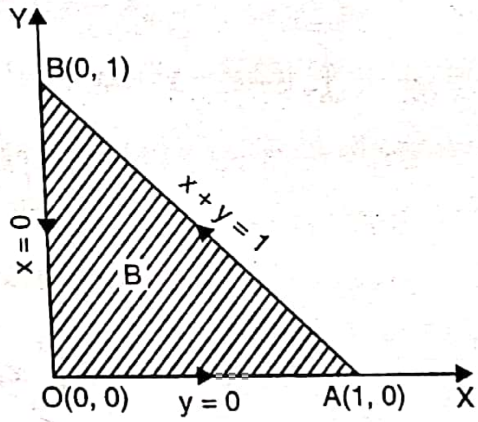
$$= \int_1^0 (3y^4 - 8y^2)\,2ydy + (4y - 6y^2\,.\,y)\,dy$$

$$= \int_1^0 (4y - 22y^3 + 6y^5)\,dy = \left[2y^2 - \frac{11}{2}y^4 + y^6\right]_1^0 = \frac{5}{2}$$

$\therefore$ Line integral along $C = -1 + \frac{5}{2} = \frac{3}{2}$ i.e., $\oint_C (Mdx + Ndy) = \frac{3}{2}$ ...(2)

The equality of (1) and (2) verifies Green's theorem in the plane.

(b) Here $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_0^1 \int_0^{1-x} 10y\,dy\,dx$



$$= \int_0^1 5\left[y^2\right]_0^{1-x} dx$$

$$= 5\int_0^1 (1-x)^2\,dx = 5\left[\frac{(1-x)^3}{-3}\right]_0^1$$

$$= -\frac{5}{3}(0-1) = \frac{5}{3} \qquad \text{...(1)}$$

Along OA, $y = 0$ $\therefore$ $dy = 0$ and the limits of x are from 0 to 1.

$\therefore$ Line integral along OA $= \int_0^1 3x^2\,dx = \left[x^3\right]_0^1$

Along AB, $y = 1 - x$ $\therefore$ $dy = -dx$ and the limits of x are from 1 to 0.

$\therefore$ Line integral along AB $= \int_1^0 [3x^2 - 8(1-x)^2]\,dx + [4(1-x) - 6x(1-x)](-dx)$

$$= \int_1^0 (3x^2 - 8 + 16x - 8x^2 - 4 + 4x + 6x - 6x^2)\,dx = \int_1^0 (-12 + 26x - 11x^2)\,dx$$

$$= \left[-12x + 13x^2 - \frac{11}{3}x^3\right]_1^0 = -\left[-12 + 13 - \frac{11}{3}\right] = \frac{8}{3}$$

Along BO, $x = 0$ $\therefore$ $dx = 0$ and the limits of y are from 1 to 0.

$\therefore$ Line integral along BO $= \int_1^0 4y\,dy = \left[2y^2\right]_1^0 = -2$

$\therefore$ Line integral along C (*i.e.*, along OABO) $= 1 + \frac{8}{3} - 2 = \frac{5}{3}$

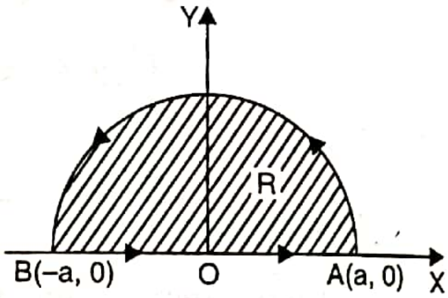
i.e., $$\oint_C (M\,dx + N\,dy) = \frac{5}{3} \qquad ...(2)$$

The equality of (1) and (2) verifies Green's theorem in the plane.

Example 2. *Use Green's theorem in a plane to evaluate the integral* $\oint_C [(2x^2 - y^2)\,dx + (x^2 + y^2)\,dy]$ *where C is the boundary in the xy-plane of the area enclosed by the x-axis and the semi circle* $x^2 + y^2 = a^2$ *in the upper half xy-plane.*

Sol. If R is the region bounded by the closed curve C, then by Green's theorem in the plane, we have

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$



Here $\quad M = 2x^2 - y^2,\ N = x^2 + y^2$

$$\frac{\partial M}{\partial y} = -2y, \frac{\partial N}{\partial x} = 2x$$

$$\therefore \quad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2(x + y)$$

The region R is bounded by

$$x = -a, x = a, y = 0, y = \sqrt{a^2 - x^2}$$

$$\therefore \quad \oint_C [(2x^2 - y^2)\,dx + (x^2 + y^2)\,dy] = \iint_R 2(x + y)\,dx\,dy$$

$$= \int_{-a}^{a} \int_0^{\sqrt{a^2 - x^2}} 2(x + y)\,dy\,dx = \int_{-a}^{a} \left[2xy + y^2\right]_0^{\sqrt{a^2 - x^2}} dx$$

$$= \int_{-a}^{a} [2x\sqrt{a^2 - x^2} + (a^2 - x^2)]\,dx$$

$$= 2\int_{-a}^{a} x\sqrt{a^2 - x^2}\,dx + \int_{-a}^{a} (a^2 - x^2)\,dx$$

$$= 2\,(0) + 2\int_0^{a} (a^2 - x^2)\,dx$$

[$\because$ $x\sqrt{a^2 - x^2}$ is an odd function and $(a^2 - x^2)$ is an even function]

$$= 2\left[a^2 x - \frac{x^3}{3}\right]_0^a = 2\left(a^3 - \frac{a^3}{3}\right) = \frac{4a^3}{3}.$$

EXERCISE 13.6

1. Verify Green's theorem in the plane for $\oint_C (xy + y^2)\,dx + x^2\,dy$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

2. Verify Green's theorem in the plane for $\oint_C (2xy - x^2)\,dx + (x^2 + y^2)\,dy$ where C is the boundary of the region enclosed by $y = x^2$ and $y^2 = x$.