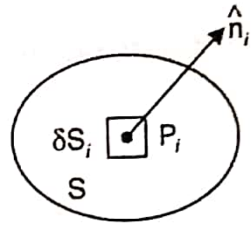
13.25. SURFACE INTEGRALS

Any integral which is to be evaluated over a surface is called a surface integral.

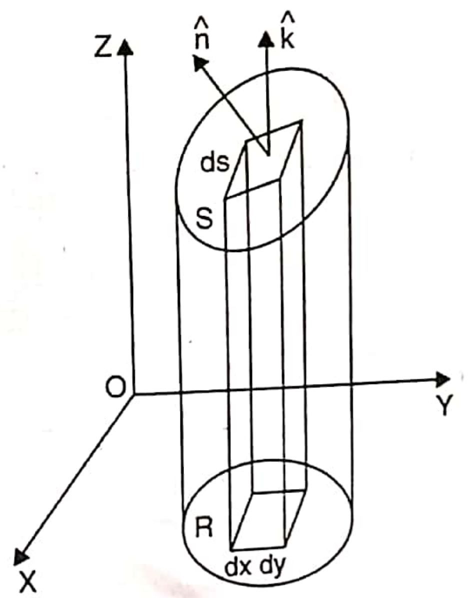
Let $\vec{F}(P)$ be a continuous vector point function and S a two sided surface. Divide S into a finite number of sub-surfaces $\delta S_1, \delta S_2, \ldots\ldots \delta S_k$. Let P_i be any point in δS_i and $\hat{n}_i$ be the unit vector at P in the direction of outward drawn normal to the surface at P_i. Then the limit of the sum



$\sum_{i=1}^{k} \vec{F}(P_i) \cdot \hat{n}_i \, \delta S_i$, as $k \to \infty$ and each $\delta S_i \to 0$ is called the *normal surface integral* of $\vec{F}(P)$ over S and is denoted by $\iint_S \vec{F} \cdot \hat{n} \, dS$.

The surface element $\vec{\delta S}$ surrounding any point P can be regarded as a vector whose magnitude is area δS and the direction that of the outward drawn normal $\hat{n}$ *i.e.* $\vec{\delta S} = \hat{n} \, \delta S$. The surface integral may alternatively be written as $\iint_S \vec{F} \cdot d\vec{S}$.

If $\vec{F}$ represents the velocity of a fluid at any point P on a closed surface S, then $\vec{F} \cdot \hat{n}$ is the normal component of $\vec{F}$ at P and $\iint_S \vec{F} \cdot \hat{n} \, dS = \iint_S \vec{F} \cdot d\vec{S}$ is a measure of volume emerg-



ing from S per unit time, *i.e.* it measures the *flux of $\vec{F}$ over S.*

Other types of surface integrals are $\iint_S \vec{F} \times d\vec{S}, \iint_S \phi \, d\vec{S}$.

Note. In order to evaluate surface integrals, it is convenient to express them as double integrals taken over the orthogonal projection of S on one of the coordinate planes. But this is possible only when any line perpendicular to the coordinate plane chosen meets the surface S in not more than one point.

Let R be the orthogonal projection of S on the *xy*-plane.

Let $\hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$ where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of $\hat{n}$.

Now, $dxdy$ = projection of dS on the *xy*-plane $= dS \cos\gamma \Rightarrow dS = \dfrac{dx\,dy}{\cos\gamma}$

Also $|\hat{k}\cdot\hat{n}| = \cos\gamma \quad \therefore \quad dS = \dfrac{dxdy}{|\hat{k}\cdot\hat{n}|}$

Hence $\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_R \vec{F}\cdot\hat{n}\dfrac{dxdy}{|\hat{k}\cdot\hat{n}|}$.

13.26. VOLUME INTEGRALS

Any integral which is to be evaluated over a volume is called a volume integral.

If V is a volume bounded by a surface S, then the triple integrals

$$\iiint_V \phi\, dV \quad \text{and} \quad \iiint_V \vec{F}\, dV$$

are called volume integrals. The first of these is a scalar and the second is a vector.

If we sub-divide the volume V into small cuboids by drawing planes parallel to the coordinate planes, then $dV = dx\,dy\,dz$.

$$\therefore \quad \iiint_V \phi\, dV = \iiint_V \phi(x, y, z)\, dx\,dy\,dz$$

If $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$, then

$$\iiint_V \vec{F}\, dV = \hat{i}\iiint_V F_1(x, y, z)\, dx\,dy\,dz + \hat{j}\iiint_V F_2(x, y, z)\, dx\,dy\,dz$$
$$+ \hat{k}\iiint_V F_3(x, y, z)\, dx\,dy\,dz.$$

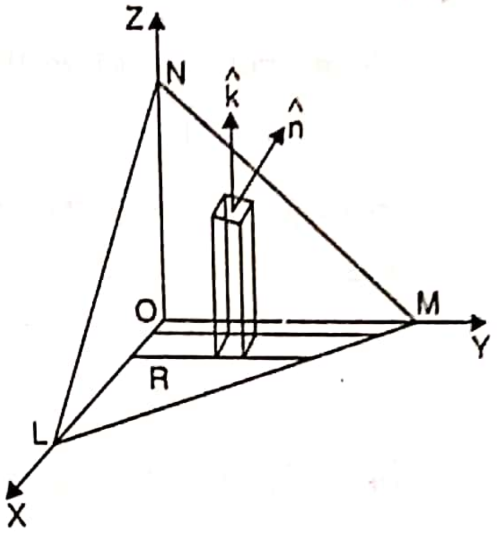
ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate* $\iint_S \vec{A}\cdot\hat{n}\,dS$ *where* $\vec{A} = (x + y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ *and S is the surface of the plane* $2x + y + 2z = 6$ *in the first octant.*

Sol. A vector normal to the surface S is given by

$$\nabla(2x + y + 2z) = 2\hat{i} + \hat{j} + 2\hat{k}$$

$\therefore \quad \hat{n}$ = a unit vector normal to surface S



$$= \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{(2)^2 + 1^2 + (2)^2}} = \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}$$

$$\hat{k}\,.\,\hat{n} = \hat{k}\,.\left(\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}\right) = \frac{2}{3}$$

$$\therefore \quad \iint_S \vec{A}\,.\,\hat{n}\, dS = \iint_R \vec{A}\,.\,\hat{n}\,\frac{dxdy}{|\hat{k}\,.\,\hat{n}|},$$

where R is the projection of S, *i.e.* triangle LMN on the xy-plane. The region R, *i.e.* triangle OLM is bounded by x-axis, y-axis and the line $2x + y = 6$, $z = 0$.

$$\text{Now} \quad \vec{A}\,.\,\hat{n} = [(x + y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}]\,.\left(\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}\right)$$

$$= \frac{2}{3}(x + y^2) - \frac{2}{3}x + \frac{4}{3}yz = \frac{2}{3}y^2 + \frac{4}{3}yz$$

$$= \frac{2}{3}y^2 + \frac{4}{3}y\left(\frac{6 - 2x - y}{2}\right) \quad \left(\because \text{ on the plane } 2x + y + 2z = 6,\ z = \frac{6 - 2x - y}{2}\right)$$

$$= \frac{2}{3}y(y + 6 - 2x - y) = \frac{4}{3}y(3 - x)$$

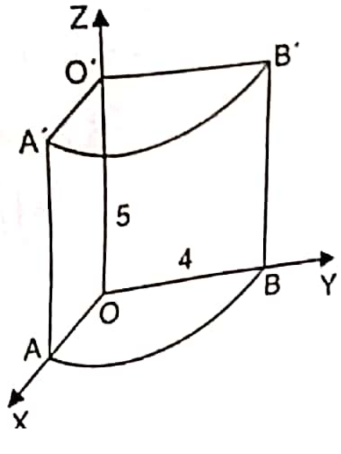
$$\text{Hence} \iint_S \vec{A}\,.\,\hat{n}\, dS = \iint_R \vec{A}\,.\,\hat{n}\,\frac{dxdy}{|\hat{k}\,.\,\hat{n}|}$$

$$= \iint_R \frac{4}{3}y\,(3 - x)\,.\,\frac{3}{2}\,dxdy = \int_0^3 \int_0^{6-2x} 2y(3 - x)\, dydx$$

$$= \int_0^3 2\,(3 - x)\,.\left[\frac{y^2}{2}\right]_2^{6-2x} dx = \int_0^3 (3 - x)\,(6 - 2x)^2\, dx$$

$$= 4\int_0^3 (3 - x)^3\, dx = 4\,.\left[\frac{(3 - x)^4}{4\,(-1)}\right]_0^3 = -(0 - 81) = 81.$$

Example 2. *Evaluate* $\iint_S \vec{A}\,.\,\hat{n}\, dS$, *where* $\vec{A} = z\hat{i} + x\hat{j} - 3y^2z\hat{k}$ *and S is the surface of the cylinder* $x^2 + y^2 = 16$ *included in the first octant between* $z = 0$ *and* $z = 5$. (M...



Sol. A vector normal to the surface S is given by

$$\nabla\,(x^2 + y^2) = 2x\hat{i} + 2y\hat{j}$$

$\therefore$ $\quad \hat{n}$ = a unit vector normal to surface S

$$= \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{(2x)^2 + (2y)^2}} = \frac{x\hat{i} + y\hat{j}}{\sqrt{x^2 + y^2}} = \frac{x\hat{i} + y\hat{j}}{4}$$

($\because$ on the surface of cylinder, $x^2 + y^2 = 16$)

Let R be the projection of S on yz-plane, then

$$\iint_S \vec{A}\cdot\hat{n}\, dS = \iint_R \vec{A}\cdot\hat{n}\frac{dy\,dz}{|\hat{i}\cdot\hat{n}|}$$

The region R is OBB′O′ enclosed by $y = 0$ to $y = 4$ and $z = 0$ to $z = 5$.

Now $\quad \hat{i}\cdot\hat{n} = \hat{i}\cdot(\frac{1}{4}x\hat{i} + \frac{1}{4}y\hat{j}) = \frac{1}{4}x$

$$\vec{A}\cdot\hat{n} = (z\hat{i} + x\hat{j} - 3y^2z\hat{k})\cdot(\tfrac{1}{4}x\hat{i} + \tfrac{1}{4}y\hat{j})$$

$$= \tfrac{1}{4}zx + \tfrac{1}{4}xy = \tfrac{1}{4}x(y+z).$$

Hence $\displaystyle\iint_S \vec{A}\cdot\hat{n}\, dS = \iint_R \vec{A}\cdot\hat{n}\frac{dy\,dz}{|\hat{i}\cdot\hat{n}|} = \iint_R \frac{1}{4}x(y+z)\frac{dy\,dz}{\frac{1}{4}x} = \int_0^5\int_0^4 (y+z)\,dy\,dz$

$$= \int_0^5 \left[\frac{y^2}{2} + zy\right]_0^4 dz = \int_0^5 (8+4z)\,dz = \left[8z + 2z^2\right]_0^5 = 40 + 50 = 90.$$

Example 3. *If* $\vec{F} = (2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$, *then evaluate* $\iiint_V \nabla\cdot\vec{F}\, dV$, *where V is bounded by the planes* $x = 0$, $y = 0$, $z = 0$ *and* $2x + 2y + z = 4$.

Sol. $\displaystyle\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(2x^2 - 3z) + \frac{\partial}{\partial y}(-2xy) + \frac{\partial}{\partial z}(-4x) = 4x - 2x = 2x$

$$\therefore \quad \iiint_V \nabla\cdot\vec{F}\, dV = \iiint_V 2x\, dx\, dy\, dz$$

$$= \int_0^2\int_0^{2-x}\int_0^{4-2x-2y} 2x\, dz\, dy\, dx = \int_0^2\int_0^{2-x} 2x\Big[z\Big]_0^{4-2x-2y} dy\,dx$$

$$= \int_0^2\int_0^{2-x} 2x(4-2x-2y)\,dy\,dx = \int_0^2\int_0^{2-x} [4x(2-x) - 4xy]\,dy\,dx$$

$$= \int_0^2 \Big[4x(2-x)y - 2xy^2\Big]_0^{2-x} dx = \int_0^2 [4x(2-x)^2 - 2x(2-x)^2]\,dx$$

$$= \int_0^2 2x(2-x)^2\,dx = 2\int_0^2 (4x - 4x^2 + x^3)\,dx$$

$$= 2\left[2x^2 - 4\frac{x^3}{3} + \frac{x^4}{4}\right]_0^2 = 2\left(8 - \frac{32}{3} + 4\right) = \frac{8}{3}.$$

Example 4. *If* $\vec{A} = 2xz\hat{i} - x\hat{j} + y^2\hat{k}$, *evaluate* $\iiint_V \vec{A}\, dV$, *where V is the region bounded by the surface* $x = 0$, $y = 0$, $x = 2$, $y = 6$, $z = x^2$, $z = 4$.

Sol. $\displaystyle\iiint_V \vec{A}\, dV = \iiint_V (2xz\hat{i} - x\hat{j} + y^2\hat{k})\, dx\, dy\, dz$

$$= \int_0^2\int_0^6\int_{x^2}^4 (2xz\hat{i} - x\hat{j} + y^2\hat{k})\, dz\, dy\, dx$$

$$= \hat{i}\int_0^2\int_0^6\int_{x^2}^4 2xz\,dz\,dy\,dx - \hat{j}\int_0^2\int_0^6\int_{x^2}^4 x\,dz\,dy\,dx + \hat{k}\int_0^2\int_0^6\int_{x^2}^4 y^2\,dz\,dy\,dx$$

$$= \hat{i}\int_0^2\int_0^6\left[xz^2\right]_{x^2}^4 dy\,dx - \hat{j}\int_0^2\int_0^6\left[xz\right]_{x^2}^4 dy\,dx + \hat{k}\int_0^2\int_0^6\left[y^2z\right]_{x^2}^4 dy\,dx$$

$$= \hat{i}\int_0^2\int_0^6 (16x - x^5)\,dy\,dx - \hat{j}\int_0^2\int_0^6 (4x - x^3)\,dy\,dx + \hat{k}\int_0^2\int_0^6 y^2(4 - x^2)\,dy\,dx$$

$$= \hat{i}\int_0^2 (16x - x^5)\left[y\right]_0^6 dx - \hat{j}\int_0^2 (4x - x^3)\left[y\right]_0^6 dx + \hat{k}\int_0^2 (4 - x^2)\left[\frac{y^3}{3}\right]_0^6 dx$$

$$= 6\hat{i}\int_0^2 (16x - x^5)\,dx - 6\hat{j}\int_0^2 (4x - x^3)\,dx + 72\hat{k}\int_0^2 (4 - x^2)\,dx$$

$$= 6\hat{i}\left[8x^2 - \frac{x^6}{6}\right]_0^2 - 6\hat{j}\left[2x^2 - \frac{x^4}{4}\right]_0^2 + 72\hat{k}\left[4x - \frac{x^3}{3}\right]_0^2 = 128\hat{i} - 24\hat{j} + 384\hat{k}.$$

EXERCISE 13.4

1. If $\vec{f}(t) = t\hat{i} + (t^2 - 2t)\hat{j} + (3t^2 + 3t^3)\hat{k}$, find $\int_0^1 \vec{f}(t)\,dt$.

2. If $\vec{r} = t\hat{i} - t^2\hat{j} + (t-1)\hat{k}$ and $\vec{S} = 2t^2\hat{i} + 6t\hat{k}$, evaluate

   (i) $\int_0^2 \vec{r}\cdot\vec{S}\,dt$ (ii) $\int_0^2 \vec{r}\times\vec{S}\,dt$.

3. Find the value of $\vec{r}$ satisfying the equation $\frac{d^2\vec{r}}{dt^2} = 6t\hat{i} - 24t^2\hat{j} + 4\sin t\hat{k}$, given that $\vec{r} = 2\hat{i} + \hat{j}$ and $\frac{d\vec{r}}{dt} = -\hat{i} - 3\hat{k}$ at $t = 0$.

4. The acceleration of a particle at any time t is given by $\vec{a} = 12\cos 2t\hat{i} - 8\sin 2t\hat{j} + 16t\hat{k}$. If the velocity $\vec{v}$ and displacement $\vec{r}$ are zero at $t = 0$, find $\vec{v}$ and $\vec{r}$ at any time t.

5. (a) If $\phi = 2xyz^2$, $\vec{F} = xy\hat{i} - z\hat{j} + x^2\hat{k}$, and C is the curve $x = t^2, y = 2t, z = t^3$ from $t = 0$ to $t = 1$, evaluate the line integrals

   (i) $\int_C \phi\,d\vec{r}$ (ii) $\int_C \vec{F}\times d\vec{r}$.