13.15. DIVERGENCE OF A VECTOR POINT FUNCTION

The divergence of a differentiable vector point function $\vec{V}$ is denoted by div $\vec{V}$ and is defined as

$$\text{div } \vec{V} = \nabla \cdot \vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \vec{V} = \hat{i} \cdot \frac{\partial \vec{V}}{\partial x} + \hat{j} \cdot \frac{\partial \vec{V}}{\partial y} + \hat{k} \cdot \frac{\partial \vec{V}}{\partial z}.$$

Obviously, the divergence of a vector point function is a scalar point function.

If $\quad \vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$, then

$$\text{div } \vec{V} = \nabla \cdot \vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot [V_1\hat{i} + V_2\hat{j} + V_3\hat{k}] = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}.$$

since $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$ and $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$.

13.16. CURL OF A VECTOR POINT FUNCTION

The curl (or rotation) of a differentiable vector point function $\vec{V}$ is denoted by curl $\vec{V}$ and is defined as

$$\text{curl } \vec{V} = \nabla \times \vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \vec{V} = \hat{i} \times \frac{\partial \vec{V}}{\partial x} + \hat{j} \times \frac{\partial \vec{V}}{\partial y} + \hat{k} \times \frac{\partial \vec{V}}{\partial z}.$$

Obviously, the curl of a vector point function is a vector point function.

If $\quad \vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$

then $\quad \text{curl } \vec{V} = \nabla \times \vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times [V_1\hat{i} + V_2\hat{j} + V_3\hat{k}]$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \hat{j}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right).$$

ILLUSTRATIVE EXAMPLES

Example 1. *If* $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, *show that*

(i) div $\vec{r} = 3$ $\qquad$ *(ii) curl* $\vec{r} = \vec{0}$.

Sol. (*i*) div $\vec{r} = \nabla \cdot \vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3.$

$$(ii)\ \text{curl}\ \vec{r} = \nabla \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right] + \hat{j}\left[\frac{\partial}{\partial z}(x) - \frac{\partial}{\partial x}(z)\right] + \hat{k}\left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right]$$

$$= \hat{i}(0) + \hat{j}(0) + \hat{k}(0) = \vec{0}.$$

Example 2. *Find the divergence and curl of the vector* $\vec{V} = (xyz)\hat{i} + (3x^2y)\hat{j} + (xz^2 - y^2z)\hat{k}$ *at the point* $(2, -1, 1)$.

Sol. $$\text{div}\ \vec{V} = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial z}(xz^2 - y^2z)$$

$$= yz + 3x^2 + 2xz - y^2 = -1 + 12 + 4 - 1 = 14 \text{ at } (2, -1, 1)$$

$$\text{curl}\ \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix} = \hat{i}(-2yz - 0) + \hat{j}(xy - z^2) + \hat{k}(6xy - xz)$$

$$= 2\hat{i} - 3\hat{j} - 14\hat{k} \text{ at } (2, -1, 1).$$

Example 3. *Find div* $\vec{F}$ *and curl* $\vec{F}$ *where* $\vec{F} = \text{grad}\,(x^3 + y^3 + z^3 - 3xyz)$.

Sol. Let $\phi = x^3 + y^3 + z^3 - 3xyz$, then

$$\vec{F} = \text{grad}\ \phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^3 + y^3 + z^3 - 3xyz)$$

$$= (3x^2 - 3yz)\hat{i} + (3y^2 - 3zx)\hat{j} + (3z^2 - 3xy)\hat{k}$$

$\therefore$ $$\text{div}\ \vec{F} = \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3zx) + \frac{\partial}{\partial x}(3z^2 - 3xy)$$

$$= 6x + 6y + 6z = 6(x + y + z)$$

and $$\text{curl}\ \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3zx & 3z^2 - 3xy \end{vmatrix}$$

$$= \hat{i}(-3x + 3x) + \hat{j}(-3y + 3y) + \hat{k}(-3z + 3z) = \vec{0}.$$

Example 4. *Find curl (curl* $\vec{V}$*) where* $\vec{V} = (2xz^2)\hat{i} - yz\hat{j} + 3xz^3\hat{k}$, *at (1, 1, 1).*

Sol. Here, $$\vec{V} = (2xz^2)\hat{i} - yz\hat{j} + 3xz^3\hat{k}$$

$\therefore$ $$\text{curl}\ \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz^2 & -yz & 3xz^3 \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(3xz^3) - \frac{\partial}{\partial z}(-yz)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(3xz^3) - \frac{\partial}{\partial z}(2xz^2)\right\} + \hat{k}\left\{\frac{\partial}{\partial x}(-yz) - \frac{\partial}{\partial y}(2xz^2)\right\}$$

$$= \hat{i}(0 + y) - \hat{j}(3z^3 - 4xz) + \hat{k}(0 - 0) = y\hat{i} + (4xz - 3z^3)\hat{j}$$

$$\text{curl}(\text{curl}\,\vec{V}) = \text{curl}\,\{y\hat{i} + (4xz - 3z^3)\hat{j}\}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 4xz - 3z^3 & 0 \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(4xz - 3z^3)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(y)\right\} + \hat{k}\left\{\frac{\partial}{\partial x}(4xz - 3z^3) - \frac{\partial}{\partial y}(y)\right\}$$

$$= \hat{i}\left(0 - (4x - 9z^2)\right) - \hat{j}(0 - 0) + \hat{k}(4z - 1)$$

$$= (9z^2 - 4x)\hat{i} + (4z - 1)\hat{k} = 5\hat{i} + 3\hat{k} \text{ at } (1, 1, 1).$$

Example 5. *Let* $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ *and* $\vec{a}$ *be a constant vector, find the value of div* $\left(\frac{\vec{a} \times \vec{r}}{r^n}\right)$.

Sol. $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \Rightarrow r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$.

$$\vec{a} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = (a_2z - a_3y)\,\hat{i} + (a_3x - a_1z)\,\hat{j} + (a_1y - a_2x)\,\hat{k}$$

$$\frac{\vec{a} \times \vec{r}}{r^n} = \frac{(a_2z - a_3y)\hat{i} + (a_3x - a_1z)\hat{j} + (a_1y - a_2x)\hat{k}}{(x^2 + y^2 + z^2)^{n/2}}$$

$$\therefore \quad \text{div}\left(\frac{\vec{a} \times \vec{r}}{r^n}\right) = \nabla \cdot \frac{\vec{a} \times \vec{r}}{r^n}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \frac{(a_2z - a_3y)\hat{i} + (a_3x - a_1z)\hat{j} + (a_1y - a_2x)\hat{k}}{(x^2 + y^2 + z^2)^{n/2}}$$

$$= \frac{\partial}{\partial x}\left\{\frac{a_2z - a_3y}{(x^2 + y^2 + z^2)^{n/2}}\right\} + \frac{\partial}{\partial y}\left\{\frac{a_3x - a_1z}{(x^2 + y^2 + z^2)^{n/2}}\right\} + \frac{\partial}{\partial z}\left\{\frac{a_1y - a_2x}{(x^2 + y^2 + z^2)^{n/2}}\right\}$$

$$= (a_2z - a_3y) \cdot \left(-\frac{n}{2}\right)(x^2 + y^2 + z^2)^{-\frac{n}{2} - 1} . 2x$$

$$+ (a_3x - a_1z)\left(-\frac{n}{2}\right)(x^2 + y^2 + z^2)^{-\frac{n}{2} - 1} . 2y + (a_1y - a_2x)\left(-\frac{n}{2}\right)(x^2 + y^2 + z^2)^{-\frac{n}{2} - 1} . 2z$$

$$= \frac{-n}{(x^2+y^2+z^2)^{\frac{n}{2}+1}} [(a_2z - a_3y)\,x + (a_3x - a_1z)\,y + (a_1y - a_2x)\,z]$$

$$= \frac{-n}{(x^2+y^2+z^2)^{\frac{n}{2}+1}} [0] = 0$$

Hence, $\text{div}\left(\dfrac{\vec{a} \times \vec{r}}{r^n}\right) = 0.$

Example 6. *Find the directional derivative of div $(\vec{u})$ at the point (1, 2, 2) in the direction of the outer normal of the sphere $x^2 + y^2 + z^2 = 9$ for $\vec{u} = x^4\hat{i} + y^4\hat{j} + z^4\hat{k}$.*

Sol. Here, $\vec{u} = x^4\hat{i} + y^4\hat{j} + z^4\hat{k}$

$$\therefore \quad \text{div}\,(\vec{u}) = \nabla \cdot \vec{u} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (x^4\hat{i} + y^4\hat{j} + z^4\hat{k})$$

$$= \frac{\partial}{\partial x}(x^4) + \frac{\partial}{\partial y}(y^4) + \frac{\partial}{\partial z}(z^4)$$

$$= 4\,(x^3 + y^3 + z^3)$$

Directional derivative of div $\vec{u} = \nabla\,(4x^3 + 4y^3 + 4z^3)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (4x^3 + 4y^3 + 4z^3)$$

$$= 12\,(x^2\hat{i} + y^2\hat{j} + z^2\hat{k})$$

$$= 12\,(\hat{i} + 4\hat{j} + 4\hat{k}) \text{ at } (1, 2, 2)$$

Outer normal to the sphere $= \nabla\,(x^2 + y^2 + z^2 - 9)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2 - 9)$$

$$= \hat{i}\,(2x) + \hat{j}\,(2y) + \hat{k}\,(2z)$$

$$= 2(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= 2(\hat{i} + 2\hat{j} + 2\hat{k}) \text{ at } (1, 2, 2)$$

$$= 2\hat{i} + 4\hat{j} + 4\hat{k}$$

Unit outer normal to the sphere at (1, 2, 2) is

$$\hat{n} = \frac{2\hat{i} + 4\hat{j} + 4\hat{k}}{\sqrt{4 + 16 + 16}} = \frac{2\hat{i} + 4\hat{j} + 4\hat{k}}{6}$$

$\therefore$ Directional derivative of div $\vec{u}$ at (1, 2, 2) in the direction of outer normal

$$= 12\,(\hat{i} + 4\hat{j} + 4\hat{k}) \cdot \frac{2\hat{i} + 4\hat{j} + 4\hat{k}}{6} = 2\,(2 + 16 + 16) = 68$$

Note: If Div of vector V = 0, then vector V is called solenoidal vector point function.

Note. If curl $\vec{V} = \vec{0}$, then $\vec{V}$ is said to be an **irrotational vector,** otherwise **rotational.**

13.19. PROPERTIES OF DIVERGENCE AND CURL

1. For a constant vector $\vec{a}$, div $\vec{a} = 0$, curl $\vec{a} = \vec{0}$

2. $\mathbf{div}\,(\vec{\mathbf{A}}+\vec{\mathbf{B}}) = \mathbf{div}\,\vec{\mathbf{A}} + \mathbf{div}\,\vec{\mathbf{B}}$ **or** $\nabla\,.\,(\vec{\mathbf{A}}+\vec{\mathbf{B}}) = \nabla\,.\,\vec{\mathbf{A}} + \nabla\,.\,\vec{\mathbf{B}}$

3. $\mathbf{curl}\,(\vec{\mathbf{A}}+\vec{\mathbf{B}}) = \mathbf{curl}\,\vec{\mathbf{A}} + \mathbf{curl}\,\vec{\mathbf{B}}$ **or** $\nabla\times(\vec{\mathbf{A}}+\vec{\mathbf{B}}) = \nabla\times\vec{\mathbf{A}} + \nabla\times\vec{\mathbf{B}}$

4. If $\vec{A}$ is a vector function and ϕ is a scalar function, then

   $\mathbf{div}\,(\phi\vec{\mathrm{A}}) = \phi\,\mathbf{div}\,\vec{\mathbf{A}} + (\mathbf{grad}\,\phi)\,.\,\vec{\mathrm{A}}$ **or** $\nabla\,.\,(\phi\vec{\mathrm{A}}) = \phi\,(\nabla\,.\,\vec{\mathrm{A}}) + (\nabla\phi)\,.\,\vec{\mathrm{A}}$

5. If A is a vector function and ϕ is a scalar function, then

   $\mathbf{curl}\,(\phi\,\vec{\mathbf{A}}) = (\mathbf{grad}\,\phi)\times\vec{\mathrm{A}} + \phi\,\mathbf{curl}\,\vec{\mathbf{A}}$ **or** $\nabla\times(\phi\vec{\mathbf{A}}) = (\nabla\phi)\times\vec{\mathbf{A}} + \phi\,(\nabla\times\vec{\mathbf{A}})$

6. $\nabla\,(\vec{\mathbf{A}}\,.\,\vec{\mathbf{B}}) = (\vec{\mathbf{A}}\,.\,\nabla)\,\vec{\mathbf{B}} + (\vec{\mathbf{B}}\,.\,\nabla)\,\vec{\mathbf{A}} + \vec{\mathbf{A}}\times(\nabla\times\vec{\mathbf{B}}) + \vec{\mathbf{B}}\times(\nabla\times\vec{\mathbf{A}})$

7. $\nabla\,.\,(\vec{\mathbf{A}}\times\vec{\mathbf{B}}) = \vec{\mathbf{B}}\,.\,(\nabla\times\vec{\mathbf{A}}) - \vec{\mathbf{A}}\,.\,(\nabla\times\vec{\mathbf{B}})$

   Or

   $\mathrm{div}\,(\vec{\mathbf{A}}\times\vec{\mathbf{B}}) = \vec{\mathbf{B}}\,.\,\mathrm{curl}\,\vec{\mathbf{A}} - \vec{\mathbf{A}}\,.\,\mathrm{curl}\,\vec{\mathbf{B}}$

8. $\nabla\times(\vec{\mathbf{A}}\times\vec{\mathbf{B}}) = (\nabla\,.\,\vec{\mathbf{B}})\vec{\mathbf{A}} - (\nabla\,.\,\vec{\mathbf{A}})\vec{\mathbf{B}} + (\vec{\mathbf{B}}\,.\,\nabla)\vec{\mathbf{A}} - (\vec{\mathbf{A}}\,.\,\nabla)\vec{\mathbf{B}}$

13.20. REPEATED OPERATIONS BY ∇

Let $\phi\,(x, y, z)$ and $\vec{V}\,(x, y, z)$ be scalar and vector point functions respectively.

Since grad ϕ and curl $\vec{V}$ are also vector point functions, we can find their divergence as well as curl, whereas div $\vec{V}$ being a scalar point function, we can find its gradient only.

1. **Div (grad** ϕ**)** $= \nabla^2\phi$ **where** $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$

2. **Curl (grad** ϕ**)** $= \nabla\times\nabla\phi = \vec{\mathbf{0}}$

3. **Div (curl $\vec{V}$) = $\nabla . (\nabla \times \vec{V}) = 0$.**

4. **Curl (curl $\vec{V}$) = grad div $\vec{V} - \nabla^2 \vec{V}$**

or $$\nabla \times (\nabla \times \vec{V}) = \nabla (\nabla . \vec{V}) - \nabla^2 \vec{V}.$$

Note 1. The above result can also be written as grad (div $\vec{V}$) = curl (curl $\vec{V}$) + $\nabla^2 \vec{V}$

or $$\nabla(\nabla . \vec{V}) = \nabla \times (\nabla \times \vec{V}) + \nabla^2 \vec{V}.$$

Note 2. Treating ∇ as a vector, the results of repeated application of ∇ can be easily written down. Thus

$$\nabla . \nabla\phi = \nabla^2\phi \qquad (\because \vec{a} . \vec{a} = a^2)$$

$$\nabla \times \nabla\phi = \vec{0} \qquad (\because \vec{a} \times \vec{a} = \vec{0})$$

$$\nabla . (\nabla \times \vec{V}) = 0 \qquad (\because \vec{a} . (\vec{a} \times \vec{b}) = [\vec{a}\ \vec{a}\ \vec{b}] = 0)$$

$$\nabla \times (\nabla \times \vec{V}) = \nabla(\nabla . \vec{V}) - \nabla^2 \vec{V} \qquad \text{(By expanding as a vector triple product)}$$

ILLUSTRATIVE EXAMPLES

Example 1. *A vector field is given by $\vec{A} = (x^2 + xy^2)\hat{i} + (y^2 + x^2y)\hat{j}$. Show that the field is irrotational.*

Sol. Field $\vec{A}$ is irrotational if curl $\vec{A} = \vec{0}$

Now $$\text{curl } \vec{A} = \nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + xy^2 & y^2 + x^2y & 0 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(2xy - 2xy) = \vec{0}.$$

$\therefore$ Field $\vec{A}$ is irrotational.

Example 2. *If the vector* $\vec{F} = (ax^2y + yz)\hat{i} + (xy^2 - xz^2)\hat{j} + (2xyz - 2x^2y^2)\hat{k}$ *is solenoidal, find the value of a. Find also the curl of this solenoidal vector.*

Sol. Here $\quad \vec{F} = (ax^2y + yz)\hat{i} + (xy^2 - xz^2)\hat{j} + (2xyz - 2x^2y^2)\hat{k}$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left[(ax^2y + yz)\hat{i} + (xy^2 - xz^2)\hat{j} + (2xyz - 2x^2y^2)\hat{k}\right]$$

$$= \frac{\partial}{\partial x}(ax^2y + yz) + \frac{\partial}{\partial y}(xy^2 - xz^2) + \frac{\partial}{\partial z}(2xyz - 2x^2y^2)$$

$$= 2axy + 2xy + 2xy = 2(a + 2)xy$$

Since $\vec{F}$ is solenoidal, div $\vec{F} = 0$

$\Rightarrow \; 2(a + 2)xy = 0 \qquad \therefore \; a = -2$

Now $\quad \vec{F} = (-2x^2y + yz)\hat{i} + (xy^2 - xz^2)\hat{j} + (2xyz - 2x^2y^2)\hat{k}$

$$\therefore \quad \text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -2x^2y + yz & xy^2 - xz^2 & 2xyz - 2x^2y^2 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(2xyz - 2x^2y^2) - \frac{\partial}{\partial z}(xy^2 - xz^2)\right] - \hat{j}\left[\frac{\partial}{\partial x}(2xyz - 2x^2y^2) - \frac{\partial}{\partial z}(-2x^2y + yz)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(xy^2 - xz^2) - \frac{\partial}{\partial y}(-2x^2y + yz)\right]$$

$$= \hat{i}(2xz - 4x^2y + 2xz) - \hat{j}(2yz - 4xy^2 - y) + \hat{k}(y^2 - z^2 + 2x^2 - z)$$

$$= 4x(z - xy)\hat{i} + (y + 4xy^2 - 2yz)\hat{j} + (2x^2 + y^2 - z^2 - z)\hat{k}$$

Example 3. *Show that* $r^\alpha \vec{R}$ *is an irrotational vector for any value of* α *but it is solenoidal if* $\alpha + 3 = 0$ *where* $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$ *and r is the magnitude of* $\vec{R}$.

Sol. Let $\vec{V} = r^\alpha \vec{R} = (x^2 + y^2 + z^2)^{\frac{\alpha}{2}}(x\hat{i} + y\hat{j} + z\hat{k})$

$$= x(x^2 + y^2 + z^2)^{\alpha/2}\hat{i} + y(x^2 + y^2 + z^2)^{\alpha/2}\hat{j} + z(x^2 + y^2 + z^2)^{\alpha/2}\hat{k}$$

$$\therefore \quad \text{curl } \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x(x^2 + y^2 + z^2)^{\alpha/2} & y(x^2 + y^2 + z^2)^{\alpha/2} & z(x^2 + y^2 + z^2)^{\alpha/2} \end{vmatrix}$$

$$= \sum i \left\{ \frac{\alpha z}{2} (x^2 + y^2 + z^2)^{\frac{\alpha}{2} - 1} . 2y - \frac{\alpha y}{2} (x^2 + y^2 + z^2)^{\frac{\alpha}{2} - 1} . 2z \right\}$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$$

$\Rightarrow$ $\vec{V} = r^\alpha \vec{R}$ is irrotational for any value of α.

Now, $$\text{div } \vec{V} = \nabla . (r^\alpha \vec{R})$$

$$= r^\alpha (\text{div } \vec{R}) + \text{grad } r^\alpha . \vec{R} \qquad ...(1)$$

$$\left[\because \text{ div } (\phi \vec{A}) = \phi (\text{div } \vec{A}) + \text{grad } \phi . \vec{A} \right]$$

and $$\text{div } (\vec{R}) = \nabla . (x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3$$

Also, $$r^2 = x^2 + y^2 + z^2 \text{ so that } \frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\text{grad } r = \hat{i} \frac{\partial r}{\partial x} + \hat{j} \frac{\partial r}{\partial y} + \hat{k} \frac{\partial r}{\partial z} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r} = \frac{\vec{R}}{r}$$

$\therefore$ $$\text{grad } r^\alpha = \alpha\, r^{\alpha - 1} \text{ grad } r = \alpha r^{\alpha - 1} \frac{\vec{R}}{r} = \alpha r^{\alpha - 2} \vec{R}$$

$\therefore$ From (1), we have

$$\text{div } \vec{V} = r^\alpha (3) + \alpha r^{\alpha - 2} \vec{R} . \vec{R} = 3r^\alpha + \alpha r^{\alpha - 2} (x^2 + y^2 + z^2)$$

$$= 3r^\alpha + \alpha r^{\alpha - 2} (r^2) = (3 + \alpha) r^\alpha$$

Now, $\vec{V}$ is solenoidal if div $\vec{V} = 0$ *i.e.,* $(3 + \alpha) r^\alpha = 0$

$\Rightarrow$ $r^\alpha \vec{R}$ is solenoidal if $\alpha + 3 = 0$.

Example 4. *If $\vec{a}$ is a constant vector and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that curl $(\vec{a} \times \vec{r}) = 2\vec{a}$.*

Sol. Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, where a_1, a_2, a_3 are constants.

$$\vec{a} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = (a_2 z - a_3 y)\hat{i} + (a_3 x - a_1 z)\hat{j} + (a_1 y - a_2 x)\hat{k}$$

$$\text{curl } (\vec{a} \times \vec{r}) = \nabla \times (\vec{a} \times \vec{r}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_2 z - a_3 y & a_3 x - a_1 z & a_1 y - a_2 x \end{vmatrix}$$

$$= (a_1 + a_1)\hat{i} + (a_2 + a_2)\hat{j} + (a_3 + a_3)\hat{k} = 2(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) = 2\vec{a}.$$

Example 5. *Prove that*

$$(i)\ \nabla(\vec{a}\cdot\vec{u}) = (\vec{a}\cdot\nabla)\vec{u} + \vec{a}\times(\nabla\times\vec{u}) \qquad (ii)\ \nabla\times(\vec{a}\times\vec{u}) = (\nabla\cdot\vec{u})\vec{a} - (\vec{a}\cdot\nabla)\vec{u}$$

where $\vec{a}$ *is a constant vector.*

Sol. (*i*) $$\nabla(\vec{a}\cdot\vec{u}) = \sum i\frac{\partial}{\partial x}(\vec{a}\cdot\vec{u}) = \sum i\left(\vec{a}\cdot\frac{\partial\vec{u}}{\partial x}\right) \qquad \ldots(1)$$

Now $$\vec{a}\times\left(\hat{i}\times\frac{\partial\vec{u}}{\partial x}\right) = \left(\vec{a}\cdot\frac{\partial\vec{u}}{\partial x}\right)i - (\vec{a}\cdot\hat{i})\frac{\partial\vec{u}}{\partial x}$$

$$\Rightarrow \quad \left(\vec{a}\cdot\frac{\partial\vec{u}}{\partial x}\right)\hat{i} = \vec{a}\times\left(\hat{i}\times\frac{\partial\vec{u}}{\partial x}\right) + (\vec{a}\cdot\hat{i})\frac{\partial\vec{u}}{\partial x}$$

$\therefore$ From (1), we have $$\nabla(\vec{a}\cdot\vec{u}) = \sum\vec{a}\times\left(\hat{i}\times\frac{\partial\vec{u}}{\partial x}\right) + \sum(\vec{a}\cdot\hat{i})\frac{\partial\vec{u}}{\partial x}$$

$$= \vec{a}\times(\nabla\times\vec{u}) + (\vec{a}\cdot\nabla)\vec{u} = (\vec{a}\cdot\nabla)\vec{u} + \vec{a}\times(\nabla\times\vec{u}).$$

(*ii*) $$\nabla\times(\vec{a}\times\vec{u}) = \sum\hat{i}\frac{\partial}{\partial x}\times(\vec{a}\times\vec{u}) = \sum\hat{i}\times\left(\vec{a}\times\frac{\partial\vec{u}}{\partial x}\right)$$

$$= \sum\left(\hat{i}\cdot\frac{\partial\vec{u}}{\partial x}\right)\vec{a} - \sum(\hat{i}\cdot\vec{a})\frac{\partial\vec{u}}{\partial x} = (\nabla\cdot\vec{u})\vec{a} - (\vec{a}\cdot\nabla)\vec{u}.$$

Example 6. *Prove that* $\nabla\cdot(\phi\nabla\psi - \psi\nabla\phi) = \phi\nabla^2\psi - \psi\nabla^2\phi.$

Sol. For a scalar function f and a vector function $\vec{G}$, we know that

$$\nabla.(f\vec{G}) = f(\nabla.\vec{G}) + (\nabla f).\vec{G}$$

Also $$\nabla.(\vec{F} - \vec{G}) = \nabla.\vec{F} - \nabla.\vec{G}$$

$$\begin{aligned}\therefore \quad \nabla\cdot(\phi\nabla\psi - \psi\nabla\phi) &= \nabla.(\phi\nabla\psi) - \nabla.(\psi\nabla\phi)\\ &= [\phi(\nabla.\nabla\psi) + \nabla\phi\cdot\nabla\psi] - [\psi(\nabla.\nabla\phi) + \nabla\psi.\nabla\phi]\\ &= \phi\nabla^2\psi + \nabla\phi.\nabla\psi - \psi\nabla^2\phi - \nabla\psi.\nabla\phi\\ &= \phi\nabla^2\psi - \psi\nabla^2\phi \qquad [\because \text{ dot product is commutative}]\end{aligned}$$

Example 7. *Prove that*

(*i*) $div\left(\dfrac{\vec{r}}{r^3}\right) = 0$ (*ii*) $\nabla^2 (r^n) = n(n+1)\, r^{n-2}$.

where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Sol. Here $r^2 = x^2 + y^2 + z^2$ so that $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}, \dfrac{\partial r}{\partial y} = \dfrac{y}{r}, \dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

$$\text{grad } r = \hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z} = \frac{\vec{r}}{r}$$

(*i*) Since $\text{div}\,(\phi\vec{A}) = \phi(\text{div}\,\vec{A}) + \text{grad}\,\phi\,.\,\vec{A}$

$$\therefore \quad \text{div}\left(\frac{\vec{r}}{r^3}\right) = \text{div}\,(r^{-3}\,\vec{r}\,) = r^{-3}\,(\text{div}\,\vec{r}\,) + (\text{grad}\,r^{-3})\,.\,\vec{r}$$

$$= 3r^{-3} + (-\,3r^{-4}\,\text{grad}\,r)\,.\,\vec{r} \qquad [\because \ \text{div}\,\vec{r} = 3]$$

$$= 3r^{-3} + \left(-\,3r^{-4}\frac{\vec{r}}{r}\right).\,\vec{r} = 3r^{-3} - 3r^{-5}\,(\vec{r}\,.\,\vec{r}\,) = 3r^{-3} - 3r^{-5}\,(r^2) = 0.$$

(*ii*) $$\nabla^2 (r^n) = \nabla.\,(\nabla r^n) = \nabla.\left(nr^{n-1}\frac{\vec{r}}{r}\right) = n\nabla.\,(r^{n-2}\,\vec{r}\,)$$

$$= n\,[(\nabla\, r^{n-2})\,.\,\vec{r} + r^{n-2}\,(\nabla.\,\vec{r}\,)] \quad [\because \quad \nabla\,.\,(\phi\vec{A}) = (\nabla\,\phi)\,.\,\vec{A} + \phi\,(\nabla\,.\,\vec{A})]$$

$$= n\left[(n-2)\,r^{n-3}\,\frac{\vec{r}}{r}.\,\vec{r} + r^{n-2}\,(3)\right] \qquad [\because \ \nabla.\,\vec{r} = 3]$$

$$= n\,[(n-2)\,r^{n-4}\,(r^2) + 3r^{n-2}] \qquad [\because \ \vec{r}\,.\,\vec{r} = r^2]$$

$$= n(n+1)r^{n-2}.$$

Second Method

$$\nabla^2 (r^n) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) r^n$$

$$= \sum \frac{\partial^2}{\partial x^2}(r^n) = \sum \frac{\partial}{\partial x}\left(\frac{\partial r^n}{\partial x}\right)$$

$$= \sum \frac{\partial}{\partial x}\left(nr^{n-1}\frac{\partial r}{\partial x}\right) = \sum \frac{\partial}{\partial x}\left(nr^{n-1}\frac{x}{r}\right) = \sum n\frac{\partial}{\partial x}\,(r^{n-2}\,x)$$

$$= n\sum\left[(n-2)\,r^{n-3}\,\frac{\partial r}{\partial x}.x + r^{n-2}\right] = n\sum\left[(n-2)\,r^{n-3}\,\frac{x}{r}.x + r^{n-2}\right]$$

$$= n \sum [(n-2)\, r^{n-4}\, x^2 + r^{n-2}] = n[(n-2)\, r^{n-4}\, (x^2+y^2+z^2) + 3r^{n-2}]$$

$$= n\, [(n-2)\, r^{n-4}(r^2) + 3r^{n-2}] = n(n+1)r^{n-2}.$$

Example 8. *Prove that the vector $f(r)\, \vec{r}$ is irrotational.*

Sol. The vector $f(r)\, \vec{r}$ will be irrotational if curl $[f(r)\vec{r}\,] = \vec{0}$

Since $\quad \text{curl}\,(\phi\vec{A}) = (\text{grad}\ \phi) \times \vec{A} + \phi\ \text{curl}\ \vec{A}$

$$\therefore \quad \text{curl}\,[f(r)\ \vec{r}\,] = [\text{grad}\ f(r)] \times \vec{r} + f(r)\ \text{curl}\ \vec{r}$$

$$= [f'(r)\ \text{grad}\ r] \times \vec{r} + f(r)\ \vec{0} \qquad [\because\ \text{curl}\ \vec{r} = \vec{0}]$$

$$= \left[f'(r)\frac{\vec{r}}{r}\right] \times \vec{r} = \frac{f'(r)}{r}(\vec{r} \times \vec{r}) = \vec{0}, \text{ since } \vec{r} \times \vec{r} = \vec{0}.$$

$\therefore$ The vector $f(r)\ \vec{r}$ is irrotational.

Example 9. *Prove that* $\nabla^2 f(r) = f''(r) + \dfrac{2}{r} f'(r)$.

Sol. $\nabla^2 f(r) = \nabla \cdot \{\nabla f(r)\} = \text{div}\ \{\text{grad}\ f(r)\} = \text{div}\ \{f'(r)\ \text{grad}\ r\} = \text{div}\left\{f'(r)\dfrac{\vec{r}}{r}\right\}$

$$= \text{div}\left\{\frac{1}{r}f'(r)\vec{r}\right\} = \frac{1}{r}f'(r)\ \text{div}\ \vec{r} + \vec{r}\,.\ \text{grad}\left\{\frac{1}{r}f'(r)\right\}$$

$$= \frac{3}{r}f'(r) + \vec{r}\,.\left[\frac{d}{dr}\left(\frac{1}{r}f'(r)\right)\text{grad}\ r\right] = \frac{3}{r}f'(r) + \vec{r}\,.\left[\left\{-\frac{1}{r^2}f'(r) + \frac{1}{r}f''(r)\right\}\frac{\vec{r}}{r}\right]$$

$$= \frac{3}{r}f'(r) + \left[-\frac{1}{r^3}f'(r) + \frac{1}{r^2}f''(r)\right](\vec{r}\,.\,\vec{r}) = \frac{3}{r}f'(r) + \left[-\frac{1}{r^3}f'(r) + \frac{1}{r^2}f''(r)\right]r^2$$

$$= \frac{3}{r}f'(r) - \frac{1}{r}f'(r) + f''(r) = f''(r) + \frac{2}{r}f'(r).$$