Let $y = f(x)$ be any function of x. On its graph take two points $P(a, f(a))$ and $Q\ (a + h, f\ (a + h))$ close to each other. Draw PL and QM perpendiculars to x-axis and $PN \perp QM$.

Now, $\quad OL = a, \quad OM = a + h$

and, $\quad PL = f(a), \ QM = f(a + h)$

Join PQ.

We know that

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

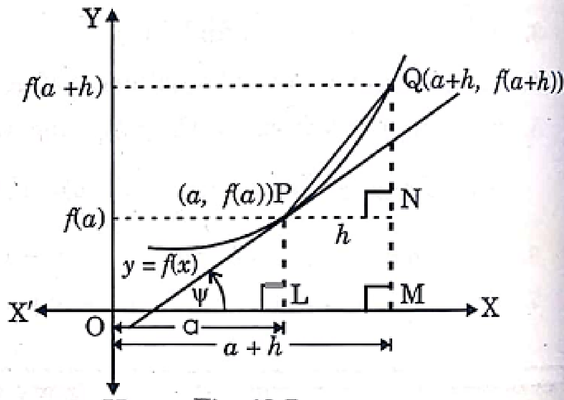

Fig. 13.7

Clearly, from Δ PQN, the ratio whose limit we are taking is equal to tan (QPN), which in fact is the slope of the chord PQ. In the limiting process as $h \to 0$, the point Q tends to P along the curve.

$$\therefore \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{QM - PL}{OM - OL}$$

$$= \lim_{h \to 0} \frac{QM - NM}{LM} = \lim_{h \to 0} \frac{QN}{PN}$$

$\therefore$ The chord PQ takes the form of tangent at P of the curve $y = f(x)$. The angle QPN takes the position of angle ψ, which the tangent at P makes with the positive direction of x-axis.

Hence, $\quad f'(a) = \tan \psi$

Thus we can say that *the derivative at any point represents the slope of tangent to the curve at that point.*

3.1 Tangent Lines and the Derivative at a Point

In this section we define the slope and tangent to a curve at a point, and the derivative of a function at a point. The derivative gives a way to find both the slope of a graph and the instantaneous rate of change of a function.

Finding a Tangent Line to the Graph of a Function

To find a tangent line to an arbitrary curve $y = f(x)$ at a point $P(x_0, f(x_0))$, we use the procedure introduced in Section 2.1. We calculate the slope of the secant line through P and a nearby point $Q(x_0 + h, f(x_0 + h))$. We then investigate the limit of the slope as $h \to 0$ (Figure 3.1). If the limit exists, we call it the slope of the curve at P and define the tangent line at P to be the line through P having this slope.

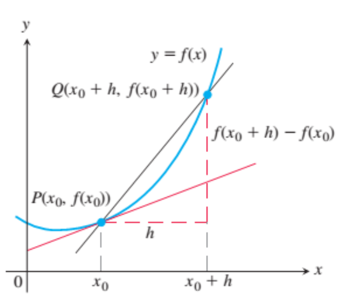

FIGURE 3.1 The slope of the tangent line at P is $\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$.

DEFINITIONS The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \qquad \text{(provided the limit exists).}$$

The **tangent line** to the curve at P is the line through P with this slope.

EXAMPLE 1

(a) Find the slope of the curve $y = 1/x$ at any point $x = a \neq 0$. What is the slope at the point $x = -1$?

(b) Where does the slope equal $-1/4$?

(c) What happens to the tangent line to the curve at the point $(a, 1/a)$ as a changes?

Solution

(a) Here $f(x) = 1/x$. The slope at $(a, 1/a)$ is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\dfrac{1}{a+h} - \dfrac{1}{a}}{h} = \lim_{h \to 0} \frac{1}{h} \frac{a - (a+h)}{a(a+h)}$$

$$= \lim_{h \to 0} \frac{-h}{ha(a+h)} = \lim_{h \to 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2}.$$

Notice how we had to keep writing "$\lim_{h \to 0}$" before each fraction until the stage at which we could evaluate the limit by substituting $h = 0$. The number a may be positive or negative, but not 0. When $a = -1$, the slope is $-1/(-1)^2 = -1$ (Figure 3.2).

(b) The slope of $y = 1/x$ at the point where $x = a$ is $-1/a^2$. It will be $-1/4$ provided that

$$-\frac{1}{a^2} = -\frac{1}{4}.$$

This equation is equivalent to $a^2 = 4$, so $a = 2$ or $a = -2$. The curve has slope $-1/4$ at the two points $(2, 1/2)$ and $(-2, -1/2)$ (Figure 3.3).

(c) The slope $-1/a^2$ is always negative if $a \neq 0$. As $a \to 0^+$, the slope approaches $-\infty$ and the tangent line becomes increasingly steep (Figure 3.2). We see this situation again as $a \to 0^-$. As a moves away from the origin in either direction, the slope approaches 0 and the tangent line levels off, becoming more and more horizontal. ■

Rates of Change: Derivative at a Point

The expression

$$\frac{f(x_0 + h) - f(x_0)}{h}, \quad h \neq 0$$

is called the **difference quotient of f at x_0 with increment h**. If the difference quotient has a limit as h approaches zero, that limit is given a special name and notation.

DEFINITION The **derivative of a function f at a point x_0**, denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided this limit exists.

3.2 The Derivative as a Function

HISTORICAL ESSAY
The Derivative
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In the last section we defined the derivative of $y = f(x)$ at the point $x = x_0$ to be the limit

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

We now investigate the derivative as a *function* derived from f by considering the limit at each point x in the domain of f.

DEFINITION The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

provided the limit exists.

Differentiable on an Interval; One-Sided Derivatives

A function $y = f(x)$ is **differentiable on an open interval** (finite or infinite) if it has a derivative at each point of the interval. It is **differentiable on a closed interval** $[a, b]$ if it is differentiable on the interior (a, b) and if the limits

$$\lim_{h \to 0^+} \frac{f(a + h) - f(a)}{h} \qquad \textbf{Right-hand derivative at } \boldsymbol{a}$$

$$\lim_{h \to 0^-} \frac{f(b + h) - f(b)}{h} \qquad \textbf{Left-hand derivative at } \boldsymbol{b}$$

exist at the endpoints (Figure 3.7).

Right-hand and left-hand derivatives may or may not be defined at any point of a function's domain. Because of Theorem 6, Section 2.4, a function has a derivative at an interior point if and only if it has left-hand and right-hand derivatives there, and these one-sided derivatives are equal.

EXAMPLE 4 Show that the function $y = |x|$ is differentiable on $(-\infty, 0)$ and on $(0, \infty)$ but has no derivative at $x = 0$.

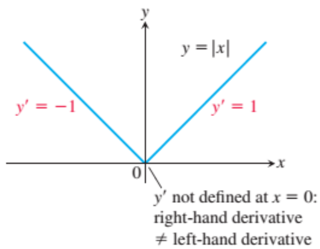

FIGURE 3.8 The function $y = |x|$ is not differentiable at the origin where the graph has a "corner" (Example 4).

Solution From Section 3.1, the derivative of $y = mx + b$ is the slope m. Thus, to the right of the origin, when $x > 0$,

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(x) = \frac{d}{dx}(1 \cdot x) = 1. \qquad \frac{d}{dx}(mx + b) = m,\ |x| = x \text{ since } x > 0$$

To the left, when $x < 0$,

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(-x) = \frac{d}{dx}(-1 \cdot x) = -1 \qquad |x| = -x \text{ since } x < 0$$

(Figure 3.8). The two branches of the graph come together at an angle at the origin, forming a non-smooth corner. There is no derivative at the origin because the one-sided derivatives differ there:

$$\begin{aligned}
\text{Right-hand derivative of } |x| \text{ at zero} &= \lim_{h \to 0^+} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0^+} \frac{|h|}{h} \\
&= \lim_{h \to 0^+} \frac{h}{h} \qquad |h| = h \text{ when } h > 0 \\
&= \lim_{h \to 0^+} 1 = 1 \\
\text{Left-hand derivative of } |x| \text{ at zero} &= \lim_{h \to 0^-} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0^-} \frac{|h|}{h} \\
&= \lim_{h \to 0^-} \frac{-h}{h} \qquad |h| = -h \text{ when } h < 0 \\
&= \lim_{h \to 0^-} -1 = -1.
\end{aligned}$$

Differentiable Functions Are Continuous

A function is continuous at every point where it has a derivative.

THEOREM 1—Differentiability Implies Continuity If f has a derivative at $x = c$, then f is continuous at $x = c$.

Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

Derivative of a Positive Integer Power

If n is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

EXAMPLE 1 Differentiate the following powers of x.

(a) x^3 **(b)** $x^{2/3}$ **(c)** $x^{\sqrt{2}}$ **(d)** $\dfrac{1}{x^4}$ **(e)** $x^{-4/3}$ **(f)** $\sqrt{x^{2+\pi}}$

Derivative Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

EXAMPLE 3 Find the derivative of the polynomial $y = x^3 + \frac{4}{3}x^2 - 5x + 1$.

Derivative Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + \frac{du}{dx}v.$$

EXAMPLE 5 Find the derivative of $y = (x^2 + 1)(x^3 + 3)$.

Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

EXAMPLE 6 Find the derivative of $y = \dfrac{t^2 - 1}{t^3 + 1}$.

Second- and Higher-Order Derivatives

If $y = f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function. If f' is also differentiable, then we can differentiate f' to get a new function of x denoted by f''. So $f'' = (f')'$. The function f'' is called the **second derivative** of f because it is the derivative of the first derivative. It is written in several ways:

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x^2 f(x).$$

Find the derivatives of the following functions using the first principle :

(i) $\cos ax$ (ii) $\tan 2x$ (iii) $\cos(3x + 2)$

(iv) $\sin x + \cos x$ (v) $\dfrac{\sin x}{x}$ (vi) $\sin\sqrt{x}$

(vii) $\sin x^2$ (viii) $\cos^2 x$ (ix) $\sin^3 x$

(x) $\sqrt{\tan x}$ (xi) $\sqrt{\operatorname{cosec} x}$