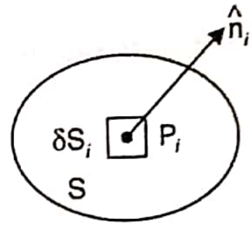
13.25. SURFACE INTEGRALS

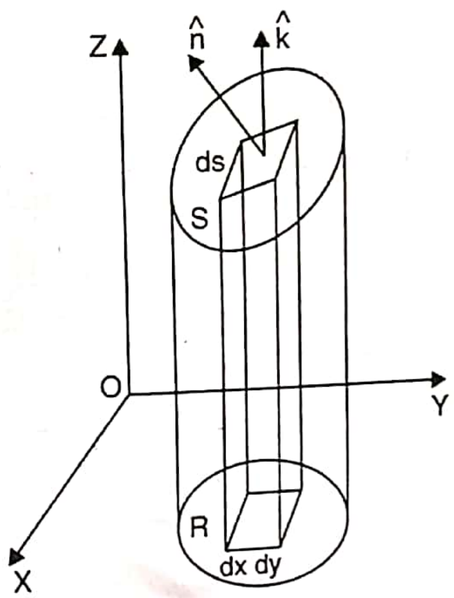
Any integral which is to be evaluated over a surface is called a surface integral.

Let $\vec{F}(P)$ be a continuous vector point function and S a two sided surface. Divide S into a finite number of sub-surfaces $\delta S_1, \delta S_2, \ldots\ldots \delta S_k$. Let P_i be any point in δS_i and $\hat{n}_i$ be the unit vector at P in the direction of outward drawn normal to the surface at P_i. Then the limit of the sum



$\sum_{i=1}^{k} \vec{F}(P_i) \cdot \hat{n}_i \, \delta S_i$, as $k \to \infty$ and each $\delta S_i \to 0$ is called the *normal surface integral* of $\vec{F}(P)$ over S and is denoted by $\iint_S \vec{F} \cdot \hat{n} \, dS$.

The surface element $\vec{\delta S}$ surrounding any point P can be regarded as a vector whose magnitude is area δS and the direction that of the outward drawn normal $\hat{n}$ *i.e.* $\vec{\delta S} = \hat{n} \, \delta S$. The surface integral may alternatively be written as $\iint_S \vec{F} \cdot d\vec{S}$.



If $\vec{F}$ represents the velocity of a fluid at any point P on a closed surface S, then $\vec{F} \cdot \hat{n}$ is the normal component of $\vec{F}$ at P and $\iint_S \vec{F} \cdot \hat{n} \, dS = \iint_S \vec{F} \cdot d\vec{S}$ is a measure of volume emerging from S per unit time, *i.e.* it measures the *flux of* $\vec{F}$ *over* S.

Other types of surface integrals are $\iint_S \vec{F} \times d\vec{S}, \iint_S \phi\, d\vec{S}$.

Note. In order to evaluate surface integrals, it is convenient to express them as double integrals taken over the orthogonal projection of S on one of the coordinate planes. But this is possible only when any line perpendicular to the coordinate plane chosen meets the surface S in not more than one point.

Let R be the orthogonal projection of S on the xy-plane.

Let $\hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$ where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of $\hat{n}$.

Now, $dxdy$ = projection of dS on the xy-plane $= d\text{S}\cos\gamma \Rightarrow d\text{S} = \dfrac{dx\,dy}{\cos\gamma}$

Also $|\hat{k}\cdot\hat{n}| = \cos\gamma \quad \therefore \quad d\text{S} = \dfrac{dxdy}{|\hat{k}\cdot\hat{n}|}$

Hence $\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_R \vec{F}\cdot\hat{n}\,\dfrac{dxdy}{|\hat{k}\cdot\hat{n}|}$.

13.26. VOLUME INTEGRALS

Any integral which is to be evaluated over a volume is called a volume integral.

If V is a volume bounded by a surface S, then the triple integrals

$$\iiint_V \phi\, d\text{V} \quad \text{and} \quad \iiint_V \vec{F}\, d\text{V}$$

are called volume integrals. The first of these is a scalar and the second is a vector.

If we sub-divide the volume V into small cuboids by drawing planes parallel to the coordinate planes, then $d\text{V} = dx\,dy\,dz$.

$$\therefore \quad \iiint_V \phi\, d\text{V} = \iiint_V \phi(x, y, z)\, dx\,dy\,dz$$

If $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$, then

$$\iiint_V \vec{F}\, d\text{V} = \hat{i}\iiint_V F_1(x, y, z)\, dx\,dy\,dz + \hat{j}\iiint_V F_2(x, y, z)\, dx\,dy\,dz + \hat{k}\iiint_V F_3(x, y, z)\, dx\,dy\,dz.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate* $\iint_S \vec{A}\cdot\hat{n}\, dS$ *where* $\vec{A} = (x + y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ *and S is the surface of the plane* $2x + y + 2z = 6$ *in the first octant.*

Sol. A vector normal to the surface S is given by

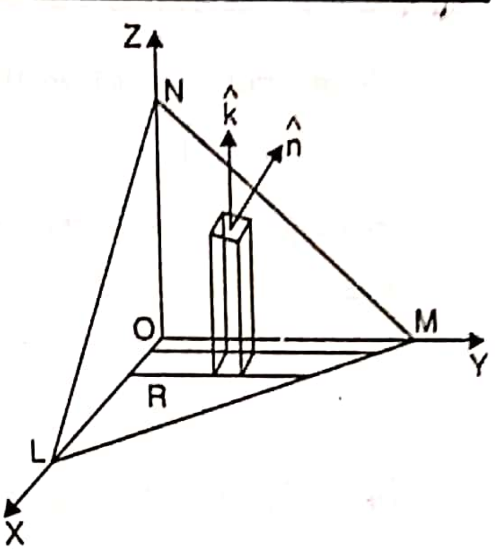
$$\nabla(2x + y + 2z) = 2\hat{i} + \hat{j} + 2\hat{k}$$

$\therefore \quad \hat{n}$ = a unit vector normal to surface S

$$= \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{(2)^2 + 1^2 + (2)^2}} = \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}$$

$$\hat{k}\,.\,\hat{n} = \hat{k}\,.\left(\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}\right) = \frac{2}{3}$$

$$\therefore \quad \iint_S \vec{A}\,.\,\hat{n}\, dS = \iint_R \vec{A}\,.\,\hat{n}\, \frac{dxdy}{|\hat{k}\,.\,\hat{n}|},$$



where R is the projection of S, *i.e.* triangle LMN on the xy-plane. The region R, *i.e.* triangle OLM is bounded by x-axis, y-axis and the line $2x + y = 6$, $z = 0$.

Now $\vec{A}\,.\,\hat{n} = [(x + y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}]\,.\left(\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}\right)$

$$= \frac{2}{3}(x + y^2) - \frac{2}{3}x + \frac{4}{3}yz = \frac{2}{3}y^2 + \frac{4}{3}yz$$

$$= \frac{2}{3}y^2 + \frac{4}{3}y\left(\frac{6 - 2x - y}{2}\right) \quad \left(\because \text{ on the plane } 2x + y + 2z = 6,\ z = \frac{6 - 2x - y}{2}\right)$$

$$= \frac{2}{3}y(y + 6 - 2x - y) = \frac{4}{3}y(3 - x)$$

Hence $\displaystyle\iint_S \vec{A}\,.\,\hat{n}\, dS = \iint_R \vec{A}\,.\,\hat{n}\, \frac{dxdy}{|\hat{k}\,.\,\hat{n}|}$

$$= \iint_R \frac{4}{3}y\,(3 - x)\,.\,\frac{3}{2}\,dxdy = \int_0^3 \int_0^{6-2x} 2y(3 - x)\, dydx$$

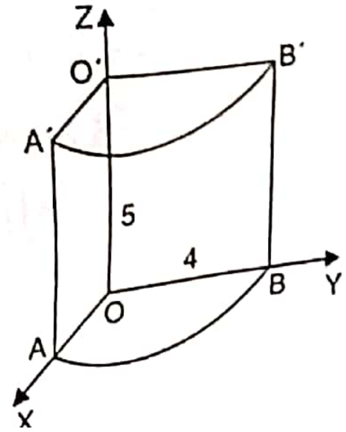
$$= \int_0^3 2\,(3 - x)\,.\left[\frac{y^2}{2}\right]_2^{6-2x} dx = \int_0^3 (3 - x)\,(6 - 2x)^2\, dx$$

$$= 4\int_0^3 (3 - x)^3\, dx = 4\,.\left[\frac{(3 - x)^4}{4(-1)}\right]_0^3 = -(0 - 81) = 81.$$

Example 2. *Evaluate* $\iint_S \vec{A}\,.\,\hat{n}\, dS$, *where* $\vec{A} = z\hat{i} + x\hat{j} - 3y^2z\hat{k}$ *and S is the surface of the cylinder* $x^2 + y^2 = 16$ *included in the first octant between* $z = 0$ *and* $z = 5$. (M...

Sol. A vector normal to the surface S is given by

$$\nabla\,(x^2 + y^2) = 2x\hat{i} + 2y\hat{j}$$



$\therefore \quad \hat{n}$ = a unit vector normal to surface S

$$= \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{(2x)^2 + (2y)^2}} = \frac{x\hat{i} + y\hat{j}}{\sqrt{x^2 + y^2}} = \frac{x\hat{i} + y\hat{j}}{4}$$

($\because$ on the surface of cylinder, $x^2 + y^2 = 16$)

Let R be the projection of S on yz-plane, then

$$\iint_S \vec{A}\,.\,\hat{n}\,dS = \iint_R \vec{A}\,.\,\hat{n}\,\frac{dydz}{|\hat{i}\,.\,\hat{n}|}$$

The region R is OBB′O′ enclosed by $y = 0$ to $y = 4$ and $z = 0$ to $z = 5$.

Now $\quad \hat{i}\,.\,\hat{n} = \hat{i}\,.\,(\frac{1}{4}x\hat{i} + \frac{1}{4}y\hat{j}) = \frac{1}{4}x$

$$\vec{A}\,.\,\hat{n} = (z\hat{i} + x\hat{j} - 3y^2z\hat{k})\,.\,(\tfrac{1}{4}x\hat{i} + \tfrac{1}{4}y\hat{j})$$

$$= \tfrac{1}{4}zx + \tfrac{1}{4}xy = \tfrac{1}{4}x(y+z).$$

Hence $\displaystyle\iint_S \vec{A}\,.\,\hat{n}\,dS = \iint_R \vec{A}\,.\,\hat{n}\,\frac{dydz}{|\hat{i}\,.\,\hat{n}|} = \iint_R \frac{1}{4}x(y+z)\frac{dydz}{\frac{1}{4}x} = \int_0^5\int_0^4 (y+z)\,dy\,dz$

$$= \int_0^5 \left[\frac{y^2}{2} + zy\right]_0^4 dz = \int_0^5 (8 + 4z)\,dz = \left[8z + 2z^2\right]_0^5 = 40 + 50 = 90.$$

Example 3. *If* $\vec{F} = (2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$, *then evaluate* $\iiint_V \nabla\,.\,\vec{F}\,dV$, *where V is bounded by the planes* $x = 0, y = 0, z = 0$ *and* $2x + 2y + z = 4$.

Sol. $\quad \nabla\,.\,\vec{F} = \dfrac{\partial}{\partial x}(2x^2 - 3z) + \dfrac{\partial}{\partial y}(-2xy) + \dfrac{\partial}{\partial z}(-4x) = 4x - 2x = 2x$

$$\therefore \quad \iiint_V \nabla\,.\,\vec{F}\,dV = \iiint_V 2x\,dx\,dy\,dz$$

$$= \int_0^2\int_0^{2-x}\int_0^{4-2x-2y} 2x\,dz\,dy\,dx = \int_0^2\int_0^{2-x} 2x\left[z\right]_0^{4-2x-2y} dy\,dx$$

$$= \int_0^2\int_0^{2-x} 2x(4 - 2x - 2y)\,dy\,dx = \int_0^2\int_0^{2-x} [4x(2-x) - 4xy]\,dy\,dx$$

$$= \int_0^2 \left[4x(2-x)y - 2xy^2\right]_0^{2-x} dx = \int_0^2 [4x(2-x)^2 - 2x(2-x)^2]\,dx$$

$$= \int_0^2 2x(2-x)^2\,dx = 2\int_0^2 (4x - 4x^2 + x^3)\,dx$$

$$= 2\left[2x^2 - 4\frac{x^3}{3} + \frac{x^4}{4}\right]_0^2 = 2\left(8 - \frac{32}{3} + 4\right) = \frac{8}{3}.$$

Example 4. *If* $\vec{A} = 2xz\hat{i} - x\hat{j} + y^2\hat{k}$, *evaluate* $\iiint_V \vec{A}\,dV$, *where V is the region bounded by the surface* $x = 0, y = 0, x = 2, y = 6, z = x^2, z = 4$.

Sol. $\displaystyle\iiint_V \vec{A}\,dV = \iiint_V (2xz\hat{i} - x\hat{j} + y^2\hat{k})\,dx\,dy\,dz$

$$= \int_0^2\int_0^6\int_{x^2}^4 (2xz\hat{i} - x\hat{j} + y^2\hat{k})\,dz\,dy\,dx$$

$$= \hat{i}\int_0^2\int_0^6\int_{x^2}^4 2xz\,dz\,dy\,dx - \hat{j}\int_0^2\int_0^6\int_{x^2}^4 x\,dz\,dy\,dx + \hat{k}\int_0^2\int_0^6\int_{x^2}^4 y^2\,dz\,dy\,dx$$

$$= \hat{i}\int_0^2\int_0^6\left[xz^2\right]_{x^2}^4 dy\,dx - \hat{j}\int_0^2\int_0^6\left[xz\right]_{x^2}^4 dy\,dx + \hat{k}\int_0^2\int_0^6\left[y^2 z\right]_{x^2}^4 dy\,dx$$

$$= \hat{i}\int_0^2\int_0^6 (16x - x^5)\,dy\,dx - \hat{j}\int_0^2\int_0^6 (4x - x^3)\,dy\,dx + \hat{k}\int_0^2\int_0^6 y^2(4 - x^2)\,dy\,dx$$

$$= \hat{i}\int_0^2 (16x - x^5)\left[y\right]_0^6 dx - \hat{j}\int_0^2 (4x - x^3)\left[y\right]_0^6 dx + \hat{k}\int_0^2 (4 - x^2)\left[\frac{y^3}{3}\right]_0^6 dx$$

$$= 6\hat{i}\int_0^2 (16x - x^5)\,dx - 6\hat{j}\int_0^2 (4x - x^3)\,dx + 72\hat{k}\int_0^2 (4 - x^2)\,dx$$

$$= 6\hat{i}\left[8x^2 - \frac{x^6}{6}\right]_0^2 - 6\hat{j}\left[2x^2 - \frac{x^4}{4}\right]_0^2 + 72\hat{k}\left[4x - \frac{x^3}{3}\right]_0^2 = 128\hat{i} - 24\hat{j} + 384\hat{k}.$$

EXERCISE 13.4

1. If $\vec{f}(t) = t\hat{i} + (t^2 - 2t)\hat{j} + (3t^2 + 3t^3)\hat{k}$, find $\int_0^1 \vec{f}(t)\,dt$.

2. If $\vec{r} = t\hat{i} - t^2\hat{j} + (t-1)\hat{k}$ and $\vec{S} = 2t^2\hat{i} + 6t\hat{k}$, evaluate

   (i) $\int_0^2 \vec{r}\cdot\vec{S}\,dt$ (ii) $\int_0^2 \vec{r}\times\vec{S}\,dt$.

3. Find the value of $\vec{r}$ satisfying the equation $\frac{d^2\vec{r}}{dt^2} = 6t\hat{i} - 24t^2\hat{j} + 4\sin t\hat{k}$, given that $\vec{r} = 2\hat{i} + \hat{j}$ and $\frac{d\vec{r}}{dt} = -\hat{i} - 3\hat{k}$ at $t = 0$.

4. The acceleration of a particle at any time t is given by $\vec{a} = 12\cos 2t\hat{i} - 8\sin 2t\hat{j} + 16t\hat{k}$. If the velocity $\vec{v}$ and displacement $\vec{r}$ are zero at $t = 0$, find $\vec{v}$ and $\vec{r}$ at any time t.

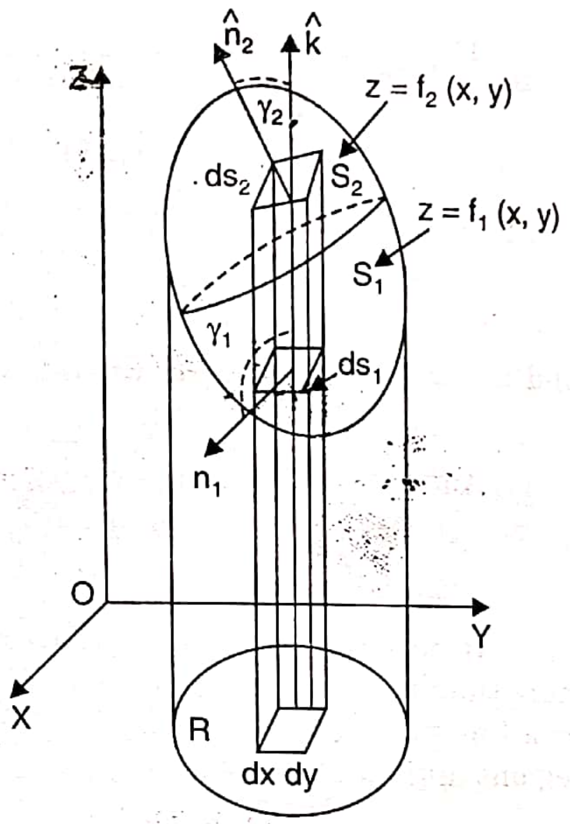
5. (a) If $\phi = 2xyz^2$, $\vec{F} = xy\hat{i} - z\hat{j} + x^2\hat{k}$, and C is the curve $x = t^2, y = 2t, z = t^3$ from $t = 0$ to $t = 1$, evaluate the line integrals

   (i) $\int_C \phi\,d\vec{r}$ (ii) $\int_C \vec{F}\times d\vec{r}$.

13.27. DIVERGENCE THEOREM OF GAUSS (*Relation between surface and volume integrals*)

If $\vec{F}$ is a vector point function having continuous first order partial derivatives in the **region V bounded by a closed surface S,** *then* $\iiint_V \nabla \cdot \vec{F}\, dV = \iint_S \vec{F} \cdot \hat{n}\, dS$, *where $\hat{n}$ is the outwards drawn unit normal vector to the surface S.*

[*i.e.*, the volume integral of the divergence of a vector point function $\vec{F}$ taken over the volume V enclosed by a surface S, is equal to the surface integral of the normal component of $\vec{F}$ taken over the closed surface S].



Let $\quad \vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$, then

$$\nabla \cdot \vec{F} = \text{div}\,\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Let α, β, γ be the angles which the outwards drawn unit normal vector $\hat{n}$ makes with the positive directions of x, y, z-axes respectively, then $\cos\alpha, \cos\beta, \cos\gamma$ are the direction cosines of $\hat{n}$ and $\hat{n} = \cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k}$.

$$\therefore\ \vec{F} \cdot \hat{n} = (F_1\hat{i} + F_2\hat{j} + F_3\hat{k}) \cdot (\cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k})$$
$$= F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma$$

Therefore, the cartesian equivalent of divergence theorem is $\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz$

$$= \iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)\, dS \qquad ...(1)$$
$$= \iint_S (F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy)$$

since $\cos\alpha\, dS = dy\, dz$, etc.

Suppose that S is such a closed surface that a line parallel to the co-ordinate axes meets it in two points only. Let S_1 and S_2 denote the lower and upper portions of S with equations $z = f_1(x, y)$ and $z = f_2(x, y)$ respectively.

Let R be the projection of S on the xy-plane, then

$$\iiint_V \frac{\partial F_3}{\partial z}\, dx\, dy\, dz = \iint_R \left[\int_{z=f_1(x,y)}^{z=f_2(x,y)} \frac{\partial F_3}{\partial z}\, dz\right] dx\, dy$$
$$= \iint_R \Big[F_3(x, y, z)\Big]_{f_1}^{f_2} dx\, dy = \iint_R [F_3(x, y, f_2) - F_3(x, y, f_1)]\, dx\, dy$$
$$= \iint_R F_3(x, y, f_2)\, dx\, dy - \iint_R F_3(x, y, f_1)\, dx\, dy \qquad ...(2)$$

Now for the upper portion S_2 of S, the normal $\hat{n}_2$ to S_2 makes an acute angle γ_2 with $\hat{k}$.

$$\therefore\quad dx\, dy = \cos\gamma_2\, dS_2 = \hat{k} \cdot \hat{n}_2\, dS_2$$

For the lower portion S_1 of S, the normal $\hat{n}_1$ to S_1 makes an obtuse angle γ_1 with $\hat{k}$.

$$\therefore \quad dx\,dy = -\cos\gamma_1\, dS_1 = -\hat{k}\cdot\hat{n}_1\, dS_1$$

$$\therefore \quad \iint_R F_3(x, y, f_2)\, dx\, dy = \iint_{S_2} F_3\, \hat{k}\cdot\hat{n}_2\, dS_2 \qquad \ldots(3)$$

and
$$\iint_R F_3(x, y, f_1)\, dxdy = -\iint_{S_1} F_3\, \hat{k}\cdot\hat{n}_1\, dS_1 \qquad \ldots(4)$$

Using (3) and (4), (2) becomes $\iiint_V \frac{\partial F_3}{\partial z}\, dx\,dy\,dz = \iint_{S_2} F_3\,\hat{k}\cdot\hat{n}_2\, dS_2 + \iint_{S_1} F_3\,\hat{k}\cdot\hat{n}_1\, dS_1$

$$= \iint_S F_3\hat{k}\cdot\hat{n}\, dS = \iint_R F_3\cos\gamma\, dS \qquad \ldots(5)$$

Similarly, by considering the projection of S on yz and zx-planes, we have

$$\iiint_V \frac{\partial F_1}{\partial x}\, dx\,dy\,dz = \iint_S F_1\hat{i}\cdot\hat{n}\, dS = \iint_S F_1\cos\alpha\, dS \qquad \ldots(6)$$

and
$$\iiint_V \frac{\partial F_2}{\partial y}\, dx\,dy\,dz = \iint_S F_2\hat{j}\cdot\hat{n}\, dS = \iint_S F_2\cos\beta\, dS \qquad \ldots(7)$$

Adding (5), (6) and (7), we get (1) *i.e.*, $\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz$

$$= \iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)\, dS \quad \text{or} \quad \iiint_V \nabla\cdot\vec{F}\, dV = \iint_R \vec{F}\cdot\hat{n}\, dS$$

In case the region be such that the lines drawn parallel to the coordinate axes meet it in more than two points, then we divide the region into various sub-regions each of which is met by a line parallel to any axis in only two points. Applying the theorem to each of these sub-regions and adding the results, we get the volume integral over the whole region.

ILLUSTRATIVE EXAMPLES

Example 1. *For any closed surface S, prove that* $\iint_S curl\,\vec{F}\cdot\hat{n}\, dS = 0$.

Sol. By the divergence theorem, we have $\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\, dS = \iiint_V (\text{div curl}\,\vec{F})\, dV$,

where V is the volume enclosed by S = 0. Since div curl $\vec{F} = 0$, therefore, $\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\, dS = 0$

Example 2. *Evaluate* $\iint_S \vec{r}\cdot\hat{n}\, dS$, *where S is a closed surface.*

Sol. By the divergence theorem, we have $\iint_S \vec{r}\cdot\hat{n}\, dS = \iiint_V \nabla\cdot\vec{r}\, dV$,

where V is the volume enclosed by S

$$= \iiint_V 3dV, \text{ since } \nabla\cdot\vec{r} = \text{div}\,\vec{r} = 3$$

$$= 3V.$$

Example 3. *Use divergence theorem to show that* $\oint_S \nabla r^2\, d\vec{S} = 6V$, *where S is any closed surface enclosing a volume V.*

Sol. By the divergence theorem, we have $\oint_S \nabla r^2 \, d\vec{S} = \int_V \text{div}\,(\nabla r^2)\, dV$

$$= \int_V \nabla \cdot (\nabla r^2)\, dV = \int_V \nabla^2 r^2 \, dV$$

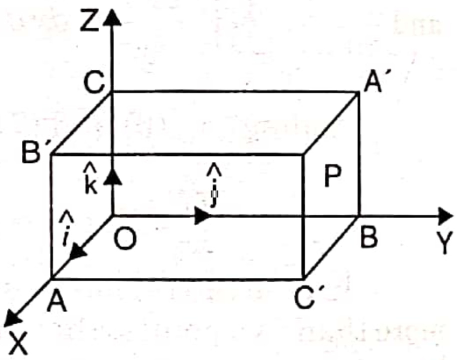
$$= \int_V \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) (x^2 + y^2 + z^2)\, dV = \int_V 6 dV = 6V.$$

Example 4. *Verify divergence theorem for $\vec{F} = (x^2 - yz)\,\hat{i} + (y^2 - zx)\,\hat{j} + (z^2 - xy)\,\hat{k}$ taken over the rectangular parallelopiped $0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c$.*

Sol. For verification of divergence theorem, we shall evaluate the volume and surface integrals separately and show that they are equal.

Now $\quad \text{div}\,\vec{F} = \dfrac{\partial}{\partial x}(x^2 - yz) + \dfrac{\partial}{\partial y}(y^2 - zx) + \dfrac{\partial}{\partial z}(z^2 - xy)$

$$= 2(x + y + z)$$



$\therefore \quad \iiint_V \text{div}\,\vec{F}\, dV$

$$= \int_0^c \int_0^b \int_0^a 2(x + y + z)\, dx dy dz$$

$$= \int_0^c \int_0^b 2\left[\left(\frac{x^2}{2} + yx + zx\right)\right]_0^a dy\, dz$$

$$= \int_0^c \int_0^b 2\left(\frac{a^2}{2} + ya + za\right) dy\, dz = \int_0^c 2\left[\left(\frac{a^2}{2} y + \frac{ay^2}{2} + azy\right)\right]_0^b dz$$

$$= 2\int_0^c \left(\frac{a^2 b}{2} + \frac{ab^2}{2} + abz\right) dz = 2\left[\frac{a^2 b}{2} z + \frac{ab^2}{2} z + ab\frac{z^2}{2}\right]_0^c$$

$$= a^2bc + ab^2c + abc^2 = abc\,(a + b + c) \qquad ...(1)$$

To evaluate the surface integral, divide the closed surface S of the rectangular parallelopiped into 6 parts.

S_1 : the face OAC′B, S_2 : the face CB′PA′, S_3 : the face OBA′C,
S_4 : the face AC′PB′, S_5 : the face OCB′A, S_6 : the face BA′PC′.

Also $\iint_S \vec{F} \cdot \hat{n}\, dS = \iint_{S_1} \vec{F} \cdot \hat{n}\, dS + \iint_{S_2} \vec{F} \cdot \hat{n}\, dS + \iint_{S_3} \vec{F} \cdot \hat{n}\, dS$

$$+ \iint_{S_4} \vec{F} \cdot \hat{n}\, dS + \iint_{S_5} \vec{F} \cdot \hat{n}\, dS + \iint_{S_6} \vec{F} \cdot \hat{n}\, dS$$

On S_1 $(z = 0)$, we have $\hat{n} = -\hat{k}$, $\vec{F} = x^2\hat{i} + y^2\hat{j} - xy\hat{k}$

so that $\quad \vec{F} \cdot \hat{n} = (x^2\hat{i} + y^2\hat{j} - xy\hat{k}) \cdot (-\hat{k}) = xy$

$$\therefore \quad \iint_S \vec{F} \cdot \hat{n}\, dS = \int_0^b \int_0^a xy\, dx\, dy = \int_0^b \left[y\frac{x^2}{2}\right]_0^a dy = \frac{a^2}{2}\int_0^b y\, dy = \frac{a^2 b^2}{4}$$

On S_2 $(z = c)$, we have $\hat{n} = \hat{k}$, $\vec{F} = (x^2 - cy)\hat{i} + (y^2 - cx)\hat{j} + (c^2 - xy)\hat{k}$

so that $$\vec{F}.\hat{n} = [(x^2 - cy)\hat{i} + (y^2 - cx)\hat{j} + (c^2 - xy)\hat{k}] \,.\, \hat{k} = c^2 - xy$$

$$\iint_{S_2} \vec{F}.\hat{n}\, dS = \int_0^b \int_0^a (c^2 - xy)\, dx\, dy = \int_0^b \left(c^2 a - \frac{a^2}{2} y\right) dy = abc^2 - \frac{a^2 b^2}{4}$$

On S_3 $(x = 0)$, we have $\hat{n} = -\hat{i}$, $\vec{F} = -yz\hat{i} + y^2\hat{j} + z^2\hat{k}$

so that $$\vec{F}.\hat{n} = (-yz\hat{i} + y^2\hat{j} + z^2\hat{k}) \,.\, (-\hat{i}) = yz$$

$$\therefore \quad \iint_{S_3} \vec{F}.\hat{n}\, dS = \int_0^c \int_0^b yz\, dy\, dz = \int_0^c \frac{b^2}{2} z\, dz = \frac{b^2 c^2}{4}$$

On S_4 $(x = a)$, we have $\hat{n} = \hat{i}$, $\vec{F} = (a^2 - yz)\hat{i} + (y^2 - az)\hat{j} + (z^2 - ay)\hat{k}$

so that $$\vec{F}.\hat{n} = [(a^2 - yz)\hat{i} + (y^2 - az)\hat{j} + (z^2 - ay)\hat{k}] \,.\, \hat{i} = a^2 - yz$$

$$\therefore \quad \iint_{S_4} \vec{F}.\hat{n}\, dS = \int_0^c \int_0^b (a^2 - yz)\, dy\, dz = \int_0^c \left(a^2 b - \frac{b^2}{2} z\right) dz = a^2 bc - \frac{b^2 c^2}{4}$$

On S_5 $(y = 0)$, we have $\hat{n} = -\hat{j}$, $\vec{F} = x^2\hat{i} - zx\hat{j} + z^2\hat{k}$

so that $$\vec{F}.\hat{n} = (x^2\hat{i} - zx\hat{j} + z^2\hat{k}).(-\hat{j}) = zx$$

$$\therefore \quad \iint_{S_5} \vec{F}.\hat{n}\, dS = \int_0^a \int_0^c zx\, dz\, dx = \int_0^a \frac{c^2}{2} x\, dx = \frac{a^2 c^2}{4}$$

On S_6 $(y = b)$, we have $\hat{n} = \hat{j}$, $\vec{F} = (x^2 - bz)\hat{i} + (b^2 - zx)\hat{j} + (z^2 - bx)\hat{k}$

so that $$\vec{F}.\hat{n} = [(x^2 - bz)\hat{i} + (b^2 - zx)\hat{j} + (z^2 - bx)\hat{k}] \,.\, \hat{j} = b^2 - zx$$

$$\therefore \quad \iint_{S_6} \vec{F}.\hat{n}\, dS = \int_0^a \int_0^c (b^2 - zx)\, dz\, dx = \int_0^a \left(b^2 c - \frac{c^2}{2} x\right) dx = ab^2 c - \frac{a^2 c^2}{4}$$

$$\therefore \quad \iint_S \vec{F}.\hat{n}\, dS = \frac{a^2 b^2}{4} + abc^2 - \frac{a^2 b^2}{4} + \frac{b^2 c^2}{4} + a^2 bc - \frac{b^2 c^2}{4} + \frac{a^2 c^2}{4} + ab^2 c - \frac{a^2 c^2}{4}$$

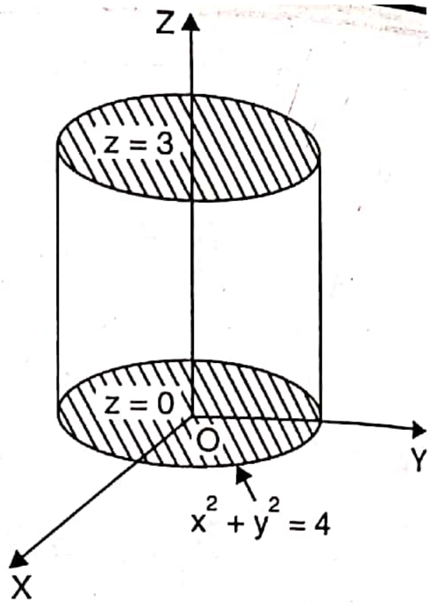
$$= abc\,(a + b + c) \qquad \ldots(2)$$

The equality of (1) and (2) verifies divergence theorem.

Example 5. *Verify divergence theorem for $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ taken over the region bounded by the cylinder $x^2 + y^2 = 4$, $z = 0$, $z = 3$.*

Sol. Since div $\vec{F} = \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2) = 4 - 4y + 2z$

$$\therefore \quad \iiint_V \text{div}\, \vec{F}\, dV = \iiint_V (4 - 4y + 2z)\, dx\, dy\, dz$$



$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^3 (4 - 4y + 2z)\, dzdydx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \Big[4z - 4yz + z^2\Big]_0^3 dydx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (12 - 12y + 9)\, dy\, dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 21\, dy\, dx,$$

(Since $12y$ is an odd function, $\int_{-a}^{a} 12y\, dy = 0$)

$$= \int_{-2}^{2} 42\sqrt{4-x^2}\, dx = 84\int_0^2 \sqrt{4-x^2}\, dx = 84\left[\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2 = 84[2\sin^{-1} 1]$$

$$= 84\left[2 \times \frac{\pi}{2}\right] = 84\pi \qquad \text{...(1)}$$

To evaluate the surface integral, divide the closed surface S of the cylinder into 3 parts.

S_1 : the circular base in the plane $z = 0$

S_2 : the circular top in the plane $z = 3$

S_3 : the curved surface of the cylinder, given by the equation $x^2 + y^2 = 4$.

Also $\iint_S \vec{F}.\hat{n}\, dS = \iint_{S_1} \vec{F}.\hat{n}\, dS + \iint_{S_2} \vec{F}.\hat{n}\, dS + \iint_{S_3} \vec{F}.\hat{n}\, dS$

On S_1 ($z = 0$), we have $\hat{n} = -\hat{k}, \vec{F} = 4x\hat{i} - 2y^2\hat{j}$

so that $\vec{F}.\hat{n} = (4x\hat{i} - 2y^2\hat{j}).(-\hat{k}) = 0$

$$\therefore \quad \iint_{S_1} \vec{F}.\hat{n}\, dS = 0$$

On S_2 ($z = 3$), we have $\hat{n} = \hat{k}, \vec{F} = 4x\hat{i} - 2y^2\hat{j} + 9\hat{k}$

so that $\vec{F}.\hat{n} = (4x\hat{i} - 2y^2\hat{j} + 9\hat{k}).\hat{k} = 9$

$$\therefore \quad \iint_{S_2} \vec{F}.\hat{n}\, dS = \iint_{S_2} 9dx\, dy = 9\iint_{S_2} dx\, dy$$

$$= 9 \times \text{area of surface } S_2 = 9\,(\pi\,.\,2^2) = 36\pi$$

On S_3, $x^2 + y^2 = 4$

A vector normal to the surface S_3 is given by $\nabla(x^2 + y^2) = 2x\hat{i} + 2y\hat{j}$

$\therefore \quad \hat{n}$ = a unit vector normal to surface S_3

$$= \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{4x^2 + 4y^2}} = \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{4 \times 4}}, \text{ since } x^2 + y^2 = 4$$

$$= \frac{x\hat{i} + y\hat{j}}{2}$$

$$\vec{F} \cdot \hat{n} = (4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}) \cdot \left(\frac{x\hat{i} + y\hat{j}}{2}\right) = 2x^2 - y^3$$

Also, on S_3, *i.e.*, $x^2 + y^2 = 4$, $x = 2\cos\theta$, $y = 2\sin\theta$ and $dS = 2d\theta\, dz$.

To cover the whole surface S_3, z varies from 0 to 3 and θ varies from 0 to 2π.

$$\therefore \quad \iint_{S_3} \vec{F} \cdot \hat{n}\, dS = \int_0^{2\pi}\int_0^3 [2(2\cos\theta)^2 - (2\sin\theta)^3]\, 2dz\, d\theta$$

$$= \int_0^{2\pi} 16(\cos^2\theta - \sin^3\theta) \times 3\, d\theta = 48\int_0^{2\pi}(\cos^2\theta - \sin^3\theta)\, d\theta = 48\pi$$

$$\left(\text{since} \int_0^{2\pi}\cos^2\theta\, d\theta = 2\int_0^{\pi}\cos^2\theta\, d\theta = 4\int_0^{\pi/2}\cos^2 d\theta = 4 \times \frac{1}{2} \times \frac{\pi}{2} = \pi, \int_0^{2\pi}\sin^3\theta\, d\theta = 0\right)$$

$$\therefore \quad \iint_S \vec{F} \cdot \hat{n}\, dS = 0 + 36\pi + 48\pi = 84\pi \qquad ...(2)$$

The equality of (1) and (2) verifies divergence theorem.

Example 6. *Find* $\iint_S \vec{F} \cdot \hat{n}\, dS$, *where* $\vec{F} = (2x + 3z)\hat{i} - (xz + y)\hat{j} + (y^2 + 2z)\hat{k}$ *and S is the surface of the sphere having centre at (3, – 1, 2) and radius 3.*

Sol. Let V be the volume enclosed by the surface S. Then by Gauss divergence theorem, we have

$$\iint_S \vec{F} \cdot d\mathrm{S} = \iiint_V \operatorname{div}\vec{F}\, dV$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(2x + 3z) + \frac{\partial}{\partial y}(-xz - y) + \frac{\partial}{\partial z}(y^2 + 2z)\right] dV$$

$$= \iiint_V (2 - 1 + 2)\, dV = 3\iiint_V dV = 3V$$

But V is the volume of a sphere of radius 3.

$$\therefore \quad V = \frac{4}{3}\pi(3)^3 = 36\pi.$$

Hence $\iint_S \vec{F} \cdot dS = 3 \times 36\pi = 108\pi$.

Example 7. *Evaluate* $\iint_S (y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k}) \cdot \hat{n}\, dS$, *where S is the part of the sphere* $x^2 + y^2 + z^2 = 1$ *above the xy-plane and bounded by this plane.*

Sol. Let V be the volume enclosed by the surface S. Then by divergence theorem, we have

$$\iint_S (y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k}) \cdot \hat{n}\, dS = \iiint_V \operatorname{div}(y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k})\, dV$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(y^2z^2) + \frac{\partial}{\partial y}(z^2x^2) + \frac{\partial}{\partial z}(z^2y^2)\right] dV$$

$$= \iiint_V 2zy^2\, dV = 2\iiint_V zy^2\, dV$$

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Changing to spherical polar coordinates by putting

$$x = r \sin\theta \cos\phi,\ y = r \sin\theta \sin\phi,\ z = r\cos\theta$$

$$dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

To cover V, the limits of r will be 0 to 1, those of θ will be 0 to $\frac{\pi}{2}$ and those of ϕ will be 0 to 2π.

$$\therefore \quad 2\iiint_V zy^2 \, dV = 2\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 (r\cos\theta)(r^2\sin^2\theta\sin^2\phi)\, r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= 2\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 r^5 \sin^3\theta\cos\theta\sin^2\phi \, dr\, d\theta\, d\phi$$

$$= 2\int_0^{2\pi}\int_0^{\pi/2} \sin^3\theta\cos\theta\sin^2\phi \left[\frac{r^6}{6}\right]_0^1 d\theta\, d\phi$$

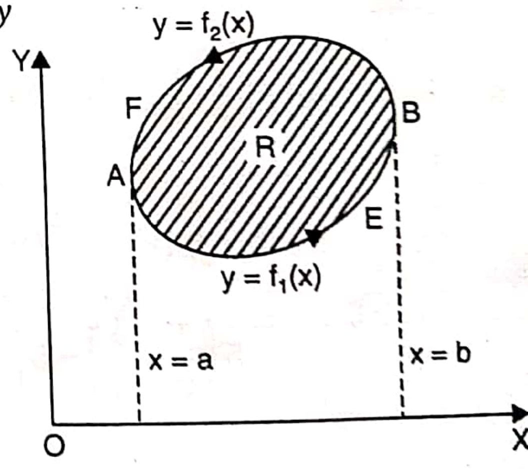
$$= \frac{2}{6}\int_0^{2\pi} \sin^2\phi \cdot \frac{2}{4.2}\, d\phi = \frac{1}{12}\int_0^{2\pi} \sin^2\phi \, d\phi = \frac{\pi}{12}.$$

13.28. GREEN'S THEOREM IN THE PLANE

If $M(x, y)$ and $N(x, y)$ be continuous functions of x and y having continuous partial derivatives $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ in a region R of the xy-plane bounded by a closed curve C, then

$$\oint_C (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

where C is traversed in the counterclockwise direction.



Let us assume that the region R is such that any line parallel to either axes meets the boundary curve C in at most two points.

[The proof can be easily extended to other cases.]

Suppose the region R is bounded between the lines $x = a$, $x = b$ and two arcs AEB and BFA whose equations are $y = f_1(x)$ and $y = f_2(x)$ respectively such that $f_2(x) > f_1(x)$.

Now $\iint_R \frac{\partial M}{\partial y}\, dx\, dy = \int_a^b \left[\int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y}\, dy\right] dx$

$$= \int_a^b \left[M(x, y)\right]_{f_1(x)}^{f_2(x)} dx = \int_a^b [M(x, f_2) - M(x, f_1)]\, dx$$

$$= \int_a^b M(x, f_2)\, dx - \int_a^b M(x, f_1)\, dx = -\int_b^a M(x, f_2)\, dx - \int_a^b M(x, f_1)\, dx$$

$$= -\left[\int_a^b M(x, f_1)\, dx + \int_b^a M(x, f_2)\, dx\right] = -\oint_C M dx$$

or $$\oint_C M dx = -\iint_R \frac{\partial M}{\partial y}\, dx\, dy \qquad ...(1)$$

Similarly, we can show that $\oint_C N dy = \iint_R \frac{\partial N}{\partial x}\, dx\, dy$ ...(2)

Adding (1) and (2), we have $\oint_C (M dx + N dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$

This theorem is useful for changing a line integral around a closed curve C into a double integral over the region R enclosed by C.

ILLUSTRATIVE EXAMPLES

Example 1. *Verify Green's theorem in the plane for* $\oint_C (3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy$ *where C is the boundary of the region defined by*

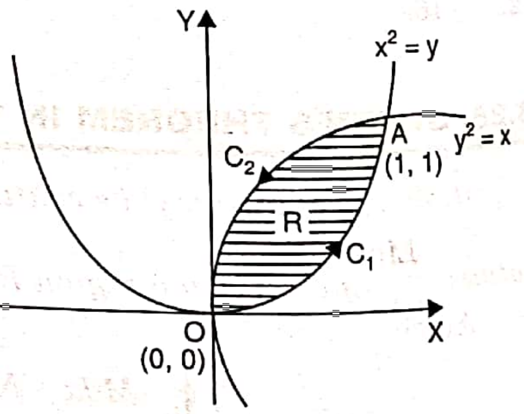
(a) $y = \sqrt{x}$, $y = x^2$ *(b)* $x = 0, y = 0, x + y = 1.$

Sol. *(a)* $y = \sqrt{x}$ *i.e.,* $y^2 = x$ and $y = x^2$ are two parabolas intersecting at O(0, 0) and A(1, 1).

Here $M = 3x^2 - 8y^2$, $N = 4y - 6xy$

$$\frac{\partial M}{\partial y} = -16y, \quad \frac{\partial N}{\partial x} = -6y$$

$$\therefore \quad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 10y$$



If R is the region bounded by C, then

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} 10y\, dy\, dx = \int_0^1 5\left[y^2\right]_{x^2}^{\sqrt{x}} dx$$

$$= 5\int_0^1 (x - x^4)\, dx = 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$$

$$= 5\left(\frac{1}{2} - \frac{1}{5}\right) = 5\left(\frac{3}{10}\right) = \frac{3}{2} \qquad ...(1)$$

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Also, $\oint_C (Mdx + Ndy) = \int_{C_1} (Mdx + Ndy) + \int_{C_2} (Mdx + Ndy)$

Along C_1, $x^2 = y$, $\therefore$ $2x\,dx = dy$ and the limits of x are from 0 to 1.

$\therefore$ Line integral along $C_1 = \int_{C_1} (Mdx + Ndy)$

$$= \int_0^1 (3x^2 - 8x^4)\,dx + (4x^2 - 6x\,.\,x^2)\,2x\,dx = \int_0^1 (3x^2 + 8x^3 - 20x^4)\,dx$$

$$= \left[x^3 + 2x^4 - 4x^5\right]_0^1 = -1$$

Along C_2, $y^2 = x$, $\therefore$ $2y\,dy = dx$ and the limits of y are from 1 to 0.

$\therefore$ Line integral along $C_2 = \int_{C_2} (Mdx + Ndy)$

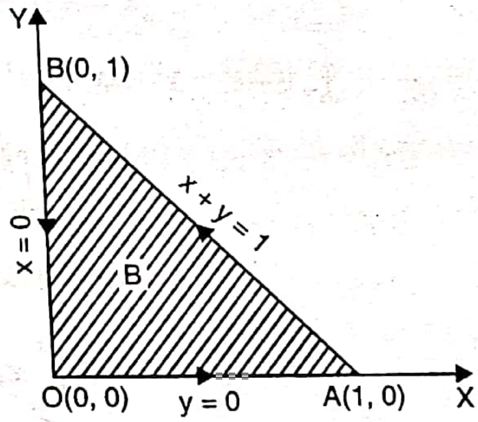
$$= \int_1^0 (3y^4 - 8y^2)\,2ydy + (4y - 6y^2\,.\,y)\,dy$$

$$= \int_1^0 (4y - 22y^3 + 6y^5)\,dy = \left[2y^2 - \frac{11}{2}y^4 + y^6\right]_1^0 = \frac{5}{2}$$

$\therefore$ Line integral along $C = -1 + \frac{5}{2} = \frac{3}{2}$ i.e., $\oint_C (Mdx + Ndy) = \frac{3}{2}$ ...(2)

The equality of (1) and (2) verifies Green's theorem in the plane.

(b) Here $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_0^1 \int_0^{1-x} 10y\,dy\,dx$



$$= \int_0^1 5\left[y^2\right]_0^{1-x} dx$$

$$= 5\int_0^1 (1-x)^2\,dx = 5\left[\frac{(1-x)^3}{-3}\right]_0^1$$

$$= -\frac{5}{3}(0 - 1) = \frac{5}{3} \qquad \text{...(1)}$$

Along OA, $y = 0$ $\therefore$ $dy = 0$ and the limits of x are from 0 to 1.

$\therefore$ Line integral along OA $= \int_0^1 3x^2\,dx = \left[x^3\right]_0^1$

Along AB, $y = 1 - x$ $\therefore$ $dy = -dx$ and the limits of x are from 1 to 0.

$\therefore$ Line integral along AB $= \int_1^0 [3x^2 - 8(1-x)^2]\,dx + [4(1-x) - 6x(1-x)]\,(-dx)$

$$= \int_1^0 (3x^2 - 8 + 16x - 8x^2 - 4 + 4x + 6x - 6x^2)\,dx = \int_1^0 (-12 + 26x - 11x^2)\,dx$$

$$= \left[-12x + 13x^2 - \frac{11}{3}x^3\right]_1^0 = -\left[-12 + 13 - \frac{11}{3}\right] = \frac{8}{3}$$

Along BO, $x = 0$ $\therefore$ $dx = 0$ and the limits of y are from 1 to 0.

$\therefore$ Line integral along BO $= \int_1^0 4y\,dy = \left[2y^2\right]_1^0 = -2$

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$\therefore$ Line integral along C (*i.e.*, along OABO) $= 1+\frac{8}{3}-2=\frac{5}{3}$

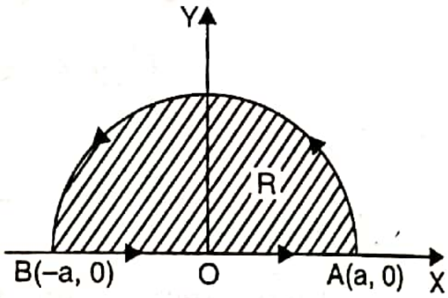
i.e.,
$$\oint_C (\mathrm{M}dx + \mathrm{N}dy) = \frac{5}{3} \qquad ...(2)$$

The equality of (1) and (2) verifies Green's theorem in the plane.

Example 2. *Use Green's theorem in a plane to evaluate the integral* $\oint_C [(2x^2-y^2)\,dx + (x^2+y^2)\,dy]$ *where C is the boundary in the xy-plane of the area enclosed by the x-axis and the semi circle* $x^2+y^2=a^2$ *in the upper half xy-plane.*

Sol. If R is the region bounded by the closed curve C, then by Green's theorem in the plane, we have

$$\oint_C (\mathrm{M}dx + \mathrm{N}dy) = \iint_R \left(\frac{\partial \mathrm{N}}{\partial x} - \frac{\partial \mathrm{M}}{\partial y}\right) dx\,dy$$



Here $\quad \mathrm{M} = 2x^2 - y^2,\ \mathrm{N} = x^2 + y^2$

$$\frac{\partial \mathrm{M}}{\partial y} = -2y,\ \frac{\partial \mathrm{N}}{\partial x} = 2x$$

$$\therefore \quad \frac{\partial \mathrm{N}}{\partial x} - \frac{\partial \mathrm{M}}{\partial y} = 2(x+y)$$

The region R is bounded by

$$x = -a,\ x = a,\ y = 0,\ y = \sqrt{a^2-x^2}$$

$$\therefore \quad \oint_C [(2x^2-y^2)\,dx + (x^2+y^2)\,dy] = \iint_R 2(x+y)\,dx\,dy$$

$$= \int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} 2(x+y)\,dy\,dx = \int_{-a}^{a}\left[2xy+y^2\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_{-a}^{a} [2x\sqrt{a^2-x^2} + (a^2-x^2)]\,dx$$

$$= 2\int_{-a}^{a} x\sqrt{a^2-x^2}\,dx + \int_{-a}^{a}(a^2-x^2)\,dx$$

$$= 2\,(0) + 2\int_0^a (a^2-x^2)\,dx$$

[$\because$ $x\sqrt{a^2-x^2}$ is an odd function and (a^2-x^2) is an even function]

$$= 2\left[a^2x - \frac{x^3}{3}\right]_0^a = 2\left(a^3 - \frac{a^3}{3}\right) = \frac{4a^3}{3}.$$

EXERCISE 13.6

1. Verify Green's theorem in the plane for $\oint_C (xy+y^2)\,dx + x^2\,dy$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

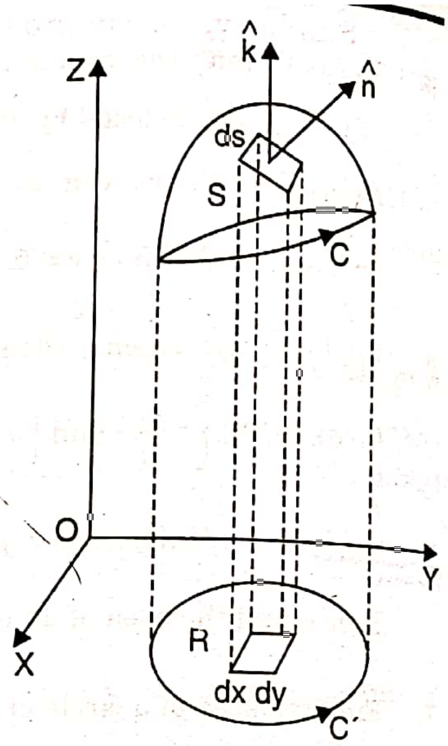
2. Verify Green's theorem in the plane for $\oint_C (2xy - x^2)\,dx + (x^2+y^2)\,dy$ where C is the boundary of the region enclosed by $y = x^2$ and $y^2 = x$.

13.29. STOKE'S THEOREM (*Relation between line and surface integrals*)

If S be an open surface bounded by a closed curve C and $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ *be any vector point function having continuous first order partial derivatives, then* $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \hat{n}\, dS$ *where* $\hat{n}$ *is a unit normal vector at any point of S drawn in the sense in which a right handed screw would advance when rotated in the sense of description of C.*

Let $\hat{n}$ make angles α, β, γ with positive directions of x, y, z axes, then $\hat{n} = \cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k}$.

Also $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ so that $d\vec{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$



$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$$

$$\therefore \quad \text{curl } \vec{F} \cdot \hat{n} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha$$

$$+ \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma$$

Also $\vec{F} \cdot d\vec{r} = (F_1\hat{i} + F_2\hat{j} + F_3\hat{k}) \times (\hat{i}dx + \hat{j}dy + \hat{k}dz)$

$$= F_1\,dx + F_2\,dy + F_3\,dz$$

$\therefore$ Stoke's theorem can be written as $\oint_C (F_1dx + F_2dy + F_3dz)$

$$= \iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma\right] dS \quad \text{...(1)}$$

Let $z = \phi(x, y)$ be the equation of the surface S whose projection on the xy-plane is R. Then the projection of C on the xy-plane is the curve C′ which bounds the region R.

$$\therefore \quad \oint_C F_1(x, y, z)\,dx = \oint_{C'} F_1(x, y, \phi)\,dx = \oint_{C'} [F_1(x, y, \phi)\,dx + 0\,dy]$$

$$= -\iint_R \frac{\partial}{\partial y} F_1(x, y, \phi)\,dx\,dy \quad \text{by Green's theorem in plane for the region R}$$

$$= -\iint_R \left[\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial \phi}{\partial y}\right] dx\,dy \quad \text{...(2)}$$

Now the direction ratios of the normal $\hat{n}$ to the surface $z = \phi(x, y)$ are $\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, -1$

$$\Rightarrow \quad \frac{\cos\alpha}{\frac{\partial \phi}{\partial x}} = \frac{\cos\beta}{\frac{\partial \phi}{\partial y}} = \frac{\cos\gamma}{-1} \quad \Rightarrow \quad \frac{\partial \phi}{\partial y} = -\frac{\cos\beta}{\cos\gamma}$$

Since $dx\,dy$ is the projection of dS on the xy-plane

$\therefore \quad dxdy = \cos\gamma\, dS$

From (2), we have $\oint_C F_1(x, y, z)\,dx = -\iint_S \left[\frac{\partial F_1}{\partial y} - \frac{\partial F_1}{\partial z} \cdot \frac{\cos\beta}{\cos\gamma}\right] \cos\gamma\, dS$

$$= -\iint_S \left[\frac{\partial F_1}{\partial y}\cos\gamma - \frac{\partial F_1}{\partial z}\cos\beta\right] dS$$

$$= \iint_S \left[\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right] dS \quad \text{...(3)}$$

Similarly, we can prove that $\oint_C F_2(x, y, z)\, dy$

$$= \iint_S \left[\frac{\partial F_2}{\partial x}\cos\gamma - \frac{\partial F_2}{\partial z}\cos\alpha\right] dS \qquad ...(4)$$

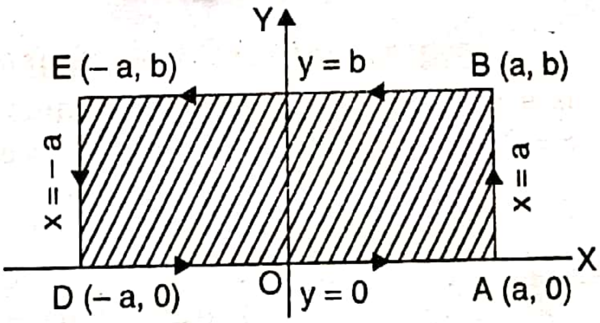
and $$\oint_C F_3(x, y, z)\, dz = \iint_S \left[\frac{\partial F_3}{\partial y}\cos\alpha - \frac{\partial F_3}{\partial x}\cos\beta\right] dS \qquad ...(5)$$

Adding (3), (4) and (5), we get (1). Hence the theorem is proved. In words, Stoke's theorem states that "The line integral of the tangential component of a vector point function $\vec{F}$ taken around a closed curve C is equal to the surface integral of the normal component of $\vec{F}$ taken over any surface S having C as its boundary".

ILLUSTRATIVE EXAMPLES

Example 1. *Verify Stoke's theorem for* $\vec{F} = (x^2 + y^2)\hat{i} - 2xy\,\hat{j}$ *taken round the rectangle bounded by the lines* $x = \pm a, y = 0, y = b$.

Sol. Let C denote the boundary of the rectangle ABED, then



$$\oint_C \vec{F}.d\vec{r} = \oint_C [(x^2 + y^2)\hat{i} - 2xy\hat{j}].(\hat{i}\,dx + \hat{j}\,dy)$$

$$= \oint_C [(x^2 + y^2)\,dx - 2xy\,dy]$$

The curve C consists of four lines AB, BE, ED and DA.

Along AB, $x = a$, $dx = 0$ and y varies from 0 to b.

$$\therefore \int_{AB} [(x^2 + y^2)\,dx - 2xy\,dy] = \int_0^b -2ay\,dy = -a\left[y^2\right]_0^b = -ab^2 \qquad ...(1)$$

Along BE, $y = b$, $dy = 0$ and x varies from a to $-a$.

$$\therefore \int_{BE} [(x^2 + y^2)\,dx - 2xy\,dy] = \int_a^{-a} (x^2 + b^2)\,dx = \left[\frac{x^3}{3} + b^2 x\right]_a^{-a} = -\frac{2a^2}{3} - 2ab^2 \qquad ...(2)$$

Along ED, $x = -a$, $dx = 0$ and y varies from b to 0.

$$\therefore \int_{ED} [(x^2 + y^2)\,dx - 2xy\,dy] = \int_a^0 2ay\,dy = a\left[y^2\right]_b^0 = -ab^2 \qquad ...(3)$$

Along DA, $y = 0$, $dy = 0$ and x varies from $-a$ to a.

$$\therefore \int_{DA} [(x^2 + y^2)\,dx - 2dx\,dy] = \int_{-a}^{a} x^2\,dx = \frac{2a^3}{3} \qquad ...(4)$$

Adding (1), (2), (3) and (4), we get

$$\oint_C \vec{F}.d\vec{r} = -ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2 + \frac{2a^3}{3} = -4ab^2 \qquad ...(5)$$

Now $\quad \text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix} = (-2y - 2y)\hat{k} = -4y\hat{k}$

For the surface S, $\hat{n} = \hat{k}$

$$\text{curl } \vec{F} . \hat{n} = -4y\hat{k} . \hat{k} = -4y$$

$$\therefore \quad \iint_S \text{curl } \vec{F} . \hat{n}\, dS = \int_0^b \int_{-a}^a -4y\, dx\, dy = \int_0^b -4y \left[x \right]_{-a}^a dy$$

$$= -8a \int_0^b y\, dy = -8a \left[\frac{y^2}{2} \right]_0^b = -4ab^2 \quad \ldots(6)$$

The equality of (5) and (6) verifies Stoke's Theorem.

Example 2. *Verify Stoke's Theorem for the vector field* $\vec{F} = (2x - y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$ *over the upper half surface of* $x^2 + y^2 + z^2 = 1$, *bounded by its projection on the xy-plane.*

(M.D.U. May 2006, May 2007, Dec. 2007; Madras 2006)

Sol. Let S be the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$. The boundary C of S is a circle in the xy-plane of radius unity and centre O. The equations of C are $x^2 + y^2 = 1$, $z = 0$ whose parametric form is $x = \cos t, y = \sin t, z = 0, 0 \le t < 2\pi$.

$$\oint_C \vec{F} . d\vec{r} = \oint_C [(2x - y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}] . (\hat{i}\, dx + \hat{j}\, dy + \hat{k}\, dz)$$

$$= \oint_C [(2x - y)\, dx - yz^2\, dy - y^2z\, dz]$$

$$= \oint_C (2x - y)\, dx, \text{ since on C, } z = 0, dz = 0$$

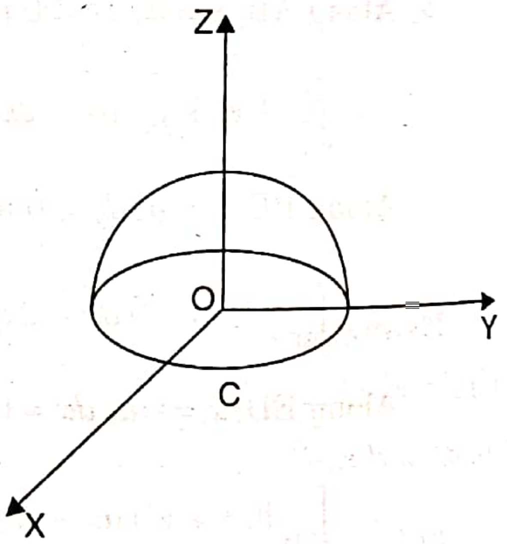
$$= \int_0^{2\pi} (2\cos t - \sin t) \frac{dx}{dt}\, dt$$

$$= \int_0^{2\pi} (2\cos t - \sin t)(-\sin t)\, dt$$

$$= \int_0^{2\pi} (-\sin 2t + \sin^2 t)\, dt$$

$$= \int_0^{2\pi} \left(-\sin 2t + \frac{1 - \cos 2t}{2} \right) dt$$

$$= \left[\frac{\cos 2t}{2} + \frac{t}{2} - \frac{\sin 2t}{4} \right]_0^{2\pi} = \tfrac{1}{2} + \pi - \tfrac{1}{2} = \pi$$



Also $\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2z \end{vmatrix} = (-2yz + 2yz)\hat{i} + (0 - 0)\hat{j} + (0 + 1)\hat{k} = \hat{k}$

$$\text{curl } \vec{F} . \hat{n} = \hat{k} . \hat{n} = \hat{n} . \hat{k}$$

$$\iint_S \text{curl } \vec{F} \cdot \hat{n}\, dS = \iint_S \hat{n} \cdot \hat{k}\, dS = \iint_R \hat{n} \cdot \hat{k} \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$$

where R is the projection of S on xy-plane.

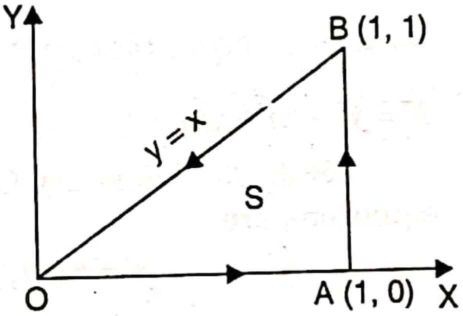
$$= \iint_R dx\,dy = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\,dx = \int_{-1}^{1} 2\sqrt{1-x^2}\, dx = 4\int_0^1 \sqrt{1-x^2}\, dx$$

$$= 4\left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1} x\right]_0^1 = 4\left[\frac{1}{2} \cdot \frac{\pi}{2}\right] = \pi$$

Since $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \hat{n}\, dS$, Stoke's theorem is verified.

Example 3. *Evaluate $\oint_C \vec{F} \cdot d\vec{r}$ by Stoke's Theorem, where $\vec{F} = y^2\hat{i} + x^2\hat{j} - (x+z)\hat{k}$ and C is the bour lary of the triangle with vertices at (0, 0, 0), (1, 0, 0) and (1, 1, 0).*

Sol. Since z-coordinates of each vertex of the triangle is zero, therefore, the triangle lies in the -plane and $\hat{n} = \hat{k}$



$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix} = \hat{j} + 2(x-y)\hat{k}$$

$\therefore$ curl $\vec{F} \cdot \hat{n} = [\hat{j} + 2(x-y)\hat{k}] \cdot \hat{k} = 2(x-y)$

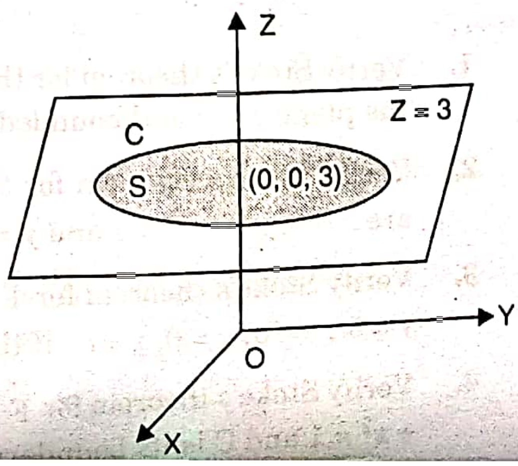
The equation of line OB is $y = x$.

By Stoke's theorem, $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \hat{n}\, dS$

$$= \int_0^1 \int_0^x 2(x-y)\, dy dx$$

$$= \int_0^1 2\left[xy - \frac{y^2}{2}\right]_0^x dx = 2\int_0^1 \left(x^2 - \frac{x^2}{2}\right) dx = \int_0^1 x^2\, dx = \frac{1}{3}.$$

Example 4. *Using Stoke's theorem for the vector function $\vec{F} = (2x + y - 2z)\,\hat{i} + (2x - 4y + z^2)\,\hat{j} + (x - 2y + z^2)\,\hat{k}$ evaluate the integral $\oint_C \vec{F} \cdot d\vec{R}$, where C is the circle with centre at (0, 0, 3) and radius 5 in the plane z = 3.*



Sol. Here $\vec{F} = (2x + y - 2z)\hat{i} + (2x - 4y + z^2)\hat{j} + (x - 2y + z^2)dz$ and C is the curve given by $(x-0)^2 + (y-0)^2 + (z-3)^2 = 5^2, z = 3$ *i.e.*, $x^2 + y^2 + (z-3)^2 = 25, z = 3$

[It is a great circle with centre at (0, 0, 3) and radius 5]

By Stoke's theorem

$$\oint_C \vec{F} \cdot d\vec{R} = \iint_S \text{curl } \vec{F} \cdot \hat{n}\, dS \qquad ...(1)$$

Now $\quad \text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x + y - 2z & 2x - 4y + z^2 & x - 2y + z^2 \end{vmatrix}$

$$= \hat{i}\,\{-2 - 2z\} - \hat{j}\,\{1 - (-2)\} + \hat{k}\,\{2 - 1\}$$

$$= (-2 - 2z)\,\hat{i} - 3\hat{j} + \hat{k}$$

$\hat{n}$ = a unit vector normal to the plane $z = 3$

i.e., $\quad \hat{n} = \hat{k}$

$\therefore$ From (1), $\oint_C \vec{F} \cdot d\vec{R} = \iint_S \left[(-2 - 2z)\,\hat{i} - 3\hat{j} + \hat{k}\right] . \hat{k} dS = \iint_S i dS = S$

$$= \text{area of circle} = \pi \times 5^2 = 25\pi.$$

Example 5. *Evaluate the surface integral $\iint_S \text{curl } \vec{F} \,.\, \hat{n}\, dS$ by transforming it into a line integral, S being that part of the surface of the paraboloid $z = 1 - x^2 - y^2$, for which $z \geq 0$ and $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$.*

Sol. The boundary C of the surface S is the circle $x^2 + y^2 = 1, z = 0$ whose parametric equations are

$$x = \cos t,\ y = \sin t,\ z = 0,\ 0 \leq t < 2\pi.$$

By Stoke's theorem, we have $\iint_S \text{curl } \vec{F} \,.\, \hat{n}\, dS = \oint_C \vec{F} \,.\, d\vec{r}$

$$= \oint_C (y\hat{i} + z\hat{j} + x\hat{k}) \,.\, (\hat{i}\, dx + \hat{j}\, dy + \hat{k}\, dz)$$

$$= \oint_C (y\, dx + z\, dy + x\, dz) = \oint_C y\, dx, \text{ since on C, } z = 0,\ dz = 0$$

$$= \oint_C y \frac{dx}{dt} dt = \int_0^{2\pi} \sin t(-\sin t)\, dt = -\int_0^{2\pi} \sin^2 t\, dt$$

$$= -2 \int_0^{\pi} \sin^2 t\, dt = -4 \int_0^{\pi/2} \sin^2 t\, dt = -4 \times \frac{1}{2} \times \frac{\pi}{2} = -\pi.$$