REDUCTION OF SURFACE INTEGRAL TO LINE INTEGRAL

5.4. Stoke's Theorem

Statement : *Let S be a piecewise smooth open surface bounded by a piecewise smooth simple curve C. If* $\mathbf{f}(x, y, z)$ *be a continuous vector function which has continuous first partial derivative in a region of space which contains S, then* $\oint_C \mathbf{f} \cdot d\mathbf{r} = \iint_S curl\ \mathbf{f} \cdot \mathbf{n}\, dS$,

where n is the unit normal vector at any point of S and C is traversed in positive direction.

Direction of C is positive if an observer walking on the boundary of S in this direction with its head pointing in the direction of outward normal $\mathbf{n}$ to S has the surface on the left.

We may put the statement of Stoke's theorem in words as under :

The line integral of the tangential component of a vector f taken around a simple closed curve C is equal to the surface integral of normal component of curl of f taken over S having C as its boundary.

5.5. Stoke's Theorem in Cartesian Form

(a) Cartesian form of Stoke's theorem in plane

Choose system of coordinate axes such that the plane of the surface is in xy plane and normal to the surface S lies along the z axis. Normal vector is constant in this case.

Suppose $\qquad \mathbf{f} = f_1\,\mathbf{i} + f_2\,\mathbf{j} + f_3\,\mathbf{k}$

$$\therefore \quad \oint_C \mathbf{f} \cdot d\mathbf{r} = \oint \mathbf{f} \cdot \frac{d\mathbf{r}}{ds}\, ds = \oint \mathbf{f} \cdot \mathbf{t}\, ds$$

where $\mathbf{t} = \dfrac{d\mathbf{r}}{ds}$ is unit vector tangent to C.

$$\therefore \quad \oint_C \mathbf{f} \cdot d\mathbf{r} = \oint_C (f_1\,\mathbf{i} + f_2\,\mathbf{j} + f_3\,\mathbf{k}) \cdot \left(\mathbf{i}\frac{dx}{ds} + \mathbf{j}\frac{dy}{ds} + \mathbf{k}\frac{dz}{ds}\right) . ds\ *$$

$$= \oint_C \left(f_1\frac{dx}{ds} + f_2\frac{dy}{ds} + f_3\frac{dz}{ds}\right) ds$$

But tangent at any point lies in the xy plane, so $\dfrac{dz}{ds} = 0$

$$\therefore \quad \oint_C \mathbf{f} \cdot d\mathbf{r} = \oint_C \left(f_1\frac{dx}{ds} + f_2\frac{dy}{ds}\right) ds \qquad ...(1)$$

Now, $\iint_S \text{curl } \mathbf{f} \cdot \mathbf{n} \, dS = \iint_S \text{curl } \mathbf{f} \cdot \mathbf{k} \, dS$ *[Here normal is along z axis]*

$$= \iint_S \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx \, dy \quad \ldots(2)$$

Using (1) and (2), Stoke's theorem is

$$\oint_C (f_1 dx + f_2 dy) = \iint_S \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx \, dy.$$

Note : *This form of Stoke's theorem is also known as Green's theorem in plane.*

(b) Cartesian form of Stoke's theorem in space

Suppose $\mathbf{f} = f_1 \mathbf{i} + f_2 \mathbf{j} + f_3 \mathbf{k}$ and $\mathbf{n}$ is a outward drawn normal vector of S making angle α, β, γ with positive direction of axes.

$$\therefore \quad \mathbf{n} = \cos \alpha \, \mathbf{i} + \cos \beta \, \mathbf{j} + \cos \gamma \, \mathbf{k}$$

Now
$$\nabla \times \mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

i.e.,
$$\text{curl } \mathbf{f} = \left[\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) \mathbf{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \mathbf{k} \right]$$

$$\therefore \quad \text{curl } \mathbf{f} \cdot \mathbf{n} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \cos \alpha + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) \cos \beta + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \cos \gamma \quad \ldots(1)$$

Also
$$\mathbf{f} \cdot d\mathbf{r} = (f_1 \mathbf{i} + f_2 \mathbf{j} + f_3 \mathbf{k}) \cdot (dx \, \mathbf{i} + dy \, \mathbf{j} + dz \, \mathbf{k})$$

or
$$\mathbf{f} \cdot d\mathbf{r} = f_1 \, dx + f_2 \, dy + f_3 \, dz$$

Then Stoke's theorem is

$$\oint_C (f_1 \, dx + f_2 \, dy + f_3 \, dz) = \iint_S \left[\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \cos \alpha + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) \cos \beta + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \cos \gamma \right] dS.$$

Example 1. *Show that* $\oint_C \mathbf{r} \cdot d\mathbf{r} = 0$.

Solution. By Stoke's theorem,

$$\oint_C \mathbf{r} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{r} \cdot \mathbf{n} \, dS$$

$$= 0. \quad [\because \textit{curl } \mathbf{r} = 0]$$

Example 2. *Show that* $\oint_C \phi\, \nabla\phi \,.\, d\mathbf{r} = 0$, *C being a closed curve.*

Solution. By Stoke's theorem,

$$\oint_C \phi\, \nabla\phi \,.\, d\mathbf{r} = \iint_S \text{curl}\,(\phi\, \nabla\phi) \,.\, \mathbf{n}\, dS$$

$$= \iint_S [\phi\, \text{curl}\, \nabla\phi + \nabla\phi \times \nabla\phi] \,.\, \mathbf{n}\, dS$$

$$= \iint_S 0 \,.\, \mathbf{n}\, dS$$

$$= 0. \qquad [\because \text{ curl } \nabla\phi = 0 \text{ and } \nabla\phi \times \nabla\phi = 0]$$

Example 3. *Show that* $\oint_C \phi\, \nabla\psi \,.\, d\mathbf{r} = -\oint_C \psi\, \nabla\phi \,.\, d\mathbf{r}$.

Solution.

$$\oint_C \nabla(\phi\psi) \,.\, d\mathbf{r} = \iint_S \text{curl}\,(\nabla\phi\psi) \,.\, \mathbf{n}\, dS$$

$$\oint_C \nabla(\phi\psi) \,.\, d\mathbf{r} = 0 \qquad \text{...(1)} \; [\because \text{ curl grad } \phi\psi = 0]$$

We know that $\nabla(\phi\psi) = \phi\, \nabla\psi + \psi\, \nabla\phi$

Putting this value in (1), we have

$$\oint_C (\phi\, \nabla\psi + \psi \nabla\phi) \,.\, d\mathbf{r} = 0$$

$$\therefore \quad \oint_C \phi\, \nabla\psi \,.\, d\mathbf{r} = -\oint_C \psi\, \nabla\phi \,.\, d\mathbf{r}$$

Example 5. *Verify Stoke's theorem for the function* $\mathbf{f} = (x^2 + y^2)\,\mathbf{i} - 2xy\,\mathbf{j}$ *taken round the rectangle bounded by* $x = \pm a,\; y = 0,\; y = b$.

Solution. Given $\mathbf{f} = (x^2 + y^2)\,\mathbf{i} - 2xy\,\mathbf{j}$

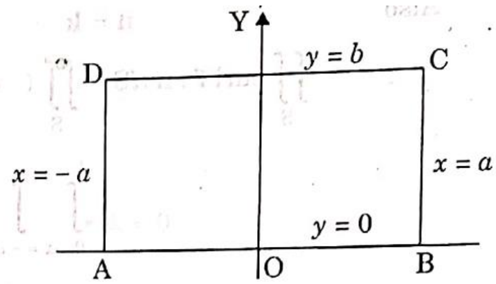

Fig. 5.3

$$\therefore \quad \mathbf{f}\,.\,d\mathbf{r} = [(x^2 + y^2)\,\mathbf{i} - 2xy\,\mathbf{j}]\,.\,(dx\,\mathbf{i} + dy\,\mathbf{j})$$

$$= (x^2 + y^2)\,dx - 2xy\,dy$$

$$\therefore \quad \oint_C \mathbf{f}\,.\,d\mathbf{r} = \oint_C (x^2 + y^2)\,dx - 2xy\,dy$$

$$= \int_{DA} [(x^2 + y^2)\,dx - 2xy\,dy] + \int_{AB} [(x^2 + y^2)\,dx - 2xy\,dy]$$

$$+ \int_{BC} [(x^2 + y^2)\,dx - 2xy\,dy] + \int_{CD} [(x^2 + y^2)\,dx - 2xy\,dy] \quad \text{...(1)}$$

On DA : $x = -a$, $\therefore\ dx = 0$

$$\therefore \quad \int_{DA} [(x^2 + y^2)\,dx - 2xy\,dy] = \int_{DA} -2(-a)y\,dy = \int_{y=b}^{0} 2ay\,dy = \left[ay^2\right]_b^0 = -ab^2$$

On AB : $y = 0$, $\therefore\ dy = 0$

$$\therefore \quad \int_{AB} [(x^2 + y^2)\,dx - 2xy\,dy] = \int_{AB} x^2 dx = \int_{-a}^{a} x^2 dx = 2\int_0^a x^2 dx = \frac{2}{3}a^3$$

On BC : $x = a$, $\therefore\ dx = 0$

$$\therefore \quad \int_{BC} [(x^2 + y^2)\,dx - 2xy\,dy] = \int_{BC} -2ay\,dy = \int_0^b (-2ay)\,dy = -ab^2$$

On CD : $y = b$, $\therefore\ dy = 0$

$$\therefore \quad \int_{CD} [(x^2 + y^2)\,dx - 2xy\,dy] = \int_{CD} (x^2 + b^2)\,dx = \int_a^{-a} (x^2 + b^2)\,dx$$

$$= \left[\frac{x^3}{3} + b^2 x\right]_a^{-a} = -2\left(\frac{a^3}{3} + b^2 a\right)$$

Substituting these values in (1), we get

$$\oint_C \mathbf{f}\,.\,d\mathbf{r} = -ab^2 + \frac{2}{3}a^3 - ab^2 - 2\left(\frac{a^3}{3} + b^2 a\right)$$

$$= -4ab^2 \quad \text{...(2)}$$

Now
$$\text{curl}\,\mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix}$$

$$= 0\,\mathbf{i} + 0\,\mathbf{j} + (-2y - 2y)\,\mathbf{k} = -4y\,\mathbf{k}$$

Also $\mathbf{n} = \mathbf{k}$ [*Surface lies in the xy plane*]

$$\therefore \iint_S \text{curl}\,\mathbf{f}\,.\,\mathbf{n}\,dS = \iint_S (-4y\,\mathbf{k})\,.\,\mathbf{k}\,dS$$

$$= \int_0^b \int_{x=-a}^{a} -4y\,dy\,dx$$

$$= -4ab^2$$ [*On simplification*] ...(3)

Hence from (2) and (3), theorem is verified.

Example 6. *Evaluate by Stoke's theorem* $\oint_C (e^x\,dx + 2y\,dy - dz)$, *where C is the curve* $x^2 + y^2 = 4,\ z = 2$.

Solution. $$\oint_C (e^x\,dx + 2y\,dy - dz) = \oint_C (e^x\,\mathbf{i} + 2y\,\mathbf{j} + (-1)\,\mathbf{k})\,.\,(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \oint_C \mathbf{f}\,.\,d\mathbf{r}, \text{ where } \mathbf{f} = e^x\mathbf{i} + 2y\,\mathbf{j} - \mathbf{k}$$

Now $$\text{curl}\,\mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix}$$

$$= (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + (0-0)\,\mathbf{k} = 0$$

$$\therefore \oint_C (e^x\,dx + 2y\,dy - dz) = \oint_C \mathbf{f}\,.\,d\mathbf{r}$$

$$= \iint_S \text{curl}\,\mathbf{f}\,.\,\mathbf{n}\,dS = 0.$$ [$\because$ *curl* $\mathbf{f} = 0$]

Example 7. *Evaluate by Stoke's theorem* $\oint_C (yz\,dx + xz\,dy + xy\,dz)$, *where C is the curve* $x^2 + y^2 = 1,\ z = y^2$.

Solution. $$\oint_C (yz\,dx + xz\,dy + xy\,dz) = \oint_C (yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k})\,.\,(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \oint_C \mathbf{f}\,.\,d\mathbf{r}, \text{ where } \mathbf{f} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$$

Now $$\operatorname{curl} \mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= (x - x)\mathbf{i} + (y - y)\mathbf{j} + (z - z)\mathbf{k} = 0$$

$\therefore$ $$\oint_C \mathbf{f} \,.\, d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{f} \,.\, \mathbf{n} \, dS = 0 \,. \qquad [\because \textit{curl}\, \mathbf{f} = 0]$$

Example 8. *Evaluate* $\oint_C \mathbf{f} \,.\, d\mathbf{r}$ *by Stoke's theorem, where* $\mathbf{f} = y^2\,\mathbf{i} + x^2\,\mathbf{j} - (x + z)\,\mathbf{k}$ *and C is the undary of triangle with vertices at (0, 0, 0), (1, 0, 0), (1, 1, 0).*

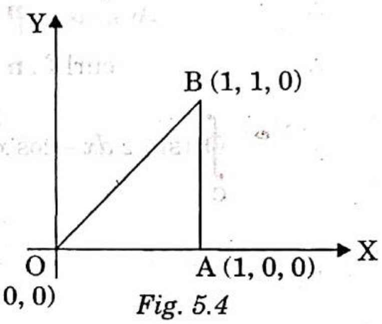

Fig. 5.4

Solution. Here $$\operatorname{curl} \mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x + z) \end{vmatrix}$$

$$= (0 - 0)\mathbf{i} + (0 + 1)\mathbf{j} + (2x - 2y)\,\mathbf{k}$$

$$= \mathbf{j} + 2(x - y)\,\mathbf{k}$$

Here triangle is in the xy plane as z co-ordinate of each vertex of the triangle is zero.

$\therefore$ $$\mathbf{n} = \mathbf{k}$$

$\therefore$ $$\operatorname{curl} \mathbf{f} \,.\, \mathbf{n} = [\,\mathbf{j} + 2\,(x - y)\,\mathbf{k}] \,.\, \mathbf{k} = 2(x - y)$$

By Stoke's theorem, $$\oint_C \mathbf{f} \,.\, d\mathbf{r} = \iint_S (\operatorname{curl} \mathbf{f} \,.\, \mathbf{n})\, dS$$

$$= \iint_S 2(x - y)\, dy\, dx$$

Note here the equation of OB is $y = x$, thus for S, x varies from 0 to 1 and y from 0 to x.

$\therefore$ $$\oint_C \mathbf{f} \,.\, d\mathbf{r} = \int_0^1 \int_{y=0}^{y=x} 2(x - y)\, dy\, dx$$

$$= \int_0^1 2\left(xy - \frac{y^2}{2}\right)_0^x dx$$

$$= \int_0^1 2\left(x^2 - \frac{x^2}{2}\right) dx = \int_0^1 x^2 dx = \frac{1}{3}\,.$$