13.19. PROPERTIES OF DIVERGENCE AND CURL

1. For a constant vector $\vec{a}$, div $\vec{a} = 0$, curl $\vec{a} = \vec{0}$

2. **div** $(\vec{\mathbf{A}} + \vec{\mathbf{B}}) =$ **div** $\vec{\mathbf{A}} +$ **div** $\vec{\mathbf{B}}$ **or** $\nabla . (\vec{\mathbf{A}} + \vec{\mathbf{B}}) = \nabla . \vec{\mathbf{A}} + \nabla . \vec{\mathbf{B}}$

$$\text{div}\,(\vec{A} + \vec{B}) = \nabla . (\vec{A} + \vec{B}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) . (\vec{A} + \vec{B})$$

$$= \hat{i} . \frac{\partial}{\partial x}(\vec{A} + \vec{B}) + \hat{j} . \frac{\partial}{\partial y}(\vec{A} + \vec{B}) + \hat{k} . \frac{\partial}{\partial z}(\vec{A} + \vec{B})$$

$$= \hat{i} . \left(\frac{\partial \vec{A}}{\partial x} + \frac{\partial \vec{B}}{\partial x}\right) + \hat{j} . \left(\frac{\partial \vec{A}}{\partial y} + \frac{\partial \vec{B}}{\partial y}\right) + \hat{k}\left(\frac{\partial \vec{A}}{\partial z} + \frac{\partial \vec{B}}{\partial z}\right)$$

$$= \left(\hat{i} . \frac{\partial \vec{A}}{\partial x} + \hat{j} . \frac{\partial \vec{A}}{\partial y} + \hat{k} . \frac{\partial \vec{A}}{\partial z}\right) + \left(\hat{i} . \frac{\partial \vec{B}}{\partial x} + \hat{j} . \frac{\partial \vec{B}}{\partial y} + \hat{k} . \frac{\partial \vec{B}}{\partial z}\right)$$

$$= \nabla . \vec{A} + \nabla . \vec{B} = \text{div}\,\vec{A} + \text{div}\,\vec{B}.$$

3. **curl** $(\vec{\mathbf{A}} + \vec{\mathbf{B}}) =$ **curl** $\vec{\mathbf{A}} +$ **curl** $\vec{\mathbf{B}}$ **or** $\nabla \times (\vec{\mathbf{A}} + \vec{\mathbf{B}}) = \nabla \times \vec{\mathbf{A}} + \nabla \times \vec{\mathbf{B}}$

$$\text{curl}\,(\vec{A} + \vec{B}) = \nabla \times (\vec{A} + \vec{B}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times (\vec{A} + \vec{B})$$

$$= \sum \hat{i} \times \frac{\partial}{\partial x}(\vec{A} + \vec{B}) = \sum \hat{i} \times \left(\frac{\partial \vec{A}}{\partial x} + \frac{\partial \vec{B}}{\partial x}\right)$$

$$= \sum \hat{i} \times \frac{\partial \vec{A}}{\partial x} + \sum \hat{i} \times \frac{\partial \vec{B}}{\partial x} = \nabla \times \vec{A} + \nabla \times \vec{B} = \text{curl}\,\vec{A} + \text{curl}\,\vec{B}.$$

4. If $\vec{A}$ is a vector function and ϕ is a scalar function, then

$$\textbf{div}\,(\phi\vec{A}) = \phi\,\textbf{div}\,\vec{\mathbf{A}} + (\textbf{grad}\,\phi)\,.\,\vec{A} \quad \textbf{or} \quad \nabla\,.\,(\phi\vec{A}) = \phi\,(\nabla\,.\,\vec{\mathbf{A}}) + (\nabla\,\phi)\,.\,\vec{A}$$

(U.P.T.U. 2006)

$$\text{div}\,(\phi\vec{A}) = \nabla\,.\,(\phi\,\vec{A}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(\phi\,\vec{A}) = \hat{i}\,.\frac{\partial}{\partial x}(\phi\vec{A}) + \hat{j}\,.\frac{\partial}{\partial y}(\phi\vec{A}) + \hat{k}\,.\frac{\partial}{\partial z}(\phi\vec{A})$$

$$= \sum \hat{i}\,.\frac{\partial}{\partial x}(\phi\vec{A}) = \sum \hat{i}\,.\left(\phi\frac{\partial\vec{A}}{\partial x} + \frac{\partial\phi}{\partial x}\vec{A}\right) = \phi\sum\left(\hat{i}\,.\frac{\partial\vec{A}}{\partial x}\right) + \sum\left(\hat{i}\frac{\partial\phi}{\partial x}\right).\vec{A}$$

$$= \phi\,(\nabla\,.\,\vec{A}) + (\nabla\phi)\,.\,\vec{A} = \phi\,\text{div}\,\vec{A} + (\text{grad}\,\phi)\,.\,\vec{A}.$$

5. If A is a vector function and ϕ is a scalar function, then

$$\textbf{curl}\,(\phi\,\vec{\mathbf{A}}) = (\textbf{grad}\,\phi)\times\vec{A} + \phi\,\textbf{curl}\,\vec{\mathbf{A}} \quad \textbf{or} \quad \nabla\times(\phi\vec{\mathbf{A}}) = (\nabla\phi)\times\vec{\mathbf{A}} + \phi\,(\nabla\times\vec{\mathbf{A}})$$

$$\text{curl}\,(\phi\vec{A}) = \nabla\times(\phi\vec{A}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times(\phi\vec{A})$$

$$= \sum\hat{i}\times\frac{\partial}{\partial x}(\phi\vec{A}) = \sum\hat{i}\times\left(\frac{\partial\phi}{\partial x}\vec{A} + \phi\frac{\partial\vec{A}}{\partial x}\right) = \sum\hat{i}\times\frac{\partial\phi}{\partial x}\vec{A} + \phi\sum\hat{i}\times\frac{\partial\vec{A}}{\partial x}$$

$$= \sum\frac{\partial\phi}{\partial x}\hat{i}\times\vec{A} + \phi\sum\hat{i}\times\frac{\partial\vec{A}}{\partial x} \qquad [\because\ \vec{a}\times(m\vec{b}) = (m\vec{a})\times\vec{b} = m(\vec{a}\times\vec{b})]$$

$$= (\nabla\phi)\times\vec{A} + \phi\,(\nabla\times\vec{A}) = (\text{grad}\,\phi)\times\vec{A} + \phi\,\text{curl}\,\vec{A}.$$

6. $$\nabla\,(\vec{\mathbf{A}}\,.\,\vec{\mathbf{B}}) = (\vec{\mathbf{A}}\,.\,\nabla)\,\vec{\mathbf{B}} + (\vec{\mathbf{B}}\,.\,\nabla)\,\vec{\mathbf{A}} + \vec{\mathbf{A}}\times(\nabla\times\vec{\mathbf{B}}) + \vec{\mathbf{B}}\times(\nabla\times\vec{\mathbf{A}})$$

$$\nabla(\vec{A}\,.\,\vec{B}) = \sum\hat{i}\frac{\partial}{\partial x}(\vec{A}\,.\,\vec{B}) = \sum\hat{i}\left(\vec{A}\,.\frac{\partial\vec{B}}{\partial x} + \frac{\partial\vec{A}}{\partial x}.\,\vec{B}\right)$$

$$= \sum\left(\vec{A}\,.\frac{\partial\vec{B}}{\partial x}\right)\hat{i} + \sum\left(\vec{B}\,.\frac{\partial\vec{A}}{\partial x}\right)\hat{i} \qquad ...(1)$$

Now, we know that $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\,.\,\vec{c})\vec{b} - (\vec{a}\,.\,\vec{b})\vec{c}$

$$\Rightarrow \qquad (\vec{a}\,.\,\vec{b})\vec{c} = (\vec{a}\,.\,\vec{c})\vec{b} - \vec{a}\times(\vec{b}\times\vec{c})$$

$$\therefore \qquad \left(\vec{A}\,.\frac{\partial\vec{B}}{\partial x}\right)\hat{i} = (\vec{A}\,.\,\hat{i})\frac{\partial\vec{B}}{\partial x} - \vec{A}\times\left(\frac{\partial\vec{B}}{\partial x}\times\hat{i}\right) = (\vec{A}\,.\,\hat{i})\frac{\partial\vec{B}}{\partial x} + \vec{A}\times\left(\hat{i}\times\frac{\partial\vec{B}}{\partial x}\right)$$

$$\sum\left(\vec{A}\,.\frac{\partial\vec{B}}{\partial x}\right)\hat{i} = \left(\vec{A}\,.\sum\hat{i}\frac{\partial}{\partial x}\right)\vec{B} + \vec{A}\times\sum\left(\hat{i}\times\frac{\partial\vec{B}}{\partial x}\right)$$

$$= (\vec{A}\,.\,\nabla)\vec{B} + \vec{A}\times(\nabla\times\vec{B}) \qquad ...(2)$$

Interchanging $\vec{A}$ and $\vec{B}$, $\sum \left(\vec{B} \cdot \frac{\partial \vec{A}}{\partial x} \right) \hat{i} = (\vec{B} \cdot \nabla)\vec{A} + \vec{B} \times (\nabla \times \vec{A})$ ...(3)

Substituting the values from (2) and (3) in (1), we get

$$\nabla(\vec{A} \cdot \vec{B}) = (\vec{A} \cdot \nabla)\vec{B} + (\vec{B} \cdot \nabla)\vec{A} + \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A}).$$

7. $\nabla \cdot (\vec{\mathbf{A}} \times \vec{\mathbf{B}}) = \vec{\mathbf{B}} \cdot (\nabla \times \vec{\mathbf{A}}) - \vec{\mathbf{A}} \cdot (\nabla \times \vec{\mathbf{B}})$

Or

$\text{div}\,(\vec{\mathbf{A}} \times \vec{\mathbf{B}}) = \vec{\mathbf{B}} \cdot \text{curl}\,\vec{\mathbf{A}} - \vec{\mathbf{A}} \cdot \text{curl}\,\vec{\mathbf{B}}$

$$\nabla \cdot (\vec{A} \times \vec{B}) = \sum \hat{i} \cdot \frac{\partial}{\partial x}(\vec{A} \times \vec{B}) = \sum \hat{i} \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B} + \vec{A} \times \frac{\partial \vec{B}}{\partial x} \right)$$

$$= \sum \hat{i} \cdot \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B} \right) + \sum \hat{i} \cdot \left(\vec{A} \times \frac{\partial \vec{B}}{\partial x} \right) = \sum \hat{i} \cdot \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B} \right) - \sum \hat{i} \cdot \left(\frac{\partial \vec{B}}{\partial x} \times \vec{A} \right)$$

$$= \sum \left(\hat{i} \times \frac{\partial \vec{A}}{\partial x} \right) \cdot \vec{B} - \sum \left(\hat{i} \times \frac{\partial \vec{B}}{\partial x} \right) \cdot \vec{A} \qquad [\text{Since } \vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}]$$

$$= (\nabla \times \vec{A}) \cdot \vec{B} - (\nabla \times \vec{B}) \cdot \vec{A} = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B}).$$

8. $\nabla \times (\vec{\mathbf{A}} \times \vec{\mathbf{B}}) = (\nabla \cdot \mathbf{B})\vec{\mathbf{A}} - (\nabla \cdot \mathbf{A})\vec{\mathbf{B}} + (\mathbf{B} \cdot \nabla)\vec{\mathbf{A}} - (\mathbf{A} \cdot \nabla)\vec{\mathbf{B}}$

$$\nabla \times (\vec{A} \times \vec{B}) = \sum \hat{i} \times \frac{\partial}{\partial x}(\vec{A} \times \vec{B}) = \sum \hat{i} \times \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B} + \vec{A} \times \frac{\partial \vec{B}}{\partial x} \right)$$

$$= \sum \hat{i} \times \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B} \right) + \sum \hat{i} \times \left(\vec{A} \times \frac{\partial \vec{B}}{\partial x} \right)$$

$$= \sum \left[(\hat{i} \cdot \vec{B}) \frac{\partial \vec{A}}{\partial x} - \left(\hat{i} \cdot \frac{\partial \vec{A}}{\partial x} \right) \vec{B} \right] + \sum \left[\left(\hat{i} \cdot \frac{\partial \vec{B}}{\partial x} \right) \vec{A} - (\hat{i} \cdot \vec{A}) \frac{\partial \vec{B}}{\partial x} \right]$$

$$[\text{Since } \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}]$$

$$= \sum (\vec{B} \cdot \hat{i}) \frac{\partial \vec{A}}{\partial x} - \left(\sum \hat{i} \cdot \frac{\partial \vec{A}}{\partial x} \right) \vec{B} + \left(\sum \hat{i} \cdot \frac{\partial \vec{B}}{\partial x} \right) \vec{A} - \sum (\vec{A} \cdot \hat{i}) \frac{\partial \vec{B}}{\partial x}$$

$$= \left(\vec{B} \cdot \sum \hat{i} \frac{\partial}{\partial x} \right) \vec{A} - (\nabla \cdot \vec{A})\vec{B} + (\nabla \cdot \vec{B})\vec{A} - \left(\mathrm{A} \cdot \sum \hat{i} \frac{\partial}{\partial x} \right) \vec{B}$$

$$= (\vec{B} \cdot \nabla)\vec{A} - (\nabla \cdot \vec{A})\vec{B} + (\nabla \cdot \vec{B})\vec{A} - (\vec{A} \cdot \nabla)\vec{B}$$

$$= (\nabla \cdot \vec{B})\vec{A} - (\nabla \cdot \vec{A})\vec{B} + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}.$$

13.20. REPEATED OPERATIONS BY ∇

Let $\phi(x, y, z)$ and $\vec{V}(x, y, z)$ be scalar and vector point functions respectively.

Since grad ϕ and curl $\vec{V}$ are also vector point functions, we can find their divergence as well as curl, whereas div $\vec{V}$ being a scalar point function, we can find its gradient only.

1. **Div (grad ϕ) = $\nabla^2\phi$ where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$**

$$\text{Div}(\text{grad }\phi) = \nabla \cdot (\nabla\phi) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z}\right) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi$$

$$= \nabla^2\phi \quad \text{where} \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

∇^2 is called the *Laplacian operator* and $\nabla^2\phi = 0$ is called *Laplace's equation*.

2. **Curl (grad ϕ) = $\nabla \times \nabla\phi = \vec{0}$** *(V.T.U. 2007)*

$$\text{Curl}(\text{grad }\phi) = \nabla \times \nabla\phi = \nabla \times \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix} = \Sigma \hat{i}\left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}\right) = \vec{0}$$

3. **Div (curl $\vec{V}$) = $\nabla \cdot (\nabla \times \vec{V}) = 0$.**

Let $\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$, then Curl $\vec{V} = \nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$

$$= \hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \hat{j}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

$\therefore$ div (curl $\vec{V}$) = $\nabla \cdot (\nabla \times \vec{V})$

$$= \frac{\partial}{\partial x}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

$$= \left(\frac{\partial^2 V_3}{\partial x \partial y} - \frac{\partial^2 V_2}{\partial x \partial z}\right) + \left(\frac{\partial^2 V_1}{\partial y \partial z} - \frac{\partial^2 V_3}{\partial y \partial x}\right) + \left(\frac{\partial^2 V_2}{\partial z \partial x} - \frac{\partial^2 V_1}{\partial z \partial y}\right) = 0.$$

4. **Curl (curl $\vec{V}$) = grad div $\vec{V} - \nabla^2 \vec{V}$** *(K.U.K. 2005 ; Madras 2006; P.T.U. 2005)*

or $$\nabla \times (\nabla \times \vec{V}) = \nabla (\nabla \cdot \vec{V}) - \nabla^2 \vec{V}.$$

Let $$\vec{V} = V_1 \hat{i} + V_2 \hat{j} + V_3 \hat{k}$$

then $$\text{curl } \vec{V} = \nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \hat{j}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

$$\therefore \quad \text{Curl (curl } \vec{V}) = \nabla \times (\nabla \times \vec{V}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial V_3}{\partial y} - \dfrac{\partial V_2}{\partial z} & \dfrac{\partial V_1}{\partial z} - \dfrac{\partial V_3}{\partial x} & \dfrac{\partial V_2}{\partial x} - \dfrac{\partial V_1}{\partial y} \end{vmatrix}$$

$$= \Sigma \hat{i} \left\{ \frac{\partial}{\partial y}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) \right\}$$

$$\frac{\partial^2 V_1}{\partial x^2}$$

$$= \Sigma \hat{i} \left\{ \frac{\partial}{\partial x}\left(\frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\right) - \left(\frac{\partial^2 V_1}{\partial y^2} + \frac{\partial^2 V_1}{\partial z^2}\right) \right\}$$

$$= \Sigma \hat{i} \left\{ \frac{\partial}{\partial x}\left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\right) - \left(\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} + \frac{\partial^2 V_1}{\partial z^2}\right) \right\}$$

$$\left[\because \frac{\partial}{\partial x}\left(\frac{\partial V_1}{\partial x}\right) = \frac{\partial^2 V_1}{\partial x^2} \right]$$

$$= \Sigma \hat{i} \left\{ \frac{\partial}{\partial x}(\nabla \cdot \vec{V}) - (\nabla^2 V_1) \right\} = \Sigma \hat{i} \frac{\partial}{\partial x}(\nabla \cdot \vec{V}) - \nabla^2 \Sigma \hat{i} V_1$$

$$= \nabla(\nabla \cdot \vec{V}) - \nabla^2 \vec{V} = \text{grad (div } \vec{V}) - \nabla^2 \vec{V}.$$

Note 1. The above result can also be written as grad (div $\vec{V}$) = curl (curl $\vec{V}$) + $\nabla^2 \vec{V}$

or $$\nabla(\nabla \cdot \vec{V}) = \nabla \times (\nabla \times \vec{V}) + \nabla^2 \vec{V}.$$

Note 2. Treating ∇ as a vector, the results of repeated application of ∇ can be easily written down. Thus

$$\nabla \cdot \nabla \phi = \nabla^2 \phi \qquad (\because \vec{a} \cdot \vec{a} = a^2)$$

$$\nabla \times \nabla \phi = \vec{0} \qquad (\because \vec{a} \times \vec{a} = \vec{0})$$

$$\nabla \cdot (\nabla \times \vec{V}) = 0 \qquad (\because \vec{a} \cdot (\vec{a} \times \vec{b}) = [\vec{a}\ \vec{a}\ \vec{b}] = 0)$$

$$\nabla \times (\nabla \times \vec{V}) = \nabla(\nabla \cdot \vec{V}) - \nabla^2 \vec{V}$$ (By expanding as a vector triple product)

ILLUSTRATIVE EXAMPLES

Example 1. *A vector field is given by* $\vec{A} = (x^2 + xy^2)\hat{i} + (y^2 + x^2y)\hat{j}$. *Show that the field is irrotational.*

Sol. Field $\vec{A}$ is irrotational if curl $\vec{A} = \vec{0}$

Now $\quad$ $\text{curl } \vec{A} = \nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 + xy^2 & y^2 + x^2y & 0 \end{vmatrix}$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(2xy - 2xy) = \vec{0}.$$

$\therefore$ Field $\vec{A}$ is irrotational.

Example 2. *If the vector* $\vec{F} = (ax^2y + yz)\hat{i} + (xy^2 - xz^2)\hat{j} + (2xyz - 2x^2y^2)\hat{k}$ *is solenoidal, find the value of a. Find also the curl of this solenoidal vector.*

Sol. Here $\qquad \vec{F} = (ax^2y + yz)\hat{i} + (xy^2 - xz^2)\hat{j} + (2xyz - 2x^2y^2)\hat{k}$

$$\text{div } \vec{F} = \nabla . \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left[(ax^2y + yz)\hat{i} + (xy^2 - xz^2)\hat{j} + (2xyz - 2x^2y^2)\hat{k}\right]$$

$$= \frac{\partial}{\partial x}(ax^2y + yz) + \frac{\partial}{\partial y}(xy^2 - xz^2) + \frac{\partial}{\partial z}(2xyz - 2x^2y^2)$$

$$= 2axy + 2xy + 2xy = 2(a+2)xy$$

Since $\vec{F}$ is solenoidal, div $\vec{F} = 0$

$\Rightarrow \quad 2(a+2)xy = 0 \qquad \therefore \quad a = -2$

Now $\quad \vec{F} = (-2x^2y + yz)\hat{i} + (xy^2 - xz^2)\hat{j} + (2xyz - 2x^2y^2)\hat{k}$

$\therefore \quad \text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -2x^2y + yz & xy^2 - xz^2 & 2xyz - 2x^2y^2 \end{vmatrix}$

$$= \hat{i}\left[\frac{\partial}{\partial y}(2xyz - 2x^2y^2) - \frac{\partial}{\partial z}(xy^2 - xz^2)\right] - \hat{j}\left[\frac{\partial}{\partial x}(2xyz - 2x^2y^2) - \frac{\partial}{\partial z}(-2x^2y + yz)\right]$$
$$+ \hat{k}\left[\frac{\partial}{\partial x}(xy^2 - xz^2) - \frac{\partial}{\partial y}(-2x^2y + yz)\right]$$

$$= \hat{i}(2xz - 4x^2y + 2xz) - \hat{j}(2yz - 4xy^2 - y) + \hat{k}(y^2 - z^2 + 2x^2 - z)$$

$$= 4x(z - xy)\hat{i} + (y + 4xy^2 - 2yz)\hat{j} + (2x^2 + y^2 - z^2 - z)\hat{k}$$

Example 3. *Show that* $r^\alpha \vec{R}$ *is an irrotational vector for any value of* α *but it is solenoidal if* $\alpha + 3 = 0$ *where* $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$ *and r is the magnitude of* $\vec{R}$.

(V.T.U., 2006; Kottayam 2005; M.D.U. Dec. 2010)

Sol. Let $\vec{V} = r^{\alpha} \vec{R} = (x^2 + y^2 + z^2)^{\frac{\alpha}{2}} (x\hat{i} + y\hat{j} + z\hat{k})$

$$= x(x^2 + y^2 + z^2)^{\alpha/2} \hat{i} + y(x^2 + y^2 + z^2)^{\alpha/2} \hat{j} + z(x^2 + y^2 + z^2)^{\alpha/2} \hat{k}$$

$$\therefore \quad \text{curl } \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x(x^2 + y^2 + z^2)^{\alpha/2} & y(x^2 + y^2 + z^2)^{\alpha/2} & z(x^2 + y^2 + z^2)^{\alpha/2} \end{vmatrix}$$

$$= \sum i \left\{ \frac{\alpha z}{2} (x^2 + y^2 + z^2)^{\frac{\alpha}{2} - 1} . 2y - \frac{\alpha y}{2} (x^2 + y^2 + z^2)^{\frac{\alpha}{2} - 1} . 2z \right\}$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$$

$\Rightarrow$ $\vec{V} = r^{\alpha} \vec{R}$ is irrotational for any value of α.

Now, $\quad \text{div } \vec{V} = \nabla . (r^{\alpha} \vec{R})$

$$= r^{\alpha} (\text{div } \vec{R}) + \text{grad } r^{\alpha} . \vec{R} \qquad ...(1)$$

$$\left[\because \ \text{div} (\phi \vec{A}) = \phi (\text{div } \vec{A}) + \text{grad } \phi . \vec{A} \right]$$

and $\quad \text{div } (\vec{R}) = \nabla . (x\hat{i} + y\hat{j} + z\hat{k})$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3$$

Also, $\quad r^2 = x^2 + y^2 + z^2$ so that $\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$

$$\text{grad } r = \hat{i} \frac{\partial r}{\partial x} + \hat{j} \frac{\partial r}{\partial y} + \hat{k} \frac{\partial r}{\partial z} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r} = \frac{\vec{R}}{r}$$

$$\therefore \quad \text{grad } r^{\alpha} = \alpha\, r^{\alpha - 1} \text{grad } r = \alpha r^{\alpha - 1} \frac{\vec{R}}{r} = \alpha r^{\alpha - 2} \vec{R}$$

$\therefore$ From (1), we have

$$\text{div } \vec{V} = r^{\alpha} (3) + \alpha\, r^{\alpha - 2} \vec{R} . \vec{R} = 3r^{\alpha} + \alpha\, r^{\alpha - 2} (x^2 + y^2 + z^2)$$

$$= 3r^{\alpha} + \alpha\, r^{\alpha - 2} (r^2) = (3 + \alpha)\, r^{\alpha}$$

Now, $\vec{V}$ is solenoidal if div $\vec{V} = 0$ *i.e.*, $(3 + \alpha)\, r^{\alpha} = 0$

$\Rightarrow$ $r^{\alpha} \vec{R}$ is solenoidal if $\alpha + 3 = 0$.

Example 4. *If $\vec{a}$ is a constant vector and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that curl $(\vec{a} \times \vec{r}) = 2\vec{a}$.*

(K.U.K. 2009)

Sol. Let $\quad \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, where a_1, a_2, a_3 are constants.

$$\vec{a} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = (a_2 z - a_3 y)\hat{i} + (a_3 x - a_1 z)\hat{j} + (a_1 y - a_2 x)\hat{k}$$

$$\text{curl}\,(\vec{a}\times\vec{r}) = \nabla\times(\vec{a}\times\vec{r}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_2z - a_3y & a_3x - a_1z & a_1y - a_2x \end{vmatrix}$$

$$= (a_1 + a_1)\hat{i} + (a_2 + a_2)\hat{j} + (a_3 + a_3)\hat{k} = 2(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) = 2\vec{a}.$$

Example 5. *Prove that*

(i) $\nabla(\vec{a}\,.\,\vec{u}) = (\vec{a}\,.\,\nabla)\vec{u} + \vec{a}\times(\nabla\times\vec{u})$ (ii) $\nabla\times(\vec{a}\times\vec{u}) = (\nabla\,.\,\vec{u})\vec{a} - (\vec{a}\,.\,\nabla)\vec{u}$

where $\vec{a}$ *is a constant vector.*

Sol. (i) $\nabla(\vec{a}\,.\,\vec{u}) = \sum i\dfrac{\partial}{\partial x}(\vec{a}\,.\,\vec{u}) = \sum i\left(\vec{a}\,.\,\dfrac{\partial\vec{u}}{\partial x}\right)$ ...(1)

Now $\vec{a}\times\left(\hat{i}\times\dfrac{\partial\vec{u}}{\partial x}\right) = \left(\vec{a}\,.\,\dfrac{\partial\vec{u}}{\partial x}\right)i - (\vec{a}\,.\,\hat{i})\dfrac{\partial\vec{u}}{\partial x}$

$\Rightarrow$ $\left(\vec{a}\,.\,\dfrac{\partial\vec{u}}{\partial x}\right)\hat{i} = \vec{a}\times\left(\hat{i}\times\dfrac{\partial\vec{u}}{\partial x}\right) + (\vec{a}\,.\,\hat{i})\dfrac{\partial\vec{u}}{\partial x}$

$\therefore$ From (1), we have $\nabla(\vec{a}\,.\,\vec{u}) = \sum\vec{a}\times\left(\hat{i}\times\dfrac{\partial\vec{u}}{\partial x}\right) + \sum(\vec{a}\,.\,\hat{i})\dfrac{\partial\vec{u}}{\partial x}$

$$= \vec{a}\times(\nabla\times\vec{u}) + (\vec{a}\,.\,\nabla)\vec{u} = (\vec{a}\,.\,\nabla)\vec{u} + \vec{a}\times(\nabla\times\vec{u}).$$

(ii) $\nabla\times(\vec{a}\times\vec{u}) = \sum\hat{i}\dfrac{\partial}{\partial x}\times(\vec{a}\times\vec{u}) = \sum\hat{i}\times\left(\vec{a}\times\dfrac{\partial\vec{u}}{\partial x}\right)$

$$= \sum\left(\hat{i}\,.\,\frac{\partial\vec{u}}{\partial x}\right)\vec{a} - \sum(\hat{i}\,.\,\vec{a})\frac{\partial\vec{u}}{\partial x} = (\nabla\,.\,\vec{u})\vec{a} - (\vec{a}\,.\,\nabla)\vec{u}.$$

Example 6. *Prove that* $\nabla\,.\,(\phi\nabla\psi - \psi\nabla\phi) = \phi\nabla^2\psi - \psi\nabla^2\phi.$

Sol. For a scalar function f and a vector function $\vec{G}$, we know that

$$\nabla.(f\vec{G}) = f(\nabla.\vec{G}) + (\nabla f).\vec{G}$$

Also $\nabla.(\vec{F} - \vec{G}) = \nabla.\vec{F} - \nabla.\vec{G}$

$\therefore$ $\nabla\,.\,(\phi\nabla\psi - \psi\nabla\phi) = \nabla.(\phi\nabla\psi) - \nabla.(\psi\nabla\phi)$

$= [\phi(\nabla.\nabla\psi) + \nabla\phi\,.\,\nabla\psi] - [\psi(\nabla.\nabla\phi) + \nabla\psi.\nabla\phi]$

$= \phi\nabla^2\psi + \nabla\phi.\nabla\psi - \psi\nabla^2\phi - \nabla\psi.\nabla\phi$

$= \phi\nabla^2\psi - \psi\nabla^2\phi$ [$\because$ dot product is commutative]

Example 7. *Prove that*

(i) $div\left(\frac{\vec{r}}{r^3}\right) = 0$ (ii) $\nabla^2 (r^n) = n(n+1)\, r^{n-2}$.

where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. (P.T.U. 2007, 2008 ; J.N.T.U. 2006 ; U.P.T.U. 2005)

Sol. Here $r^2 = x^2 + y^2 + z^2$ so that $\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$

$$\text{grad } r = \hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z} = \frac{\vec{r}}{r}$$

(i) Since $\text{div}(\phi\vec{A}) = \phi(\text{div }\vec{A}) + \text{grad }\phi \,.\, \vec{A}$

$$\therefore \quad \text{div}\left(\frac{\vec{r}}{r^3}\right) = \text{div}\,(r^{-3}\,\vec{r}) = r^{-3}\,(\text{div }\vec{r}) + (\text{grad } r^{-3})\,.\,\vec{r}$$

$$= 3r^{-3} + (-\,3r^{-4}\text{ grad } r)\,.\,\vec{r} \qquad [\because \quad \text{div }\vec{r} = 3]$$

$$= 3r^{-3} + \left(-\,3r^{-4}\frac{\vec{r}}{r}\right).\,\vec{r} = 3r^{-3} - 3r^{-5}\,(\vec{r}\,.\,\vec{r}) = 3r^{-3} - 3r^{-5}\,(r^2) = 0.$$

(ii) $$\nabla^2 (r^n) = \nabla.\,(\nabla r^n) = \nabla.\left(nr^{n-1}\frac{\vec{r}}{r}\right) = n\nabla.\,(r^{n-2}\,\vec{r}\,)$$

$$= n\,[(\nabla\, r^{n-2})\,.\,\vec{r} + r^{n-2}\,(\nabla.\,\vec{r}\,)] \quad [\because \quad \nabla\,.\,(\phi\vec{A}) = (\nabla\,\phi)\,.\,\vec{A} + \phi\,(\nabla\,.\,\vec{A})]$$

$$= n\left[(n-2)\,r^{n-3}\,\frac{\vec{r}}{r}\,.\,\vec{r} + r^{n-2}\,(3)\right] \qquad [\because \quad \nabla.\,\vec{r} = 3]$$

$$= n\,[(n-2)\,r^{n-4}\,(r^2) + 3r^{n-2}] \qquad [\because \quad \vec{r}\,.\,\vec{r} = r^2]$$

$$= n(n+1)r^{n-2}.$$

Second Method

$$\nabla^2 (r^n) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) r^n$$

$$= \sum \frac{\partial^2}{\partial x^2}(r^n) = \sum \frac{\partial}{\partial x}\left(\frac{\partial r^n}{\partial x}\right)$$

$$= \sum \frac{\partial}{\partial x}\left(nr^{n-1}\frac{\partial r}{\partial x}\right) = \sum \frac{\partial}{\partial x}\left(nr^{n-1}\frac{x}{r}\right) = \sum n\frac{\partial}{\partial x}\,(r^{n-2}\,x)$$

$$= n\sum\left[(n-2)\,r^{n-3}\,\frac{\partial r}{\partial x}.x + r^{n-2}\right] = n\sum\left[(n-2)\,r^{n-3}\,\frac{x}{r}.x + r^{n-2}\right]$$

$$= n \sum [(n-2)\, r^{n-4}\, x^2 + r^{n-2}] = n[(n-2)\, r^{n-4}\, (x^2+y^2+z^2) + 3r^{n-2}]$$

$$= n\, [(n-2)\, r^{n-4}(r^2) + 3r^{n-2}] = n(n+1)r^{n-2}.$$

Example 8. *Prove that the vector $f(r)\,\vec{r}$ is irrotational.*

Sol. The vector $f(r)\,\vec{r}$ will be irrotational if curl $[f(r)\,\vec{r}\,] = \vec{0}$

Since $\quad \text{curl}\,(\phi \vec{A}) = (\text{grad}\,\phi) \times \vec{A} + \phi\,\text{curl}\,\vec{A}$

$\therefore \quad \text{curl}\,[f(r)\,\vec{r}\,] = [\text{grad}\,f(r)] \times \vec{r} + f(r)\,\text{curl}\,\vec{r}$

$$= [f'(r)\,\text{grad}\,r] \times \vec{r} + f(r)\,\vec{0} \qquad [\because\ \text{curl}\,\vec{r} = \vec{0}]$$

$$= \left[f'(r)\frac{\vec{r}}{r}\right] \times \vec{r} = \frac{f'(r)}{r}(\vec{r} \times \vec{r}) = \vec{0}, \text{ since } \vec{r} \times \vec{r} = \vec{0}.$$

$\therefore$ The vector $f(r)\,\vec{r}$ is irrotational.

Example 9. *Prove that $\nabla^2 f(r) = f''(r) + \dfrac{2}{r} f'(r)$.* (M.D.U., Dec. 2005 ; Kerala 2005)

Sol. $\nabla^2 f(r) = \nabla \cdot \{\nabla f(r)\} = \text{div}\,\{\text{grad}\,f(r)\} = \text{div}\,\{f'(r)\,\text{grad}\,r\} = \text{div}\left\{f'(r)\dfrac{\vec{r}}{r}\right\}$

$$= \text{div}\left\{\frac{1}{r} f'(r)\,\vec{r}\right\} = \frac{1}{r} f'(r)\,\text{div}\,\vec{r} + \vec{r}\,.\,\text{grad}\left\{\frac{1}{r} f'(r)\right\}$$

$$= \frac{3}{r} f'(r) + \vec{r}\,.\left[\frac{d}{dr}\left(\frac{1}{r} f'(r)\right)\text{grad}\,r\right] = \frac{3}{r} f'(r) + \vec{r}\,.\left[\left\{-\frac{1}{r^2} f'(r) + \frac{1}{r} f''(r)\right\}\frac{\vec{r}}{r}\right]$$

$$= \frac{3}{r} f'(r) + \left[-\frac{1}{r^3} f'(r) + \frac{1}{r^2} f''(r)\right](\vec{r}\,.\,\vec{r}) = \frac{3}{r} f'(r) + \left[-\frac{1}{r^3} f'(r) + \frac{1}{r^2} f''(r)\right] r^2$$

$$= \frac{3}{r} f'(r) - \frac{1}{r} f'(r) + f''(r) = f''(r) + \frac{2}{r} f'(r).$$

Example 10. *If $u\vec{F} = \nabla v$, where u, v are scalar fields and $\vec{F}$ is a vector field, show that $\vec{F}\,.\,\text{curl}\,\vec{F} = 0$.*

$$\left[\because\ \vec{F} = \frac{1}{u}\nabla v\right]$$

Sol. $\quad \text{curl}\,\vec{F} = \nabla \times \left(\dfrac{1}{u}\nabla v\right)$

$$= \nabla\frac{1}{u} \times \nabla v + \frac{1}{u}\,\nabla \times (\nabla v) \qquad [\because\ \nabla \times (\phi\vec{A}) = \nabla\phi \times \vec{A} + \phi\nabla \times \vec{A}]$$

$$= \nabla\frac{1}{u} \times \nabla v \qquad [\because\ \nabla \times \nabla v = \vec{0}]$$

$$\therefore \quad \vec{F} \,.\, \text{curl}\, \vec{F} = \frac{1}{u} \nabla v \,.\left(\nabla \frac{1}{u} \times \nabla v \right) = 0$$

being the scalar triple product in which two factors are equal.

Example 11. *Prove that* $\vec{a}.\nabla\left(\vec{b}.\nabla\frac{1}{r}\right) = \frac{3(\vec{a}.\vec{r})(\vec{b}.\vec{r})}{r^5} - \frac{\vec{a}.\vec{b}}{r^3}$ *where* $\vec{a}$ *and* $\vec{b}$ *are constant vectors.*

(M.D.U. May 2006, May 2007)

Sol. We know that $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

$$\therefore \quad \nabla \frac{1}{r} = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) (x^2 + y^2 + z^2)^{-\frac{1}{2}}$$

$$= \frac{-1}{2(x^2 + y^2 + z^2)^{\frac{3}{2}}} (2x\hat{i} + 2y\hat{j} + 2z\hat{k}) = -\frac{\vec{r}}{r^3}$$

$$\Rightarrow \quad \nabla\left(\vec{b}.\nabla\frac{1}{r}\right) = \nabla\left[\vec{b}.\left(-\frac{\vec{r}}{r^3}\right)\right] = -\nabla\left[\frac{1}{r^3}(\vec{b}.\vec{r})\right] \qquad \text{[Form } \nabla(fg)]$$

$$= -\left[\frac{1}{r^3}\nabla(\vec{b}.\vec{r}) + (\vec{b}.\vec{r})\nabla\frac{1}{r^3}\right]$$

$$= -\left[\frac{1}{r^3}\{(\vec{b}.\nabla)\vec{r} + (\vec{r}.\nabla)\vec{b} + \vec{b}\times(\nabla\times\vec{r}) + \vec{r}\times(\nabla\times\vec{b})\}\right.$$

$$\left. + (\vec{b}.\vec{r})\left\{\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right\}(x^2 + y^2 + z^2)^{-\frac{3}{2}}\right] \qquad \text{...(1)}$$

Let $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ where b_1, b_2, b_3 are independent of x, y, z (since $\vec{b}$ is a constant vector)

$$(\vec{b}.\nabla)\vec{r} = (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}).\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\vec{r}$$

$$= \left(b_1\frac{\partial}{\partial x} + b_2\frac{\partial}{\partial y} + b_3\frac{\partial}{\partial z}\right)(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = \vec{b}$$

$$(\vec{r}.\nabla)\vec{b} = (x\hat{i} + y\hat{j} + z\hat{k}).\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\vec{b}$$

$$= \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) = \vec{0}$$

$$\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)^{-\frac{3}{2}} = \frac{-3}{2(x^2 + y^2 + z^2)^{\frac{5}{2}}}(2x\hat{i} + 2y\hat{j} + 2z\hat{k})$$

$$= -\frac{3\vec{r}}{r^5}$$

Also $\quad \nabla \times \vec{r} = \vec{0}, \nabla \times \vec{b} = \vec{0}$

$\therefore$ From (1), we have

$$\nabla\left(\vec{b}.\nabla\frac{1}{r}\right) = -\left[\frac{1}{r^3}\left\{\vec{b}+\vec{0}+\vec{0}+\vec{0}\right\}+(\vec{b}.\vec{r})\left(\frac{-3\vec{r}}{r^5}\right)\right] = -\frac{\vec{b}}{r^3}+\frac{3(\vec{b}.\vec{r})\vec{r}}{r^5}$$

$$\Rightarrow \quad \vec{a}.\nabla\left(\vec{b}.\nabla\frac{1}{r}\right) = \vec{a}.\left[-\frac{\vec{b}}{r^3}+\frac{3(\vec{b}.\vec{r})\vec{r}}{r^5}\right] = -\frac{\vec{a}.\vec{b}}{r^3}+\frac{3(\vec{b}.\vec{r})(\vec{a}.\vec{r})}{r^5}$$

$$= \frac{3(\vec{a}.\vec{r})(\vec{b}.\vec{r})}{r^5}-\frac{\vec{a}.\vec{b}}{r^3}$$

Example 12. *If r is the distance of a point (x, y, z) from the origin, prove that* $curl\left(\hat{k}\times grad\frac{1}{r}\right)+grad\left(\hat{k}.grad\frac{1}{r}\right)=\vec{0}$*, where* $\hat{k}$ *is the unit vector in the direction of OZ.*

Sol. Here, $\quad r = \sqrt{x^2+y^2+z^2} \quad$ so that $r^2 = x^2+y^2+z^2$

and $$\frac{\partial r}{\partial x}=\frac{x}{r}, \frac{\partial r}{\partial y}=\frac{y}{r}, \frac{\partial r}{\partial z}=\frac{z}{r}$$

$$\therefore \quad \text{grad } r = \hat{i}\frac{\partial r}{\partial x}+\hat{j}\frac{\partial r}{\partial y}+\hat{k}\frac{\partial r}{\partial z} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{r}$$

$$\text{grad }\frac{1}{r} = -\frac{1}{r^2}\text{ grad } r = -\frac{x\hat{i}+y\hat{j}+z\hat{k}}{r^3}$$

$$\Rightarrow \quad \hat{k}\times\text{grad }\frac{1}{r} = -\frac{x(\hat{k}\times\hat{i})+y(\hat{k}\times\hat{j})+z(\hat{k}\times\hat{k})}{r^3}$$

$$= -\frac{x\hat{j}-y\hat{i}}{r^3} = \frac{y\hat{i}-x\hat{j}}{(x^2+y^2+z^2)^{3/2}}$$

and $$\hat{k}.\text{grad }\frac{1}{r} = -\frac{\hat{k}.(x\hat{i}+y\hat{j}+z\hat{k})}{r^3} = -\frac{z}{(x^2+y^2+z^2)^{3/2}}$$

$$\therefore \quad \text{curl}\left(\hat{k}\times\text{grad }\frac{1}{r}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{y}{(x^2+y^2+z^2)^{3/2}} & \frac{-x}{(x^2+y^2+z^2)^{3/2}} & 0 \end{vmatrix}$$

$$= \hat{i}\frac{\partial}{\partial z}\left\{\frac{x}{(x^2+y^2+z^2)^{3/2}}\right\}+\hat{j}\frac{\partial}{\partial z}\left\{\frac{y}{(x^2+y^2+z^2)^{3/2}}\right\}$$

$$+\hat{k}\left[\frac{\partial}{\partial x}\left\{\frac{-x}{(x^2+y^2+z^2)^{3/2}}\right\}-\frac{\partial}{\partial y}\left\{\frac{y}{(x^2+y^2+z^2)^{3/2}}\right\}\right]$$

$$= \hat{i}\left[-\frac{3}{2}x(x^2+y^2+z^2)^{-\frac{5}{2}}.2z\right] + \hat{j}\left[-\frac{3}{2}y(x^2+y^2+z^2)^{-\frac{5}{2}}.2z\right]$$

$$+\hat{k}\left[\left\{-(x^2+y^2+z^2)^{-\frac{3}{2}} + \frac{3}{2}x(x^2+y^2+z^2)^{-\frac{5}{2}}.2x\right\}\right.$$

$$\left. - \left\{(x^2+y^2+z^2)^{-\frac{3}{2}} - \frac{3}{2}y(x^2+y^2+z^2)^{-\frac{5}{2}}.2y\right\}\right]$$

$$= \frac{-3xz\hat{i}}{(x^2+y^2+z^2)^{5/2}} - \frac{3yz\hat{j}}{(x^2+y^2+z^2)^{5/2}}$$

$$+\hat{k}\left[\frac{-(x^2+y^2+z^2)+3x^2-(x^2+y^2+z^2)+3y^2}{(x^2+y^2+z^2)^{5/2}}\right]$$

$$= \frac{-3xz\hat{i} - 3yz\hat{j} + (x^2+y^2-2z^2)\hat{k}}{r^5} \qquad \text{...(1)}$$

Also, $\text{grad}\left(\hat{k}\,.\,\text{grad}\,\frac{1}{r}\right) = \nabla\left[\frac{-z}{(x^2+y^2+z^2)^{3/2}}\right]$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left[\frac{-z}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$= \hat{i}\left[\frac{3}{2}z(x^2+y^2+z^2)^{-\frac{5}{2}}.2x\right] + \hat{j}\left[\frac{3}{2}z(x^2+y^2+z^2)^{-\frac{5}{2}}.2y\right]$$

$$+\hat{k}\left[-(x^2+y^2+z^2)^{-\frac{3}{2}} + \frac{3}{2}z(x^2+y^2+z^2)^{-\frac{5}{2}}.2z\right]$$

$$= \frac{3xz\hat{i}}{(x^2+y^2+z^2)^{5/2}} + \frac{3yz\hat{j}}{(x^2+y^2+z^2)^{5/2}} + \frac{\{-(x^2+y^2+z^2)+3z^2\}\hat{k}}{(x^2+y^2+z^2)^{5/2}}$$

$$= \frac{3xz\hat{i} + 3yz\hat{j} + (2z^2-x^2-y^2)\hat{k}}{r^5} \qquad \text{...(2)}$$

Adding (1) and (2), we have

$$\text{curl}\left(\hat{k}\times\text{grad}\,\frac{1}{r}\right) + \text{grad}\left(\hat{k}\,.\,\text{grad}\,\frac{1}{r}\right) = \vec{0}$$

EXERCISE 13.3

1. Evaluate

(i) div $(3x^2\hat{i} + 5xy^2\hat{j} + xyz^3\hat{k})$ at the point (1, 2, 3).

(ii) div $[(xy\sin z)\hat{i} + (y^2\sin x)\hat{j} + (z^2\sin xy)\hat{k}]$ at the point $\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right)$.

(iii) curl $[e^{xyz}(\hat{i} + \hat{j} + \hat{k})]$

13.21. INTEGRATION OF VECTOR FUNCTIONS

Let $\vec{f}(t)$ and $\vec{F}(t)$ be two vector functions of a scalar variable t such that $\frac{d}{dt}\vec{F}(t) = \vec{f}(t)$, then $\vec{F}(t)$ is called an integral of $\vec{f}(t)$ with respect to t and we write $\int \vec{f}(t)dt = \vec{F}(t)$

If $\vec{c}$ is any *arbitrary constant vector* independent of t, then $\frac{d}{dt}\{\vec{F}(t) + \vec{c}\} = \vec{f}(t)$

This is equivalent to $\int \vec{f}(t)\,dt = \vec{F}(t) + \vec{c}$

$\vec{F}(t)$ is called the *indefinite integral* of $\vec{f}(t)$. The constant vector $\vec{c}$ is called the *constant of integration* and can be determined if some initial conditions are given.

The *definite integral* of $\vec{f}(t)$ between the limits $t = a$ and $t = b$ is written as

$$\int_a^b \vec{f}(t)\,dt = \left[\vec{F}(t)\right]_a^b = \vec{F}(b) - \vec{F}(a).$$

Note 1. If $\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$, then

$$\int \vec{f}(t)\,dt = \hat{i}\int f_1(t)\,dt + \hat{j}\int f_2(t)\,dt + \hat{k}\int f_3(t)\,dt.$$

Thus in order to integrate a vector function, integrate its components.

Note 2. We can obtain some standard results for integration of vector functions by considering the derivatives of suitable vector functions. For example,

(i) $\frac{d}{dt}(\vec{r}\cdot\vec{s}) = \frac{d\vec{r}}{dt}\cdot\vec{s} + \vec{r}\cdot\frac{d\vec{s}}{dt} \Rightarrow \int\left(\frac{d\vec{r}}{dt}\cdot\vec{s} + \vec{r}\cdot\frac{d\vec{s}}{dt}\right)dt = \vec{r}\cdot\vec{s} + c$

Here c is a scalar quantity since the integrand is a scalar.

(ii) $\frac{d}{dt}(\vec{r}^2) = 2\vec{r}\cdot\frac{d\vec{r}}{dt} \Rightarrow \int\left(2\vec{r}\cdot\frac{d\vec{r}}{dt}\right)dt = \vec{r}^2 + c$ where c is a scalar quantity.

(iii) $\frac{d}{dt}\left(\vec{r}\times\frac{d\vec{r}}{dt}\right) = \frac{d\vec{r}}{dt}\times\frac{d\vec{r}}{dt} + \vec{r}\times\frac{d^2\vec{r}}{dt^2} = \vec{r}\times\frac{d^2\vec{r}}{dt^2} \Rightarrow \int\left(\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right)dt = \vec{r}\times\frac{d\vec{r}}{dt} + \vec{c}$

Here $\vec{c}$ is a vector quantity since the integrand is a vector.

(iv) If $\vec{a}$ is a constant vector, then $\frac{d}{dt}(\vec{a}\times\vec{r}) = \frac{d\vec{a}}{dt}\times\vec{r} + \vec{a}\times\frac{d\vec{r}}{dt} = \vec{a}\times\frac{d\vec{r}}{dt}$

$\Rightarrow \int\left(\vec{a}\times\frac{d\vec{r}}{dt}\right)dt = \vec{a}\times\vec{r} + \vec{c}$, where $\vec{c}$ is a vector quantity.

ILLUSTRATIVE EXAMPLES

Example 1. *The acceleration of a particle at time t is given by* $\vec{a} = 18\cos 3t\,\hat{i} - 8\sin 2t\,\hat{j} + 6t\,\hat{k}$.

If the velocity $\vec{v}$ *and displacement* $\vec{r}$ *be zero at* $t = 0$, *find* $\vec{v}$ *and* $\vec{r}$ *at any point* t.

Sol. Here $\quad \vec{a} = \dfrac{d^2\vec{r}}{dt^2} = 18\cos 3t\hat{i} - 8\sin 2t\hat{j} + 6t\hat{k}$

Integrating, we have $\vec{v} = \dfrac{d\vec{r}}{dt} = \hat{i}\int 18\cos 3t\,dt + \hat{j}\int -8\sin 2t\,dt + \hat{k}\int 6t\,dt$

$$= 6\sin 3t\hat{i} + 4\cos 2t\hat{j} + 3t^2\hat{k} + \vec{c}$$

At $\quad t = 0, \vec{v} = \vec{0} \quad \Rightarrow \quad \vec{0} = 4\hat{j} + \vec{c} \quad \Rightarrow \quad \vec{c} = -4\hat{j}$

$$\therefore \qquad \vec{v} = \frac{d\vec{r}}{dt} = 6\sin 3t\hat{i} + 4(\cos 2t - 1)\hat{j} + 3t^2\hat{k}$$

Integrating again, we have $\vec{r} = \hat{i}\int 6\sin 3t\,dt + \hat{j}\int 4(\cos 2t - 1)\,dt + \hat{k}\int 3t^2\,dt$

$$= -2\cos 3t\hat{i} + (2\sin 2t - 4t)\hat{j} + t^3\hat{k} + \vec{c}$$

At $\quad t = 0, \vec{r} = \vec{0} \quad \Rightarrow \quad \vec{0} = -2\hat{i} + \vec{c} \quad \Rightarrow \quad \vec{c} = 2\hat{i}$

$$\therefore \qquad \vec{r} = 2(1 - \cos 3t)\hat{i} + 2(\sin 2t - 2t)\hat{j} + t^3\hat{k}.$$

Example 2. *If* $\vec{A}(t) = (3t^2 - 2t)\hat{i} + (6t - 4)\hat{j} + 4t\hat{k}$, *evaluate* $\int_2^3 \vec{A}(t)\,dt$.

Sol. $\displaystyle\int_2^3 \vec{A}(t)\,dt = \int_2^3 [(3t^2 - 2t)\hat{i} + (6t - 4)\hat{j} + 4t\hat{k}]dt$

$$= \hat{i}\int_2^3 (3t^2 - 2t)\,dt + \hat{j}\int_2^3 (6t - 4)\,dt + \hat{k}\int_2^3 4t\,dt$$

$$= \hat{i}\left[t^3 - t^2\right]_2^3 + \hat{j}\left[3t^2 - 4t\right]_2^3 + \hat{k}\left[2t^2\right]_2^3 = 14\hat{i} + 11\hat{j} + 10\hat{k}.$$

Example 3. *If* $\vec{r}(t) = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$, *prove that* $\displaystyle\int_1^2 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt = -14\hat{i} + 75\hat{j} - 15\hat{k}$.

Sol. Since $\dfrac{d}{dt}\left(\vec{r} \times \dfrac{d\vec{r}}{dt}\right) = \dfrac{d\vec{r}}{dt} \times \dfrac{d\vec{r}}{dt} + \vec{r} \times \dfrac{d^2\vec{r}}{dt^2} = \vec{r} \times \dfrac{d^2\vec{r}}{dt^2}$

$$\therefore \qquad \int_1^2 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt = \left[\vec{r} \times \frac{d\vec{r}}{dt}\right]_1^2 \qquad \text{...(1)}$$

Let us now find $\vec{r} \times \dfrac{d\vec{r}}{dt}$.

$$\vec{r} \times \frac{d\vec{r}}{dt} = (5t^2\hat{i} + t\hat{j} - t^3\hat{k}) \times (10t\hat{i} + \hat{j} - 3t^2\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5t^2 & t & -t^3 \\ 10t & 1 & -3t^2 \end{vmatrix} = -2t^3\hat{i} + 5t^4\hat{j} - 5t^2\hat{k}$$

$\therefore$ From (1), we have $\displaystyle\int_1^2 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt = \left[-2t^3\hat{i} + 5t^4\hat{j} - 5t^2\hat{k}\right]_1^2$

$$= \left[-2t^3\right]_1^2 \hat{i} + \left[5t^4\right]_1^2 \hat{j} - \left[-5t^2\right]_1^2 \hat{k} = -14\hat{i} + 75\hat{j} - 15\hat{k}.$$

Example 4. *Given that* $\vec{r}(t) = \begin{cases} 2\hat{i} - \hat{j} + 2\hat{k}, & \textit{when } t = 2 \\ 4\hat{i} - 2\hat{j} + 3\hat{k}, & \textit{when } t = 3 \end{cases}$,

show that $\displaystyle\int_2^3 \left(\vec{r} \cdot \frac{d\vec{r}}{dt}\right) dt = 10.$

Sol. Since $\dfrac{d}{dt}(\vec{r}^2) = 2\vec{r} \cdot \dfrac{d\vec{r}}{dt}$

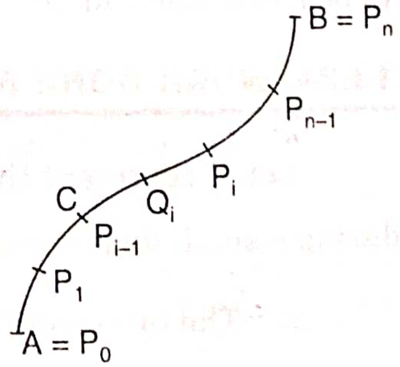
$$\therefore \quad \int_2^3 \left(\vec{r} \cdot \frac{d\vec{r}}{dt}\right) dt = \frac{1}{2}\left[\vec{r}^2\right]_2^3 \qquad \ldots(1)$$

When $t = 3$, $\quad \vec{r}^2 = (4\hat{i} - 2\hat{j} + 3\hat{k}) . (4\hat{i} - 2\hat{j} + 3\hat{k}) = 16 + 4 + 9 = 29$

When $t = 2$, $\quad \vec{r}^2 = (2\hat{i} - \hat{j} + 2\hat{k}) . (2\hat{i} - \hat{j} + 2\hat{k}) = 4 + 1 + 4 = 9$

$\therefore$ From (1), we have $\displaystyle\int_2^3 \left(\vec{r} \cdot \frac{d\vec{r}}{dt}\right) dt = \frac{1}{2}[29 - 9] = 10.$

13.22. LINE INTEGRALS



Any integral which is to be evaluated along a curve is called a line integral.

Let $\vec{F}(P)$ be a continuous vector point function defined at every point of a curve C is space. Divide the curve C into n parts by the points

$$A = P_0, P_1, P_2, \ldots\ldots, P_n = B$$

and let $\vec{R}_0, \vec{R}_1, \vec{R}_2, \ldots\ldots, \vec{R}_n$ be the position vectors of these points. Let Q_i be any point on the arc $P_{i-1} P_i$. Then the limit of the sum

$$\sum_{i=1}^{n} \vec{F}(Q_i) . \delta\vec{R}_i \quad \text{where} \quad \delta\vec{R}_i = \vec{R}_i - \vec{R}_{i-1} \qquad ...(1)$$

as $n \to \infty$ and every $|\delta\vec{R}_i| \to 0$, if it exists, is called a line integral of $\vec{F}$ along C and is denoted by

$$\int_C \vec{F} . d\vec{R} \quad \text{or} \quad \int_C \vec{F} . \frac{d\vec{R}}{dt} dt.$$

Clearly, it is a scalar. It is called the *tangential line integral* of $\vec{F}$ along C.

If the scalar products in (1) are replaced by vector products, then the corresponding line integral is defined as $\int_C \vec{F} \times d\vec{R}$ which is a vector.

If the vector function $\vec{F}$ is replaced by a scalar function ϕ, then the corresponding line integral is defined as $\int_C \phi \, d\vec{R}$, which is a vector.

If $\quad \vec{F}(x, y, z) = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ and $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$ then $d\vec{R} = \hat{i}dx + \hat{j}dy + \hat{k}dz$

$$\therefore \quad \int_C \vec{F} . d\vec{R} = \int_C (f_1 dx + f_2 dy + f_3 dz)$$

If the parametric equations of the curve C are $x = x(t), y = y(t), z = z(t)$ and $t = t_1$ at A, $t = t_2$ at B, then

$$\int_C \vec{F} . d\vec{R} = \int_{t_1}^{t_2} \left(f_1 \frac{dx}{dt} + f_2 \frac{dy}{dt} + f_3 \frac{dz}{dt} \right) dt$$

If C is a closed curve, then the integral sign $\int_C$ is replaced by $\oint_C$.

13.23. CIRCULATION

In fluid dynamics, if $\vec{V}$ represents the velocity of a fluid particle and C is a closed curve, then the integral $\oint_C \vec{V} . d\vec{R}$ is called the *circulation* of $\vec{V}$ around the curve C.

If the circulation of $\vec{V}$ around every closed curve in a region D vanishes, then $\vec{V}$ is said to be *irrotational* in D.

13.24. WORK DONE BY A FORCE

Let $\vec{F}$ represent the force acting on a particle moving along an arc AB. The work done during a small displacement $\delta\vec{R}$ is $\vec{F} . \delta\vec{R}$.

$\therefore$ The total work done by $\vec{F}$ during displacement from A to B is given by $\int_A^B \vec{F} . d\vec{R}$

If the force $\vec{F}$ is conservative, then there exists a scalar function ϕ such that

$$\vec{F} = \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$\therefore$ The work done by $\vec{F}$ during displacement from A to B $= \int_A^B \vec{F} \cdot d\vec{R}$

$$= \int_A^B \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z} \right) \times (\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= \int_A^B \left(\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz \right) = \int_A^B d\phi = \Big[\phi\Big]_A^B = \phi_B - \phi_A$$

Thus, in a conservative field, the work done depends on the value of ϕ at the end points A and B, and not on the path joining A and B.

ILLUSTRATIVE EXAMPLES

Example 1. *If* $\vec{F} = 3xy\hat{i} - y^2\hat{j}$, *evaluate* $\int_C \vec{F} \cdot d\vec{r}$, *where* C *is the arc of the parabola* $y = 2x^2$ *from* $(0, 0)$ *to* $(1, 2)$.

Sol. Since the integration is performed in the xy-plane ($z = 0$), we take

$$\vec{r} = x\hat{i} + y\hat{j} \quad \text{so that } d\vec{r} = dx\hat{i} + dy\hat{j}$$

$$\therefore \quad \vec{F} \cdot d\vec{r} = (3xy\hat{i} - y^2\hat{j}) \cdot (dx\hat{i} + dy\hat{j}) = 3xy\,dx - y^2\,dy$$

On C : $y = 2x^2$ from $(0, 0)$ to $(1, 2)$

$$\vec{F} \cdot d\vec{r} = 3x(2x^2)\,dx - (2x^2)^2\,(4x\,dx) = (6x^3 - 16x^5)\,dx$$

Also, x varies from 0 to 1.

$$\therefore \quad \int_C \vec{F} \cdot d\vec{r} = \int_0^1 (6x^3 - 16x^5)\,dx = \left[\frac{6x^4}{4} - \frac{16x^6}{6} \right]_0^1 = \frac{3}{2} - \frac{8}{3} = -\frac{7}{6}$$

Note that if the curve is traversed in the opposite sense, *i.e.*, from (1, 2) to (0, 0), the value of the integral would be $\frac{7}{6}$.

Second Method

Let $x = t$, then the parametric equations of the parabola $y = 2x^2$ are $x = t$, $y = 2t^2$.

At the point (0, 0), $\quad x = 0$ and so $t = 0$

At the point (1, 2), $\quad x = 1$ and so $t = 1$

If $\vec{r}$ is the position vector of any point (x, y) on C, then $\vec{r} = x\hat{i} + y\hat{j} = t\hat{i} + 2t^2\hat{j}$

Also in terms of t, $\vec{F} = (3t \times 2t^2)\hat{i} - (2t^2)^2\hat{j} = 6t^3\hat{i} - 4t^4\hat{j}$

$$\therefore \quad \int_C \vec{F} \cdot d\vec{r} = \int_C \left(\vec{F} \cdot \frac{d\vec{r}}{dt} \right) dt = \int_0^1 (6t^3\hat{i} - 4t^4\hat{j}) \cdot (\hat{i} + 4t\hat{j})\,dt$$

$$= \int_0^1 (6t^3 - 16t^5)\,dt = \left[\frac{6t^4}{4} - \frac{16t^6}{6} \right]_0^1 = \frac{3}{2} - \frac{8}{3} = -\frac{7}{6}.$$

Example 2. *A vector field is given by $\vec{F} = (\sin y)\hat{i} + x(1 + \cos y)\hat{j}$. Evaluate the line integral over the circular path given by $x^2 + y^2 = a^2$, $z = 0$.* (M.D.U., Dec. 2006 ; K.U.K. 2006)

Sol. The parametric equations of the circular path are $x = a \cos t$, $y = a \sin t$, $z = 0$ where t varies from 0 to 2π.

Since the particle moves in the xy-plane ($z = 0$), we can take $\vec{r} = x\hat{i} + y\hat{j}$ so that $\vec{dr} = dx\hat{i} + dy\hat{j}$

$$\therefore \quad \oint_C \vec{F}\cdot\vec{dr} = \oint_C [\sin y\hat{i} + x(1+\cos y)\hat{j}]\cdot(dx\hat{i} + dy\hat{j})$$

$$= \oint_C [\sin y\, dx + x\,(1+\cos y)\, dy] = \oint_C [(\sin y\, dx + x\cos y\, dy) + x\, dy]$$

$$= \oint_C d\,(x\sin y) + \oint_C x\, dy = \int_0^{2\pi} d\,[a\cos t\sin(a\sin t)] + \int_0^{2\pi} a\cos t\,.\,a\cos t\, dt$$

$$= \Big[a\cos t\sin(a\sin t)\Big]_0^{2\pi} + a^2\int_0^{2\pi}\cos^2 t\, dt$$

$$= \frac{a^2}{2}\int_0^{2\pi}(1+\cos 2t)\, dt = \frac{a^2}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi} = \frac{a^2}{2}(2\pi) = \pi a^2.$$

Example 3. *If $\vec{F} = 2y\hat{i} - z\hat{j} + x\hat{k}$, evaluate $\oint_C \vec{F}\times\vec{dr}$ along the curve $x = \cos t$, $y = \sin t$, $z = 2\cos t$ from $t = 0$ to $t = \frac{\pi}{2}$.*

Sol. $$\vec{F}\times\vec{dr} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2y & -z & x \\ dx & dy & dz \end{vmatrix} = (-z dz - x\, dy)\hat{i} + (x\, dx - 2y\, dz)\hat{j} + (2y\, dy + z\, dx)\hat{k}$$

In terms of t,

$$\vec{F}\times\vec{dr} = [-2\cos t(-2\sin t)\, dt - \cos t(\cos t)\, dt]\hat{i}$$
$$+ [\cos t(-\sin t)\, dt - 2\sin t(-2\sin t)\, dt]\hat{j} + [2\sin t(\cos t)\, dt + 2\cos t(-\sin t)\, dt]\hat{k}$$
$$= [(4\cos t\sin t - \cos^2 t)\hat{i} + (4\sin^2 t - \cos t\sin t)\hat{j}]\, dt$$

$$\therefore \quad \int_C \vec{F}\times\vec{dr} = \int_0^{\pi/2}[(4\cos t\sin t - \cos^2 t)\hat{i} + (4\sin^2 t - \cos t\sin t)\hat{j})\, dt$$

$$= \left(4.\frac{1}{2} - \frac{1}{2}.\frac{\pi}{2}\right)\hat{i} + \left(4.\frac{1}{2}.\frac{\pi}{2} - \frac{1}{2}\right)\hat{j} = \left(2 - \frac{\pi}{4}\right)\hat{i} + \left(\pi - \frac{1}{2}\right)\hat{j}.$$

Example 4. *Compute $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F} = \dfrac{\hat{i}y - \hat{j}x}{x^2 + y^2}$ and C is the circle $x^2 + y^2 = 1$ traversed counter clockwise.*

Sol. For the circle $x^2 + y^2 = 1$ in the xy-plane, we take $\vec{r} = x\hat{i} + y\hat{j}$ so that $d\vec{r} = dx\hat{i} + dy\hat{j}$

$$\int_C \vec{F}.d\vec{r} = \int_C \left(\frac{\hat{i}y - \hat{j}x}{x^2+y^2}\right).(dx\hat{i} + dy\hat{j})$$

$$= \int_C \frac{y\,dx - x\,dy}{x^2+y^2} = \int_C (y\,dx - x\,dy) \qquad [\because \; x^2+y^2=1]$$

Parametric equation of the circle C : $x^2 + y^2 = 1$ is $x = \cos\theta$, $y = \sin\theta$

so that $dx = -\sin\theta d\theta$, $dy = \cos\theta\, d\theta$ and θ varies from 0 to 2π.

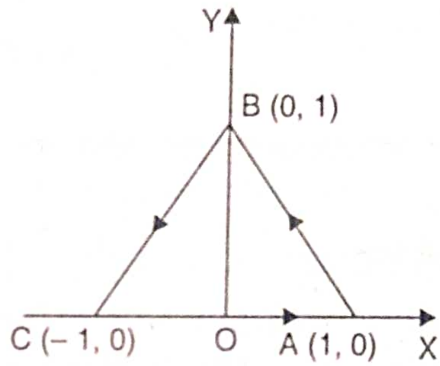
$$\therefore \quad \int_C \vec{F}.d\vec{r} = \int_0^{2\pi} \sin\theta(-\sin\theta\, d\theta) - \cos\theta(\cos\theta\, d\theta)$$

$$= -\int_0^{2\pi} (\sin^2\theta + \cos^2\theta)\, d\theta = -\int_0^{2\pi} d\theta$$

$$= -[\theta]_0^{2\pi} = -2\pi.$$

Example 5. *Compute the line integral* $\int_C (y^2dx - x^2dy)$ *about the triangle whose vertices are* (1, 0), (0, 1) and (– 1, 0).

Sol. Here C is the triangle ABC.



On AB Equation of AB is

$$y - 0 = \frac{1-0}{0-1}(x-1) \quad \text{or} \quad y = 1 - x$$

$\therefore \quad dy = -dx$ and x varies from 1 to 0.

On BC Equation of BC is

$$y - 1 = \frac{0-1}{-1-0}(x-0) \quad \text{or} \quad y = 1 + x$$

$\therefore \quad dy = dx$ and x varies from 0 to – 1.

On CA $y = 0 \quad \therefore \quad dy = 0$ and x varies from – 1 to 1.

$$\therefore \int_C (y^2dx - x^2dy) = \int_{AB}(y^2dx - x^2dy) + \int_{BC}(y^2dx - x^2dy) + \int_{CA}(y^2dx - x^2dy)$$

$$= \int_1^0 [(1-x)^2dx - x^2(-dx)] + \int_0^{-1} [(1+x)^2dx - x^2dx] + \int_{-1}^1 0\,dx$$

$$= \int_1^0 (2x^2 - 2x + 1)dx + \int_0^{-1}(2x+1)dx + 0$$

$$= \left[\frac{2x^3}{3} - \frac{2x^2}{2} + x\right]_1^0 + \left[\frac{2x^2}{2} + x\right]_0^{-1}$$

$$= \left(-\frac{2}{3} + 1 - 1\right) + (1-1) = -\frac{2}{3}$$

Example 6. *Find the work done in moving a particle in the force field* $\vec{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$ *along*

(*i*) *the straight line from (0, 0, 0) to (2, 1, 3)* (S.V.T.U. 2007; P.T.U. 2005)

(*ii*) *the curve defined by* $x^2 = 4y$, $3x^3 = 8z$ *from* $x = 0$ *to* $x = 2$.

Sol. Work done $= \int_C \vec{F}.d\vec{r} = \int_C [3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}].(dx\hat{i} + dy\hat{j} + dz\hat{k})$

$$= \int_C [3x^2dx + (2xz - y)dy + zdz] \qquad \ldots(1)$$

(*i*) Equation of straight line from (0, 0, 0) to (2, 1, 3) are

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0} \quad \text{or} \quad \frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t \quad \text{(say)}$$

$\therefore$ C : $x = 2t, \quad y = t, \quad z = 3t$

so that $dx = 2dt, dy = dt, dz = 3dt$

The points (0, 0, 0) and (2, 1, 3) correspond to $t = 0$ and $t = 1$ respectively.

$\therefore$ From (1), we have

$$\text{Work done} = \int_0^1 [3(2t)^2\, 2dt + \{2(2t)(3t) - t\}\, dt + (3t)\, 3dt]$$

$$= \int_0^1 (24t^2 + 12t^2 - t + 9t)dt = \int_0^1 (36t^2 + 8t)dt$$

$$= \left[12t^3 + 4t^2\right]_0^1 = 16$$

(*ii*) Let $x = t$, then C : $x = t, y = \dfrac{t^2}{4}, z = \dfrac{3t^3}{8}$

so that $dx = dt, dy = \dfrac{t}{2}dt, dz = \dfrac{9t^2}{8}dt$

From $x = 0$ to $x = 2$, t varies from 0 to 2. $\qquad (\because \ t = x)$

$\therefore$ From (1), we have

$$\text{Work done} = \int_0^2 \left[3t^2dt + \left\{2(t)\left(\frac{3t^3}{8}\right) - \frac{t^2}{4}\right\}\left(\frac{t}{2}dt\right) + \frac{3t^3}{8}\left(\frac{9t^2}{8}dt\right)\right]$$

$$= \int_0^2 \left(3t^2 + \frac{3}{8}t^5 - \frac{1}{8}t^3 + \frac{27}{64}t^5\right)dt = \int_0^2 \left(\frac{51}{64}t^5 - \frac{1}{8}t^3 + 3t^2\right)dt$$

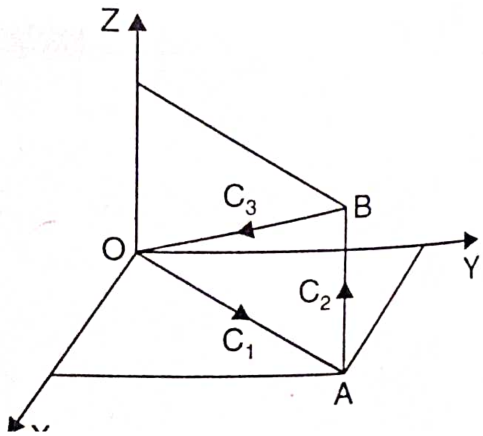
$$= \left[\frac{51}{64}\cdot\frac{t^6}{6} - \frac{t^4}{32} + t^3\right]_0^2 = \frac{17}{2} - \frac{1}{2} + 8 = 16.$$

Example 7. *Find the work done by the force $\vec{F} = x\hat{i} - z\hat{j} + 2y\hat{k}$ in the displacement along the closed path C consisting of the segments C_1, C_2 and C_3 where*

on C_1, $\quad 0 \le x \le 1, \quad y = x, \quad z = 0$

on C_2, $\quad 0 \le z \le 1, \quad x = 1, \quad y = 1$

on C_3, $\quad 1 \ge x \ge 0, \quad y = z = x.$



Sol. Total work done

$$= \oint_C \vec{F}.d\vec{r} = \oint_C (x\hat{i} - z\hat{j} + 2y\hat{k}).(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \oint_C (xdx - zdy + 2ydz)$$

$$= \oint_{C_1} (xdx - zdy + 2ydz) + \oint_{C_2} (xdx - zdy + 2ydz) + \oint_{C_3} (xdx - zdy + 2ydz)$$

$$= W_1 + W_2 + W_3$$

where W_1, W_2, W_3 denote the work done in displacement along C_1, C_2 and C_3 respectively.

On C_1, $\quad 0 \le x \le 1, \quad y = x, \quad z = 0, \quad dy = dx, \quad dz = 0$

$$\therefore \quad W_1 = \int_{C_1} x\,dx = \int_0^1 x\,dx = \frac{1}{2}$$

On C_2, $\quad 0 \le z \le 1, \quad x = 1, \quad y = 1, \quad dx = 0, \quad dy = 0$

$$\therefore \quad W_2 = \int_{C_2} 2dz = 2\int_0^1 dz = 2$$

On C_3, $\quad 1 \ge x \ge 0, \quad y = z = x, \quad dy = dz = dx$

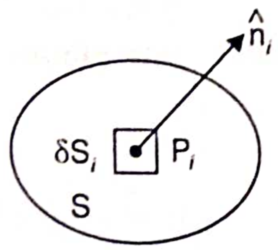
$$\therefore \quad W_3 = \int_{C_3} (x\,dx - x\,dx + 2\,xdx) = 2\int_1^0 x\,dx = -1.$$

$$\text{Total work done} = W_1 + W_2 + W_3 = \frac{1}{2} + 2 - 1 = \frac{3}{2}.$$

13.25. SURFACE INTEGRALS

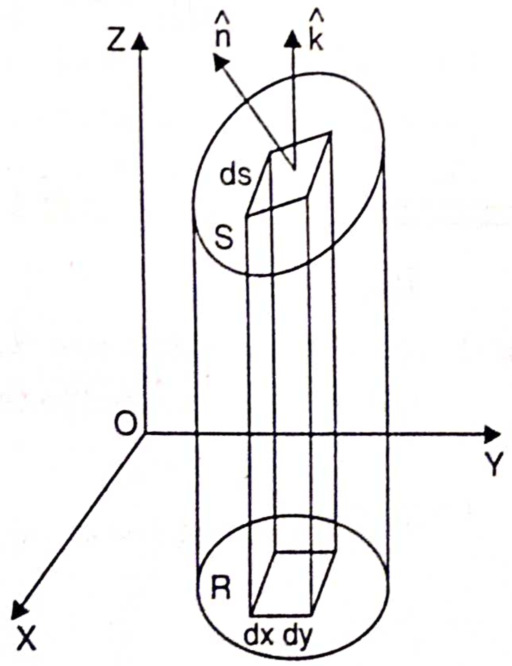
Any integral which is to be evaluated over a surface is called a surface integral.

Let $\vec{F}(P)$ be a continuous vector point function and S a two sided surface. Divide S into a finite number of sub-surfaces $\delta S_1, \delta S_2, \ldots\ldots \delta S_k$. Let P_i be any point in δS_i and $\hat{n}_i$ be the unit vector at P in the direction of outward drawn normal to the surface at P_i. Then the limit of the sum



$\sum_{i=1}^{k} \vec{F}(P_i) \cdot \hat{n}_i \, \delta S_i$, as $k \to \infty$ and each $\delta S_i \to 0$ is called the *normal surface integral* of $\vec{F}(P)$ over S and is denoted by $\iint_S \vec{F} \cdot \hat{n}\, dS$.

The surface element $\delta\vec{S}$ surrounding any point P can be regarded as a vector whose magnitude is area δS and the direction that of the outward drawn normal $\hat{n}$ *i.e.* $\delta\vec{S} = \hat{n}\,\delta S$. The surface integral may alternatively be written as $\iint_S \vec{F} \cdot d\vec{S}$.



If $\vec{F}$ represents the velocity of a fluid at any point P on a closed surface S, then $\vec{F} \cdot \hat{n}$ is the normal component of $\vec{F}$ at P and $\iint_S \vec{F} \cdot \hat{n}\, dS = \iint_S \vec{F} \cdot d\vec{S}$ is a measure of volume emerg-