Definition. *Let X and Y be two non-empty sets. A function or mapping 'f' from X into Y written as $f: X \to Y$ is a rule by which each element $x \in X$ is associated to a unique element $y \in Y$. We then say that f is a mapping of X into Y or f is a function of X to Y if the elements of X are mapped on the elements of Y by the rule f.*

The number y is called the value of the function f at x and is written as $y = f(x)$.

Note : Usually the word function is used when X and Y are the sets of numbers while the word mapping is used in the case of general sets of any type. Thus function is a special case of mapping.

Elements $y \in Y$ to which the elements $x \in X$ is associated under the mapping f is called ***f*- image** of x or **image** of x or the value of the function at x and x is called **pre-image** of y.

The set X is called **domain** and Y is called **co-domain** of the function. Clearly **range** is the subset of Y and is denoted by $f(x)$.

The numbers x and y, where y is an image of x are also denoted by ordered pairs (x, y) or $[x, f(x)]$. The mapping can be illustrated by the following examples :

Illustration 1. Let $X = \{2, 3, 4\}$, $Y = \{3, 4, 5, 6, 7\}$ and let f be a function defined as $f(2) = 3$, $f(3) = 5, f(4) = 7$.

Here $f(2) = 3,\ f(3) = 5,\ f(4) = 7$.

Let us draw its diagram.

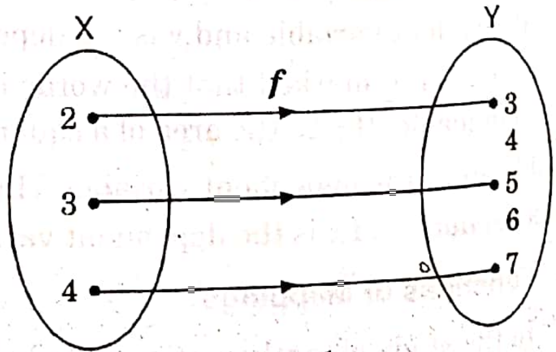


Fig. 1

We observe that every element of X has an image in Y (or equivalently, we do not have any element in X which does not have an image in Y) and image of each element of X is unique (or equivalently no element of X has more than one image.)

$\therefore \quad f: X \to Y$ is a function.

Illustration 2. Let X = {a, b, c}, Y = {2, 3, 4, 5}

Let us define $f(a) = 2$, $f(b) = 4$

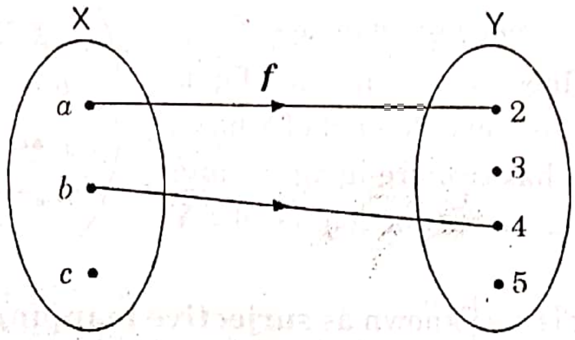


Fig. 2

Here f is not a function as $c \in$ X does not have its image.

Illustration 3. Let X = {a, b, c} and Y = {2, 3, 4, 5} and let f be a function defined by

$$f(a) = 2, \quad f(b) = 3, \quad f(b) = 4, \quad f(c) = 5.$$

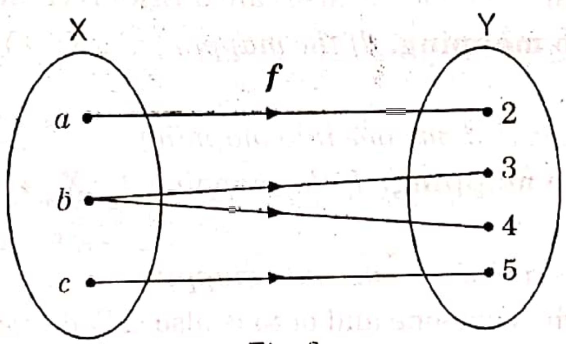


Fig. 3

Here every element of X has an image in Y but there is one element $b \in$ X which has more than one image *i.e.,* the image is not unique.

Hence f is not a function.

Remark. The important thing to remember is that when the value of x is given, there is exactly one value of $f(x)$. If f assigns two or more values of y to a single value of x, then f is not a function.

For example, $f(x) = \sqrt{x}$ or $f(x) = -\sqrt{x}$ defines a function f, whereas $f(x) = \pm\sqrt{x}$ does not define a function according to this definition.

1.5. Classification of Mappings

1. Into mapping. *If f is a mapping from X to Y i.e., $f : X \to Y$ defined in such a way that there exists atleast one element in $y \in Y$ which has no pre-image in X, then the mapping f is called* **into mapping**.

Thus f is a mapping from X to Y. We observe that in an into function $\{f(x)\} \subset$ Y for all $x \in$ X.

The mapping defined in the adjoining fig. is *an into mapping.*

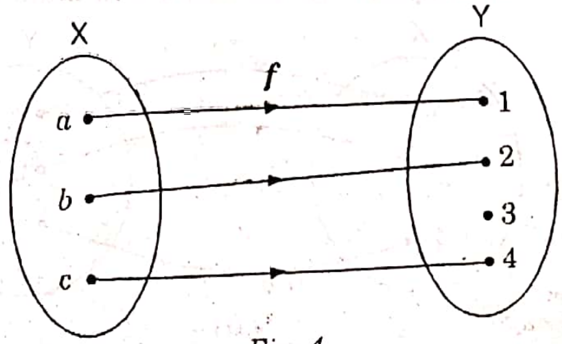


Fig. 4

2. Onto mapping. *A mapping f from X to Y i.e., $f: X \to Y$ is said to be an* **onto mapping** *if every element $y \in Y$, has atleast one pre-image $x \in X$.*

The mapping defined in the adjoining fig. is an onto mapping because every element of Y has a pre-image in X (here 3 has two pre-images *c* and *d*). We observe that in an onto function $\{f(x)\} = Y$ for all $x \in X$.

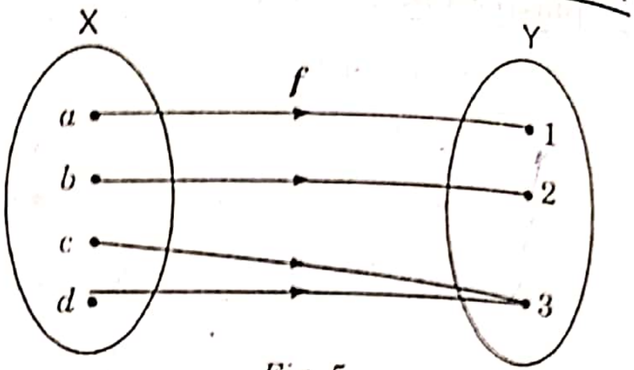


Fig. 5

An onto mapping is also known as **surjective mapping .**

Thus, $f: X \to Y$ is surjective if $f(X) = Y$.

3. One-one mapping. *Let f be a mapping from X to Y i.e., $f: X \to Y$ defined in such a way that different elements of X are mapped on different elements of Y, then f is said to be* **one-one mapping** (No two different elements of X have the same image in Y)

A mapping which is one-one is also called **injective mapping.**

4. One-one into mapping. *If the mapping $f: X \to Y$ is*

(i) one-one and

(ii) into, then f is called one-one into mapping.

5. One-one onto mapping. *If the mapping $f: X \to Y$ is*

(i) one-one and

(ii) onto, then f is called one-one onto mapping.

A mapping which is one-one and onto is also called *bijective mapping* or *bijection.*

6. Many-one mapping. *A mapping $f: X \to Y$ is said to be many one mapping if two or more elements of X have the same image in Y.*

7. Many-one into mapping. *If the mapping $f: X \to Y$ is*

(i) many one and

(ii) into, then f is called many-one into mapping.

In this case atleast one element $y \in Y$ has no pre-image.

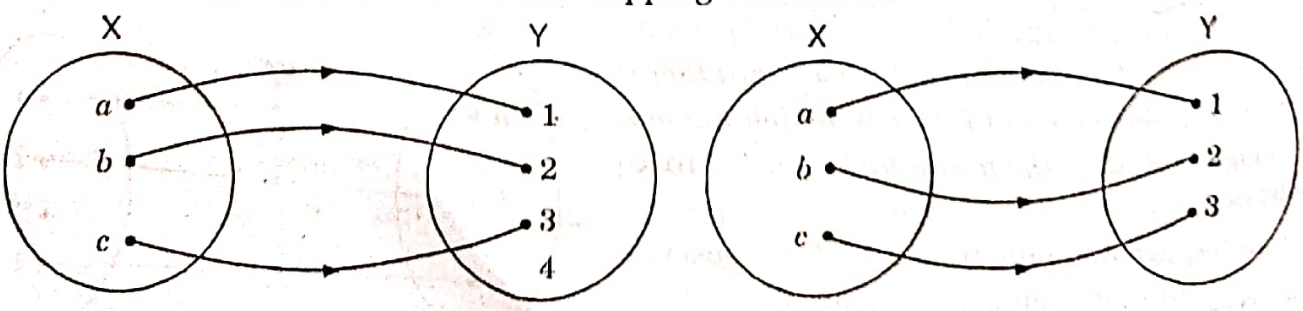
8. Many-one onto mapping. *If the mapping $f: X \to Y$ is*

(i) many one and

(ii) onto, then f is called many-one onto mapping.

In this case each element of Y has atleast one pre-image.

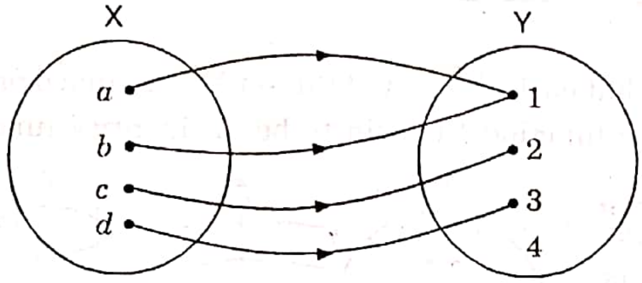
The diagrams given below will make mapping more clear.



(i) *One-one into function*

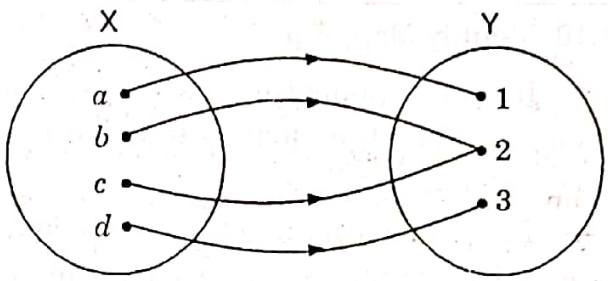
(ii) *One-one onto mapping*

Fig. 6



(iii)

Many-one into function



(iv)

Many-one onto mapping

Fig. 6

1.6. Ordered Pairs of Elements of a Set

Let A and B be the two given non-empty sets.

Let $a \in$ A and $b \in$ B, then a pair of the form (a, b) is called an **ordered pair**.

Thus, in ordered pair (a, b); 'a' is regarded as '*the first element*' and 'b' as the '*second element*'.

The set $\{a, b\}$ is the same as $\{b, a\}$ but $(a, b) \neq (b, a)$.

1.7. Cartesian Product of Two Sets

Let A and B be two non-empty sets. The cartesian product of A and B is denoted by A × B, and is defined as the set of all ordered pairs (a, b), where $a \in$ A and $b \in$ B.

Symbolically, $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

Examples : 1. Suppose A = {1, 3, 5} and B = (x, y)

Then $A \times B = \{(1, x), (3, x), (5, x), (1, y), (3, y), (5, y)\}$

and $B \times A = \{(x, 1), (x, 3), (x, 5), (y, 1), (y, 3), (y, 5)\}$

In general, $A \times B \neq B \times A$.

2. Let A = {1, 2, 3, 4} and B = ϕ then $A \times B = \phi$

Cor. (1) If $n(A) = m$ and $n(B) = n$, then

$n(A \times B) = n(B \times A) = mn$

(2) A × B and B × A are equivalent sets.

(3) $A \times B \times C = \{(a, b, c); a \in A, \ b \in B, \ c \in C\}$

1.8. Mapping in Terms of Ordered Pairs

Every subset C of A × B in which every element of A appears once and only once as the first element of some ordered pair is called a mapping of A into B.

Thus, a function f from A to B is a sub-set of A × B. In set notation, it is described as $\{(x, y) : y = f(x)\}$ where $x \in$ A and $y \in$ B and x appears once and only once as first element of an ordered pair.

1.9. Operators or Transformation

If the domain and co-domain of a function f are the same i.e., $f : A \to A$ then f is called an **operator** *or* **transformation** *on A.*

1.10. Identity Mapping

If f is a mapping from a set $X \to X$ such that each element of the set X be mapped on itself, then f is called the identity function *i.e.*, the function f is said to be an identity function if $f(x) = x$ for all $x \in X$.

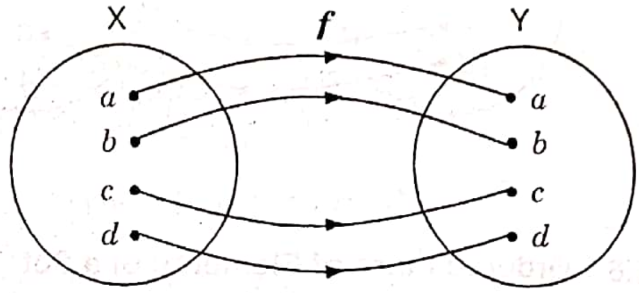


Fig. 7

Thus in identity mapping, the domain set is equal to range set, and each element is mapped on itself. We denote this function by I_x.

Therefore, we have

$$I_x(x) = x \text{ for all } x \in X$$

Let $\quad X = \{a, b, c, d\}$

and $\quad f(a) = a,\ f(b) = b,\ f(c) = c,\ f(d) = d$

Then $\quad f = \{(a, a), (b, b), (c, c), (d, d)\}$ is an identity mapping.

1.11. Equality of Mappings

Two functions f and g of $S \to T$ are said to be equal, if and only if $f(x) = g(x)$ for all $x \in S$ and we write $f = g$.

If $f \neq g$, then there must exist atleast one element $x \in S$ such that $f(x) \neq g(x)$.

Example. Let f be a function defined by fig. 8.

Let $g(x) = 2x$ be a function whose domain is $\{1, 2, 3\}$. Then it is clear from the diagram that f and g have the same domain and both assign the same image to each element in the domain.

Therefore $f = g$.

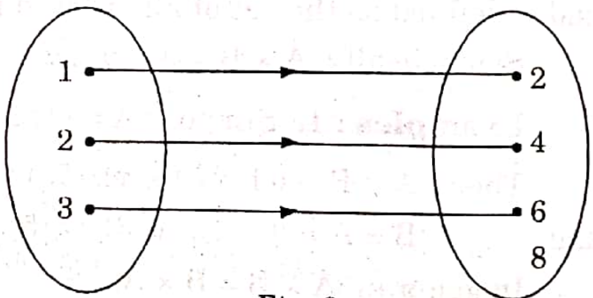


Fig. 8

1.12. Constant Function

If f is a function defined on a set A such that $y = f(x) = c$ for each $x \in A$, where c is a real number, then f is called a **constant function.**

The graph of this function is set of all points (x, c) *i.e.*, the straight line parallel to x-axis.

Domain = all reals

Range = $\{c\}$

Thus a mapping $f : A \to B$ is called a constant function if each element of A is mapped into a single element of B.

Example. *Let $f : R \to R$ be defined by $f(x) = 7$.*

Thus f is a constant function since 7 is assigned to every element.

1.13. Inverse Function

Let f be a one-one function of X onto Y. Since the function f is one-one and onto, therefore there is one and only one element in Y corresponding to each element in X.

Here we have

$$f(x_1) = y_1, \qquad f(x_2) = y_2, \qquad f(x_3) = y_3, \qquad f(x_4) = y_4$$

Let us denote

$$x_1 = f^{-1}(y_1), \quad x_2 = f^{-1}(y_2), \quad x_3 = f^{-1}(y_3), \quad x_4 = f^{-1}(y_4)$$

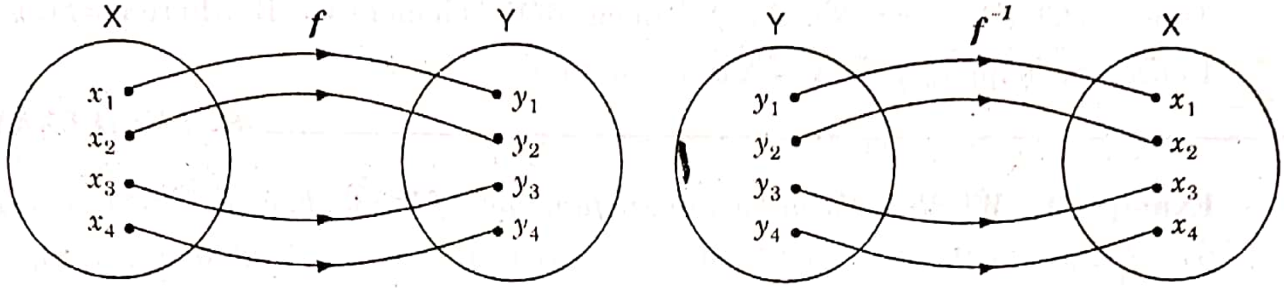


Fig. 9

Thus we see that there exists a one-one correspondence f^{-1} which maps elements of Y onto the elements of X.

Definition of inverse mapping : *If $f: X \to Y$ be a one-one onto mapping, then the mapping $f^{-1}: Y \to X$ which associates to each element $y \in Y$, the element $x \in X$ whose image was $y \in Y$, is called the inverse of the mapping $f: X \to Y$.*

i.e., *if $f^{-1}: Y \to X : f^{-1}(y) = x \Leftrightarrow f(x) = y$, where $y \in Y$ and $x \in X$, then f^{-1} is called inverse of f.*

Remark. ***A function is invertible if and only if f is one-one onto function.***

Note : 1. (*i*) f^{-1} should not be confused with $\frac{1}{f}$ as f^{-1} simply denotes the inverse of f and not $\frac{1}{f}$.

Thus, if f is one-one and onto and $y = f(x)$, then $x = f^{-1}(y)$.

2. The range of f is the domain of f^{-1} and the domain of f is the range of f^{-1}.

1.14. Inverse Image of a Set

Let f be a mapping from A into B and let S be any sub-set of B *i.e.*, $S \subseteq B$. Then the inverse image of S under f, denoted by $f^{-1}(S)$, consists of those elements in A, which are mapped into some elements in S. Thus $f^{-1}(S) = \{x : f(x) \in S\}$, where $x \in A$ *i.e.*, $f^{-1}(S)$ is the set of those elements of A whose f - images belongs to S.

1.15. Theorem

If $f: A \to B$ is one-one and onto then $f^{-1}: B \to A$ is also one-one and onto.

Proof. Let $x_1, x_2 \in A$ and $y_1, y_2 \in B$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$

Since f^{-1} denotes the inverse of f, we have

$$f^{-1}(y_1) = x_1 \quad \text{and} \quad f^{-1}(y_2) = x_2$$

Now $\quad f^{-1}(y_1) = f^{-1}(y_2) \quad \Rightarrow \quad x_1 = x_2$

$\Rightarrow \quad f(x_1) = f(x_2) \quad$ [$\because$ *f is one-one mapping from A to B*]

$\Rightarrow \quad y_1 = y_2$

Thus, f^{-1} is a one-one mapping.

Again, let x be any element of A. Since f is a mapping from A to B, therefore there exists an element $y \in B$ such that $y = f(x)$ or $x = f^{-1}(y)$.

Thus, each element $x \in A$ is the f^{-1} image of the element $y \in B$, where $y = f(x)$.

Hence, the mapping $f^{-1}: B \to A$ is also onto.

SOLVED EXAMPLES

Example 1. *Which of the following are functions if $X = \{a, b, c, d\}$, $Y = \{1, 2, 3, 4, 5\}$.*

(i) $f_1 = \{(a, 1), (b, 1), (c, 3), (d, 4)\}$ (ii) $f_2 = \{(a, 1), (b, 2), (c, 4), (a, 2), (d, 5)\}$

(iii) $f_3 = \{(a, 2), (b, 1), (c, 4), (d, 5)\}$.

Solution. Here $X = \{a, b, c, d\}$ and $Y = \{1, 2, 3, 4, 5\}$

(i) f_1 is a function because each element of X has a unique image, *viz.*

$$f(a) = 1, \quad f(b) = 1, \quad f(c) = 3, \quad f(d) = 4.$$

(ii) f_2 is not a function as $a \in X$ has two images *viz.*,

$$f(a) = 1 \quad \text{and} \quad f(a) = 2.$$

(iii) f_3 is a function because each element of X has a unique image, *viz.*,

$$f(a) = 2, \quad f(b) = 1, \quad f(c) = 4, \quad f(d) = 5.$$

Example 2. *Examine the relation and state whether it is a function or not :*

(i) $y = \pm\sqrt{1 - x^2}$, *for* $x \le 1$ (ii) $y = \sqrt{x}$, *for* $x \ge 0$

(iii) $y = -(x)$, *for* $x \in R$ (iv) $y = \dfrac{1}{x-1}$, *for* $x \ge 0$

Solution. (i) Since $\pm\sqrt{1-x^2}$ is not defined for $x < -1$

Thus the points $x = -2, -3, \ldots\ldots$, lying in the domain have no image.

$\therefore$ It is not a function.

(ii) Yes, $y = \sqrt{x}$, for $x \ge 0$ is a function because for every value of x, we can find a corresponding y.

(iii) Yes, because for every value of x, we can find a corresponding y.

(iv) No, $y = \dfrac{1}{x-1}$ is not a function, since 1 lies in the domain but $f(1)$ does not exist.

Example 3. *Let $A = \{3, 4, 5, 6\}$ and $B = \{1, 2, 4\}$. If R_1 is from A to B, draw its figure and state the type of function where $R_1 = \{(3, 1), (4, 2), (5, 2), (6, 4)\}$.*

Solution. Here $A = \{3, 4, 5, 6\}$ and $B = \{1, 2, 4\}$

The function $R_1 = \{(3, 1), (4, 2), (5, 2), (6, 4)\}$ is *many-one onto* as two elements 4 and 5 of A have the same image 2 in B and every element of B has atleast one pre-image in A as shown in fig.10.

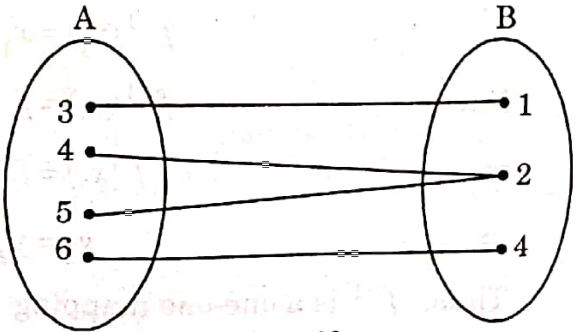


Fig. 10