3..2. ▶ DEFINITE INTEGRAL AS A LIMIT OF A SUM

If $f(x)$ is a continuous and single valued function in the closed interval $[a, b]$, $a < b$, then

$$\lim_{h \to 0} h.[f(a) + f(a+h) + f(a+2h) + \ldots\ldots.. + f(a+(n-1)h)],$$

where $nh = b - a$ is called the **definite integral** *of $f(x)$ between the limits a and b and is written as $\int_a^b f(x)\,dx$.*

2 = (1)

Thus, $\int_a^b f(x)\,dx = \lim_{h \to 0} h.[f(a) + f(a+h) + f(a+2h) + \ldots. + f(a+(n-1)h)],$

where $nh = b - a$.

Remarks.

1. This method of evaluating $\int_a^b f(x)\,dx$ is called the **integral as the limit of a sum** or **integration from definition** or **integration from first principle.**
2. Students should recall to their memory the following results, which we shall be using frequently in solved examples :

(i) $1+2+3+\ldots\ldots+(n-1)=\dfrac{n(n-1)}{2}$

(ii) $1^2+2^2+3^2+\ldots\ldots+(n-1)^2=\dfrac{(n-1)\,n\,(2n-1)}{6}$

(iii) $1^3+2^3+3^3+\ldots\ldots+(n-1)^3=\left[\dfrac{n(n-1)}{2}\right]^2$

(iv) $a+ar+ar^2+\ldots\ldots+ar^{n-1}=\dfrac{a\,(r^n-1)}{(r-1)}$.

SOLVED EXAMPLES

Example 1.

Evaluate the following integrals as limit of a sum :

(i) $\displaystyle\int_0^2 (x+5)\,dx$ (ii) $\displaystyle\int_1^2 (3x-2)\,dx$

Solution. By definition,

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a)+f(a+h)+f(a+2h)+\ldots\ldots+f(a+(n-1)h)]$$

(i) Let $f(x)=x+5$, $a=0$, $b=2$ and $nh=b-a=2-0=2$

$$\therefore \quad \int_0^2 (x+5)\,dx = \lim_{h\to 0} h[f(0)+f(0+h)+f(0+2h)+\ldots\ldots+f(0+(n-1)h)]$$

$$= \lim_{h\to 0} h\,[5+(h+5)+(2h+5)+\ldots\ldots+((n-1)h+5)]$$

$$= \lim_{h\to 0} h\,[(5+5+5+\ldots\ldots+5)+h+2h+3h+\ldots\ldots+(n-1)h]$$

$$= \lim_{h\to 0} h\,[5n+h(1+2+3+\ldots\ldots+(n-1))]$$

$$= \lim_{h\to 0} h\left[5n+h\frac{n(n-1)}{2}\right] = \lim_{h\to 0}\left[5nh+\frac{nh\,(nh-h)}{2}\right]$$

$$= \lim_{h\to 0}\left[5.2+\frac{2(2-h)}{2}\right] = 10+2 = \mathbf{12}. \qquad [\because nh=2]$$

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(*ii*) Let $f(x) = 3x - 2$, $a = 1$, $b = 2$ and $nh = b - a = 2 - 1 = 1$

$$\therefore \int_1^2 (3x-2)\,dx = \lim_{h\to 0} h\,[f(1) + f(1+h) + f(1+2h) + \ldots\ldots + f(1+(n-1)h)]$$

$$= \lim_{h\to 0} h\,[\{3.1-2\} + \{3(1+h)-2\} + \{3(1+2h)-2\} + \ldots\ldots + \{3(1+(n-1)h)-2\}]$$

$$= \lim_{h\to 0} h[1 + (1+3h) + (1+6h) + \ldots\ldots + (1+3(n-1)\,h)]$$

$$= \lim_{h\to 0} h[(1+1+1+\ldots\ldots+1) + (3h+6h+\ldots\ldots+3(n-1)h)]$$

$$= \lim_{h\to 0} h[n + 3h\,\{1+2+\ldots\ldots+(n-1)\}] = \lim_{h\to 0} h\left[n + 3h\,.\,\frac{n(n-1)}{2}\right]$$

$$= \lim_{h\to 0}\left[nh + \frac{3nh(nh-h)}{2}\right] = \left[1 + \frac{3.1.1}{2}\right] = 1 + \frac{3}{2} = \frac{5}{2}. \qquad [\because nh = 1]$$

Example 2.

Evaluate $\int_a^b x^2 dx$ *as a limit of a sum.*

Solution. By definition,

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f(a+(n-1)\,h)]$$

Here $\quad f(x) = x^2, \qquad nh = b - a$

$\therefore \quad f(a) = a^2,\ \ f(a+h) = (a+h)^2, \ldots\ldots, f(a+(n-1)\,h) = (a+(n-1)\,h)^2$

$$\therefore \int_a^b x^2 dx = \lim_{h\to 0} h\,[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f(a+(n-1)\,h)]$$

$$= \lim_{h\to 0} h[a^2 + (a+h)^2 + (a+2h)^2 + \ldots\ldots + (a+(n-1)\,h)^2]$$

$$= \lim_{h\to 0} h[a^2 + (a^2 + 2ah + h^2) + (a^2 + 4ah + 4h^2) + \ldots\ldots$$
$$\ldots\ldots + \{a^2 + 2a(n-1)h + (n-1)^2h^2\}]$$

$$= \lim_{h\to 0} h[a^2 + a^2 + \ldots\ldots \text{ to } n \text{ terms}] + 2ah\,\{1+2+3+\ldots\ldots+(n-1)\}$$
$$+ h^2\{1^2 + 2^2 + 3^2 + \ldots\ldots + (n-1)^2\}]$$

$$= \lim_{h\to 0} h\left[na^2 + 2ah.\frac{n(n-1)}{2} + h^2\,\frac{(n-1)\,n\,(2n-1)}{6}\right]$$

$$= \lim_{h \to 0} \left[nh\, a^2 + anh\,(nh - h) + \frac{1}{6}(nh - h)\, nh\,(2nh - h) \right]$$

$$= \lim_{h \to 0} [(b - a)a^2 + a\,(b - a)\,(b - a - h)$$

$$+ \frac{1}{6}(b - a - h)\,(b - a)\,(2(b - a) - h)] \; [\because nh = b - a]$$

$$= (b - a)\, a^2 + a(b - a)\,(b - a) + \frac{1}{6}(b - a)\,(b - a)\, 2\,(b - a)$$

$$= (b - a)\left[a^2 + ab - a^2 + \frac{1}{3}(b^2 + a^2 - 2ab) \right]$$

$$= \frac{b - a}{3}[3ab + b^2 + a^2 - 2ab]$$

$$= \frac{b - a}{3}[a^2 + b^2 + ab] = \frac{1}{3}(b - a)\,(a^2 + b^2 + ab)$$

$$= \mathbf{\frac{1}{3}(b^3 - a^3).}$$

Example 3.

Find $\int_0^2 (x^2 + 1)\; dx$ *as the limit of a sum.*

Solution. By definition,

$$\int_a^b f(x)\, dx = \lim_{h \to 0} h\,.\,[f(a) + f(a + h) + \ldots.. + f(a + (n - 1)\, h)]$$

Let $f(x) = x^2 + 1$, $a = 0$, $b = 2$ and $nh = b - a = 2 - 0 = 2$

$$\therefore \int_0^2 (x^2 + 1)\, dx = \lim_{h \to 0} h\,[f(0) + f(0 + h) + f(0 + 2h) + \ldots.. + f(0 + (n - 1)\, h)]$$

$$= \lim_{h \to 0} h\,[1 + \{h^2 + 1\} + \{(2h)^2 + 1\} + \ldots.. + \{((n - 1)\, h)^2 + 1\}]$$

$$= \lim_{h \to 0} h\,[(1 + 1 + 1 + \ldots.. + 1) + \{h^2 + (2h)^2 + \ldots.. + ((n - 1)\, h)^2\}]$$

$$= \lim_{h \to 0} h\,[n + h^2\, \{1^2 + 2^2 + \ldots.. + (n - 1)^2\}]$$

$$= \lim_{h \to 0} h\left[n + h^2\,.\,\frac{(n - 1)\, n\,(2n - 1)}{6} \right]$$

$$= \lim_{h \to 0}\left[nh + \frac{(nh - h)\, nh\,(2nh - h)}{6} \right]$$

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$$= \lim_{h \to 0} \left[2 + \frac{(2-h)\, 2\, (2.2 - h)}{6} \right] \qquad [\because nh = 2]$$

$$= 2 + \frac{2.2.4}{6} = 2 + \frac{8}{3} = \frac{\mathbf{14}}{\mathbf{3}}.$$

Example 4.

Evaluate $\int_1^4 (x^2 - x)\, dx$ *as limit of a sum.*

Solution. By definition,

$$\int_a^b f(x)\, dx = \lim_{h \to 0} h\, [f(a) + f(a+h) + f(a+2h) + \ldots.. + f(a + (n-1)\, h)]$$

Let $f(x) = x^2 - x$, $a = 1$, $b = 4$ and $nh = 4 - 1 = 3$

$$\therefore \quad \int_1^4 (x^2 - x)\, dx = \lim_{h \to 0} h\, [f(1) + f(1+h) + f(1+2h) + \ldots.. + f(1 + (n-1)\, h)]$$

$$= \lim_{h \to 0} h\, [0 + \{(1+h)^2 - (1+h)\} + \{(1+2h)^2 - (1+2h)\} + \ldots. + \{(1+(n-1)h)^2 - (1+(n-1)\, h)\}]$$

$$= \lim_{h \to 0} h\, [\{1 + 2h + h^2 - 1 - h\} + \{1 + 4h + (2h)^2 - 1 - 2h\} + \ldots.. \ldots.. + \{1 + 2(n-1)\, h + ((n-1)h)^2 - 1 - (n-1)\, h\}]$$

$$= \lim_{h \to 0} h\, [\{h + h^2\} + \{2h + (2h)^2\} + \ldots.. + \{(n-1)\, h + ((n-1)\, h)^2\}]$$

$$= \lim_{h \to 0} h\, [h\, \{1 + 2 + 3 + \ldots.. + (n-1)\} + h^2 \{1^2 + 2^2 + 3^2 + \ldots.. + (n-1)^2\}]$$

$$= \lim_{h \to 0} h \left[h \cdot \frac{n\,(n-1)}{2} + h^2 \cdot \frac{(n-1)\, n\, (2n-1)}{6} \right]$$

$$= \lim_{h \to 0} \left[\frac{(nh)\,(nh-h)}{2} + \frac{(nh-h)\,(nh)\,(2nh-h)}{6} \right] \qquad [\because nh = 3]$$

$$= \frac{3.3}{2} + \frac{3.3.6}{6} = \frac{9}{2} + 9 = \frac{\mathbf{27}}{\mathbf{2}}.$$