1.7. ▶ SECOND DERIVATIVE TEST FOR MAXIMA AND MINIMA

Although the first derivative test is quiet useful in finding the local maximum or local minimum but it is a lengthy process as we have to verify how $f'(x)$ changes sign as x passes through the points given by $f'(x) = 0$. We now give another test known as **second derivative test** which enables us to find the points of local maxima and local minima.

1.7.1. Theorem. (*Second Derivative Test*)

Let $f(x)$ be a differentiable function on I and let $x_0 \in I$. Let $f''(x)$ be continuous at x_0. Then

(i) x_0 is **local maximum** *if $f'(x_0) = 0$ and $f''(x_0) < 0$.*

(ii) x_0 is **local minimum** *if $f'(x_0) = 0$ and $f''(x_0) > 0$.*

Proof. The proof of this theorem is beyond the scope of this book.

Remarks.

1. The second derivative test fails if $f''(x_0) = 0$. In that case we either revert back to the first derivative test or proceed to higher order derivative test which is given in art 1.7.3.
2. If $f''(x_0) = 0$ and x_0 is not a point of local maximum or local minimum, then x_0 is a **point of inflexion**.

1.7.2. Working Rule to find points of local maximum and local minimum by second derivative test.

(1) (*i*) Find $f'(x)$ for the function $y = f(x)$.

(*ii*) Put $f'(x) = 0$ and solve this equation to obtain different values of x, say a, b, c...

To test at $x = a$.

(2) Find $f''(x)$ and determine the sign of $f''(x)$ at $x = a$.

(3) (*i*) If at $x = a$, $f'(a) = 0$ and $f''(a) > 0$, then we conclude that **x = a is a point of local minimum.**

(*ii*) If at $x = a$, $f'(a) = 0$ and $f''(a) < 0$, then we conclude that **x = a is a point of local maximum.**

Similar test holds for the points $x = b$, $x = c$, etc.

1.7.3. Higher Order Derivative Test

Let a be a critical point *i.e.*, $f'(a)$ exists and $f''(a) = 0$.

Suppose $n \geq 2$ is the smallest positive integer such that $f^{(n)}(a) \neq 0$. Then the following table describes the behaviour of the function :

n	*Sign of $f^{(n)}(a)$*	*Nature of the critical point 'a'*
Odd	+ ve or – ve	Neither maximum nor minimum
Even	+ ve	Minimum
Even	– ve	Maximum

The higher order derivative test can also be stated as :

If f be a differentiable function on an interval I and 'a' be an interior point of I such at

(*i*) $f'(a) = f''(a) = f'''(a) = \ldots\ldots = f^{n-1}(a) = 0$ *and*

(*ii*) $f^n(a)$ *exists and is non-zero, then*

$x = a$ *is a point of local maximum if n is even and* $f^n(a) < 0$

$x = a$ *is a point of local minimum if n is even and* $f^n(a) > 0$

$x = a$ *is neither a point of local maximum nor a point of local minimum if n is odd.*

1.7.4. Working rule to find points of local maximum and local minimum by higher derivative test.

1. Find $f'(x)$
2. Put $f'(x) = 0$ and solve this equation to obtain different values of x say $a_1, a_2, \dots, a_n$ which are the stationary values of x and the points where the function can attain a local maximum or a local minimum.

To test at $x = a$:

3. Find $f''(x)$ at $x = a$

   If $f''(a_1) < 0$, then $x = a_1$ is a point of local maximum.

   If $f''(a_1) > 0$, then $x = a_1$ is a point of local minimum.

   If $f''(a_1) = 0$, then we find $f'''(x)$ at $x = a_1$.

   If $f'''(a_1) \neq 0$, then $x = a_1$ is neither a point of local maximum nor a point of local minimum and is called the point of inflexion.

   If $f'''(a_1) = 0$, we find $f^{iv}(x)$ at $x = a_1$

   If $f^{iv}(a_1) < 0$, then $x = a_1$ is a point of local maximum

   If $f^{iv}(a_1) > 0$, then $x = a_1$ is a point of local minimum.

   If $f^{iv}(x) = 0$, we find $f^{v}(x)$ and the above process is repeated.

Similar tests hold for points $x = a_2$, $x = a_3$, etc.

Example 7.

Find the points of local maximum and local minimum of the following functions. Also find the local maximum and local minimum values.

(i) $f(x) = x^3 - 6x^2 + 9x + 15$ (ii) $f(x) = x^4 - 62x^2 + 120x + 9$.

Solution. (i) Here $f(x) = x^3 - 6x^2 + 9x + 15$

$\therefore$ $f'(x) = 3x^2 - 12x + 9$

For local maximum or local minimum, $f'(x) = 0$

$\Rightarrow$ $3x^2 - 12x + 9 = 0 \Rightarrow 3(x^2 - 4x + 3) = 0 \Rightarrow 3(x-1)(x-3) = 0$

$\therefore$ either $x = 1$ or $x = 3$

Now, $f''(x) = 6x - 12$

At $x = 1$: $f''(x) = 6(1) - 12 = -6 < 0 \Rightarrow f(x)$ has a local maximum at $x = 1$.

At $x = 3$: $f''(x) = 6(3) - 12 = 6 > 0 \Rightarrow f(x)$ has local minimum at $x = 3$.

Local maximum value $= f(1) = 1 - 6 + 9 + 15 = \mathbf{19}$.

Local minimum value $= f(3) = 27 - 54 + 27 + 15 = \mathbf{15}$.

(*ii*) Here $f(x) = x^4 - 62x^2 + 120x + 9$

$\therefore \quad f'(x) = 4x^3 - 124x + 120$

For local maximum or local minimum, $f'(x) = 0$

$\Rightarrow \quad 4(x^3 - 31x + 30) = 0 \quad \Rightarrow \quad x^3 - x - 30x + 30 = 0$

$\Rightarrow \quad x(x^2 - 1) - 30(x - 1) = 0 \quad \Rightarrow \quad (x - 1)(x^2 + x - 30) = 0$

$\Rightarrow \quad (x - 1)(x - 5)(x + 6) = 0 \quad \Rightarrow \quad x = 1, 5, -6$

Now, $\quad f''(x) = 12x^2 - 124$

At $x = 1$: $f''(x) = 12 - 124 = -112 < 0, \quad \Rightarrow$ $f(x)$ has a local maximum at $x = 1$.

At $x = 5$: $f''(x) = 12(25) - 124 = 176 > 0 \quad \Rightarrow$ $f(x)$ has a local minimum at $x = 5$.

At $x = -6$: $f''(x) = 12(36) - 124 = 308 > 0 \quad \Rightarrow$ $f(x)$ has a local minimum at $x = -6$

Local maximum value (at $x = 1$) $= f(1) = 1 - 62 + 120 + 9 = \mathbf{68}$.

Local minimum value (at $x = 5$) $= f(5) = (5)^4 - 62(5)^2 + 120(5) + 9 = \mathbf{-316}$.

Local minimum value (at $x = -6$) $= f(-6) = (-6)^4 - 62(-6)^2 + 120(-6) + 9 = \mathbf{-1647}$.

Example 8.

Determine the point where the function $f(x) = \sin x + \cos x$ is maximum in $0 < x < \pi/2$.

Solution. Here $f(x) = \sin x + \cos x$

$\therefore \quad f'(x) = \cos x - \sin x$

For local maximum or local minimum, $f'(x) = 0$

$\Rightarrow \quad \cos x - \sin x = 0 \quad \Rightarrow \quad \tan x = 1 \quad \Rightarrow \quad x = \frac{\pi}{4} \qquad [\because 0 < x < \pi/2]$

Now, $\quad f''(x) = -\sin x - \cos x$

At $x = \frac{\pi}{4}$, $\quad f''(x) = -\sin\frac{\pi}{4} - \cos\frac{\pi}{4} = -\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) < 0$.

Thus $f(x)$ has a local maximum at $x = \frac{\pi}{4}$

Local maximum value $= f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$.

Example 9.

Find the local maximum and local minimum values of the function :

(*i*) $f(x) = \sin 2x$ *in* $0 < x < \pi$. (*ii*) $f(x) = \sin^4 x + \cos^4 x$ *in* $0 < x < \pi/2$.

(*iii*) $f(x) = e^x \sin x$ *in* $0 < x < 2\pi$

Solution. (*i*) $f(x) = \sin 2x$

$\therefore \quad f'(x) = 2\cos 2x$

For local maximum or local minimum $f'(x) = 0$

$$\Rightarrow \quad 2\cos 2x = 0 \Rightarrow \cos 2x = 0 \Rightarrow 2x = \frac{\pi}{2}, \frac{3\pi}{2}$$

$\therefore \quad x = \frac{\pi}{4}, \frac{3\pi}{4}$, both of which lie between 0 and π.

Now, $\quad f''(x) = -4\sin 2x$

At $x = \frac{\pi}{4}$, $f''(x) = -4\sin\frac{\pi}{2} = -4 < 0 \Rightarrow f(x)$ has a local maximum at $x = \frac{\pi}{4}$

$\therefore$ Local maximum value $= f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{2} = \mathbf{1}$.

At $x = \frac{3\pi}{4}$, $f''(x) = -4\sin\frac{3\pi}{2} = 4 > 0 \Rightarrow f(x)$ has a local minimum at $x = \frac{3\pi}{4}$

$\therefore$ Local minimum value $= f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{2} = \mathbf{-1}$.

(*ii*) $\quad f(x) = \sin^4 x + \cos^4 x$

$\therefore \quad f'(x) = 4\sin^3 x\cos x + 4\cos^3 x(-\sin x)$

$$= 4\sin x\cos x(\sin^2 x - \cos^2 x) = 2\sin 2x(-\cos 2x) = -\sin 4x$$

For local maximum or local minimum, $f'(x) = 0$

$$\Rightarrow \quad -\sin 4x = 0 \Rightarrow 4x = 0, \pi, 2\pi \Rightarrow x = 0, \frac{\pi}{4}, \frac{\pi}{2}.$$

But x lies between 0 and $\frac{\pi}{2}$, hence $x = 0$ and $x = \frac{\pi}{2}$ are rejected.

Now, $\quad f''(x) = -4\cos 4x$

At $x = \frac{\pi}{4}$, $\quad f''(x) = -4\cos\pi = 4 > 0$

$\Rightarrow \quad f(x)$ has a local minimum at $x = \frac{\pi}{4}$

$\therefore$ Local minimum value $= f\left(\frac{\pi}{4}\right) = \left(\sin\frac{\pi}{4}\right)^4 + \left(\cos\frac{\pi}{4}\right)^4$

$$= \left(\frac{1}{\sqrt{2}}\right)^4 + \left(\frac{1}{\sqrt{2}}\right)^4 = \frac{1}{4} + \frac{1}{4} = \frac{\mathbf{1}}{\mathbf{2}}.$$

(iii) $f(x) = e^x \sin x; \;\; 0 < x < 2\pi$

$\therefore \quad f'(x) = e^x \cos x + e^x \sin x$

$= e^x (\sin x + \cos x)$

For local maximum or local minimum, $f'(x) = 0$

Now, $\quad f'(x) = 0 \;\; \text{at} \;\; x = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}$ *[As $0 < x < 2\pi$]*

$$f''(x) = e^x (\cos x - \sin x) + e^x (\sin x + \cos x) = 2e^x \cos x$$

At $x = \dfrac{3\pi}{4}$, $\quad f''\left(\dfrac{3\pi}{4}\right) = 2e^{\frac{3\pi}{4}} \cos \dfrac{3\pi}{4} = -\text{ ve}$ *[$\because \cos\theta$ is –ve in second quadrant]*

$\Rightarrow f(x)$ has a local maximum at $x = \dfrac{3\pi}{4}$

$\therefore$ Local maximum value $= f\left(\dfrac{3\pi}{4}\right) = e^{\frac{3\pi}{4}} \sin \dfrac{3\pi}{4} = \dfrac{1}{\sqrt{2}} e^{\frac{3\pi}{4}}$

At $x = \dfrac{7\pi}{4}$, $\quad f''\left(\dfrac{7\pi}{4}\right) = 2e^{\frac{7\pi}{4}} \cos \dfrac{7\pi}{4} = +\text{ve}$ *[$\because \cos\theta$ is +ve in fourth quadrant]*

$\Rightarrow f(x)$ has a local minimum at $x = \dfrac{7\pi}{4}$

$\therefore$ Local minimum value $= f\left(\dfrac{7\pi}{4}\right) = e^{\frac{7\pi}{4}} \sin \dfrac{7\pi}{4} = -\dfrac{1}{\sqrt{2}} e^{\frac{7\pi}{4}}$.

Example 10.

Test for local maxima or local minima, if any, for the function $f(x) = (x - 3)^4$.

Solution. Here $f(x) = (x - 3)^4 \;\Rightarrow\; f'(x) = 4(x - 3)^3$

For local maximum or local minimum, $f'(x) = 0$

$\Rightarrow \quad 4(x - 3)^3 = 0 \;\Rightarrow\; (x - 3)^3 = 0 \;\Rightarrow\; x = 3$

Now, $\quad f''(x) = 12(x - 3)^2$

At $x = 3$**:** $\quad f''(x) = 0.$

Hence the second derivative test fails and thus we revert back to first derivative test.

Let us take $x = 2{\cdot}9$ which is to the left of $x = 3$ and $x = 3{\cdot}1$, which is to the right of $x = 3$ and find the values of $f'(x)$ at these points.

$$f'(2{\cdot}9) = 4(2{\cdot}9 - 3)^3 = -\text{ ve}$$
$$f'(3{\cdot}1) = 4(3{\cdot}1 - 3)^3 = +\text{ ve}$$

Thus $f'(x)$ changes sign from **negative to positive** as x increases through 3.

$\therefore f(x)$ has **local minimum** at $x = 3$.

Second Method. (*Using higher order derivative test*)

At $x = 3$, $f''(x) = 0$, which does not give any inference. So we will find higher order derivatives.

Now, $f'''(x) = 24(x-3)$

At $x = 3$, $f'''(x) = 0$

Again, $f^{iv}(x) = 24$

At $x = 3$, $f^{iv}(x) = 24 > 0$

$\therefore$ $x = 3$ is a point of local **minima** and minimum value of $f(x) = (3-3)^4 = \mathbf{0}$.

Example 11.

Find all the points of local maxima and local minima of the function

$$f(x) = 2x^3 - 6x^2 + 6x + 5.$$

Solution. We have $f(x) = 2x^3 - 6x^2 + 6x + 5$

$\therefore$ $f'(x) = 6x^2 - 12x + 6 = 6(x-1)^2$

and $f''(x) = 12(x-1)$

For local maximum or local minimum, $f'(x) = 0$

$\Rightarrow$ $6(x-1)^2 = 0 \quad \Rightarrow \quad x = 1$

At $x = 1$ **:** $f''(x) = 0$. Thus the second derivative test fails and so we use first derivative test.

We observe that $f'(x) > 0$ for values close to 1 and to the left and to the right of 1 *i.e.*, $f'(x)$ does not change sign as x increases through 1. Hence, by first derivative test, the point $x = 1$ is neither a point of local maxima nor a point of local minima *i.e.*, x is a **point of inflexion.**

Example 12.

If $f(x) = a \log |x| + bx^2 + x$ has extreme values at $x = -1$ and at $x = 2$, then find a and b.

Solution. We have $f(x) = a \log |x| + bx^2 + x$ ...(1)

Clearly, $f(x)$ is not defined at $x = 0$, thus its domain is $R - \{0\}$

Differentiating (1) w.r.t. x, we have

$$f'(x) = \frac{a}{x} + 2bx + 1$$

It is given that $f(x)$ has extreme values at $x = -1$ and $x = 2$ *i.e.*, $f'(x) = 0$ at $x = -1$ and $x = 2$.

Now, $f'(-1) = 0 \quad \Rightarrow \quad -a - 2b + 1 = 0 \quad \Rightarrow \quad a + 2b = 1$ ...(2)

and $f'(2) = 0 \quad \Rightarrow \quad \frac{a}{2} + 4b + 1 = 0 \quad \Rightarrow \quad a + 8b = -2$ ...(3)

Subtracting (2) from (3), we get

$$6b = -3 \quad \Rightarrow \quad b = -\frac{1}{2}$$

Then from (2), $\quad a + 2\left(-\frac{1}{2}\right) = 1 \quad \Rightarrow \quad a = 2$

Hence, $\quad \boldsymbol{a = 2, \quad b = -\frac{1}{2}}$.

Example 13.

Examine whether the function $x^{1/x}$ $(x > 0)$ possesses a maximum or a minimum. If yes, then determine it.

Solution. Let $\quad y = x^{1/x}, \quad x > 0$

Taking logarithm of both sides, we have

$$\log y = \frac{1}{x} \log x$$

Differentiating w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} \cdot \frac{1}{x} + \log x \cdot \left(-\frac{1}{x^2}\right)$$

$$= \frac{1 - \log x}{x^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = y\left(\frac{1 - \log x}{x^2}\right) \quad \ldots(1)$$

For maximum or minimum values, $\frac{dy}{dx} = 0$

$\Rightarrow \quad 1 - \log_e x = 0 \quad \Rightarrow \quad \log_e x = 1 \quad \Rightarrow \quad x = e \qquad [\because y = x^{1/x} > 0 \text{ as } x > 0]$

Differentiating (1) w.r.t. x, we have

$$\frac{d^2y}{dx^2} = y\left[\frac{x^2\left(-\frac{1}{x}\right) - (1 - \log x)(2x)}{x^4}\right] - \left(\frac{1 - \log x}{x^2}\right) \cdot \frac{dy}{dx}$$

$$= y\left[\frac{-3 + 2\log x}{x^3}\right] - \left(\frac{1 - \log x}{x^2}\right)\frac{dy}{dx}$$

At $x = e$: $\quad \frac{d^2y}{dx^2} = x^{1/x}\left[\frac{-3 + 2\log x}{x^3}\right]_{x=e} - 0 \qquad \left[\because \frac{dy}{dx} = 0 \text{ at } x = e\right]$

$$= e^{1/e}\left[\frac{-3+2\log e}{e^3}\right] = \frac{-e^{1/e}}{e^3} = -\text{ve} \qquad [\because \log e = 1 \text{ and } 2 < e < 3]$$

$\therefore$ $x = e$ is a point of local maxima and local maximum value $= e^{1/e}$.