Vector Calculus

13.1. VECTOR FUNCTIONS

If to each value of a scalar variable t, there corresponds a value of a vector $\vec{r}$, then $\vec{r}$ is called a vector function of the scalar variable t and we write $\vec{r} = \vec{r}(t)$ or $\vec{r} = \vec{f}(t)$.

For example, the position vector $\vec{r}$ of a particle moving along a curved path is a vector function of time t, a scalar.

Since every vector can be uniquely expressed as a linear combination of three fixed non-coplanar vectors, therefore, we may write $\vec{f}(t) = f_1(t)\,\hat{i} + f_2(t)\,\hat{j} + f_3(t)\,\hat{k}$ where $\hat{i}, \hat{j}, \hat{k}$ denote unit vectors along the axis of x, y, z respectively, $f_1(t)$, $f_2(t)$ and $f_3(t)$ are called the components of the vector $\vec{f}(t)$ along the coordinate axes.

13.2. DERIVATIVE OF A VECTOR FUNCTION WITH RESPECT TO A SCALAR

Let $\vec{r} = \vec{f}(t)$ be a vector function of the scalar variable t. Let δt be a small increment in t and $\delta\vec{r}$, the corresponding increment in $\vec{r}$.

Then $\vec{r} + \delta\vec{r} = \vec{f}(t + \delta t)$ so that $\delta\vec{r} = \vec{f}(t + \delta t) - \vec{f}(t)$

and
$$\frac{\delta\vec{r}}{\delta t} = \frac{\vec{f}(t + \delta t) - \vec{f}(t)}{\delta t}$$

If $\lim\limits_{\delta t \to 0} = \dfrac{\delta\vec{r}}{\delta t} = \lim\limits_{\delta t \to 0} \dfrac{\vec{f}(t + \delta t) - \vec{f}(t)}{\delta t}$ exists, then the value of this limit is denoted by $\dfrac{d\vec{r}}{dt}$ and is called the derivative of $\vec{r}$ with respect to t.

Since $\dfrac{d\vec{r}}{dt}$ is itself a vector function of t, its derivative is denoted by $\dfrac{d^2\vec{r}}{dt^2}$ and is called the second derivative of $\vec{r}$ with respect to t. Similarly, we can define higher order derivatives of $\vec{r}$.

13.3. GENERAL RULES FOR DIFFERENTIATION

If $\vec{a}, \vec{b}$ and $\vec{c}$ are vector functions of a scalar t and ϕ is a scalar function of t, then

(i) $\frac{d}{dt}(\vec{a} \pm \vec{b}) = \frac{d\vec{a}}{dt} \pm \frac{d\vec{b}}{dt}$ (ii) $\frac{d}{dt}(\vec{a} \cdot \vec{b}) = \vec{a} \cdot \frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt} \cdot \vec{b}$

(iii) $\frac{d}{dt}(\vec{a} \times \vec{b}) = \vec{a} \times \frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt} \times \vec{b}$ (iv) $\frac{d}{dt}(\phi \vec{a}) = \phi \frac{d\vec{a}}{dt} + \frac{d\phi}{dt}\vec{a}$

(v) $\frac{d}{dt}[\vec{a}\ \vec{b}\ \vec{c}] = \left[\frac{d\vec{a}}{dt}\ \vec{b}\ \vec{c}\right] + \left[\vec{a}\ \frac{d\vec{b}}{dt}\ \vec{c}\right] + \left[\vec{a}\ \vec{b}\ \frac{d\vec{c}}{dt}\right]$

(vi) $\frac{d}{dt}\{\vec{a} \times (\vec{b} \times \vec{c})\} = \frac{d\vec{a}}{dt} \times (\vec{b} \times \vec{c}) + \vec{a} \times \left(\frac{d\vec{b}}{dt} \times \vec{c}\right) + \vec{a} \times \left(\vec{b} \times \frac{d\vec{c}}{dt}\right)$.

Proof. (*i*) $\frac{d}{dt}(\vec{a} + \vec{b}) = \lim_{\delta t \to 0} \frac{\{(\vec{a} + \delta\vec{a}) + (\vec{b} + \delta\vec{b})\} - (\vec{a} + \vec{b})}{\delta t}$

$$= \lim_{\delta t \to 0} \frac{\delta\vec{a} + \delta\vec{b}}{\delta t} = \lim_{\delta t \to 0} \left(\frac{\delta\vec{a}}{\delta t} + \frac{\delta\vec{b}}{\delta t}\right)$$

$$= \lim_{\delta t \to 0} \frac{\delta\vec{a}}{\delta t} + \lim_{\delta t \to 0} \frac{\delta\vec{b}}{\delta t} = \frac{d\vec{a}}{dt} + \frac{d\vec{b}}{dt}$$

Similarly, $\frac{d}{dt}(\vec{a} - \vec{b}) = \frac{d\vec{a}}{dt} - \frac{d\vec{b}}{dt}$

(*ii*) $\frac{d}{dt}(\vec{a} \cdot \vec{b}) = \lim_{\delta t \to 0} \frac{(\vec{a} + \delta\vec{a}) \cdot (\vec{b} + \delta\vec{b}) - \vec{a} \cdot \vec{b}}{\delta t}$

$$= \lim_{\delta t \to 0} \frac{\vec{a} \cdot \vec{b} + \vec{a} \cdot \delta\vec{b} + \delta\vec{a} \cdot \vec{b} + \delta\vec{a} \cdot \delta\vec{b} - \vec{a} \cdot \vec{b}}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\vec{a} \cdot \delta\vec{b} + \delta\vec{a} \cdot \vec{b} + \delta\vec{a} \cdot \delta\vec{b}}{\delta t} = \lim_{\delta t \to 0} \left\{\vec{a} \cdot \frac{\delta\vec{b}}{\delta t} + \frac{\delta\vec{a}}{\delta t} \cdot \vec{b} + \frac{\delta\vec{a}}{\delta t} \cdot \delta\vec{b}\right\}$$

$$= \vec{a} \cdot \frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt} \cdot \vec{b} + \frac{d\vec{a}}{dt} \cdot \vec{0}, \quad \text{since } \delta\vec{b} \to \vec{0} \text{ as } \delta t \to 0$$

$$= \vec{a} \cdot \frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt} \cdot \vec{b} \qquad \left[\because \frac{d\vec{a}}{dt} \cdot \vec{0} = 0\right]$$

Note. Since $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$, while evaluating $\frac{d}{dt}(\vec{a} \cdot \vec{b})$, the order of factors is immaterial.

(*iii*) $\frac{d}{dt}(\vec{a} \times \vec{b}) = \lim_{\delta t \to 0} \frac{(\vec{a} + \delta\vec{a}) \times (\vec{b} + \delta\vec{b}) - \vec{a} \times \vec{b}}{\delta t}$

$$= \lim_{\delta t \to 0} \frac{\vec{a} \times \vec{b} + \vec{a} \times \delta\vec{b} + \delta\vec{a} \times \vec{b} + \delta\vec{a} \times \delta\vec{b} - \vec{a} \times \vec{b}}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\vec{a} \times \delta\vec{b} + \delta\vec{a} \times \vec{b} + \delta\vec{a} \times \delta\vec{b}}{\delta t} = \lim_{\delta t \to 0} \left\{ \vec{a} \times \frac{\delta\vec{b}}{\delta t} + \frac{\delta\vec{a}}{\delta t} \times \vec{b} + \frac{\delta\vec{a}}{\delta t} \times \delta\vec{b} \right\}$$

$$= \vec{a} \times \frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt} \times \vec{b} + \frac{d\vec{a}}{dt} \times \vec{0} \quad \text{since } \delta\vec{b} \to \vec{0} \text{ as } \delta t \to 0$$

$$= \vec{a} \times \frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt} \times \vec{b} \qquad \left[\because \frac{d\vec{a}}{dt} \times \vec{0} = \vec{0}\right]$$

Note. Since $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$, while evaluating $\frac{d}{dt}(\vec{a} \times \vec{b})$, the order of factors $\vec{a}$ and $\vec{b}$ must be maintained.

$$(iv) \quad \frac{d}{dt}(\phi\,\vec{a}) = \lim_{\delta t \to 0} \frac{(\phi + \delta\phi)(\vec{a} + \delta\vec{a}) - \phi\vec{a}}{\delta t} = \lim_{\delta t \to 0} \frac{\phi\,\vec{a} + \phi\delta\,\vec{a} + \delta\phi\vec{a} + \delta\phi\delta\vec{a} - \phi\vec{a}}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\phi\delta\vec{a} + \delta\phi\vec{a} + \delta\phi\delta\vec{a}}{\delta t} = \lim_{\delta t \to 0} \left\{ \phi\frac{\delta\vec{a}}{\delta t} + \frac{\delta\phi}{\delta t}\vec{a} + \frac{\delta\phi}{\delta t}\delta\vec{a} \right\}$$

$$= \phi\frac{d\vec{a}}{dt} + \frac{d\phi}{dt}\vec{a} + \frac{d\phi}{dt}\vec{0}, \quad \text{since } \delta\vec{a} \to \vec{0} \text{ as } \delta t \to 0$$

$$= \phi\frac{d\vec{a}}{dt} + \frac{d\phi}{dt}\vec{a} \qquad \left[\because \frac{d\phi}{dt}\vec{0} = \vec{0}\right]$$

Note. $\phi\vec{a}$ is the product of a vector by a scalar. We usually write the scalar in the first position and the vector in the second position.

$$(v) \quad \frac{d}{dt}[\vec{a}\ \vec{b}\ \vec{c}] = \frac{d}{dt}[\vec{a}\,.(\vec{b} \times \vec{c})] = \vec{a}\,.\frac{d}{dt}(\vec{b} \times \vec{c}) + \frac{d\vec{a}}{dt}.(\vec{b} \times \vec{c}) \qquad \text{[by rule (ii)]}$$

$$= \vec{a}\,.\left(\vec{b} \times \frac{d\vec{c}}{dt} + \frac{d\vec{b}}{dt} \times \vec{c}\right) + \frac{d\vec{a}}{dt}.(\vec{b} \times \vec{c}) \qquad \text{[by rule (iii)]}$$

$$= \vec{a}\,.\left(\vec{b} \times \frac{d\vec{c}}{dt}\right) + \vec{a}\,.\left(\frac{d\vec{b}}{dt} \times \vec{c}\right) + \frac{d\vec{a}}{dt}.(\vec{b} \times \vec{c})$$

$$= \left[\vec{a}\ \vec{b}\ \frac{d\vec{c}}{dt}\right] + \left[\vec{a}\ \frac{d\vec{b}}{dt}\ \vec{c}\right] + \left[\frac{d\vec{a}}{dt}\ \vec{b}\ \vec{c}\right] = \left[\frac{d\vec{a}}{dt}\ \vec{b}\ \vec{c}\right] + \left[\vec{a}\ \frac{d\vec{b}}{dt}\ \vec{c}\right] + \left[\vec{a}\ \vec{b}\ \frac{d\vec{c}}{dt}\right]$$

Note. $[\vec{a}\ \vec{b}\ \vec{c}]$ is the scalar product of three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$. While evaluating $\frac{d}{dt}[\vec{a}\ \vec{b}\ \vec{c}]$, the cyclic order of factors must be maintained.

(vi) $\frac{d}{dt}\{\vec{a} \times (\vec{b} \times \vec{c})\} = \vec{a} \times \frac{d}{dt}(\vec{b} \times \vec{c}) + \frac{d\vec{a}}{dt} \times (\vec{b} \times \vec{c})$ [by rule (iii)]

$$= \vec{a} \times \left(\vec{b} \times \frac{d\vec{c}}{dt} + \frac{d\vec{b}}{dt} \times \vec{c}\right) + \frac{d\vec{a}}{dt} \times (\vec{b} \times \vec{c})$$

$$= \frac{d\vec{a}}{dt} \times (\vec{b} \times \vec{c}) + \vec{a} \times \left(\frac{d\vec{b}}{dt} \times \vec{c}\right) + \vec{a} \times \left(\vec{b} \times \frac{d\vec{c}}{dt}\right).$$

13.4. DERIVATIVE OF A CONSTANT VECTOR

A vector is said to be constant if both its magnitude and direction are fixed. If either of these changes, the vector is not constant.

Let $\vec{r}$ be a constant vector function of the scalar variable t.

Let $\vec{r} = \vec{f}(t)$, then $\vec{r} = \vec{f}(t + \delta t)$ so that $\vec{f}(t + \delta t) - \vec{f}(t) = \vec{0}$

$$\therefore \quad \frac{d\vec{r}}{dt} = \lim_{\delta t \to 0} \frac{\vec{f}(t+\delta t) - \vec{f}(t)}{\delta t} = \lim_{\delta t \to 0} \vec{0} = \vec{0}$$

Thus, the derivative of a constant vector is equal to the null vector.

Note. $\hat{i}, \hat{j}, \hat{k}$ being fixed unit vectors are constant vectors.

$$\therefore \quad \frac{d\hat{i}}{dt} = \frac{d\hat{j}}{dt} = \frac{d\hat{k}}{dt} = \vec{0}.$$

13.5. DERIVATIVE OF A VECTOR FUNCTION IN TERMS OF ITS COMPONENTS

Let $\vec{r}$ be a vector function of the scalar variable t.

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, where the components x, y, z are scalar function of t.

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}(x\hat{i}) + \frac{d}{dt}(y\hat{j}) + \frac{d}{dt}(z\hat{k}) = x\frac{d\hat{i}}{dt} + \frac{dx}{dt}\hat{i} + y\frac{d\hat{j}}{dt} + \frac{dy}{dt}\hat{j} + z\frac{d\hat{k}}{dt} + \frac{dz}{dt}\hat{k}$$

$$= \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}, \quad \text{since} \quad \frac{d\hat{i}}{dt} = \frac{d\hat{j}}{dt} = \frac{d\hat{k}}{dt} = \vec{0}.$$

If $\quad x = f_1(t), y = f_2(t), z = f_3(t)$; then $\vec{r} = f_1(t)\,\hat{i} + f_2(t)\,\hat{j} + f_3(t)\,\hat{k}$

$$\Rightarrow \quad \frac{d\vec{r}}{dt} = f_1'(t)\,\hat{i} + f_2'(t)\,\hat{j} + f_3'(t)\,\hat{k}$$

Therefore to differentiate a vector, differentiate its components.

13.6. IF $\vec{F}(t)$ HAS A CONSTANT MAGNITUDE, THEN $\vec{F} \cdot \frac{d\vec{F}}{dt} = 0$

$\vec{F}(t)$ has a constant magnitude $\Rightarrow$ $|\vec{F}(t)|$ = constant

$$\therefore \quad \vec{F}(t) \cdot \vec{F}(t) = |\vec{F}(t)|^2 = \text{constant} \quad \Rightarrow \quad \frac{d}{dt}(\vec{F} \cdot \vec{F}) = 0$$

$$\Rightarrow \quad \vec{F}\cdot\frac{d\vec{F}}{dt}+\frac{d\vec{F}}{dt}\cdot\vec{F}=0 \quad\Rightarrow\quad 2\vec{F}\cdot\frac{d\vec{F}}{dt}=0 \quad\Rightarrow\quad \vec{F}\cdot\frac{d\vec{F}}{dt}=0$$

Note. $\quad \vec{F}\cdot\frac{d\vec{F}}{dt}=0 \Rightarrow \frac{d\vec{F}}{dt}\perp\vec{F}.$

13.7. IF $\vec{F}$ (t) HAS A CONSTANT DIRECTION, THEN $\vec{F}\times\frac{d\vec{F}}{dt}=\vec{0}$

Let $|\vec{F}(t)|=f(t)$. Let $\hat{G}(t)$ be a unit vector in the direction of $\vec{F}(t)$ so that $\vec{F}(t)=f(t)\,\hat{G}(t)$

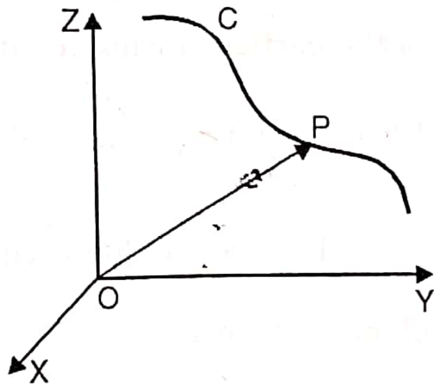
$$\therefore \quad \frac{d\vec{F}}{dt}=f\frac{d\hat{G}}{dt}+\frac{df}{dt}\hat{G} \qquad \text{...(1)}$$

If $\vec{F}(t)$ has constant direction, so has $\hat{G}(t)$. Thus, $\hat{G}(t)$ is a constant vector and $\frac{d\hat{G}}{dt}=\vec{0}$

From (1), $\quad \frac{d\vec{F}}{dt}=\frac{df}{dt}\hat{G}$

$$\therefore \quad \vec{F}\times\frac{d\vec{F}}{dt}=f\,\hat{G}\times\left(\frac{df}{dt}\hat{G}\right)=f\frac{df}{dt}\hat{G}\times\hat{G}=\vec{0}.$$

13.8. GEOMETRICAL INTERPRETATION OF $\frac{d\vec{r}}{dt}$

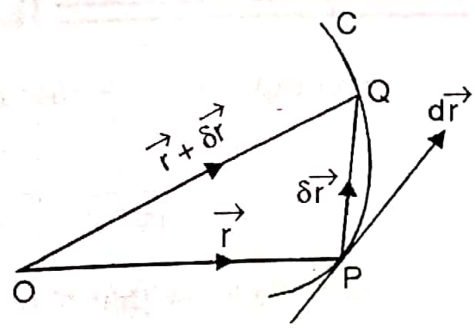


Let O be the origin of reference. Let the position vector of a point P be given by $\vec{r}=\vec{f}(t)$. As t varies continuously, P traces out a curve C as shown in the figure. Thus, a vector function $\vec{f}(t)$ represents a curve in space.

For example, (*i*) the vector equation $\vec{r}=at^2\hat{i}+2at\hat{j}$ represents the parabola $y^2=4ax$ in the xy-plane because its parametric equations are

$$x=at^2,\ y=2at.$$

(*ii*) the vector equation $\vec{r}=a\cos t\,\hat{i}+b\sin t\,\hat{j}$ represents the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in the xy-plane because its parametric equations are $x=a\cos t,\ y=b\sin t$.



Now, let $\vec{r}=\vec{f}(t)$ be the vector equation of a curve C in space. Let $\vec{r}$ and $\vec{r}+\delta\vec{r}$ be the position vectors of two neighbouring points P and Q on this curve.

$$\vec{PQ} = \vec{OQ} - \vec{OP} = (\vec{r} + \delta\vec{r}) - \vec{r} = \delta\vec{r}$$

$\therefore \quad \dfrac{\delta\vec{r}}{\delta t}$ is directed along the chord PQ.

As $\delta t \to 0$, $Q \to P$, chord PQ $\Rightarrow$ tangent to the curve at P.

$\therefore \quad \lim\limits_{\delta t \to 0} \dfrac{\delta\vec{r}}{\delta t} = \dfrac{d\vec{r}}{dt}$ *is a vector along the tangent to the curve at P.*

Suppose the scalar parameter t is replaced by s, where s denotes the arc length from any convenient point A on the curve upto P. Thus, arc AP $= s$ and arc AQ $= s + \delta s$ so that δs = arc PQ. In this case $\dfrac{d\vec{r}}{ds}$ will be a vector along the tangent at P. Also

$$\left|\frac{d\vec{r}}{ds}\right| = \lim_{\delta s \to 0}\left|\frac{\delta\vec{r}}{\delta s}\right| = \lim_{Q \to P} \frac{\text{chord PQ}}{\text{arc PQ}} = 1.$$

Thus, $\dfrac{d\vec{r}}{ds}$ *is the unit vector* $\hat{T}$ *along the tangent at P.*

13.9. VELOCITY AND ACCELERATION

If the scalar variable t denotes the time and $\vec{r}$ is the position vector of a moving particle P, then $\delta\vec{r}$ is the displacement of the particle in time δt. The vector $\dfrac{\delta\vec{r}}{\delta t}$ is the average velocity of the particle during the interval δt. If $\vec{v}$ represents the velocity vector of the particle at P, then $\vec{v} = \lim\limits_{\delta t \to 0} \dfrac{\delta\vec{r}}{\delta t} = \dfrac{d\vec{r}}{dt}$ and its direction is along the tangent at P.

If $\delta\vec{v}$ be the change in velocity $\vec{v}$ during the time δt, then $\dfrac{\delta\vec{v}}{\delta t}$ is the average acceleration of the particle during the interval δt. If $\vec{a}$ represents the acceleration of the particle at P, then

$$\vec{a} = \lim_{\delta t \to 0} \frac{\delta\vec{v}}{\delta t} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d^2\vec{r}}{dt^2}.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Show that if* $\vec{r} = \vec{a} \sin \omega t + \vec{b} \cos \omega t$, *where* $\vec{a}, \vec{b}, \omega$ *are constants, then*

$$\frac{d^2\vec{r}}{dt^2} = -\omega^2\vec{r} \quad \textit{and} \quad \vec{r} \times \frac{d\vec{r}}{dt} = -\omega\vec{a} \times \vec{b}. \qquad \text{(U.P.T.U. 2007)}$$

Sol. We know that if $\vec{r} = \phi\vec{f}$, where ϕ is a scalar function of t, then $\dfrac{d\vec{r}}{dt} = \phi\dfrac{d\vec{f}}{dt} + \dfrac{d\phi}{dt}\vec{f}$.

But if $\vec{f}$ is a constant vector, then $\dfrac{d\vec{f}}{dt} = \vec{0}$

$$\therefore \qquad \frac{d\vec{r}}{dt} = \frac{d\phi}{dt}\vec{f}$$

Now $\qquad \vec{r} = \vec{a}\sin\omega t + \vec{b}\cos\omega t \quad \Rightarrow \quad \dfrac{d\vec{r}}{dt} = \vec{a}\,\omega\cos\omega t - \vec{b}\,\omega\sin\omega t$

$$\therefore \qquad \frac{d^2\vec{r}}{dt^2} = -\vec{a}\,\omega^2\sin\omega t - \vec{b}\,\omega^2\cos\omega t = -\omega^2(\vec{a}\sin\omega t + \vec{b}\cos\omega t) = -\omega^2\vec{r}$$

Also $\quad \vec{r}\times\dfrac{d\vec{r}}{dt} = (\vec{a}\sin\omega t + \vec{b}\cos\omega t)\times(\vec{a}\,\omega\cos\omega t - \vec{b}\,\omega\sin\omega t)$

$$= \omega\,(-\vec{a}\times\vec{b}\sin^2\omega t + \vec{b}\times\vec{a}\cos^2\omega t) \qquad [\because \ \vec{a}\times\vec{a} = \vec{0} = \vec{b}\times\vec{b}\,]$$

$$= \omega\,(-\vec{a}\times\vec{b}\sin^2\omega t - \vec{a}\times\vec{b}\cos^2\omega t) = -\,\omega\,\vec{a}\times\vec{b}.$$

Example 2. *If* $\vec{r} = a\cos t\,\hat{i} + a\sin t\,\hat{j} + at\tan\alpha\,\hat{k}$, *find* $\left|\dfrac{d\vec{r}}{dt}\times\dfrac{d^2\vec{r}}{dt^2}\right|$ *and* $\left[\dfrac{d\vec{r}}{dt}\ \dfrac{d^2\vec{r}}{dt^2}\ \dfrac{d^3\vec{r}}{dt^3}\right]$.

Sol. $\qquad \vec{r} = a\cos t\hat{i} + a\sin t\hat{j} + at\tan\alpha\,\hat{k}$

$$\Rightarrow \qquad \frac{d\vec{r}}{dt} = -a\sin t\hat{i} + a\cos t\hat{j} + a\tan\alpha\,\hat{k}$$

$$\frac{d^2\vec{r}}{dt^2} = -a\cos t\hat{i} - a\sin t\hat{j}$$

$$\frac{d^3\vec{r}}{dt^3} = a\sin t\hat{i} - a\cos t\hat{j}$$

$$\therefore \qquad \frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin t & a\cos t & a\tan\alpha \\ -\cos t & -a\sin t & 0 \end{vmatrix}$$

$$= a^2\sin t\tan\alpha\,\hat{i} - a^2\cos t\tan\alpha\,\hat{j} + a^2\hat{k}$$

$$\therefore \qquad \left|\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right| = \sqrt{a^4\sin^2 t\tan^2\alpha + a^4\cos^2 t\tan^2\alpha + a^4}$$

$$= a^2\sqrt{\tan^2\alpha + 1} = a^2\sec\alpha$$

Also $\quad \left[\dfrac{d\vec{r}}{dt}\ \dfrac{d^2\vec{r}}{dt^2}\ \dfrac{d^3\vec{r}}{dt^3}\right] = \begin{vmatrix} -a\sin t & a\cos t & a\tan\alpha \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix}$ $\quad$ (expanding by third column)

$$= a\tan\alpha(a^2\cos^2 t + a^2\sin^2 t) = a^3\tan\alpha.$$

Example 3. *If* $\dfrac{d\vec{u}}{dt} = \vec{w} \times \vec{u}$ *and* $\dfrac{d\vec{v}}{dt} = \vec{w} \times \vec{v}$, *prove that* $\dfrac{d}{dt}(\vec{u} \times \vec{v}) = \vec{w} \times (\vec{u} \times \vec{v})$.

Sol.
$$\frac{d}{dt}(\vec{u} \times \vec{v}) = \vec{u} \times \frac{d\vec{v}}{dt} + \frac{d\vec{u}}{dt} \times \vec{v} = \vec{u} \times (\vec{w} \times \vec{v}) + (\vec{w} \times \vec{u}) \times \vec{v}$$
$$= (\vec{u} \cdot \vec{v})\vec{w} - (\vec{u} \cdot \vec{w})\vec{v} + (\vec{v} \cdot \vec{w})\vec{u} - (\vec{v} \cdot \vec{u})\vec{w}$$
$$= (\vec{v} \cdot \vec{w})\vec{u} - (\vec{u} \cdot \vec{w})\vec{v} \qquad [\because \vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}]$$
$$= (\vec{w} \cdot \vec{v})\vec{u} - (\vec{w} \cdot \vec{u})\vec{v} = \vec{w} \times (\vec{u} \times \vec{v}).$$

Example 4. *If* $\hat{R}$ *is a unit vector in the direction of* $\vec{r}$, *prove that* $\hat{R} \times \dfrac{d\hat{R}}{dt} = \dfrac{\vec{r}}{r^2} \times \dfrac{d\vec{r}}{dt}$, *where* $r = |\vec{r}|$.

Sol. We have $\vec{r} = r\hat{R}$ so that $\hat{R} = \dfrac{1}{r}\vec{r} \Rightarrow \dfrac{d\hat{R}}{dt} = \dfrac{1}{r}\dfrac{d\vec{r}}{dt} - \dfrac{1}{r^2}\dfrac{dr}{dt}\vec{r}$

$$\therefore \quad \hat{R} \times \frac{d\hat{R}}{dt} = \frac{\vec{r}}{r} \times \left(\frac{1}{r}\frac{d\vec{r}}{dt} - \frac{1}{r^2}\frac{dr}{dt}\vec{r}\right) = \frac{\vec{r}}{r^2} \times \frac{d\vec{r}}{dt} - \frac{1}{r^3}\frac{dr}{dt}\vec{r} \times \vec{r}$$
$$= \frac{\vec{r}}{r^2} \times \frac{d\vec{r}}{dt} \qquad (\because \vec{r} \times \vec{r} = \vec{0})$$

Example 5. *If* $\vec{r}$ *is a vector function of a scalar t and* $\vec{a}$ *is a constant vector, differentiate the following with respect to t :*

(i) $\dfrac{\vec{r} \times \vec{a}}{\vec{r} \cdot \vec{a}}$ (ii) $\dfrac{\vec{r} + \vec{a}}{\vec{r}^2 + \vec{a}^2}$.

Sol. (*i*) Let $\vec{R} = \dfrac{\vec{r} \times \vec{a}}{\vec{r} \cdot \vec{a}}$

Here $\vec{r} \cdot \vec{a}$ is a scalar function of t and $\dfrac{d\vec{a}}{dt} = \vec{0}$

$$\therefore \quad \frac{d\vec{R}}{dt} = \frac{1}{\vec{r} \cdot \vec{a}}\frac{d}{dt}(\vec{r} \times \vec{a}) + \left\{\frac{d}{dt}\left(\frac{1}{\vec{r} \cdot \vec{a}}\right)\right\}(\vec{r} \times \vec{a})$$
$$= \frac{1}{\vec{r} \cdot \vec{a}}\left(\vec{r} \times \frac{d\vec{a}}{dt} + \frac{d\vec{r}}{dt} \times \vec{a}\right) - \frac{\frac{d}{dt}(\vec{r} \cdot \vec{a})}{(\vec{r} \cdot \vec{a})^2}(\vec{r} \times \vec{a}) \qquad \left[\because \frac{d}{dt}\left(\frac{1}{f(t)}\right) = -\frac{f'(t)}{(f(t))^2}\right]$$
$$= \frac{\frac{d\vec{r}}{dt} \times \vec{a}}{\vec{r} \cdot \vec{a}} - \frac{\vec{r} \cdot \frac{d\vec{a}}{dt} + \frac{d\vec{r}}{dt} \cdot \vec{a}}{(\vec{r} \cdot \vec{a})^2}(\vec{r} \times \vec{a}) = \frac{\frac{d\vec{r}}{dt} \times \vec{a}}{\vec{r} \cdot \vec{a}} - \frac{\frac{d\vec{r}}{dt} \cdot \vec{a}}{(\vec{r} \cdot \vec{a})^2}(\vec{r} \times \vec{a}).$$

(ii) Let $\vec{R} = \dfrac{\vec{r} + \vec{a}}{\vec{r}^2 + \vec{a}^2}$

Here $\vec{r}^2 = |\vec{r}|^2$ is a scalar function of t

$\vec{a}^2 = |\vec{a}|^2$ is a constant, independent of t

$\therefore$ $\vec{r}^2 + \vec{a}^2$ is a scalar function of t

Also $\dfrac{d}{dt}(\vec{r}^2) = \dfrac{d}{dt}(\vec{r}\cdot\vec{r}) = \vec{r}\cdot\dfrac{d\vec{r}}{dt} + \dfrac{d\vec{r}}{dt}\cdot\vec{r} = 2\vec{r}\cdot\dfrac{d\vec{r}}{dt}$

$$\therefore \quad \frac{d\vec{R}}{dt} = \frac{1}{\vec{r}^2 + \vec{a}^2}\frac{d}{dt}(\vec{r} + \vec{a}) + \left\{\frac{d}{dt}\left(\frac{1}{\vec{r}^2 + \vec{a}^2}\right)\right\}(\vec{r} + \vec{a})$$

$$= \frac{1}{\vec{r}^2 + \vec{a}^2}\left(\frac{d\vec{r}}{dt} + \frac{d\vec{a}}{dt}\right) - \frac{\frac{d}{dt}(\vec{r}^2 + \vec{a}^2)}{(\vec{r}^2 + \vec{a}^2)^2}(\vec{r} + \vec{a}) = \frac{\frac{d\vec{r}}{dt}}{\vec{r}^2 + \vec{a}^2} - \frac{2\vec{r}\cdot\frac{d\vec{r}}{dt}}{(\vec{r}^2 + \vec{a}^2)^2}(\vec{r} + \vec{a}).$$

Example 6. *Find*

(i) $\dfrac{d^2}{dt^2}\left[\vec{r}\ \dfrac{d\vec{r}}{dt}\ \dfrac{d^2\vec{r}}{dt^2}\right]$ (ii) $\dfrac{d}{dt}\left[\vec{r} \times \left(\dfrac{d\vec{r}}{dt} \times \dfrac{d^2\vec{r}}{dt^2}\right)\right]$.

Sol. (i) Let $R = \left[\vec{r}\ \dfrac{d\vec{r}}{dt}\ \dfrac{d^2\vec{r}}{dt^2}\right]$, then R is the scalar triple product of three vectors $\vec{r}, \dfrac{d\vec{r}}{dt}\ \dfrac{d^2\vec{r}}{dt^2}$

$$\therefore \quad \frac{dR}{dt} = \left[\frac{d\vec{r}}{dt}\ \frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\right] + \left[\vec{r}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^2\vec{r}}{dt^2}\right] + \left[\vec{r}\ \frac{d\vec{r}}{dt}\ \frac{d^3\vec{r}}{dt^3}\right] = \left[\vec{r}\ \frac{d\vec{r}}{dt}\ \frac{d^3\vec{r}}{dt^3}\right],$$

scalar triple products having two equal vectors vanish.

Differentiating again, we have $\dfrac{d^2R}{dt^2} = \left[\dfrac{d\vec{r}}{dt}\ \dfrac{d\vec{r}}{dt}\ \dfrac{d^3\vec{r}}{dt^3}\right] + \left[\vec{r}\ \dfrac{d^2\vec{r}}{dt^2}\ \dfrac{d^3\vec{r}}{dt^3}\right] + \left[\vec{r}\ \dfrac{d\vec{r}}{dt}\ \dfrac{d^4\vec{r}}{dt^4}\right]$

$$= \left[\vec{r}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right] + \left[\vec{r}\ \frac{d\vec{r}}{dt}\ \frac{d^4\vec{r}}{dt^4}\right]$$

(ii) Let $\vec{R} = \vec{r} \times \left(\dfrac{d\vec{r}}{dt} \times \dfrac{d^2\vec{r}}{dt^2}\right)$, then $\vec{R}$ is the vector triple product of three vectors.

$$\therefore \quad \frac{d\vec{R}}{dt} = \frac{d\vec{r}}{dt} \times \left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right) + \vec{r} \times \left(\frac{d^2\vec{r}}{dt^2} \times \frac{d^2\vec{r}}{dt^2}\right) + \vec{r} \times \left(\frac{d\vec{r}}{dt} \times \frac{d^3\vec{r}}{dt^3}\right)$$

$$= \frac{d\vec{r}}{dt} \times \left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right) + \vec{r} \times \left(\frac{d\vec{r}}{dt} \times \frac{d^3\vec{r}}{dt^3}\right) \quad \text{since } \frac{d^2\vec{r}}{dt^2} \times \frac{d^2\vec{r}}{dt^2} = \vec{0}.$$

Example 7. *Find the unit tangent vector at any point on the curve $x = t^2 + 2$, $y = 4t - 5$, $z = 2t^2 - 6t$, where t is any variable. Also determine the unit tangent vector at the point $t = 2$.*

Sol. If $\vec{r}$ is the position vector of any point (x, y, z) on the given curve, then $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\Rightarrow \qquad \vec{r} = (t^2 + 2)\hat{i} + (4t - 5)\hat{j} + (2t^2 - 6t)\hat{k}$$

The vector $\frac{d\vec{r}}{dt}$ is along the tangent at the point (x, y, z) to the given curve.

Now $$\frac{d\vec{r}}{dt} = 2t\hat{i} + 4\hat{j} + (4t - 6)\hat{k}$$

and $$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{(2t)^2 + (4)^2 + (4t - 6)^2} = \sqrt{20t^2 - 48t + 52} = 2\sqrt{5t^2 - 12t + 13}$$

$\therefore$ The unit tangent vector $$\hat{T} = \frac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|} = \frac{t\hat{i} + 2\hat{j} + (2t - 3)\hat{k}}{\sqrt{5t^2 - 12t + 13}}.$$

Also the unit tangent vector at the point $t = 2$ is $\frac{2\hat{i} + 2\hat{j} + (2 \times 2 - 3)\hat{k}}{\sqrt{5 \times 4 - 12 \times 2 + 13}} = \frac{1}{3}(2\hat{i} + 2\hat{j} + \hat{k})$.

Example 8. *Find the angle between the tangents to the curve $\vec{r} = t^2\hat{i} + 2t\hat{j} - t^3\hat{k}$ at the points $t = \pm 1$.*

Sol. $\frac{d\vec{r}}{dt} = 2t\hat{i} + 2\hat{j} - 3t^2\hat{k}$ is a vector along the tangent at any point 't'.

If $\vec{T_1}$ and $\vec{T_2}$ are the vectors along the tangents at $t = 1$ and $t = -1$ respectively, then

$$\vec{T_1} = 2\hat{i} + 2\hat{j} - 3\hat{k} \quad \text{and} \quad \vec{T_2} = -2\hat{i} + 2\hat{j} - 3\hat{k}$$

If θ is the angle between $\vec{T_1}$ and $\vec{T_2}$, then

$$\cos\theta = \frac{\vec{T_1} \cdot \vec{T_2}}{|\vec{T_1}||\vec{T_2}|} = \frac{2(-2) + 2(2) - 3(-3)}{\sqrt{4+4+9} \cdot \sqrt{4+4+9}} = \frac{9}{17}$$

$$\therefore \qquad \theta = \cos^{-1}\left(\frac{9}{17}\right).$$

Example 9. *A particle moves on the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$ where t is the time. Find the components of velocity and acceleration at time $t = 1$ in the direction $\hat{i} - 3\hat{j} + 2\hat{k}$.*

Sol. If $\vec{r}$ is the position vector of any point (x, y, z) on the given curve, then

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = 2t^2\hat{i} + (t^2 - 4t)\hat{j} + (3t - 5)\hat{k}$$

Velocity $$\vec{v} = \frac{d\vec{r}}{dt} = 4t\hat{i} + (2t - 4)\hat{j} + 3\hat{k} = 4\hat{i} - 2\hat{j} + 3\hat{k} \quad \text{at } t = 1$$

Acceleration $$\vec{a} = \frac{d^2\vec{r}}{dt^2} = 4\hat{i} + 2\hat{j} = 4\hat{i} + 2\hat{j} \quad \text{at } t = 1$$

Now the unit vector in the given direction $\hat{i} - 3\hat{j} + 2\hat{k}$

$$= \frac{\hat{i} - 3\hat{j} + 2\hat{k}}{|\hat{i} - 3\hat{j} + 2\hat{k}|} = \frac{\hat{i} - 3\hat{j} + 2\hat{k}}{\sqrt{14}} = \hat{n} \quad \text{(say)}$$

$\therefore$ The component of velocity in the given direction

$$= \vec{v} \cdot \hat{n} = (4\hat{i} - 2\hat{j} + 3\hat{k}) \cdot \frac{\hat{i} - 3\hat{j} + 2\hat{k}}{\sqrt{14}}$$

$$= \frac{4(1) - 2(-3) + 3(2)}{\sqrt{14}} = \frac{16\sqrt{14}}{14} = \frac{8\sqrt{14}}{7}$$

and the component of acceleration in the given direction

$$= \vec{a} \cdot \hat{n} = (4\hat{i} + 2\hat{j}) \cdot \frac{\hat{i} - 3\hat{j} + 2\hat{k}}{\sqrt{14}} = \frac{-2}{\sqrt{14}} = -\frac{\sqrt{14}}{7}.$$

1. If $\vec{r} = \sin t\hat{i} + \cos t\hat{j} + t\hat{k}$, find $\left|\dfrac{d^2\vec{r}}{dt^2}\right|$.

2. If $\vec{r} = (\cos nt)\hat{i} + (\sin nt)\hat{j}$, where n is a constant and t varies, show that $\vec{r} \times \dfrac{d\vec{r}}{dt} = n\hat{k}$.

3. Show that $\vec{r} = \vec{a}\, e^{mt} + \vec{b}\, e^{nt}$ is the solution of the differential equation

$$\frac{d^2\vec{r}}{dt^2} - (m+n)\frac{d\vec{r}}{dt} + mn\vec{r} = \vec{0}.$$

[**Hint.** $\vec{a}$ and $\vec{b}$ are constant vectors.]

4. If $\vec{r}$ is a vector function of a scalar t and $\vec{a}$ is a constant vector, differentiate the following with respect to t :

(i) $\vec{r} \cdot \vec{a}$ (ii) $\vec{r} \times \vec{a}$ (iii) $\vec{r} \times \dfrac{d\vec{r}}{dt}$ (iv) $\vec{r} \cdot \dfrac{d\vec{r}}{dt}$.

5. Prove the following :

(i) $\dfrac{d}{dt}\left[\vec{a} \cdot \dfrac{d\vec{b}}{dt} - \dfrac{d\vec{a}}{dt} \cdot \vec{b}\right] = \vec{a} \cdot \dfrac{d^2\vec{b}}{dt^2} - \dfrac{d^2\vec{a}}{dt^2} \cdot \vec{b}$ (ii) $\dfrac{d}{dt}\left[\vec{a} \times \dfrac{d\vec{b}}{dt} - \dfrac{d\vec{a}}{dt} \times \vec{b}\right] = \vec{a} \times \dfrac{d^2\vec{b}}{dt^2} - \dfrac{d^2\vec{a}}{dt^2} \times \vec{b}$.

6. (a) Verify the formula, $\dfrac{d}{dt}(\vec{A} \cdot \vec{B}) = \vec{A} \cdot \dfrac{d\vec{B}}{dt} + \dfrac{d\vec{A}}{dt} \cdot \vec{B}$ for $\vec{A} = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$, $\vec{B} = \sin t\hat{i} - \cos t\hat{j}$.

(b) If $\vec{A} = 2t\hat{i} - t^2\hat{j} + t^3\hat{k}$, $\vec{B} = -t\hat{i} + t^2\hat{k}$, $\vec{C} = t^3\hat{i} - 2t\hat{k}$, find $\dfrac{d}{dt}(\vec{A} \cdot \vec{B} \times \vec{C})$ at $t = 1$.

(c) If $\vec{A} = \sin t\,\hat{i} - \cos t\,\hat{j} + t\,\hat{k}$, $\vec{B} = \cos t\,\hat{i} - \sin t\,\hat{j} - 3\,\hat{k}$ and $\vec{C} = 2\hat{i} + 3\hat{j} - \hat{k}$, find $\dfrac{d}{dt}[\vec{A} \times (\vec{B} \times \vec{C})]$ at $t = 0$.

7. Find the unit tangent vector at any point on the curve $x = 3\cos t$, $y = 3\sin t$, $z = 4t$.

8. Find the angle between the tangents to the curve $x = t$, $y = t^2$, $z = t^3$, at $t = \pm 1$.

9. A particle moves along the curve $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$, where t is the time. Determine its velocity and acceleration vectors and also the magnitudes of velocity and acceleration at $t = 0$.

10. The position vector of a particle at time t is $\vec{r} = \cos(t-1)\,\hat{i} + \sinh(t-1)\hat{j} + \alpha t^3\hat{k}$. Find the condition imposed on α by requiring that at time $t = 1$, the acceleration is normal to the position vector.

11. A particle moves along the curve $x = t^3 + 1$, $y = t^2$, $z = 2t + 5$ where t is the time. Find the components of its velocity and acceleration at $t = 1$ in the direction $\hat{i} - \hat{j} + 3\hat{k}$.