13.10. SCALAR AND VECTOR FIELDS

A variable quantity whose value at any point in a region of space depends upon the position of the point, is called a *point function*. There are two types of point functions.

(*a*) *Scalar Point Function*. Let R be a region of space at each point of which a scalar $\phi = \phi(x, y, z)$ is given, then ϕ is called a *scalar function* and R is called a *scalar field*.

The temperature distribution in a medium, the distribution of atmospheric pressure in space are examples of scalar point functions.

(*b*) *Vector Point Function*. Let R be a region of space at each point of which a vector $\vec{v} = \vec{v}(x, y, z)$ is given, then $\vec{v}$ is called a *vector point function* and R is called a *vector field*. Each vector $\vec{v}$ of the field is regarded as a localised vector attached to the corresponding point (x, y, z).

The velocity of a moving fluid at any instant, the gravitational force are examples of vector point functions.

13.11. GRADIENT OF A SCALAR FIELD

Let $\phi(x, y, z)$ be a function defining a scalar field, then the vector $\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$ is called the **gradient** of the scalar field ϕ and is denoted by grad ϕ.

Thus, $$\text{grad } \phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

The gradient of scalar field ϕ is obtained by operating on ϕ by the vector operator

$$\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

This operator is denoted by the symbol ∇, read as **del** (also called nabla).

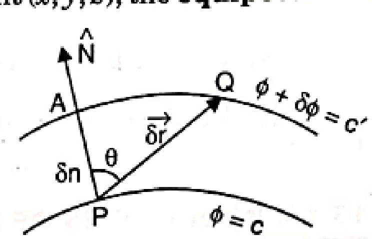
Thus, grad $\phi = \nabla\phi$.

13.12. GEOMETRICAL INTERPRETATION OF GRADIENT

If a surface $\phi(x, y, z) = c$ is drawn through any point P such that at each point on the surface, the function has the same value as at P, then such a surface is called a *level surface* through P. For example, if $\phi(x, y, z)$ represents potential at the point (x, y, z), the **equipotential surface** $\phi(x, y, z) = c$ is a level surface.

Through any point passes one and only one level surface. Moreover, no two level surfaces can intersect.

Consider the level surface through P at which the function has value ϕ and another level surface through a neighbouring point Q where the value is $\phi + \delta\phi$.



Let $\vec{r}$ and $\vec{r}+\delta\vec{r}$ be the position vectors of P and Q respectively, then $\overrightarrow{PQ} = \delta\vec{r}$.

Now $$\nabla\phi.\,\delta\vec{r} = \left(\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}\right).(\hat{i}\delta x+\hat{j}\delta y+\hat{k}\delta z)$$

$$= \frac{\partial\phi}{\partial x}\delta x+\frac{\partial\phi}{\partial y}\delta y+\frac{\partial\phi}{\partial z}\delta z = \delta\phi \qquad ...(1)$$

If Q lies on the same level surface as P, then $\delta\phi = 0$,

$\therefore$ (1) reduces to $\nabla\phi\,.\,\delta\vec{r} = 0$.

Thus, $\nabla\phi$ is perpendicular to every $\delta\vec{r}$ lying in the surface.

Hence $\nabla\phi$ **is normal to the surface** $\phi(x, y, z) = c$.

Let $\nabla\phi = |\nabla\phi|\,\hat{N}$, where $\hat{N}$ is a unit vector normal to the surface. Let PA $= \delta n$ be the perpendicular distance between the two level surfaces through P and Q. Then the rate of change of ϕ in the direction of normal to the surface through P is

$$\frac{\partial\phi}{\partial n} = \lim_{\delta n\to 0}\frac{\partial\phi}{\partial n} = \lim_{\partial n\to 0}\frac{\nabla\phi\,.\,\delta\vec{r}}{\delta n} \qquad \text{[by (1)]}$$

$$= \lim_{\delta n\to 0}\frac{|\nabla\phi|\hat{N}\,.\,\delta\vec{r}}{\delta n} = |\nabla\phi| \quad (\because\ \hat{N}\,.\,\delta\vec{r} = |\hat{N}||\delta\vec{r}|\cos\theta = |\delta\vec{r}|\cos\theta = \delta n)$$

$\therefore \quad |\nabla\phi| = \dfrac{\partial\phi}{\partial n}$.

Hence the gradient of a scalar field ϕ is a vector normal to the surface $\phi = c$ and has a magnitude equal to the rate of change of ϕ along this normal.

13.13. DIRECTIONAL DERIVATIVE

Let PQ $= \delta r$, then $\lim\limits_{\delta r\to 0}\dfrac{\delta\phi}{\delta r} = \dfrac{\partial\phi}{\partial r}$ is called the directional derivative of ϕ at P in the direction PQ.

Let $\hat{N}'$ be a unit vector in the direction PQ, then $\delta r = \dfrac{\delta n}{\cos\theta} = \dfrac{\delta n}{\hat{N}\,.\,\hat{N}'}$

$$\therefore \quad \frac{\partial\phi}{\partial r} = \lim_{\delta r\to 0}\left[\hat{N}\,.\,\hat{N}'\frac{\delta\phi}{\delta n}\right] = \hat{N}\,.\,\hat{N}'\frac{\partial\phi}{\partial n}$$

$$= \hat{N}'\,.\,\hat{N}\frac{\partial\phi}{\partial n} = \hat{N}'\,.\,\hat{N}|\nabla\phi| = \hat{N}'\,.\,\nabla\phi \qquad \left(\because\ |\nabla\phi| = \frac{\partial\phi}{\partial n} \text{ and } \hat{N}|\nabla\phi| = \nabla\phi\right)$$

Thus, the directional derivative $\dfrac{\partial\phi}{\partial r}$ is the resolved part of $\nabla\phi$ in the direction $\hat{N}'$.

Since $$\frac{\partial\phi}{\partial r} = \hat{N}'\,.\,\nabla\phi = |\nabla\phi|\cos\theta \le |\nabla\phi|.$$

$\therefore$ *$\nabla\phi$ gives the maximum rate of change of ϕ and the magnitude of this maximum is* $|\nabla\phi|$.

13.14. PROPERTIES OF GRADIENT

(*a*) If ϕ is a constant scalar point function, then $\nabla\phi = \vec{0}$

(*b*) If ϕ_1 and ϕ_2 are two scalar point functions, then

(*i*) $\nabla(\phi_1 \pm \phi_2) = \nabla\phi_1 \pm \nabla\phi_2$

(*ii*) $\nabla(c_1\phi_1 + c_2\phi_2) = c_1\nabla\phi_1 + c_2\nabla\phi_2$, where c_1, c_2 are constant

(*iii*) $\nabla(\phi_1\phi_2) = \phi_1\nabla\phi_2 + \phi_2\nabla\phi_1$

(*iv*) $\nabla\left(\dfrac{\phi_1}{\phi_2}\right) = \dfrac{\phi_2\nabla\phi_1 - \phi_1\nabla\phi_2}{\phi_2^{\,2}}$, $\phi_2 \neq 0$.

ILLUSTRATIVE EXAMPLES

Example 1. *Find grad ϕ when ϕ is given by $\phi = 3x^2y - y^3z^2$ at the point (1, – 2, – 1).*

Sol. Grad $\phi = \nabla\phi = \left(\hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}\right)(3x^2y - y^3z^2)$

$$= \hat{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \hat{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \hat{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= \hat{i}\,(6xy) + \hat{j}(3x^2 - 3y^2z^2) + \hat{k}(-2y^3z)$$

$$= -12\hat{i} - 9\hat{j} - 16\hat{k} \text{ at the point } (1, -2, -1).$$

Example 2. *If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, show that*

(*i*) *grad* $r = \dfrac{\vec{r}}{r}$ (*ii*) *grad* $\left(\dfrac{1}{r}\right) = -\dfrac{\vec{r}}{r^3}$ (*iii*) $\nabla r^n = nr^{n-2}\,\vec{r}$

(*iv*) $\nabla(\vec{a}\,.\,\vec{r}) = \vec{a}$, *where $\vec{a}$ is a constant vector.*

Sol. $r = |\,\vec{r}\,| = \sqrt{x^2 + y^2 + z^2}$, or $r^2 = x^2 + y^2 + z^2$

Differentiating partially w.r.t. x, we have $2r\dfrac{\partial r}{\partial x} = 2x$ or $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

Similarly, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$ and $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

(*i*) grad $r = \nabla r = \left(\hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}\right)r = \hat{i}\dfrac{\partial r}{\partial x} + \hat{j}\dfrac{\partial r}{\partial y} + \hat{k}\dfrac{\partial r}{\partial z}$

$$= \hat{i}\left(\frac{x}{r}\right) + \hat{j}\left(\frac{y}{r}\right) + \hat{k}\left(\frac{z}{r}\right) = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r} = \frac{\vec{r}}{r}.$$

(*ii*) grad $\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{r}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)$

$$= \hat{i}\left(-\frac{1}{r^2}\cdot\frac{\partial r}{\partial x}\right) + \hat{j}\left(-\frac{1}{r^2}\cdot\frac{\partial r}{\partial y}\right) + \hat{k}\left(-\frac{1}{r^2}\cdot\frac{\partial r}{\partial z}\right)$$

$$= \hat{i}\left(-\frac{1}{r^2}\cdot\frac{x}{r}\right) + \hat{j}\left(-\frac{1}{r^2}\cdot\frac{y}{r}\right) + \hat{k}\left(-\frac{1}{r^2}\cdot\frac{z}{r}\right)$$

$$= -\frac{1}{r^3}(x\hat{i} + y\hat{j} + z\hat{k}) = -\frac{\vec{r}}{r^3}.$$

(*iii*) $\nabla r^n = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)r^n = \hat{i}\left(nr^{n-1}\frac{\partial r}{\partial x}\right) + \hat{j}\left(nr^{n-1}\frac{\partial r}{\partial y}\right) + \hat{k}\left(nr^{n-1}\frac{\partial r}{\partial z}\right)$

$$= \hat{i}\left(nr^{n-1}\cdot\frac{x}{r}\right) + \hat{j}\left(nr^{n-1}\cdot\frac{y}{r}\right) + \hat{k}\left(nr^{n-1}\cdot\frac{z}{r}\right) = nr^{n-2}(x\hat{i} + y\hat{j} + z\hat{k}) = nr^{n-2}\vec{r}.$$

(*iv*) Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, where a_1, a_2, a_3 are constants.

$$\vec{a}\cdot\vec{r} = a_1x + a_2y + a_3z$$

$$\therefore \quad \nabla(\vec{a}\cdot\vec{r}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(a_1x + a_2y + a_3z)$$

$$= \hat{i}\frac{\partial}{\partial x}(a_1x + a_2y + a_3z) + \hat{j}\frac{\partial}{\partial y}(a_1x + a_2y + a_3z) + \hat{k}\frac{\partial}{\partial z}(a_1x + a_2y + a_3z)$$

$$= a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = \vec{a}.$$

Example 3. *Find a unit vector normal to the surface $x^3 + y^3 + 3xyz = 3$ at the point $(1, 2, -1)$.*

Sol. Let $\phi = x^3 + y^3 + 3xyz = 3$, then $\frac{\partial\phi}{\partial x} = 3x^2 + 3yz$, $\frac{\partial\phi}{\partial y} = 3y^2 + 3xz$, $\frac{\partial\phi}{\partial z} = 3xy$

$$\therefore \quad \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = (3x^2 + 3yz)\hat{i} + (3y^2 + 3xz)\hat{j} + (3xy)\hat{k}$$

At $(1, 2, -1)$, $\nabla\phi = -3\hat{i} + 9\hat{j} + 6\hat{k}$

which is a vector normal to the given surface at $(1, 2, -1)$.

Hence a unit vector normal to the given surface at $(1, 2, -1)$

$$= \frac{-3\hat{i} + 9\hat{j} + 6\hat{k}}{\sqrt{[(-3)^2 + (9)^2 + (6)^2]}} = \frac{-3\hat{i} + 9\hat{j} + 6\hat{k}}{3\sqrt{14}} = \frac{1}{\sqrt{14}}(-\hat{i} + 3\hat{j} + 2\hat{k}).$$

Example 4. *Find the directional derivative of the function $f = x^2 - y^2 + 2z^2$ at the point P(1, 2, 3) in the direction of the line PQ where Q is the point (5, 0, 4).*

In what direction it will be maximum ? Find also the magnitude of this maximum.

Sol. We have $\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z} = 2x\hat{i} - 2y\hat{j} + 4z\hat{k} = 2\hat{i} - 4\hat{j} + 12\hat{k}$ at P(1, 2, 3)

Also $$\vec{PQ} = \vec{OQ} - \vec{OP} = (5\hat{i} + 4\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = 4\hat{i} - 2\hat{j} + \hat{k}$$

If $\hat{n}$ is a unit vector in the direction $\vec{PQ}$, then $\hat{n} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{16+4+1}} = \frac{1}{\sqrt{21}}(4\hat{i} - 2\hat{j} + \hat{k})$

$\therefore$ Directional derivative of f in the direction $\vec{PQ} = (\nabla f) \cdot \hat{n}$

$$= (2\hat{i} - 4\hat{j} + 12\hat{k}) \cdot \frac{1}{\sqrt{21}}(4\hat{i} - 2\hat{j} + \hat{k}) = \frac{1}{\sqrt{21}}[2(4) - 4(-2) + 12(1)]$$

$$= \frac{28}{\sqrt{21}} = \frac{4}{3}\sqrt{21}$$

The directional derivative of f is maximum in the direction of the normal to the given surface *i.e.*, in the direction of $\nabla f = 2\hat{i} - 4\hat{j} + 12\hat{k}$

The maximum value of this directional derivative $= |\nabla f|$

$$= \sqrt{(2)^2 + (-4)^2 + (12)^2} = \sqrt{164} = 2\sqrt{41}.$$

Example 5. *Find the directional derivative of $\phi = 5x^2y - 5y^2z + \frac{5}{2}z^2x$ at the point P(1, 1, 1) in the direction of the line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$.*

Sol. Here, $$\phi = 5x^2y - 5y^2z + \frac{5}{2}z^2x$$

$\therefore$ $$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \left(10xy + \frac{5}{2}z^2\right)\hat{i} + (5x^2 - 10yz)\hat{j} + (-5y^2 + 5zx)\hat{k}$$

$$= \frac{25}{2}\hat{i} - 5\hat{j} \qquad \text{at P(1, 1, 1)}$$

The direction of the given line is $\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$

$\Rightarrow$ $$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{2\hat{i} - 2\hat{j} + \hat{k}}{3}$$

$\therefore$ **The required directional derivative**

$$= (\nabla\phi) . \hat{a} = \left(\frac{25}{2}\hat{i} - 5\hat{j}\right) . \left(\frac{2\hat{i} - 2\hat{j} + \hat{k}}{3}\right)$$

$$= \left(\frac{25}{2}\right)\left(\frac{2}{3}\right) + (-5)\left(-\frac{2}{3}\right) + (0)\left(\frac{1}{3}\right) = \frac{35}{3}.$$

Example 7. *Find the values of constants a, b and c so that the maximum value of the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at (1, 2, – 1) has a magnitude 64 in the direction parallel to z-axis.*

Sol. Here, $\phi = axy^2 + byz + cz^2x^3$

$\therefore$ $$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= (ay^2 + 3cz^2x^2)\,\hat{i} + (2axy + bz)\,\hat{j} + (by + 2czx^3)\,\hat{k}$$

$$= (4a + 3c)\,\hat{i} + (4a - b)\,\hat{j} + (2b - 2c)\,\hat{k} \text{ at } (1, 2, -1)$$

Now, the directional derivative of ϕ is maximum in the direction of the normal to the given surface *i.e.*, in the direction of $\nabla\phi$. But we are given that the directional derivative of ϕ is maximum in the direction parallel to z-axis *i.e.*, parallel to $\hat{k}$.

Hence co-efficients of $\hat{i}$ and $\hat{j}$ in $\nabla\phi$ should be zero and the co-efficient of $\hat{k}$ positive.

Thus, $4a + 3c = 0$ ...(1)

$4a - b = 0$ ...(2)

and $2b - 2c > 0$ *i.e.*, $b > c$ ...(3)

Then, $\nabla\phi = 2(b - c)\,\hat{k}$

Also maximum value of directional derivative $= |\nabla\phi|$

$\therefore$ $|2\,(b - c)\,\hat{k}\,| = 64$ (given)

$\Rightarrow$ $2(b - c) = 64$ or $b - c = 32$ ...(4)

Solving (1), (2) and (4), we have

$$a = 6, b = 24, c = -8.$$

Example 8. *Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point (2, – 1, 2).* (Kottayam 2005)

Sol. Angle between two surfaces at a point is the angle between the normals to the surfaces at that point.

Let $\phi_1 = x^2 + y^2 + z^2 = 9$ and $\phi_2 = x^2 + y^2 - z = 3$

Then grad $\phi_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$ and grad $\phi_2 = 2x\hat{i} + 2y\hat{j} - \hat{k}$

Let $\vec{n_1}$ = grad ϕ_1 at the point (2, – 1, 2) and $\vec{n_2}$ = grad ϕ_2 at the point (2, – 1, 2). Then

$$\vec{n_1} = 4\hat{i} - 2\hat{j} + 4\hat{k} \quad \text{and} \quad \vec{n_2} = 4\hat{i} - 2\hat{j} - \hat{k}$$

The vectors $\vec{n_1}$ and $\vec{n_2}$ are along normals to the two surfaces at the point (2, – 1, 2). If θ is the angle between these vectors, then

$$\cos\theta = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|} = \frac{4(4)-2(-2)+4(-1)}{\sqrt{16+4+16}\,.\sqrt{16+4+1}} = \frac{16}{6\sqrt{21}}$$

$$\therefore \qquad \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right).$$

EXERCISE

1. Find grad ϕ when ϕ is given by
   (i) $\phi = x^2 + yz$ (ii) $\phi = x^3 + y^3 + 3xyz$ (iii) $\phi = \log(x^2 + y^2 + z^2)$.
2. If $r = |\vec{r}|$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that
   (i) $\nabla f(r) = f'(r)\,\nabla r$ (ii) $\nabla \log r = \frac{\vec{r}}{r^2}$
   (iii) $\nabla(e^{r^2}) = 2e^{r^2}\,\vec{r}$ (iv) grad $|\vec{r}|^2 = 2\vec{r}$
   (v) $\text{grad}\left(\frac{1}{r^2}\right) = -\frac{2\vec{r}}{r^4}$
3. If $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = yz + zx + xy$, prove that :
   (i) (grad u) . [(grad v) × (grad w)] = 0
   (ii) grad u, grad v and grad w are coplanar vectors.
   [**Hint.** Three vectors are coplanar if their scalar triple product is zero].
4. Find a unit vector normal to the surface
   (i) $xy^3z^2 = 4$ at the point (– 1, – 1, 2) (ii) $x^2y + 2xz = 4$ at the point (2, – 2, 3).
5. Find the directional derivative of the function
   (i) $f(x, y, z) = xy^2 + yz^3$ at the point (2, – 1, 1) in the direction of the vector $\hat{i} + 2\hat{j} + 2\hat{k}$.
   (ii) $f(x, y, z) = 2xy + z^2$ at the point (1, – 1, 3) in the direction of the vector $\hat{i} + 2\hat{j} + 2\hat{k}$.
   (iii) $\phi = x^2yz + 4xz^2$ at the point (1, – 2, – 1) in the direction of the vector $2\hat{i} - \hat{j} - 2\hat{k}$.