3.3. ▶ DEFINITE INTEGRAL

If $f(x)$ is a continuous function defined on closed interval $[a, b]$ and $F(x)$ is the indefinite integral of $f(x)$ and $x = a$ and $x = b$ be the two given values of x, then $[F(b) - F(a)]$ is called the **definite integral** *of $f(x)$ between the limits a and b.*

It is denoted by $\int_a^b f(x)\, dx$ and read as integral of $f(x)$ from a to b.

Here 'a' is called the **lower limit** and 'b' the **upper limit** of integration.

Thus $\int_a^b \boldsymbol{f(x)\, dx} = [\mathbf{F}(\boldsymbol{x})]_a^b = \mathbf{F}(\boldsymbol{b}) - \mathbf{F}(\boldsymbol{a})$ where $\int f(x) = F(x)$.

Note.

1. *After integration, we first substitute x = upper limit and then subtract x = lower limit from it.*
2. *While evaluating definite integrals, arbitrary constant 'c' is not added.*

3.4. ▶ FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

We have two definitions of definite integral, namely :

1. $\int_a^b f(x)\, dx = F(b) - F(a)$, where $F'(x) = f(x)$.

2. $\int_a^b f(x)\,dx = \lim_{h\to 0} h\,[f(a)+f(a+h)+f(a+2h)+\ldots\ldots+f(a+(n-1)h)]$, where $nh = b-a$.

The fundamental theorem of integral calculus establishes the equivalence of the above two definitions of a definite integral.

Statement. *If $f(x)$ is a continuous and single-valued function of x in the interval (a, b), where a and b are finite and $a < b$, then*

$$\int_a^b f(x)\,dx = F(b) - F(a), \textit{ where } F'(x) = f(x).$$

Proof. Let AB be the graph of curve $y = f(x)$ and AL and BM be the ordinates $x = a$, $x = b$.

Then $S = \text{Area ALMB} = F(b) - F(a)$ ...(1)

Now divide LM into n equal parts each equal to h by means of points $A_1, A_2, A_3, \ldots A_{n-1}$, so that

$$\begin{aligned} b-a &= LM \\ &= LA_1 + A_1A_2 + \ldots + A_{n-1}M \\ &= h + h + \ldots \text{ to } n \text{ terms} \\ &= nh \Rightarrow h = \frac{b-a}{n} \end{aligned}$$

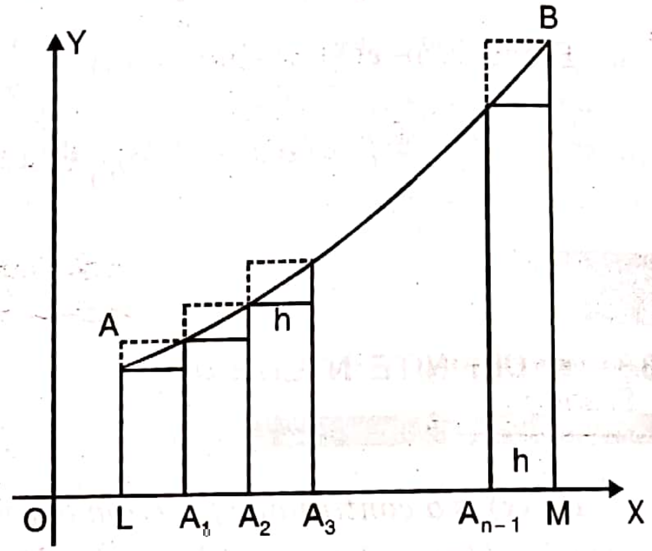

Fig.3.1

As $n \to \infty$, $h \to 0$

Since $y = f(x)$

$\therefore$ Ordinates L, $A_1, A_2, \ldots, A_{n-1}$, M are $f(a), f(a+h), f(a+2h), \ldots\ldots$

$\ldots\ldots f(a+(n-1)h), f(a+nh)$ respectively. $[\because b = a + nh]$

Now areas of inner rectangles are

$$h\,.f(a),\ h\,.f(a+h),\ h\,.f(a+2h),\ldots\ldots,\ h\,.(a+(n-1)h)$$

Since breadth of each inner rectangle is h therefore, sum of these areas

$$= h\,.\,[f(a)+f(a+h)+\ldots+f(a+(n-1)h)] \quad \ldots(2)$$
$$= S_n \text{ (say)}$$

Also, the heights of outer rectangles are the lengths of the ordinates at the points $A_1, A_2, \ldots, A_{n-1}$, M and the width of each outer rectangle is h.

$\therefore$ Area of the outer rectangles are

$$h\,.f(a+h),\ h\,.f(a+2h),\ldots\ldots,\ h\,.f(a+nh)$$

$\therefore$ Sum of these areas

$$\begin{aligned} &= h\,.\,[f(a+h)+f(a+2h)+\ldots\ldots+f(a+(n-1)h)+f(a+nh)] \\ &= h\,[f(a)+f(a+h)+f(a+2h)+\ldots\ldots+f(a+(n-1)h)] + h\,[f(a+nh)-f(a)] \\ &= S_n + h\,[f(b)-f(a)] \end{aligned}$$

...(3) $[\because a + nh = b]$

From Fig. 3.1, it is obvious that the area ALMB lies between the sum of inner and sum of outer rectangles. Hence from (2) and (3), we get

$$S_n < \text{Area ALMB} < S_n + h\,[f(b) - f(a)]$$

i.e.,

$$S_n < S < S_n + h\,[f(b) - f(a)]$$

or

$$0 < S - S_n < h\,[f(b) - f(a)] \quad \text{...(4)}$$

Now as $h \to 0$, $S - S_n$ lies between 0 and a quantity which tends towards zero.

$$\therefore \quad \lim_{h \to 0} S - S_n = 0$$

$$\therefore \quad S = \lim_{h \to 0} S_n = \int_a^b f(x)\,dx \quad \text{...(5)}$$

$\therefore$ From (1) and (5), we have

$$\int_a^b \boldsymbol{f(x)\,dx} = \mathbf{F\,(b) - F\,(a)}, \text{ where } F'(x) = f(x).$$

SOLVED EXAMPLES

Example 1.

Evaluate :

(i) $\int_0^4 (\sqrt{x} - 2x + x^2)\,dx$ (ii) $\int_0^{\pi/2} \sin x\,dx$

(iii) $\int_0^1 \frac{dx}{1+x^2}$ (iv) $\int_a^b \frac{1}{x}\,dx$

Solution. (i) $$\int_0^4 (\sqrt{x} - 2x + x^2)\,dx = \left[\frac{x^{3/2}}{3/2} - 2.\frac{x^2}{2} + \frac{x^3}{3}\right]_0^4 = \left[\frac{2}{3}x^{3/2} - x^2 + \frac{1}{3}x^3\right]_0^4$$

$$= \left[\left(\frac{2}{3}.(4)^{3/2} - (4)^2 + \frac{1}{3}.(4)^3\right) - 0\right] = \frac{2}{3}.8 - 16 + \frac{64}{3} = \mathbf{\frac{32}{3}}.$$

(ii) $$\int_0^{\pi/2} \sin x\,dx = [-\cos x]_0^{\pi/2} = -[\cos x]_0^{\pi/2}$$

$$= -\left[\cos\frac{\pi}{2} - \cos 0\right] = -[0 - 1] = \mathbf{1}.$$

(iii) $$\int_0^1 \frac{dx}{1+x^2} = [\tan^{-1} x]_0^1 = \tan^{-1}(1) - \tan^{-1} 0 = \frac{\pi}{4} - 0 = \mathbf{\frac{\pi}{4}}.$$

(iv) $$\int_a^b \frac{1}{x}\,dx = [\log x]_a^b = \log b - \log a = \mathbf{\log\left(\frac{b}{a}\right)}.$$

Example 2.

Evaluate :

(i) $\int_0^2 \sqrt{6x+4}\, dx$ (ii) $\int_0^1 \frac{x}{\sqrt{1+x}}\, dx$ (iii) $\int_0^1 \frac{1-x}{1+x}\, dx.$

Solution. (i) $\int_0^2 \sqrt{6x+4}\, dx = \int_0^2 (6x+4)^{1/2}\, dx = \left[\frac{(6x+4)^{3/2}}{6.\frac{3}{2}}\right]_0^2$

$$= \frac{1}{9}\left[(6x+4)^{3/2}\right]_0^2 = \frac{1}{9}[(16)^{3/2} - (4)^{3/2}]$$

$$= \frac{1}{9}[64-8] = \mathbf{\frac{56}{9}}.$$

(ii) $\int_0^1 \frac{x}{\sqrt{1+x}}\, dx = \int_0^1 \frac{1+x-1}{\sqrt{1+x}}\, dx = \int_0^1 [(1+x)^{1/2} - (1+x)^{-1/2}]\, dx$

$$= \left[\frac{2}{3}(1+x)^{3/2} - 2(1+x)^{1/2}\right]_0^1 = \left[\left(\frac{2}{3}(2)^{3/2} - 2(2)^{1/2}\right) - \left(\frac{2}{3} - 2\right)\right]$$

$$= \frac{4\sqrt{2}}{3} - 2\sqrt{2} + \frac{4}{3} = \mathbf{\frac{2\sqrt{2}}{3}(\sqrt{2}-1).}$$

(iii) $\int_0^1 \frac{1-x}{1+x}\, dx = -\int_0^1 \frac{x-1}{x+1}\, dx = -\int_0^1 \frac{x+1-2}{x+1}\, dx$

$$= -\int_0^1 \left(1 - \frac{2}{x+1}\right) dx = -[x - 2\log|x+1|]_0^1$$

$$= -[(1-2\log 2) - (0 - 2\log 1)] = \mathbf{2\log 2 - 1.} \qquad [\because \log 1 = 0]$$

Example 3.

Evaluate :

(i) $\int_0^{\pi/2} \sin^2 x\, dx$ (ii) $\int_0^{\pi/2} \cos^3 x\, dx$

(iii) $\int_0^{\pi/2} \frac{dx}{1+\cos x}$ (iv) $\int_0^{\pi/2} \sqrt{1+\sin x}\, dx$

(v) $\int_0^{\pi/2} \sqrt{1-\cos 2x}\, dx$ (vi) $\int_0^{\pi/4} \sqrt{1-\sin 2x}\, dx$

Solution. (*i*) $$\int_0^{\pi/2} \sin^2 x\, dx = \int_0^{\pi/2} \left(\frac{1-\cos 2x}{2}\right) dx$$

$$= \frac{1}{2}\left[\int_0^{\pi/2} 1.\, dx - \int_0^{\pi/2} \cos 2x\, dx\right]$$

$$= \frac{1}{2}[\,x\,]_0^{\pi/2} - \frac{1}{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - 0\right] - \frac{1}{4}[\sin \pi - \sin 0] = \frac{\pi}{4}.$$

(*ii*) $$\int_0^{\pi/2} \cos^3 x\, dx = \int_0^{\pi/2} \left(\frac{3}{4}\cos x + \frac{1}{4}\cos 3x\right) dx \qquad [\because \cos 3A = 4\cos^3 A - 3\cos A]$$

$$= \left[\frac{3}{4}\sin x + \frac{1}{4}.\frac{\sin 3x}{3}\right]_0^{\pi/2}$$

$$= \left[\left(\frac{3}{4}\sin\frac{\pi}{2} + \frac{1}{12}\sin\frac{3\pi}{2}\right) - \left(\frac{3}{4}\sin 0 + \frac{1}{12}\sin 0\right)\right]$$

$$= \left[\left(\frac{3}{4} - \frac{1}{12}\right) - 0\right] = \frac{8}{12} = \frac{2}{3}.$$

(*iii*) $$\int_0^{\pi/2} \frac{dx}{1+\cos x} = \int_0^{\pi/2} \frac{dx}{2\cos^2 x/2} = \frac{1}{2}\int_0^{\pi/2} \sec^2\frac{x}{2}\, dx$$

$$= \frac{1}{2}\left[\frac{\tan\frac{x}{2}}{\frac{1}{2}}\right]_0^{\pi/2} = \left[\tan\frac{x}{2}\right]_0^{\pi/2} = \tan\frac{\pi}{4} - 0 = 1.$$

(*iv*) $$\int_0^{\pi/2} \sqrt{1+\sin x}\, dx = \int_0^{\pi/2} \left(\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}\right)^{1/2} dx$$

$$= \int_0^{\pi/2} \left[\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2\right]^{1/2} dx = \int_0^{\pi/2} \left(\cos\frac{x}{2} + \sin\frac{x}{2}\right) dx$$

$$= \left[\frac{\sin\frac{x}{2}}{\frac{1}{2}} + \frac{\left(-\cos\frac{x}{2}\right)}{\frac{1}{2}}\right]_0^{\pi/2} = \left[2\sin\frac{x}{2} - 2\cos\frac{x}{2}\right]_0^{\pi/2}$$

$$= 2\left[\left(\sin\frac{\pi}{4} - \cos\frac{\pi}{4}\right) - (\sin 0 - \cos 0)\right]$$

$$= 2\left[\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) - (0 - 1)\right] = 2\,[0 + 1] = \mathbf{2.}$$

(v) $$\int_0^{\pi/2} \sqrt{1 - \cos 2x}\, dx = \int_0^{\pi/2} \sqrt{2\sin^2 x}\, dx$$

$$= \sqrt{2}\int_0^{\pi/2} \sin x\, dx = \sqrt{2}\,[-\cos x]_0^{\pi/2} = -\sqrt{2}\left[\cos\frac{\pi}{2} - \cos 0\right]$$

$$= -\sqrt{2}\,[0 - 1] = \boldsymbol{\sqrt{2}}.$$

(vi) $$\int_0^{\pi/4} \sqrt{1 - \sin 2x}\, dx = \int_0^{\pi/4} \sqrt{(\cos^2 x + \sin^2 x - 2\sin x\cos x)}\, dx$$

$$= \int_0^{\pi/4} \sqrt{(\cos x - \sin x)^2}\, dx$$

$$= \int_0^{\pi/4} (\cos x - \sin x)\, dx = [\sin x + \cos x]_0^{\pi/4}$$

$$= \left[\left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) - (\sin 0 + \cos 0)\right] = \left[\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0 + 1)\right]$$

$$= \frac{2}{\sqrt{2}} - 1 = \boldsymbol{\sqrt{2} - 1}.$$

Example 4.

Evaluate :

(i) $$\int_{\pi/6}^{\pi/3} \frac{\sin x}{\cos^2 x}\, dx$$ (ii) $$\int_0^{\pi/4} 2\tan^3 x\, dx$$ (iii) $$\int_0^{\pi/2} \frac{\sin^2\theta}{(1 + \cos\theta)^2}\, d\theta$$

Solution. (i) $$\int_{\pi/6}^{\pi/3} \frac{\sin x}{\cos^2 x}\, dx = \int_{\pi/6}^{\pi/3} \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}\, dx$$

$$= \int_{\pi/6}^{\pi/3} \sec x . \tan x\, dx = [\sec x]_{\pi/6}^{\pi/3}$$

$$= \left(\sec\frac{\pi}{3} - \sec\frac{\pi}{6}\right) = \mathbf{2 - \frac{2}{\sqrt{3}}}.$$

(ii) $$\int_0^{\pi/4} 2\tan^3 x\,dx = 2\int_0^{\pi/4} \tan x \tan^2 x\,dx$$

$$= 2\int_0^{\pi/4} \tan x(\sec^2 x - 1)\,dx$$

$$= 2\int_0^{\pi/4} \tan x \sec^2 x\,dx - 2\int_0^{\pi/4} \tan x\,dx$$

$$= 2.\left[\frac{\tan^2 x}{2}\right]_0^{\pi/4} + 2\left[\log\cos x\right]_0^{\pi/4}$$

$$= [1-0] + 2\left[\log\cos\frac{\pi}{4} - \log 1\right] = 1 + 2\log\frac{1}{\sqrt{2}}$$

$$= 1 + 2(\log 1 - \log\sqrt{2}) = 1 + 2\log 1 - \log(\sqrt{2})^2 = \mathbf{1 - \log 2.}$$

(iii) $$\int_0^{\pi/2} \frac{\sin^2\theta}{(1+\cos\theta)^2}\,d\theta = \int_0^{\pi/2} \frac{\left(2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}\right)^2}{\left(2\cos^2\frac{\theta}{2}\right)^2}\,d\theta$$

$$= \int_0^{\pi/2} \frac{4\sin^2\frac{\theta}{2}.\cos^2\frac{\theta}{2}}{4\cos^4\frac{\theta}{2}}\,d\theta = \int_0^{\pi/2} \tan^2\frac{\theta}{2}\,d\theta$$

$$= \int_0^{\pi/2} \left(\sec^2\frac{\theta}{2} - 1\right)d\theta = \left[2\tan\frac{\theta}{2} - \theta\right]_0^{\pi/2}$$

$$= \left[\left(2\tan\frac{\pi}{4} - \frac{\pi}{2}\right) - (2\tan 0 - 0)\right] = \left[\left(2 - \frac{\pi}{2}\right) - 0\right] = \mathbf{2 - \frac{\pi}{2}.}$$

Example 5.

Evaluate :

(i) $\displaystyle\int_0^1 \frac{x^5}{1+x^6}\,dx$ (ii) $\displaystyle\int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\,dx$ (iii) $\displaystyle\int_1^2 \frac{dx}{x(1+\log x)}$

Solution. (i) Let $\displaystyle I = \int_0^1 \frac{x^5}{1+x^6}\,dx$

Put $x^6 = t$ so that $6x^5\,dx = dt \Rightarrow x^5\,dx = \frac{1}{6}\,dt$

Now when $x = 0, t = 0$ and when $x = 1,\ t = 1$

$$\therefore \quad I = \frac{1}{6}\int_0^1 \frac{dt}{1+t} = \frac{1}{6}\Big[\log|1+t|\Big]_0^1 = \frac{1}{6}[\log 2 - \log 1] = \mathbf{\frac{1}{6}\log 2.}$$

(*ii*) Let $\quad I = \displaystyle\int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\,dx$

Put $1 - x^2 = t$ so that $-2x\,dx = dt \;\Rightarrow\; x\,dx = -\frac{1}{2}\,dt$

Now when $x = 0, t = 1$ and when $x = \frac{1}{2}, t = \frac{3}{4}$

$$\therefore \quad I = -\frac{1}{2}\int_1^{3/4} \frac{dt}{\sqrt{t}} = -\frac{1}{2}\int_1^{3/4} t^{-1/2}\,dt = -\frac{1}{2}\Big[2t^{1/2}\Big]_1^{3/4}$$

$$= -\left[\left(\frac{3}{4}\right)^{1/2} - (1)^{1/2}\right] = 1 - \frac{\sqrt{3}}{2} = \mathbf{\frac{2-\sqrt{3}}{2}}.$$

(*iii*) Let $\quad I = \displaystyle\int_1^2 \frac{dx}{x(1+\log x)}$.

Put $1 + \log x = t$ so that $\frac{1}{x}\,dx = dt$

Now when $x = 1, t = 1 + \log 1 = 1$ and when $x = 2, t = 1 + \log 2$

$$\therefore \quad I = \int_1^{1+\log 2} \frac{dt}{t} = \Big[\log|t|\Big]_1^{1+\log 2}$$

$$= [\log(1+\log 2) - \log 1] = \mathbf{\log(1+\log 2).}$$

Example 6.

Evaluate :

(*i*) $\displaystyle\int_0^{\pi/2} \frac{\sin\theta}{\sqrt{1+\cos\theta}}\,d\theta$ (*ii*) $\displaystyle\int_0^1 \frac{(\tan^{-1}x)^2}{1+x^2}\,dx$

(*iii*) $\displaystyle\int_0^{\pi/3} \frac{\sec x\tan x}{1+\sec^2 x}\,dx$ (*iv*) $\displaystyle\int_0^{\sqrt[3]{\pi^2}} \sqrt{x}\cos^2(x^{3/2})\,dx$

Solution. (*i*) Let $I = \displaystyle\int_0^{\pi/2} \frac{\sin\theta}{\sqrt{1+\cos\theta}}\,d\theta$

Put $1 + \cos\theta = t$ so that $-\sin\theta\, d\theta = dt \Rightarrow \sin\theta\, d\theta = -dt$

Now when $\theta = 0$, $t = 1 + \cos 0 = 2$ and when $x = \frac{\pi}{2}$, $t = 1 + \cos\frac{\pi}{2} = 1$

$$\therefore \quad I = -\int_2^1 \frac{dt}{\sqrt{t}} = -\left[2t^{1/2}\right]_2^1 = -2\left[(1)^{1/2} - (2)^{1/2}\right] = \mathbf{2(\sqrt{2} - 1)}.$$

(ii) Let $\quad I = \int_0^1 \frac{(\tan^{-1} x)^2}{1 + x^2}\, dx$

Put $\tan^{-1} x = t$ so that $\frac{1}{1+x^2}\, dx = dt$

Now when $x = 0$, $t = \tan^{-1} 0 = 0$ and when $x = 1$, $t = \tan^{-1} 1 = \frac{\pi}{4}$

$$\therefore \quad I = \int_0^{\pi/4} t^2\, dt = \left[\frac{t^3}{3}\right]_0^{\pi/4} = \frac{1}{3}\left[\left(\frac{\pi}{4}\right)^3 - 0\right] = \frac{1}{3}\cdot\frac{\pi^3}{64} = \frac{\boldsymbol{\pi^3}}{\mathbf{192}}.$$

(iii) Let $\quad I = \int_0^{\pi/3} \frac{\sec x \tan x}{1 + \sec^2 x}\, dx.$

Put $\sec x = t$ so that $\sec x \tan x\, dx = dt$

Now when $x = 0$, $t = \sec 0 = 1$ and when $x = \frac{\pi}{3}$, $t = \sec\frac{\pi}{3} = 2$

$$\therefore \quad I = \int_1^2 \frac{dt}{1+t^2} = \left[\tan^{-1} t\right]_1^2 = \tan^{-1} 2 - \tan^{-1} 1 = \mathbf{\tan^{-1} 2 - \frac{\pi}{4}}.$$

(iv) $\quad$ Let $I = \int_0^{\sqrt[3]{\pi^2}} \sqrt{x} \cos^2 (x^{3/2})\, dx$

Put $x^{3/2} = t$ so that $\frac{3}{2} x^{1/2}\, dx = dt \Rightarrow \sqrt{x}\, dx = \frac{2}{3}\, dt$

Now when $x = 0$, $t = 0$ and when $x = \sqrt[3]{\pi^2}$, $t = (\pi^{2/3})^{3/2} = \pi$

$$\therefore \quad I = \frac{2}{3}\int_0^{\pi} \cos^2 t\, dt = \frac{2}{3} \times \frac{1}{2}\int_0^{\pi} (1 + \cos 2t)\, dt$$

$$= \frac{1}{3}\left[t + \frac{\sin 2t}{2}\right]_0^{\pi}$$

$$= \frac{1}{3}\left[\left(\pi + \frac{\sin 2\pi}{2}\right) - \left(0 + \frac{\sin 0}{2}\right)\right] = \frac{1}{3}\left[(\pi + 0) - 0 + 0\right] = \boldsymbol{\frac{\pi}{3}}.$$

Example 7.

Evaluate :

(i) $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\cos x}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}\,dx$ (ii) $\displaystyle\int_0^{\pi/2} \frac{1}{1+\cos^2 x}\,dx$

(iii) $\displaystyle\int_0^{\pi/2} \frac{dx}{4\sin^2 x+5\cos^2 x}$ (iv) $\displaystyle\int_0^{\pi/4} \frac{\sin 2\theta}{\sin^4\theta+\cos^4\theta}\,d\theta$

Solution. (i) Let I $= \displaystyle\int_{\pi/4}^{\pi/2} \frac{\cos x}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}\,dx$

Now, $\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2 = \cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\cos\frac{x}{2}\sin\frac{x}{2} = 1+\sin x$

$$\therefore \qquad I = \int_{\pi/4}^{\pi/2} \frac{\cos x}{1+\sin x}\,dx$$

Put $1+\sin x = t$ so that $\cos x\,dx = dt$

Now when $x = \frac{\pi}{4}, t = 1+\sin\frac{\pi}{4} = 1+\frac{1}{\sqrt{2}}$ and when $x = \frac{\pi}{2}, t = 1+\sin\frac{\pi}{2} = 1+1 = 2$

$$\therefore \qquad I = \int_{1+\frac{1}{\sqrt{2}}}^{2} \frac{dt}{t} = \left[\log|t|\right]_{1+\frac{1}{\sqrt{2}}}^{2}$$

$$= \left[\log 2 - \log\left(1+\frac{1}{\sqrt{2}}\right)\right] = \mathbf{log}\left(\frac{2\sqrt{2}}{\sqrt{2}+1}\right).$$

(ii) Let $\displaystyle I = \int_0^{\pi/2} \frac{dx}{1+\cos^2 x} = \int_0^{\pi/2} \frac{dx}{\sin^2 x+\cos^2 x+\cos^2 x}$

$$= \int_0^{\pi/2} \frac{dx}{2\cos^2 x+\sin^2 x} = \int_0^{\pi/2} \frac{\sec^2 x}{2+\tan^2 x}\,dx$$

[*Dividing the numerator and denominator by* $\cos^2 x$]

Put $\tan x = t$ so that $\sec^2 x\,dx = dt$

Now when $x = 0, t = \tan 0 = 0$ and when $x = \frac{\pi}{2}, t = \tan\frac{\pi}{2} = \infty$

$$\therefore \quad I = \int_0^{\infty} \frac{dt}{2+t^2} = \int_0^{\infty} \frac{dt}{(\sqrt{2})^2 + t^2} = \frac{1}{\sqrt{2}} \left[\tan^{-1} \frac{t}{\sqrt{2}} \right]_0^{\infty}$$

$$= \frac{1}{\sqrt{2}} [\tan^{-1} \infty - \tan^{-1} 0] = \frac{1}{\sqrt{2}} \left[\frac{\pi}{2} - 0 \right] = \frac{\pi}{2\sqrt{2}}.$$

(*iii*) Let $I = \int_0^{\pi/2} \frac{dx}{4 \sin^2 x + 5 \cos^2 x} = \int_0^{\pi/2} \frac{\sec^2 x \, dx}{4 \tan^2 x + 5}$

[*Dividing the numerator and denominator by* $\cos^2 x$]

Put $\tan x = t$ so that $\sec^2 x \, dx = dt$

Now when $x = 0$, $t = 0$ and when $x = \frac{\pi}{2}$, $t = \infty$

$$\therefore \quad I = \int_0^{\infty} \frac{dt}{4t^2 + 5} = \frac{1}{4} \int_0^{\infty} \frac{dt}{t^2 + \frac{5}{4}} = \frac{1}{4} \int_0^{\infty} \frac{dt}{t^2 + \left(\frac{\sqrt{5}}{2} \right)^2}$$

$$= \frac{1}{4} \cdot \frac{1}{\frac{\sqrt{5}}{2}} \left[\tan^{-1} \left(\frac{t}{\frac{\sqrt{5}}{2}} \right) \right]_0^{\infty}$$

$$= \frac{1}{2\sqrt{5}} \left[\tan^{-1} \infty - \tan^{-1} 0 \right] = \frac{1}{2\sqrt{5}} \cdot \left[\frac{\pi}{2} - 0 \right] = \frac{\pi}{4\sqrt{5}}.$$

(*iv*) Let $I = \int_0^{\pi/4} \frac{\sin 2\theta}{\sin^4 \theta + \cos^4 \theta} d\theta = \int_0^{\pi/4} \frac{2 \sin \theta \cos \theta}{\cos^4 \theta \, (1 + \tan^4 \theta)} d\theta$

$$= 2 \int_0^{\pi/4} \frac{\tan \theta \, . \sec^2 \theta}{\tan^4 \theta + 1} d\theta$$

Put $\tan^2 \theta = t$ so that $2 \tan \theta \sec^2 \theta = dt$

Now when $\theta = 0$, $t = 0$ and when $\theta = \frac{\pi}{4}$, $t = 1$

$$\therefore \quad I = \int_0^{1} \frac{dt}{t^2 + 1} = \left[\tan^{-1} t \right]_0^1 = \left[\tan^{-1} 1 - \tan^{-1} 0 \right] = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$