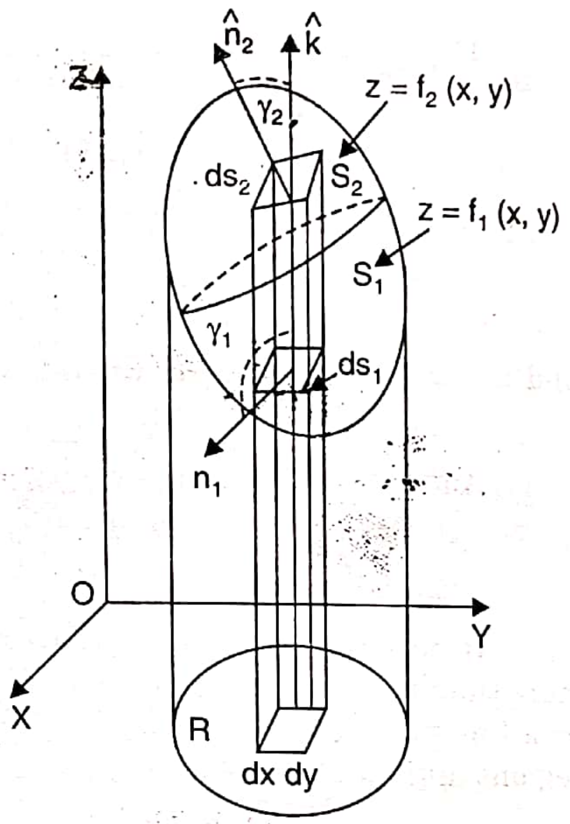
13.27. DIVERGENCE THEOREM OF GAUSS (*Relation between surface and volume integrals*)

If $\vec{F}$ is a vector point function having continuous first order partial derivatives in the ***region V bounded by a closed surface S,*** *then* $\iiint_V \nabla . \vec{F}\, dV = \iint_S \vec{F} . \hat{n}\ dS$, *where $\hat{n}$ is the outwards drawn unit normal vector to the surface S.*

[*i.e.*, the volume integral of the divergence of a vector point function $\vec{F}$ taken over the volume V enclosed by a surface S, is equal to the surface integral of the normal component of $\vec{F}$ taken over the closed surface S].



Let $\quad \vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$, then

$$\nabla . \vec{F} = \text{div } \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Let α, β, γ be the angles which the outwards drawn unit normal vector $\hat{n}$ makes with the positive directions of x, y, z-axes respectively, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of $\hat{n}$ and $\hat{n} = \cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k}$.

$$\therefore\ \vec{F} . \hat{n} = (F_1\hat{i} + F_2\hat{j} + F_3\hat{k}) . (\cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k})$$
$$= F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma$$

Therefore, the cartesian equivalent of divergence theorem is $\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\, dy\, dz$

$$= \iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)\, dS \qquad ...(1)$$

$$= \iint_S (F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy)$$

since $\cos\alpha\, dS = dy\, dz$, etc.

Suppose that S is such a closed surface that a line parallel to the co-ordinate axes meets it in two points only. Let S_1 and S_2 denote the lower and upper portions of S with equations $z = f_1(x, y)$ and $z = f_2(x, y)$ respectively.

Let R be the projection of S on the xy-plane, then

$$\iiint_V \frac{\partial F_3}{\partial z}\, dx\, dy\, dz = \iint_R \left[\int_{z=f_1(x,y)}^{z=f_2(x,y)} \frac{\partial F_3}{\partial z}\, dz\right] dx\, dy$$

$$= \iint_R \Big[F_3(x, y, z)\Big]_{f_1}^{f_2} dx\, dy = \iint_R [F_3(x, y, f_2) - F_3(x, y, f_1)]\, dx\, dy$$

$$= \iint_R F_3(x, y, f_2)\, dx\, dy - \iint_R F_3(x, y, f_1)\, dx\, dy \qquad ...(2)$$

Now for the upper portion S_2 of S, the normal $\hat{n}_2$ to S_2 makes an acute angle γ_2 with $\hat{k}$.

$$\therefore\quad dx\, dy = \cos\gamma_2\, dS_2 = \hat{k} . \hat{n}_2\, dS_2$$

For the lower portion S_1 of S, the normal $\hat{n}_1$ to S_1 makes an obtuse angle γ_1 with $\hat{k}$.

$\therefore$ $$dx\, dy = -\cos\gamma_1\, dS_1 = -\hat{k}\cdot\hat{n}_1\, dS_1$$

$\therefore$ $$\iint_R F_3(x, y, f_2)\, dx\, dy = \iint_{S_2} F_3\, \hat{k}\cdot\hat{n}_2\, dS_2 \qquad ...(3)$$

and $$\iint_R F_3(x, y, f_1)\, dxdy = -\iint_{S_1} F_3\, \hat{k}\cdot\hat{n}_1\, dS_1 \qquad ...(4)$$

Using (3) and (4), (2) becomes $\iiint_V \frac{\partial F_3}{\partial z}\, dx\, dy\, dz = \iint_{S_2} F_3\, \hat{k}\cdot\hat{n}_2\, dS_2 + \iint_{S_1} F_3\, \hat{k}\cdot\hat{n}_1\, dS_1$

$$= \iint_S F_3\hat{k}\cdot\hat{n}\, dS = \iint_R F_3\cos\gamma\, dS \qquad ...(5)$$

Similarly, by considering the projection of S on yz and zx-planes, we have

$$\iiint_V \frac{\partial F_1}{\partial x}\, dx\, dy\, dz = \iint_S F_1\hat{i}\cdot\hat{n}\, dS = \iint_S F_1\cos\alpha\, dS \qquad ...(6)$$

and $$\iiint_V \frac{\partial F_2}{\partial y}\, dx\, dy\, dz = \iint_S F_2\hat{j}\cdot\hat{n}\, dS = \iint_S F_2\cos\beta\, dS \qquad ...(7)$$

Adding (5), (6) and (7), we get (1) *i.e.*, $\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\, dy\, dz$

$$= \iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)\, dS \quad \text{or} \quad \iiint_V \nabla\cdot\vec{F}\, dV = \iint_R \vec{F}\cdot\hat{n}\, dS$$

In case the region be such that the lines drawn parallel to the coordinate axes meet it in more than two points, then we divide the region into various sub-regions each of which is met by a line parallel to any axis in only two points. Applying the theorem to each of these sub-regions and adding the results, we get the volume integral over the whole region.

ILLUSTRATIVE EXAMPLES

Example 1. *For any closed surface S, prove that* $\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\, dS = 0$.

Sol. By the divergence theorem, we have $\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\, dS = \iiint_V (\text{div curl}\,\vec{F})\, dV$,

where V is the volume enclosed by S = 0. Since div curl $\vec{F} = 0$, therefore, $\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\, dS = 0$

Example 2. *Evaluate* $\iint_S \vec{r}\cdot\hat{n}\, dS$, *where S is a closed surface.*

Sol. By the divergence theorem, we have $\iint_S \vec{r}\cdot\hat{n}\, dS = \iiint_V \nabla\cdot\vec{r}\, dV$,

where V is the volume enclosed by S

$$= \iiint_V 3dV, \text{ since } \nabla\cdot\vec{r} = \text{div}\,\vec{r} = 3$$

$$= 3V.$$

Example 3. *Use divergence theorem to show that* $\oint_S \nabla r^2\, d\vec{S} = 6V$, *where S is any closed surface enclosing a volume V.*

Sol. By the divergence theorem, we have $\oint_S \nabla r^2 \, d\vec{S} = \int_V \text{div}\,(\nabla r^2)\, dV$

$$= \int_V \nabla \cdot (\nabla r^2)\, dV = \int_V \nabla^2 r^2 \, dV$$

$$= \int_V \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) (x^2 + y^2 + z^2)\, dV = \int_V 6dV = 6V.$$

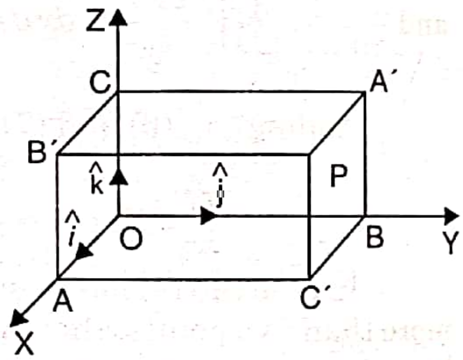
Example 4. *Verify divergence theorem for $\vec{F} = (x^2 - yz)\,\hat{i} + (y^2 - zx)\,\hat{j} + (z^2 - xy)\,\hat{k}$ taken over the rectangular parallelopiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$.*

(... Dec. 20...)

Sol. For verification of divergence theorem, we shall evaluate the volume and surface integrals separately and show that they are equal.

Now $\quad \text{div}\,\vec{F} = \dfrac{\partial}{\partial x}(x^2 - yz) + \dfrac{\partial}{\partial y}(y^2 - zx) + \dfrac{\partial}{\partial z}(z^2 - xy)$

$$= 2(x + y + z)$$



$\therefore \quad \iiint_V \text{div}\,\vec{F}\, dV$

$$= \int_0^c \int_0^b \int_0^a 2(x + y + z)\, dx dy dz$$

$$= \int_0^c \int_0^b 2\left[\left(\frac{x^2}{2} + yx + zx\right)\right]_0^a dy\, dz$$

$$= \int_0^c \int_0^b 2\left(\frac{a^2}{2} + ya + za\right) dy\, dz = \int_0^c 2\left[\left(\frac{a^2}{2}y + \frac{ay^2}{2} + azy\right)\right]_0^b dz$$

$$= 2\int_0^c \left(\frac{a^2 b}{2} + \frac{ab^2}{2} + abz\right) dz = 2\left[\frac{a^2 b}{2}z + \frac{ab^2}{2}z + ab\frac{z^2}{2}\right]_0^c$$

$$= a^2bc + ab^2c + abc^2 = abc\,(a + b + c) \qquad \text{...(1)}$$

To evaluate the surface integral, divide the closed surface S of the rectangular parallelopiped into 6 parts.

S_1 : the face OAC′B, S_2 : the face CB′PA′, S_3 : the face OBA′C,
S_4 : the face AC′PB′, S_5 : the face OCB′A, S_6 : the face BA′PC′.

Also $\displaystyle \iint_S \vec{F}\cdot\hat{n}\, dS = \iint_{S_1} \vec{F}\cdot\hat{n}\, dS + \iint_{S_2} \vec{F}\cdot\hat{n}\, dS + \iint_{S_3} \vec{F}\cdot\hat{n}\, dS$

$$+ \iint_{S_4} \vec{F}\cdot\hat{n}\, dS + \iint_{S_5} \vec{F}\cdot\hat{n}\, dS + \iint_{S_6} \vec{F}\cdot\hat{n}\, dS$$

On S_1 $(z = 0)$, we have $\hat{n} = -\hat{k}$, $\vec{F} = x^2\hat{i} + y^2\hat{j} - xy\hat{k}$

so that $\quad \vec{F}\cdot\hat{n} = (x^2\hat{i} + y^2\hat{j} - xy\hat{k})\cdot(-\hat{k}) = xy$

$$\therefore \quad \iint_S \vec{F}\cdot\hat{n}\, dS = \int_0^b \int_0^a xy\, dx\, dy = \int_0^b \left[y\frac{x^2}{2}\right]_0^a dy = \frac{a^2}{2}\int_0^b y\, dy = \frac{a^2 b^2}{4}$$

On S_2 $(z = c)$, we have $\hat{n} = \hat{k}, \vec{F} = (x^2 - cy)\hat{i} + (y^2 - cx)\hat{j} + (c^2 - xy)\hat{k}$

so that $$\vec{F} \cdot \hat{n} = [(x^2 - cy)\hat{i} + (y^2 - cx)\hat{j} + (c^2 - xy)\hat{k}] \cdot \hat{k} = c^2 - xy$$

$$\iint_{S_2} \vec{F} \cdot \hat{n}\, dS = \int_0^b \int_0^a (c^2 - xy)\, dx\, dy = \int_0^b \left(c^2 a - \frac{a^2}{2} y \right) dy = abc^2 - \frac{a^2 b^2}{4}$$

On S_3 $(x = 0)$, we have $\hat{n} = -\hat{i}, \vec{F} = -yz\hat{i} + y^2\hat{j} + z^2\hat{k}$

so that $$\vec{F} \cdot \hat{n} = (-yz\hat{i} + y^2\hat{j} + z^2\hat{k}) \cdot (-\hat{i}) = yz$$

$$\therefore \quad \iint_{S_3} \vec{F} \cdot \hat{n}\, dS = \int_0^c \int_0^b yz\, dy\, dz = \int_0^c \frac{b^2}{2} z\, dz = \frac{b^2 c^2}{4}$$

On S_4 $(x = a)$, we have $\hat{n} = \hat{i}, \vec{F} = (a^2 - yz)\hat{i} + (y^2 - az)\hat{j} + (z^2 - ay)\hat{k}$

so that $$\vec{F} \cdot \hat{n} = [(a^2 - yz)\hat{i} + (y^2 - az)\hat{j} + (z^2 - ay)\hat{k}] \cdot \hat{i} = a^2 - yz$$

$$\therefore \quad \iint_{S_4} \vec{F} \cdot \hat{n}\, dS = \int_0^c \int_0^b (a^2 - yz)\, dy\, dz = \int_0^c \left(a^2 b - \frac{b^2}{2} z \right) dz = a^2 bc - \frac{b^2 c^2}{4}$$

On S_5 $(y = 0)$, we have $\hat{n} = -\hat{j}, \vec{F} = x^2\hat{i} - zx\hat{j} + z^2\hat{k}$

so that $$\vec{F} \cdot \hat{n} = (x^2\hat{i} - zx\hat{j} + z^2\hat{k}) \cdot (-\hat{j}) = zx$$

$$\therefore \quad \iint_{S_5} \vec{F} \cdot \hat{n}\, dS = \int_0^a \int_0^c zx\, dz\, dx = \int_0^a \frac{c^2}{2} x\, dx = \frac{a^2 c^2}{4}$$

On S_6 $(y = b)$, we have $\hat{n} = \hat{j}, \vec{F} = (x^2 - bz)\hat{i} + (b^2 - zx)\hat{j} + (z^2 - bx)\hat{k}$

so that $$\vec{F} \cdot \hat{n} = [(x^2 - bz)\hat{i} + (b^2 - zx)\hat{j} + (z^2 - bx)\hat{k}] \cdot \hat{j} = b^2 - zx$$

$$\therefore \quad \iint_{S_6} \vec{F} \cdot \hat{n}\, dS = \int_0^a \int_0^c (b^2 - zx)\, dz\, dx = \int_0^a \left(b^2 c - \frac{c^2}{2} x \right) dx = ab^2 c - \frac{a^2 c^2}{4}$$

$$\therefore \quad \iint_S \vec{F} \cdot \hat{n}\, dS = \frac{a^2 b^2}{4} + abc^2 - \frac{a^2 b^2}{4} + \frac{b^2 c^2}{4} + a^2 bc - \frac{b^2 c^2}{4} + \frac{a^2 c^2}{4} + ab^2 c - \frac{a^2 c^2}{4}$$

$$= abc\,(a + b + c) \qquad \text{...(2)}$$

The equality of (1) and (2) verifies divergence theorem.

Example 5. *Verify divergence theorem for $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ taken over the region bounded by the cylinder $x^2 + y^2 = 4$, $z = 0$, $z = 3$.*

Sol. Since div $\vec{F} = \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2) = 4 - 4y + 2z$

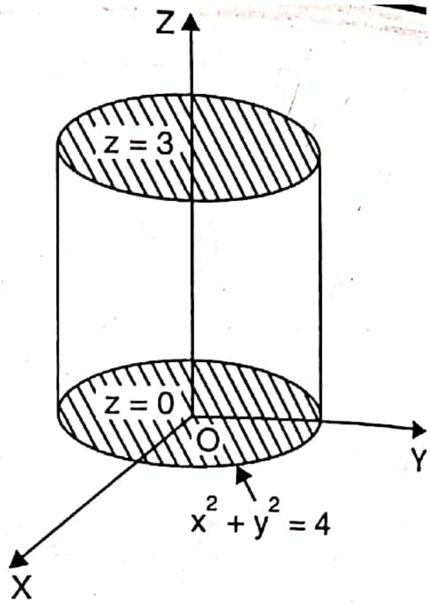
$$\therefore \quad \iiint_V \operatorname{div} \vec{F}\, dV = \iiint_V (4 - 4y + 2z)\, dx\, dy\, dz$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^3 (4 - 4y + 2z)\, dz dy dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4z - 4yz + z^2\right]_0^3 dy dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (12 - 12y + 9)\, dy\, dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 21\, dy\, dx,$$



(Since $12y$ is an odd function, $\int_{-a}^{a} 12y\, dy = 0$)

$$= \int_{-2}^{2} 42\sqrt{4 - x^2}\, dx = 84\int_0^2 \sqrt{4 - x^2}\, dx = 84\left[\frac{x\sqrt{4 - x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2 = 84[2\sin^{-1} 1]$$

$$= 84\left[2 \times \frac{\pi}{2}\right] = 84\pi \qquad \text{...(1)}$$

To evaluate the surface integral, divide the closed surface S of the cylinder into 3 parts.

S_1 : the circular base in the plane $z = 0$

S_2 : the circular top in the plane $z = 3$

S_3 : the curved surface of the cylinder, given by the equation $x^2 + y^2 = 4$.

Also $\quad \iint_S \vec{F} . \hat{n}\, dS = \iint_{S_1} \vec{F} . \hat{n}\, dS + \iint_{S_2} \vec{F} . \hat{n}\, dS + \iint_{S_3} \vec{F} . \hat{n}\, dS$

On S_1 ($z = 0$), we have $\hat{n} = -\hat{k}, \vec{F} = 4x\hat{i} - 2y^2\hat{j}$

so that $\quad \vec{F} . \hat{n} = (4x\hat{i} - 2y^2\hat{j}) . (-\hat{k}) = 0$

$$\therefore \quad \iint_{S_1} \vec{F} . \hat{n}\, dS = 0$$

On S_2 ($z = 3$), we have $\hat{n} = \hat{k}, \vec{F} = 4x\hat{i} - 2y^2\hat{j} + 9\hat{k}$

so that $\quad \vec{F} . \hat{n} = (4x\hat{i} - 2y^2\hat{j} + 9\hat{k}) . \hat{k} = 9$

$$\therefore \quad \iint_{S_2} \vec{F} . \hat{n}\, dS = \iint_{S_2} 9 dx\, dy = 9\iint_{S_2} dx\, dy$$

$$= 9 \times \text{area of surface } S_2 = 9\,(\pi\,.\,2^2) = 36\pi$$

On S_3, $x^2 + y^2 = 4$

A vector normal to the surface S_3 is given by $\nabla\,(x^2 + y^2) = 2x\hat{i} + 2y\hat{j}$

$\therefore \quad \hat{n}$ = a unit vector normal to surface S_3

$$= \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{4x^2 + 4y^2}} = \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{4 \times 4}}, \text{ since } x^2 + y^2 = 4$$

$$= \frac{x\hat{i} + y\hat{j}}{2}$$

$$\vec{F} \cdot \hat{n} = (4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}) \cdot \left(\frac{x\hat{i} + y\hat{j}}{2}\right) = 2x^2 - y^3$$

Also, on S_3, *i.e.*, $x^2 + y^2 = 4$, $x = 2\cos\theta$, $y = 2\sin\theta$ and $dS = 2d\theta\, dz$.
To cover the whole surface S_3, z varies from 0 to 3 and θ varies from 0 to 2π.

$$\therefore \quad \iint_{S_3} \vec{F} \cdot \hat{n}\, dS = \int_0^{2\pi}\int_0^3 [2(2\cos\theta)^2 - (2\sin\theta)^3]\, 2dz\, d\theta$$

$$= \int_0^{2\pi} 16(\cos^2\theta - \sin^3\theta) \times 3\, d\theta = 48\int_0^{2\pi}(\cos^2\theta - \sin^3\theta)\, d\theta = 48\pi$$

$$\left(\text{since} \int_0^{2\pi}\cos^2\theta\, d\theta = 2\int_0^{\pi}\cos^2\theta\, d\theta = 4\int_0^{\pi/2}\cos^2 d\theta = 4 \times \frac{1}{2} \times \frac{\pi}{2} = \pi, \int_0^{2\pi}\sin^3\theta\, d\theta = 0\right)$$

$$\therefore \quad \iint_S \vec{F} \cdot \hat{n}\, dS = 0 + 36\pi + 48\pi = 84\pi \qquad \text{...(2)}$$

The equality of (1) and (2) verifies divergence theorem.

Example 6. *Find* $\iint_S \vec{F} \cdot \hat{n}\, dS$, *where* $\vec{F} = (2x + 3z)\hat{i} - (xz + y)\hat{j} + (y^2 + 2z)\hat{k}$ *and S is the surface of the sphere having centre at* $(3, -1, 2)$ *and radius 3.*

Sol. Let V be the volume enclosed by the surface S. Then by Gauss divergence theorem, we have

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_V \operatorname{div}\vec{F}\, dV$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(2x + 3z) + \frac{\partial}{\partial y}(-xz - y) + \frac{\partial}{\partial z}(y^2 + 2z)\right] dV$$

$$= \iiint_V (2 - 1 + 2)\, dV = 3\iiint_V dV = 3V$$

But V is the volume of a sphere of radius 3.

$$\therefore \quad V = \frac{4}{3}\pi(3)^3 = 36\pi.$$

Hence $\iint_S \vec{F} \cdot d\vec{S} = 3 \times 36\pi = 108\pi$.

Example 7. *Evaluate* $\iint_S (y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k}) \cdot \hat{n}\, dS$, *where S is the part of the sphere* $x^2 + y^2 + z^2 = 1$ *above the xy-plane and bounded by this plane.*

Sol. Let V be the volume enclosed by the surface S. Then by divergence theorem, we have

$$\iint_S (y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k}) \cdot \hat{n}\, dS = \iiint_V \operatorname{div}(y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k})\, dV$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(y^2z^2) + \frac{\partial}{\partial y}(z^2x^2) + \frac{\partial}{\partial z}(z^2y^2)\right] dV$$

$$= \iiint_V 2zy^2\, dV = 2\iiint_V zy^2\, dV$$

Changing to spherical polar coordinates by putting

$$x = r \sin\theta \cos\phi,\ y = r \sin\theta \sin\phi,\ z = r\cos\theta$$

$$dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

To cover V, the limits of r will be 0 to 1, those of θ will be 0 to $\frac{\pi}{2}$ and those of ϕ will be 0 to 2π.

$$\therefore\quad 2\iiint_V zy^2\, dV = 2\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 (r\cos\theta)(r^2\sin^2\theta\sin^2\phi)\, r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= 2\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 r^5 \sin^3\theta\cos\theta\sin^2\phi\, dr\, d\theta\, d\phi$$

$$= 2\int_0^{2\pi}\int_0^{\pi/2} \sin^3\theta\cos\theta\sin^2\phi \left[\frac{r^6}{6}\right]_0^1 d\theta\, d\phi$$

$$= \frac{2}{6}\int_0^{2\pi} \sin^2\phi \cdot \frac{2}{4.2}\, d\phi = \frac{1}{12}\int_0^{2\pi}\sin^2\phi\, d\phi = \frac{\pi}{12}.$$

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2. Verify divergence theorem for $\vec{F} = x^2\hat{i} + z\hat{j} + yz\hat{k}$ taken over the cube bounded by $x = 0, x = 1$; $y = 0$; $y = 1$; $z = 0, z = 1$.

3. Verify divergence theorem for $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ taken over the cube bounded by $x = 0, x = 1$; $y = 0, y = 1$; $z = 0, z = 1$. *(Madras 2006; M.D.U. May 2005; U.P.T.U. 2009)*

4. Verify divergence theorem for $\vec{F} = (x^3 - yz)\hat{i} - 2x^2y\hat{j} + 2\hat{k}$ taken over the cube bounded by the planes $x = 0, x = a$; $y = 0, y = a$; $z = 0, z = a$.

5. (*a*) Verify divergence theorem for $\vec{F} = y\hat{i} + x\hat{j} + z^2\hat{k}$ over the cylindrical region bounded by $x^2 + y^2 = a^2$, $z = 0$ and $z = h$.

(*b*) Verify Gauss divergence theorem for the function $\vec{F} = y\hat{i} + x\hat{j} + z^2\hat{k}$ over the cylindrical region bounded by $x^2 + y^2 = 9$, $z = 0$ and $z = 2$. *(M.D.U. Dec. 2005)*

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