End Sem Exam

19MAT111 — MVC

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6. Given that

a, b, c — are unit vectors

$a\times(b\times c) = b$

We know that

$$a\times(b\times c) = (a\cdot c)b - (a\cdot b)c \quad — \text{①}$$

from the above formulae

$$(a\cdot c)b - (a\cdot b)c = b$$

$$(a\cdot c)b - b = (a\cdot b)c$$

$$(a\cdot c - 1)b = (a\cdot b)c$$

But b and c are non parallel

$$a \cdot c - 1 = 0 \quad \text{—} (2)$$

$$a \cdot b = 0 \quad \text{—} (3)$$

from (2)

$$a \cdot c = 1$$

$$|a||c| \cos\theta = 1$$

[Given a, b, c are unit vectors, then magnitude of $a, b, c = 1$

$$1 \cdot 1 \cos\theta = 1$$

$$\cos\theta = 1$$

$$\theta = 0°$$

Angle between a and c is $0°$

from (3)

$$a \cdot b = 0$$

$$|a| |b| \cos\theta = 0$$

$$1 \cdot 1 \cos\theta = 0$$

$$\cos\theta = 0$$

$$\theta = \pi/2$$

Angle b/w a and b is $\pi/2$

7. Given;

$\vec{a} = \sin\theta\,\hat{i} + \cos\theta\,\hat{j} + \theta\,\hat{k}$, $\vec{b} = \cos\theta\,\hat{i} - \sin\theta\,\hat{j} - 3\hat{k}$

$\vec{c} = 2\hat{i} + 3\hat{j} - \hat{k}$

Find :- $\frac{d}{d\theta}[\vec{a} \times (\vec{b} \times \vec{c})]$ at $\theta = \pi$

Sol

$$\vec{b} \times \vec{c} = (\cos\theta\,\hat{i} - \sin\theta\,\hat{j} - 3\hat{k}) \times (2\hat{i} + 3\hat{j} - \hat{k})$$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & -\sin\theta & -3 \\ 2 & 3 & -1 \end{vmatrix}$$

$$\vec{b} \times \vec{c} = \hat{i}(\sin\theta + 9) - \hat{j}(-\cos\theta + 6) + \hat{k}(3\cos\theta + 2\sin\theta)$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\sin\theta\,\hat{i} + \cos\theta\,\hat{j} + \theta\,\hat{k}) \times ((\sin\theta + 9)\hat{i} - (-\cos\theta + 6)\hat{j} + (3\cos\theta + 2\sin\theta)\hat{k})$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sin\theta & \cos\theta & \theta \\ \sin\theta + 9 & \cos\theta - 6 & 3\cos\theta + 2\sin\theta \end{vmatrix}$$

$$= \hat{i}(3\cos^2\theta + 2\sin\theta\cos\theta - \theta\cos\theta + 6\theta) - \hat{j}(3\sin\theta\cos\theta + 2\sin^2\theta - \theta\sin\theta - 9\theta) + \hat{k}(\sin\theta\cos\theta - 6\sin\theta - \sin\theta\cos\theta - 9\cos\theta)$$

$$\Rightarrow \vec{a}\times(\vec{b}\times\vec{c}) = \hat{i}(3\cos^2\theta + \sin 2\theta - \theta\cos\theta + 6\theta) - \hat{j}(3\sin\theta\cos\theta + 2\sin^2\theta - \theta\sin\theta - 2\theta) + \hat{k}(-6\sin\theta - 9\cos\theta)$$

$$\frac{d}{d\theta}[\vec{a}\times(\vec{b}\times\vec{c})] = \frac{d}{d\theta}[\hat{i}(3\cos^2\theta + \sin 2\theta - \theta\cos\theta + 6\theta)] - \frac{d}{d\theta}[\hat{j}(3\sin\theta\cos\theta + 2\sin^2\theta - \theta\sin\theta - 2\theta)] + \frac{d}{d\theta}[\hat{k}(-6\sin\theta - 9\cos\theta)]$$

Now

$$\hat{i}\frac{d}{d\theta}(3\cos^2\theta + \sin 2\theta - \theta\cos\theta + 6\theta)$$

$$= (6\cos\theta(-\sin\theta) + 2\cos 2\theta + \theta\sin\theta - \cos\theta + 6)\hat{i}$$

$$= (-3\sin 2\theta + 2\cos 2\theta + \theta\sin\theta - \cos\theta + 6)\hat{i}$$

$$\hat{j}\frac{d}{d\theta}\left(\frac{3}{2}\sin 2\theta + 2\sin^2\theta - \theta\sin\theta - 9\theta\right)$$

$$= \left(\frac{3}{2}\cos 2\theta \cdot 2 + 4\sin\theta\cos\theta - \sin\theta - \theta\cos\theta - 9\right)\hat{j}$$

$\hat{k}^* = (3\cos 2\theta + 2\sin 2\theta - \sin\theta - \theta\cos\theta - 9)\hat{j}$

$\hat{k}\frac{d}{d\theta}(-6\sin\theta - 9\cos\theta)$

$= (-6\cos\theta + 9\sin\theta)\hat{k}$

$\therefore \frac{d}{d\theta}[\vec{a}\times(\vec{b}\times\vec{c})] = \hat{i}(-3\sin 2\theta + 2\cos 2\theta + \theta\sin\theta - \cos\theta + 6)$

$- \hat{j}(3\cos 2\theta + 2\sin 2\theta - \sin\theta - \theta\cos\theta - 9)$

$+ \hat{k}(-6\cos\theta + 9\sin\theta)$

$\therefore$ At $\theta = \pi$

$\frac{d}{d\theta}[\vec{a}\times(\vec{b}\times\vec{c})] = \cancel{\hat{i}(-3(0) + 2}$

$\hat{i}(-3\sin 2(\pi) + 2\cos 2(\pi) + \pi\sin(\pi)$

$- \cos(\pi) + 6]$

$-\hat{j}(3\cos 2(\pi) + 2\sin 2(\pi) - \sin(\pi) - \pi\cos(\pi) - 9)$

$+\hat{k}(-6\cos(\pi) + 9\sin(\pi)$

$= \hat{i}(-3(0) + 2(1) + \pi(0) - (-1) + 6]$

$+ \hat{j}(3(1) + 2(0) - 0 - \pi(-1) - 9]$

$+ \hat{k}(-6(-1) + 9(0)]$

$= \hat{i}(0 + 2 + 0 + 1 + 6] + \hat{j}[3 + \pi - 9] + \hat{k}[6]$

$= 9\hat{i} + \cancel{g}[\pi - 6]\hat{j} + 6\hat{k}$

8. Given,

$$f(x,y,z) = 4x^2 + 25y^2 + 9z^2$$

$$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$= \hat{i}\frac{\partial}{\partial x}(4x^2+25y^2+9z^2) + \hat{j}\frac{\partial}{\partial y}(4x^2+25y^2+9z^2)$$

$$+ \hat{k}\frac{\partial}{\partial z}(4x^2+25y^2+9z^2)$$

$$= \hat{i}(8x) + \hat{j}(50y] + \hat{k}(18z)$$

$$= 8x\,\hat{i} + 50y\,\hat{j} + 18z\,\hat{k}$$

∇f at $P(5,0,0)$

$$= 8(5)\,\hat{i} + 50(0)\,\hat{j} + 18(0)\,\hat{k}$$

$$= 40\,\hat{i}$$

$\nabla f \cdot \vec{n}$ = directional derivative

$$\nabla f(x,y,z) = \langle 8x, 50y, 18z \rangle$$

$$\nabla f(5,0,0) = \langle 40, 0, 0 \rangle$$

P in the direction of a vector $\vec{a}=[0,1,1]$

$Q=(0,1,1)$

P(5,0,0) Q(0,1,1)

$$\vec{PQ} = \vec{OQ} - \vec{OP}$$

$$\vec{PQ} = -5\hat{i} + \hat{j} + \hat{k}$$

$\hat{n}$ is the unit vector in the direction $\vec{PQ}$

then

$$\hat{n} = \frac{-5\hat{i}+\hat{j}+\hat{k}}{\sqrt{25+1+1}} = \frac{1}{\sqrt{27}}[-5\hat{i}+\hat{j}+\hat{k}]$$

Directional derivative of f in the direction of $\vec{PQ} = (\nabla f)\cdot\hat{n}$

$$= (40\hat{i})\frac{1}{\sqrt{27}}[-5\hat{i}+\hat{j}+\hat{k}]$$

$$= \frac{1}{\sqrt{27}}[-5(40)]$$

$$= \frac{-200}{\sqrt{27}}$$

9. $\vec{f} = (2x-y+2z)\hat{i} + (x+y-z)\hat{j} + (3x-2y-5z)\hat{k}$.

Equation of circle $\Rightarrow x^2+y^2 = \alpha^2$

Let parametric equations of

$$x = \alpha\cos\theta, \quad y = \alpha\sin\theta$$

$dx = \alpha\sin\theta \cdot d\theta, \quad dy = \alpha\cos\theta\, d\theta, \quad z=0 \Rightarrow dz = 0$

$W = \int \vec{F} \cdot d\vec{r}$

$= \int (F_1\,dx + F_2\,dy + F_3\,dx) = \int (2x-y)dx + (x+4)\,dy$

$= \int_0^{2\pi} (2\alpha\cos\theta - \alpha\sin\theta)(-\alpha\sin\theta\, d\theta) + (\alpha\cos\theta + \alpha\sin\theta)(\alpha\cos\theta\, d\theta)$

$= \int_0^{2\pi} (-2\alpha^2\sin\theta\cos\theta + \alpha^2\sin^2\theta + \alpha^2\cos^2\theta + \alpha^2\sin\theta\cos\theta)d\theta$

$= \int_0^{2\pi} (\alpha^2 - \alpha^2\sin\theta\cos\theta)d\theta$

$\Rightarrow \alpha^2 \cdot \int_0^{2\pi} \left(1 - \frac{\sin 2\theta}{2}\right) d\theta$

$= \alpha^2\left(\theta + \frac{\cos 2\theta}{4}\right)_0^{2\pi}$

$= \alpha^2\left(2\pi + \frac{\cos 4\pi}{4} - 0 - \frac{\cos 0}{4}\right)$

$= \alpha^2(2\pi) = 2\pi\alpha^2$

10. $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$

$$\iint_S \vec{F}\cdot d\vec{s} = \iiint_V \operatorname{div}\vec{F}\, dv$$

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \frac{d}{dx}(4xz) + \frac{d}{dy}(-y^2) + \frac{d}{dz}(yz)$$

$$= 4z - 2y + y = 4z - y$$

The range of x, y, z all are from 0 to 1

$$\iiint_V \operatorname{div}\vec{F}\, dv = \int_0^1\int_0^1\int_0^1 (4z - \cancel{2}y)\, dx\, dy\, dz$$

$$= \int_0^1\int_0^1 \left[\frac{4z^2}{2} - yz\right]_0^1 dx\, dy$$

$$= \int_0^1\int_0^1 [2\cancel{z} - y]\, dx\, dy$$

$$\cancel{\int_0^1\int_0^1} = \int_0^1 \left[2\cancel{z}y - \frac{y^2}{2}\right]_0^1 dx$$

$$= \int_0^1 \left[2\cancel{z} - \frac{1}{2}\right] dx$$

$$= \left[2x - \frac{1}{2}x\right]_0^1$$

$$= 2 - \frac{1}{2} = 3/2$$

For plane $x=1$

$$\iint_S \vec{F}\cdot\vec{dS_1} = \iint \vec{F}\cdot i\, dA$$

$$= \int_0^1\int_0^1 4xz \cdot dz\, dy$$

$$= \int_0^1 \cancel{\left[\frac{4xz^2}{2}\right]_0^1}$$

$$= \int_0^1 \int_0^1 4z\, dz\, dy$$

$$= \int_0^1 \left[\frac{4z^2}{2}\right]_0^1 dy$$

$$= \int_0^1 \frac{4}{2}\, dy$$

$$[2y]_0^1$$

$$= 2$$