3.5. ▶ SOME PROPERTIES OF DEFINITE INTEGRAL

Property 1. $\displaystyle\int_a^b f(x)\,dx = \int_a^b f(z)\,dz.$

Proof. Let $\displaystyle\int f(x)\,dx = F(x)$

$\therefore$ $\displaystyle\int f(z)\,dz = F(z)$

$$\text{L.H.S.} = \int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a) \quad \ldots(1)$$

$$\text{R.H.S.} = \int_a^b f(z)\,dz = \Big[F(z)\Big]_a^b = F(b) - F(a) \quad \ldots(2)$$

From (1) and (2), we have $\displaystyle\int_a^b f(x)\,dx = \int_a^b f(z)\,dz.$

Property 2. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$

Proof. Let $\displaystyle\int f(x)\,dx = F(x)$

$$\therefore \quad \int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a) \quad \ldots(1)$$

and

$$\int_b^a f(x)\,dx = \Big[F(x)\Big]_b^a = F(a) - F(b) = -[F(b) - F(a)] \quad \ldots(2)$$

From (1) and (2), we have $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$

Property 3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, *where* $a < c < b$.

Proof. Let $\int f(x)\,dx = F(x)$

$$\text{L.H.S.} = \int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a) \quad \text{...(1)}$$

$$\text{R.H.S.} = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= \Big[F(x)\Big]_a^c + \Big[F(x)\Big]_c^b$$

$$= [F(c) - F(a)] + [F(b) - F(c)] = F(b) - F(a) \quad \text{...(2)}$$

From (1) and (2), we have

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Cor. Generalisation of Property 3.

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \int_{c_2}^{c_3} f(x)\,dx + \ldots + \int_{c_{n-1}}^{c_n} f(x)\,dx + \int_{c_n}^{b} f(x)\,dx,$$

where $a < c_1 < c_2 < \ldots < c_n < b$.

Property 4. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

Proof. Put $a - x = z$ so that $-dx = dz \Rightarrow dx = -dz$

Now when $x = 0, z = a$ and when $x = a, z = 0$

$$\therefore \quad \int_0^a f(a-x)\,dx = \int_a^0 f(z)(-dz) = -\int_a^0 f(z)\,dz$$

$$= \int_0^a f(z)\,dz \quad \text{[Using Property 2]}$$

$$= \int_0^a f(x)\,dx \quad \text{[Using Property 1]}$$

Hence $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Remark.

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

Putting $a+b-x=z$ in the above proof, we get the desired result.

Property 5. $\displaystyle\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$

Proof. Let $\displaystyle\int f(x)\,dx = F(x)$ and $\displaystyle\int g(x)\,dx = G(x)$

Now, $\displaystyle\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx = F(x) + G(x)$

$$\therefore \quad \int_a^b [f(x)+g(x)]\,dx = \Big[F(x)+G(x)\Big]_a^b = [F(b)+G(b)] - [F(a)+G(a)]$$

$$= [F(b)-F(a)] + [G(b)-G(a)] = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

Hence $\displaystyle\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$

Property 6. $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$ ***when*** $f(x)$ ***is an odd function of*** x.

$\displaystyle = 2\int_0^a f(x)\,dx,$ ***when*** $f(x)$ ***is an even function of*** x.

Proof. $\displaystyle\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$ ...(1) [*Using Property 3*]

In the first integral on R.H.S. of (1), put $x=-z$, so that $dx=-dz$

Now when $x=-a$, $z=a$ and when $x=0$, $z=0$

$$\therefore \quad \int_{-a}^{0} f(x)\,dx = \int_a^0 f(-z)(-dz) = -\int_a^0 f(-z)\,dz$$

$$= \int_0^a f(-z)\,dz = \int_0^a f(-x)\,dx$$

$$\therefore \quad \int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx \qquad ...(2)$$

Case I. When $f(x)$ is an *odd function* of x, then $f(-x) = -f(x)$

Hence from (2), we get $\displaystyle\int_{-a}^{a} f(x)\,dx = -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = \mathbf{0}.$

Case II. When $f(x)$ is an *even function* of x, then $f(-x) = f(x)$

Hence from (2), we get

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

Property 7. $\displaystyle\int_{0}^{2a} \boldsymbol{f(x)\,dx} = \int_{0}^{a} \boldsymbol{f(x)\,dx} + \int_{0}^{a} \boldsymbol{f(2a-x)\,dx}.$

Proof. We know that $\displaystyle\int_{0}^{2a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{a}^{2a} f(x)\,dx$ ...(1) [Property 3]

Now in $\displaystyle\int_{a}^{2a} f(x)\,dx$, put $x = 2a - z$ so that $dx = -dz$

Now, when $x = a$, $z = a$ and when $x = 2a$, $z = 0$

$$\therefore \quad \int_{a}^{2a} f(x)\,dx = \int_{a}^{0} f(2a-z)(-dz) = -\int_{a}^{0} f(2a-z)\,dz$$

$$= \int_{0}^{a} f(2a-z)\,dz = \int_{0}^{a} f(2a-x)\,dx \qquad ...(2)$$

Putting it in (1), we have

$$\int_{0}^{2a} f(x)\,dx = \int_{0}^{a} \boldsymbol{f(x)\,dx} + \int_{0}^{a} \boldsymbol{f(2a-x)\,dx}.$$

Property 8. $\displaystyle\int_{0}^{2a} \boldsymbol{f(x)\,dx} = 2\int_{0}^{a} \boldsymbol{f(x)\,dx}$ ***if*** $\boldsymbol{f(2a-x) = f(x)}$

$\qquad = \boldsymbol{0}$ ***if*** $\boldsymbol{f(2a-x) = -f(x)}$.

Proof. From property (7), we have

$$\int_{0}^{2a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(2a-x)\,dx$$

$$= 2\int_{0}^{a} f(x)\,dx, \text{ when } f(2a-x) = f(x)$$

$$= 0, \text{ when } f(2a-x) = -f(x).$$

SOLVED EXAMPLES

Example 1.

Evaluate :

(i) $\displaystyle\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx$ [CBSE 2002 C]

(ii) $\displaystyle\int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$

(iii) $\displaystyle\int_0^{\pi/2} \frac{\sin^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x}\,dx$

(iv) $\displaystyle\int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}}\,dx$

Solution. (*i*) Let $$I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx \qquad ...(1)$$

Also, $$I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}\,dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\,dx \qquad ...(2)$$

Adding (1) and (2), we get

$$I + I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx + \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x}\,dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x}\,dx = \int_0^{\pi/2} 1.dx = \left[x\right]_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$\Rightarrow \quad 2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}.$$

(*ii*) Let $$I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx \qquad ...(1)$$

Also, $$I = \int_0^{\pi/2} \frac{\sqrt{\cos\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}}\,dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}}\, dx \quad \text{...(2)}$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx + \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}}\, dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos x} + \sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}}\, dx = \int_0^{\pi/2} 1.dx = [x]_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$\therefore \quad 2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}.$$

(*iii*) Let $I = \int_0^{\pi/2} \frac{\sin^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x}\, dx$ ...(1)

Also, $$I = \int_0^{\pi/2} \frac{\sin^{3/2}\left(\frac{\pi}{2} - x\right)}{\sin^{3/2}\left(\frac{\pi}{2} - x\right) + \cos^{3/2}\left(\frac{\pi}{2} - x\right)}\, dx$$

$$= \int_0^{\pi/2} \frac{\cos^{3/2} x}{\cos^{3/2} x + \sin^{3/2} x}\, dx \quad \text{...(2)}$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{\sin^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x}\, dx + \int_0^{\pi/2} \frac{\cos^{3/2} x}{\cos^{3/2} x + \sin^{3/2} x}\, dx$$

$$= \int_0^{\pi/2} \frac{\sin^{3/2} x + \cos^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x}\, dx = \int_0^{\pi/2} 1.dx = [x]_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$\therefore \quad 2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}.$$

(*iv*) Let $I = \int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}}\, dx$ ...(1)

Also, $$I = \int_0^{\pi} \frac{e^{\cos(\pi - x)}}{e^{\cos(\pi - x)} + e^{-\cos(\pi - x)}}\, dx \quad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx\right]$$

$$= \int_0^{\pi} \frac{e^{-\cos x}}{e^{-\cos x} + e^{\cos x}}\, dx \quad \text{...(2)}$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}}\,dx + \int_0^{\pi} \frac{e^{-\cos x}}{e^{-\cos x} + e^{\cos x}}\,dx = \int_0^{\pi} \frac{e^{\cos x} + e^{-\cos x}}{e^{\cos x} + e^{-\cos x}}\,dx$$

$$\therefore \quad 2I = \int_0^{\pi} 1\,dx = \left[x\right]_0^{\pi} = \pi \;\Rightarrow\; I = \frac{\pi}{2}.$$

Example 2.

Evaluate :

(i) $\int_0^1 x(1-x)^n\,dx$ (ii) $\int_0^1 \cot^{-1}(1-x+x^2)\,dx$

Solution. (i) Let $I = \int_0^1 x(1-x)^n\,dx$

$$= \int_0^1 (1-x)[1-(1-x)]^n\,dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^1 (1-x)x^n\,dx = \int_0^1 x^n\,dx - \int_0^1 x^{n+1}\,dx$$

$$= \left[\frac{x^{n+1}}{n+1}\right]_0^1 - \left[\frac{x^{n+2}}{n+2}\right]_0^1 = \frac{1}{n+1} - \frac{1}{n+2} = \frac{1}{(n+1)(n+2)}.$$

(ii) Let $I = \int_0^1 \cot^{-1}(1-x+x^2)\,dx = \int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right)dx$

$$\left[\because \cot^{-1} x = \tan^{-1}\frac{1}{x} \text{ for } x > 0\right]$$

$$= \int_0^1 \tan^{-1}\left(\frac{x+(1-x)}{1-x(1-x)}\right)dx = \int_0^1 [\tan^{-1} x + \tan^{-1}(1-x)]\,dx$$

$$= \int_0^1 \tan^{-1} x\,dx + \int_0^1 \tan^{-1}[1-(1-x)]\,dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^1 \tan^{-1} x\,dx + \int_0^1 \tan^{-1} x\,dx = 2\int_0^1 \tan^{-1} x \,.\, 1\,dx$$

$$= 2\left[\tan^{-1} x \,.\, x\right]_0^1 - 2\int_0^1 \frac{1}{1+x^2}\,.\,x\,dx$$

$$= 2\left[x\tan^{-1} x\right]_0^1 - \int_0^1 \frac{2x}{1+x^2}\,dx$$

$$= 2\,[\tan^{-1} 1 - 0] - \left[\log|1+x^2|\right]_0^1$$

$$= 2\,.\,\frac{\pi}{4} - [\log 2 - \log 1] = \frac{\pi}{2} - \log 2.$$

Example 3.

Using properties of definite integral, evaluate $\int_0^{\pi/2} \sin^2 x\,dx$.

Solution. Let $\quad I = \int_0^{\pi/2} \sin^2 x\,dx \quad$ ...(1)

Also, $\quad I = \int_0^{\pi/2} \sin^2\left(\frac{\pi}{2} - x\right) dx \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$$= \int_0^{\pi/2} \cos^2 x\,dx \quad ...(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} (\sin^2 x + \cos^2 x)\,dx$$

$$= \int_0^{\pi/2} 1.dx = \left[x\right]_0^{\pi/2} = \frac{\pi}{2}$$

$\therefore \quad 2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}.$

Example 4.

Evaluate $\int_0^{\pi/2} \log(\tan x)\,dx$.

Solution. Let $\quad I = \int_0^{\pi/2} \log(\tan x)\,dx \quad$ ...(1)

Also $\quad I = \int_0^{\pi/2} \log\left[\tan\left(\frac{\pi}{2} - x\right)\right] dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$$= \int_0^{\pi/2} \log(\cot x)\,dx \qquad ...(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \log(\tan x)\,dx + \int_0^{\pi/2} \log(\cot x)\,dx$$

$$= \int_0^{\pi/2} [\log(\tan x) + \log(\cot x)]\,dx$$

$$= \int_0^{\pi/2} \log[\tan x \cot x]\,dx = \int_0^{\pi/2} \log 1 . dx = 0 \qquad [\because \log 1 = 0]$$

$\therefore \quad 2I = 0 \Rightarrow \mathbf{I = 0.}$

Example 5.

Evaluate :

(i) $\int_0^{\pi/2} (2 \log \sin x - \log \sin 2x)\,dx$ (ii) $\int_0^{\pi/2} \log(\sin x)\,dx$ [CBSE 20...]

(iii) $\int_0^{\pi} \log(1 + \cos x)\,dx$

Solution. (*i*) Let $I = \int_0^{\pi/2} (2 \log \sin x - \log \sin 2x)\,dx \qquad ...(1)$

Also, $\quad I = \int_0^{\pi/2} \left[2 \log \sin\left(\frac{\pi}{2} - x\right) - \log \sin 2\left(\frac{\pi}{2} - x\right)\right] dx$

$$= \int_0^{\pi/2} [2 \log \cos x - \log \sin(\pi - 2x)]\,dx$$

$$= \int_0^{\pi/2} (2 \log \cos x - \log \sin 2x)\,dx \qquad ...(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} (2 \log \sin x - \log \sin 2x)\,dx + \int_0^{\pi/2} (2 \log \cos x - \log \sin 2x)\,dx$$

i.e.,
$$2I = \int_0^{\pi/2} [2(\log \sin x + \log \cos x) - 2\log \sin 2x]\, dx$$

$\therefore$
$$I = \int_0^{\pi/2} [\log(\sin x \cos x) - \log \sin 2x]\, dx$$
$$= \int_0^{\pi/2} \left(\log \frac{2\sin x \cos x}{2} - \log \sin 2x\right) dx$$
$$= \int_0^{\pi/2} \left(\log \frac{\sin 2x}{2} - \log \sin 2x\right) dx$$
$$= \int_0^{\pi/2} (\log \sin 2x - \log 2 - \log \sin 2x)\, dx$$
$$= \int_0^{\pi/2} (-\log 2)\, dx = -\log 2\, \left[x\right]_0^{\pi/2}$$
$$= -\log 2\left(\frac{\pi}{2} - 0\right) = -\frac{\pi}{2}\log 2.$$

(*ii*) Let
$$I = \int_0^{\pi/2} \log(\sin x)\, dx \quad ...(1)$$

Also,
$$I = \int_0^{\pi/2} \log\left[\sin\left(\frac{\pi}{2} - x\right)\right] dx \qquad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

i.e.,
$$I = \int_0^{\pi/2} \log(\cos x)\, dx \quad ...(2)$$

Adding (1) and (2), we get
$$2I = \int_0^{\pi/2} [\log(\sin x) + \log(\cos x)]\, dx$$
$$= \int_0^{\pi/2} \log(\sin x \cos x)\, dx = \int_0^{\pi/2} \log\left(\frac{2\sin x \cos x}{2}\right) dx$$
$$= \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right) dx = \int_0^{\pi/2} \log(\sin 2x)\, dx - \int_0^{\pi/2} (\log 2)\, dx$$

Put $2x = t$ in the first integral so that $dx = \frac{1}{2}\, dt$

Now when $x = 0$, $t = 0$ and when $x = \frac{\pi}{2}$, $t = \pi$

$$\therefore \quad 2I = \frac{1}{2}\int_0^{\pi} \log \sin t \, dt - \log 2 \int_0^{\pi/2} dx$$

$$= \frac{1}{2}\int_0^{\pi} \log \sin x \, dx - \log 2 \, [x]_0^{\pi/2} \qquad \left[\because \int_a^b f(x)\,dx = \int_a^b f(t)\,dt\right]$$

$$= \frac{1}{2} \times 2 \int_0^{\pi/2} \log \sin x \, dx - \log 2 \left(\frac{\pi}{2} - 0\right)$$

$$= \int_0^{\pi/2} \log \sin x \, dx - \frac{\pi}{2}\log 2 = I - \frac{\pi}{2}\log 2$$

$$\therefore \quad \mathbf{I = -\frac{\pi}{2}\log 2}.$$

(iii) Let $I = \int_0^{\pi} \log(1 + \cos x)\,dx$

$$= \int_0^{\pi} \log\left(2\cos^2 \frac{x}{2}\right) dx \qquad \left[\because 1 + \cos x = 2\cos^2 \frac{x}{2}\right]$$

$$= \int_0^{\pi} \left[\log 2 + \log\left(\cos \frac{x}{2}\right)^2\right] dx$$

$$= \int_0^{\pi} \left[\log 2 + 2\log\left(\cos \frac{x}{2}\right)\right] dx$$

$$= \int_0^{\pi} \log 2 \, dx + 2\int_0^{\pi} \log \cos \frac{x}{2} \, dx$$

$$= \log 2 \, [x]_0^{\pi} + 2\int_0^{\pi/2} \log(\cos t) \,.\, 2dt \qquad \left[\text{Putting } \frac{x}{2} = t \text{ so that } dx = 2dt\right]$$

$$= \log 2 \, [\pi - 0] + 4\int_0^{\pi/2} \log(\cos t)\,dt$$

$$= \pi \log 2 + 4\int_0^{\pi/2} \log \cos\left(\frac{\pi}{2} - t\right) dt$$

$$= \pi \log 2 + 4\int_0^{\pi/2} \log(\sin t)\,dt$$

$$= \pi \log 2 + 4\left[-\frac{\pi}{2}\log 2\right] \qquad \text{[See Example 5 (ii)]}$$

$$= \pi \log 2 - 2\pi \log 2 = \mathbf{-\pi \log 2.}$$