13.21. INTEGRATION OF VECTOR FUNCTIONS

Let $\vec{f}(t)$ and $\vec{F}(t)$ be two vector functions of a scalar variable t such that $\frac{d}{dt}\vec{F}(t) = \vec{f}(t)$, then $\vec{F}(t)$ is called an integral of $\vec{f}(t)$ with respect to t and we write $\int \vec{f}(t)\,dt = \vec{F}(t)$

If $\vec{c}$ is any *arbitrary constant vector* independent of t, then $\frac{d}{dt}\{\vec{F}(t) + \vec{c}\} = \vec{f}(t)$

This is equivalent to $\int \vec{f}(t)\,dt = \vec{F}(t) + \vec{c}$

$\vec{F}(t)$ is called the *indefinite integral* of $\vec{f}(t)$. The constant vector $\vec{c}$ is called the *constant of integration* and can be determined if some initial conditions are given.

The *definite integral* of $\vec{f}(t)$ between the limits $t = a$ and $t = b$ is written as

$$\int_a^b \vec{f}(t)\,dt = \left[\vec{F}(t)\right]_a^b = \vec{F}(b) - \vec{F}(a).$$

Note 1. If $\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$, then

$$\int \vec{f}(t)\,dt = \hat{i}\int f_1(t)\,dt + \hat{j}\int f_2(t)\,dt + \hat{k}\int f_3(t)\,dt.$$

Thus in order to integrate a vector function, integrate its components.

Note 2. We can obtain some standard results for integration of vector functions by considering the derivatives of suitable vector functions. For example,

(i) $\frac{d}{dt}(\vec{r}\cdot\vec{s}) = \frac{d\vec{r}}{dt}\cdot\vec{s} + \vec{r}\cdot\frac{d\vec{s}}{dt} \Rightarrow \int\left(\frac{d\vec{r}}{dt}\cdot\vec{s} + \vec{r}\cdot\frac{d\vec{s}}{dt}\right)dt = \vec{r}\cdot\vec{s} + c$

Here c is a scalar quantity since the integrand is a scalar.

(ii) $\frac{d}{dt}(\vec{r}^{\,2}) = 2\vec{r}\cdot\frac{d\vec{r}}{dt} \Rightarrow \int\left(2\vec{r}\cdot\frac{d\vec{r}}{dt}\right)dt = \vec{r}^{\,2} + c$ where c is a scalar quantity.

(iii) $\frac{d}{dt}\left(\vec{r}\times\frac{d\vec{r}}{dt}\right) = \frac{d\vec{r}}{dt}\times\frac{d\vec{r}}{dt} + \vec{r}\times\frac{d^2\vec{r}}{dt^2} = \vec{r}\times\frac{d^2\vec{r}}{dt^2} \Rightarrow \int\left(\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right)dt = \vec{r}\times\frac{d\vec{r}}{dt} + \vec{c}$

Here $\vec{c}$ is a vector quantity since the integrand is a vector.

(iv) If $\vec{a}$ is a constant vector, then $\frac{d}{dt}(\vec{a}\times\vec{r}) = \frac{d\vec{a}}{dt}\times\vec{r} + \vec{a}\times\frac{d\vec{r}}{dt} = \vec{a}\times\frac{d\vec{r}}{dt}$

$\Rightarrow \int\left(\vec{a}\times\frac{d\vec{r}}{dt}\right)dt = \vec{a}\times\vec{r} + \vec{c}$, where $\vec{c}$ is a vector quantity.

ILLUSTRATIVE EXAMPLES

Example 1. *The acceleration of a particle at time t is given by* $\vec{a} = 18\cos 3t\,\hat{i} - 8\sin 2t\,\hat{j} + 6t\,\hat{k}$.

If the velocity $\vec{v}$ *and displacement* $\vec{r}$ *be zero at* $t = 0$, *find* $\vec{v}$ *and* $\vec{r}$ *at any point t.*

Sol. Here $\qquad \vec{a} = \dfrac{d^2\vec{r}}{dt^2} = 18\cos 3t\hat{i} - 8\sin 2t\hat{j} + 6t\hat{k}$

Integrating, we have $\vec{v} = \dfrac{d\vec{r}}{dt} = \hat{i}\int 18\cos 3t\,dt + \hat{j}\int -8\sin 2t\,dt + \hat{k}\int 6t\,dt$

$$= 6\sin 3t\hat{i} + 4\cos 2t\hat{j} + 3t^2\hat{k} + \vec{c}$$

At $\quad t = 0,\ \vec{v} = \vec{0} \quad \Rightarrow \quad \vec{0} = 4\hat{j} + \vec{c} \quad \Rightarrow \quad \vec{c} = -4\hat{j}$

$\therefore$
$$\vec{v} = \frac{d\vec{r}}{dt} = 6\sin 3t\hat{i} + 4(\cos 2t - 1)\hat{j} + 3t^2\hat{k}$$

Integrating again, we have $\vec{r} = \hat{i}\int 6\sin 3t\,dt + \hat{j}\int 4(\cos 2t - 1)\,dt + \hat{k}\int 3t^2\,dt$

$$= -2\cos 3t\hat{i} + (2\sin 2t - 4t)\hat{j} + t^3\hat{k} + \vec{c}$$

At $\quad t = 0,\ \vec{r} = \vec{0} \quad \Rightarrow \quad \vec{0} = -2\hat{i} + \vec{c} \quad \Rightarrow \quad \vec{c} = 2\hat{i}$

$\therefore$
$$\vec{r} = 2(1 - \cos 3t)\hat{i} + 2(\sin 2t - 2t)\hat{j} + t^3\hat{k}.$$

Example 2. *If* $\vec{A}(t) = (3t^2 - 2t)\hat{i} + (6t - 4)\hat{j} + 4t\hat{k}$, *evaluate* $\int_2^3 \vec{A}(t)\,dt$.

Sol. $\quad \int_2^3 \vec{A}(t)\,dt = \int_2^3 [(3t^2 - 2t)\hat{i} + (6t - 4)\hat{j} + 4t\hat{k}]dt$

$$= \hat{i}\int_2^3 (3t^2 - 2t)\,dt + \hat{j}\int_2^3 (6t - 4)\,dt + \hat{k}\int_2^3 4t\,dt$$

$$= \hat{i}\left[t^3 - t^2\right]_2^3 + \hat{j}\left[3t^2 - 4t\right]_2^3 + \hat{k}\left[2t^2\right]_2^3 = 14\hat{i} + 11\hat{j} + 10\hat{k}.$$

Example 3. *If* $\vec{r}(t) = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$, *prove that* $\int_1^2 \left(\vec{r} \times \dfrac{d^2\vec{r}}{dt^2}\right) dt = -14\hat{i} + 75\hat{j} - 15\hat{k}$.

Sol. Since $\dfrac{d}{dt}\left(\vec{r} \times \dfrac{d\vec{r}}{dt}\right) = \dfrac{d\vec{r}}{dt} \times \dfrac{d\vec{r}}{dt} + \vec{r} \times \dfrac{d^2\vec{r}}{dt^2} = \vec{r} \times \dfrac{d^2\vec{r}}{dt^2}$

$\therefore$
$$\int_1^2 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt = \left[\vec{r} \times \frac{d\vec{r}}{dt}\right]_1^2 \qquad \ldots(1)$$

Let us now find $\vec{r} \times \dfrac{d\vec{r}}{dt}$.

$$\vec{r} \times \frac{d\vec{r}}{dt} = (5t^2\hat{i} + t\hat{j} - t^3\hat{k}) \times (10t\hat{i} + \hat{j} - 3t^2\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5t^2 & t & -t^3 \\ 10t & 1 & -3t^2 \end{vmatrix} = -2t^3\hat{i} + 5t^4\hat{j} - 5t^2\hat{k}$$

$\therefore$ From (1), we have $\displaystyle\int_1^2 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt = \Big[-2t^3\hat{i} + 5t^4\hat{j} - 5t^2\hat{k}\Big]_1^2$

$$= \Big[-2t^3\Big]_1^2 \hat{i} + \Big[5t^4\Big]_1^2 \hat{j} - \Big[-5t^2\Big]_1^2 \hat{k} = -14\hat{i} + 75\hat{j} - 15\hat{k}.$$

Example 4. *Given that* $\vec{r}(t) = \begin{cases} 2\hat{i} - \hat{j} + 2\hat{k}, & \textit{when } t = 2 \\ 4\hat{i} - 2\hat{j} + 3\hat{k}, & \textit{when } t = 3 \end{cases}$,

show that $\displaystyle\int_2^3 \left(\vec{r} \cdot \frac{d\vec{r}}{dt}\right) dt = 10.$

Sol. Since $\dfrac{d}{dt}(\vec{r}^2) = 2\vec{r} \cdot \dfrac{d\vec{r}}{dt}$

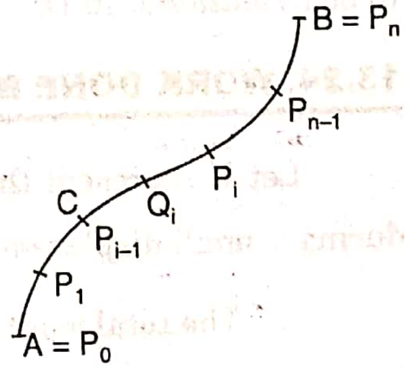
$$\therefore \quad \int_2^3 \left(\vec{r} \cdot \frac{d\vec{r}}{dt}\right) dt = \frac{1}{2}\Big[\vec{r}^2\Big]_2^3 \qquad \ldots(1)$$

When $t = 3$, $\quad \vec{r}^2 = (4\hat{i} - 2\hat{j} + 3\hat{k}) \cdot (4\hat{i} - 2\hat{j} + 3\hat{k}) = 16 + 4 + 9 = 29$

When $t = 2$, $\quad \vec{r}^2 = (2\hat{i} - \hat{j} + 2\hat{k}) \cdot (2\hat{i} - \hat{j} + 2\hat{k}) = 4 + 1 + 4 = 9$

$\therefore$ From (1), we have $\displaystyle\int_2^3 \left(\vec{r} \cdot \frac{d\vec{r}}{dt}\right) dt = \frac{1}{2}[29 - 9] = 10.$

13.22. LINE INTEGRALS



Any integral which is to be evaluated along a curve is called a line integral.

Let $\vec{F}(P)$ be a continuous vector point function defined at every point of a curve C is space. Divide the curve C into n parts by the points

$$A = P_0, P_1, P_2, \ldots\ldots, P_n = B$$

and let $\vec{R}_0, \vec{R}_1, \vec{R}_2, \ldots\ldots, \vec{R}_n$ be the position vectors of these points. Let Q_i be any point on the arc $P_{i-1} P_i$. Then the limit of the sum

$$\sum_{i=1}^{n} \vec{F}(Q_i).\delta\vec{R}_i \quad \text{where} \quad \delta\vec{R}_i = \vec{R}_i - \vec{R}_{i-1} \qquad ...(1)$$

as $n \to \infty$ and every $|\delta\vec{R}_i| \to 0$, if it exists, is called a line integral of $\vec{F}$ along C and is denoted by

$$\int_C \vec{F}.d\vec{R} \quad \text{or} \quad \int_C \vec{F}.\frac{d\vec{R}}{dt}\,dt.$$

Clearly, it is a scalar. It is called the *tangential line integral* of $\vec{F}$ along C.

If the scalar products in (1) are replaced by vector products, then the corresponding line integral is defined as $\int_C \vec{F} \times d\vec{R}$ which is a vector.

If the vector function $\vec{F}$ is replaced by a scalar function ϕ, then the corresponding line integral is defined as $\int_C \phi\, d\vec{R}$, which is a vector.

If $\quad \vec{F}(x, y, z) = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ and $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$ then $d\vec{R} = \hat{i}\,dx + \hat{j}dy + \hat{k}dz$

$$\therefore \quad \int_C \vec{F}.d\vec{R} = \int_C (f_1dx + f_2dy + f_3dz)$$

If the parametric equations of the curve C are $x = x(t), y = y(t), z = z(t)$ and $t = t_1$ at A, $t = t_2$ at B, then

$$\int_C \vec{F}.d\vec{R} = \int_{t_1}^{t_2} \left(f_1\frac{dx}{dt} + f_2\frac{dy}{dt} + f_3\frac{dz}{dt}\right)dt$$

If C is a closed curve, then the integral sign $\int_C$ is replaced by $\oint_C$.

13.23. CIRCULATION

In fluid dynamics, if $\vec{V}$ represents the velocity of a fluid particle and C is a closed curve, then the integral $\oint_C \vec{V}.d\vec{R}$ is called the *circulation* of $\vec{V}$ around the curve C.

If the circulation of $\vec{V}$ around every closed curve in a region D vanishes, then $\vec{V}$ is said to be *irrotational* in D.

13.24. WORK DONE BY A FORCE

Let $\vec{F}$ represent the force acting on a particle moving along an arc AB. The work done during a small displacement $\delta\vec{R}$ is $\vec{F}.\delta\vec{R}$.

$\therefore$ The total work done by $\vec{F}$ during displacement from A to B is given by $\int_A^B \vec{F}.d\vec{R}$

If the force $\vec{F}$ is conservative, then there exists a scalar function ϕ such that

$$\vec{F} = \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$\therefore$ The work done by $\vec{F}$ during displacement from A to B $= \int_A^B \vec{F} \cdot d\vec{R}$

$$= \int_A^B \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z} \right) \times (\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= \int_A^B \left(\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz \right) = \int_A^B d\phi = \Big[\phi\Big]_A^B = \phi_B - \phi_A$$

Thus, in a conservative field, the work done depends on the value of ϕ at the end points A and B, and not on the path joining A and B.

ILLUSTRATIVE EXAMPLES

Example 1. *If $\vec{F} = 3xy\hat{i} - y^2\hat{j}$, evaluate $\int_C \vec{F} \cdot d\vec{r}$, where C is the arc of the parabola $y = 2x^2$ from (0, 0) to (1, 2).*

Sol. Since the integration is performed in the xy-plane ($z = 0$), we take

$$\vec{r} = x\hat{i} + y\hat{j} \quad \text{so that } d\vec{r} = dx\hat{i} + dy\hat{j}$$

$$\therefore \quad \vec{F} \cdot d\vec{r} = (3xy\hat{i} - y^2\hat{j}) \cdot (dx\hat{i} + dy\hat{j}) = 3xy\,dx - y^2\,dy$$

On C : $y = 2x^2$ from (0, 0) to (1, 2)

$$\vec{F} \cdot d\vec{r} = 3x(2x^2)\,dx - (2x^2)^2\,(4x\,dx) = (6x^3 - 16x^5)\,dx$$

Also, x varies from 0 to 1.

$$\therefore \quad \int_C \vec{F} \cdot d\vec{r} = \int_0^1 (6x^3 - 16x^5)\,dx = \left[\frac{6x^4}{4} - \frac{16x^6}{6} \right]_0^1 = \frac{3}{2} - \frac{8}{3} = -\frac{7}{6}$$

Note that if the curve is traversed in the opposite sense, *i.e.*, from (1, 2) to (0, 0), the value of the integral would be $\frac{7}{6}$.

Second Method

Let $x = t$, then the parametric equations of the parabola $y = 2x^2$ are $x = t$, $y = 2t^2$.

At the point (0, 0), $\quad x = 0$ and so $t = 0$

At the point (1, 2), $\quad x = 1$ and so $t = 1$

If $\vec{r}$ is the position vector of any point (x, y) on C, then $\vec{r} = x\hat{i} + y\hat{j} = t\hat{i} + 2t^2\hat{j}$

Also in terms of t, $\vec{F} = (3t \times 2t^2)\hat{i} - (2t^2)^2\hat{j} = 6t^3\hat{i} - 4t^4\hat{j}$

$$\therefore \quad \int_C \vec{F} \cdot d\vec{r} = \int_C \left(\vec{F} \cdot \frac{d\vec{r}}{dt} \right) dt = \int_0^1 (6t^3\hat{i} - 4t^4\hat{j}) \cdot (\hat{i} + 4t\hat{j})\,dt$$

$$= \int_0^1 (6t^3 - 16t^5)\,dt = \left[\frac{6t^4}{4} - \frac{16t^6}{6} \right]_0^1 = \frac{3}{2} - \frac{8}{3} = -\frac{7}{6}.$$

Example 2. *A vector field is given by* $\vec{F} = (\sin y)\hat{i} + x(1 + \cos y)\hat{j}$. *Evaluate the line integral over the circular path given by* $x^2 + y^2 = a^2$, $z = 0$.

Sol. The parametric equations of the circular path are $x = a \cos t, y = a \sin t, z = 0$ where t varies from 0 to 2π.

Since the particle moves in the xy-plane ($z = 0$), we can take $\vec{r} = x\hat{i} + y\hat{j}$ so that $d\vec{r} = dx\hat{i} + dy\hat{j}$

$$\therefore \oint_C \vec{F} \cdot d\vec{r} = \oint_C [\sin y\hat{i} + x(1+\cos y)\hat{j}] \cdot (dx\hat{i} + dy\hat{j})$$

$$= \oint_C [\sin y\, dx + x(1+\cos y)\, dy] = \oint_C [(\sin y\, dx + x \cos y\, dy) + x\, dy]$$

$$= \oint_C d(x \sin y) + \oint_C x\, dy = \int_0^{2\pi} d[a \cos t \sin(a \sin t)] + \int_0^{2\pi} a \cos t \cdot a \cos t\, dt$$

$$= \Big[a \cos t \sin(a \sin t)\Big]_0^{2\pi} + a^2 \int_0^{2\pi} \cos^2 t\, dt$$

$$= \frac{a^2}{2} \int_0^{2\pi} (1+\cos 2t)\, dt = \frac{a^2}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi} = \frac{a^2}{2}(2\pi) = \pi a^2.$$

Example 3. *If* $\vec{F} = 2y\hat{i} - z\hat{j} + x\hat{k}$, *evaluate* $\oint_C \vec{F} \times d\vec{r}$ *along the curve* $x = \cos t$, $y = \sin t$, $z = 2 \cos t$ *from* $t = 0$ *to* $t = \frac{\pi}{2}$.

Sol. $$\vec{F} \times d\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2y & -z & x \\ dx & dy & dz \end{vmatrix} = (-z dz - x\, dy)\hat{i} + (x\, dx - 2y\, dz)\hat{j} + (2y\, dy + z\, dx)\hat{k}$$

In terms of t,

$$\vec{F} \times d\vec{r} = [-2 \cos t(-2 \sin t)\, dt - \cos t(\cos t)\, dt]\hat{i}$$
$$+ [\cos t(-\sin t)\, dt - 2 \sin t(-2 \sin t)\, dt]\hat{j} + [2 \sin t(\cos t)\, dt + 2 \cos t(-\sin t)\, dt]\hat{k}$$

$$= [(4 \cos t \sin t - \cos^2 t)\hat{i} + (4 \sin^2 t - \cos t \sin t)\hat{j}]\, dt$$

$$\therefore \int_C \vec{F} \times d\vec{r} = \int_0^{\pi/2} [(4 \cos t \sin t - \cos^2 t)\hat{i} + (4 \sin^2 t - \cos t \sin t)\hat{j}]\, dt$$

$$= \left(4 \cdot \frac{1}{2} - \frac{1}{2} \cdot \frac{\pi}{2}\right)\hat{i} + \left(4 \cdot \frac{1}{2} \cdot \frac{\pi}{2} - \frac{1}{2}\right)\hat{j} = \left(2 - \frac{\pi}{4}\right)\hat{i} + \left(\pi - \frac{1}{2}\right)\hat{j}.$$

Example 4. *Compute* $\int_C \vec{F} \cdot d\vec{r}$, *where* $\vec{F} = \dfrac{\hat{i}y - \hat{j}x}{x^2 + y^2}$ *and C is the circle* $x^2 + y^2 = 1$ *traversed counter clockwise.*

Sol. For the circle $x^2 + y^2 = 1$ in the xy-plane, we take $\vec{r} = x\hat{i} + y\hat{j}$ so that $d\vec{r} = dx\hat{i} + dy\hat{j}$.

$$\int_C \vec{F}.d\vec{r} = \int_C \left(\frac{\hat{i}y - \hat{j}x}{x^2+y^2}\right).(dx\,\hat{i} + dy\,\hat{j})$$

$$= \int_C \frac{y\,dx - x\,dy}{x^2+y^2} = \int_C (y\,dx - x\,dy) \qquad [\because\ x^2+y^2=1]$$

Parametric equation of the circle C : $x^2 + y^2 = 1$ is $x = \cos\theta$, $y = \sin\theta$

so that $dx = -\sin\theta d\theta$, $dy = \cos\theta\, d\theta$ and θ varies from 0 to 2π.

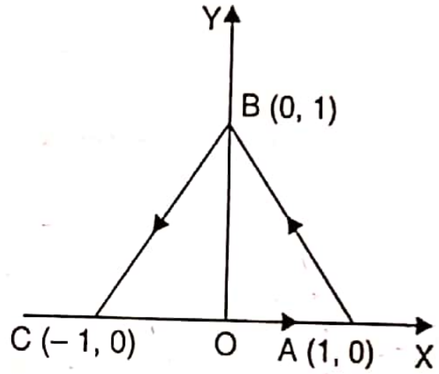
$$\therefore \quad \int_C \vec{F}.d\vec{r} = \int_0^{2\pi} \sin\theta(-\sin\theta\, d\theta) - \cos\theta(\cos\theta\, d\theta)$$

$$= -\int_0^{2\pi} (\sin^2\theta + \cos^2\theta)\, d\theta = -\int_0^{2\pi} d\theta$$

$$= -[\theta]_0^{2\pi} = -2\pi.$$

Example 5. *Compute the line integral* $\int_C (y^2dx - x^2dy)$ *about the triangle whose vertices are* $(1, 0)$, $(0, 1)$ *and* $(-1, 0)$.

Sol. Here C is the triangle ABC.



On AB Equation of AB is

$$y - 0 = \frac{1-0}{0-1}(x-1) \quad \text{or} \quad y = 1 - x$$

$\therefore$ $dy = -dx$ and x varies from 1 to 0.

On BC Equation of BC is

$$y - 1 = \frac{0-1}{-1-0}(x-0) \quad \text{or} \quad y = 1 + x$$

$\therefore$ $dy = dx$ and x varies from 0 to -1.

On CA $y = 0$ $\therefore$ $dy = 0$ and x varies from -1 to 1.

$$\therefore \int_C (y^2dx - x^2dy) = \int_{AB} (y^2dx - x^2dy) + \int_{BC} (y^2dx - x^2dy) + \int_{CA} (y^2dx - x^2dy)$$

$$= \int_1^0 [(1-x)^2dx - x^2(-dx)] + \int_0^{-1} [(1+x)^2dx - x^2dx] + \int_{-1}^1 0\,dx$$

$$= \int_1^0 (2x^2 - 2x + 1)\,dx + \int_0^{-1} (2x+1)\,dx + 0$$

$$= \left[\frac{2x^3}{3} - \frac{2x^2}{2} + x\right]_1^0 + \left[\frac{2x^2}{2} + x\right]_0^{-1}$$

$$= \left(-\frac{2}{3} + 1 - 1\right) + (1-1) = -\frac{2}{3}$$

Example 6. *Find the work done in moving a particle in the force field* $\vec{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$ *along*

(i) the straight line from $(0, 0, 0)$ *to* $(2, 1, 3)$

(ii) the curve defined by $x^2 = 4y$, $3x^3 = 8z$ *from* $x = 0$ *to* $x = 2$.

Sol. Work done $= \int_C \vec{F}.d\vec{r} = \int_C [3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}].(dx\hat{i} + dy\hat{j} + dz\hat{k})$

$$= \int_C [3x^2dx + (2xz - y)dy + zdz] \qquad ...(1)$$

(*i*) Equation of straight line from (0, 0, 0) to (2, 1, 3) are

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0} \quad \text{or} \quad \frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t \quad \text{(say)}$$

$\therefore$ C : $x = 2t, \quad y = t, \quad z = 3t$

so that $\qquad dx = 2dt, dy = dt, dz = 3dt$

The points (0, 0, 0) and (2, 1, 3) correspond to $t = 0$ and $t = 1$ respectively.

$\therefore$ From (1), we have

$$\text{Work done} = \int_0^1 [3(2t)^2\, 2dt + \{2(2t)(3t) - t\}\, dt + (3t)\, 3dt]$$

$$= \int_0^1 (24t^2 + 12t^2 - t + 9t)dt = \int_0^1 (36t^2 + 8t)dt$$

$$= \left[12t^3 + 4t^2\right]_0^1 = 16$$

(*ii*) Let $x = t$, then C : $x = t, y = \frac{t^2}{4}, z = \frac{3t^3}{8}$

so that $\qquad dx = dt, dy = \frac{t}{2}\,dt, dz = \frac{9t^2}{8}\,dt$

From $x = 0$ to $x = 2$, t varies from 0 to 2. $\qquad (\because \quad t = x)$

$\therefore$ From (1), we have

$$\text{Work done} = \int_0^2 \left[3t^2dt + \left\{2(t)\left(\frac{3t^3}{8}\right) - \frac{t^2}{4}\right\}\left(\frac{t}{2}\,dt\right) + \frac{3t^3}{8}\left(\frac{9t^2}{8}\,dt\right)\right]$$

$$= \int_0^2 \left(3t^2 + \frac{3}{8}t^5 - \frac{1}{8}t^3 + \frac{27}{64}t^5\right)dt = \int_0^2 \left(\frac{51}{64}t^5 - \frac{1}{8}t^3 + 3t^2\right)dt$$

$$= \left[\frac{51}{64}.\frac{t^6}{6} - \frac{t^4}{32} + t^3\right]_0^2 = \frac{17}{2} - \frac{1}{2} + 8 = 16.$$

DEFINITION The **length** of a smooth curve $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, $a \leq t \leq b$, that is traced exactly once as t increases from $t = a$ to $t = b$, is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt. \tag{1}$$

EXAMPLE 1 A glider is soaring upward along the helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$. How long is the glider's path from $t = 0$ to $t = 2\pi$?

Solution The path segment during this time corresponds to one full turn of the helix (Figure 13.13). The length of this portion of the curve is

$$\begin{aligned} L &= \int_a^b |\mathbf{v}|\, dt = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2}\, dt \\ &= \int_0^{2\pi} \sqrt{2}\, dt = 2\pi\sqrt{2} \text{ units of length.} \end{aligned}$$