3.6. ▶ DEFINITE INTEGRAL AS AREA UNDER A CURVE

Theorem. ***If $f(x)$ be a continuous non-negative function in $a \le x \le b$, then the area bounded by the curve $y = f(x)$, the x-axis and ordinates $x = a$ and $x = b$ is given by the definite integral $\int_a^b f(x)\, dx$.***

Proof. Let AB be an arc on the given curve $y = f(x)$ and DA and CB be the ordinates, $x = a$ and $x = b$ respectively. Let $P(x, y)$ be any point on the curve and consider a point $Q(x + \delta x, y + \delta y)$ near to $P(x, y)$ on the given curve. Draw PE and QF perpendiculars on x-axis. Complete the rectangle EFQS. Also, draw PR parallel to x-axis.

Let us denote the area DEPA and DFQA by $A(x)$ and $A(x + \delta x)$ respectively.

Then, area (EFQP) = $A(x + \delta x) - A(x)$.

Now, area (rect. PEFR) < area (EFQP) < area (rect. EFQS)

i.e., $$y \cdot \delta x < A(x + \delta x) - A(x) < (y + \delta y) \cdot \delta x$$

i.e., $$y < \frac{A(x + \delta x) - A(x)}{\delta x} < (y + \delta y)$$

i.e., $$\lim_{\delta x \to 0} y < \lim_{\delta x \to 0} \frac{A(x + \delta x) - A(x)}{\delta x} < \lim_{\delta y \to 0} (y + \delta y)$$

i.e., $$y < \frac{dA}{dx} < y \quad \Rightarrow \quad \frac{dA}{dx} = y = f(x)$$

$$\therefore \int_a^b f(x)\, dx = \int_a^b \frac{dA}{dx}\, dx = [A]_a^b = [A]_{x=b} - [A]_{x=a}$$

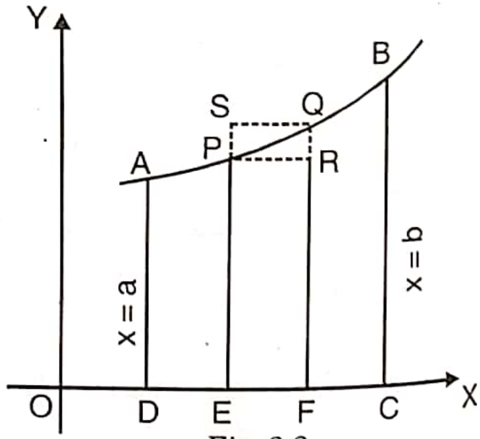

Fig. 3.2

Now from fig. 3.2. it is clear that when $x = a$, A = 0 when $x = b$, A = area ABCD

$$\therefore \quad \int_a^b f(x)\,dx = \text{area ABCD}$$

Thus the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$ is $\int_a^b f(x)\,dx$.

Remarks.

1. Area lying below the x-axis.

If $f(x) \le 0$ for $a \le x \le b$, then the graph of $y = f(x)$ lies below the x-axis.

$\therefore$ Area bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$ and $x = b$ is given by

$$\text{Area ABCD} = \int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx.$$

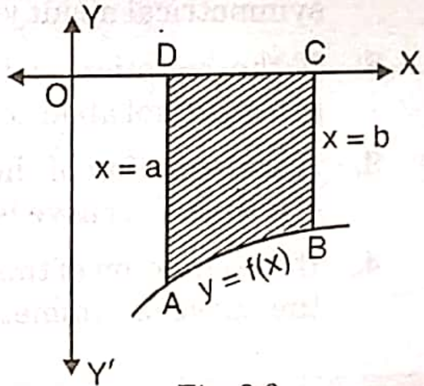

Fig. 3.3

2. Area lying above as well as below the x-axis.

If $f(x) \ge 0$ for $a \le x \le c$ and $f(x) \le 0$ for $c \le x \le b$, then the area bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$ and $x = b$ is equal to

$$\int_a^c f(x)\,dx + \int_c^b -f(x)\,dx$$

i.e.,
$$\int_a^c f(x)\,dx - \int_c^b f(x)\,dx$$

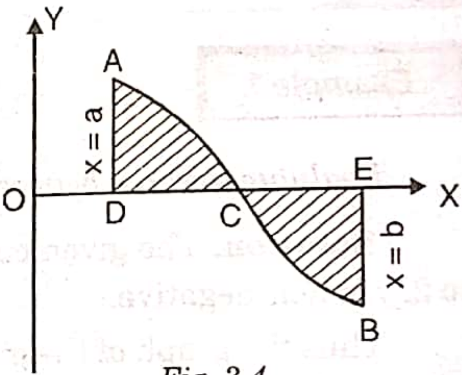

Fig. 3.4

3. The area bounded by the curve $x = f(y)$, the y-axis and the abscissae $y = c$ and $y = d$ is equal to

$$\int_c^d x\,dy \qquad \text{i.e.,} \qquad \int_c^d f(y)\,dy$$

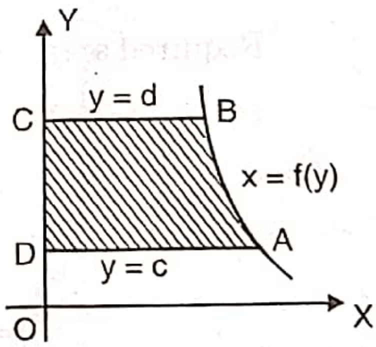

Fig. 3.5

4. If the curve $x = f(y)$ lies to the left of y-axis, then the area bounded by the curve $x = f(y)$, the y-axis and the abscissae $y = c$ and $y = d$ is equal to

$$\int_c^d (-x)\,dy \qquad \text{i.e.,} \qquad \int_c^d -f(y)\,dy.$$

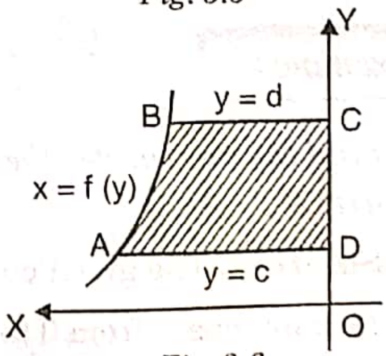

Fig. 3.6

Note on Rough Sketch :

To find the area under the curve, we first draw a rough sketch of the area which gives us an idea as to whether the region lies above or below the x-axis. To draw the rough sketch, some points (x, y) satisfying the equation of the curve are found. Then these points are plotted on graph and joined with free hand which gives the rough sketch of the graph.

Note on Symmetry of the Curve :

Following points are to be noted for the symmetry of the curve :

1. If the equation of the curve contains only even powers of x, then the curve is symmetrical about y-axis.
2. If the equation of the curve contains only even powers of y, then the curve is symmetrical about x-axis.
3. If the equation of the curve remains unchanged when x is replaced by $-x$ and y by $-y$, then the curve is symmetrical in opposite quadrants.
4. If the equation of the curve remains unchanged when x and y are interchanged, then the curve is symmetrical about the line $y = x$.

SOLVED EXAMPLES

Example 1.

Evaluate the area between the curve $y = x^2$, x-axis and the lines $x = 0$ and $x = 2$.

Solution. The given curve is $y = x^2$. Now for $0 \le x \le 2$, $y > 0$ *i.e.*, as x increases from 0 to 2, y is non-negative.

Thus the graph of the given curve lies above the x-axis.

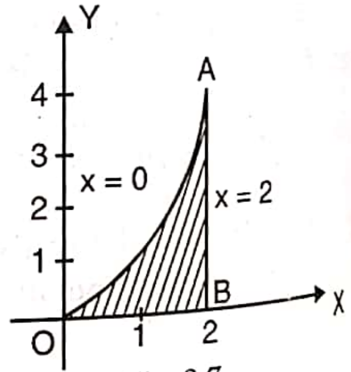

Fig. 3.7

$$\therefore \quad \text{Required area} = \int_0^2 y\, dx$$

$$= \int_0^2 x^2\, dx = \left[\frac{x^3}{3}\right]_0^2$$

$$= \left(\frac{8}{3} - 0\right) = \frac{8}{3} \textbf{ sq. units.}$$

Example 2.

Find the area under the curve, $y = (x^2 + 2)^2 + 2x$, between the ordinates $x = 0$ and $x = 2$ and the x-axis.

Solution. The given curve is $\quad y = (x^2 + 2)^2 + 2x$

As x increases from 0 to 2, the value of y is non-negative.

Thus the given curve lies above the x-axis.

$\therefore$ Required area $= \int_0^2 y\,dx = \int_0^2 [(x^2+2)^2 + 2x]\,dx$

$$= \int_0^2 (x^4 + 4x^2 + 4 + 2x)\,dx = \left[\frac{x^5}{5} + 4.\frac{x^3}{3} + 4x + 2.\frac{x^2}{2}\right]_0^2$$

$$= \left[\frac{1}{5}x^5 + \frac{4}{3}x^3 + x^2 + 4x\right]_0^2$$

$$= \left[\left(\frac{1}{5}(2)^5 + \frac{4}{3}(2)^3 + (2)^2 + 4(2)\right) - 0\right]$$

$$= \frac{32}{5} + \frac{32}{3} + 4 + 8 = \frac{436}{15} \text{ sq. units.}$$

Example 3.

Using integration, find the area of the region bounded by the line $2y = 5x + 7$, x-axis and the lines $x = 2$ and $x = 8$.

Solution. Here $2y = 5x + 7$

$\Rightarrow$ $y = \frac{5}{2}x + \frac{7}{2}$, which represents a straight line.

Now as x increases from 2 to 8, the value of y is non-negative.

Thus the graph of the curve lies above the x-axis.

The following table gives some values of x and y satisfying the given equation :

x :	2	4	8
y :	8.5	13.5	23.5

Plotting these points and joining them with a free hand, we get the graph of the given line as shown in fig. 3.8. The shaded portion in the figure is the required area.

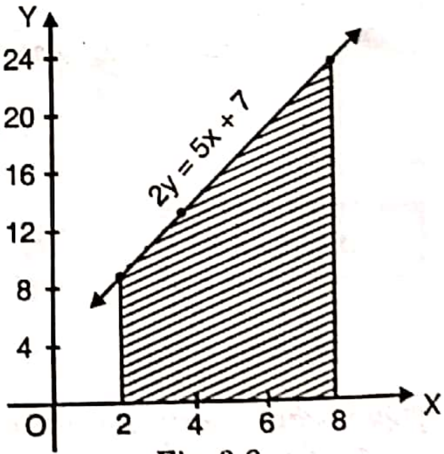

Fig. 3.8

$\therefore$ Required area $= \int_2^8 y\,dx = \int_2^8 \left(\frac{5}{2}x + \frac{7}{2}\right) dx$

$$= \left[\frac{5}{4}.x^2 + \frac{7}{2}x\right]_2^8 = \left[\left(\frac{5}{4}\times 64 + \frac{7}{2}\times 8\right) - \left(\frac{5}{4}\times 4 + \frac{7}{2}\times 2\right)\right]$$

$$= [(80 + 28) - (5 + 7)] = 108 - 12 = \mathbf{96 \text{ sq. units.}}$$

Example 4.

Draw a rough sketch of the curve $y = x^2 - 9$ and find the area bounded by the curve, x-axis and the lines $x = 0$, $x = 5$. [F...

Solution. Here $y = x^2 - 9 = (x + 3)(x - 3)$

$\therefore$ For $0 \le x \le 3$, $y \le 0$ *i.e.*, the graph of the given curve lies below the x-axis.

For $3 \le x \le 5$, $y \ge 0$ *i.e.*, the graph of the given curve lies above the x-axis.

The following table gives some values of x and y satisfying the given equation:

x :	0	1	2	3	4	5
y :	-9	-8	-5	0	7	16

Plotting these points and joining them with a free hand, we get the rough sketch as shown in fig. 3.9. The shaded portion in the figure is the required area.

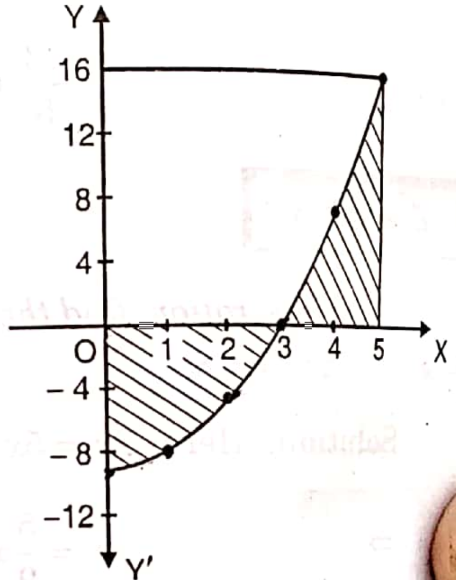

Fig. 3.9

$$\therefore \text{ Required area } = \int_0^3 -y\,dx + \int_3^5 y\,dx$$

$$= \int_0^3 -(x^2-9)\,dx + \int_3^5 (x^2-9)\,dx$$

$$= \int_0^3 (9-x^2)\,dx + \int_3^5 (x^2-9)\,dx = \left[9x - \frac{x^3}{3}\right]_0^3 + \left[\frac{x^3}{3} - 9x\right]_3^5$$

$$= [(27-9)-(0-0)] + \left[\left(\frac{125}{3} - 45\right) - (9-27)\right]$$

$$= 18 + \frac{125}{3} - 45 + 18 = \frac{125}{3} - 9 = \frac{98}{3} \textbf{ sq. units.}$$

Example 5.

Make a rough sketch of the graph of the function $y = \sin^2 x$, $0 \le x \le \frac{\pi}{2}$ and find the area enclosed between the curve and the x-axis. [HB 20..

Solution. The given curve is $y = \sin^2 x$

Now we know that $\sin x$ is always positive for $0 \le x \le \frac{\pi}{2}$.

Thus the curve will lie above the x-axis for $0 \le x \le \frac{\pi}{2}$.

The following table gives some values of x and y satisfying the given equation :

x :	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
y :	0	0.25	0.5	0.75	1

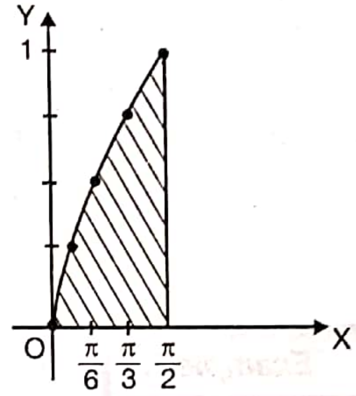

Fig. 3.10

Plotting these points and joining them with a free hand, we get the rough sketch as shown in fig. 3.10. The shaded portion in the figure is the required area.

$$\therefore \text{ Required area } = \int_0^{\pi/2} y\,dx = \int_0^{\pi/2} \sin^2 x\,dx$$

$$= \int_0^{\pi/2} \left(\frac{1-\cos 2x}{2}\right) dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} - \frac{\sin \pi}{2}\right) - (0-0)\right] = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4} \textbf{ sq. units.}$$

Example 6.

Draw a rough sketch of the function $y = 2\sqrt{1-x^2}$; $x \in [0, 1]$ and evaluate the area enclosed between the curve and the x-axis.

Solution. The given function is $y = 2\sqrt{1-x^2}$, which is non-negative for $x \in [0, 1]$.

Thus the curve lies above the x-axis.

The following table gives some values of x and y satisfying the given equation :

x :	0	0.5	1
y :	2	1.72	0

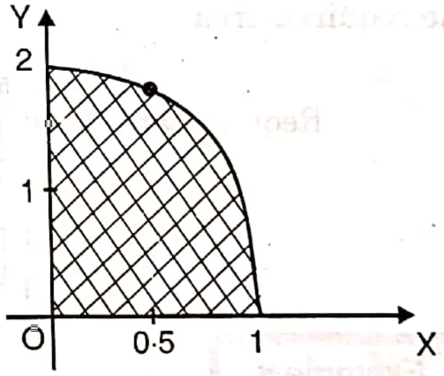

Fig. 3.11

Plotting these points and joining them with a free hand, we get the rough sketch as shown in fig. 3.11. The shaded portion in the figure is the required area.

$$\therefore \text{ Required area } = \int_0^1 y\,dx = \int_0^1 2\sqrt{1-x^2}\,dx$$

Put $x = \sin\theta$ so that $dx = \cos\theta\,d\theta$

Now when $x = 0, \theta = 0$ and when $x = 1, \theta = \frac{\pi}{2}$

$$\therefore \text{ Required area } = \int_0^{\pi/2} 2\sqrt{1-\sin^2\theta}\,.\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} 2\cos^2\theta\, d\theta = \int_0^{\pi/2} (1 + \cos 2\theta)\, d\theta$$

$$= \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \left[\left(\frac{\pi}{2} + \frac{\sin \pi}{2}\right) - \left(0 + \frac{\sin 0}{2}\right)\right]$$

$$= \left(\frac{\pi}{2} + 0\right) - (0 + 0) = \frac{\pi}{2} \textbf{ sq. units.}$$

Example 7.

Find the area bounded by the curve $y = x^2 - 4$ and the lines $y = 0$ and $y = 5$.

Solution. The given curve is

$$y = x^2 - 4 \Rightarrow x^2 = y + 4$$

Since the curve contains only even powers of x, thus it is symmetrical about y-axis.

The following table gives some values of x and y satisfying the given equation:

y:	0	2	3	4	5
x:	± 2	$\pm\sqrt{6}$	$\pm\sqrt{7}$	$\pm\sqrt{8}$	± 3

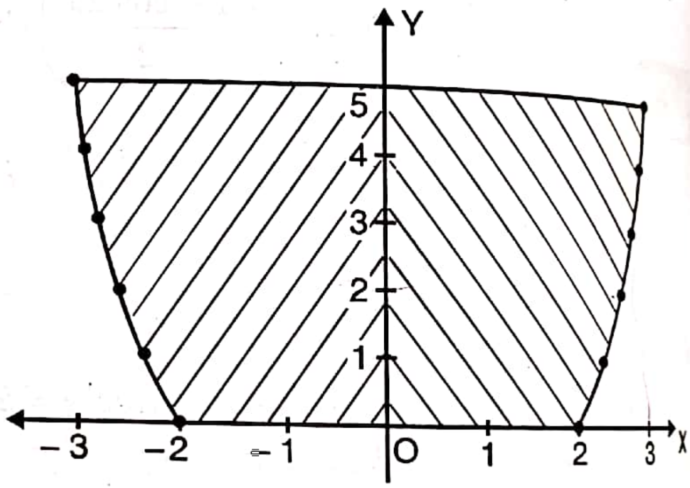

Fig. 3.12

Plotting these points and joining them with a free hand, we get the rough sketch as shown in fig. 3.12. The shaded portion in the figure is the required area.

$\therefore$ Required area $= 2\int_0^5 x\, dy = 2\int_0^5 \sqrt{y+4}\, dy = 2\left[\frac{(y+4)^{3/2}}{3/2}\right]_0^5$

$$= \frac{4}{3}\left[(9)^{3/2} - (4)^{3/2}\right] = \frac{4}{3}(27 - 8) = \frac{\mathbf{76}}{\mathbf{3}} \textbf{ sq. units.}$$

Example 8.

Find the area bounded by the curve $y^2 = 2y - x$ and the y-axis.

Solution. The given curve is $y^2 = 2y - x \quad \Rightarrow \quad x = 2y - y^2$

The following table gives some values of x and y satisfying the given equation.

y:	0	1/2	1	3/2	2
x:	0	3/4	1	3/4	0

Plotting these points and joining them with a free hand, we get the rough sketch as shown in fig. 3.13. The shaded portion in the figure is the required area.

Now the curve cuts the y-axis at the points where $x = 0$

i.e., $$2y - y^2 = 0 \Rightarrow y(2-y) = 0$$

$$\Rightarrow \quad y = 0 \text{ and } y = 2$$

$$\therefore \text{ Required area } = \int_0^2 x\,dy = \int_0^2 (2y - y^2)\,dy$$

$$= \left[y^2 - \frac{y^3}{3}\right]_0^2 = \left[\left(4 - \frac{8}{3}\right) - 0\right]$$

$$= \frac{4}{3} \textbf{ sq. units.}$$

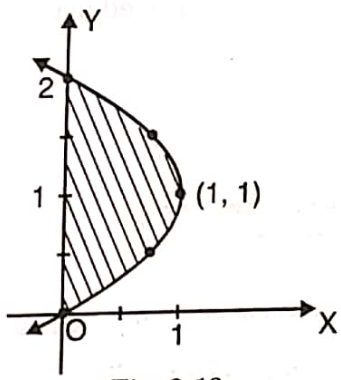

Fig. 3.13

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