Chapter 4

Applications of differentiation

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Critical number

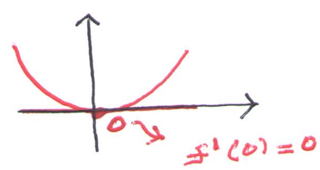
A critical number of a function f is a number c in the domain of f such that $f'(c)=0$ or $f'(c)$ does not exist.

Example: $f(x) = x^2$

$f'(x) = 2x$

$f'(x) = 0 \Rightarrow 2x = 0 \Rightarrow \boxed{x=0}$

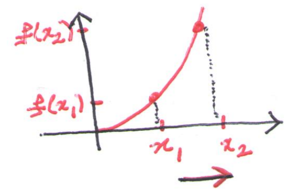
The critical number is $\boxed{0}$



Increasing function:

A function f is increasing on an Interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, $x_1, x_2 \in I$

Example: $f(x) = x^2$, $I = (0, \infty)$



$x_1 = 1$, $x_2 = 2$

$\underbrace{f(1)}_{f(x_1)} = 1 = 1^2 < 4 = \underbrace{f(2)}_{f(x_2)} = 2^2$

Result: If $f'(x) > 0$ on an interval, then f is increasing on that interval.

Remark: If $f(x) = x^2$, then $f'(x) = 2x$.

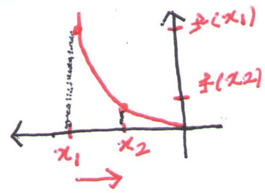
Then $f'(x) > 0 \Leftrightarrow 2x > 0$ if $x > 0$.

This means that f is increasing on $(0, \infty)$.

Decreasing Function

A function f is decreasing on an interval I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$, $x_1, x_2 \in I$

Example: $f(x) = x^2$, $I = (-\infty, 0)$



$x_1 = -2$ $\quad$ $x_2 = -1$

$f(x_1) = (-2)^2 = 4 > 1 = f(-1) = (-1)^2$ $\quad$ $f(x_2)$

Result: If $f'(x) < 0$, on an interval, then f is decreasing on that interval.

Remark: If $f(x) = x^2$, then $f'(x) = 2x$.

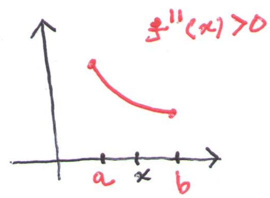
Then $f'(x) < 0 \Leftrightarrow 2x < 0$ if $x < 0$.

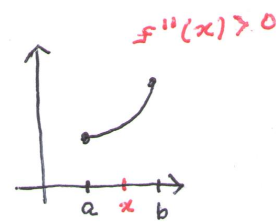
This means that f is decreasing on $(-\infty, 0)$

Concave upward

If the graph of f lies above all of its tangents on an interval, then it is called concave upward on I.

Example:

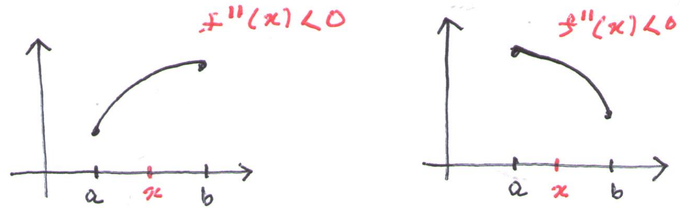




Concave downward

If the graph of f lies below all of its tangents on an interval, then it is called concave downward on I

Example:

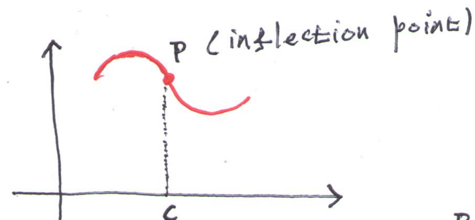


Concavity test

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

(c) If a curve changes from concave upward to concave downward or from concave downward to concave upward at P, then P is called an inflection point.



f has inflection point P at c

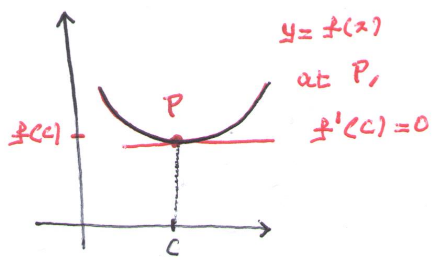
The second derivative test

Suppose that f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

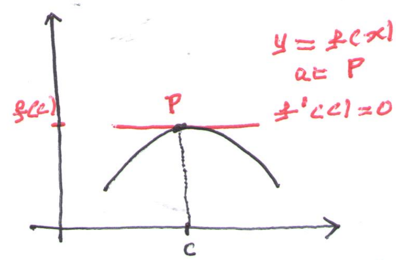
(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

f has <u>local</u> <u>minimum</u> at c



$f''(c) > 0$, f is concave upward

f has <u>local</u> <u>maximum</u> at c



$f''(c) < 0$, f is concave downward

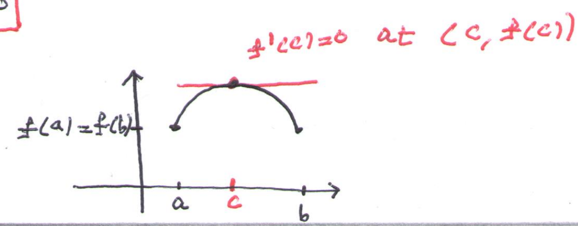
<u>Rolle's</u> <u>theorem</u>: Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$
2. f is differentiable on the open interval (a, b)
3. $f(a) = f(b)$

Then there is a number $\boxed{c}$ in (a, b) such that

$$\boxed{f'(c) = 0}$$

<u>Example</u>:



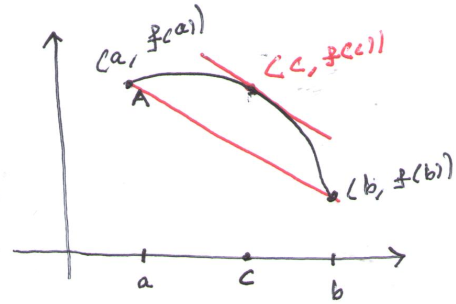
<u>Mean value</u> <u>theorem</u>: Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$
2. f is differentiable on the open interval (a, b)

Then there is a number $\boxed{c}$ in (a, b) such that

$$\boxed{f'(c) = \frac{f(b) - f(a)}{b - a}}$$

Example:



$$\left.\begin{array}{c}\text{The slope of the Line AB} \\ \dfrac{f(b)-f(a)}{b-a}\end{array}\right\} = \begin{array}{c}\text{slope of the} \\ \text{tangent at } (c, f(c)) \\ f'(c)\end{array}$$

Example: (a) Find the intervals on which f is increasing or decreasing

(b) Find the local maximum and local minimum values of f.

(c) Find the intervals of concavity and the inflection points of the following functions:

(1) $f(x) = x^4 - 4x^3$ (2) $f(x) = 2x^3 + 3x^2 - 36x$

(3) $f(x) = x^4 - 2x^2 + 3$ (4) $f(x) = xe^{-x}$

solution: (1) (a)

$$f(x) = x^4 - 4x^3$$

$$f'(x) = 4x^3 - 12x^2$$

$$f'(x) = 0 \Rightarrow 4x^3 - 12x^2 = 0$$

$$4x^2(x-3) = 0$$

$$x^2 = 0, \quad x - 3 = 0$$

$$\boxed{x=0} \qquad \boxed{x=3}$$

The critical numbers are $\boxed{0}$ and $\boxed{3}$

I_1 I_2 I_3

0 3

	4	x^2	$x-3$	$f'(x) = 4x^2(x-3)$	
$x<0$	+	+	−	− <0	decreasing
$0<x<3$	+	+	−	− <0	decreasing
$x>3$	+	+	+	+ >0	increasing

On $(-\infty, 0)$ f is decreasing

on $(0, 3)$ f is decreasing

on $(3, \infty)$ f is increasing

1 b $\quad f'(x) = 4x^3 - 12x^2$

$$f''(x) = 12x^2 - 24x$$

$$f''(x) = 12x(x-2)$$

$f''(\boxed{0}) = 0$ (Don't consider the point 0)

$f''(3) = 12 \cdot 3(3-2) = 36 \cdot 1 = 36 > 0$

f has local minimum at $\boxed{3}$

The local minimum value is $f(3)$

$$f(3) = 3^4 - 4 \cdot 3^3$$

$$= 3^3(3-4) = -27$$

$$\boxed{f(3) = -27}$$

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1 c $\quad f''(x) = 12x(x-2)$

$\boxed{x=0}$

$x - 2 = 0$

$\boxed{x=2}$

$I_1 \quad I_2 \quad I_3$

0 2

	12	x	$x-2$	$f''(x)$	
$x<0$	+	−	−	+ >0	CU
$0<x<2$	+	+	−	− <0	CD
$x>2$	+	+	+	+ >0	CU

on $(-\infty, 0)$, f is concave upward

on $(0, 2)$, f is concave downward

on $(2, \infty)$, f is concave upward

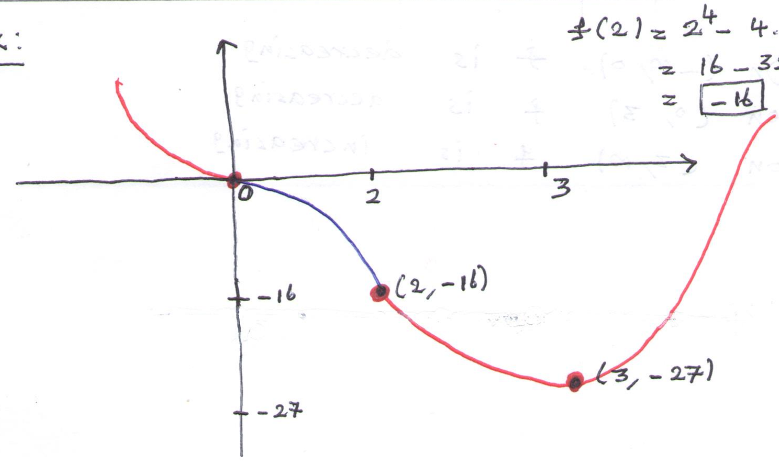
and f has points of inflection at $\boxed{0}$ and $\boxed{2}$

Remark:

$f(2) = 2^4 - 4 \cdot 2^3$

$= 16 - 32$

$= \boxed{-16}$



2 a

$f(x) = 2x^3 + 3x^2 - 36x$

$f'(x) = 6x^2 + 6x - 36$

$= 6(x^2 + x - 6)$

$= 6(x+3)(x-2)$

$f'(x) = 0 \Rightarrow 6(x+3)(x-2) = 0$

$\Rightarrow x+3 = 0, \quad x-2 = 0$

$\boxed{x = -3} \quad \boxed{x = 2}$

The critical numbers are $\boxed{-3, 2}$

I_1		I_2		I_3
	-3		2	

	6	$x+3$	$x-2$	$f'(x)$		
$x < -3$	+	−	−	+	>0	increasing
$-3 < x < 2$	+	+	−	−	<0	decreasing
$x > 2$	+	+	+	+	>0	increasing

on $(-\infty, -3)$, f is increasing

on $(-3, 2)$, f is decreasing

on $(2, \infty)$, f is increasing

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2 b

$f'(x) = 6x^2 + 6x - 36$

$f''(x) = 12x + 6$

$f''(-3) = 12(-3) + 6 = -36 + 6 = -30 < 0$

f has local maximum at $\boxed{-3}$

The local maximum value is $f(-3)$

$f(-3) = 2(-3)^3 + 3(-3)^2 - 36(-3)$

$= -54 + 27 + 108 = 81$

$\boxed{f(-3) = 81}$

$f''(2) = 12(2) + 6 = 24 + 6 = 30 > 0$

f has local minimum at $\boxed{2}$

The local minimum value is $f(2)$

$f(2) = 2 \cdot 2^3 + 3 \cdot 2^2 - 36(2)$

$= 16 + 12 - 72 = -44$

$\boxed{f(2) = -44}$

(2) (c) $f''(x) = 12x+6$

$12x+6=0$

$12x=-6$

$x=\frac{-6}{12}$

$\boxed{x=-\frac{1}{2}}$

I_1 I_2

$-\frac{1}{2}$

	$12x+6$	$f''(x)$		
$x<-\frac{1}{2}$	$-$	$-$	<0	CD
$x>-\frac{1}{2}$	$+$	$+$	>0	CU

on $(-\infty, -\frac{1}{2})$, f is concave downward

on $(-\frac{1}{2}, \infty)$, f is concave upward

and f has point of inflection at $\boxed{-\frac{1}{2}}$

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(3) (a) $f(x) = x^4 - 2x^2 + 3$

$f'(x) = 4x^3 - 4x$

$= 4x(x^2-1)$

$4x(x-1)(x+1)$

$f'(x)=0 \Rightarrow 4x(x^2-1)=0$

$\Rightarrow x=0,\quad x^2-1=0 \Rightarrow x^2=1 \Rightarrow x=\pm\sqrt{1}$

$\boxed{x=0}$ $\boxed{x=\pm 1}$

The critical numbers are $\boxed{0,\ 1,\ -1}$

I_1 I_2 I_3 I_4

-1 0 1

	4	x	$x-1$	$x+1$	$f'(x)=4x(x^2-1)$		
$x<-1$	$+$	$-$	$-$	$-$	$-$	<0	dereasing
$-1<x<0$	$+$	$-$	$-$	$+$	$+$	>0	Increasing
$0<x<1$	$+$	$+$	$-$	$+$	$-$	<0	decreasing
$x>1$	$+$	$+$	$+$	$+$	$+$	>0	increasing

on $(-\infty, -1)$ f is decreasing

on $(-1, 0)$ f is increasing

on $(0, 1)$ f is decreasing

on $(1, \infty)$ f is increasing

(3) (b) $f''(x) = 12x^2 - 4$

$f''(0) = 0 - 4 = -4 < 0$

f has local maximum at 0

The local maximum value is $f(0)$

$f(0) = 0 - 0 + 3$

$\boxed{f(0) = 3}$

$f''(1) = 12 \cdot 1 - 4 = 8 > 0$

f has local minimum at 1

The local minimum value is $f(1)$

$f(1) = 1 - 2 + 3 = 2$

$\boxed{f(1) = 2}$

$f''(-1) = 12(-1)^2 - 4 = 12 - 4 = 8 > 0$

f has local minimum at -1

The local minimum value is $f(-1)$

$f(-1) = (-1)^4 - 2(-1)^2 + 3$

$= 1 - 2 + 3 = 2$

$\boxed{f(-1) = 2}$

③ ⓒ

$f'(x) = 4x^3 - 4x$

$f''(x) = 12x^2 - 4$

$12x^2 - 4 = 0$

$12x^2 = 4$

$x^2 = \frac{4}{12}$

$x^2 = \frac{1}{3}$

$x = \pm\sqrt{\frac{1}{3}}$

$\boxed{x = \pm\frac{1}{\sqrt{3}}}$

I_1 $\quad -\frac{1}{\sqrt{3}} \quad$ I_2 $\quad \frac{1}{\sqrt{3}} \quad$ I_3

	$12x^2-4$	$f''(x)$		
$x < -\frac{1}{\sqrt{3}}$	+	+	>0	CU
$-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$	−	−	<0	CD
$x > \frac{1}{\sqrt{3}}$	+	+	>0	CU

on $(-\infty, -\frac{1}{\sqrt{3}})$ f is concave upward

on $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ f is concave downward

on $(\frac{1}{\sqrt{3}}, \infty)$ f is concave upward

and f has points of inflection at $\boxed{\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}}$

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④ ⓐ

$f(x) = xe^{-x}$

$f'(x) = x\left(e^{-x}(-1)\right) + e^{-x}\cdot 1$

$= -xe^{-x} + e^{-x}$

$= e^{-x}(-x+1)$

$f'(x) = 0 \Rightarrow e^{-x}(-x+1) = 0$ $\quad$ (since $e^{-x} \neq 0$)

$\Rightarrow -x+1 = 0$

$\boxed{1 = x}$

The critical number is $\boxed{1}$

I_1 $\quad 1 \quad$ I_2

	e^{-x}	$-x+1$	$f'(x)$		
$x<1$	+	+	+	>0	increasing
$x>1$	+	−	−	<0	decreasing

on $(-\infty, 1)$ f is increasing

on $(1, \infty)$ f is decreasing

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4 b

$$f'(x) = e^{-x}(-x+1)$$

$$f''(x) = e^{-x}(-1) + (-x+1)(e^{-x}(-1))$$

$$= -e^{-x} - e^{-x}(-x+1)$$

$$= e^{-x}(-1+x-1)$$

$$= e^{-x}(x-2)$$

$$f''(1) = e^{-1}(1-2) = \frac{-1}{e} < 0$$

f has local maximum at $\boxed{1}$

The local maximum value is $f(1)$

$$f(1) = 1 \cdot e^{-1} = \frac{1}{e}$$

$$\boxed{f(1) = \frac{1}{e}}$$

---

4 c

$$f''(x) = e^{-x}(x-2)$$

$x - 2 = 0$

$\boxed{x = 2}$

$e^{-x} \neq 0$

I_1 | 2 | I_2

	e^{-x}	$x-2$	$f''(x)$		
$x<2$	+	−	−	<0	CD
$x>2$	+	+	+	>0	CU

on $(-\infty, 2)$ f is concave downward

on $(2, \infty)$ f is concave upward

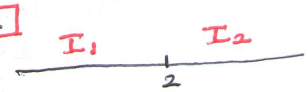
and f has point of inflection at $\boxed{2}$

Example: Find the interval on which $f(x) = x^2 - 4x + 3$ is increasing or decreasing

solution:

$$f(x) = x^2 - 4x + 3$$
$$f'(x) = 2x - 4$$
$$f'(x) = 0 \Rightarrow 2x - 4 = 0 \Rightarrow 2x = 4$$
$$x = \frac{4}{2}$$
$$\boxed{x = 2}$$

The critical number is $\boxed{2}$



	$2x-4$	$f'(x)$	
$x<2$	$-$	$-$ <0	decreasing
$x>2$	$+$	$+$ >0	increasing

on $(-\infty, 2)$, f is decreasing

on $(2, \infty)$ f is increasing

Example: Verify that the function satisfies the three hypothesis of Rolle's theorem on the given interval. Find the number c that satisfy the conclusion of Rolle's theorem:

(i) $f(x) = 5 - 12x + 3x^2$ on $[1, 3]$

(ii) $f(x) = \sqrt{x} - \frac{x}{3}$ on $[0, 9]$

solution: (i) $f(x) = 5 - 12x + 3x^2$

$$f'(x) = -12 + 6x$$

Since f and f' are polynomial,
f is continuous on $[1, 3]$ and
f is differentiable on $(1, 3)$. and

$f(1) = 5 - 12 + 3 = -4$ $\boxed{f(1) = -4}$

$f(3) = 5 - 36 + 27 = -4$ $\boxed{f(3) = -4}$

$f(1) = f(3)$

By Rolle's theorem, there is a number c in $(1,3)$ such that

$$f'(c) = 0 \qquad \Big| \quad f'(x) = -12+6x$$

$$-12+6c = 0$$

$$6c = 12$$

$$c = \frac{12}{6}$$

$$\boxed{c = 2}$$

(ii) $f(x) = \sqrt{x} - \frac{x}{3}$ on $[0, 9]$

$f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{3} \quad (x > 0)$

Since $\sqrt{x}$ and $\frac{x}{3}$ are continuous on $[0, 9]$, f is continuous on $[0, 9]$ and f is differentiable on $(0, 9)$ (note that $\sqrt{x}$ is not differentiable at 0)

$f(0) = 0 - 0 = 0 \Rightarrow \boxed{f(0) = 0}$

$f(9) = \sqrt{9} - \frac{9}{3} = 3 - 3 = 0 \Rightarrow f(9) = 0$

$$\boxed{f(0) = f(9)}$$

By Rolle's theorem, there is a number c in $(0,9)$ such that

$$f'(c) = 0$$

$$\frac{1}{2\sqrt{c}} - \frac{1}{3} = 0$$

$$\Rightarrow \frac{1}{2\sqrt{c}} = \frac{1}{3}$$

$$\frac{3}{2} = \sqrt{c}$$

$$\left(\frac{3}{2}\right)^2 = c \Rightarrow \boxed{c = \frac{9}{4}}$$

Example: Verify that the functions satisfies the hypothesis of the Mean Value theorem on the given interval. Find the number c that satisfy the Conclusion of the Mean Value theorem:

(i) $f(x) = 3x^2 + 2x + 5$ on $[-1, 1]$

(ii) $f(x) = e^{-2x}$ on $[0, 3]$

solution: (i) $f(x) = 3x^2 + 2x + 5$

$f'(x) = 6x + 2$

since f is polynomial, f is Continuous on $[-1, 1]$ and f is differentiable on $(-1,1)$

By Mean value theorem, there is a number c such that

$$f'(c) = \frac{f(1) - f(-1)}{1-(-1)}$$

$f(1) = 3+2+5 = \boxed{10}$

$f(-1) = 3(-1)^2 + 2(-1) + 5 = 3 - 2 + 5 = \boxed{6}$

$$6c+2 = \frac{10-6}{1+1}$$

$$6c+2 = \frac{\cancel{4}^{2}}{\cancel{2}_{1}}$$

$$6c = 2-2$$

$$\boxed{c = 0}$$

(ii) $f(x) = e^{-2x}$, $f'(x) = e^{-2x}(-2) = -2e^{-2x}$

f is Continuous on $[0, 3]$

f is differentiable on $(0, 3)$

By Mean value theorm,

$f'(x) = -2e^{-2x}$

$f'(c) = -2e^{-2c}$

$f(3) = e^{-6}$

$f(0) = e^{0} = 1$

$$f'(c) = \boxed{-2e^{-2c}} = \boxed{\frac{e^{-6}-1}{3-0}} = \frac{f(3) - f(0)}{3-0}$$

$$-2e^{-2c} = \frac{e^{-6}-1}{3}$$

$$-2e^{-2c} = \frac{e^{-6}-1}{3}$$

$$e^{-2c} = \frac{e^{-6}-1}{-6}$$

$$e^{-2c} = \frac{1-e^{-6}}{6}$$

$$\ln\left(e^{-2c}\right) = \ln\left(\frac{1-e^{-6}}{6}\right)$$

$$\ln(e^{x}) = x$$

$$-2c = \ln\left(\frac{1-e^{-6}}{6}\right)$$

$$\boxed{c = -\frac{1}{2}\ln\left(\frac{1-e^{-6}}{6}\right)}$$