Example 1. *Show that*

(i) $\lim\limits_{x \to \infty} \frac{1}{x} = 0$ (ii) $\lim\limits_{x \to 0} \frac{1}{|x|} = \infty$ (iii) $\lim\limits_{x \to \infty} \frac{x-1}{x+1} = 1$

Solution. (*i*) Here $x \to \infty$. As x assumes larger and larger values, $\frac{1}{x}$ becomes smaller and smaller and comes very close to zero.

$$\therefore \qquad \lim_{x \to \infty} \frac{1}{x} = \mathbf{0}.$$

(*ii*) Here $x \to 0$. As x becomes very small in magnitude, $|x|$ is positive and is also very small.

$\therefore$ $\frac{1}{|x|}$ is positive and becomes very large *i.e.*, as $x \to 0$, $\frac{1}{|x|} \to \infty$.

$$\therefore \qquad \lim_{x \to 0} \frac{1}{|x|} = \infty.$$

(*iii*) $$\lim_{x \to \infty} \frac{x-1}{x+1} = \lim_{x \to \infty} \frac{1-\frac{1}{x}}{1+\frac{1}{x}} = \frac{1-0}{1+0} = \mathbf{1} \qquad \left[\because \frac{1}{x} \to 0 \text{ as } x \to \infty\right]$$

Example 2. *Evaluate* :

(i) $\lim\limits_{x \to \infty} \left(\frac{2x^2 - 5x + 7}{3x^2 + 2x - 5}\right)$ (ii) $\lim\limits_{x \to \infty} \frac{(2x-3)(3x+5)(4x-6)}{3x^3 + x - 1}$.

Solution. (*i*) Dividing the numerator and denominator by highest degree of x *i.e.*, x^2, we have

$$\lim_{x \to \infty} \left(\frac{2x^2 - 5x + 7}{3x^2 + 2x - 5}\right) = \lim_{x \to \infty} \left(\frac{2 - \frac{5}{x} + \frac{7}{x^2}}{3 + \frac{2}{x} - \frac{5}{x^2}}\right) = \frac{2-0+0}{3+0-0} = \frac{\mathbf{2}}{\mathbf{3}}.$$

$$\left[\because \text{As } x \to \infty; \frac{1}{x}, \frac{1}{x^2} \to 0\right]$$

(*ii*) **First Method :** Dividing the numerator and denominator by x^3 (the highest degree term in the numerator and denominator), we have

$$\lim_{x\to\infty} \frac{(2x-3)(3x+5)(4x-6)}{3x^3+x-1} = \lim_{x\to\infty} \frac{\left(2-\frac{3}{x}\right)\left(3+\frac{5}{x}\right)\left(4-\frac{6}{x}\right)}{3+\frac{1}{x^2}-\frac{1}{x^3}}$$

$$= \frac{2.3.4}{3} = \mathbf{8}. \qquad \left[\because \text{ As } x\to\infty;\ \frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3} \to 0\right]$$

Second Method : Put $x = \frac{1}{y}$. Now as $x \to \infty, y \to 0$.

Then the given limit becomes

$$\lim_{y\to 0} \frac{\left(\frac{2}{y}-3\right)\left(\frac{3}{y}+5\right)\left(\frac{4}{y}-6\right)}{\frac{3}{y^3}+\frac{1}{y}-1} = \lim_{y\to 0} \frac{(2-3y)(3+5y)(4-6y)}{(3+y^2-y^3)} \qquad [\textit{Simplifying}]$$

$$= \frac{(2)(3)(4)}{3} = \mathbf{8}. \qquad \left[\textit{Not of } \frac{0}{0} \textit{ form}; \therefore \textit{ put } y=0\right]$$

Example 3. *Evaluate the following limits :*

(*i*) $\displaystyle\lim_{x\to\infty} \frac{x^2-5}{\sqrt{3x^6+4x^2+2}}$ (*ii*) $\displaystyle\lim_{x\to\infty} \frac{\sqrt{3x^2-1}-\sqrt{2x^2-1}}{4x+3}$

(*iii*) $\displaystyle\lim_{x\to-\infty} \sqrt{4x^2+7x}+2x$

Solution. (*i*) Dividing the numerator and denominator by highest degree of x *i.e.*, x^3, we have

$$\lim_{x\to\infty} \frac{x^2-5}{\sqrt{3x^6+4x^2+2}} = \lim_{x\to\infty} \left(\frac{\frac{x^2}{x^3}-\frac{5}{x^3}}{\frac{\sqrt{3x^6+4x^2+2}}{x^3}}\right) = \lim_{x\to\infty} \left(\frac{\frac{1}{x}-\frac{5}{x^3}}{\sqrt{\frac{3x^6}{x^6}+\frac{4x^2}{x^6}+\frac{2}{x^6}}}\right)$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}-\frac{5}{x^3}}{\sqrt{3+\frac{4}{x^4}+\frac{2}{x^6}}} = \frac{0-0}{\sqrt{3+0+0}} = \frac{0}{\sqrt{3}} = \mathbf{0}.$$

(*ii*) Dividing the numerator and denominator by x, we have

$$\lim_{x\to\infty} \frac{\sqrt{3x^2-1}-\sqrt{2x^2-1}}{4x+3} = \lim_{x\to\infty}\left(\frac{\dfrac{\sqrt{3x^2-1}-\sqrt{2x^2-1}}{x}}{\dfrac{4x+3}{x}}\right)$$

$$= \lim_{x\to\infty}\left(\frac{\sqrt{\dfrac{3x^2-1}{x^2}}-\sqrt{\dfrac{2x^2-1}{x^2}}}{4+\dfrac{3}{x}}\right) = \lim_{x\to\infty}\frac{\sqrt{3-\dfrac{1}{x^2}}-\sqrt{2-\dfrac{1}{x^2}}}{4+\dfrac{3}{x}}$$

$$= \frac{\sqrt{3}-\sqrt{2}}{4}.$$

(*iii*) Put $y=-x$. Now as $x\to-\infty$, $y\to\infty$

$$\therefore \quad \lim_{x\to-\infty}(\sqrt{4x^2+7x}+2x) = \lim_{y\to\infty}(\sqrt{4y^2-7y}-2y)\times\frac{\sqrt{4y^2-7y}+2y}{\sqrt{4y^2-7y}+2y}$$

$$= \lim_{y\to\infty}\frac{4y^2-7y-4y^2}{\sqrt{4y^2-7y}+2y} = \lim_{y\to\infty}\frac{-7y}{y\left[\sqrt{4-\dfrac{7}{y}}+2\right]}$$

$$= \lim_{y\to\infty}\frac{-7}{\sqrt{4-\dfrac{7}{y}}+2} = \frac{-7}{\sqrt{4-0}+2} = \frac{-7}{2+2} = -\frac{7}{4}.$$

Example 4. *Evaluate* $\lim\limits_{n\to\infty}\dfrac{\Sigma n^3}{2n^4}$.

Solution. $$\lim_{n\to\infty}\frac{\Sigma n^3}{2n^4} = \lim_{n\to\infty}\frac{\left[\dfrac{n(n+1)}{2}\right]^2}{2n^4} \qquad \left[\because \Sigma n^3 = \left(\frac{n(n+1)}{2}\right)^2\right]$$

$$= \lim_{n\to\infty}\frac{n^2(n+1)^2}{8n^4} = \lim_{n\to\infty}\frac{(n+1)^2}{8n^2} = \lim_{n\to\infty}\frac{\left(1+\dfrac{1}{n}\right)^2}{8}$$

[*Dividing the numerator and denominator by* n^2]

$$= \frac{1}{8}. \qquad \left[\because \frac{1}{n}\to 0 \text{ as } n\to\infty\right]$$

Example 5. *Prove that* $\lim\limits_{x\to\infty}(\sqrt{x^2+x+1}-x) \neq \lim\limits_{x\to\infty}(\sqrt{x^2+1}-x)$.

Solution. L.H.S. $= \lim_{x \to \infty} (\sqrt{x^2+x+1} - x) \times \frac{(\sqrt{x^2+x+1}+x)}{(\sqrt{x^2+x+1}+x)}$ [*Rationalising*]

$$= \lim_{x \to \infty} \frac{(x^2+x+1) - x^2}{\sqrt{x^2+x+1}+x} = \lim_{x \to \infty} \frac{x+1}{\sqrt{x^2+x+1}+x}$$

$$= \lim_{x \to \infty} \frac{x\left(1+\frac{1}{x}\right)}{x\left[\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+1\right]} = \frac{1}{2} \qquad \left[\because \textit{As } x \to \infty,\ \frac{1}{x} \to 0\right]$$

R.H.S. $= \lim_{x \to \infty} (\sqrt{x^2+1} - x) \times \left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}\right)$ [*Rationalising*]

$$= \lim_{x \to \infty} \frac{(x^2+1) - x^2}{\sqrt{x^2+1}+x} = \lim_{x \to \infty} \frac{1}{\sqrt{x^2+1}+x} = \lim_{x \to \infty} \frac{1}{x\left(\sqrt{1+\frac{1}{x^2}}+1\right)} = 0$$

Hence, $\lim_{x \to \infty} (\sqrt{x^2+x+1} - x) \neq \lim_{x \to \infty} (\sqrt{x^2+1} - x)$.