2.5 Continuity

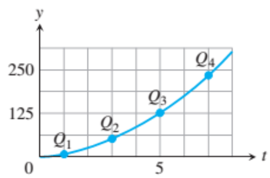

FIGURE 2.34 Connecting plotted points.

When we plot function values generated in a laboratory or collected in the field, we often connect the plotted points with an unbroken curve to show what the function's values are likely to have been at the points we did not measure (Figure 2.34). In doing so, we are assuming that we are working with a *continuous function*, so its outputs vary regularly and consistently with the inputs, and do not jump abruptly from one value to another without taking on the values in between. Intuitively, any function $y = f(x)$ whose graph can be sketched over its domain in one unbroken motion is an example of a continuous function. Such functions play an important role in the study of calculus and its applications.

Continuity at a Point

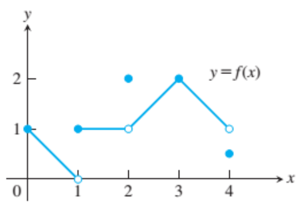

FIGURE 2.35 The function is not continuous at $x = 1$, $x = 2$, and $x = 4$ (Example 1).

To understand continuity, it helps to consider a function like that in Figure 2.35, whose limits we investigated in Example 2 in the last section.

EXAMPLE 1 At which numbers does the function f in Figure 2.35 appear to be not continuous? Explain why. What occurs at other numbers in the domain?

Solution First we observe that the domain of the function is the closed interval $[0, 4]$, so we will be considering the numbers x within that interval. From the figure, we notice right away that there are breaks in the graph at the numbers $x = 1$, $x = 2$, and $x = 4$. The break at $x = 1$ appears as a jump, which we identify later as a "jump discontinuity." The break at $x = 2$ is called a "removable discontinuity" since by changing the function definition at that one point, we can create a new function that is continuous at $x = 2$. Similarly $x = 4$ is a removable discontinuity.

Numbers at which the graph of f has breaks:

At the interior point $x = 1$, the function fails to have a limit. It does have both a left-hand limit, $\lim_{x\to 1^-} f(x) = 0$, as well as a right-hand limit, $\lim_{x\to 1^+} f(x) = 1$, but the limit values are different, resulting in a jump in the graph. The function is not continuous at $x = 1$. However the function value $f(1) = 1$ is equal to the limit from the right, so the function *is* continuous from the right at $x = 1$.

At $x = 2$, the function does have a limit, $\lim_{x\to 2} f(x) = 1$, but the value of the function is $f(2) = 2$. The limit and function values are not the same, so there is a break in the graph and f is not continuous at $x = 2$.

At $x = 4$, the function does have a left-hand limit at this right endpoint, $\lim_{x\to 4^-} f(x) = 1$, but again the value of the function $f(4) = \frac{1}{2}$ differs from the value of the limit. We see again a break in the graph of the function at this endpoint and the function is not continuous from the left.

Numbers at which the graph of f has no breaks:

> **DEFINITIONS** Let c be a real number that is either an interior point or an endpoint of an interval in the domain of f.
>
> The function f is **continuous at c** if
> $$\lim_{x\to c} f(x) = f(c).$$
> The function f is **right-continuous at c (or continuous from the right)** if
> $$\lim_{x\to c^+} f(x) = f(c).$$
> The function f is **left-continuous at c (or continuous from the left)** if
> $$\lim_{x\to c^-} f(x) = f(c).$$

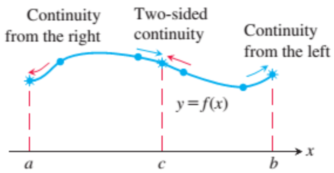

FIGURE 2.36 Continuity at points a, b, and c.

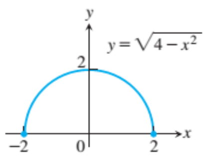

FIGURE 2.37 A function that is continuous over its domain (Example 2).

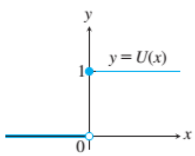

FIGURE 2.38 A function that has a jump discontinuity at the origin (Example 3).

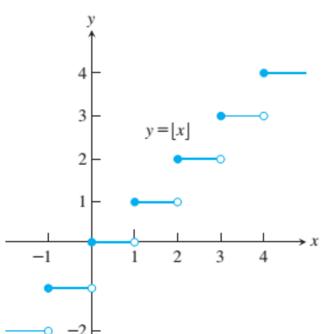

FIGURE 2.39 The greatest integer function is continuous at every noninteger point. It is right-continuous, but not left-continuous, at every integer point (Example 4).

The function f in Example 1 is continuous at every x in $[0, 4]$ except $x = 1, 2$, and 4. It is right-continuous but not left-continuous at $x = 1$, neither right- nor left-continuous at $x = 2$, and not left-continuous at $x = 4$.

From Theorem 6, it follows immediately that a function f is continuous at an *interior* point c of an interval in its domain if and only if it is both right-continuous and left-continuous at c (Figure 2.36). We say that a function is **continuous over a closed interval** $[a, b]$ if it is right-continuous at a, left-continuous at b, and continuous at all interior points of the interval. This definition applies to the infinite closed intervals $[a, \infty)$ and $(-\infty, b]$ as well, but only one endpoint is involved. If a function is not continuous at point c of its domain, we say that f is **discontinuous at c**, and that f has a discontinuity at c. Note that a function f can be continuous, right-continuous, or left-continuous only at a point c for which $f(c)$ is defined.

EXAMPLE 2 The function $f(x) = \sqrt{4 - x^2}$ is continuous over its domain $[-2, 2]$ (Figure 2.37). It is right-continuous at $x = -2$, and left-continuous at $x = 2$. ■

EXAMPLE 3 The unit step function $U(x)$, graphed in Figure 2.38, is right-continuous at $x = 0$, but is neither left-continuous nor continuous there. It has a jump discontinuity at $x = 0$. ■

At an interior point or an endpoint of an interval in its domain, a function is continuous at points where it passes the following test.

> **Continuity Test**
> A function $f(x)$ is continuous at a point $x = c$ if and only if it meets the following three conditions.
>
> **1.** $f(c)$ exists (c lies in the domain of f).
> **2.** $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$).
> **3.** $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value).

For one-sided continuity, the limits in parts 2 and 3 of the test should be replaced by the appropriate one-sided limits.

EXAMPLE 4 The function $y = \lfloor x \rfloor$ introduced in Section 1.1 is graphed in Figure 2.39. It is discontinuous at every integer n, because the left-hand and right-hand limits are not equal as $x \to n$:

$$\lim_{x \to n^-} \lfloor x \rfloor = n - 1 \quad \text{and} \quad \lim_{x \to n^+} \lfloor x \rfloor = n.$$

Since $\lfloor n \rfloor = n$, the greatest integer function is right-continuous at every integer n (but not left-continuous).

The greatest integer function is continuous at every real number other than the integers. For example,

$$\lim_{x \to 1.5} \lfloor x \rfloor = 1 = \lfloor 1.5 \rfloor.$$

In general, if $n - 1 < c < n$, n an integer, then

$$\lim_{x \to c} \lfloor x \rfloor = n - 1 = \lfloor c \rfloor.$$ ■

Figure 2.40 displays several common ways in which a function can fail to be continuous. The function in Figure 2.40a is continuous at $x = 0$. The function in Figure 2.40b does not contain $x = 0$ in its domain. It would be continuous if its domain were extended so that $f(0) = 1$. The function in Figure 2.40c would be continuous if $f(0)$ were 1 instead of 2. The discontinuity in Figure 2.40c is **removable**. The function has a limit as $x \to 0$, and we can remove the discontinuity by setting $f(0)$ equal to this limit.

The discontinuities in Figure 2.40d through f are more serious: $\lim_{x\to 0} f(x)$ does not exist, and there is no way to improve the situation by appropriately defining f at 0. The step function in Figure 2.40d has a **jump discontinuity**: The one-sided limits exist but have different values. The function $f(x) = 1/x^2$ in Figure 2.40e has an **infinite discontinuity**. The function in Figure 2.40f has an **oscillating discontinuity**: It oscillates so much that its values approach each number in $[-1, 1]$ as $x \to 0$. Since it does not approach a single number, it does not have a limit as x approaches 0.

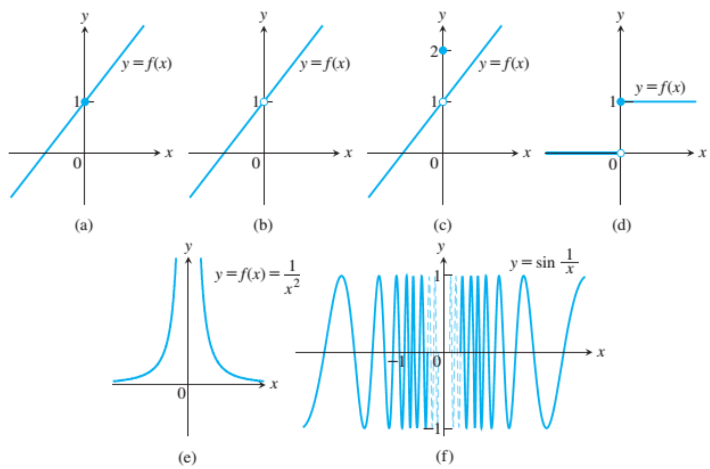

FIGURE 2.40 The function in (a) is continuous at $x = 0$; the functions in (b) through (f) are not.

Continuous Functions

We now describe the continuity behavior of a function throughout its entire domain, not only at a single point. We define a **continuous function** to be one that is continuous at every point in its domain. This is a property of the *function*. A function always has a specified domain, so if we change the domain then we change the function, and this may change its continuity property as well. If a function is discontinuous at one or more points of its domain, we say it is a **discontinuous** function.

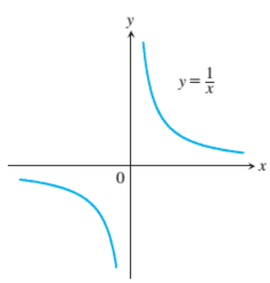

FIGURE 2.41 The function $f(x) = 1/x$ is continuous over its natural domain. It is not defined at the origin, so it is not continuous on any interval containing $x = 0$ (Example 5).

EXAMPLE 5

(a) The function $f(x) = 1/x$ (Figure 2.41) is a continuous function because it is continuous at every point of its domain. The point $x = 0$ is not in the domain of the function f, so f is not continuous on any interval containing $x = 0$. Moreover, there is no way to extend f to a new function that is defined and continuous at $x = 0$. The function f does not have a removable discontinuity at $x = 0$.

(b) The identity function $f(x) = x$ and constant functions are continuous everywhere by Example 3, Section 2.3. ■

Algebraic combinations of continuous functions are continuous wherever they are defined.

THEOREM 8—Properties of Continuous Functions

If the functions f and g are continuous at $x = c$, then the following algebraic combinations are continuous at $x = c$.

1. *Sums:*	$f + g$
2. *Differences:*	$f - g$
3. *Constant multiples:*	$k \cdot f$, for any number k
4. *Products:*	$f \cdot g$
5. *Quotients:*	f/g, provided $g(c) \neq 0$
6. *Powers:*	f^n, n a positive integer
7. *Roots:*	$\sqrt[n]{f}$, provided it is defined on an interval containing c, where n is a positive integer

Most of the results in Theorem 8 follow from the limit rules in Theorem 1, Section 2.2. For instance, to prove the sum property we have

$$\begin{aligned}
\lim_{x \to c}(f + g)(x) &= \lim_{x \to c}(f(x) + g(x)) \\
&= \lim_{x \to c} f(x) + \lim_{x \to c} g(x) && \text{Sum Rule, Theorem 1} \\
&= f(c) + g(c) && \text{Continuity of } f, g \text{ at } c \\
&= (f + g)(c).
\end{aligned}$$

This shows that $f + g$ is continuous.

EXAMPLE 6

(a) Every polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is continuous because $\lim_{x \to c} P(x) = P(c)$ by Theorem 2, Section 2.2.

(b) If $P(x)$ and $Q(x)$ are polynomials, then the rational function $P(x)/Q(x)$ is continuous wherever it is defined ($Q(c) \neq 0$) by Theorem 3, Section 2.2. ■

EXAMPLE 7 The function $f(x) = |x|$ is continuous. If $x > 0$, we have $f(x) = x$, a polynomial. If $x < 0$, we have $f(x) = -x$, another polynomial. Finally, at the origin, $\lim_{x \to 0} |x| = 0 = |0|$. ■

The functions $y = \sin x$ and $y = \cos x$ are continuous at $x = 0$ by Example 11 of Section 2.2. Both functions are continuous everywhere (see Exercise 64). It follows from Theorem 8 that all six trigonometric functions are continuous wherever they are defined. For example, $y = \tan x$ is continuous on $\cdots \cup (-\pi/2, \pi/2) \cup (\pi/2, 3\pi/2) \cup \cdots$.

Continuity of Compositions of Functions

Functions obtained by composing continuous functions are continuous. If $f(x)$ is continuous at $x = c$ and $g(x)$ is continuous at $x = f(c)$, then $g \circ f$ is also continuous at $x = c$ (Figure 2.42). In this case, the limit of $g \circ f$ as $x \to c$ is $g(f(c))$.

THEOREM 9—Compositions of Continuous Functions

If f is continuous at c and g is continuous at $f(c)$, then the composition $g \circ f$ is continuous at c.

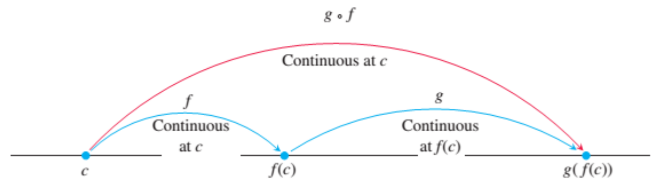

FIGURE 2.42 Compositions of continuous functions are continuous.

Intuitively, Theorem 9 is reasonable because if x is close to c, then $f(x)$ is close to $f(c)$, and since g is continuous at $f(c)$, it follows that $g(f(x))$ is close to $g(f(c))$.

The continuity of compositions holds for any finite number of compositions of functions. The only requirement is that each function be continuous where it is applied. An outline of a proof of Theorem 9 is given in Exercise 6 in Appendix 4.

EXAMPLE 8 Show that the following functions are continuous on their natural domains.

(a) $y = \sqrt{x^2 - 2x - 5}$ **(b)** $y = \dfrac{x^{2/3}}{1 + x^4}$

(c) $y = \left|\dfrac{x - 2}{x^2 - 2}\right|$ **(d)** $y = \left|\dfrac{x \sin x}{x^2 + 2}\right|$

Solution

(a) The square root function is continuous on $[0, \infty)$ because it is a root of the continuous identity function $f(x) = x$ (Part 7, Theorem 8). The given function is then the composition of the polynomial $f(x) = x^2 - 2x - 5$ with the square root function $g(t) = \sqrt{t}$, and is continuous on its natural domain.

(b) The numerator is the cube root of the identity function squared; the denominator is an everywhere-positive polynomial. Therefore, the quotient is continuous.

(c) The quotient $(x - 2)/(x^2 - 2)$ is continuous for all $x \neq \pm\sqrt{2}$, and the function is the composition of this quotient with the continuous absolute value function (Example 7).

(d) Because the sine function is everywhere-continuous (Exercise 64), the numerator term $x \sin x$ is the product of continuous functions, and the denominator term $x^2 + 2$ is an everywhere-positive polynomial. The given function is the composite of a quotient of continuous functions with the continuous absolute value function (Figure 2.43). ■

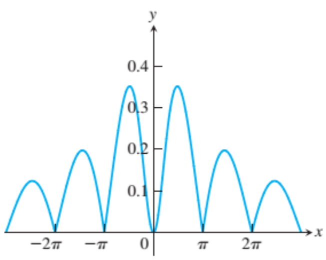

FIGURE 2.43 The graph suggests that $y = |(x \sin x)/(x^2 + 2)|$ is continuous (Example 8d).

Theorem 9 is actually a consequence of a more general result, which we now prove. It states that if the limit of $f(x)$ as x approaches c is equal to b, then the limit of the composition function $g \circ f$ as x approaches c is equal to $g(b)$.