1.4. ▶ MAXIMA AND MINIMA

In our daily life we come across many situations where we wish to find the maximum and minimum values of various functions. In this section, we shall find the maximum and minimum values of different functions using the concept of differentiation.

1.4.1. Maxima

A function $f(x)$ is said to have a maximum value in an interval I at x_0, if x_0 is in I and if $f(x_0) \geq f(x)$ for all x in I. The number $f(x_0)$ is called **maximum value** *of $f(x)$ in I and x_0 is called* **point of maximum** *of $f(x)$ in I.*

From fig. 1.9, it is evident that $x = x_0$ is the point of maximum of $f(x)$ in the interval (a, b). Alternatively, we can also state that a *function $f(x)$ is said to be maximum at $x = x_0$ if the function changes from* **increasing** *to* **decreasing** *in the neighbourhood of x_0.*

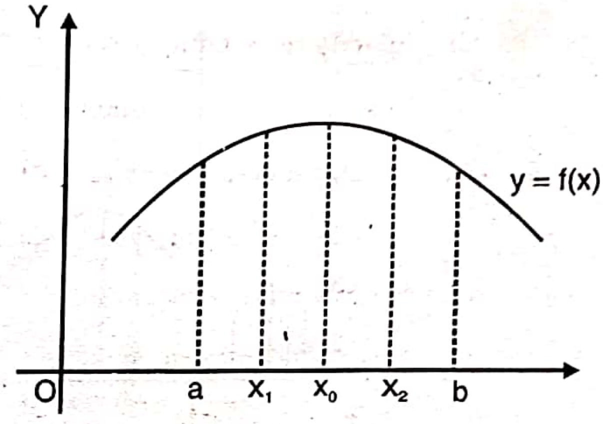

Fig. 1.9

1.4.2. Minima

A function $f(x)$ is said to have a minimum value in an interval I at x_0, if x_0 is in I and if $f(x_0) \leq f(x)$ for all x in I. The number $f(x_0)$ is called the **minimum value** *of $f(x)$ in I and x_0 is called a* **point of minimum** *of $f(x)$ in I.*

From fig. 1.10, it is evident that $x = x_0$ is the point of minimum of $f(x)$ in the interval (a, b).

Alternatively, we can also state that a *function $f(x)$ is said to be minimum at $x = x_0$ if it changes from* **decreasing** *to* **increasing** *in the neighbourhood of x_0.*

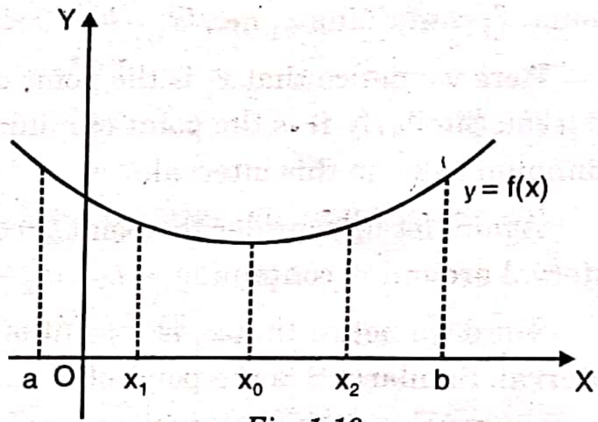

Fig. 1.10

Remark.

The number $f(x_0)$ is called the *extreme value* of f in I and the point x_0 is called an *extreme point or point of extremum.*

1.5. ▶ LOCAL MAXIMA AND LOCAL MINIMA

Let $y = f(x)$ be the given function whose graph is given by PQRS. The points P, Q, R, S are special points on the graph, as the graph takes a turn at each of these points. These points are called the **turning points** of the function. (At these points the tangent is parallel to x-axis and these are also known as **stationary points.**)

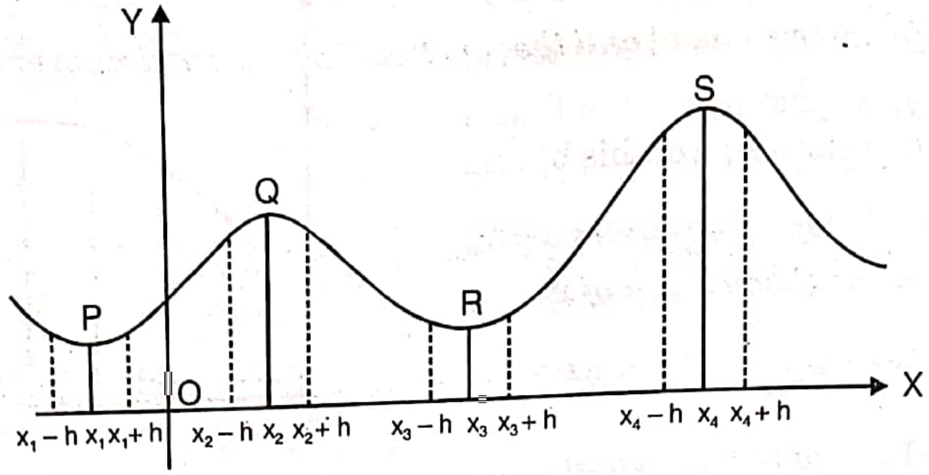

Fig. 1.11

Let us consider the point P corresponding to $x = x_1$. Let $(x_1 - h, x_1 + h)$ be a small interval around x_1 containing x_1 *i.e.*, $(x_1 - h) < x < (x_1 + h)$.

Here we notice that x_1 is the point of minimum and $f(x_1)$ is the minimum value in this interval. Similarly R is the point of minimum in the interval $(x_3 - h, x_3 + h)$ and $f(x_3)$ is the minimum value in this interval.

Again, let us consider the point Q corresponding to $x = x_2$. Let $(x_2 - h, x_2 + h)$ be a small interval around x_2 containing x_2 *i.e.*, $(x_2 - h) < x < (x_2 + h)$.

Here we notice that x_2 is a point of maximum and $f(x_2)$ is the maximum value in this interval. Similarly S is the point of maximum in the interval $(x_4 - h, x_4 + h)$ and $f(x_4)$ is the maximum value in this interval.

Thus we observe that P and R are the points of minimum in the small interval around their corresponding points $x = x_1$ and $x = x_3$ respectively. These points are called **Local Minima.** Similarly, Q and S are the points of maximum in the small interval around their corresponding points $x = x_2$ and $x = x_4$ respectively. These points are called **Local Maxima.**

We now proceed on to define Local Minima and Local Maxima.

(*i*) Local Maxima

Let f be a real function and let x_0 be an interior point in the domain of f. We then say that x_0 is a local maximum of f, if there is an open interval containing x_0 such that $f(x_0) > f(x)$ for every x in that open interval and $f(x_0)$ is called the **local maximum value of f(x).**

(*ii*) Local Minima

Let f be a real function and let x_0 be an interior point in the domain of f. We then say that x_0 is a local minimum of f, if there is an open interval containing x_0 such that $f(x_0) < f(x)$ for every x in that open interval and $f(x_0)$ is called the **local minimum value of f(x).**

Note.

The maximum and minimum values are also called the ***Extremum values.***

1.5.1. Theorem.

If $f'(x)$ exists in the interval $[a, b]$ and the function $f(x)$ has maximum or minimum value at $x = c \in (a, b)$ then $f'(c) = 0$.

(The proof is beyond the scope of this book).

Remark.

The point c in the domain of a function at which $f'(c) = 0$ is called a **critical point.**

1.6. ▶ FIRST DERIVATIVE TEST

Let $f(x)$ be a differentiable function of I and let $x_0 \in I$; then

(1) x_0 is a **point of local maximum** of $f(x)$ if

(i) $f'(x_0) = 0$

(ii) $f'(x) > 0$ at every point close to and to the left of x_0; and $f'(x) < 0$ at every point close to and to the right of x_0.

(2) x_0 **is a point of local minimum** of $f(x)$ if

(i) $f'(x_0) = 0$

(ii) $f'(x) < 0$ at every point close to and the left of x_0; and $f'(x) > 0$ at every point close to and to the right of x_0.

(3) If $f'(x_0) = 0$, but $f'(x)$ does not change sign as x increases through x_0; then x_0 is neither a point of local maximum nor a point of local minimum but is a **point of inflexion.**

1.6.1. Working Rule to Find Local Maximum and Local Minimum Values.

(i) Find $f'(x)$ for the function $f(x)$.

(ii) Put $f'(x) = 0$ and solve this equation and obtain different values of x say a, b, c...

To test the point $x = a$:

(iii) Determine the sign of $f'(x)$ for values of x slightly less than and slightly greater than a. Now :

(a) *If $f'(x)$ changes sign from* **positive to negative,** *then $x = a$ is the* point of local maximum and $f(a)$ is a local maximum value.

(b) *If $f'(x)$ changes sign from* **negative to positive,** *then $x = a$ is the point of local minimum and $f(a)$ is a local minimum value.*

(c) *If $f'(x)$ does not changes its sign, then $x = a$ is a* **point of inflexion.**

Note.

Similar tests hold for the points 'b' and 'c' etc.

Example 1.

Find the point of local maximum or local minimum for the following functions, using the first derivative test. Also find the local maximum and local minimum values.

(i) $x^3 - 3x$ (ii) $\frac{x}{2} + \frac{2}{x}, x > 0.$

Solution. (i) Let $f(x) = x^3 - 3x$

$\therefore \quad f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x - 1)(x + 1)$

Now, $f'(x) = 0 \Rightarrow 3(x - 1)(x + 1) = 0 \Rightarrow$ either $x = 1$ or $x = -1$.

Let us test the nature of the function at the points $x = 1$ and $x = -1$.

At x = 1 :

Let us take $x = 0{\cdot}9$ to the left of $x = 1$ and $x = 1{\cdot}1$ to the right of $x = 1$ and find the signs of $f'(x)$ at these points.

$$f'(0{\cdot}9) = 3(0{\cdot}9 - 1)(0{\cdot}9 + 1) = -\text{ve}$$
$$f'(1{\cdot}1) = 3(1{\cdot}1 - 1)(1{\cdot}1 + 1) = +\text{ve}.$$

Thus $f'(x)$ changes sign from **negative to positive** as x increases through 1 and hence $x = 1$ is a point of **local minimum.**

Local minimum value $= f(1) = (1)^3 - 3(1) = \mathbf{-2}$.

At x = − 1 :

Let us take $x = -1{\cdot}1$ to the left of $x = -1$ and $x = -0{\cdot}9$ to the right of $x = -1$.

$$\therefore \quad f'(-1{\cdot}1) = 3(-1{\cdot}1 - 1)(-1{\cdot}1 + 1) = +\text{ve}$$
$$f'(-0{\cdot}9) = 3(-0{\cdot}9 - 1)(-0{\cdot}9 + 1) = -\text{ve}$$

Thus $f'(x)$ changes sign from **positive to negative** as x increases through -1 and hence $x = -1$ is a point of **local maximum.**

Local maximum value $= f(-1) = (-1)^3 - 3(-1) = \mathbf{2}$.

(ii) Let $$f(x) = \frac{x}{2} + \frac{2}{x} \Rightarrow f'(x) = \frac{1}{2} - \frac{2}{x^2} = \frac{x^2 - 4}{2x^2}$$

Now, $$f'(x) = 0 \Rightarrow \frac{x^2 - 4}{2x^2} = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$$

Since it is given that $x > 0$, hence $x = -2$ is rejected.

Let us test the nature of the function at the point $x = 2$.

At x = 2 :

Let us take $x = 1{\cdot}9$ to the left of $x = 2$ and $x = 2{\cdot}1$ to the right of $x = 2$.

Now, $$f'(1{\cdot}9) = \frac{(1{\cdot}9)^2 - 4}{2(1{\cdot}9)^2} = -\text{ve}$$

$$f'(2{\cdot}1) = \frac{(2{\cdot}1)^2 - 4}{2(2{\cdot}1)^2} = +\text{ve}$$

Thus $f'(x)$ changes sign from **negative to positive** as x increases through 2. Hence $f(x)$ has a **local minimum** at $x = 2$.

Local minimum value $= f(2) = \frac{2}{2} + \frac{2}{2} = 1 + 1 = \mathbf{2}$.

Example 2.

Determine the local maximum and local minimum values for the following functions :

(*i*) $x^3 - 3x^2 - 9x - 7$ (*ii*) $x\sqrt{32 - x^2}$

Solution. (*i*) Let $f(x) = x^3 - 3x^2 - 9x - 7$

$\therefore$ $f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3)$

Now $f'(x) = 0 \Rightarrow 3(x^2 - 2x - 3) = 0 \Rightarrow x^2 - 2x - 3 = 0$

$\Rightarrow (x - 3)(x + 1) = 0 \Rightarrow$ either $x = -1$ or $x = 3$

Let us test the nature of the function at the points $x = -1, 3$.

At $x = -1$:

When x is slightly < -1, $f'(x) = 3(-\text{ve})(-\text{ve}) = +\text{ve}$

and when x is slightly > -1, $f'(x) = 3(-\text{ve})(+\text{ve}) = -\text{ve}$

Thus $f'(x)$ changes sign from **positive to negative** as x increases through -1.

$\therefore f(x)$ has a **local maximum** at $x = -1$ and the local maximum value is

$$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) - 7$$
$$= -1 - 3 + 9 - 7 = \mathbf{-2}.$$

At $x = 3$:

When x is slightly < 3, $f'(x) = 3(-\text{ve})(+\text{ve}) = -\text{ve}$

and when x is slightly > 3, $f'(x) = 3(+\text{ve})(+\text{ve}) = +\text{ve}$

Thus $f'(x)$ changes sign from **negative to positive** as x increases through 3.

$\therefore f(x)$ has a **local minimum** at $x = 3$

Local minimum value $= f(3) = (3)^3 - 3(3)^2 - 9(3) - 7$
$$= 27 - 27 - 27 - 7 = \mathbf{-34}.$$

Note.

By x slightly less than or slightly greater than 3 we mean that x is say 2·9 or 3·1 respectively.

(ii) Let $f(x) = x\sqrt{32-x^2}$

$$\therefore \quad f'(x) = x \cdot \frac{-2x}{2\sqrt{32-x^2}} + \sqrt{32-x^2} \cdot 1$$

$$= \frac{-x^2+32-x^2}{\sqrt{32-x^2}} = \frac{2(16-x^2)}{\sqrt{32-x^2}} = \frac{-2(x-4)(x+4)}{\sqrt{32-x^2}}$$

For local maximum or local minimum, $f'(x) = 0$

Now, $f'(x) = 0$ at $x = 4, -4$

At $x = 4$:

When x is slightly < 4, $f'(x) = \frac{(-)(-)(+)}{(+)} = +\text{ve}$

and when x is slightly > 4, $f'(x) = \frac{(-)(+)(+)}{(+)} = -\text{ve}$

$\therefore$ $f'(x)$ changes sign from **positive to negative** as x increases through 4.

$\Rightarrow$ $f(x)$ has a **local maximum** at $x = 4$.

Local maximum value $= f(4) = 4\sqrt{32-16} = \mathbf{16}$.

At $x = -4$:

When x is slightly < -4, $f'(x) = -\text{ve}$

and when x is slightly > -4, $f'(x) = +\text{ve}$

$\therefore$ $f'(x)$ changes sign from **negative to positive** as x increases through -4.

$\therefore$ $f(x)$ has a **local minimum** at $x = -4$

Local minimum value $= f(-4) = -4\sqrt{32-16} = \mathbf{-16}$.

Example 3.

Examine $y = (x-2)^3(x-3)^2$ for local maximum and local minimum values. Also find the point of inflexion, if any.

Solution. Let $f(x) = y = (x-2)^3(x-3)^2$ ...(1)

$$\therefore \quad \frac{dy}{dx} = (x-2)^3 \cdot 2(x-3) + (x-3)^2 \cdot 3(x-2)^2$$

$$= (x-2)^2(x-3)[2x-4+3x-9] = (x-2)^2(x-3)(5x-13)$$

For local maximum or minimum, $\frac{dy}{dx} = 0$

$$\therefore \quad (x-2)^2(x-3)(5x-13) = 0 \Rightarrow x = 2, 3 \text{ or } \frac{13}{5}.$$

At $x = 2$: When x is slightly < 2, $\frac{dy}{dx} = (+)(-)(-) = +\text{ve}$

When x is slightly > 2, $\frac{dy}{dx} = (+)(-)(-) = +\text{ve}$

Thus $\frac{dy}{dx}$ does not change sign as x increases through 2. Hence $x = 2$ is a **point of inflexion.**

At $x = 3$: When x is slightly < 3, $\frac{dy}{dx} = (+)(-)(+) = -\text{ve}$

When x is slightly > 3, $\frac{dy}{dx} = (+)(+)(+) = +\text{ve}$

Thus $\frac{dy}{dx}$ changes sign from **negative to positive** as x increases through 3.

$\therefore$ $f(x)$ has a **local minimum** at $x = 3$

Local minimum value $= f(3) = (3-2)^3 (3-3)^2 = 0$

At $x = \frac{13}{5}$: When x is slightly $< \frac{13}{5}$, $\frac{dy}{dx} = (+)(-)(-) = +\text{ve}$

When x is slightly $> \frac{13}{5}$, $\frac{dy}{dx} = (+)(-)(+) = -\text{ve}$

Thus $\frac{dy}{dx}$ changes sign from **positive to negative** as x increases through $\frac{13}{5}$.

$\therefore$ $f(x)$ has a **local maximum** at $x = \frac{13}{5}$

Local maximum value $= f\left(\frac{13}{5}\right) = \left(\frac{13}{5} - 2\right)^3 \left(\frac{13}{5} - 3\right)^2 = \frac{27}{125} \times \frac{4}{25} = \mathbf{\frac{108}{3125}}$.

Example 4.

Find the local maximum and local minimum values, if any, of the function

$$y = \frac{x^4}{x-1}, \quad x \neq 1.$$

Solution. Let $f(x) = \frac{x^4}{x-1}$, $x \neq 1$

$$\therefore \quad \frac{dy}{dx} = \frac{(x-1).4x^3 - x^4.1}{(x-1)^2} = \frac{3x^4 - 4x^3}{(x-1)^2} = \frac{x^3(3x-4)}{(x-1)^2}$$

For local maximum or minimum, $\frac{dy}{dx} = 0$

$$\therefore \quad \frac{x^3(3x-4)}{(x-1)^2} = 0 \Rightarrow x^3(3x-4) = 0 \Rightarrow x = 0, \frac{4}{3} \qquad [\because x \neq 1]$$

At $x = 0$: When x is slightly < 0, $\frac{dy}{dx} = \frac{(-)(-)}{(+)} = +\text{ ve}$

When x is slightly > 0, $\frac{dy}{dx} = \frac{(+)(-)}{(+)} = -\text{ ve}$

Thus $\frac{dy}{dx}$ changes sign from + ve to – ve as x increases through 0.

$\therefore$ $f(x)$ has a local maximum at $x = 0$

Local maximum value $= f(0) = \frac{(0)^4}{0-1} = \mathbf{0.}$

At $x = \frac{4}{3}$: When x is slightly $< \frac{4}{3}$, $\frac{dy}{dx} = \frac{(+)(-)}{(+)} = -$ ve

When x is slightly $> \frac{4}{3}$, $\frac{dy}{dx} = \frac{(+)(+)}{(+)} = +$ ve

Thus $\frac{dy}{dx}$ changes sign from – ve to + ve as x increases through $\frac{4}{3}$.

$\therefore$ $f(x)$ has a local minimum at $x = \frac{4}{3}$

Local minimum value $= f\left(\frac{4}{3}\right) = \frac{(4/3)^4}{\left(\frac{4}{3}-1\right)} = \frac{\frac{256}{81}}{\frac{1}{3}} = \mathbf{\frac{256}{27}}$.

Example 5.

Find the local maximum or local minimum values of the constant function α.

Solution. Let $f(x) = \alpha$

$\therefore$ $f'(x) = 0$ for all x. [$\because$ *α is constant*]

Let c be any real number, then $f'(c) = 0$

When x is slightly $< c$, $f'(x) = 0$

When x is slightly $> c$, $f'(x) = 0$

$\therefore$ $f'(x)$ does not changes sign. Thus c is neither a point of local maximum nor a point of local minimum. Hence $f(x)$ has neither local maximum nor local minimum.

Example 6.

Find the local maximum or local minimum value of the function

$$\sin x - \cos x \; ; \; 0 < x < 2\pi.$$

Solution. Let $f(x) = \sin x - \cos x$

$\therefore$ $f'(x) = \cos x + \sin x$

For local maximum or local minimum, $f'(x) = 0$

$\Rightarrow$ $\cos x + \sin x = 0 \Rightarrow \cos x = -\sin x$

$\Rightarrow$ $\tan x = -1 \Rightarrow \tan x = -\tan\frac{\pi}{4}$

$$\Rightarrow \qquad \tan x = \tan\left(\pi - \frac{\pi}{4}\right), \ \tan\left(2\pi - \frac{\pi}{4}\right) \quad \Rightarrow \quad x = \frac{3\pi}{4}, \frac{7\pi}{4}$$

At $\boldsymbol{x = \frac{3\pi}{4}}$ **:**

For values of x slightly less than $\frac{3\pi}{4}$, $f'(x)$ is + ve.

For values of x slightly greater than $\frac{3\pi}{4}$, $f'(x)$ is – ve

Thus $f'(x)$ changes sign from **positive to negative** as x increases through $\frac{3\pi}{4}$

$\therefore f(x)$ has a **local maximum** at $x = \frac{3\pi}{4}$.

Local maximum value $= f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4} - \cos\frac{3\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$.

At $\boldsymbol{x = \frac{7\pi}{4}}$ **:**

For values of x slightly less than $\frac{7\pi}{4}$, $f'(x)$ is – ve.

For values of x slightly greater than $\frac{7\pi}{4}$, $f'(x)$ is + ve

Thus $f'(x)$ changes sign from **negative to positive** as x increases through $\frac{7\pi}{4}$

$\therefore f(x)$ has a **local minimum** at $x = \frac{7\pi}{4}$.

Local minimum value $= f\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{4} - \cos\frac{7\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}$.