NNTI. A2

By. Honglu Ma #7055053
and Camilo Martínez Martínez #7057573

Ex 1

a) $A = UDV^T$

$\Rightarrow A^TA = (UDV^T)^T UDV^T$

$\Rightarrow A^TA = VD^TU^TUDV^T$

$\Rightarrow A^TA = VD^TDV^T$ ①

A^TA is symmetric

$\Rightarrow A^TA = V\Lambda V^T$ where $\Lambda = \operatorname{diag} \lambda$ ②

① & ② $\Rightarrow D^TD = \operatorname{diag}\lambda$

$\Rightarrow \operatorname{diag} \sigma^2 = \operatorname{diag}\lambda$ □

b)

$\det(A - \lambda I) = 0$

$\Rightarrow \det\left(\begin{bmatrix} 3-\lambda & 1 \\ 0 & 4-\lambda \end{bmatrix}\right) = 0$

$\Rightarrow (3-\lambda)(4-\lambda) = 0$

$\Rightarrow \lambda = 3$ or 4

when $\lambda = 3$

$\begin{bmatrix} 3-3 & 1 \\ 0 & 4-3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix}$ for $t \in R$

eigenvector: $\begin{bmatrix} \pm 1 \\ 0 \end{bmatrix}$

when $\lambda = 4$

$\begin{bmatrix} 3-4 & 1 \\ 0 & 4-4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

let $y = t$, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

for $t \in R$

eigenvector: $\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ or $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$

c) Let λ be an eigenvalue of AB, by definition

$\Rightarrow ABv = \lambda v$ for some vector v,

let $v' = Bv$

$BAv' = B(ABv) = B(\lambda v) = \lambda Bv = \lambda v'$

thus λ is an eigenvalue of BA.

Ex 2. a) $f(x) = w^Tx = \sum_{i=1}^{n} w_i x_i$ where w_i are elements of w

and x_i are elements of x, $1 \le i \le n$

$\frac{\partial f}{\partial x_i} = w_i \Rightarrow \nabla_x f(x) = w$

b) $f(x) = x^T A x = \sum_{j=1}^{n} \left(\left(\sum_{i=1}^{n} a_{ij} x_i \right) x_j \right)$

the terms contains x_i in f are $a_{ii} x_i^2 + \sum_{k=1, k \neq i}^{n} a_{ik} x_i x_k + \sum_{l=1, l \neq i}^{n} a_{li} x_l x_i$

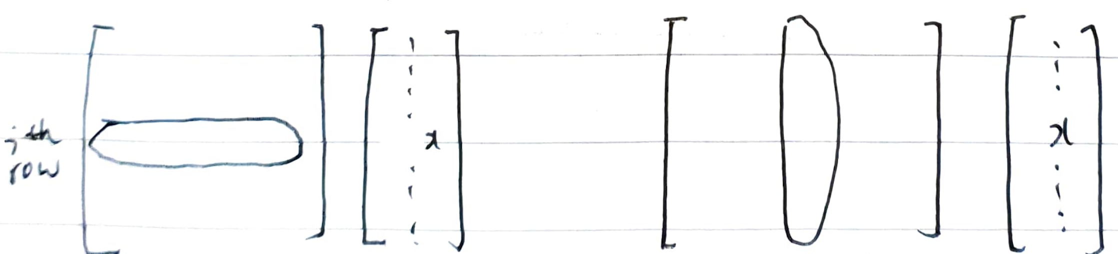
$$= a_{ii} x_i^2 + \sum_{j=1, j \neq i}^{n} (a_{ij} + a_{ji}) x_i x_j$$

So $\frac{\partial f}{\partial x_i} = 2 a_{ii} x_i + \sum_{j=1, j \neq i}^{n} (a_{ij} + a_{ji}) x_j$

$$= 2 a_{ii} x_i + \sum_{k=1, k \neq i}^{n} a_{ik} x_k + \sum_{l=1, l \neq i}^{n} a_{li} x_l$$

$$= \left(a_{ii} x_i + \sum_{k=1, k \neq i}^{n} a_{ik} x_k \right) + \left(a_{ii} x_i + \sum_{l=1, l \neq i}^{n} a_{li} x_l \right)$$

$$= \underbrace{\sum_{p=1}^{n} a_{ip} x_p} + \underbrace{\sum_{q=1}^{n} a_{qi} x_q}$$



dot product of the ith row in A with x,

jth column.
dot product of the ith column in A with x

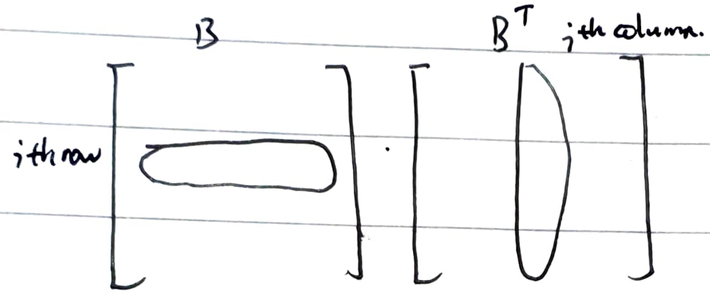
thus $= Ax + A^T x$

c) $f(x) = \|Bx\|_2^2 = \sum_{j=1}^{n} \sum_{i=1}^{n} (b_{ij})^2 x_i^2$

$= x_i^2 \, (b_{i1}^2 + b_{i2}^2 + \cdots + b_{in}^2)$

$$\frac{\partial f}{\partial x_i} = 2x_i \underbrace{\left(\sum_{j=1}^{n} b_{ij}^2 \right)}$$

this is the same as ith row in B dot product with itself so



thus $=$ $2(B^TB)x$

d). $f(x) = \|Bx - c\|_2^2 = \sum_{j=1}^{n} \sum_{i=1}^{n} (b_{ij} x_i - c_i)^2$

$= \sum_{j=1}^{n} \sum_{i=1}^{n} (b_{ij}^2 x_i^2 - 2 b_{ij} x_i c_i + c_i^2)$

$$\frac{\partial f}{\partial x_i} = \underbrace{2x_i \left(\sum_{j=1}^{n} b_{ij}^2 \right)}_{2(B^TB)x_i} - 2 \underbrace{\sum_{j=1}^{n} b_{ij} c_i}$$

similarily, this is the ith row in B

$= B^T c_i$

thus $= 2(B^TB)x - 2B^T c$

$= 2B^T(Bx - c)$