## 1.3. ▶ INCREASING AND DECREASING FUNCTIONS

### 1.3.1. Increasing Function

*If $y = f(x)$ is a function of $x$ in the open interval $(a, b)$ and if $y$ increases as $x$ increases in the interval $(a, b)$, then $y$ is called the* **increasing function** *of $x$ on the interval $(a, b)$.*

In other words, a function $f(x)$ is increasing on an open interval $(a, b)$ if for

$$x_1, x_2 \in (a, b); \quad x_1 < x_2 \quad \Rightarrow \quad f(x_1) \le f(x_2).$$

A function $f(x)$ is said to be **strictly increasing** on an open interval $(a, b)$ if

$$x_1 < x_2 \quad \Rightarrow \quad f(x_1) < f(x_2)$$

The following figures will explain the concept of increasing and strictly increasing functions clearly.

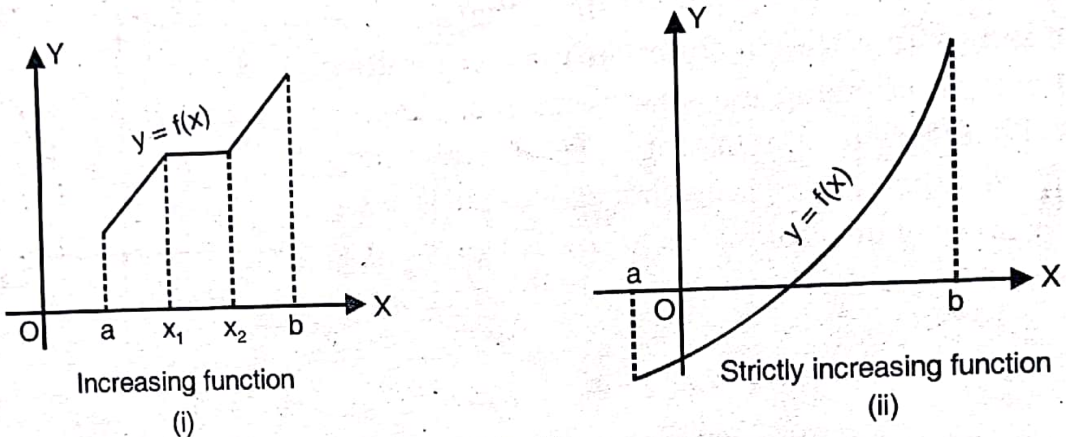



*Fig. 1.7*

Observe that in fig. 1.7($i$), although $f(x)$ is increasing on $(a, b)$, but it is strictly increasing only on intervals $(a, x_1)$ and $(x_2, b)$.

### 1.3.2. Decreasing Function

*If $y = f(x)$ is a function of $x$ in the open interval $(a, b)$ and if $y$ decreases as $x$ increases in the interval $(a, b)$, then $y$ is called the* **decreasing function** *of $x$ on the interval $(a, b)$.*

In other words, a function $f(x)$ is decreasing on an open interval $(a, b)$ if for

$$x_1, x_2 \in (a, b); \; x_1 < x_2 \quad \Rightarrow \quad f(x_1) \ge f(x_2)$$

A function $f(x)$ is said to be **strictly decreasing** on an open interval $(a, b)$ if

$$x_1 < x_2 \quad \Rightarrow \quad f(x_1) > f(x_2)$$

The following figures will explain the concept of decreasing and strictly decreasing functions clearly.

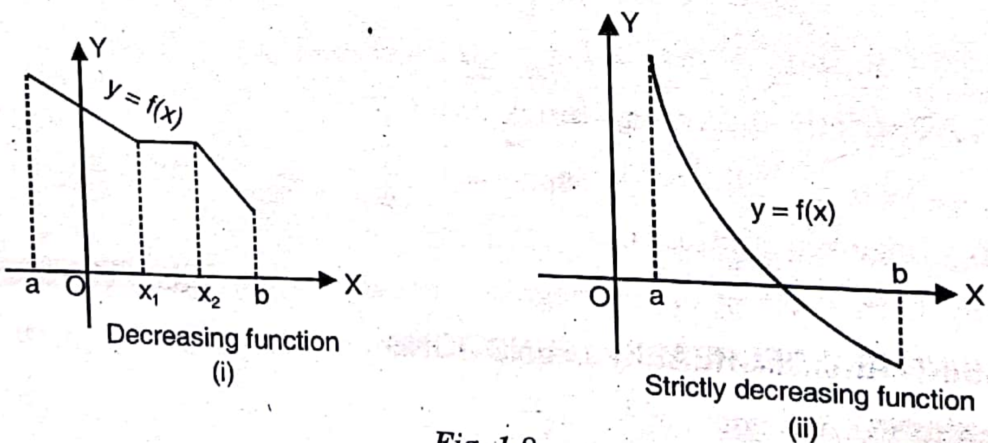
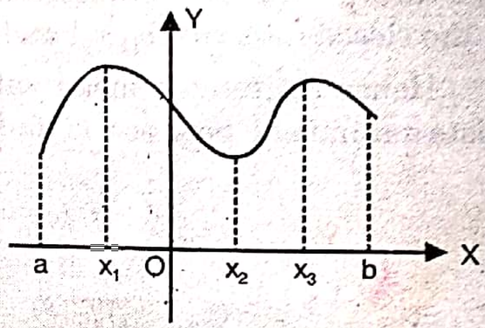



*Fig. 1.8*

Observe that in fig. 1.8($i$), although $f(x)$ is decreasing on $(a, b)$, but it is strictly decreasing only on intervals $(a, x_1)$ and $(x_2, b)$.

**Remarks.**

1. A function is said to be **monotonic** on an interval if it is either increasing or decreasing in the interval.
2. A strictly increasing (decreasing) function on an open interval is always increasing (decreasing) on that interval; although every increasing (decreasing) function may not be strictly increasing (decreasing).
3. It is possible that a function may be neither strictly increasing nor strictly decreasing on a given interval. For example, in the adjoining figure, function $f(x)$ is neither strictly increasing nor strictly decreasing on $(a, b)$. However, it is increasing on the sub-intervals $(a, x_1)$, $(x_2, x_3)$ and decreasing on the intervals $(x_1, x_2)$ and $(x_3, b)$.




## SOLVED EXAMPLES

**Example 1.**

*Without using derivatives show that the function*
*(i)* $f(x) = 2x + 5$ *is a strictly increasing function on* $\mathbf{R}$
*(ii)* $f(x) = x^2$ *is a strictly decreasing function on* $(-\infty, 0]$.

**Solution.** (*i*) Let $x_1, x_2 \in \mathbf{R}$ and let $x_1 < x_2$

Now, $x_1 < x_2 \Rightarrow 2x_1 < 2x_2$

$\Rightarrow$ $2x_1 + 5 < 2x_2 + 5$

$\Rightarrow$ $f(x_1) < f(x_2)$

Therefore, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in \mathbf{R}$

Hence, $f$ is a strictly increasing function on $\mathbf{R}$.

(*ii*) Let $x_1, x_2 \in (-\infty, 0]$ and let $x_1 < x_2$ ...(1)

where $x_1, x_2$ are negative numbers.

Multiplying both sides of (1) by negative number $x_1$, we get $x_1^2 > x_1x_2$ ...(2)

Multiplying both sides of (1) by negative number $x_2$, we get $x_1x_2 > x_2^2$ ...(3)

From (2) and (3), we get $x_1^2 > x_2^2$

*i.e.,* $f(x_1) > f(x_2)$

Therefore, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in (-\infty, 0]$

Hence, $f(x) = x^2$ is a strictly decreasing function on $(-\infty, 0]$.

### 1.3.3. Condition for Monotonicity of Functions

**Theorem** *If $f(x)$ is a differentiable real function defined on an open interval $(a, b)$, then*

*(i) $f(x)$ is strictly increasing on $(a, b)$ iff $f'(x) > 0$ for all $x \in (a, b)$.*

*(ii) $f(x)$ is strictly decreasing on $(a, b)$ iff $f'(x) < 0$ for all $x \in (a, b)$.*

**Proof.** Let $x_1$ and $x_2$ be any two numbers $\in (a, b)$ such that $x_1 < x_2$. Consider the sub-interval $[x_1, x_2]$. Since $f$ is differentiable on $(a, b)$ and $[x_1, x_2] \subset (a, b)$, therefore $f(x)$ is continuous on the closed interval $[x_1, x_2]$ and derivable on the open interval $(x_1, x_2)$.

Hence Lagrange's mean value theorem is applicable to $f(x)$ in $[x_1, x_2]$ and hence there exists a number $c$ between $x_1$ and $x_2$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\Rightarrow \quad (x_2 - x_1) f'(c) = f(x_2) - f(x_1) \qquad \text{...(1)}$$

Now, $\quad x_1 < x_2 \quad \Rightarrow \quad x_2 - x_1 > 0$

Also, $\quad f'(c) > 0 \qquad [\because f'(x) > 0 \text{ for all real } x \in (a, b)]$

$\therefore \quad (x_2 - x_1) f'(c) > 0$

$\therefore$ From (1), we have

$$f(x_2) - f(x_1) > 0 \quad \Rightarrow \quad f(x_2) > f(x_1) \quad \Rightarrow \quad f(x_1) < f(x_2)$$

Therefore, $\quad x_1 < x_2 \quad \Rightarrow \quad f(x_1) < f(x_2)$

Hence, $f(x)$ is a strictly increasing function of $x$.

This shows that $f(x)$ is **strictly increasing** on $(a, b)$ if $f'(x) > 0$ for all $x \in (a, b)$

Similarly, we can show that $f(x)$ is **strictly decreasing** on $(a, b)$ if $f'(x) < 0$ for all $x \in (a, b)$.

**Remarks.**

1. A function is *strictly increasing* on an open interval where its derivative is *positive* and *strictly decreasing* on an open interval where its derivative is *negative*.
2. A function is *increasing* on an open interval where its derivative is *non-negative* and *decreasing* on an open interval where its derivative is *non-positive*.
3. The values of $x$ where $f'(x) = 0$ are called the **critical values** of the function $f(x)$.

**Example 2.**

*Prove that the function $x^3 - 6x^2 + 12x - 18$ is increasing on* $\mathbf{R}$.

**Solution.** Let $f(x) = x^3 - 6x^2 + 12x - 18$

$\therefore \quad f'(x) = 3x^2 - 12x + 12 = 3(x^2 - 4x + 4) = 3(x-2)^2$

Now, $(x-2)^2$ being a square quantity, can never be negative and therefore $f'(x) \geq 0$.

Hence the function $x^3 - 6x^2 + 12x - 18$ is increasing on $\mathbf{R}$.

**Example 3.**

*Test whether the function $f(x) = x^2 - 6x + 3$ is increasing in the interval [4, 6].*

**Solution.** Here $f(x) = x^2 - 6x + 3 \Rightarrow f'(x) = 2x - 6$

For increasing function $f'(x) > 0 \Rightarrow 2x - 6 > 0 \Rightarrow x > 3$

$\therefore f(x)$ is an increasing function for $x > 3$.

Since [4, 6] is a sub-interval of $x > 3$, therefore $f(x)$ is an increasing function on [4, 6].

**Example 4.** ✓

*Determine whether the function $f(x) = x + \sin x$, for all $x$ is increasing or decreasing.*

**Solution.** Here $f(x) = x + \sin x \Rightarrow f'(x) = 1 + \cos x$

Now, the least value of $\cos x$ is $-1$ at $x = \pi$.

$\therefore \quad f'(x) = 1 + \cos x \geq 0$, for all $x$

Hence, $f(x)$ is an increasing function for all $x$.

**Example 5.** ✓

*Find the value of $b$ for which the function $f(x) = \sin x - bx + c$ is a decreasing function on R.*

**Solution.** Here

$$f(x) = \sin x - bx + c$$

$$\therefore \quad f'(x) = \cos x - b$$

Now, $f(x)$ is decreasing on $\mathbf{R}$ if $f'(x) \leq 0$ for all $x \in \mathbf{R}$

$$\Rightarrow \quad \cos x - b \leq 0, \quad x \in \mathbf{R}$$

$$\Rightarrow \quad \cos x \leq b, \quad x \in \mathbf{R}$$

$$\Rightarrow \quad b \geq 1.$$

**Example 6.**

*Let I be any interval disjoint from (–1, 1). Prove that the function $x + \dfrac{1}{x}$ is strictly increasing on I.*

**Solution.** Let

$$f(x) = x + \frac{1}{x}$$

$$\therefore \quad f'(x) = 1 - \frac{1}{x^2}$$

Now, $f(x)$ will be strictly increasing if $f'(x) > 0$ for all $x$

i.e., if $\quad 1 - \frac{1}{x^2} > 0 \quad$ i.e., if $\quad \frac{x^2 - 1}{x^2} > 0$

i.e., if $\quad x^2 - 1 > 0 \qquad$ [$\because x^2$ *is always positive*]

i.e., if $\quad (x+1)(x-1) > 0$

i.e., if $\quad (x+1) > 0$ and $(x-1) > 0 \quad$ or $\quad (x+1) < 0$ and $(x-1) < 0$

i.e., if $\quad x > 1 \quad$ or $\quad x < -1$ i.e., any interval I except $(-1, 1)$.

Hence, $f(x)$ will be strictly increasing in any interval I disjoint from $(-1, 1)$.

**Example 7.**

*Find the intervals in which the following functions are strictly increasing or strictly decreasing*

(i) $f(x) = x^2 + 2x - 5$ $\qquad$ (ii) $f(x) = 2x^3 - 15x^2 + 36x + 1$

[NCERT]

**Solution.** (i) Here $\quad f(x) = x^2 + 2x - 5$

$\therefore \quad f'(x) = 2x + 2 = 2(x+1)$

The function $f(x)$ will be strictly increasing if $f'(x) > 0$

i.e., if $\quad 2(x+1) > 0 \quad$ i.e., $\quad x + 1 > 0 \quad$ i.e., $\quad x > -1$

The function $f(x)$ will be strictly decreasing if $f'(x) < 0$

i.e., $\quad 2(x+1) < 0 \quad$ i.e., $\quad (x+1) < 0 \quad$ i.e., $\quad x < -1$

Hence, $f(x)$ **is strictly increasing on** $(-1, \infty)$ and **strictly decreasing on** $(-\infty, -1)$.

(ii) Here $\quad f(x) = 2x^3 - 15x^2 + 36x + 1$

$\therefore \quad f'(x) = 6x^2 - 30x + 36 = 6(x^2 - 5x + 6) = 6(x-2)(x-3)$

Now, $\quad f'(x) = 0 \Rightarrow 6(x-2)(x-3) = 0 \Rightarrow x = 2, 3$, which are the critical values.

These values of $x$ give rise to following intervals :

(a) $x < 2$ $\qquad$ (b) $2 < x < 3$ $\qquad$ (c) $x > 3$

**When $x < 2$,** $(x-2)$ is –ve and $(x-3)$ is – ve $\Rightarrow f'(x) = 6(-\text{ve})(-\text{ve}) = +\text{ve}$

$\therefore f(x)$ is strictly increasing for $x < 2$.

**When $2 < x < 3$,** $(x-2)$ is + ve and $(x-3)$ is – ve $\Rightarrow f'(x) = 6(+\text{ve})(-\text{ve}) = -\text{ve}$

$\therefore f(x)$ is strictly decreasing for $2 < x < 3$.

**When $x > 3$,** $(x-2)$ is + ve and $(x-3)$ is + ve $\Rightarrow f'(x) = 6(+\text{ve})(+\text{ve}) = +\text{ve}$

$\therefore f(x)$ is strictly increasing for $x > 3$.

Hence, $f(x)$ is **strictly increasing** for $x < 2$ or $x > 3$ i.e., on $(-\infty, 2) \cup (3, \infty)$ and **strictly decreasing** for $2 < x < 3$ i.e., on $(2, 3)$.

**Example 8.**

*Find the intervals in which the function $f(x) = 5 + 36x + 3x^2 - 2x^3$ is*

*(i) strictly increasing* *(ii) strictly decreasing.*

**Solution.** Here $f(x) = 5 + 36x + 3x^2 - 2x^3$

$\therefore$ $f'(x) = 36 + 6x - 6x^2 = -6(x^2 - x - 6) = -6(x-3)(x+2)$

The function $f(x)$ will be strictly increasing if $f'(x) > 0$

*i.e.*, if $-6(x-3)(x+2) > 0$ *i.e.*, if $(x-3)(x+2) < 0$

$\Rightarrow$ either $(x-3) > 0$ and $(x+2) < 0$ | or $(x-3) < 0$ and $(x+2) > 0$

$\Rightarrow$ $x > 3$ and $x < -2$ | $\Rightarrow$ $x < 3$ and $x > -2$

which is not possible | $\Rightarrow$ $-2 < x < 3$

$\therefore f(x)$ is strictly increasing for $-2 < x < 3$ *i.e.*, on $\mathbf{(-2, 3)}$

The function $f(x)$ will be strictly decreasing if $f'(x) < 0$

*i.e.*, if $-6(x-3)(x+2) < 0$ $\Rightarrow$ $(x-3)(x+2) > 0$

$\Rightarrow$ either $(x-3) > 0$ and $(x+2) > 0$ | or $(x-3) < 0$ and $(x+2) < 0$

$\Rightarrow$ $x > 3$ and $x > -2$ | $\Rightarrow$ $x < 3$ and $x < -2$

$\Rightarrow$ $x > 3$ | $\Rightarrow$ $x < -2$

$\therefore f(x)$ is strictly decreasing for $x < -2$ or $x > 3$ *i.e.*, on $\mathbf{(-\infty, -2) \cup (3, \infty)}$.

**Note.**

*To find the intervals where the function is increasing or decreasing, any one of the method used in example 7 (ii) or example 8 can be employed.*

**Example 9.**

*Find the intervals in which $f(x) = (x-1)^3 (x-2)^2$ is strictly increasing or strictly decreasing.*

**Solution.** Here $f(x) = (x-1)^3 (x-2)^2$

$\therefore$
$$f'(x) = (x-1)^3 \,.\, 2(x-2) + (x-2)^2 \,.\, 3(x-1)^2$$
$$= (x-1)^2 (x-2)[2(x-1) + 3(x-2)] = (x-1)^2 (x-2)(5x-8)$$

Now, $f'(x) = 0 \Rightarrow (x-1)^2 (x-2)(5x-8) = 0$

$$\Rightarrow x = 1, 2, \frac{8}{5}, \text{ which are the critical values}$$

These values of $x$ give rise to following intervals :

(a) $x < 1$ (b) $1 < x < \frac{8}{5}$ (c) $\frac{8}{5} < x < 2$ (d) $x > 2$

**When $x < 1$,** $f'(x) = (+\text{ ve})(-\text{ ve})(-\text{ ve}) = +\text{ ve}$

$\therefore$ $f(x)$ is strictly increasing for $x < 1$.

**When $1 < x < \frac{8}{5}$,** $f'(x) = (+\text{ ve})(-\text{ ve})(-\text{ ve}) = +\text{ ve}$

$\therefore$ $f(x)$ is strictly increasing for $1 < x < \frac{8}{5}$

**When $\frac{8}{5} < x < 2$,** $f'(x) = (+\text{ ve})(-\text{ ve})(+\text{ ve}) = -\text{ ve}$

$\therefore$ $f(x)$ is strictly decreasing for $\frac{8}{5} < x < 2$

**When $x > 2$,** $f'(x) = (+\text{ ve})(+\text{ ve})(+\text{ ve}) = +\text{ ve}$

$\therefore$ $f(x)$ is strictly increasing for $x > 2$

Hence, $f(x)$ is strictly increasing for $x < 1$, $1 < x < \frac{8}{5}$ and $x > 2$

*i.e.*, on $(-\infty, 1) \cup \left(1, \frac{8}{5}\right) \cup (2, \infty)$ and strictly decreasing for $\frac{8}{5} < x < 2$ *i.e.*, on $\left(\frac{8}{5}, 2\right)$

**Note.**

1. *If there are n values of x obtained by putting $f'(x) = 0$, then there will be, in general, $(n+1)$ intervals in which we are to discuss the increasing or decreasing nature of $f(x)$.*
2. *Always consider the critical values in ascending order of magnitude before arranging them into intervals.*

**Example 10.**

*Determine for what values of x, the function $f(x) = x^4 - \frac{x^3}{3}$ is strictly increasing or strictly decreasing. At what points is the tangent parallel to x-axis ?*

**Solution.** Here $f(x) = x^4 - \frac{1}{3}x^3$

$$\therefore \quad f'(x) = 4x^3 - \frac{1}{3} \cdot 3x^2 = x^2(4x - 1)$$

Now, $f'(x) = 0 \Rightarrow x^2(4x - 1) = 0 \Rightarrow x = 0, \frac{1}{4}$, which are the critical values.

These values of $x$ give rise to following intervals :

(a) $x < 0$ (b) $0 < x < \frac{1}{4}$ (c) $x > \frac{1}{4}$

**When $x < 0$,** $f'(x) = (+\text{ ve})(-\text{ ve}) = -\text{ ve}$

$\therefore$ $f(x)$ is strictly decreasing for $x < 0$.

**When $0 < x < \frac{1}{4}$,** $f'(x) = (+\text{ ve})(-\text{ ve}) = -\text{ ve}$

$\therefore$ $f(x)$ is strictly decreasing for $0 < x < \frac{1}{4}$.

**When $x > \frac{1}{4}$,** $f'(x) = (+\text{ ve})(+\text{ ve}) = +\text{ ve}$

$\therefore$ $f(x)$ is strictly increasing for $x > \frac{1}{4}$.

Hence, $f(x)$ is strictly increasing on $\left(\frac{1}{4}, \infty\right)$ and strictly decreasing on $(-\infty, 0) \cup \left(0, \frac{1}{4}\right)$.

*Tangent parallel to x-axis.*

The tangents will be parallel to $x$-axis, if $f'(x) = 0$

*i.e.,* $x^2(4x - 1) = 0 \Rightarrow x = 0, \frac{1}{4}$

When $x = 0$, $f(0) = 0$

and when $x = \frac{1}{4}$, $f\left(\frac{1}{4}\right) = \frac{1}{256} - \frac{1}{64 \times 3} = \frac{3-4}{768} = -\frac{1}{768}$.

Hence, the points where tangents are parallel to $x$-axis are **(0, 0)** and $\left(\frac{1}{4}, -\frac{1}{768}\right)$.

**Example 11.**

*Find the intervals for which the function $f(x) = x^4 - 2x^2$ is strictly increasing or strictly decreasing.*

**Solution.** Here $f(x) = x^4 - 2x^2$

Differentiating w.r.t. $x$, we get

$$f'(x) = 4x^3 - 4x = 4x(x + 1)(x - 1).$$

Now, $f'(x) = 0 \Rightarrow 4x(x + 1)(x - 1) = 0$

$\Rightarrow$ $x = 0, -1, 1$, which are the critical values.

These values of $x$ give rise to the following four intervals :

(a) $x < -1$, (b) $-1 < x < 0$, (c) $0 < x < 1$, (d) $x > 1$

We now discuss the nature of $f'(x)$ in all these intervals.

(a) **When $x < -1$,** $f'(x) = (4)(-)(-)(-) = -\text{ ve} \Rightarrow f'(x) < 0$

$\therefore f(x)$ is strictly decreasing for $x < -1$.

(b) **When $-1 < x < 0$,** $f'(x) = (4)(-)(+)(-) = +\text{ve} \Rightarrow f'(x) > 0$

$\therefore$ $f(x)$ is strictly increasing for $-1 < x < 0$.

(c) **When $0 < x < 1$,** $f'(x) = (4)(+)(+)(-) = -\text{ve} \Rightarrow f'(x) < 0$

$\therefore$ $f(x)$ is strictly decreasing for $0 < x < 1$.

(d) **When $x > 1$,** $f'(x) = (4)(+)(+)(+) = +\text{ve} \Rightarrow f'(x) > 0$

$\therefore$ $f(x)$ is strictly increasing for $x > 1$.

Thus the function $f(x)$ is strictly increasing for $-1 < x < 0, x > 1$ *i.e.*, on $\mathbf{(-1, 0) \cup (1, \infty)}$ and strictly decreasing for $x < -1, 0 < x < 1$ *i.e.*, on $\mathbf{(-\infty, -1) \cup (0, 1)}$.

**Example 12.**

*Find the intervals in which the function $f(x) = (x + 2)e^{-x}$ is strictly increasing or strictly decreasing.*

**Solution.** Here $f(x) = (x + 2)e^{-x}$

$$\therefore \quad f'(x) = (x + 2)e^{-x}(-1) + (e^{-x}).1$$

$$= e^{-x}[-x - 2 + 1] = -\frac{(x + 1)}{e^x}$$

The function $f(x)$ will be strictly increasing if $f'(x) > 0$

*i.e.*, if $-\dfrac{(x+1)}{e^x} > 0 \Rightarrow -(x + 1) > 0$ [$\because$ *$e^x$ is always $> 0$ for all $x \in \mathbf{R}$*]

$\Rightarrow \quad -x > 1 \Rightarrow x < -1$

Hence, $f(x)$ is strictly increasing on $\mathbf{(-\infty, -1)}$

The function $f(x)$ will be strictly decreasing if $f'(x) < 0$

*i.e.*, if $-\dfrac{(x+1)}{e^x} < 0 \Rightarrow -(x + 1) < 0$

$\Rightarrow \quad -x < 1 \Rightarrow x > -1$

Hence, $f(x)$ is strictly decreasing on $\mathbf{(-1, \infty)}$.

**Example 13.**

*(i) Show that $f(x) = (x - 1)e^x + 1$ is a strictly increasing function for $x > 0$.*

*(ii) Show that $f(x) = e^{1/x}, x \neq 0$ is a strictly decreasing function.*

**Solution.** (*i*) We have $f(x) = (x - 1)e^x + 1$

$\therefore \quad f'(x) = (x - 1)e^x + e^x . 1 = xe^x - e^x + e^x = xe^x$

For all $x > 0$, $\quad xe^x > 0$

$\therefore \quad f'(x) > 0$ for all $x > 0$ [$\because$ *$x > 0$ and $e^x > 0$ for $x > 0$*]

Hence, $f(x)$ is a strictly increasing function for all $x > 0$.