Example 4. *Find the domain of existence of the function :*

(a) $y = \sqrt{(x-3)(5-x)}$ (b) $y = \dfrac{1}{\sqrt{(2-x)(x-3)}}$.

Solution. (a) In this case y is defined for only those values of x which make the product $(x-3)(5-x)$ non-negative.

There are two possibilities :

(i) $(x-3)$ and $(5-x)$ are both positive or zero, so that

$$x \geq 3 \text{ and } 5 \geq x \text{ i.e., } 5 \geq x \geq 3 \text{ or } 3 \leq x \leq 5.$$

$\therefore$ y is defined in closed interval [3, 5].

(ii) $(x-3)$ and $(5-x)$ are both negative or zero, so that $x \leq 3$ and $5 \leq x$

This is impossible because x cannot be greater than or equal to 5 when it is less than or equal to 3.

Hence the function is defined only in the closed interval [3, 5] which is the domain of the function.

(b) In this case y is not defined when $2 - x = 0$ or $x - 3 = 0$ *i.e.*, when $x = 2$ or $x = 3$

$\therefore$ y is not defined for $x = 2$ or $x = 3$

For $2 < x < 3$, the expression $(2-x)(x-3)$ is positive and y can be determined.

$\therefore$ Open interval (2, 3) is the domain of the function.

Example 5. *Find the range of the function*

$$f(x) = 2 + \cos 3x \text{ if } -\frac{\pi}{6} \leq x \leq \frac{\pi}{6}.$$

Solution. We are given $-\dfrac{\pi}{6} \leq x \leq \dfrac{\pi}{6}$

i.e., $$-\frac{\pi}{2} \leq 3x \leq \frac{\pi}{2}$$ [*Multiplying by* 3]

When $3x = -\dfrac{\pi}{2}$, $\cos 3x = \cos\left(-\dfrac{\pi}{2}\right) = \cos\dfrac{\pi}{2} = 0$

When $3x = 0$, $\cos 3x = \cos 0 = 1$

When $3x = \dfrac{\pi}{2}$, $\cos 3x = \cos\dfrac{\pi}{2} = 0$

Thus as x varies from $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$, the least value of $\cos x$ is 0 and greatest value is 1.

$\therefore$ $$0 \leq \cos 3x \leq 1$$

Adding 2 to each number, $2 \leq 2 + \cos 3x \leq 3$ *i.e.*, $2 \leq f(x) \leq 3$

Hence range of $f(x)$ is the closed interval [2, 3].

Example 6. *Find whether the following function f admits of an inverse f^{-1}. If so find the inverse f^{-1} and state its domain.*

(i) $f(x) = \dfrac{1}{(x+1)^2}$, $x \neq -1$ (ii) $f(x) = \dfrac{x}{x+1}$, $x \neq -1$

Solution. (*i*) We have $f(x) = \dfrac{1}{(x+1)^2}$, $x \neq -1$

The domain of f is the set of all real numbers except -1 and the range is the set of all positive real numbers.

Putting $f(x) = y$, we have $y = \dfrac{1}{(x+1)^2}$

Solving for x, we get : $\quad x = -1 \pm \dfrac{1}{\sqrt{y}}$

Since for each positive value of y in the range of f, x is not unique, thus f does not admit of an inverse, *i.e.*, f^{-1} does not exist.

(*ii*) Here we have $f(x) = \dfrac{x}{x+1}$, $\quad x \neq -1$

Domain of f is the set of all numbers, except -1

Putting $y = f(x)$, we have $y = \dfrac{x}{x+1}$

Solving for x, we get $\quad x = \dfrac{y \text{ is here}}{1-y}$

Since for each value of $y \neq 1$ in the range of f, x is unique, thus f is one-one and hence admits of an inverse which is given by $x = f^{-1}(y) = \dfrac{y \text{ is here}}{1-y}$

The domain of f^{-1} is the range of f *i.e.*, set of all real numbers except 1.

Example 7. *If $f : R \to R$ is defined as $f(x) = \dfrac{2x+3}{4}$, $x \in R$; prove that f is a bijective function and hence find the inverse of f.*

Solution. Let $x_1, x_2 \in \mathrm{R}$

$$\therefore \qquad f(x_1) = f(x_2) \Rightarrow \frac{2x_1+3}{4} = \frac{2x_2+3}{4} \Rightarrow x_1 = x_2$$

$\therefore$ f is one-one or injective function.

Again, let $y \in \mathrm{R}$

$$\therefore \qquad f(x) = y = \frac{2x+3}{4} \Rightarrow x = \frac{4y-3}{2}$$

Now there exists $x \in \mathrm{R}$ for all $y \in \mathrm{R}$

$\therefore$ f is onto or surjective function.

Also since $\quad x = \dfrac{4y-3}{2} \Rightarrow f^{-1}(y) = \dfrac{4y-3}{2} \quad [\because f(x) = y \Rightarrow x = f^{-1}(y)]$

Hence $f^{-1} : \mathrm{R} \to \mathrm{R}$ is defined as $f^{-1}(x) = \dfrac{4x-3}{2}$.

Example 8. *Let Q be the set of rational numbers. Let $f: Q \to Q$ be defined by $f(x) = 5x + 7$ for all $x \in Q$. Show that f is one-one. Also find f^{-1}.*

Solution. Let x_1 and x_2 be any two different elements in Q

Then $x_1 \neq x_2$

$\Rightarrow \quad 5x_1 \neq 5x_2 \quad \Rightarrow \quad 5x_1 + 7 \neq 5x_2 + 7 \quad \Rightarrow \quad f(x_1) \neq f(x_2)$

Thus, different elements in Q have different f-images in Q.

$\therefore$ f is one-one

Consider $y = 5x + 7 \Rightarrow x = \frac{1}{5}(y - 7)$

Thus for each $y \in Q$, there exists $x \in Q$

$\therefore$ f is onto

$\therefore$ f is one-one and onto and hence invertible.

Since $x = \frac{1}{5}(y-7) \Rightarrow f^{-1}(y) = \frac{1}{5}(y-7)$ $\quad [\because f(x) = y \Rightarrow x = f^{-1}(y)]$

Example 9. *Let $X = \left\{x : x \in R \text{ and } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\right\}$ i.e., let $X = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $Y = \{y : Y \in R \text{ and } -1 \leq y \leq 1\}$ i.e., let $Y = [-1, 1]$; show that the function $f: X \to Y$ defined by $f(x) = \sin x$, $(x \in X)$ is one-one. Also find f^{-1}.*

Solution. Let x_1 and x_2 be any two different real numbers belonging to the closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Then $x_1 \neq x_2 \quad \Rightarrow \quad \sin x_1 \neq \sin x_2 \quad \Rightarrow \quad f(x_1) \neq f(x_2)$

Thus, f is one-one.

Let y be any arbitrary real number lying in the closed interval $[-1, 1]$. Now there exists a real number x lying in the closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $y = \sin x$.

Hence every element $y \in Y$ is the f-image of some element $x \in X$ and so f is onto.

$\therefore$ $f: X \to Y$ is one-one onto and hence invertible.

Let $y = \sin x \Rightarrow x = \sin^{-1} y \Rightarrow f^{-1}(y) = \sin^{-1} y$ $\quad [\because f(x) = y \Rightarrow x = f^{-1}(y)]$

Example 10. *If $f: R \to R$ be a function on R defined by $f(x) = x^2 + 8x + 3$; find $f^{-1}(-12)$.*

Solution. Here $f(x) = x^2 + 8x + 3$

Then $f^{-1}(-12)$ is given by $x^2 + 8x + 3 = -12$

or $\quad x^2 + 8x + 15 = 0$

or $\quad (x+5)(x+3) = 0 \Rightarrow x = -5, -3$

$\therefore \quad f^{-1}(-12) = \{-5, -3\}$.