Thus, $\quad \text{grad}\ \phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$

The gradient of scalar field ϕ is obtained by operating on ϕ by the vector operator

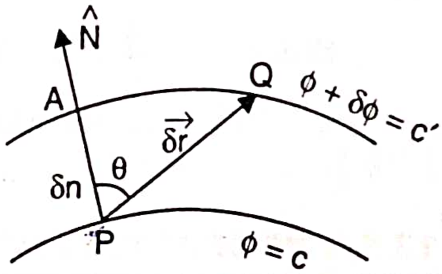
$$\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

This operator is denoted by the symbol ∇, read as **del** (also called nabla).

Thus, $\quad \text{grad}\ \phi = \nabla\phi$.

13.12. GEOMETRICAL INTERPRETATION OF GRADIENT *(K.U.K., 2009)*

If a surface $\phi(x, y, z) = c$ is drawn through any point P such that at each point on the surface, the function has the same value as at P, then such a surface is called a *level surface* through P. For example, if $\phi(x, y, z)$ represents potential at the point (x, y, z), the **equipotential surface** $\phi(x, y, z) = c$ is a level surface.



Through any point passes one and only one level surface. Moreover, no two level surfaces can intersect.

Consider the level surface through P at which the function has value ϕ and another level surface through a neighbouring point Q where the value is $\phi + \delta\phi$.

Let $\vec{r}$ and $\vec{r} + \delta\vec{r}$ be the position vectors of P and Q respectively, then $\overrightarrow{PQ} = \delta\vec{r}$.

Now $\quad \nabla\phi . \delta\vec{r} = \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right).(\hat{i}\,\delta x + \hat{j}\,\delta y + \hat{k}\,\delta z)$

$$= \frac{\partial \phi}{\partial x}\delta x + \frac{\partial \phi}{\partial y}\delta y + \frac{\partial \phi}{\partial z}\delta z = \delta\phi \quad \text{...(1)}$$

If Q lies on the same level surface as P, then $\delta\phi = 0$,

$\therefore$ (1) reduces to $\nabla\phi . \delta\vec{r} = 0$.

Thus, $\nabla\phi$ is perpendicular to every $\delta\vec{r}$ lying in the surface.

Hence $\nabla\phi$ **is normal to the surface** $\phi(x, y, z) = c$.

Let $\nabla\phi = |\nabla\phi|\,\hat{N}$, where $\hat{N}$ is a unit vector normal to the surface. Let PA $= \delta n$ be the perpendicular distance between the two level surfaces through P and Q. Then the rate of change of ϕ in the direction of normal to the surface through P is

$$\frac{\partial \phi}{\partial n} = \lim_{\delta n \to 0}\frac{\partial \phi}{\partial n} = \lim_{\partial n \to 0}\frac{\nabla\phi . \delta\vec{r}}{\delta n} \quad \text{[by (1)]}$$

$$= \lim_{\delta n \to 0}\frac{|\nabla\phi|\hat{N} . \delta\vec{r}}{\delta n} = |\nabla\phi| \quad (\because \ \hat{N} . \delta\vec{r} = |\hat{N}||\delta\vec{r}|\cos\theta = |\delta\vec{r}|\cos\theta = \delta n)$$

$\therefore \quad |\nabla\phi| = \frac{\partial \phi}{\partial n}$.

Hence the gradient of a scalar field ϕ is a vector normal to the surface $\phi = c$ and has a magnitude equal to the rate of change of ϕ along this normal.

13.13. DIRECTIONAL DERIVATIVE

Let $PQ = \delta r$, then $\lim\limits_{\delta r \to 0} \dfrac{\delta \phi}{\delta r} = \dfrac{\partial \phi}{\partial r}$ is called the directional derivative of ϕ at P in the direction PQ.

Let $\hat{N}'$ be a unit vector in the direction PQ, then $\delta r = \dfrac{\delta n}{\cos\theta} = \dfrac{\delta n}{\hat{N}\,.\,\hat{N}'}$

$$\therefore \qquad \frac{\partial \phi}{\partial r} = \lim_{\delta r \to 0}\left[\hat{N}\,.\,\hat{N}'\frac{\delta \phi}{\delta n}\right] = \hat{N}\,.\,\hat{N}'\frac{\partial \phi}{\partial n}$$

$$= \hat{N}'\,.\,\hat{N}\frac{\partial \phi}{\partial n} = \hat{N}'\,.\,\hat{N}\,|\nabla\phi| = \hat{N}'\,.\,\nabla\phi \qquad \left(\because\ |\nabla\phi| = \frac{\partial \phi}{\partial n} \text{ and } \hat{N}\,|\nabla\phi| = \nabla\phi\right)$$

Thus, the directional derivative $\dfrac{\partial \phi}{\partial r}$ *is the resolved part of* $\nabla\phi$ *in the direction* $\hat{N}'$.

Since $\qquad \dfrac{\partial \phi}{\partial r} = \hat{N}'\,.\,\nabla\phi = |\nabla\phi|\cos\theta \le |\nabla\phi|$.

$\therefore$ $\nabla\phi$ *gives the maximum rate of change of* ϕ *and the magnitude of this maximum is* $|\nabla\phi|$.

13.14. PROPERTIES OF GRADIENT

(*a*) If ϕ is a constant scalar point function, then $\nabla\phi = \vec{0}$

(*b*) If ϕ_1 and ϕ_2 are two scalar point functions, then

(*i*) $\nabla(\phi_1 \pm \phi_2) = \nabla\phi_1 \pm \nabla\phi_2$

(*ii*) $\nabla(c_1\phi_1 + c_2\phi_2) = c_1\nabla\phi_1 + c_2\nabla\phi_2$, where c_1, c_2 are constant

(*iii*) $\nabla(\phi_1\phi_2) = \phi_1\nabla\phi_2 + \phi_2\nabla\phi_1$

(*iv*) $\nabla\left(\dfrac{\phi_1}{\phi_2}\right) = \dfrac{\phi_2\nabla\phi_1 - \phi_1\nabla\phi_2}{\phi_2^{\,2}}$, $\phi_2 \ne 0$.

All the above results can be easily proved. For example

$$(iii)\quad \nabla(\phi_1\phi_2) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(\phi_1\phi_2) = \hat{i}\frac{\partial}{\partial x}(\phi_1\phi_2) + \hat{j}\frac{\partial}{\partial y}(\phi_1\phi_2) + \hat{k}\frac{\partial}{\partial z}(\phi_1\phi_2)$$

$$= \hat{i}\left(\phi_1\frac{\partial \phi_2}{\partial x} + \phi_2\frac{\partial \phi_1}{\partial x}\right) + \hat{j}\left(\phi_1\frac{\partial \phi_2}{\partial y} + \phi_2\frac{\partial \phi_1}{\partial y}\right) + \hat{k}\left(\phi_1\frac{\partial \phi_2}{\partial z} + \phi_2\frac{\partial \phi_1}{\partial z}\right)$$

$$= \phi_1\left(\hat{i}\frac{\partial \phi_2}{\partial x} + \hat{j}\frac{\partial \phi_2}{\partial y} + \hat{k}\frac{\partial \phi_2}{\partial z}\right) + \phi_2\left(\hat{i}\frac{\partial \phi_1}{\partial x} + \hat{j}\frac{\partial \phi_1}{\partial y} + \hat{k}\frac{\partial \phi_1}{\partial z}\right)$$

$$= \phi_1\nabla\phi_2 + \phi_2\nabla\phi_1.$$

$$(iv)\quad \nabla\left(\frac{\phi_1}{\phi_2}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{\phi_1}{\phi_2}\right) = \hat{i}\frac{\partial}{\partial x}\left(\frac{\phi_1}{\phi_2}\right) + \hat{j}\frac{\partial}{\partial y}\left(\frac{\phi_1}{\phi_2}\right) + \hat{k}\frac{\partial}{\partial z}\left(\frac{\phi_1}{\phi_2}\right)$$

$$= \hat{i}\,\frac{\phi_2\dfrac{\partial \phi_1}{\partial x} - \phi_1\dfrac{\partial \phi_2}{\partial x}}{\phi_2^2} + \hat{j}\,\frac{\phi_2\dfrac{\partial \phi_1}{\partial y} - \phi_1\dfrac{\partial \phi_2}{\partial y}}{\phi_2^2} + \hat{k}\,\frac{\phi_2\dfrac{\partial \phi_1}{\partial z} - \phi_1\dfrac{\partial \phi_2}{\partial z}}{\phi_2^2}$$

$$= \frac{1}{\phi_2^2}\left[\phi_2\left(\hat{i}\frac{\partial\phi_1}{\partial x}+\hat{j}\frac{\partial\phi_1}{\partial y}+\hat{k}\frac{\partial\phi_1}{\partial z}\right)-\phi_1\left(\hat{i}\frac{\partial\phi_2}{\partial x}+\hat{j}\frac{\partial\phi_2}{\partial y}+\hat{k}\frac{\partial\phi_2}{\partial z}\right)\right]$$

$$= \frac{\phi_2\nabla\phi_1-\phi_1\nabla\phi_2}{\phi_2^2}.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Find grad ϕ when ϕ is given by $\phi = 3x^2y - y^3z^2$ at the point (1, – 2, – 1).*

Sol. Grad $\phi = \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(3x^2y-y^3z^2)$ (U.P.T.U. 2007)

$$= \hat{i}\frac{\partial}{\partial x}(3x^2y-y^3z^2)+\hat{j}\frac{\partial}{\partial y}(3x^2y-y^3z^2)+\hat{k}\frac{\partial}{\partial z}(3x^2y-y^3z^2)$$

$$= \hat{i}\,(6xy)+\hat{j}(3x^2-3y^2z^2)+\hat{k}(-2y^3z)$$

$$= -12\hat{i}-9\hat{j}-16\hat{k} \text{ at the point } (1,-2,-1).$$

Example 2. *If $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$, show that*

(*i*) *grad* $r = \frac{\vec{r}}{r}$ (*ii*) *grad* $\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$ (*iii*) $\nabla r^n = nr^{n-2}\,\vec{r}$

(*iv*) $\nabla(\vec{a}\,.\,\vec{r}) = \vec{a}$, *where $\vec{a}$ is a constant vector.* (U.P.T.U. 2008)

Sol. $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$, or $r^2 = x^2+y^2+z^2$

Differentiating partially w.r.t. x, we have $2r\frac{\partial r}{\partial x} = 2x$ or $\frac{\partial r}{\partial x} = \frac{x}{r}$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

(*i*) grad $r = \nabla r = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)r = \hat{i}\frac{\partial r}{\partial x}+\hat{j}\frac{\partial r}{\partial y}+\hat{k}\frac{\partial r}{\partial z}$

$$= \hat{i}\left(\frac{x}{r}\right)+\hat{j}\left(\frac{y}{r}\right)+\hat{k}\left(\frac{z}{r}\right) = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{r} = \frac{\vec{r}}{r}.$$

(*ii*) grad $\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{r}\right) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)$

$$= \hat{i}\left(-\frac{1}{r^2}\cdot\frac{\partial r}{\partial x}\right)+\hat{j}\left(-\frac{1}{r^2}\cdot\frac{\partial r}{\partial y}\right)+\hat{k}\left(-\frac{1}{r^2}\cdot\frac{\partial r}{\partial z}\right)$$

$$= \hat{i}\left(-\frac{1}{r^2}\cdot\frac{x}{r}\right)+\hat{j}\left(-\frac{1}{r^2}\cdot\frac{y}{r}\right)+\hat{k}\left(-\frac{1}{r^2}\cdot\frac{z}{r}\right)$$

$$= -\frac{1}{r^3}(x\hat{i}+y\hat{j}+z\hat{k}) = -\frac{\vec{r}}{r^3}.$$

(iii) $\nabla r^n = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) r^n = \hat{i}\left(nr^{n-1}\frac{\partial r}{\partial x}\right) + \hat{j}\left(nr^{n-1}\frac{\partial r}{\partial y}\right) + \hat{k}\left(nr^{n-1}\frac{\partial r}{\partial z}\right)$

$= \hat{i}\left(nr^{n-1}\cdot\frac{x}{r}\right) + \hat{j}\left(nr^{n-1}\cdot\frac{y}{r}\right) + \hat{k}\left(nr^{n-1}\cdot\frac{z}{r}\right) = nr^{n-2}\,(x\hat{i} + y\hat{j} + z\hat{k}) = nr^{n-2}\,\vec{r}.$

(iv) Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, where a_1, a_2, a_3 are constants.

$\vec{a}\,.\,\vec{r} = a_1x + a_2y + a_3z$

$\therefore \quad \nabla(\vec{a}\,.\,\vec{r}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(a_1x + a_2y + a_3z)$

$= \hat{i}\frac{\partial}{\partial x}(a_1x + a_2y + a_3z) + \hat{j}\frac{\partial}{\partial y}(a_1x + a_2y + a_3z) + \hat{k}\frac{\partial}{\partial z}(a_1x + a_2y + a_3z)$

$= a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = \vec{a}.$

Example 3. *Find a unit vector normal to the surface $x^3 + y^3 + 3xyz = 3$ at the point (1, 2, – 1).*

Sol. Let $\quad \phi = x^3 + y^3 + 3xyz = 3$, then $\frac{\partial\phi}{\partial x} = 3x^2 + 3yz, \frac{\partial\phi}{\partial y} = 3y^2 + 3xz, \frac{\partial\phi}{\partial z} = 3xy$

$\therefore \quad \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = (3x^2 + 3yz)\hat{i} + (3y^2 + 3xz)\hat{j} + (3xy)\hat{k}$

At (1, 2, – 1), $\nabla\phi = -3\hat{i} + 9\hat{j} + 6\hat{k}$

which is a vector normal to the given surface at (1, 2, – 1).

Hence a unit vector normal to the given surface at (1, 2, – 1)

$$= \frac{-3\hat{i} + 9\hat{j} + 6\hat{k}}{\sqrt{[(-3)^2 + (9)^2 + (6)^2]}} = \frac{-3\hat{i} + 9\hat{j} + 6\hat{k}}{3\sqrt{14}} = \frac{1}{\sqrt{14}}(-\hat{i} + 3\hat{j} + 2\hat{k}).$$

Example 4. *Find the directional derivative of the function $f = x^2 - y^2 + 2z^2$ at the point P(1, 2, 3) in the direction of the line PQ where Q is the point (5, 0, 4).*

In what direction it will be maximum ? Find also the magnitude of this maximum.

Sol. We have $\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z} = 2x\hat{i} - 2y\hat{j} + 4z\hat{k} = 2\hat{i} - 4\hat{j} + 12\hat{k}$ at P(1, 2, 3)

Also $\quad \vec{PQ} = \vec{OQ} - \vec{OP} = (5\hat{i} + 4\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = 4\hat{i} - 2\hat{j} + \hat{k}$

If $\hat{n}$ is a unit vector in the direction $\vec{PQ}$, then $\hat{n} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{16 + 4 + 1}} = \frac{1}{\sqrt{21}}(4\hat{i} - 2\hat{j} + \hat{k})$

$\therefore$ Directional derivative of f in the direction $\vec{PQ} = (\nabla f)\,.\,\hat{n}$

$$= (2\hat{i} - 4\hat{j} + 12\hat{k})\,.\,\frac{1}{\sqrt{21}}(4\hat{i} - 2\hat{j} + \hat{k}) = \frac{1}{\sqrt{21}}[2(4) - 4(-2) + 12(1)]$$

$$= \frac{28}{\sqrt{21}} = \frac{4}{3}\sqrt{21}$$

The directional derivative of f is maximum in the direction of the normal to the given surface *i.e.*, in the direction of $\nabla f = 2\hat{i} - 4\hat{j} + 12\hat{k}$

The maximum value of this directional derivative = $|\nabla f|$

$$= \sqrt{(2)^2 + (-4)^2 + (12)^2} = \sqrt{164} = 2\sqrt{41}.$$

Example 5. *Find the directional derivative of* $\phi = 5x^2y - 5y^2z + \frac{5}{2}z^2x$ *at the point* $P(1, 1, 1)$ *in the direction of the line* $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$.

Sol. Here,
$$\phi = 5x^2y - 5y^2z + \frac{5}{2}z^2x$$

$$\therefore \quad \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \left(10xy + \frac{5}{2}z^2\right)\hat{i} + (5x^2 - 10yz)\hat{j} + (-5y^2 + 5zx)\hat{k}$$

$$= \frac{25}{2}\hat{i} - 5\hat{j} \qquad \text{at P(1, 1, 1)}$$

The direction of the given line is $\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$

$$\Rightarrow \quad \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{2\hat{i} - 2\hat{j} + \hat{k}}{3}$$

$\therefore$ The required directional derivative

$$= (\nabla\phi).\hat{a} = \left(\frac{25}{2}\hat{i} - 5\hat{j}\right).\left(\frac{2\hat{i} - 2\hat{j} + \hat{k}}{3}\right)$$

$$= \left(\frac{25}{2}\right)\left(\frac{2}{3}\right) + (-5)\left(-\frac{2}{3}\right) + (0)\left(\frac{1}{3}\right) = \frac{35}{3}.$$

Example 6. *If the directional derivative of* $\phi = ax^2y + by^2z + cz^2x$ *at the point* $(1, 1, 1)$ *has maximum magnitude 15 in the direction parallel to the line* $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$, *find the values of* a, b *and* c. (M.D.U. Dec. 2010)

Sol. Here,
$$\phi = ax^2y + by^2z + cz^2x$$

$$\therefore \quad \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= (2axy + cz^2)\hat{i} + (ax^2 + 2byz)\hat{j} + (by^2 + 2czx)\hat{k}$$

$$= (2a + c)\hat{i} + (a + 2b)\hat{j} + (b + 2c)\hat{k} \text{ at } (1, 1, 1)$$

Now, the directional derivative of ϕ is maximum in the direction of the normal to the given surface *i.e.*, in the direction of $\nabla\phi$.

But we are given that the directional derivative of ϕ is maximum in the direction parallel to the line.

$$\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1} \quad \text{i.e., parallel to the vector } 2\hat{i} - 2\hat{j} + \hat{k}.$$

$$\therefore \quad \frac{2a+c}{2} = \frac{a+2b}{-2} = \frac{b+2c}{1}$$

[Two vectors are parallel if the corresponding scalar components are proportional].

$$\Rightarrow \quad \frac{2a+c}{2} = \frac{a+2b}{-2} \quad \text{and} \quad \frac{a+2b}{-2} = \frac{b+2c}{1}$$

$$\Rightarrow \quad 2a + c = -a - 2b \quad \text{and} \quad a + 2b = -2b - 4c$$

$$\Rightarrow \quad 3a + 2b + c = 0 \quad \text{and} \quad a + 4b + 4c = 0$$

By cross-multiplication, we have

$$\frac{a}{8-4} = \frac{b}{1-12} = \frac{c}{12-2}$$

or

$$\frac{a}{4} = \frac{b}{-11} = \frac{c}{10} = \lambda \quad \text{(say)}$$

$$\Rightarrow \quad a = 4\lambda, b = -11\lambda, c = 10\lambda$$

The maximum value of directional derivative of ϕ

$$= |\nabla\phi| = \sqrt{(2a+c)^2 + (a+2b)^2 + (b+2c)^2}$$

Since it is given to be 15, we have

$$\sqrt{(2a+c)^2 + (a+2b)^2 + (b+2c)^2} = 15$$

$$\Rightarrow \quad (8\lambda + 10\lambda)^2 + (4\lambda - 22\lambda)^2 + (-11\lambda + 20\lambda)^2 = 225$$

$$\Rightarrow \quad (324 + 324 + 81)\lambda^2 = 225 \Rightarrow \lambda^2 = \frac{225}{729} = \frac{25}{81}$$

$$\Rightarrow \quad \lambda = \pm 5/9$$

$$\therefore \quad a = \pm\frac{20}{9}, b = \mp\frac{55}{9}, c = \pm\frac{50}{9}$$

Example 7. *Find the values of constants a, b and c so that the maximum value of the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at (1, 2, – 1) has a magnitude 64 in the direction parallel to z-axis.*

(Rajasthan 2006)

Sol. Here,
$$\phi = axy^2 + byz + cz^2x^3$$

$$\therefore \quad \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= (ay^2 + 3cz^2x^2)\,\hat{i} + (2axy + bz)\,\hat{j} + (by + 2czx^3)\,\hat{k}$$

$$= (4a + 3c)\,\hat{i} + (4a - b)\,\hat{j} + (2b - 2c)\,\hat{k} \text{ at } (1, 2, -1)$$

Now, the directional derivative of ϕ is maximum in the direction of the normal to the given surface *i.e.*, in the direction of $\nabla\phi$. But we are given that the directional derivative of ϕ is maximum in the direction parallel to z-axis *i.e.*, parallel to $\hat{k}$.

Hence co-efficients of $\hat{i}$ and $\hat{j}$ in $\nabla\phi$ should be zero and the co-efficient of $\hat{k}$ positive.

Thus, $4a + 3c = 0$ ...(1)

$4a - b = 0$ ...(2)

and $2b - 2c > 0$ *i.e.,* $b > c$ ...(3)

Then, $\nabla\phi = 2(b - c)\,\hat{k}$

Also maximum value of directional derivative = $|\nabla\phi|$

$\therefore$ $|2(b-c)\,\hat{k}| = 64$ (given)

$\Rightarrow$ $2(b - c) = 64$ or $b - c = 32$ ...(4)

Solving (1), (2) and (4), we have

$$a = 6,\ b = 24,\ c = -8.$$

Example 8. *Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point (2, – 1, 2).* (Kottayam 2005)

Sol. Angle between two surfaces at a point is the angle between the normals to the surfaces at that point.

Let $\phi_1 = x^2 + y^2 + z^2 = 9$ and $\phi_2 = x^2 + y^2 - z = 3$

Then grad $\phi_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$ and grad $\phi_2 = 2x\hat{i} + 2y\hat{j} - \hat{k}$

Let $\vec{n_1}$ = grad ϕ_1 at the point (2, – 1, 2) and $\vec{n_2}$ = grad ϕ_2 at the point (2, – 1, 2). Then

$$\vec{n_1} = 4\hat{i} - 2\hat{j} + 4\hat{k} \quad \text{and} \quad \vec{n_2} = 4\hat{i} - 2\hat{j} - \hat{k}$$

The vectors $\vec{n_1}$ and $\vec{n_2}$ are along normals to the two surfaces at the point (2, – 1, 2). If θ is the angle between these vectors, then

$$\cos\theta = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|} = \frac{4(4) - 2(-2) + 4(-1)}{\sqrt{16+4+16}\,.\sqrt{16+4+1}} = \frac{16}{6\sqrt{21}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right).$$

EXERCISE 13.2

1. Find grad ϕ when ϕ is given by

(*i*) $\phi = x^2 + yz$ (*ii*) $\phi = x^3 + y^3 + 3xyz$ (*iii*) $\phi = \log(x^2 + y^2 + z^2)$.

2. If $r = |\vec{r}|$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that

(*i*) $\nabla f(r) = f'(r)\,\nabla r$ (*ii*) $\nabla \log r = \dfrac{\vec{r}}{r^2}$

(*iii*) $\nabla(e^{r^2}) = 2e^{r^2}\,\vec{r}$ (*iv*) grad $|\vec{r}|^2 = 2\vec{r}$

(*v*) grad$\left(\dfrac{1}{r^2}\right) = -\dfrac{2\vec{r}}{r^4}$

9. (*i*) $a = c = 2, b = -2$ (*ii*) 1 10. $\cos^{-1}\left(\frac{1}{\sqrt{22}}\right)$ 12. $a = 2.5, b = 1$

13. $\cos^{-1}\left(\frac{1}{\sqrt{30}}\right)$.

13.15. DIVERGENCE OF A VECTOR POINT FUNCTION

The divergence of a differentiable vector point function $\vec{V}$ is denoted by div $\vec{V}$ and is defined as

$$\text{div } \vec{V} = \nabla \cdot \vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \vec{V} = \hat{i} \cdot \frac{\partial \vec{V}}{\partial x} + \hat{j} \cdot \frac{\partial \vec{V}}{\partial y} + \hat{k} \cdot \frac{\partial \vec{V}}{\partial z}.$$

Obviously, the divergence of a vector point function is a scalar point function.

If $\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$, then

$$\text{div } \vec{V} = \nabla \cdot \vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot [V_1\hat{i} + V_2\hat{j} + V_3\hat{k}] = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}.$$

since $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$ and $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$.

13.16. CURL OF A VECTOR POINT FUNCTION

The curl (or rotation) of a differentiable vector point function $\vec{V}$ is denoted by curl $\vec{V}$ and is defined as

$$\text{curl } \vec{V} = \nabla \times \vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \vec{V} = \hat{i} \times \frac{\partial \vec{V}}{\partial x} + \hat{j} \times \frac{\partial \vec{V}}{\partial y} + \hat{k} \times \frac{\partial \vec{V}}{\partial z}.$$

Obviously, the curl of a vector point function is a vector point function.

If $\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$

then curl $\vec{V} = \nabla \times \vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times [V_1\hat{i} + V_2\hat{j} + V_3\hat{k}]$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \hat{j}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right).$$

ILLUSTRATIVE EXAMPLES

Example 1. *If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, show that*

(*i*) *div $\vec{r} = 3$* (*ii*) *curl $\vec{r} = \vec{0}$.* (P.T.U. 2006; U.P.T.U. 2006)

Sol. (*i*) div $\vec{r} = \nabla \cdot \vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3.$

(*ii*) $\text{curl } \vec{r} = \nabla \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$

$$= \hat{i}\left[\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right] + \hat{j}\left[\frac{\partial}{\partial z}(x) - \frac{\partial}{\partial x}(z)\right] + \hat{k}\left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right]$$

$$= \hat{i}(0) + \hat{j}(0) + \hat{k}(0) = \vec{0}.$$

Example 2. *Find the divergence and curl of the vector* $\vec{V} = (xyz)\hat{i} + (3x^2y)\hat{j} + (xz^2 - y^2z)\hat{k}$ *at the point (2, – 1, 1).*

Sol. $\text{div } \vec{V} = \dfrac{\partial}{\partial x}(xyz) + \dfrac{\partial}{\partial y}(3x^2y) + \dfrac{\partial}{\partial z}(xz^2 - y^2z)$

$$= yz + 3x^2 + 2xz - y^2 = -1 + 12 + 4 - 1 = 14 \text{ at } (2, -1, 1)$$

$$\text{curl } \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix} = \hat{i}(-2yz - 0) + \hat{j}(xy - z^2) + \hat{k}(6xy - xz)$$

$$= 2\hat{i} - 3\hat{j} - 14\hat{k} \text{ at } (2, -1, 1).$$

Example 3. *Find div* $\vec{F}$ *and curl* $\vec{F}$ *where* $\vec{F} = \text{grad}\,(x^3 + y^3 + z^3 - 3xyz)$. (K.U.K. 2006)

Sol. Let $\phi = x^3 + y^3 + z^3 - 3xyz$, then

$$\vec{F} = \text{grad } \phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^3 + y^3 + z^3 - 3xyz)$$

$$= (3x^2 - 3yz)\hat{i} + (3y^2 - 3zx)\hat{j} + (3z^2 - 3xy)\hat{k}$$

$\therefore$ $\text{div } \vec{F} = \dfrac{\partial}{\partial x}(3x^2 - 3yz) + \dfrac{\partial}{\partial y}(3y^2 - 3zx) + \dfrac{\partial}{\partial x}(3z^2 - 3xy)$

$$= 6x + 6y + 6z = 6(x + y + z)$$

and $\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3zx & 3z^2 - 3xy \end{vmatrix}$

$$= \hat{i}(-3x + 3x) + \hat{j}(-3y + 3y) + \hat{k}(-3z + 3z) = \vec{0}.$$

Example 4. *Find curl (curl* $\vec{V}$*) where* $\vec{V} = (2xz^2)\hat{i} - yz\hat{j} + 3xz^3\hat{k}$*, at (1, 1, 1).* (P.T.U., 2006)

Sol. Here, $\vec{V} = (2xz^2)\hat{i} - yz\hat{j} + 3xz^3\hat{k}$

$\therefore$ $\text{curl } \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xz^2 & -yz & 3xz^3 \end{vmatrix}$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(3xz^3) - \frac{\partial}{\partial z}(-yz)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(3xz^3) - \frac{\partial}{\partial z}(2xz^2)\right\}$$
$$+ \hat{k}\left\{\frac{\partial}{\partial x}(-yz) - \frac{\partial}{\partial y}(2xz^2)\right\}$$
$$= \hat{i}(0+y) - \hat{j}(3z^3 - 4xz) + \hat{k}(0-0) = y\hat{i} + (4xz - 3z^3)\hat{j}$$

$$\text{curl}(\text{curl}\,\vec{V}) = \text{curl}\,\{y\hat{i} + (4xz - 3z^3)\hat{j}\}$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 4xz - 3z^3 & 0 \end{vmatrix}$$
$$= \hat{i}\left\{\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(4xz - 3z^3)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(y)\right\}$$
$$+ \hat{k}\left\{\frac{\partial}{\partial x}(4xz - 3z^3) - \frac{\partial}{\partial y}(y)\right\}$$
$$= \hat{i}\left(0 - (4x - 9z^2)\right) - \hat{j}(0-0) + \hat{k}(4z - 1)$$
$$= (9z^2 - 4x)\hat{i} + (4z-1)\hat{k} = 5\hat{i} + 3\hat{k} \text{ at } (1, 1, 1).$$

Example 5. *Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{a}$ be a constant vector, find the value of div* $\left(\dfrac{\vec{a} \times \vec{r}}{r^n}\right)$.

Sol. $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \Rightarrow r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$.

$$\vec{a} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = (a_2z - a_3y)\,\hat{i} + (a_3x - a_1z)\,\hat{j} + (a_1y - a_2x)\,\hat{k}$$

$$\frac{\vec{a} \times \vec{r}}{r^n} = \frac{(a_2z - a_3y)\hat{i} + (a_3x - a_1z)\hat{j} + (a_1y - a_2x)\hat{k}}{(x^2 + y^2 + z^2)^{n/2}}$$

$$\therefore \text{ div}\left(\frac{\vec{a} \times \vec{r}}{r^n}\right) = \nabla \cdot \frac{\vec{a} \times \vec{r}}{r^n}$$
$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \frac{(a_2z - a_3y)\hat{i} + (a_3x - a_1z)\hat{j} + (a_1y - a_2x)\hat{k}}{(x^2 + y^2 + z^2)^{n/2}}$$
$$= \frac{\partial}{\partial x}\left\{\frac{a_2z - a_3y}{(x^2 + y^2 + z^2)^{n/2}}\right\} + \frac{\partial}{\partial y}\left\{\frac{a_3x - a_1z}{(x^2 + y^2 + z^2)^{n/2}}\right\} + \frac{\partial}{\partial z}\left\{\frac{a_1y - a_2x}{(x^2 + y^2 + z^2)^{n/2}}\right\}$$
$$= (a_2z - a_3y) \cdot \left(-\frac{n}{2}\right)(x^2 + y^2 + z^2)^{-\frac{n}{2} - 1} \cdot 2x$$
$$+ (a_3x - a_1z)\left(-\frac{n}{2}\right)(x^2 + y^2 + z^2)^{-\frac{n}{2} - 1} \cdot 2y + (a_1y - a_2x)\left(-\frac{n}{2}\right)(x^2 + y^2 + z^2)^{-\frac{n}{2} - 1} \cdot 2z$$

$$= \frac{-n}{(x^2+y^2+z^2)^{\frac{n}{2}+1}} [(a_2z - a_3y)\,x + (a_3x - a_1z)\,y + (a_1y - a_2x)\,z]$$

$$= \frac{-n}{(x^2+y^2+z^2)^{\frac{n}{2}+1}} [0] = 0$$

Hence, $\text{div}\left(\frac{\vec{a} \times \vec{r}}{r^n}\right) = 0.$

Example 6. *Find the directional derivative of div $(\vec{u})$ at the point (1, 2, 2) in the direction of the outer normal of the sphere $x^2 + y^2 + z^2 = 9$ for $\vec{u} = x^4\hat{i} + y^4\hat{j} + z^4\hat{k}$.*

Sol. Here, $\vec{u} = x^4\hat{i} + y^4\hat{j} + z^4\hat{k}$

$$\therefore \quad \text{div}(\vec{u}) = \nabla \cdot \vec{u} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (x^4\hat{i} + y^4\hat{j} + z^4\hat{k})$$

$$= \frac{\partial}{\partial x}(x^4) + \frac{\partial}{\partial y}(y^4) + \frac{\partial}{\partial z}(z^4)$$

$$= 4\,(x^3 + y^3 + z^3)$$

Directional derivative of div $\vec{u} = \nabla\,(4x^3 + 4y^3 + 4z^3)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (4x^3 + 4y^3 + 4z^3)$$

$$= 12\,(x^2\hat{i} + y^2\hat{j} + z^2\hat{k})$$

$$= 12\,(\hat{i} + 4\hat{j} + 4\hat{k}) \text{ at } (1, 2, 2)$$

Outer normal to the sphere $= \nabla\,(x^2 + y^2 + z^2 - 9)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2 - 9)$$

$$= \hat{i}\,(2x) + \hat{j}\,(2y) + \hat{k}\,(2z)$$

$$= 2(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= 2(\hat{i} + 2\hat{j} + 2\hat{k}) \text{ at } (1, 2, 2)$$

$$= 2\hat{i} + 4\hat{j} + 4\hat{k}$$

Unit outer normal to the sphere at (1, 2, 2) is

$$\hat{n} = \frac{2\hat{i} + 4\hat{j} + 4\hat{k}}{\sqrt{4 + 16 + 16}} = \frac{2\hat{i} + 4\hat{j} + 4\hat{k}}{6}$$

$\therefore$ Directional derivative of div $\vec{u}$ at (1, 2, 2) in the direction of outer normal

$$= 12\,(\hat{i} + 4\hat{j} + 4\hat{k}) \cdot \frac{2\hat{i} + 4\hat{j} + 4\hat{k}}{6} = 2\,(2 + 16 + 16) = 68$$